# JUDICIOUS PARTITIONS OF GRAPHS AND HYPERGRAPHS 

A Thesis<br>Presented to<br>The Academic Faculty

by
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In Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the School of Mathematics

Georgia Institute of Technology
August 2011

## JUDICIOUS PARTITIONS OF GRAPHS AND HYPERGRAPHS

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To my parents and Gigi,
for their support and love

## ACKNOWLEDGEMENTS

First of all, I would like to express my sincere gratitude to my advisor Professor Xingxing Yu for his enthusiasm, patience and encouragement. Throughout my Ph.D. studies, he has provided knowledgable mentorship and his sincere suggestions have inspired me, not only in mathematics, but also in becoming a better person. I could not have imagined having a better advisor and mentor.

I would like to thank Professors Asaf Shapira, Prasad Tetali, Robin Thomas and Eric Vigoda for being members of my committee. I would like to convey my sincere thanks to Professors Asaf Shapira and Robin Thomas for their insightful instruction and inspiring lectures which broadened my research scope. I am also grateful to Professors Wenan Zang and Ken-ichi Kawarabayashi for giving me the opportunities to visit the University of Hong Kong and the National Institute of Informatics of Japan. Finally, I want to give my special thanks to Professors Béla Bollobás and Benny Sudakov for their help and encouragement.

It is my pleasure to thank Professor Luca Dieci for recruiting me to Georgia Tech and providing help whenever needed. I also would like to thank Klara Grodzinsky and Cathy Jacobson for their help on my teaching and language skills throughout the years.

I am indebted to many of my colleagues for helpful discussions and for providing a stimulating and fun environment in which to learn and grow. I am especially grateful to Arash Asadi, Hao Deng, Sarah Fletcher, Lu Nan, Lei Wang, Benjamin Webb, Paul Wollan, Hua Xu, Tianjun Ye and Pei-Lan Yen.

Lastly and most importantly, I wish to thank my parents, Guoqi Ma and Meifen Yao, for giving birth to me in the first place, and my girlfriend, Gigi Song, for always guiding me to stay on task as well as her daily care and encouragement. Without their continuous support, it would have been impossible to achieve my goals. To them, I dedicate this thesis.

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## SUMMARY

Classical partitioning problems, like the Max-Cut problem, ask for partitions that optimize one quantity, which are important to such fields as VLSI design, combinatorial optimization, and computer science. Judicious partitioning problems on graphs or hypergraphs ask for partitions that optimize several quantities simultaneously. In this dissertation, we work on judicious partitions of graphs and hypergraphs, and solve or asymptotically solve several open problems of Bollobás and Scott on judicious partitions, using the probabilistic method and extremal techniques.

We establish a conjecture of Bollobás and Scott in [12], by showing that: for any integer $k \geq 2$ and any hypergraph $G$ with $m_{i}$ edges of size $i, i=1,2$, there is a partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k, V_{i}$ contains at most $m_{1} / k+m_{2} / k^{2}-o\left(m_{2}\right)$ edges. This is best possible since the expected bound in a random partition is $m_{1} / k+m_{2} / k^{2}$. We also prove that: for integer $k \geq 3$, any hypergraph with $m_{i}$ edges of size $i, i=1,2$, has a partition $V_{1}, \ldots, V_{k}$ such that each $V_{i}$ meets at least $m_{1} / k+m_{2} /(k-1)-o\left(m_{2}\right)$ edges. This result implies for large graphs the conjecture of Bollobás and Scott [9] that every graph with $m$ edges admits a partition $V_{1}, \ldots, V_{k}$ such that each $V_{i}$ meets at least $2 m /(2 k-1)$ edges. For $k=2$, we prove that $V(G)$ admits a partition into two sets each meeting at least $m_{1} / 2+3 m_{2} / 4-o\left(m_{2}\right)$ edges, which solves a special case of a more general problem of Bollobás and Scott in [12].

Bollobás and Scott [12] asked for the smallest $f(k, m)$ such that for any integer $k \geq 2$ and any graph $G$ with $m$ edges, there is a partition $V(G)=\bigcup_{i=1}^{k} V_{i}$ such that for $1 \leq i \neq j \leq k$, $e\left(V_{i} \cup V_{j}\right) \leq f(k, m)$. They conjectured that $f(k, m) \leq \frac{12 m}{(k+1)(k+2)}+O(n)$ for general graphs, and $f(k, m) \leq \frac{12 m}{(k+1)(k+2)}$ for dense graphs. We obtain a general bound on $f(k, m)$, and prove conjecture for dense graphs and for $k=3,4,5$ asymptotically.

We also work on a long standing conjecture of Bollobás and Thomason (see $[7,9,11$, 12]): for any integer $r \geq 3$, the vertex set of any $r$-uniform hypergraph with $m$ edges admits a partition $V_{1}, \ldots, V_{r}$ such that for $i=1, \ldots, r$, each $V_{i}$ meets at least $\frac{r}{2 r-1} m$ edges. We prove the bound $0.65 m-o(m)$ for $r=3$, which for large graph, is better than $0.6 m$ suggested by this conjecture.

## CHAPTER I

## INTRODUCTION

We study judicious partitioning problems on graphs and hypergraphs. We solve or asymptotically solve several open problems of Bollobás and Scott on judicious partitions, using probabilistic method and extremal techniques. In this chapter we provide notation and terminology necessary for the subsequent chapters.

### 1.1 Notation

Let $G$ be a graph or hypergraph, and let $S \subseteq V(G)$. We use $G[S]$ to denote the subgraph of $G$ consisting of $S$ and all edges of $G$ with all incident vertices in $S$. Let $A, B$ be subsets of $V(G)$ or subgraphs of $G$, we use $(A, B)$ to denote the set of edges of $G$ that have incident vertices in both $A$ and $B$. For an edge (or hyperedge) $e$ of $G$, we use $V(e)$ to denote the set of incident vertices of $e$. We write $e_{G}(S):=|\{e \in E(G): V(e) \subseteq S\}|, e_{G}(S, T):=\mid\{e \in E(G)$ : $V(e) \cap S \neq \emptyset \neq V(e) \cap T\} \mid$ for any $T \subseteq V(G)$, and $d_{G}(S):=|\{e \in E(G): V(e) \cap S \neq \emptyset\}|$. When understood, the reference to $G$ in the subscript may be dropped. Let $k \geq 2$ be an integer, a $k$-partition of $V(G)$ is a collection of subsets of $V(G), V_{1}, V_{2}, \ldots, V_{k}$, such that $V_{1} \cup V_{2} \cup \ldots \cup V_{k}=V(G)$ and $V_{i} \cap V_{j}=\emptyset$ for any $1 \leq i<j \leq k$. We use $b(G)$ to denote the maximum number of edges in a bipartite subgraph of $G$.

We will also prove several results for weighted graphs. Let $G$ be a graph and let $w$ : $V(G) \cup E(G) \rightarrow \mathbf{R}^{+}$, where $\mathbf{R}^{+}$represents the nonnegative reals. For $S \subseteq V(G)$ we write

$$
w_{G}(S)=\sum_{u \in S} w(u)+\sum_{\{e \in E(G): V(e) \subseteq S\}} w(e)
$$

and

$$
\tau_{G}(S)=\sum_{u \in S} w(u)+\sum_{\{e \in E(G): V(e) \cap S \neq 0\}} w(e) .
$$

If $G$ is understood, we use $\tau(S), w(S)$ instead of $\tau_{G}(S), w_{G}(S)$, respectively. We point out that if $H$ is an induced subgraph of $G$, then for any $S \subseteq V(H)$, we have $w_{H}(S)=w_{G}(S)$. Also, note that when $w(e)=1$ for all $e \in E(G)$ and $w(v)=0$ for all $v \in V(G)$, we have $w(S)=e(S)$ and $\tau(S)=d(S)$.

We will use the standard notation of probability theory. Given a sample space, let $X$ be a random variable and $A$ be an event. We use $\mathbb{P}(A)$ to denote the probability that $A$ occurs, $\mathbb{E}(X)$ to denote the expectation of random variable $X$, and $\mathbb{E}(X \mid A)$ to denote the expectation of $X$ conditional on $A$.

### 1.2 Background

Classical graph partitioning problems often ask for partitions of a graph that optimize a single quantity. For example, the well-known Max-Cut Problem asks for a partition $V_{1}, V_{2}$ of $V(G)$, where $G$ is a weighted graph, that maximizes the total weight of edges with one end in each $V_{i}$. This problem is NP-hard, see [29]. It is shown [6] that it is also NP-hard to approximate the Max-Cut problem on cubic graphs beyond the ratio of 0.997 . However, the Max-Cut problem is polynomial time solvable for planar graphs, see [25,36]. Goemans and Williamson [24] used semidefinite programming and hyperplane rounding to give a randomized algorithm with expected performance guarantee of 0.87856 . Feige, Karpinski and Langberg [22] gave a similar randomized algorithm that improves this bound to 0.921 for subcubic graphs; a graph is called subcubic if it has maximum degree at most three.

The unweighted version of Max-cut problem is often called the Maximum Bipartite Subgraph Problem: Given a graph $G$, find a partition $V_{1}, V_{2}$ of $V(G)$ that maximizes $e\left(V_{1}, V_{2}\right)$, the number of edges with one end in each $V_{i}$. This is also NP-hard, see [21,23]. Moreover, Yannakakis [49] showed that the Maximum Bipartite Subgraph Problem is NPhard even when restricted to triangle-free cubic graphs.

However, it is easy to prove that any graph with $m$ edges has a partition $V_{1}, V_{2}$ with
$e\left(V_{1}, V_{2}\right) \geq m / 2$ : if one randomly picks a partition $U_{1}, U_{2}$, the probability of any edge belongs to $\left(U_{1}, U_{2}\right)$ is exactly $1 / 2$, therefore $\mathbb{E}\left(e\left(U_{1}, U_{2}\right)\right)=m / 2$ and the conclusion follows. Edwards [17, 18] improved the lower bound to $m / 2+\frac{1}{4}(\sqrt{2 m+1 / 4}-1 / 2)$. This is best possible, as $K_{2 n+1}$ are extremal graphs. Alon [1] showed that for infinite many integers $m$, there exist graphs $G_{m}$ such that $b\left(G_{m}\right) \geq m / 2+\frac{\sqrt{2 m}}{4}+\Theta\left(m^{1 / 4}\right)$, where $e\left(G_{m}\right)=m$, confirming a conjecture of Erdős in [20] that the gap between Edwards' bound and the truth can be arbitrary large. (Recall that $b(G)$ is the maximum number of edges in a bipartite subgraph of $G$.)

This lower bound may be improved by forbidding a fixed graph. For example, Erdős and Lovász (see [19]), Poljak and Tuza [37] and Shearer [43] made progress on improving the lower bound for triangle-free graphs. Alon [1] finally showed that $b(G) \geq m / 2+\Theta\left(m^{4 / 5}\right)$ for any triangle-free graph $G$ with $m$ edges, which is tight up to constant. For general H free graphs, the Maximum Bipartite Subgraph Problem is studied in [4], i.e. $H$ is an even cycle or a graph obtained by connecting a single vertex to all vertices of a fixed forest. But the main term of the best lower bound of $b(G)$ is still $m / 2$, for $H$-free graph $G$ with $m$ edges, where $H$ is triangle or one of the graphs studied in [4].

For some classes of graphs, the main term of the lower bound can exceed $|E(G)| / 2$. Erdős [19] proved that if $G$ is $2 k$-colorable then $b(G) \geq \frac{k}{2 k-1}|E(G)|$. As a consequence, if $G$ is a graph with bounded maximum degree, then the lower bound can exceed $|E(G)| / 2$. In particular, Erdős' result implies that $b(G) \geq \frac{2}{3}|E(G)|$ for cubic graph $G$. Locke [31] and Stanton [44] showed that $b(G) \geq \frac{7}{9}|E(G)|$ if $G$ is cubic and $G$ is not $K_{4}$. Hopkins and Stanton [28] showed that $b(G) \geq \frac{4}{5}|E(G)|$ if $G$ is triangle-free cubic graph. More discussion on cubic (or subcubic) triangle-free graphs can be found in [12, 16, 35, 46, 48].

The Maximum Bipartite Subgraph Problem for integer weighted graphs also have been studied in [3] by N. Alon and E. Halperin. For other subsequent work of the Maximum Bipartition Subgraph Problem, we refer the reader to $[30,38,45]$.

In practice one often needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called Judicious Partitioning Problems by Bollobás and Scott [8]. One such example is the problem of finding a partition $V_{1}, V_{2}$ of the vertex set of a graph $G$ that minimizes $\max \left\{e\left(V_{1}\right), e\left(V_{2}\right)\right\}$, or equivalently, maximizes $\min \left\{d\left(V_{1}\right), d\left(V_{2}\right)\right\}$ (since $d\left(V_{i}\right)=e(G)-e\left(V_{3-i}\right)$ for $i=1,2$ ). This problem is also known as the Bottleneck Bipartition Problem, raised by Entringer (see, for example, [39, 40]). Shahrokhi and Székely [42] showed that this problem is also NP-hard. Porter [39] proved that any graph with $m$ edges has a partition of its vertex set into $V_{1}, V_{2}$ with $e\left(V_{i}\right) \leq m / 4+O(\sqrt{m})$ for $i=1,2$. Bollobás and Scott [10] improved this bound by proving

Theorem 1.2.1. (Bollobás and Scott [10]) For any graph $G$ with $m$ edges, there exists a bipartition $V_{1}, V_{2}$ of $V(G)$ such that for $i=1,2$

$$
e\left(V_{i}\right) \leq \frac{m}{4}+\frac{1}{8}(\sqrt{2 m+1 / 4}-1 / 2)
$$

They also showed that the complete graphs $K_{2 n+1}$ are the only extremal graphs (modulo isolated vertices).

Bollobás and Scott [10] further proved that for any integer $k \geq 1$ and any graph $G$ with $m$ edges, $V(G)$ has a $k$-partition $V_{1}, \ldots, V_{k}$ such that

$$
e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}(\sqrt{2 m+1 / 4}-1 / 2)
$$

for $i \in\{1,2, \ldots, k\}$. The complete graphs of order $k n+1$ are the only extremal graphs (modulo isolated vertices).

In fact, Bollobás and Scott [10] proved an even stronger result that any graph with $m$ edges has a partition $V_{1}, V_{2}$ of its vertex set such that

$$
e\left(V_{1}, V_{2}\right) \geq \frac{m}{2}+\frac{1}{4}(\sqrt{2 m+1 / 4}-1 / 2)
$$

and for $i=1,2$,

$$
e\left(V_{i}\right) \leq \frac{m}{4}+\frac{1}{8}(\sqrt{2 m+1 / 4}-1 / 2)
$$

Xu and Yu [47] recently generalized this result to $k$-partitions: any graph with $m$ edges has a $k$-partition $V_{1}, \ldots, V_{k}$ of its vertex set such that the number of edges whose incident vertices are not in the same set

$$
e\left(V_{1}, V_{2}, \ldots, V_{k}\right) \geq \frac{k-1}{k} m+\frac{1}{2 k}(\sqrt{2 m+1 / 4}-1 / 2)
$$

and for $i \in\{1,2, \ldots, k\}$,

$$
e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}(\sqrt{2 m+1 / 4}-1 / 2) .
$$

Alon et al. [2] showed that there is a connection between the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem. More precisely, they proved the following: Let $G$ be a graph with $m$ edges and largest cut of size $m / 2+\delta$. If $\delta \leq m / 30$ then $V(G)$ admits a partition $V_{1}, V_{2}$ such that for $i=1,2$,

$$
e\left(V_{i}\right) \leq m / 4-\delta / 2+10 \delta^{2} / m+3 \sqrt{m} ;
$$

and if $\delta \geq m / 30$ then $V(G)$ admits a partition $V_{1}, V_{2}$ such that for $i=1,2$,

$$
e\left(V_{i}\right) \leq m / 4-m / 100 .
$$

Bollobás and Scott [15] recently extended this result to $k$-partitions: there is also a connection between the generalized "Maximum $k$-Partite Subgraph Problem" and the generalized "Bottleneck $k$-Partition Problem".

In their paper $[7,12,13,41]$, Bollobás and Scott studied $k$-partitions $V_{1}, \ldots, V_{k}$ in a graph or hypergraph that minimize $\max \left\{e\left(V_{1}\right), e\left(V_{2}\right), \ldots, e\left(V_{k}\right)\right\}$, or minimize $\max \left\{e\left(V_{i} \cup V_{j}\right): 1 \leq\right.$ $i<j \leq k\}$, or maximize $\min \left\{d\left(V_{1}\right), d\left(V_{2}\right), \ldots, d\left(V_{k}\right)\right\}$. We have seen that when $k=2$, minimizing $\max \left\{e\left(V_{1}\right), e\left(V_{2}\right)\right\}$ is equivalent to maximizing $\min \left\{d\left(V_{1}\right), d\left(V_{2}\right)\right\}$. However, when $k \geq 3$, minimizing $\max \left\{e\left(V_{1}\right), e\left(V_{2}\right), \ldots, e\left(V_{k}\right)\right\}$ is very different from maximizing $\min \left\{d\left(V_{1}\right), d\left(V_{2}\right), \ldots, d\left(V_{k}\right)\right\}$. These problems become more difficult if one imposes restrictions on the sizes of $V_{i}, 1 \leq i \leq k$; for example, we have the Balanced Bipartition Problem when $k=2$ and $\left|\left|V_{1}\right|-\left|V_{2}\right|\right| \leq 1$. For more problems and references, we refer the reader to [12-14,41].

### 1.3 Problems and results

We discuss several judicious partitioning problems which we are interested in and present our results to those problems in this section. In Section 1.3.1, we discuss several judicious partitioning problems about graphs with requirement on edges as well as on vertices. In Section 1.3.2, we consider judicious partitioning problems for bounding the size of all pairs in a $k$-partition of a graph. In Section 1.3.3, we focus a long standing conjecture of Bollobás and Thomason on 3-uniform hypergraphs. Our results on those problems can be found in [32-34].

### 1.3.1 Hypergraphs with edge size at most 2

We discuss several judicious partitioning problems about graphs with requirement on edges as well as on vertices, and such problems are called mixed partitioning problems. We follow Bollobás and Scott [12] to use the term hypergraphs with edge size at most 2 .

Our first result is

Theorem 1.3.1. If $G$ is a hypergraph with $m_{i}$ edges of size $i, i=1,2$, then $V(G)$ admits a partition $V_{1}, V_{2}$ such that for $i=1,2$

$$
d\left(V_{i}\right) \geq m_{1} / 2+3 m_{2} / 4+o\left(m_{2}\right) .
$$

Bollobás and Scott [12] suggested the lower bound $\left(m_{1}-1\right) / 2+2 m_{2} / 3$ as a starting point for a more general problem, and Theorem 1.3.1 verifies this for large graphs. Note that if we take a partition $V_{1}, V_{2}$ randomly and uniformly, then $\mathbb{E}\left(d\left(V_{i}\right)\right)=m_{1} / 2+3 m_{2} / 4$.

Next we attempt to generalize Theorem 1.3.1 to $k$-partitions. In particular, we prove

Theorem 1.3.2. Let $k \geq 3$ be an integer and let $G$ be a hypergraph with $m_{i}$ edges of size $i$, $i=1,2$. Then there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k$,

$$
d\left(V_{i}\right) \geq \frac{m_{1}}{k}+\frac{m_{2}}{k-1}+o\left(m_{2}\right) .
$$

Note, if we take a $k$-partition $V_{1}, V_{2}, \ldots, V_{k}$ randomly and uniformly, then $\mathbb{E}\left(d\left(V_{i}\right)\right)=m_{1} / k+$ $(2 k-1) m_{2} / k^{2}$. Theorem 1.3.2 implies the following conjecture of Bollobás and Scott [11] for graphs with sufficiently many edges

Conjecture 1.3.3. (Bollobás and Scott [11]) Every graph with m edges has a partition into $k$ sets, each meeting at least $2 m /(2 k-1)$ edges.

We also consider a generalization of the Bottleneck Bipartition Problem to hypergraphs. We have

Theorem 1.3.4. Let $G$ be a hypergraph with $m_{i}$ edges of size $i, i=1,2$. Then for any integer $k \geq 1$, there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k$,

$$
e\left(V_{i}\right) \leq \frac{m_{1}}{k}+\frac{m_{2}}{k^{2}}+o\left(m_{2}\right) .
$$

Note that for a random $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$, we have $\mathbb{E}\left(e\left(V_{i}\right)\right)=m_{1} / k+m_{2} / k^{2}$. In its special case, when $m_{1}=o\left(m_{2}\right)$, Theorem 1.3.4 follows from Eq. 2 in [12]. Theorem 1.3.4 establishes a conjecture of Bollobás and Scott [12] for large graphs that: any hypergraph with $m_{i}$ edges of size $i, i=1,2$, admits a $k$-partition $V_{1}, \ldots, V_{k}$ such that for $i=1, \ldots, k$,

$$
e\left(V_{i}\right) \leq \frac{m_{1}}{k}+\frac{m_{2}}{\binom{k+1}{2}}+O(1) .
$$

In Chapter 2, we will prove weighted versions of Theorem 1.3.1,Theorem 1.3.2 and Theorem 1.3.4.

### 1.3.2 Bounds for pairs in partitions of graphs

The following judicious partitioning problem is proposed in [12]:

Problem 1.3.5. (Bollobás and Scott [12]) What is the smallest $f(k, m)$ such that for any integer $k \geq 2$, every graph $G$ with $m$ edges has a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i<j \leq k, e\left(V_{i} \cup V_{j}\right) \leq f(k, m)$ ?

Note that the case $k=2$ is trivial. For $k=3$, we see that for each permutation $i j k$ of $\{1,2,3\}, d\left(V_{i}\right)=m-e\left(V_{j} \cup V_{k}\right)$; so Problem 1.3.5 asks for a lower bound on $\min \left\{d\left(V_{i}\right): i=\right.$ $1,2,3\}$, and hence Theorem 1.3.2 provides an upper bound on $f(3, m)$. For $k \geq 4$, bounding $\max \left\{e\left(V_{i} \cup V_{j}\right): 1 \leq i<j \leq k\right\}$ is much more difficult than bounding $\max \left\{e\left(V_{i}\right): 1 \leq i \leq k\right\}$; in the former case one needs to bound $\binom{k}{2}$ quantities, while in the latter case one only needs to bound $k$ quantities.

We prove the following general bound on $f(k, m)$ :

Theorem 1.3.6. For any integer $k \geq 3, f(k, m)<1.6 m / k+o(m)$, and $f(k, m)<1.5 m / k+$ $o(m)$ for $k \geq 23$.

We now show that $f(k, m) \geq m /(k-1)$, which is close to $1.6 m / k$ when $k$ is small. For $k \geq 3$, take the graph $K_{1, n}$ with $n \geq k-1$, and let $x$ be the vertex of degree $n$. Let $V_{1}, \ldots, V_{k}$ be a $k$-partition of $V(G)$ with $x \in V_{1}$. Without loss of generality, we may assume that $\left|V_{2}\right| \geq\left(n+1-\left|V_{1}\right|\right) /(k-1)$. Now $e\left(V_{1} \cup V_{2}\right) \geq\left(n+1-\left|V_{1}\right|\right) /(k-1)+\left(\left|V_{1}\right|-1\right)=$ $\left(n+(k-2)\left(\left|V_{1}\right|-1\right)\right) /(k-1) \geq n /(k-1)=m /(k-1)$, where $m=n$ is the number of edges in $K_{1, n}$.

The complete graph $K_{k+2}$ has $m=\binom{k+2}{2}$ edges, and any $k$-partition $V_{1}, \ldots, V_{k}$ of $V\left(K_{k+2}\right)$ has two sets, say $V_{1}, V_{2}$, such that $\left|V_{1} \cup V_{2}\right|=4$. So $e\left(V_{1} \cup V_{2}\right)=6=\frac{12 m}{(k+1)(k+2)}$. This shows that $f(k, m) \geq \frac{12 m}{(k+1)(k+2)}$. For large $n$, a simple counting shows that for any $k$-partition $V_{1}, \ldots, V_{k}$ of $V\left(K_{n}\right), k \geq 2$, there exist $V_{i}, V_{j}$ such that $\left|V_{i}\right|+\left|V_{j}\right| \geq 2 n / k$, and hence $e\left(V_{i} \cup V_{j}\right) \geq$ $\binom{2 n / k}{2}$. From this, we deduce that $f(k, m) \geq 4 m / k^{2}+O(n)$, and this bound is achieved by taking a balanced $k$-partition of $V\left(K_{n}\right)$ (i.e., any two partition sets differ in size by at most one).

The consideration of $K_{1, n}$ and $K_{k+2}$ lead Bollobás and Scott [12] to the following conjecture. Note that $K_{1, n}$ is sparse, i.e. the number of edges is $O(n)$.

Conjecture 1.3.7. (Bollobás and Scott [12]) For each integer $k \geq 2$, every graph $G$ with $m$
edges and $n$ vertices has a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i<j \leq k$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{12 m}{(k+1)(k+2)}+O(n)
$$

Conjecture 1.3.7 is trivial for $k=2$, as the bound becomes $m+O(n)$. For $k=3$, Conjecture 1.3.7 is equivalent to the following problem: Find a partition $V(G)=V_{1} \cup V_{2} \cup V_{3}$ so that $d\left(V_{i}\right) \geq 2 m / 5+O(n)$. We point out that Theorem 1.3.2 implies $d\left(V_{i}\right) \geq m / 2+o(m)$; therefore Conjecture 1.3.7 holds for $k=3$ and large $m$.

We show that Conjecture 1.3.7 holds for dense graphs as well:

Theorem 1.3.8. Let $k \geq 2$ be an integer and let $\epsilon>0$. If $G$ is a graph with $m$ edges and $\delta(G) \geq \epsilon n$, then there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i<j \leq k$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{4}{k^{2}} m+o_{\epsilon}(m)
$$

Note that the main term $4 m / k^{2}$ is tight because of the complete graphs $K_{n}$. Theorem 1.3.8 implies the following conjecture of Bollobás and Scott [12] for large graphs.

Conjecture 1.3.9. (Bollobás and Scott [12]) For each $k \geq 2$ there is a constant $c_{k}>0$ such that if $G$ is a graph with $m$ edges, $n$ vertices, and minimum degree $\delta(G) \geq c_{k} n$, then there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i<j \leq k$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{12 m}{(k+1)(k+2)} .
$$

From Theorem 1.3.2, we have $f(3, m) \leq m / 2+o(m)$, which is less than $\frac{12 m}{(k+1)(k+2)}=\frac{3}{5} m$ for large $m$. We will show that $f(4, m) \leq m / 3+o(m)$, which is less than $\frac{12 m}{(k+1)(k+2)}=\frac{2}{5} m$ for large $m$. We will further show that $f(5, m) \leq 4 m / 15+o(m)$, which is less than $\frac{12 m}{(k+1)(k+2)}=\frac{2}{7} m$ for large $m$. Therefore, Conjecture 1.3.7 holds for dense graph as well as for $k=3,4,5$ and large $m$.

We also study the problem of finding a $k$-partitions $V_{1}, \ldots, V_{k}$ of $V(G)$ that satisfy bounds on both $\max \left\{e\left(V_{i}\right): 1 \leq i \leq k\right\}$ and $\max \left\{e\left(V_{i} \cup V_{j}\right): 1 \leq i<j \leq k\right\}$. It is proved in [10] that there exists a $k$-partition $V_{1}, \ldots, V_{k}$ of a graph with $m$ edges such that
$e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}(\sqrt{2 m+1 / 4}-1 / 2)$ for $1 \leq i \leq k$. Bollobás and Scott [12] asked whether it is possible to find a $k$-partition $V_{1}, \ldots, V_{k}$ such that $e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}(\sqrt{2 m+1 / 4}-1 / 2)$ for $1 \leq i \leq k$, and $e\left(V_{i} \cup V_{j}\right) \leq \frac{12 m}{(k+1)(k+2)}+O(n)$ for $1 \leq i<j \leq k$. We will show that for $k=3$ and $k=4$ one can find a partition satisfying these bounds asymptotically.

### 1.3.3 3-Uniform hypergraphs

If $V_{1}, V_{2}$ is a bipartition of a graph $G$ maximizing $e\left(V_{1}, V_{2}\right)$, then each $v \in V_{i}$ has at least as many neighbors in $V_{3-i}$ as in $V_{i}$. Summing over all vertices in $V_{i}$, we get $e\left(V_{1}, V_{2}\right) \geq 2 e\left(V_{i}\right)$ for $i=1,2$. Hence $e\left(V_{i}\right) \leq m / 3$, where $m$ is the number of edges in $G$, so $d\left(V_{i}\right) \geq m-m / 3=$ $2 m / 3$ for $i=1,2$.

In an attempt to extend the above to hypergraphs, Bollobás and Thomason made the following conjecture (see [7,9,11,12]), one of the early problems about judicious partitions.

Conjecture 1.3.10. (Bollobás and Thomason 1980s) For any integer $r \geq 3$, the vertex set of any r-uniform hypergraph with $m$ edges admits a $r$-partition $V_{1}, \ldots, V_{r}$ such that for $i=1, \ldots, r$,

$$
d\left(V_{i}\right) \geq \frac{r}{2 r-1} m
$$

The conjectured bound is best possible; the complete $r$-uniform hypergraphs on $2 r-1$ vertices are such extremal hypergraphs. To see this, note that such a hypergraph has $m=$ $\binom{2 r-1}{r}$ edges, and any $r$-partition of such a hypergraph has a partition set with just one vertex, which meets $\binom{2 r-2}{r-1}$ edges.

Bollobás, Reed and Thomason [7] proved that every 3-uniform hypergraph with $m$ edges has a partition $V_{1}, V_{2}, V_{3}$ such that $d\left(V_{i}\right) \geq(1-1 / e) m \approx 0.21 m$ (here $e$ is the base of the natural logarithm). In [11], this bound is improved to $(5 / 9) m$ by Bollobás and Scott using the following approach: find a reasonable partition, and remove vertices of one set and try to partition the remaining vertices into $r-1$ parts in a better way. They [11] also proved a bound for general case: $d\left(V_{i}\right) \geq 0.27 m$ for any integer $r \geq 3$ and $1 \leq i \leq r$. Note that the bound for $r=3$ in Conjecture 1.3.10 is 0.6 m . Halesgrave [26] extended the idea
of Bollobás and Scott in [11] and solved the case $r=3$ completely. (Bollobás informed us that Halesgrave actually did it in 2006.) For large graphs, this bound may be improved. We prove the following result, which for large $m$ gives an even better bound than what Conjecture 1.3.10 suggests for $r=3$.

Theorem 1.3.11. Every 3-uniform hypergraph with $m$ edges has a 3-partition $V_{1}, V_{2}, V_{3}$ such that for $i=1,2,3$,

$$
d\left(V_{i}\right) \geq 0.65 m-o(m) .
$$

### 1.4 Azuma-Heoffding inequality

The approach we take is similar in spirit to that of Bollobás and Scott [9, 12]. First we partition a set of large degree vertices, then we establish a random process to partition the remaining vertices, and finally we apply a concentration inequality to bound the deviations. The key is to pick the probabilities appropriately so that the expectations of the process will be in a range that we want. This will be achieved by extremal techniques.

The concentration inequality we need is the Azuma-Heoffding inequality [5,27], which bounds deviations in a random process. We use the version given in [9].

Lemma 1.4.1. (Azuma-Heoffding Inequality) Let $Z_{1}, \ldots, Z_{n}$ be independent random variables taking values in $\{1, \ldots, k\}$, let $Z:=\left(Z_{1}, \ldots, Z_{n}\right)$, and let $f:\{1, \ldots, k\}^{n} \rightarrow \mathbf{N}$ such that $\left|f(Y)-f\left(Y^{\prime}\right)\right| \leq c_{i}$ for any $Y, Y^{\prime} \in\{1, \ldots, k\}^{n}$ which differ only in the ith coordinate. Then for any $z>0$,

$$
\begin{aligned}
& \mathbb{P}(f(Z) \geq \mathbb{E}(f(Z))+z) \leq \exp \left(\frac{-2 z^{2}}{\sum_{i=1}^{n} c_{i}^{2}}\right), \\
& \mathbb{P}(f(Z) \leq \mathbb{E}(f(Z))-z) \leq \exp \left(\frac{-2 z^{2}}{\sum_{i=1}^{n} c_{i}^{2}}\right) .
\end{aligned}
$$

Before applying Lemma 1.4.1, we fix a $k$-partition $V_{1}, V_{2}, \ldots, V_{k}$ of the large degree vertices. In the application of 3-uniform hypergraphs, this $k$-partition will be chosen to satisfy certain requirements. We then order the remaining vertices as $v_{1}, v_{2}, \ldots, v_{n}$, and design a random process to assign every $v_{i}$ to $V_{j}$ with probability $p_{i}^{j}$ independently, where $p_{i}^{j}$ will be
determined, $1 \leq i \leq n, 1 \leq j \leq k$. Then the choice of $v_{i}$ goves a random variable $Z_{i}$. The quantities we are interested in, numbers of edges with certain requirements, are functions of $Z=\left(Z_{1}, \ldots, Z_{n}\right)$, which satisfy the condition in Lemma 1.4.1, namely, $\left|f(Z)-f\left(Z^{\prime}\right)\right| \leq c_{i}$ for any $Z, Z^{\prime}$ differing only in the $i$ th coordinate $Z_{i}$, where $c_{i}$ is the degree of vertex $v_{i}$ in graph (or hypergrpah). This is because that if we change $Z_{i}$, i.e. the choice of vertex $v_{i}$, the edges affected are those incident with $v_{i}$; so the quantities change by at most the degree of $v_{i}$.

We have to make sure that those probabilities $p_{i}^{j}$ can be chosen such that our random process gives us the desired expectations for the quantities we care. This turns out to be quite difficult when dealing with several quantities. We will also make sure that we can pick an appropriate set of large degree vertices, so that $\sum_{i=1}^{n} c_{i}^{2}$ is of order $o\left(m^{2}\right)$, where $m$ is the number of edges. This will guarantee that after applying Lemma 1.4.1, $z$ can be chosen to be of order $o(m)$.

We organize the rest of this dissertation as follows. In Chapter 2, we prove Theorems 1.3.1, 1.3.2 and 1.3.4. Chapter 3 concentrates on the bounds for pairs in $k$-partitions of graphs, where we will prove Theorems 1.3.6 and 1.3.8. In Chapter 4, we focus on 3uniform hypergraphs and prove Theorem 1.3.11.

## CHAPTER II

## HYPERGRAPHS WITH EDGE SIZE AT MOST 2

There are three sections in this chapter. In Section 2.1, we prove Theorem 1.3.1. In Section 2.2, we prove Theorem 1.3.2. And in Section 2.3, we prove Theorem 1.3.4.

### 2.1 Bipartitions

In this section we consider the following problem of Bollobás and Scott [12]. Given a hypergraph $G$ with $m_{i}$ edges of size $i, 1 \leq i \leq 2$, does there exist a partition of $V(G)$ into sets $V_{1}$ and $V_{2}$ such that $d\left(V_{i}\right) \geq \frac{m_{1}-1}{2}+\frac{2}{3} m_{2}$ for $i=1,2$. This problem was motivated by Conjecture 1.3.10, the Bollobás-Thomason conjecture on $r$-uniform hypergraphs. Bollobás and Scott [12] proved that if $G$ is a hypergraph with $m_{i}$ edges of size $i, i=1, \ldots, k$, then $V(G)$ admits a partition $V_{1}, V_{2}$ such that for $i=1,2$,

$$
d\left(V_{i}\right) \geq \frac{m_{1}-1}{3}+\frac{2 m_{2}}{3}+\ldots+\frac{k m_{k}}{k+1}
$$

They then used this to show that every 3-uniform hypergraph with $m$ edges can be partitioned into three sets, each of which meets at least $\frac{5}{9} m$ edges.

In [11], Bollobás and Scott suggest that the following might hold. Given a hypergraph $G$ with $m_{i}$ edges of size $i, 1 \leq i \leq k$, there exists a partition of $V(G)$ into sets $V_{1}, V_{2}$ such that for $i=1,2$,

$$
d\left(V_{i}\right) \geq \frac{m_{1}-1}{2}+\frac{2 m_{2}}{3}+\ldots+\frac{k m_{k}}{k+1} .
$$

In fact, they suggest in [12] that asymptotically the bound may be much larger, i.e. for $i=1,2$,

$$
d\left(V_{i}\right) \geq \frac{1}{2} m_{1}+\frac{3}{4} m_{2}+\ldots+\left(1-\frac{1}{2^{k}}\right) m_{k}+o\left(m_{1}+\ldots+m_{k}\right) .
$$

In this section we confirm this for $k=2$ by proving Theorem 2.1.3. Note that by taking a random bipartition $V_{1}, V_{2}$, we have $\mathbb{E}\left(d\left(V_{i}\right)\right)=\frac{m_{1}}{2}+\frac{3}{4} m_{2}+\ldots+\left(1-\frac{1}{2^{k}}\right) m_{k}$.

We need a simple lemma to be used to pick probabilities in a random process.

Lemma 2.1.1. Let $a, b, n \in \mathbf{R}^{+}$with $a+b>0$. Then there exists $p \in[0,1]$ such that

$$
\min \{(n+b) p+a,(n+a)(1-p)+b\} \geq \frac{n}{2}+\frac{3}{4}(a+b) .
$$

Proof. Setting $(n+b) p+a=(n+a)(1-p)+b$, we obtain $p=\frac{n+b}{2 n+a+b}$ and

$$
(n+b) p+a=\frac{(n+b)^{2}}{2 n+a+b}+a .
$$

Clearly $p \in[0,1]$. It is straightforward to show that

$$
\frac{(n+b)^{2}}{2 n+a+b}+a-\left(\frac{n}{2}+\frac{3}{4}(a+b)\right)=\frac{(a-b)^{2}}{4(2 n+a+b)} \geq 0 .
$$

Hence, the assertion of the lemma holds.
Remark. Note that $p=\frac{n+b}{2 n+a+b}$ works for Lemma 2.1.1.
We now prove the main result in this section. Recall the notation $\tau(X)$.

Theorem 2.1.2. Let $G$ be a graph with $n$ vertices and $m$ edges and let $w: V(G) \cup E(G) \rightarrow$ $\mathbf{R}^{+}$such that $w(e)>0$ for all $e \in E(G)$. Let $\lambda=\max \{w(x): x \in V(G) \cup E(G)\}, w_{1}=$ $\sum_{v \in V(G)} w(v)$, and $w_{2}=\sum_{e \in E(G)} w(e)$. Then there is a bipartition $X, Y$ of $V(G)$ such that

$$
\min \{\tau(X), \tau(Y)\} \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\lambda \cdot O\left(m^{4 / 5}\right) .
$$

Proof. We may assume that $G$ is connected, since otherwise we simply consider the individual components. Let $V(G)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq d\left(v_{2}\right) \geq \cdots \geq d\left(v_{n}\right)$.

First, we need to deal with an appropriate number of vertices so that the remaining vertices have small degree (and hence will be useful when applying the Azuma-Hoeffding inequality in Lemma 1.4.1). Since $G$ is connected, $n-1 \leq m<\frac{1}{2} n^{2}$. Fix $0<\alpha<\frac{1}{2}$ (to be optimized later), and let $V_{1}=\left\{v_{1}, \ldots, v_{t}\right\}$ such that $t=\left\lfloor\mathrm{cm}^{\alpha}\right\rfloor$, where $c$ is some constant and $c<\sqrt{2}$. (Note that, since $\alpha<1 / 2, c<\sqrt{2}$, and $m<\frac{1}{2} n^{2}$, we have $t<n$.) Then

$$
e\left(V_{1}\right) \leq\binom{ t}{2}<\frac{1}{2} t^{2} \leq \frac{1}{2} c^{2} m^{2 \alpha} .
$$

Since $\sum_{i=1}^{t+1} d\left(v_{i}\right) \leq 2 m$,

$$
d\left(v_{t+1}\right) \leq \frac{2 m}{t+1} \leq \frac{2}{c} m^{1-\alpha} .
$$

Let $V_{2}=V(G) \backslash V_{1}$, and rename the vertices in $V_{2}$ as $\left\{u_{1}, u_{2}, \ldots, u_{n-t}\right\}$ such that $e\left(\left\{u_{i}\right\}, V_{1} \cup\right.$ $\left.\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$; which can be done since we assume that $G$ is connected.

We now define a random process. First, fix an arbitrary partition $V_{1}=X_{0} \cup Y_{0}$, and assign color 1 to all vertices in $X_{0}$ and color 2 to all vertices in $Y_{0}$. The vertices $u_{i} \in V_{2}$ are independently colored 1 with probability $p_{i}$, and 2 with probability $1-p_{i}$. (The $p_{i}$ 's are constants to be determined recursively.) Let $Z_{i}$ denote the indicator random variable of the event of coloring $u_{i}$. Hence $Z_{i}=j, j \in\{1,2\}$, iff $u_{i}$ is assigned color $j$. When this process stops we obtain a bipartition of $V(G)$ into two sets $X, Y$, where $X$ consists of all vertices with color 1 and $Y$ consists of all vertices of color 2 (and hence $X_{0} \subseteq X$ and $Y_{0} \subseteq Y$ ).

We need additional notation to facilitate the choices of $p_{i}(1 \leq i \leq n-t)$, the computations of expectations of $\tau(X)$ and $\tau(Y)$, and the estimations of concentration bounds. Let $G_{i}=G\left[V_{1} \cup\left\{u_{1}, u_{2}, \ldots, u_{i}\right\}\right]$ for $i=1, \ldots, n-t$, let $G_{0}=G\left[V_{1}\right]$, and let the elements of $V\left(G_{i}\right) \cup E\left(G_{i}\right)$ inherit their weights from $G$. Let $x_{0}=\tau\left(X_{0}\right)$ and $y_{0}=\tau\left(Y_{0}\right)$, and define, for $i=1, \ldots, n-t$,

$$
\begin{aligned}
X_{i} & =\left\{\text { vertices of } G_{i} \text { with color } 1\right\}, \\
Y_{i} & =\left\{\text { vertices of } G_{i} \text { with color } 2\right\}, \\
x_{i} & =\tau_{G_{i}}\left(X_{i}\right), \\
y_{i} & =\tau_{G_{i}}\left(Y_{i}\right), \\
\Delta x_{i} & =x_{i}-x_{i-1}, \\
\Delta y_{i} & =y_{i}-y_{i-1}, \\
a_{i} & =\sum_{e \in\left(u_{i}, X_{i-1}\right)} w(e), \\
b_{i} & =\sum_{e \in\left(u_{i}, Y_{i-1}\right)} w(e) .
\end{aligned}
$$

Note that $x_{i}$ and $y_{i}$ are random variables which depend on only $\left(Z_{1}, Z_{2}, \ldots, Z_{i}\right)$; and $a_{i}$ and
$b_{i}$ are random variables which depend on only $\left(Z_{1}, Z_{2}, \ldots, Z_{i-1}\right)$. Thus,

$$
\begin{aligned}
& \mathbb{E}\left(\Delta x_{i} \mid Z_{1}, \ldots, Z_{i-1}\right)=p_{i}\left(w\left(u_{i}\right)+b_{i}\right)+a_{i}, \\
& \mathbb{E}\left(\Delta y_{i} \mid Z_{1}, \ldots, Z_{i-1}\right)=\left(1-p_{i}\right)\left(w\left(u_{i}\right)+a_{i}\right)+b_{i} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left(\Delta x_{i}\right) & =\mathbb{E}\left(\mathbb{E}\left(\Delta x_{i} \mid Z_{1}, \ldots, Z_{i-1}\right)\right) \\
& =\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right)\left(p_{i}\left(w\left(u_{i}\right)+b_{i}\right)+a_{i}\right) \\
& =p_{i}\left(w\left(u_{i}\right)+\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) b_{i}\right)+\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{i} .
\end{aligned}
$$

Similarly,

$$
\mathbb{E}\left(\Delta y_{i}\right)=\left(1-p_{i}\right)\left(w\left(u_{i}\right)+\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{i}\right)+\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) b_{i} .
$$

Let

$$
\begin{aligned}
& \alpha_{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{i}, \\
& \beta_{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) b_{i} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \mathbb{E}\left(\Delta x_{i}\right)=p_{i}\left(w\left(u_{i}\right)+\beta_{i}\right)+\alpha_{i}, \\
& \mathbb{E}\left(\Delta y_{i}\right)=\left(1-p_{i}\right)\left(w\left(u_{i}\right)+\alpha_{i}\right)+\beta_{i} .
\end{aligned}
$$

Note that $\alpha_{i}, \beta_{i}$ are determined by $p_{1}, \ldots, p_{i-1}$, since $a_{i}$ and $b_{i}$ are determined by $Z_{1}, \ldots, Z_{i-1}$. Also note that $e_{i}:=a_{i}+b_{i}=\sum_{e \in\left(u_{i}, G_{i-1}\right)} w(e)$ is the total weight of edges in $\left(u_{i}, V\left(G_{i-1}\right)\right)$, which is independent of $Z_{1}, \ldots, Z_{i-1}$ and is the same in both $G$ and $G_{i}$. Further, $e_{i}>0$ by
our choice of $u_{i}$ and the assumption that $w(e)>0$ for all $e \in E(G)$. Hence,

$$
\begin{aligned}
\alpha_{i}+\beta_{i} & =\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right)\left(a_{i}+b_{i}\right) \\
& =\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) e_{i} \\
& =e_{i}>0 .
\end{aligned}
$$

Let $p_{i}=\frac{w\left(u_{i}\right)+\beta_{i}}{2 w\left(u_{i}\right)+\alpha_{i}+\beta_{i}}$. Note that $p_{i}$ is recursively defined, since $\alpha_{i}$ and $\beta_{i}$ are determined by $p_{1}, \ldots, p_{i-1}$. It follows from Lemma 2.1.1 that $p_{i} \in[0,1]$ and

$$
\min \left\{\mathbb{E}\left(\Delta x_{i}\right), \mathbb{E}\left(\Delta y_{i}\right)\right\} \geq \frac{1}{2} w\left(u_{i}\right)+\frac{3}{4}\left(\alpha_{i}+\beta_{i}\right)=\frac{1}{2} w\left(u_{i}\right)+\frac{3}{4} e_{i} .
$$

We can now compute the expectations of $x_{n-t}$ and $y_{n-t}$ :

$$
\begin{aligned}
& \mathbb{E}\left(x_{n-t}\right)=\mathbb{E}\left(x_{0}\right)+\sum_{i=1}^{n-t} \mathbb{E}\left(\Delta x_{i}\right) \geq \mathbb{E}\left(x_{0}\right)+\frac{1}{2} \sum_{i=1}^{n-t} w\left(u_{i}\right)+\frac{3}{4} \sum_{i=1}^{n-t} e_{i}, \\
& \mathbb{E}\left(y_{n-t}\right)=\mathbb{E}\left(y_{0}\right)+\sum_{i=1}^{n-t} \mathbb{E}\left(\Delta y_{i}\right) \geq \mathbb{E}\left(y_{0}\right)+\frac{1}{2} \sum_{i=1}^{n-t} w\left(u_{i}\right)+\frac{3}{4} \sum_{i=1}^{n-t} e_{i} .
\end{aligned}
$$

Let $X=X_{n-t}, Y=Y_{n-t}$. Then $X \cup Y=V(G)$ and $X \cap Y=\emptyset$. Note that $\tau(X)=$ $x_{n-t}, \tau(Y)=y_{n-t}, \tau\left(X_{0}\right)=x_{0}, \tau\left(Y_{0}\right)=y_{0}, \mathbb{E}\left(x_{0}\right)=x_{0}$, and $\mathbb{E}\left(y_{0}\right)=y_{0}$. Also note that $w_{2}=\sum_{V(e) \subseteq V_{1}} w(e)+\sum_{i=1}^{n-t} e_{i}$. Hence

$$
\begin{aligned}
\mathbb{E}(\tau(X)) & \geq \frac{1}{2}\left(w_{1}-\sum_{i=1}^{t} w\left(v_{i}\right)\right)+\frac{3}{4}\left(w_{2}-\sum_{V(e) \subseteq V_{1}} w(e)\right)+\tau\left(X_{0}\right) \\
& \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\left(\frac{1}{2} \sum_{i=1}^{t} w\left(v_{i}\right)+\frac{3}{4} \sum_{V(e) \subseteq V_{1}} w(e)\right) \\
& \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\lambda\left(\frac{1}{2} t+\frac{3}{4} e\left(V_{1}\right)\right) \\
& \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\lambda\left(\frac{1}{2} c m^{\alpha}+\frac{3}{8} c^{2} m^{2 \alpha}\right) .
\end{aligned}
$$

Similarly,

$$
\mathbb{E}(\tau(Y)) \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\lambda\left(\frac{1}{2} c m^{\alpha}+\frac{3}{8} c^{2} m^{2 \alpha}\right) .
$$

Next we show that $\tau(X)$ and $\tau(Y)$ are concentrated around their respective means. Note that changing the color of some $u_{i}$ would affect $\tau(X)$ and $\tau(Y)$ by at most $d\left(u_{i}\right) \lambda+w\left(u_{i}\right) \leq$ $\left(d\left(u_{i}\right)+1\right) \lambda$. Hence by applying Lemma 1.4.1, we have

$$
\begin{aligned}
\mathbb{P}(\tau(X)<\mathbb{E}(\tau(X))-z) & \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right)^{2}}\right) \\
& \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right) \cdot\left(d\left(v_{t+1}\right)+1\right)}\right) \\
& <\exp \left(-\frac{2 z^{2}}{\lambda^{2}\left(1+\frac{2}{c} m^{1-\alpha}\right) \cdot(2 m+n-1)}\right) \\
& <\exp \left(-\frac{2 z^{2}}{2 \lambda^{2} \frac{2}{c} m^{1-\alpha} \cdot(2 m+m)}\right) \\
& =\exp \left(-\frac{c z^{2}}{6 \lambda^{2} m^{2-\alpha}}\right) .
\end{aligned}
$$

Let $z=\lambda\left(\frac{6 \ln 2}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then

$$
\mathbb{P}(\tau(X)<\mathbb{E}(\tau(X))-z)<\frac{1}{2}
$$

and

$$
\mathbb{P}(\tau(Y)<\mathbb{E}(\tau(Y))-z)<\frac{1}{2} .
$$

So there exists a partition $V(G)=X \cup Y$ such that

$$
\tau(X) \geq \mathbb{E}(\tau(X))-z \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}+\lambda \cdot o(m)
$$

and

$$
\tau(Y) \geq \mathbb{E}(\tau(Y))-z \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}+\lambda \cdot o(m) .
$$

The $o(m)$ term in the above expressions is

$$
-\left(\frac{1}{2} c m^{\alpha}+\frac{3}{8} c^{2} m^{2 \alpha}+\left(\frac{6 \ln 2}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}\right) .
$$

So picking $\alpha=2 / 5$ to minimize $\max \left\{2 \alpha, 1-\frac{\alpha}{2}\right\}$, we have

$$
\min \{\tau(X), \tau(Y)\} \geq \frac{1}{2} w_{1}+\frac{3}{4} w_{2}-\lambda \cdot O\left(m^{4 / 5}\right) .
$$

Note that a random bipartition $V_{1}, V_{2}$ shows that $\mathbb{E}\left(d\left(V_{i}\right)\right)=w_{1} / 2+3 w_{2} / 4$. When $G$ is a hypergraph whose edges are of size 1 or 2 , we may view $G$ as a weighted graph with weight function $w$ such that $w(e)=1$ for all $e \in E(G)$ with $|V(e)|=2, w(v)=1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v)=0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 2.1.2 then gives the following result which, in turn, implies Theorem 1.3.1.

Theorem 2.1.3. Let $G$ be a hypergraph with $m_{i}$ edges of size $i, i=1,2$. Then there is a partition $V_{1}, V_{2}$ of $V(G)$ such that for $i=1,2$,

$$
d\left(V_{i}\right) \geq \frac{1}{2} m_{1}+\frac{3}{4} m_{2}-O\left(m_{2}^{4 / 5}\right) .
$$

The following is a consequence of Theorem 2.1.3.

Corollary 2.1.4. Let $k \geq 2$ be an integer and $G$ be a hypergraph with $m_{i}$ edges of size $i, i=1,2, \ldots, k$. Then there is a partition $V_{1}, V_{2}$ of $V(G)$ such that for $i=1,2, d\left(V_{i}\right) \geq$ $\frac{1}{2} m_{1}+\frac{3}{4}\left(m_{2}+m_{3}+\ldots+m_{k}\right)+o\left(m_{2}+m_{3}+\ldots+m_{k}\right)$.

Proof. For each $e \in E(G)$, if $|V(e)| \leq 2$ then let $e^{\prime}:=e$; otherwise, let $e^{\prime}$ be some 2-element subset of $V(e)$. Let $G^{\prime}$ denote the hypergraph with $V\left(G^{\prime}\right)=V(G)$ and $E\left(G^{\prime}\right)=\left\{e^{\prime}: e \in\right.$ $E(G)\}$. Then $G^{\prime}$ has $m_{1}$ edges of size 1 , and $m_{2}+m_{3}+\ldots+m_{k}$ edges of size 2 .

By Theorem 2.1.3, $V\left(G^{\prime}\right)$ has a partition $V_{1}, V_{2}$ such that for $i=1,2, d\left(V_{i}\right) \geq \frac{m_{1}}{2}+$ $\frac{3}{4}\left(m_{2}+\ldots+m_{k}\right)+o\left(m_{2}+\ldots+m_{k}\right)$ edges. By the construction of $G^{\prime}$, we see that $V_{1}, V_{2}$ is the desired partition of $V(G)$.

## 2.2 -Partitions - bounding edges meeting each set

In this section, we prove Conjecture 1.3.3 for graphs with large $m$. For $k=2$, Conjecture 1.3.3 follows from the fact that every graph with $m$ edges has a bipartition $V_{1}, V_{2}$ such that for $i \in\{1,2\}$, each vertex in $V_{i}$ has at least as many neighbors in $V_{3-i}$ as in $V_{i}$.

We use the same approach as in the previous section, namely, first partition an appropriate set of vertices of larger degree, then establish a random process to compute expectations,
and finally apply the Azuma-Hoeffding inequality to bound deviations. As before, we need to pick probabilities $p_{i}$ in the process. To this end we need several lemmas. Our first lemma will be used to take care of critical points when applying Lagrange multipliers to optimize a function.

Lemma 2.2.1. Let $a_{i}=a>0$ for $i=1, \ldots, l$, and let $a_{j}=0$ for $j=l+1, \ldots, k$, where $k \geq l \geq 2$. Let $\delta \geq 0$ and $\alpha_{i}=\left(\sum_{j=1}^{k} a_{j}\right)+\delta-a_{i}$. Then

$$
1+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}} \geq\left(\frac{\delta}{k}+\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} a_{i}\right) \sum_{i=1}^{k} \frac{1}{\alpha_{i}} .
$$

Proof. By the assumption of the lemma, we have $\alpha_{i}=(l-1) a+\delta>0$ for $1 \leq i \leq l$, and $\alpha_{i}=l a+\delta>0$ for $l+1 \leq i \leq k$. Let

$$
f:=1+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}}-\left(\frac{\delta}{k}+\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} a_{i}\right) \sum_{i=1}^{k} \frac{1}{\alpha_{i}} .
$$

We need to prove $f \geq 0$. For convenience, let $\delta=a \varepsilon$. Then $\varepsilon \geq 0$ and

$$
f=1+\frac{l}{l-1+\varepsilon}-\left(\frac{\varepsilon}{k}+\frac{2 k-1}{k^{2}} l\right)\left(\frac{l}{l-1+\varepsilon}+\frac{k-l}{l+\varepsilon}\right) .
$$

A straightforward calculation shows that

$$
(l-1+\varepsilon)(l+\varepsilon) f=\frac{l}{k^{2}}(k-1)(k-l) \geq 0 .
$$

Hence the assertion of the lemmas holds.

Note that in the lemma below we are unable to require $p_{i} \geq 0$, and hence they cannot serve as probabilities in a random process. However, this lemma is needed to prove Lemma 2.2.3.

Lemma 2.2.2. Let $\delta \geq 0$ and, for $i=1, \ldots, k$, let $a_{i} \geq 0$ and $\alpha_{i}=\left(\sum_{j=1}^{k} a_{j}\right)+\delta-a_{i}$. Then there exist $p_{i}, i=1, \ldots, k$, such that $\sum_{i=1}^{k} p_{i}=1$ and, for $1 \leq i \leq k$,

$$
\alpha_{i} p_{i}+a_{i} \geq \frac{\delta}{k}+\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} a_{i} .
$$

Proof. For convenience let $f_{i}\left(p_{1}, \ldots, p_{k}\right):=\alpha_{i} p_{i}+a_{i}, i=1, \ldots, k$. If $a_{i}=0$ for $i=1, \ldots, k$, then the assertion of the lemma holds by letting $p_{i}=1 / k$ for $i=1, \ldots, k$. So without loss of generality we may assume $a_{1}>0$.

Now assume $a_{i}=0$ for $i=2, \ldots, k$. Then $f_{1}=\delta p_{1}+a_{1}$ and $f_{i}=\left(a_{1}+\delta\right) p_{i}$ for $2 \leq i \leq k$. Setting $f_{i}=f_{1}$ for $i=2, \ldots, k$, we get $p_{i}=\frac{\delta p_{1}+a_{1}}{a_{1}+\delta}$. Requiring $\sum_{i=1}^{k} p_{i}=1$, we obtain $p_{1}=\frac{(2-k) a_{1}+\delta}{a_{1}+k \delta}$. Hence for $i=1, \ldots, k$,

$$
f_{i}=\delta p_{1}+a_{1}=\frac{\left(\delta+a_{1}\right)^{2}}{a_{1}+k \delta},
$$

and so,

$$
f_{i}-\left(\frac{\delta}{k}+\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} a_{i}\right)=\frac{(k-1)^{2} a_{1}^{2}}{\left(a_{1}+k \delta\right) k^{2}} \geq 0 .
$$

Therefore, we may further assume that $a_{2}>0$. Hence $\alpha_{i}>0$ for all $i=1, \ldots, k$. Setting $f_{i}=f_{1}$ for $i=2, \ldots, k$, we get $p_{i}=\frac{\alpha_{1} p_{1}+a_{1}-a_{i}}{\alpha_{i}}$ for $i=1, \ldots, k$. Requiring $\sum_{i=1}^{k} p_{i}=1$ and noting that $a_{i}-a_{1}=\alpha_{1}-\alpha_{i}$ for $1 \leq i \leq k$, we have

$$
p_{1}=\frac{1+\sum_{i=1}^{k} \frac{a_{i}-a_{1}}{\alpha_{i}}}{\alpha_{1} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}}=\frac{1+\sum_{i=1}^{k} \frac{\alpha_{1}-\alpha_{i}}{\alpha_{i}}}{\alpha_{1} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}}=1-\frac{k-1}{\alpha_{1} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}} .
$$

Indeed, for $j=1, \ldots, k$,

$$
p_{j}=1-\frac{k-1}{\alpha_{j} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}} .
$$

Note that $\alpha_{j}+a_{j}=\alpha_{i}+a_{i}$ for any $1 \leq i, j \leq k$. Hence for $j=1,2, \ldots, k$, we have

$$
f_{j}=\alpha_{j} p_{j}+a_{j}=\frac{\sum_{i=1}^{k} \frac{\alpha_{j}+a_{j}}{\alpha_{i}}-(k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_{i}}}=\frac{\sum_{i=1}^{k} \frac{\alpha_{i}+a_{i}}{\alpha_{i}}-(k-1)}{\sum_{i=1}^{k} \frac{1}{\alpha_{i}}}=\frac{1+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}}}{\sum_{i=1}^{k} \frac{1}{\alpha_{i}}} .
$$

Now define

$$
f\left(a_{1}, a_{2}, \ldots, a_{k}\right)=1+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}}-\left(\frac{\delta}{k}+\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} a_{i}\right) \sum_{i=1}^{k} \frac{1}{\alpha_{i}} .
$$

To complete the proof of this lemma, we need to show $f\left(a_{1}, \ldots, a_{k}\right) \geq 0$.

Case 1. $\delta=0$.
Then $\alpha_{i}+a_{i}=\sum_{j=1}^{k} a_{j}$ for $j=1, \ldots, k$. Set $\alpha=\sum_{j=1}^{k} a_{j}$; then $\sum_{i=1}^{k} \alpha_{i}=(k-1) \alpha$. Moreover,

$$
\begin{aligned}
f\left(a_{1}, \ldots, a_{k}\right) & =1+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}}-\frac{(2 k-1) \alpha}{k^{2}} \sum_{i=1}^{k} \frac{1}{\alpha_{i}} \\
& =1+\sum_{i=1}^{k} \frac{\alpha-\alpha_{i}}{\alpha_{i}}-\frac{(2 k-1) \alpha}{k^{2}} \sum_{i=1}^{k} \frac{1}{\alpha_{i}} \\
& =\frac{(k-1)^{2} \alpha}{k^{2}} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}-(k-1) \\
& \geq \frac{(k-1)^{2} \alpha}{k^{2}} \frac{k^{2}}{\sum_{i=1}^{k} \alpha_{i}}-(k-1) \\
& =0 .
\end{aligned}
$$

Here the inequality follows from Cauchy-Schwarz, and the last equality follows from the face that $\sum_{i=1}^{k} \alpha_{i}=(k-1) \alpha$.

Case 2. $\delta>0$.
Then $\alpha_{i}>0$ for $i=1, \ldots, k$. (So in this case we need not require $a_{1}>0$ and $a_{2}>0$.) Set $\alpha=\sum_{j=1}^{k} a_{j}$.

Let $g_{l}\left(a_{1}, \ldots, a_{l}\right)=f\left(a_{1}, \ldots, a_{l}, 0, \ldots, 0\right)$. It then suffices to show that $g_{l}\left(a_{1}, \ldots, a_{l}\right) \geq 0$ on the domain $D_{l}:=[0, \alpha]^{l}$ for $l=1, \ldots, k$.

First, we prove that for $l \in\{1, \ldots, k\}, g_{l} \geq 0$ at all possible critical points of $g_{l}$ in $D_{l}$, subject to $\sum_{j=1}^{k} a_{j}-\alpha=0$. For $j=1, \ldots, l$,

$$
\frac{\partial g_{l}}{\partial a_{j}}=-\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{i}^{2}}+\frac{a_{j}}{\alpha_{j}^{2}}+\frac{1}{\alpha_{j}}+\frac{\delta}{k}\left(\sum_{i=1}^{k} \frac{1}{\alpha_{i}^{2}}-\frac{1}{\alpha_{j}^{2}}\right)-\frac{2 k-1}{k^{2}}\left(\sum_{i=1}^{k} \frac{1}{\alpha_{i}}-\sum_{i=1}^{k} a_{i} \sum_{i=1}^{k} \frac{1}{\alpha_{i}^{2}}+\sum_{i=1}^{k} \frac{a_{i}}{\alpha_{j}^{2}}\right) .
$$

Using the method of Lagrange multipliers, we have $\frac{\partial g_{l}}{\partial a_{j}}=\lambda$ for all $j=1, \ldots, l$. So $\frac{\partial g_{l}}{\partial a_{j}}=\frac{\partial g_{l}}{\partial a_{1}}$, which gives

$$
\frac{a_{j}}{\alpha_{j}^{2}}+\frac{1}{\alpha_{j}}-\frac{\delta}{k} \frac{1}{\alpha_{j}^{2}}-\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} \frac{a_{i}}{\alpha_{j}^{2}}=\frac{a_{1}}{\alpha_{1}^{2}}+\frac{1}{\alpha_{1}}-\frac{\delta}{k} \frac{1}{\alpha_{1}^{2}}-\frac{2 k-1}{k^{2}} \sum_{i=1}^{k} \frac{a_{i}}{\alpha_{1}^{2}} .
$$

Since $\alpha_{j}+a_{j}=\alpha_{1}+a_{1}=\sum_{i=1}^{k} a_{i}+\delta$, we have

$$
\frac{1}{\alpha_{j}^{2}}\left(\frac{(k-1)^{2}}{k^{2}} \sum_{i=1}^{n} a_{i}+\frac{k-1}{k} \delta\right)=\frac{1}{\alpha_{1}^{2}}\left(\frac{(k-1)^{2}}{k^{2}} \sum_{i=1}^{n} a_{i}+\frac{k-1}{k} \delta\right) .
$$

Hence $1 / \alpha_{j}^{2}=1 / \alpha_{1}^{2}$ for all $j=1, \ldots, l$. Therefore, $\alpha_{j}=\alpha_{1}$ for $j=1, \ldots, l$. This implies $a_{j}=a_{1}$ for $j=1, \ldots, l$. It now follows from Lemma 2.2.1 that $g_{l} \geq 0$ at all possible critical points of $g_{l}$ in $[0, \alpha]^{l}$.

We now show that $g_{l} \geq 0$ on $[0, \alpha]^{l}$ by applying induction on $l$. Suppose $l=1$. Then $\alpha=a_{1}$. So $\alpha_{1}=\delta$, and $\alpha_{i}=a_{1}+\delta$ for $i=2, \ldots, k$. Hence,

$$
g_{1}\left(a_{1}\right)=1+\frac{a_{1}}{\delta}-\left(\frac{\delta}{k}+\frac{(2 k-1) a_{1}}{k^{2}}\right)\left(\frac{1}{\delta}+\frac{k-1}{a_{1}+\delta}\right)=\frac{(k-1)^{2}}{k^{2}}\left(\frac{a_{1}^{2}}{\delta\left(a_{1}+\delta\right)}\right) \geq 0 .
$$

So we may assume $l \geq 2$ and $g_{i} \geq 0$ for all $i=1, \ldots, l-1$. We now show $g_{l} \geq 0$ on the domain $[0, \alpha]^{l}$ by proving it for all points in the boundary of $[0, \alpha]^{l}$ (since $g_{l} \geq 0$ at all possible critical points of $\left.g_{l}\right)$. Let $\left(a_{1}, \ldots, a_{l}\right)$ be in the boundary of $[0, \alpha]^{l}$. Then $a_{j}=0$ or $a_{j}=\alpha$ for some $j \in\{1, \ldots, l\}$. Note that $g_{l}$ is a symmetric function. So we may assume without loss of generality that $a_{l}=0$ or $a_{1}=\alpha$. If $a_{l}=0$, then $g_{l}\left(a_{1}, \ldots, a_{l}\right)=$ $g_{l-1}\left(a_{1}, \ldots, a_{l-1}\right) \geq 0$ by induction hypothesis. If $a_{1}=\alpha$ then $a_{j}=0$ for $j=2, \ldots, l$, and so, $g_{l}\left(a_{1}, \ldots, a_{l}\right)=g_{1}\left(a_{1}\right) \geq 0$. Again, we have $g_{l}\left(a_{1}, \ldots, a_{l}\right) \geq 0$.

Note that in the proof of Lemma 2.2.2 when $\alpha_{i}>0,1 \leq i \leq k$, we have

$$
p_{j}=1-\frac{k-1}{\alpha_{j} \sum_{i=1}^{k} \frac{1}{\alpha_{i}}}
$$

for $j=1, \ldots, k$, which may be negative. We now apply Lemma 2.2.2 to prove the next result which gives the $p_{i}$ 's needed in a random process.

Lemma 2.2.3. Let $\delta \geq 0$. For $i=1, \ldots, k$, where $k \geq 3$, let $a_{i} \geq 0$ and $\alpha_{i}=\left(\sum_{j=1}^{k} a_{j}\right)+\delta-a_{i}$. Then there exist $p_{i} \in[0,1], 1 \leq i \leq k$, such that $\sum_{i=1}^{k} p_{i}=1$ and for $1 \leq i \leq k$,

$$
\alpha_{i} p_{i}+a_{i} \geq \frac{\delta}{k}+\frac{1}{k-1} \sum_{i=1}^{k} a_{i} .
$$

Proof. If $a_{i}=0$ for $1 \leq i \leq k$, then the assertion of the lemma holds by taking $p_{i}=1 / k$, $i=1, \ldots, k$. So we may assume without loss of generality that $a_{1}>0$. If $a_{i}=0$ for $2 \leq i \leq k$ and $\delta=0$, then $\alpha_{1}=0$ and $\alpha_{i}=a_{1}$ for $2 \leq i \leq k$; and the assertion of the lemma
holds by setting $p_{1}=0$ and $p_{i}=\frac{1}{k-1}$ for $i=2, \ldots, k$. Therefore, we may further assume that $a_{2}>0$ or $\delta>0$. As a consequence, we have $\alpha_{i}>0$ for $1 \leq i \leq k$.

We prove the assertion of this lemma by induction on $k$. For $1 \leq i \leq k$, let

$$
f_{i}\left(p_{1}, \ldots, p_{k}\right):=\alpha_{i} p_{i}+a_{i}
$$

For $k=3$, it follows from Lemma 2.2.2 (and the remark following its proof) that there exist $p_{1}^{\prime}, p_{2}^{\prime}, p_{3}^{\prime}$ such that $p_{1}^{\prime}+p_{2}^{\prime}+p_{3}^{\prime}=1$ and for $i=1,2,3$,

$$
p_{i}^{\prime}=1-\frac{2}{\alpha_{i} \sum_{i=1}^{3} \frac{1}{\alpha_{j}}} \text { and } f_{i}\left(p_{1}^{\prime}, p_{2}^{\prime}, p_{3}^{\prime}\right) \geq \frac{\delta}{3}+\frac{5}{9} \sum_{i=1}^{3} a_{i}
$$

If $p_{i}^{\prime} \geq 0$ for $i=1,2,3$, then the assertion of the lemma holds by taking $p_{i}:=p_{i}^{\prime}, i=1,2,3$. So we may assume that $p_{3}^{\prime}<0$, which implies $a_{3}>\alpha_{3} p_{3}^{\prime}+a_{3}=f_{3}\left(p_{1}^{\prime}, p_{2}^{\prime}, p_{3}^{\prime}\right) \geq \frac{\delta}{3}+\frac{5}{9} \sum_{i=1}^{3} a_{i}$. By Lemma 2.1.1 (with $n:=a_{3}+\delta$ ), there exist $p_{1}, p_{2} \in[0,1]$ such that $p_{1}+p_{2}=1$ and

$$
\begin{aligned}
& f_{1}\left(p_{1}, p_{2}, 0\right)=\left(a_{2}+a_{3}+\delta\right) p_{1}+a_{1} \geq \frac{a_{3}+\delta}{2}+\frac{3}{4}\left(a_{1}+a_{2}\right), \\
& f_{2}\left(p_{1}, p_{2}, 0\right)=\left(a_{1}+a_{3}+\delta\right) p_{2}+a_{2} \geq \frac{a_{3}+\delta}{2}+\frac{3}{4}\left(a_{1}+a_{2}\right) .
\end{aligned}
$$

Now, let $p_{3}=0$. Then $p_{1}+p_{2}+p_{3}=1, p_{i} \in[0,1]$ for all $1 \leq i \leq 3$, and

$$
\begin{aligned}
f_{1}\left(p_{1}, p_{2}, p_{3}\right)=\alpha_{1} p_{1}+a_{1} & \geq \frac{\delta}{3}+\frac{1}{2}\left(a_{1}+a_{2}+a_{3}\right), \\
f_{2}\left(p_{1}, p_{2}, p_{3}\right)=\alpha_{2} p_{2}+a_{2} & \geq \frac{\delta}{3}+\frac{1}{2}\left(a_{1}+a_{2}+a_{3}\right), \\
f_{3}\left(p_{1}, p_{2}, p_{3}\right)=a_{3} & \geq \frac{\delta}{3}+\frac{1}{2}\left(a_{1}+a_{2}+a_{3}\right) .
\end{aligned}
$$

Hence Lemma 2.2.3 holds for $k=3$.
Now let $n \geq 3$ be an integer, and assume that the assertion of the lemma holds when $k=n$. We prove the assertion of the lemma also holds when $k=n+1$. By Lemma 2.2.2 (and the remark following its proof), there exist $p_{i}^{\prime}, 1 \leq i \leq n+1$, such that $\sum_{i=1}^{n+1} p_{i}^{\prime}=1$ and for $i=1, \ldots, n+1$,

$$
f_{i}\left(p_{1}^{\prime}, \ldots, p_{n+1}^{\prime}\right) \geq \frac{\delta}{n+1}+\frac{2 n+1}{(n+1)^{2}} \sum_{i=1}^{n+1} a_{i}
$$

and

$$
p_{i}^{\prime}=1-\frac{n}{\alpha_{i} \sum_{j=1}^{n+1} \frac{1}{\alpha_{j}}} \leq 1 .
$$

If $p_{i}^{\prime} \geq 0$ for $1 \leq i \leq n+1$, then let $p_{i}:=p_{i}^{\prime}$; and the lemma holds (since $\frac{2 n+1}{(n+1)^{2}}>\frac{1}{n}$ when $n \geq 3$ ). So we may assume without loss of generality that $p_{n+1}^{\prime}<0$. Then

$$
\begin{aligned}
a_{n+1} & >\alpha_{n+1} p_{n+1}^{\prime}+a_{n+1} \\
& =f_{n+1}\left(p_{1}^{\prime}, \ldots, p_{n+1}^{\prime}\right) \\
& \geq \frac{\delta}{n+1}+\frac{2 n+1}{(n+1)^{2}} \sum_{i=1}^{n+1} a_{i} \\
& \geq \frac{\delta}{n+1}+\frac{1}{n} \sum_{i=1}^{n+1} a_{i}
\end{aligned}
$$

Let $\delta^{\prime}=\delta+a_{n+1}$. Then for $1 \leq i \leq n$ we have $\alpha_{i}=\left(\sum_{j=1}^{n} a_{j}\right)+\delta^{\prime}-a_{i}$. Hence by the induction hypothesis, there exist $p_{i} \in[0,1], 1 \leq i \leq n$, such that $\sum_{i=1}^{n} p_{i}=1$ and, for $i=1, \ldots, n$,

$$
\begin{aligned}
\alpha_{i} p_{i}+a_{i} & \geq \frac{\delta^{\prime}}{n}+\frac{1}{n-1} \sum_{i=1}^{n} a_{i} \\
& =\frac{\delta}{n}+\frac{a_{n+1}}{n}+\frac{1}{n-1} \sum_{i=1}^{n} a_{i} .
\end{aligned}
$$

Let $p_{n+1}=0$. Then $\sum_{i=1}^{n+1} p_{i}=1$ and $p_{i} \in[0,1]$ for all $1 \leq i \leq n+1$. Also, for any $1 \leq i \leq n$,

$$
\begin{aligned}
& f_{i}\left(p_{1}, \ldots, p_{n+1}\right) \geq \frac{\delta}{n}+\frac{a_{n+1}}{n}+\frac{1}{n-1} \sum_{i=1}^{n} a_{i} \geq \frac{\delta}{n+1}+\frac{1}{n} \sum_{i=1}^{n+1} a_{i}, \text { and } \\
& f_{n+1}\left(p_{1}, \ldots, p_{n+1}\right)=a_{n+1} \geq \frac{\delta}{n+1}+\frac{1}{n} \sum_{i=1}^{n+1} a_{i} .
\end{aligned}
$$

Hence, Lemma 2.2.3 holds for $k=n+1$, completing the proof of this lemma.
We can now prove the following partition result on weighted graphs.

Theorem 2.2.4. Let $k \geq 3$ be an integer, let $G$ be a graph with $m$ edges, and let $w: V(G) \cup$ $E(G) \rightarrow \mathbf{R}^{+}$such that $w(e)>0$ for all $e \in E(G)$. Let $\lambda=\max \{w(x): x \in V(G) \cup E(G)\}$,
$w_{1}=\sum_{v \in V(G)} w(v)$ and $w_{2}=\sum_{e \in E(G)} w(e)$. Then there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i \leq k$,

$$
\tau\left(V_{i}\right) \geq \frac{1}{k} w_{1}+\frac{1}{k-1} w_{2}-\lambda \cdot O\left(m^{4 / 5}\right) .
$$

Proof. We may assume that $G$ is connected. We use the same notation as in the proof of Theorem 2.1.2. Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $V_{1}=\left\{v_{1}, \ldots, v_{t}\right\}$ with $t=\left\lfloor\mathrm{cm}^{\alpha}\right\rfloor$, where $0<\alpha<1 / 2$ and $0<c<\sqrt{2}$; and let $V_{2}:=$ $V(G) \backslash V_{1}=\left\{u_{1}, \ldots, u_{n-t}\right\}$ such that $e\left(u_{i}, V_{1} \cup\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$. Then

$$
e\left(V_{1}\right) \leq \frac{1}{2} c^{2} m^{2 \alpha} \quad \text { and } \quad d\left(v_{t+1}\right) \leq \frac{2}{c} m^{1-\alpha} .
$$

Fix an arbitrary partition $V_{1}=Y_{1} \cup Y_{2} \cup \cdots \cup Y_{k}$ and, for each $i \in\{1, \ldots, k\}$, assign the color $i$ to all vertices in $Y_{i}$. We extend this coloring to $V(G)$ such that each vertex $u_{i} \in V_{2}$ is independently assigned the color $j$ with probability $p_{j}^{i}, \sum_{j=1}^{k} p_{j}^{i}=1$. Let $Z_{i}$ be the indicator random variable of the event of coloring $u_{i}$, i.e., $Z_{i}=j$ iff $u_{i}$ is colored $j$. Let $G_{i}=G\left[V_{1} \cup\left\{u_{1}, \cdots, u_{i}\right\}\right]$ for $i=1, \ldots, n-t$, and let $G_{0}=G\left[V_{1}\right]$. Let $X_{j}^{0}=Y_{j}$ and $x_{j}^{0}=\tau\left(X_{j}^{0}\right)$, and for $i=1, \ldots, n-t$ and $j=1, \ldots, k$, define

$$
\begin{aligned}
X_{j}^{i} & =\left\{\text { vertices of } G_{i} \text { with color } j\right\}, \\
x_{j}^{i} & =\tau_{G_{i}}\left(X_{j}^{i}\right), \\
\Delta x_{j}^{i} & =x_{j}^{i}-x_{j}^{i-1}, \\
a_{j}^{i} & =\sum_{e \in\left(u_{i}, X_{j}^{i-1}\right)} w(e) .
\end{aligned}
$$

Note that $a_{j}^{i}$ depends on only $\left(Z_{1}, \ldots, Z_{i-1}\right)$. Hence, for $1 \leq i \leq n-t$ and $1 \leq j \leq k$,

$$
\mathbb{E}\left(\Delta x_{j}^{i} \mid Z_{1}, \ldots, Z_{i-1}\right)=p_{j}^{i}\left(\sum_{l=1}^{k} a_{l}^{i}+w\left(u_{i}\right)-a_{j}^{i}\right)+a_{j}^{i} .
$$

So

$$
\mathbb{E}\left(\Delta x_{j}^{i}\right)=p_{j}^{i}\left(\sum_{l=1}^{k} b_{l}^{i}+w\left(u_{i}\right)-b_{j}^{i}\right)+b_{j}^{i},
$$

where for $1 \leq l \leq k$,

$$
b_{l}^{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{l}^{i} .
$$

Since $a_{l}^{i}$ is determined by $\left(Z_{1}, \ldots, Z_{i-1}\right), b_{l}^{i}$ is determined by $p_{j}^{s}, 1 \leq s \leq i-1$ and $1 \leq j \leq k$.
By Lemma 2.2.3 (with $\delta=w\left(u_{i}\right)$ ), there exist $p_{j}^{i} \in[0,1], 1 \leq j \leq k$, such that $\sum_{j=1}^{k} p_{j}^{i}=$ 1 and

$$
\mathbb{E}\left(\Delta x_{j}^{i}\right) \geq \frac{w\left(u_{i}\right)}{k}+\frac{1}{k-1} \sum_{j=1}^{k} b_{j}^{i} .
$$

Clearly, each $p_{j}^{i}$ is dependent only on $b_{l}^{i}, 1 \leq l \leq k$, and hence is determined (recursively) by $p_{l}^{s}, 1 \leq l \leq k$ and $1 \leq s \leq i-1$. Note that $e_{i}:=\sum_{j=1}^{k} a_{j}^{i}=\sum_{e \in\left(u_{i}, G_{i-1}\right)} w(e)$ is the total weight of the edges in $\left(u_{i}, G_{i-1}\right)$, which is independent of $Z_{1}, \ldots, Z_{n-t}$. Thus,

$$
\begin{aligned}
\mathbb{E}\left(\Delta x_{j}^{i}\right) & \geq \frac{w\left(u_{i}\right)}{k}+\frac{1}{k-1} \sum_{j=1}^{k} \sum_{\left(Z_{1}, \ldots, z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{j}^{i} \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k-1} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)}\left(\mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) \sum_{j=1}^{k} a_{j}^{i}\right) \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k-1} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) e_{i} \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k-1} e_{i} .
\end{aligned}
$$

Therefore, noting that $w_{2}=\sum_{V(e) \subseteq V_{1}} w(e)+\sum_{i=1}^{n-t} e_{i}$, we have

$$
\begin{aligned}
\mathbb{E}\left(x_{j}^{n-t}\right) & =\sum_{i=1}^{n-t} \mathbb{E}\left(\Delta x_{j}^{i}\right)+\mathbb{E}\left(x_{j}^{0}\right) \\
& \geq \frac{1}{k} \sum_{i=1}^{n-t} w\left(u_{i}\right)+\frac{1}{k-1} \sum_{i=1}^{n-t} e_{i}+x_{j}^{0} \\
& \geq \frac{1}{k}\left(w_{1}-\sum_{i=1}^{t} w\left(v_{i}\right)\right)+\frac{1}{k-1}\left(w_{2}-\sum_{V(e) \subseteq V_{1}} w(e)\right) \\
& \geq \frac{1}{k} w_{1}+\frac{1}{k-1} w_{2}-\left(\frac{1}{k} \sum_{i=1}^{t} w\left(v_{i}\right)+\frac{1}{k-1} \sum_{V(e) \subseteq V_{1}} w(e)\right) \\
& \geq \frac{1}{k} w_{1}+\frac{1}{k-1} w_{2}-\lambda\left(\frac{1}{k} t+\frac{1}{k-1} e\left(V_{1}\right)\right) .
\end{aligned}
$$

Let $x_{j}:=x_{j}^{n-t}=\tau_{G}\left(X_{j}^{n-t}\right), j=1, \ldots, k$. Now changing the color of $u_{i}$ only affects $x_{j}$ by at
$\operatorname{most} d\left(u_{i}\right) \lambda+w\left(u_{i}\right) \leq\left(d\left(u_{i}\right)+1\right) \lambda$. Hence, by Lemma 1.4.1, we have, for $j=1, \ldots, k$,

$$
\begin{aligned}
\mathbb{P}\left(x_{j}<\mathbb{E}\left(x_{j}\right)-z\right) & \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right)^{2}}\right) \\
& \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right) \cdot\left(d\left(v_{t+1}\right)+1\right)}\right) \\
& <\exp \left(-\frac{2 z^{2}}{\lambda^{2}(2 m+n-1) \cdot \frac{4}{c} m^{1-\alpha}}\right) \\
& \leq \exp \left(-\frac{c z^{2}}{6 \lambda^{2} m^{2-\alpha}}\right) .
\end{aligned}
$$

Let $z=\lambda\left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$; then

$$
\mathbb{P}\left(x_{j}<\mathbb{E}\left(x_{j}\right)-z\right)<\exp (-\ln k)=\frac{1}{k} .
$$

So there exists a partition $V(G)=X_{1} \cup X_{2} \cup \cdots \cup X_{k}$ such that for $j=1, \ldots, k$,

$$
\begin{aligned}
x_{j} & \geq \mathbb{E}\left(x_{j}\right)-z \\
& \geq \frac{1}{k} w_{1}+\frac{1}{k-1} w_{2}-\lambda\left(\frac{1}{k} t+\frac{1}{k-1} e\left(V_{1}\right)\right)-z \\
& \geq \frac{1}{k} w_{1}+\frac{1}{k-1} w_{2}+\lambda \cdot o(m),
\end{aligned}
$$

where the $o(m)$ term in the expression is

$$
-\left(\frac{c}{k} m^{\alpha}+\frac{1}{2(k-1)} c^{2} m^{2 \alpha}+\left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}\right) .
$$

Picking $\alpha=\frac{2}{5}$ to minimize $\max \{2 \alpha, 1-\alpha / 2\}$, the $o(m)$ term becomes $O\left(m^{\frac{4}{5}}\right)$.
Suppose $G$ is a hypergraph whose edges have size 1 or 2 . We may view $G$ as a weighted graph with weight function $w$ such that $w(e)=1$ for all $e \in E(G)$ with $|V(e)|=2, w(v)=1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v)=0$ for all $v \in V(G)$ with $\{v\} \notin E(G)$. Theorem 2.2.4 then gives the following result, which implies Theorem 1.3.2.

Theorem 2.2.5. Let $k \geq 3$ be an integer and let $G$ be a hypergraph with $m_{i}$ edges of size $i$, $i=1,2$. Then there is a partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k$,

$$
d\left(V_{i}\right) \geq \frac{m_{1}}{k}+\frac{m_{2}}{k-1}-O\left(m_{2}^{4 / 5}\right) .
$$

We have the following corollary, which establishes Conjecture 1.3.3 for large graphs.

Corollary 2.2.6. Let $G$ be a graph with $m$ edges and let $k \geq 3$ be an integer. Then there is an integer $f(k)$ such that if $m \geq f(k)$ then $V(G)$ has a partition $V_{1}, \ldots, V_{k}$ such that $d\left(V_{i}\right) \geq 2 m /(2 k-1)$ for $i=1, \ldots, k$.

Note that our proof gives $f(k)=O\left(k^{10}(\log k)^{5 / 2}\right)$.

## 2.3 k-Partitions - bounding edges inside each set

Bollobás and Scott [8] proved that every graph with $m$ edges can be partitioned into $k$ sets each of which contains at most $m /\binom{k+1}{2}$ edges, with $K_{k+1}$ as the unique extremal graph. For large graphs, they proved in [10] that this bound can be improved to $(1+o(1)) m / k^{2}$. They also [12] conjectured that:

Conjecture 2.3.1. (Bollobás and Scott [12]) Any hypergraph with $m_{i}$ edges of size $i, i=$ 1,2 , admits a $k$-partition $V_{1}, \ldots, V_{k}$ such that for $i=1, \ldots, k$,

$$
e\left(V_{i}\right) \leq \frac{m_{1}}{k}+\frac{m_{2}}{\binom{k+1}{2}}+O(1) .
$$

We now prove Conjecture 2.3.1. The following two lemmas will enable us to choose the probabilities in a random process.

Lemma 2.3.2. Let $\delta \geq 0$ and, for integers $k \geq l \geq 1$, let $a_{i}=a>0$ for $i=1, \ldots, l$ and $a_{j}=0$ for $j=l+1, \ldots, k$. Suppose $\delta+a_{i}>0$ for all $1 \leq i \leq k$. Then

$$
\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}} \leq \frac{\delta}{k}+\frac{1}{k^{2}} \sum_{i=1}^{k} a_{i} .
$$

Proof. If $l=k$ then the inequality holds with equality (both sides equal to $(\delta+a) / k$ ). So we may assume $k>l$. Then $\delta>0$, since $\delta+a_{k}>0$ by assumption. Thus $\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}=\frac{l}{\delta+a}+\frac{k-l}{\delta}$ and $\sum_{i=1}^{k} a_{i}=l a$. Hence

$$
\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}}-\left(\frac{\delta}{k}+\frac{1}{k^{2}} \sum_{i=1}^{k} a_{i}\right)=\frac{-l(k-l) a^{2}}{k^{2}(k \delta+(k-l) a)} \leq 0 .
$$

Thus the assertion of the lemma holds.

Lemma 2.3.3. Let $\delta \geq 0$ and let $a_{i} \geq 0$ for $i=1, \ldots, k$. Then there exist $p_{i} \in[0,1]$, $i=1, \ldots, k$, such that $\sum_{i=1}^{k} p_{i}=1$ and, for $1 \leq i \leq k$,

$$
\left(\delta+a_{i}\right) p_{i} \leq \frac{\delta}{k}+\frac{1}{k^{2}} \sum_{i=1}^{k} a_{i} .
$$

Proof. If there exists some $1 \leq i \leq k$ such that $\delta+a_{i}=0$, then $\delta=a_{i}=0$. In this case let $p_{i}=1$ and $p_{j}=0$ for $j \neq i, 1 \leq j \leq k$. Then $\left(\delta+a_{i}\right) p_{i}=0$ for $i=1, \ldots, k$; and clearly the assertion of the lemma holds.

Therefore, we may assume that $\delta+a_{i}>0,1 \leq i \leq k$. Setting $\left(\delta+a_{i}\right) p_{i}=\left(\delta+a_{1}\right) p_{1}$ for $i=2, \ldots, k$, we have $p_{i}=\frac{\delta+a_{1}}{\delta+a_{i}} p_{1}$. Requiring $\sum_{i=1}^{k} p_{i}=1$ we have $\left(\delta+a_{1}\right) p_{1} \sum_{i=1}^{k} \frac{1}{\delta+a_{i}}=1$. Hence for $i=1, \ldots, k$,

$$
\left(\delta+a_{i}\right) p_{i}=\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}} .
$$

Let

$$
f\left(a_{1}, a_{2}, \ldots, a_{k}\right):=\frac{1}{\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}}-\left(\frac{\delta}{k}+\frac{1}{k^{2}} \sum_{i=1}^{k} a_{i}\right) .
$$

We need to show $f \leq 0$. This is clear if $a_{i}=0$ for $i=1, \ldots, k$, since $f(0, \ldots, 0)=0$.
Let $g_{l}\left(a_{1}, \ldots, a_{l}\right):=f\left(a_{1}, \ldots, a_{l}, 0, \ldots, 0\right)$ for $l=1, \ldots, k$. We now show that $g_{l} \leq 0$ on $D_{l}:=[0, \alpha]^{l}$ for all $1 \leq l \leq k$; and hence $f=g_{k} \leq 0$. We apply induction on $l$.

Suppose $l=1$. Clearly, $g_{1}(0)=f(0,0, \ldots, 0)=0$, and if $a_{1}=a>0$ then by Lemma 2.3.2, $g_{1}\left(a_{1}\right)=f\left(a_{1}, 0, \ldots, 0\right) \leq 0$.

Therefore, we may assume $l \geq 2$. It suffices to prove $g_{l}\left(a_{1}, \ldots, a_{l}\right) \leq 0$ for all points $\left(a_{1}, \ldots, a_{l}\right)$ that are on the boundary of $D_{l}$ or critical points of $g_{l}$ in $D_{l}$.

Let $\left(a_{1}, \ldots, a_{l}\right)$ be a point on the boundary of $D_{l}$. Then there exists $j \in\{1, \ldots, l\}$ such that $a_{j}=0$ or $a_{j}=\alpha$. Since $g_{l}$ is a symmetric function, we may assume that $a_{l}=0$ or $a_{1}=\alpha$. If $a_{l}=0$, then $g_{l}\left(a_{1}, \ldots, a_{l-1}, 0\right)=g_{l-1}\left(a_{1}, \ldots, a_{l-1}\right) \leq 0$, by induction hypothesis. If $a_{1}=\alpha$, then $a_{2}=\ldots=a_{k}=0$, and so $g_{l}\left(a_{1}, \ldots, a_{l}\right)=g_{1}\left(a_{1}\right) \leq 0$ by induction basis.

Hence it remains to prove $g_{l} \leq 0$ at its critical points in $D_{l}$, subject to $\sum_{j=1}^{l} a_{j}-\alpha=0$.

Note that for all $j=1, \ldots, l$,

$$
\frac{\partial f}{\partial a_{j}}=\frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}\right)^{2}} \cdot \frac{1}{\left(\delta+a_{j}\right)^{2}}-\frac{1}{k^{2}} .
$$

Also note that $\frac{\partial g_{l}}{\partial a_{j}}$ is obtained from $\frac{\partial f}{\partial a_{j}}$ by setting $a_{l+1}=\ldots=a_{k}=0$. Thus, letting $\frac{\partial g_{l}}{\partial a_{j}}=\lambda$ (the Lagrange multiplier) for $j=1, \ldots, l$, we have for $1 \leq s \neq t \leq l$,

$$
\frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}\right)^{2}} \cdot \frac{1}{\left(\delta+a_{s}\right)^{2}}-\frac{1}{k^{2}}=\frac{1}{\left(\sum_{i=1}^{k} \frac{1}{\delta+a_{i}}\right)^{2}} \cdot \frac{1}{\left(\delta+a_{t}\right)^{2}}-\frac{1}{k^{2}}
$$

As a consequence, $\left(\delta+a_{s}\right)^{2}=\left(\delta+a_{t}\right)^{2}$ which implies $a_{s}=a_{t}$ for all $1 \leq s \neq t \leq l$. Thus, if $\left(a_{1}, a_{2}, \ldots, a_{l}\right)$ is a critical point of $g_{l}$ in $D_{l}$, then there exists $a>0$ such that $a_{i}=a>0$ for $i=1, \ldots, l$. Now it follows from Lemma 2.3.2 that $g_{l} \leq 0$.

We now prove the following partition result for weighted graphs.

Theorem 2.3.4. Let $G$ be a graph with $m$ edges, and let $w: V(G) \cup E(G) \rightarrow \mathbf{R}^{+}$such that $w(e)>0$ for all $e \in E(G)$. Let $\lambda:=\max \{w(x): x \in V(G) \cup E(G)\}, w_{1}=\sum_{v \in V(G)} w(v)$ and $w_{2}=\sum_{e \in E(G)} w(e)$. Then for any integer $k \geq 1$ there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k$,

$$
e\left(V_{i}\right) \leq \frac{1}{k} w_{1}+\frac{1}{k^{2}} w_{2}+\lambda \cdot O\left(m^{4 / 5}\right)
$$

Proof. We may assume that $G$ is connected. We use the same notation as in the proof of Theorem 2.1.2. Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $V_{1}=\left\{v_{1}, \ldots, v_{t}\right\}$ with $t=\left\lfloor c m^{\alpha}\right\rfloor$, where $0<\alpha<1 / 2$ and $0<c<\sqrt{2}$; and let $V_{2}:=$ $V(G) \backslash V_{1}=\left\{u_{1}, \ldots, u_{n-t}\right\}$ such that $e\left(u_{i}, V_{1} \cup\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$. Then $e\left(V_{1}\right) \leq \frac{1}{2} c^{2} m^{2 \alpha}$ and $d\left(v_{t+1}\right) \leq \frac{2}{c} m^{1-\alpha}$.

Fix an arbitrary $k$-partition $V_{1}=Y_{1} \cup Y_{2} \cup \cdots \cup Y_{k}$, and assign each member of $Y_{i}$ the color $i, 1 \leq i \leq k$. Extend this coloring to $V(G)$, where each vertex $u_{i} \in V_{2}$ is independently assigned the color $j$ with probability $p_{j}^{i}$ and $\sum_{j=1}^{k} p_{j}^{i}=1$. Let $Z_{i}$ denote the indicator random variable of the event of coloring $u_{i}$. Hence $Z_{i}=j$ iff $u_{i}$ is assigned the color $j$.

Let $G_{i}=G\left[V_{1} \cup\left\{u_{1}, \cdots, u_{i}\right\}\right]$ for $i=1, \ldots, n-t$, and let $G_{0}=G\left[V_{1}\right]$. Let $X_{j}^{0}=Y_{j}$ and $x_{j}^{0}=w\left(X_{j}^{0}\right)$, and for $i=1, \ldots, n-t$ define

$$
\begin{aligned}
X_{j}^{i} & =\left\{\text { vertices of } G_{i} \text { with color } j\right\}, \\
x_{j}^{i} & =w\left(X_{j}^{i}\right), \\
\Delta x_{j}^{i} & =x_{j}^{i}-x_{j}^{i-1} \\
a_{j}^{i} & =\sum_{e \in\left(u_{i}, X_{j}^{i-1}\right)} w(e) .
\end{aligned}
$$

Note that $a_{j}^{i}$ depends on $\left(Z_{1}, \ldots, Z_{i-1}\right)$ only. Hence for $1 \leq i \leq n-t$ and $1 \leq j \leq k$,

$$
\mathbb{E}\left(\Delta x_{j}^{i} \mid Z_{1}, \ldots, Z_{i-1}\right)=\left(w\left(u_{i}\right)+a_{j}^{i}\right) p_{j}^{i},
$$

and so

$$
\mathbb{E}\left(\Delta x_{j}^{i}\right)=\left(w\left(u_{i}\right)+b_{j}^{i}\right) p_{j}^{i},
$$

where here

$$
b_{j}^{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{j}^{i} .
$$

Since $a_{j}^{i}$ is determined by $\left(Z_{1}, \ldots, Z_{i-1}\right), b_{j}^{i}$ is determined by $p_{j}^{s}, 1 \leq j \leq k$ and $1 \leq s \leq$ $i-1$. Note that $e_{i}:=\sum_{j=1}^{k} a_{j}^{i}=\sum_{e \in\left(u_{i}, G_{i-1}\right)} w(e)>0$, which is independent of $Z_{1}, \ldots, Z_{n-t}$. By Lemma 2.3.3, there exists $p_{j}^{i} \in[0,1], 1 \leq j \leq k$, such that $\sum_{j=1}^{k} p_{j}^{i}=1$ and, for $1 \leq i \leq n-t$ and $j=1, \ldots, k$,

$$
\begin{aligned}
\mathbb{E}\left(\Delta x_{j}^{i}\right) & \leq \frac{w\left(u_{i}\right)}{k}+\frac{1}{k^{2}} \sum_{j=1}^{k} b_{j}^{i} \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k^{2}} \sum_{j=1}^{k} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) a_{j}^{i} \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k^{2}} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)}\left(\mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) \sum_{j=1}^{k} a_{j}^{i}\right) \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k^{2}} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) e_{i} \\
& =\frac{w\left(u_{i}\right)}{k}+\frac{1}{k^{2}} e_{i} .
\end{aligned}
$$

Note that $p_{j}^{i}$ is determined by $b_{l}^{i}, 1 \leq l \leq k$; and hence $p_{j}^{i}$ is recursively defined by $p_{l}^{s}$, $1 \leq l \leq k$ and $1 \leq s \leq i-1$. Also note that $w_{2}=\sum_{e \in E\left(G_{0}\right)} w(e)+\sum_{i=1}^{n-t} e_{i}$. Now

$$
\begin{aligned}
\mathbb{E}\left(x_{j}^{n-t}\right) & =\sum_{i=1}^{n-t} \mathbb{E}\left(\Delta x_{j}^{i}\right)+\mathbb{E}\left(x_{j}^{0}\right) \\
& \leq \frac{1}{k} \sum_{i=1}^{n-t} w\left(u_{i}\right)+\frac{1}{k^{2}} \sum_{i=1}^{n-t} e_{i}+x_{j}^{0} \\
& \leq \frac{1}{k} w_{1}+\frac{1}{k^{2}} w_{2}+\sum_{i=1}^{t} w\left(v_{i}\right)+\sum_{V(e) \subseteq V_{1}} w(e) \\
& \leq \frac{1}{k} w_{1}+\frac{1}{k^{2}} w_{2}+\lambda\left(t+e\left(V_{1}\right)\right) .
\end{aligned}
$$

Clearly, changing the color of $u_{i}$ affects $x_{j}:=x_{j}^{n-t}$ by at most $d\left(u_{i}\right) \lambda+w\left(u_{i}\right) \leq\left(d\left(u_{i}\right)+1\right) \lambda$. So by Lemma 1.4.1,

$$
\begin{aligned}
\mathbb{P}\left(x_{j}>\mathbb{E}\left(x_{j}\right)+z\right) & \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right)^{2}}\right) \\
& \leq \exp \left(-\frac{2 z^{2}}{\lambda^{2} \sum_{i=1}^{n-t}\left(d\left(u_{i}\right)+1\right)\left(d\left(v_{t+1}\right)+1\right)}\right) \\
& <\exp \left(-\frac{2 z^{2}}{\lambda^{2}(2 m+n-1) \frac{4}{c} m^{1-\alpha}}\right) \\
& \leq \exp \left(-\frac{c z^{2}}{6 \lambda^{2} m^{2-\alpha}}\right) .
\end{aligned}
$$

Let $z=\lambda\left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then

$$
\mathbb{P}\left(x_{j}>\mathbb{E}\left(x_{j}\right)+z\right)<\exp (-\ln k)=\frac{1}{k} .
$$

So there exists a partition $V(G)=X_{1} \cup X_{2} \cup \cdots \cup X_{k}$, where $X_{j}:=X_{j}^{n-t}$, such that for $1 \leq j \leq k$,

$$
\begin{aligned}
x_{j} & \leq \mathbb{E}\left(x_{j}\right)+z \\
& \leq \frac{1}{k} w_{1}+\frac{1}{k^{2}} w_{2}+\lambda\left(t+e\left(V_{1}\right)\right)+z \\
& \leq \frac{1}{k} w_{1}+\frac{1}{k^{2}} w_{2}+\lambda \cdot o(m) .
\end{aligned}
$$

The $o(m)$ term in the expression is

$$
c m^{\alpha}+\frac{1}{2} c^{2} m^{2 \alpha}+\left(\frac{6 \ln k}{c}\right)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}} .
$$

Picking $\alpha=\frac{2}{5}$ to minimize $\max \{2 \alpha, 1-\alpha / 2\}$, the $o(m)$ term becomes $O\left(m^{\frac{4}{5}}\right)$.
For a hypergraph $G$ whose edges are of size 1 or 2, we may view $G$ as a weighted graph with weight function $w$ such that $w(e)=1$ for all $e \in E(G)$ with $|V(e)|=2, w(v)=1$ for all $v \in V(G)$ with $\{v\} \in E(G)$, and $w(v)=0$ for $v \in V(G)$ with $\{v\} \notin E(G)$. Then Theorem 2.3.4 gives the following result, implying Theorem 1.3.4 and establishing Conjecture 2.3.1 raised by Bollobás and Scott [12].

Theorem 2.3.5. Let $G$ be a hypergraph with $m_{i}$ edges of size $i, i=1,2$. Then for any integer $k \geq 1$, there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $i=1, \ldots, k$,

$$
e\left(V_{i}\right) \leq \frac{m_{1}}{k}+\frac{m_{2}}{k^{2}}+O\left(m_{2}^{4 / 5}\right)
$$

Note that the term $m_{1} / k+m_{2} / k^{2}$ is the expected value of $e\left(V_{i}\right)$ if $V_{1}, \ldots, V_{k}$ is a random $k$-partition. Bollobás and Scott ask in [12] whether it is possible to replace $O\left(m_{2}^{4 / 5}\right)$ in Theorem 2.3 .5 with $O\left(\sqrt{m_{1}+m_{2}}\right)$. This is still open.

## CHAPTER III

## BOUNDS FOR PAIRS IN PARTITIONS OF GRAPHS

In this chapter we study Problem 1.3.5, Conjecture 1.3.7 and Conjecture 1.3.9. Recall $f(k, m)$ in Problem 1.3.5.

In Section 3.1, we show that $f(k, m)<1.6 m / k+o(m)$, and that $f(k, m)<1.5 m / k+o(m)$ for $k \geq 23$. In Section 3.2, we prove $f(k, m) \leq 4 m / k^{2}+o(m)$ for dense graphs, which confirms Conjecture 1.3.7 for such graphs, and we establish Conjecture 1.3.9 for graphs with $\Omega\left(k^{12}(\ln k)^{3}\right)$ edges.

In Section 3.3, we show $f(4, m) \leq m / 3+o(m)$ and $f(5, m) \leq 4 m / 15+o(m)$, which imply Conjecture 1.3.7 for $k=4$ and $k=5$. In Section 3.4, we study the problem raised by Bollobás and Scott [12] that for any graph $G$ with $m$ edges, whether it is possible to find a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that

$$
e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}(\sqrt{2 m+1 / 4}-1 / 2)
$$

for $1 \leq i \leq k$, and

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{12 m}{(k+1)(k+2)}+O(n)
$$

for $1 \leq i<j \leq k$. We show that for $k=3$ and $k=4$ one can find a partition satisfying these bounds asymptotically.

### 3.1 A general bound

In this section, we prove a bound on $f(k, m)$ in Problem 1.3.5. We need a simple lemma which will also be used in Section 3.3 for finding probabilities when dealing with 4partitions.

Lemma 3.1.1. Let $a_{j} \geq 0$ for $j \in\{1,2,3,4\}$ such that $\alpha:=\sum_{j=1}^{4} a_{j}>0$, and let $f_{i j}\left(x_{i}, x_{j}\right)=$ $\left(a_{i}+a_{j}\right)\left(x_{i}+x_{j}\right)$ for $1 \leq i \neq j \leq 4$. Then there exist $p_{i} \in[0,1 / 2], 1 \leq i \leq 4$, such that $\sum_{i=1}^{4} p_{i}=1$ and, for $1 \leq i \neq j \leq 4, f_{i j}\left(p_{i}, p_{j}\right) \leq \alpha / 3$.

Proof. First, assume $a_{i} \leq \alpha / 2$ for all $1 \leq i \leq 4$. Then $p_{i}:=1 / 2-a_{i} / \alpha \in\left[0, \frac{1}{2}\right]$, and

$$
f_{i j}\left(p_{i}, p_{j}\right)=\left(a_{i}+a_{j}\right)\left(1-\frac{a_{i}+a_{j}}{\alpha}\right)=-\frac{1}{\alpha}\left(a_{i}+a_{j}-\frac{\alpha}{2}\right)^{2}+\frac{\alpha}{4} \leq \frac{\alpha}{4} .
$$

So we may assume without loss of generality that $a_{4}>\alpha / 2$. Then $a_{i}+a_{j} \leq \alpha / 2$ for all $1 \leq i \neq j \leq 3$. Let $p_{1}=p_{2}=p_{3}=1 / 3$ and $p_{4}=0$. Then for $1 \leq i \leq 3$, $f_{i 4}=\left(a_{i}+a_{4}\right) / 3 \leq \alpha / 3$; and for $1 \leq i \neq j \leq 3, f_{i j}=\left(a_{i}+a_{j}\right)(2 / 3) \leq(\alpha / 2)(2 / 3)=\alpha / 3$.

Remark. From the above proof, we see that among the $p_{i}$ satisfying the assertion of Lemma 3.1.1, we may choose $p_{i}=0$ when $a_{i}>\alpha / 2$, and $p_{i} \leq \max \left\{1 / 2-a_{i} / \alpha, 1 / 3\right\}$ when $a_{i} \leq \alpha / 2$.

We need another lemma.

Lemma 3.1.2. Let $h_{4}=1 / 3$. There exist $t_{k}, h_{k}$ for $k \geq 5$ such that

$$
\begin{aligned}
& h_{k}=\frac{2-2 t_{k}}{k-2 t_{k}}, \text { and } \\
& \frac{2-2 t_{k}}{k-2 t_{k}}=\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t_{k} .
\end{aligned}
$$

Moreover, $h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$.

Proof. We first show that there exist $t_{k} \in(0,1 / 2)$ and $h_{k} \in(1 /(k-1), 2 / k), k \geq 5$, such that

$$
\begin{aligned}
& h_{k}=\frac{2-2 t_{k}}{k-2 t_{k}} \text {, and } \\
& \frac{2-2 t_{k}}{k-2 t_{k}}=\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t_{k} .
\end{aligned}
$$

Suppose $k \geq 5$. Let

$$
f_{k}(t)=\frac{2-2 t}{k-2 t}
$$

and

$$
g_{k}(t)=\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t .
$$

It is easy to see that $f_{k}(t)$ is decreasing, and $g_{k}(t)$ is increasing. Now assume that $\frac{1}{k-2} \leq$ $h_{k-1}<\frac{2}{k-1}$ for some $k \geq 5$. Note that

$$
g_{k}(0)=\frac{k-3}{k} h_{k-1}<\frac{k-3}{k} \frac{2}{k-1}<\frac{2}{k}=f_{k}(0),
$$

and

$$
g_{k}(1 / 2)=\frac{k-2}{k} h_{k-1}+\frac{4}{k(k-1)} \geq \frac{1}{k}+\frac{4}{k(k-1)}>\frac{1}{k-1}=f_{k}(1 / 2) .
$$

Therefore, since $f_{k}(t)$ is decreasing and $g_{k}(t)$ is increasing and because both are continuous over [ $0,1 / 2$ ], there exists $t_{k} \in(0,1 / 2)$, for each $k \geq 5$, such that $f_{k}\left(t_{k}\right)=g_{k}\left(t_{k}\right)$. Let $h_{k}:=f_{k}\left(t_{k}\right)=\frac{2-2 t_{k}}{k-2 t_{k}}$. Then since $t_{k} \in(0,1 / 2), 1 /(k-1)<h_{k}<2 / k$ for $k \geq 5$.

Next, we show that $h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$. Let $h_{k}=c_{k} / k$, and it suffices to show $c_{k}<1.6$, and $c_{k}<1.5=3 / 2$ for $k \geq 23$. Since $h_{k} \in(1 /(k-1), 2 / k), c_{k} \in(1,2)$. Note that

$$
c_{k}=\frac{2-2 t_{k}}{k-2 t_{k}} k=(k-3) h_{k-1}+\left(h_{k-1}+\frac{4}{k-1}\right) 2 t_{k}=\frac{k-3}{k-1} c_{k-1}+\frac{4+c_{k-1}}{k-1} 2 t_{k} .
$$

From $c_{k}=\frac{2-2 t_{k}}{k-2 t_{k}} k$ we deduce $t_{k}=\frac{2 k-k c_{k}}{2 k-2 c_{k}}$; and so

$$
c_{k}=\frac{k-3}{k-1} c_{k-1}+\frac{\left(4+c_{k-1}\right)\left(2 k-k c_{k}\right)}{(k-1)\left(k-c_{k}\right)} .
$$

With $h_{4}=1 / 3$ (and hence $c_{4}=4 / 3$ ) and using MATLAB, we have $c_{k}<1.6$ for $k=$ $5, \ldots, 22$, and $c_{23} \approx 1.4962<3 / 2$. Now assume $k \geq 24$ and $c_{k-1}<3 / 2$. Then

$$
c_{k}<\frac{k-3}{k-1} \times \frac{3}{2}+\frac{(4+3 / 2)\left(2 k-k c_{k}\right)}{(k-1)\left(k-c_{k}\right)}
$$

and so

$$
2(k-1) c_{k}<3(k-3)+11\left(2-c_{k}\right)+11\left(2-c_{k}\right) c_{k} /\left(k-c_{k}\right)
$$

Hence, since $c_{k} \in(1,2)$,

$$
(2 k+9) c_{k}<3 k+13+\frac{11\left(2-c_{k}\right) c_{k}}{k-c_{k}}=3 k+13+\frac{11\left(1-\left(1-c_{k}\right)^{2}\right)}{k-c_{k}}<3 k+13+11 /(k-2) .
$$

Therefore,

$$
c_{k}<\frac{3 k+13}{2 k+9}+\frac{11}{(2 k+9)(k-2)} \leq 3 / 2 .
$$

The last inequality holds since we assume $k \geq 24$.

We can now prove the main lemma of this section for $k$-partitions.

Lemma 3.1.3. Let $k \geq 4$ be an integer, let $a_{j} \geq 0$ for $j \in\{1, \ldots, k\}$ such that $\alpha:=\sum_{j=1}^{k} a_{j}>$ 0 , and let $f_{i j}\left(x_{i}, x_{j}\right)=\left(a_{i}+a_{j}\right)\left(x_{i}+x_{j}\right)$ for $1 \leq i \neq j \leq k$. Then there exist $p_{i} \in[0,2 / k]$, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} p_{i}=1$ and, for $1 \leq i \neq j \leq k, f_{i j}\left(p_{i}, p_{j}\right) \leq h_{k} \alpha$, where $h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$.

Proof. We apply induction on $k$; the case $k=4$ follows from Lemma 3.1.1 (as $h_{4}=1 / 3$ ). Suppose $k \geq 5$. By Lemma 3.1.2 and since $h_{4}=1 / 3$, there exist $t_{k} \in(0,1 / 2), h_{k} \in$ $(1 /(k-1), 2 / k)$ for $k \geq 5$ such that

$$
h_{k}=\frac{2-2 t_{k}}{k-2 t_{k}}=\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t_{k},
$$

$h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$.
First, assume that there exists some $l \in\{1, \ldots, k\}$ such that $a_{l} \geq t_{k} \alpha$, say $l=k$. Let $p_{i}=x\left(x\right.$ will be determined later) for $1 \leq i<k$, with $0 \leq x \leq \frac{1}{k-1}$, and let $p_{k}=1-(k-1) x$. Then $\sum_{i=1}^{k} p_{i}=1$; for $1 \leq i \leq k-1$,

$$
f_{i k}\left(p_{i}, p_{k}\right) \leq(1-(k-2) x) \alpha ;
$$

and for $1 \leq i \neq j \leq k-1$,

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq 2 x\left(a_{i}+a_{j}\right) \leq 2 x\left(\alpha-a_{k}\right) \leq\left(1-t_{k}\right) 2 x \alpha .
$$

We wish to minimize $\max \left\{1-(k-2) x,\left(1-t_{k}\right) 2 x\right\}$. Setting $1-(k-2) x=\left(1-t_{k}\right) 2 x$, we have

$$
x=\frac{1}{k-2 t_{k}}
$$

and, for $1 \leq i \neq j \leq k$,

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{2-2 t_{k}}{k-2 t_{k}} \alpha .
$$

We point out that since $t_{k} \in(0,1 / 2)$, indeed $x \in(0,1 /(k-1)]$ and so $x$ is well-defined. Note that $p_{i} \in[0,2 / k]$ for $1 \leq i \leq k$.

Second, let us assume that $a_{i} \leq t_{k} \alpha$ for all $1 \leq i \leq k$. By the induction hypothesis, for any $l \in\{1, \ldots, k\}$ there exist $p_{i}^{l} \in[0,2 /(k-1)], i \in\{1, \ldots, k\} \backslash\{l\}$, such that $\sum_{i \in\{1, \ldots, k\} \backslash l\}} p_{i}^{l}=1$ and for any $\{i, j\} \subseteq\{1, \ldots, k\} \backslash\{l\}$,

$$
\left(a_{i}+a_{j}\right)\left(p_{i}^{l}+p_{j}^{l}\right) \leq h_{k-1}\left(\alpha-a_{l}\right) .
$$

For $1 \leq i \leq k$, let

$$
p_{i}=\frac{1}{k} \sum_{l \in\{1, \ldots, k \backslash \backslash i\}} p_{i}^{l} .
$$

Since $p_{i}^{l} \leq 2 /(k-1)$ for $i \in\{1, \ldots, k\} \backslash\{l\}$, we have $p_{i} \in[0,2 / k]$ for $1 \leq i \leq k$. Also,

$$
\sum_{i=1}^{k} p_{i}=\frac{1}{k} \sum_{i=1}^{k} \sum_{l \in\{1, \ldots, k \backslash \backslash\{i\}} p_{i}^{l}=\frac{1}{k} \sum_{l=1}^{k} \sum_{i \in\{1, \ldots, k\} \backslash\{l\}} p_{i}^{l}=\frac{1}{k} \sum_{l=1}^{k} 1=1 .
$$

Moreover, for $1 \leq i \neq j \leq k$,

$$
\begin{aligned}
f_{i j}\left(p_{i}, p_{j}\right) & =\left(a_{i}+a_{j}\right)\left(p_{i}+p_{j}\right) \\
& =\frac{1}{k}\left(a_{i}+a_{j}\right)\left(\sum_{l \in\{1, \ldots, k \backslash \backslash \backslash i\}} p_{i}^{l}+\sum_{l \in\{1, \ldots, k \backslash \backslash \backslash j\}} p_{j}^{l}\right) \\
& =\frac{1}{k}\left(\sum_{l \in\{1, \ldots, k \backslash \backslash i, j\}}\left(a_{i}+a_{j}\right)\left(p_{i}^{l}+p_{j}^{l}\right)\right)+\frac{1}{k}\left(a_{i}+a_{j}\right)\left(p_{i}^{j}+p_{j}^{i}\right) \\
& \leq \frac{h_{k-1}}{k} \sum_{l \in\{1, \ldots, k \backslash \backslash\{i, j\}}\left(\alpha-a_{l}\right)+\frac{1}{k}\left(a_{i}+a_{j}\right)\left(p_{i}^{j}+p_{j}^{i}\right) \\
& \leq \frac{h_{k-1}}{k}\left((k-3) \alpha+a_{i}+a_{j}\right)+\frac{4}{k(k-1)}\left(a_{i}+a_{j}\right) \\
& \leq\left(\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t_{k}\right) \alpha .
\end{aligned}
$$

Note that

$$
h_{k}=\frac{2-2 t_{k}}{k-2 t_{k}}=\frac{k-3}{k} h_{k-1}+\left(\frac{h_{k-1}}{k}+\frac{4}{k(k-1)}\right) 2 t_{k},
$$

$h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$. This completes the proof of the lemma.

Theorem 3.1.4. Let $k \geq 4$ be an integer. Then $f(k, m) \leq h_{k} m+O\left(m^{4 / 5}\right)$, where $h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$.

Proof. Let $G$ be a graph with $m$ edges, and we may assume that $G$ is connected (as otherwise we simply consider individual components). Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq$ $d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $V_{1}=\left\{v_{1}, \ldots, v_{t}\right\}$ with $t=\left\lfloor m^{\alpha}\right\rfloor$, where $0<\alpha<1 / 2$ and will be optimized later. Then $t<n$ since $m<n^{2} / 2$. Moreover,

$$
e\left(V_{1}\right)<t^{2} / 2 \leq \frac{1}{2} m^{2 \alpha} \quad \text { and } \quad d\left(v_{t+1}\right)<2 m^{1-\alpha},
$$

since $(t+1) d\left(v_{t+1}\right) \leq \sum_{i=1}^{t+1} d\left(v_{i}\right) \leq 2 m$.
Label the vertices in $V_{2}:=V(G) \backslash V_{1}$ as $u_{1}, \ldots, u_{n-t}$ such that $e\left(u_{i}, V_{1} \cup\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$. Note that this can be done since $G$ is connected.

Fix an arbitrary $k$-partition $V_{1}=\bigcup_{i=1}^{k} Y_{i}$, and assign each member of $Y_{i}$ the color $i$, $1 \leq i \leq k$. Extend this coloring to $V(G)$ such that each vertex $u_{i} \in V_{2}$ is independently assigned the color $j$ with probability $p_{j}^{i}$, where $\sum_{j=1}^{k} p_{j}^{i}=1$ and $p_{j}^{i}$ will be determined later. Let $Z_{i}$ denote the indicator random variable of the event of coloring $u_{i}$. Hence $Z_{i}=j$ iff $u_{i}$ is assigned the color $j$.

Let $G_{i}=G\left[V_{1} \cup\left\{u_{1}, \cdots, u_{i}\right\}\right]$ for $i=1, \ldots, n-t$, and let $G_{0}=G\left[V_{1}\right]$. Let $X_{j}^{0}=Y_{j}$ for $1 \leq j \leq k$, and $x_{j l}^{0}=e\left(X_{j}^{0} \cup X_{l}^{0}\right)$ for $1 \leq j \neq l \leq k$. For $i=1, \ldots, n-t$ and $1 \leq j, l \leq k$, define

$$
\begin{aligned}
X_{j}^{i} & :=\left\{\text { vertices of } G_{i} \text { with color } j\right\}, \\
x_{j l}^{i} & :=e\left(X_{j}^{i} \cup X_{l}^{i}\right), \\
\Delta x_{j l}^{i} & :=x_{j l}^{i}-x_{j l}^{i-1}, \\
b_{j}^{i} & :=e\left(u_{i}, X_{j}^{i-1}\right) .
\end{aligned}
$$

Note that $b_{j}^{i}$ depends on $\left(Z_{1}, \ldots, Z_{i-1}\right)$ only. Hence for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq k$,

$$
\mathbb{E}\left(\Delta x_{j l}^{i} \mid Z_{1}, \ldots, Z_{i-1}\right)=\left(b_{j}^{i}+b_{l}^{i}\right)\left(p_{j}^{i}+p_{l}^{i}\right),
$$

and so

$$
\mathbb{E}\left(\Delta x_{j l}^{i}\right)=\left(a_{j}^{i}+a_{l}^{i}\right)\left(p_{j}^{i}+p_{l}^{i}\right),
$$

where

$$
a_{j}^{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) b_{j}^{i} .
$$

Since $b_{j}^{i}$ is determined by $\left(Z_{1}, \ldots, Z_{i-1}\right), a_{j}^{i}$ is determined by $p_{j}^{s}, 1 \leq j \leq k$ and $1 \leq s \leq$ $i-1$. Note that $\sum_{j=1}^{k} b_{j}^{i}=e\left(u_{i}, G_{i-1}\right)>0$, and that $e\left(u_{i}, G_{i-1}\right)$ is independent of $Z_{1}, \ldots, Z_{n-t}$. Moreover,

$$
\begin{aligned}
\sum_{j=1}^{k} a_{j}^{i} & =\sum_{j=1}^{k} \sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) b_{j}^{i} \\
& =\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)}\left(\mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) \sum_{j=1}^{k} b_{j}^{i}\right) \\
& =\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} \mathbb{P}\left(Z_{1}, \ldots, Z_{i-1}\right) e\left(u_{i}, G_{i-1}\right) \\
& =e\left(u_{i}, G_{i-1}\right) \\
& >0 .
\end{aligned}
$$

So by Lemma 3.1.3, there exist $p_{j}^{i} \in[0,1], 1 \leq j \leq k$, such that $\sum_{j=1}^{k} p_{j}^{i}=1$ and, for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq k$,

$$
\mathbb{E}\left(\Delta x_{j l}^{i}\right) \leq h_{k} \sum_{j=1}^{k} a_{j}^{i}=h_{k} e\left(u_{i}, G_{i-1}\right),
$$

where $h_{k}<1.6 / k$, and $h_{k}<1.5 / k$ for $k \geq 23$.
Note that $p_{j}^{i}$ is determined by $a_{j}^{i}, 1 \leq i \leq k$; and hence $p_{j}^{i}$ is recursively determined by $p_{j}^{s}, 1 \leq j \leq k$ and $1 \leq s \leq i-1$. Also note that $m=e\left(G_{0}\right)+\sum_{i=1}^{n-t} e\left(u_{i}, G_{i-1}\right)$. Now

$$
\begin{aligned}
\mathbb{E}\left(x_{j l}^{n-t}\right) & =\sum_{i=1}^{n-t} \mathbb{E}\left(\Delta x_{j l}^{i}\right)+\mathbb{E}\left(x_{j l}^{0}\right) \\
& \leq h_{k} \sum_{i=1}^{n-t} e\left(u_{i}, G_{i-1}\right)+x_{j l}^{0} \\
& \leq h_{k} m+e\left(V_{1}\right) \\
& \leq h_{k} m+\frac{1}{2} m^{2 \alpha} .
\end{aligned}
$$

Clearly, changing the color of $u_{i}$ (i.e., changing $Z_{i}$ ) affects $x_{j l}:=x_{j l}^{n-t}$ by at most $d\left(u_{i}\right)$. So
by Lemma 1.4.1,

$$
\begin{aligned}
\mathbb{P}\left(x_{j l}>\mathbb{E}\left(x_{j l}\right)+z\right) & \leq \exp \left(-\frac{z^{2}}{2 \sum_{i=1}^{n-t} d\left(u_{i}\right)^{2}}\right) \\
& \leq \exp \left(-\frac{z^{2}}{2 \sum_{i=1}^{n-t} d\left(u_{i}\right) d\left(v_{t+1}\right)}\right) \\
& <\exp \left(-\frac{z^{2}}{4 m 2 m^{1-\alpha}}\right) \\
& \leq \exp \left(-\frac{z^{2}}{8 m^{2-\alpha}}\right) .
\end{aligned}
$$

Let $z=(8 \ln (k(k-1) / 2))^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then for $1 \leq j \neq l \leq k$,

$$
\mathbb{P}\left(x_{j l}>\mathbb{E}\left(x_{j l}\right)+z\right)<\exp (-\ln (k(k-1) / 2))=\frac{2}{k(k-1)} .
$$

So there exists a partition $V(G)=\bigcup_{i=1}^{k} X_{i}$ such that for $1 \leq j \neq l \leq k$,

$$
e\left(X_{j} \cup X_{l}\right) \leq \mathbb{E}\left(x_{j l}\right)+z \leq h_{k} m+\frac{1}{2} m^{2 \alpha}+z \leq h_{k} m+o(m),
$$

where the $o(m)$ term in the expression is

$$
\frac{1}{2} m^{2 \alpha}+(8 \ln (k(k-1) / 2))^{\frac{1}{2}} m^{1-\frac{\alpha}{2}} .
$$

Choosing $\alpha=\frac{2}{5}$ to minimize $\max \{2 \alpha, 1-\alpha / 2\}$, the $o(m)$ term becomes $O\left(m^{\frac{4}{5}}\right)$.

### 3.2 Dense graphs

We now prove Conjecture 1.3 .7 for graphs with large minimum degree. The approach is similar to that for proving Theorem 3.1.4, but simpler because the large minimum degree condition helps to bound $e\left(V_{1}, V_{2}\right)$. Note that the term $4 m / k^{2}$ in the theorem below is best possible (by simply taking a random $k$-partition). The following result implies Theorem 1.3.8.

Theorem 3.2.1. Let $k \geq 2$ be an integer and let $\epsilon>0$. If $G$ is a graph with $m$ edges and $\delta(G) \geq \epsilon n$, then there is a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i \neq j \leq k$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{4}{k^{2}} m+\left(\sqrt{2 / \epsilon}+\sqrt{8 \ln \frac{k(k-1)}{2}}\right) m^{5 / 6} .
$$

Proof. We may assume that $G$ is connected (otherwise it suffices to consider individual components). Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $V_{1}=$ $\left\{v_{1}, \ldots, v_{t}\right\}$ with $t=\left\lfloor m^{\alpha}\right\rfloor$, where $0<\alpha<1 / 2$. Then

$$
t<n-1, \quad e\left(V_{1}\right)<m^{2 \alpha} / 2, \quad \text { and } \quad d\left(v_{t+1}\right) \leq 2 m^{1-\alpha} .
$$

Let $V_{2}=V(G) \backslash V_{1}=\left\{u_{1}, \ldots, u_{n-t}\right\}$ such that $e\left(u_{i}, V_{1} \cup\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$. Now assume $\delta(G) \geq \epsilon n$. Then $2 m=\sum_{v \in V(G)} d(v) \geq \epsilon n^{2}$. So $n \leq \sqrt{2 m / \epsilon}$. Thus,

$$
e\left(V_{1}, V_{2}\right)+2 e\left(V_{1}\right)=\sum_{i=1}^{t} d\left(v_{i}\right)<t n \leq m^{\alpha} \sqrt{2 m / \epsilon}=\sqrt{2 / \epsilon} m^{1 / 2+\alpha} .
$$

Fix an arbitrary partition $V_{1}=Y_{1} \cup Y_{2} \cup \ldots \cup Y_{k}$ and, for each $i \in\{1, \ldots, k\}$, assign the color $i$ to all vertices in $Y_{i}$. We extend this coloring to $V(G)$ by independently assigning the color $j$ (for each $j \in\{1, \ldots, k\}$ ) to each vertex $u_{i} \in V_{2}$ with probability $1 / k$. Let $Z_{i}$ denote the indicator random variable of the event of coloring $u_{i}$.

Let $X_{i}$ be the set of vertices of $G$ with color $i$. Then $Y_{i} \subseteq X_{i}$ for $1 \leq i \leq k$; and for $1 \leq i \neq j \leq k$,

$$
\begin{aligned}
\mathbb{E}\left(e\left(X_{i} \cup X_{j}\right)\right) & =\mathbb{E}\left(e\left(\left(X_{i} \cup X_{j}\right) \cap V_{2}\right)\right)+\mathbb{E}\left(e\left(\left(X_{i} \cup X_{j}\right) \cap V_{2}, Y_{i} \cup Y_{j}\right)\right)+e\left(Y_{i} \cup Y_{j}\right) \\
& \leq(2 / k)^{2} e\left(V_{2}\right)+e\left(V_{1}, V_{2}\right)+e\left(V_{1}\right) \\
& \leq \frac{4}{k^{2}} m+\sqrt{2 / \epsilon} m^{1 / 2+\alpha} .
\end{aligned}
$$

Clearly, changing the color of $u_{i}$ (i.e., changing $Z_{i}$ ) affects $e\left(X_{i} \cup X_{j}\right)$ by at most $d\left(u_{i}\right)$. Then as in the proof of Theorem 3.1.4, we apply Lemma 1.4.1 to conclude that for any $1 \leq i \neq j \leq k$,

$$
\mathbb{P}\left(e\left(X_{i} \cup X_{j}\right)>\mathbb{E}\left(e\left(X_{i} \cup X_{j}\right)\right)+z\right) \leq \exp \left(-\frac{z^{2}}{2 \sum_{i=1}^{n-t} d\left(u_{i}\right)^{2}}\right)<\exp \left(-\frac{z^{2}}{8 m^{2-\alpha}}\right) .
$$

Let $z=\sqrt{8 \ln (k(k-1) / 2)} m^{1-\alpha / 2}$. Then for $1 \leq i \neq j \leq k$,

$$
\mathbb{P}\left(e\left(X_{i} \cup X_{j}\right)>\mathbb{E}\left(e\left(X_{i} \cup X_{j}\right)\right)+z\right)<\exp \left(-\ln \frac{k(k-1)}{2}\right)=\frac{2}{k(k-1)} .
$$

So there exists a partition $V(G)=X_{1} \cup X_{2} \cup \ldots \cup X_{k}$ such that, for $1 \leq i \neq j \leq k$,

$$
\begin{aligned}
e\left(X_{i} \cup X_{j}\right) & \leq \frac{4}{k^{2}} m+\sqrt{2 / \epsilon} m^{1 / 2+\alpha}+z \\
& \leq \frac{4}{k^{2}} m+\sqrt{2 / \epsilon} m^{1 / 2+\alpha}+\sqrt{8 \ln (k(k-1) / 2)} m^{1-\alpha / 2}
\end{aligned}
$$

Picking $\alpha=1 / 3$ to minimize $\max \{1 / 2+\alpha, 1-\alpha / 2\}$, we have the desired bound.
As a corollary, Conjecture 1.3 .9 holds for graphs with $\Omega\left(k^{12}(\ln k)^{3}\right)$ edges. Hence Conjecture 1.3.7 holds for all graphs $G$ with $\delta(G) \geq \epsilon n$, for any fixed $k \geq 2$ and $\epsilon>0$.

### 3.3 Bounds for 4-partitions and 5-partitions

In this section, we prove Conjecture 1.3.7 for 4-partitions and 5-partitions. For 4-partitions, we use Lemma 3.1.1. For 5-partitions, we need the following lemma.

Lemma 3.3.1. Let $a_{j} \geq 0$ for $j \in\{1, \ldots, 5\}$ such that $\alpha:=\sum_{j=1}^{5} a_{j}>0$, and let $f_{i j}\left(x_{i}, x_{j}\right)=$ $\left(a_{i}+a_{j}\right)\left(x_{i}+x_{j}\right)$ for $1 \leq i \neq j \leq 5$. Then there exist $p_{i} \in[0,2 / 5], 1 \leq i \leq 5$, such that $\sum_{i=1}^{5} p_{i}=1$ and, for $1 \leq i \neq j \leq 5, f_{i j}\left(p_{i}, p_{j}\right) \leq 4 \alpha / 15$.

Proof. If there exists some $l \in\{1, \ldots, 5\}$ such that $a_{l} \geq 5 \alpha / 11$, then $a_{i}+a_{j} \leq 6 \alpha / 11$ for $\{i, j\} \subseteq\{1, \ldots, 5\} \backslash\{l\}$. Let $p_{l}=1 / 45$ and let $p_{i}=11 / 45$ for $i \in\{1, \ldots, 5\} \backslash\{l\}$. Then for $i \in\{1, \ldots, 5\} \backslash\{l\}$,

$$
f_{i l}\left(p_{i}, p_{l}\right)=\left(a_{i}+a_{l}\right)\left(p_{i}+p_{l}\right) \leq \alpha\left(\frac{11}{45}+\frac{1}{45}\right)=\frac{4}{15} \alpha ;
$$

and for $\{i, j\} \subseteq\{1, \ldots, 5\} \backslash\{l\}$,

$$
f_{i j}=\left(a_{i}+a_{j}\right)\left(p_{i}+p_{j}\right) \leq \frac{6 \alpha}{11}\left(\frac{11}{45}+\frac{11}{45}\right)=\frac{4}{15} \alpha .
$$

Therefore, we may assume that $a_{i}<5 \alpha / 11$ for all $1 \leq i \leq 5$. By Lemma 3.1.1, for any $1 \leq l \leq 5$ there exist $p_{i}^{l} \in[0,1 / 2], i \in\{1, \ldots, 5\} \backslash\{l\}$, such that $\sum_{i \in\{1, \ldots, 5\} \backslash\{l\}} p_{i}^{l}=1$ and, for $\{i, j\} \subseteq\{1, \ldots, 5\} \backslash\{l\}$,

$$
\left(a_{i}+a_{j}\right)\left(p_{i}^{l}+p_{j}^{l}\right) \leq \frac{1}{3}\left(\alpha-a_{l}\right) .
$$

Indeed, by the remark following Lemma 3.1.1, we may choose $p_{i}^{l}, i \in\{1, \ldots, 5\} \backslash\{l\}$, such that $p_{i}^{l}=0$ when $a_{i}>\left(\alpha-a_{l}\right) / 2$, and $p_{i}^{l} \leq \max \left\{1 / 2-a_{i} /\left(\alpha-a_{l}\right), 1 / 3\right\}$ when $a_{i} \leq\left(\alpha-a_{l}\right) / 2$.

For $1 \leq i \leq 5$, let $p_{i}=\frac{1}{5} \sum_{l \in\{1, \ldots, 5\} \backslash\{i\}} p_{i}^{l}$. Then $p_{i} \in[0,2 / 5]$, and

$$
\sum_{i=1}^{5} p_{i}=\frac{1}{5} \sum_{i=1}^{5} \sum_{l \in\{1, \ldots, 5 \backslash \backslash\{i\}} p_{i}^{l}=\frac{1}{5} \sum_{l=1}^{5} \sum_{i \in\{1, \ldots, 5) \backslash\{l\}} p_{i}^{l}=\frac{1}{5} \sum_{l=1}^{5} 1=1 .
$$

So for $1 \leq i \neq j \leq 5$,

$$
\begin{aligned}
f_{i j}\left(p_{i}, p_{j}\right) & =\left(a_{i}+a_{j}\right)\left(p_{i}+p_{j}\right) \\
& =\frac{1}{5}\left(a_{i}+a_{j}\right)\left(\sum_{l \in\{1, \ldots, 5 \backslash \backslash \backslash i\}} p_{i}^{l}+\sum_{l \in\{1, \ldots, 5 \backslash \backslash(j\}} p_{j}^{l}\right) \\
& =\frac{1}{5}\left(\sum_{l \in\{1, \ldots, 5 \backslash \backslash i, j\}}\left(a_{i}+a_{j}\right)\left(p_{i}^{l}+p_{j}^{l}\right)\right)+\frac{1}{5}\left(a_{i}+a_{j}\right)\left(p_{i}^{j}+p_{j}^{i}\right) \\
& \leq \frac{1}{15}\left(\sum_{l \in\{1, \ldots, 5 \backslash \backslash i, j\}}\left(\alpha-a_{l}\right)\right)+\frac{1}{5}\left(a_{i}+a_{j}\right)\left(p_{i}^{j}+p_{j}^{i}\right) \\
& =\frac{1}{15}\left(2 \alpha+a_{i}+a_{j}\right)+\frac{1}{5}\left(a_{i}+a_{j}\right)\left(p_{i}^{j}+p_{j}^{i}\right) \\
& =\frac{2}{15} \alpha+\left(a_{i}+a_{j}\right)\left(\frac{1}{15}+\frac{1}{5}\left(p_{i}^{j}+p_{j}^{i}\right)\right) .
\end{aligned}
$$

We need to show that $f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{4}{15} \alpha$ for $1 \leq i \neq j \leq 5$.
If $a_{i}>\left(\alpha-a_{j}\right) / 2$ and $a_{j}>\left(\alpha-a_{i}\right) / 2$, then $p_{i}^{j}=p_{j}^{i}=0$, and hence

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{3}{15} \alpha<\frac{4}{15} \alpha .
$$

Now assume $a_{i}>\left(\alpha-a_{j}\right) / 2$ and $a_{j} \leq\left(\alpha-a_{i}\right) / 2$. Then $p_{i}^{j}=0$ and $p_{j}^{i} \leq \max \{1 / 2-$ $\left.a_{j} /\left(\alpha-a_{i}\right), 1 / 3\right\}$. Suppose $1 / 2-a_{j} /\left(\alpha-a_{i}\right)>1 / 3$. Then $a_{j}<\left(\alpha-a_{i}\right) / 6$; and hence, since $a_{i}>\left(\alpha-a_{j}\right) / 2$, we have $a_{i}>\left(\alpha-\alpha / 6+a_{i} / 6\right) / 2$. Solving this inequality for $a_{i}$, we have $a_{i}>5 \alpha / 11$ which contradicts our assumption. Therefore, $1 / 2-a_{j} /\left(\alpha-a_{i}\right) \leq 1 / 3$, and so $p_{j}^{i} \leq 1 / 3$. Hence

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{2}{15} \alpha+\left(a_{i}+a_{j}\right)\left(\frac{1}{15}+\frac{1}{5} \frac{1}{3}\right) \leq \frac{4}{15} \alpha .
$$

By symmetry, if $a_{j}>\left(\alpha-a_{i}\right) / 2$ and $a_{i} \leq\left(\alpha-a_{j}\right) / 2$, then $f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{4}{15} \alpha$.

So we are left with the case when $a_{i} \leq\left(\alpha-a_{j}\right) / 2$ and $a_{j} \leq\left(\alpha-a_{i}\right) / 2$. Then $a_{i}+a_{j} \leq$ $\alpha-\left(a_{i}+a_{j}\right) / 2$, and so $a_{i}+a_{j} \leq 2 \alpha / 3$. Moreover, $p_{i}^{j} \leq \max \left\{1 / 2-a_{i} /\left(\alpha-a_{j}\right), 1 / 3\right\}$ and $p_{j}^{i} \leq \max \left\{1 / 2-a_{j} /\left(\alpha-a_{i}\right), 1 / 3\right\}$.

If $1 / 2-a_{i} /\left(\alpha-a_{j}\right)>1 / 3$ and $1 / 2-a_{j} /\left(\alpha-a_{i}\right)>1 / 3$, then $6 a_{i}+a_{j}<\alpha$ and $6 a_{j}+a_{i}<\alpha$. Hence $a_{i}+a_{j}<2 \alpha / 7$, and so (since $p_{i}^{j} \leq 1 / 2$ and $p_{j}^{i} \leq 1 / 2$ ),

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{2}{15} \alpha+\left(a_{i}+a_{j}\right)\left(\frac{1}{15}+\frac{1}{5}\left(\frac{1}{2}+\frac{1}{2}\right)\right)<\frac{2}{15} \alpha+\frac{2}{7} \frac{4}{15} \alpha<\frac{4}{15} \alpha .
$$

If $1 / 2-a_{i} /\left(\alpha-a_{j}\right)>1 / 3$ and $1 / 2-a_{j} /\left(\alpha-a_{i}\right) \leq 1 / 3$, then $6 a_{i}+a_{j} \leq \alpha$ and $p_{j}^{i} \leq 1 / 3$. Since $a_{j} \leq\left(\alpha-a_{i}\right) / 2, a_{i}+2 a_{j} \leq \alpha$. So $11\left(a_{i}+a_{j}\right)=6 a_{i}+a_{j}+5\left(a_{i}+2 a_{j}\right) \leq 6 \alpha$, and hence $a_{i}+a_{j} \leq 6 \alpha / 11$. Then

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{2}{15} \alpha+\left(a_{i}+a_{j}\right)\left(\frac{1}{15}+\frac{1}{5}\left(\frac{1}{2}+\frac{1}{3}\right)\right) \leq \frac{2}{15} \alpha+\frac{6}{11} \frac{7}{30} \alpha<\frac{4}{15} \alpha .
$$

The case when $1 / 2-a_{i} /\left(\alpha-a_{j}\right) \leq 1 / 3$ and $1 / 2-a_{j} /\left(\alpha-a_{i}\right)>1 / 3$ is symmetric.
Therefore, we may assume that $1 / 2-a_{i} /\left(\alpha-a_{j}\right) \leq 1 / 3$ and $1 / 2-a_{j} /\left(\alpha-a_{i}\right) \leq 1 / 3$. Then $p_{i}^{j} \leq 1 / 3$ and $p_{j}^{i} \leq 1 / 3$. Recall that $a_{i}+a_{j} \leq 2 \alpha / 3$. Hence

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \frac{2}{15} \alpha+\left(a_{i}+a_{j}\right)\left(\frac{1}{15}+\frac{1}{5}\left(\frac{1}{3}+\frac{1}{3}\right)\right) \leq \frac{2}{15} \alpha+\frac{2}{3} \frac{1}{5} \alpha=\frac{4}{15} \alpha .
$$

Using the same proof of Theorem 3.1.4, with Lemma 3.1.1 and Lemma 3.3.1 in place of Lemma 3.1.3, we have the following results on 4-partitions and 5-partitions.

Theorem 3.3.2. $f(4, m) \leq m / 3+O\left(m^{4 / 5}\right)$.

Theorem 3.3.3. $f(5, m) \leq 4 m / 15+O\left(m^{4 / 5}\right)$.

Recall that the graphs $K_{1, n}$ give $f(4, m) \geq m / 3$ and $f(5, m) \geq m / 4$. When $k=4$, $12 /((k+2)(k+1))=3 / 5>1 / 3$. So as a consequence of Theorem 3.3.2, Conjecture 1.3.7 holds for $k=4$ asymptotically. When $k=5,12 m /((k+2)(k+1))=2 / 7>4 / 15$. Hence, Theorem 3.3.3 establishes Conjecture 1.3.7 for $k=5$ asymptotically.

### 3.4 Simultaneous bounds for 3-partitions and 4-partitions

In this section, we study the following problem suggested by Bollobás and Scott [12].

Problem 3.4.1. For any integer $k \geq 2$ and for any graph $G$ with $m$ edges and $n$ vertices, is it possible to find a $k$-partition $V_{1}, \ldots, V_{k}$ of $V(G)$ such that for $1 \leq i \leq k$,

$$
e\left(V_{i}\right) \leq \frac{m}{k^{2}}+\frac{k-1}{2 k^{2}}\left(\sqrt{2 m+\frac{1}{4}}-\frac{1}{2}\right),
$$

and for $1 \leq i<j \leq k$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{12 m}{(k+1)(k+2)}+O(n) ?
$$

Recall that Bollobás and Scott [10] showed the existence of a $k$-partition satisfying the above bound on $e\left(V_{i}\right)$, and $K_{k n+1}$ are the only extremal graphs. Also recall that the bound on $e\left(V_{i} \cup V_{j}\right)$ is best possible for $K_{k+2}$.

We show that for $k=3$ and $k=4$, one can find partitions that satisfy these bounds asymptotically. For large $k$, a similar approach as in the proofs of Lemma 3.1.3 and Theorem 3.1.4 may be used to give some bounds.

Note that in the proofs to follow, we will use the fact that the maximum of $x(a-x)$, $a>0$, is $a^{2} / 4$.

Lemma 3.4.2. Let $a_{j} \geq 0$ for $j=1,2,3$ such that $\alpha:=a_{1}+a_{2}+a_{3}>0$, let $f_{i j}\left(x_{i}, x_{j}\right)=$ $\left(a_{i}+a_{j}\right)\left(x_{i}+x_{j}\right)$ for $1 \leq i \neq j \leq 3$, and let $g_{i}\left(x_{i}\right)=a_{i} x_{i}$ for $1 \leq i \leq 3$. Then there exist $p_{i} \in[0,2 / 3], 1 \leq i \leq 3$, such that $\sum_{i=1}^{3} p_{i}=1, f_{i j}\left(p_{i}, p_{j}\right) \leq 5 \alpha / 9$ for $1 \leq i \neq j \leq 3$, and $g_{i}\left(p_{i}\right) \leq \alpha / 9$ for $1 \leq i \leq 3$.

Proof. First, assume that $a_{i}<2 \alpha / 3$ for all $i=1,2,3$. Let $p_{i}=2 / 3-a_{i} / \alpha$. Then $p_{i} \in$ $[0,2 / 3], i=1,2,3$, and $p_{1}+p_{2}+p_{3}=1$. Moreover, for $1 \leq i \neq j \leq 3$,

$$
f_{i j}\left(p_{i}, p_{j}\right)=\frac{a_{i}+a_{j}}{\alpha}\left(\frac{4}{3}-\frac{a_{i}+a_{j}}{\alpha}\right) \alpha \leq \frac{4}{9} \alpha<\frac{5}{9} \alpha
$$

and, for $i=1,2,3$,

$$
g_{i}\left(p_{i}\right)=\frac{a_{i}}{\alpha}\left(\frac{2}{3}-\frac{a_{i}}{\alpha}\right) \alpha \leq \frac{1}{9} \alpha .
$$

Next assume that some $a_{i}>5 \alpha / 6$, say $a_{3}>5 \alpha / 6$. So $a_{1}+a_{2} \leq \alpha / 6$. We choose $p_{1}=p_{2}=4 / 9$ and $p_{3}=1 / 9$. Then $f_{12}\left(p_{1}, p_{2}\right)<\alpha / 6<5 \alpha / 9 ; f_{i 3}\left(p_{i}, p_{3}\right) \leq 5 \alpha / 9$ for $i=1,2 ; g_{3}\left(p_{3}\right) \leq \alpha / 9 ;$ and $g_{i}\left(p_{i}\right) \leq(\alpha / 6)(4 / 9)=2 \alpha / 27<\alpha / 9$ for $i=1,2$.

Therefore, we may assume that there exists some $a_{i}$, say $a_{3}$, such that $2 \alpha / 3 \leq a_{3} \leq$ $5 \alpha / 6$. Then $\alpha / 6 \leq a_{1}+a_{2} \leq \alpha / 3$. Let $p_{3}=0$ and $p_{i}=2 / 3-a_{i} /\left(3\left(a_{1}+a_{2}\right)\right)$ for $i=1,2$. Then $p_{i} \in[0,2 / 3]$ and $p_{1}+p_{2}+p_{3}=1$.

Clearly, $g_{3}\left(p_{3}\right)=0$ and, for $i=1,2$,

$$
g_{i}\left(p_{i}\right)=\frac{a_{i}}{3\left(a_{1}+a_{2}\right)}\left(\frac{2}{3}-\frac{a_{i}}{3\left(a_{1}+a_{2}\right)}\right) 3\left(a_{1}+a_{2}\right) \leq \frac{3}{9}\left(a_{1}+a_{2}\right) \leq \frac{1}{9} \alpha .
$$

Note that $f_{12}\left(p_{1}, p_{2}\right)=a_{1}+a_{2} \leq \alpha / 3<5 \alpha / 9$. So it remains to show that $f_{13}\left(p_{1}, p_{3}\right) \leq$ $5 \alpha / 9$ and $f_{23}\left(p_{2}, p_{3}\right) \leq 5 \alpha / 9$. By symmetry we only need to prove $f_{13}\left(p_{1}, p_{3}\right) \leq 5 \alpha / 9$.

Note that $f_{13}\left(p_{1}, p_{3}\right)=\left(a_{1}+a_{3}\right)\left(2 / 3-a_{1} /\left(3\left(\alpha-a_{3}\right)\right)\right)$, which may be viewed as a function of $a_{1}, a_{3}$ (while fixing $\alpha$ ). We look for the maximal value of $h\left(a_{1}, a_{3}\right):=f_{13}\left(p_{1}, p_{3}\right)$ subject to $2 \alpha / 3 \leq a_{1}+a_{3} \leq \alpha$ and $2 \alpha / 3 \leq a_{3} \leq 5 \alpha / 6$. Taking partial derivatives and setting them to 0 , we have

$$
\frac{\partial h}{\partial a_{1}}=\frac{2}{3}-\frac{a_{1}}{3\left(\alpha-a_{3}\right)}-\frac{a_{1}+a_{3}}{3\left(\alpha-a_{3}\right)}=0,
$$

and

$$
\frac{\partial h}{\partial a_{3}}=\frac{2}{3}-\frac{a_{1}}{3\left(\alpha-a_{3}\right)}-\frac{1}{3} a_{1} \frac{a_{1}+a_{3}}{\left(\alpha-a_{3}\right)^{2}}=0 .
$$

Then $a_{1} /\left(\alpha-a_{3}\right)=1\left(\right.$ from $\frac{\partial h}{\partial a_{1}}=\frac{\partial h}{\partial a_{3}}$ ), and hence $a_{3}=0\left(\right.$ from $\left.\frac{\partial h}{\partial a_{1}}=0\right)$, a contradiction. So the maximal value of $h$ occurs on the boundary of the region defined by $2 \alpha / 3 \leq a_{1}+a_{3} \leq \alpha$ and $2 \alpha / 3 \leq a_{3} \leq 5 \alpha / 6$.

When $a_{1}+a_{3}=2 \alpha / 3$, then $a_{1}=0$ and $a_{3}=2 \alpha / 3$, and hence $h=4 \alpha / 9$. When $a_{1}+a_{3}=\alpha$ then $h=\alpha / 3$. When $a_{3}=2 \alpha / 3$ then $h=\left(a_{1}+2 \alpha / 3\right)\left(2 / 3-a_{1} / \alpha\right)=$ $\left(2 / 3+a_{1} / \alpha\right)\left(2 / 3-a_{1} / \alpha\right) \alpha \leq 4 \alpha / 9$. When $a_{3}=5 \alpha / 6$, then $h \leq\left(a_{1}+5 \alpha / 6\right)\left(2 / 3-2 a_{1} / \alpha\right)=$ $\left(5 / 6+a_{1} / \alpha\right)\left(2 / 3-2 a_{1} / \alpha\right) \alpha \leq 5 \alpha / 9$. Hence $f_{13}\left(p_{1}, p_{3}\right) \leq 5 \alpha / 9$.

The next lemma is for 4-partitions.

Lemma 3.4.3. Let $a_{j} \geq 0$ for $j=1,2,3,4$ such that $\alpha:=a_{1}+a_{2}+a_{3}+a_{4}>0$, let $f_{i j}\left(x_{i}, x_{j}\right)=\left(a_{i}+a_{j}\right)\left(x_{i}+x_{j}\right)$ for $1 \leq i \neq j \leq 4$, and let $g_{i}\left(x_{i}\right)=a_{i} x_{i}$ for $1 \leq i \leq 4$. Then there exist $p_{i} \in[0,1 / 2], 1 \leq i \leq 4$, such that $\sum_{i=1}^{4} p_{i}=1, f_{i j}\left(p_{i}, p_{j}\right) \leq 2 \alpha / 5$ for $1 \leq i \neq j \leq 4$, and $g_{i}\left(p_{i}\right) \leq \alpha / 16$ for $1 \leq i \leq 4$.

Proof. First, suppose $a_{i}<\alpha / 2$ for all $1 \leq i \leq 4$. Let $p_{i}=1 / 2-a_{i} / \alpha$. Then $p_{i} \in[0,1 / 2]$ for $1 \leq i \leq 4$, and $\sum_{i=1}^{4} p_{i}=1$. Moreover, for $1 \leq i \neq j \leq 4$,

$$
f_{i j}\left(p_{i}, p_{j}\right)=\frac{a_{i}+a_{j}}{\alpha}\left(1-\frac{a_{i}+a_{j}}{\alpha}\right) \alpha \leq \frac{1}{4} \alpha<\frac{2}{5} \alpha,
$$

and for $1 \leq i \leq 4$,

$$
g_{i}\left(p_{i}\right)=\frac{a_{i}}{\alpha}\left(\frac{1}{2}-\frac{a_{i}}{\alpha}\right) \alpha \leq \frac{1}{16} \alpha .
$$

Now assume that some $a_{i}>4 \alpha / 5$, say $a_{4}>4 \alpha / 5$. Then $a_{1}+a_{2}+a_{3} \leq \alpha / 5$. Let $p_{1}=p_{2}=p_{3}=5 / 16$ and $p_{4}=1 / 16$. Then for $i=1,2,3$,

$$
f_{i 4}\left(p_{i}, p_{4}\right) \leq 6 \alpha / 16<2 \alpha / 5 ;
$$

for $1 \leq i \neq j \leq 3$,

$$
f_{i j}\left(p_{i}, p_{j}\right) \leq \alpha / 5<2 \alpha / 5
$$

$g_{4}\left(p_{4}\right) \leq \alpha / 16$; and for $i=1,2,3, g_{i}\left(p_{i}\right) \leq(\alpha / 5)(5 / 16)=\alpha / 16$.
So we may assume that there exists some $a_{i}$, say $a_{4}$, such that $\alpha / 2 \leq a_{4} \leq 4 \alpha / 5$. Then $\alpha / 5 \leq a_{1}+a_{2}+a_{3} \leq \alpha / 2$. Let $p_{4}=0$ and $p_{i}=1 / 2-a_{i} /\left(2\left(\alpha-a_{4}\right)\right)$ for $i=1,2,3$. Then $p_{i} \in[0,1 / 2]$ and $\sum_{i=1}^{4} p_{i}=1$.

Clearly, $g_{4}\left(p_{4}\right)=0$. Note that $\alpha-a_{4} \leq \alpha / 2$. So for $i=1,2,3$

$$
g_{i}\left(p_{i}\right)=\frac{a_{i}}{2\left(\alpha-a_{4}\right)}\left(\frac{1}{2}-\frac{a_{i}}{2\left(\alpha-a_{4}\right)}\right) 2\left(\alpha-a_{4}\right) \leq \frac{1}{16} \alpha ;
$$

and for $1 \leq i \neq j \leq 3$,

$$
f_{i j}\left(p_{i}, p_{j}\right)=\frac{a_{i}+a_{j}}{2\left(\alpha-a_{4}\right)}\left(1-\frac{a_{i}+a_{j}}{2\left(\alpha-a_{4}\right)}\right) 2\left(\alpha-a_{4}\right) \leq \frac{1}{4} \alpha<\frac{2}{5} \alpha .
$$

Thus it remains to prove $f_{i 4}\left(p_{i}, p_{4}\right) \leq 2 \alpha / 5$ for $i=1,2,3$. By symmetry, we only prove $f_{14}\left(p_{1}, p_{4}\right) \leq 2 \alpha / 5$. Note that $h\left(a_{1}, a_{4}\right):=f_{14}\left(p_{1}, p_{4}\right)=\left(a_{1}+a_{4}\right)\left(1 / 2-a_{1} /\left(2\left(\alpha-a_{4}\right)\right)\right)$ may be viewed as a function of $a_{1}, a_{4}$ (while fixing $\alpha$ ), and we look for its maximal value subject to $\alpha / 2 \leq a_{1}+a_{4} \leq \alpha$ and $\alpha / 2 \leq a_{4} \leq 4 \alpha / 5$.

Taking partial derivatives and setting them to 0 , we have

$$
\frac{\partial h}{\partial a_{1}}=\frac{1}{2}-\frac{a_{1}}{2\left(\alpha-a_{4}\right)}-\frac{1}{2} \frac{a_{1}+a_{4}}{\alpha-a_{4}}=0,
$$

and

$$
\frac{\partial h}{\partial a_{4}}=\frac{1}{2}-\frac{a_{1}}{2\left(\alpha-a_{4}\right)}-\frac{1}{2} a_{1} \frac{a_{1}+a_{4}}{\left(\alpha-a_{4}\right)^{2}}=0 .
$$

Then $a_{1} /\left(\alpha-a_{4}\right)=1\left(\right.$ from $\left.\frac{\partial h}{\partial a_{1}}=\frac{\partial h}{\partial a_{4}}\right)$, and so $a_{4}<0\left(\right.$ from $\left.\frac{\partial h}{\partial a_{1}}=0\right)$, a contradiction. Thus, the maximal value of $h$ occurs when $a_{1}+a_{4} \in\{\alpha / 2, \alpha\}$ or $a_{4} \in\{\alpha / 2,4 \alpha / 5\}$.

When $a_{1}+a_{4}=\alpha / 2$, we have $a_{1}=0$ and $a_{4}=\alpha / 2$, and hence $h=\alpha / 4$. When $a_{1}+a_{4}=\alpha$, then $h=0$. When $a_{4}=\alpha / 2$ then $h=\alpha\left(1 / 2+a_{1} / \alpha\right)\left(1 / 2-a_{1} / \alpha\right) \leq \alpha / 4$. When $a_{4}=4 \alpha / 5$, then $h=\alpha\left(4 / 5+a_{1} / \alpha\right)\left(1 / 2-5 a_{1} /(2 \alpha)\right) \leq 2 \alpha / 5$. Hence $f_{14}\left(a_{1}, a_{4}\right) \leq 2 \alpha / 5$.

Now we use Lemma 3.4.2 and (essentially) the same proof of Theorem 3.1.4 to prove

Theorem 3.4.4. Let $G$ be a graph with $m$ edges. Then there is a partition $V_{1}, V_{2}, V_{3}$ of $V(G)$ such that for $1 \leq i \leq 3$,

$$
e\left(V_{i}\right) \leq \frac{1}{9} m+O\left(m^{4 / 5}\right),
$$

and for $1 \leq i \neq j \leq 3$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{5}{9} m+O\left(m^{4 / 5}\right)
$$

Proof. We may assume that $G$ is connected. Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq$ $d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $V_{1}=\left\{v_{1}, \ldots, v_{t}\right\}$ with $t=\left\lfloor m^{\alpha}\right\rfloor$, where $0<\alpha<1 / 2$. Then $t<n-1, e\left(V_{1}\right)<\frac{1}{2} m^{2 \alpha}$, and $d\left(v_{t+1}\right) \leq 2 m^{1-\alpha}$. Let $V_{2}:=V(G) \backslash V_{1}=\left\{u_{1}, \ldots, u_{n-t}\right\}$ such that $e\left(u_{i}, V_{1} \cup\left\{u_{1}, \ldots, u_{i-1}\right\}\right)>0$ for $i=1, \ldots, n-t$.

Fix an arbitrary 3-partition $V_{1}=Y_{1} \cup Y_{2} \cup Y_{3}$, and assign each member of $Y_{i}$ the color $i, 1 \leq i \leq 3$. Extend this coloring to $V(G)$ such that each vertex $u_{i} \in V_{2}$ is independently
assigned the color $j$ with probability $p_{j}^{i}$, where $\sum_{j=1}^{3} p_{j}^{i}=1$ and $p_{j}^{i}$ will be determined later. Let $Z_{i}$ denote the indicator random variable of the event of coloring $u_{i}$.

Let $G_{i}=G\left[V_{1} \cup\left\{u_{1}, \cdots, u_{i}\right\}\right]$ for $i=1, \ldots, n-t$, and let $G_{0}=G\left[V_{1}\right]$. Let $X_{j}^{0}=Y_{j}$ and $x_{j l}^{0}=e\left(X_{j}^{0} \cup X_{l}^{0}\right)$ for $1 \leq j, l \leq 3$. For $i=1, \ldots, n-t$ and $1 \leq j, l \leq 3$, define

$$
\begin{aligned}
X_{j}^{i} & :=\left\{\text { vertices of } G_{i} \text { with color } j\right\}, \\
x_{j l}^{i} & :=e\left(X_{j}^{i} \cup X_{l}^{i}\right), \\
\Delta x_{j l}^{i} & :=x_{j l}^{i}-x_{j l}^{i-1}, \\
b_{j}^{i} & :=e\left(u_{i}, X_{j}^{i-1}\right)
\end{aligned}
$$

When $j=l$, let $x_{j}^{i}:=x_{j l}^{i}$ and $\Delta x_{j}^{i}=\Delta x_{j l}^{i}$. Note that $b_{j}^{i}$ depends on $\left(Z_{1}, \ldots, Z_{i-1}\right)$ only and $\sum_{j=1}^{3} b_{j}^{i}=e\left(u_{i}, G_{i-1}\right)$ is independent of $\left(Z_{1}, \ldots, Z_{i-1}\right)$.

Let $a_{j}^{i}=\sum_{\left(Z_{1}, \ldots, Z_{i-1}\right)} P\left(Z_{1}, \ldots, Z_{i-1}\right) b_{j}^{i}$, which is determined by $p_{j}^{s}, 1 \leq j \leq 3$ and $1 \leq s \leq$ $i-1$. As in the proof of Theorem 3.1.4, for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq 3$ we have

$$
\mathbb{E}\left(\Delta x_{j l}^{i}\right)=\left(a_{j}^{i}+a_{l}^{i}\right)\left(p_{j}^{i}+p_{l}^{i}\right),
$$

and for $1 \leq i \leq n-t$ we have

$$
\mathbb{E}\left(\Delta x_{j}^{i}\right)=a_{j}^{i} p_{j}^{i}
$$

By Lemma 3.4.2, there exist $p_{j}^{i} \in[0,2 / 3], 1 \leq j \leq 3$, such that $\sum_{j=1}^{3} p_{j}^{i}=1$, for $1 \leq i \leq n-t$ and $1 \leq j \neq l \leq 3$,

$$
\mathbb{E}\left(\Delta x_{j l}^{i}\right) \leq \frac{5}{9} \sum_{j=1}^{3} a_{j}^{i}=\frac{5}{9} \sum_{j=1}^{3} b_{j}^{i}=\frac{5}{9} e\left(u_{i}, G_{i-1}\right),
$$

and for $1 \leq i \leq n-t$,

$$
\mathbb{E}\left(\Delta x_{j}^{i}\right) \leq \frac{1}{9} \sum_{j=1}^{3} a_{j}^{i}=\frac{1}{9} \sum_{j=1}^{3} b_{j}^{i}=\frac{1}{9} e\left(u_{i}, G_{i-1}\right) .
$$

Note that $p_{j}^{i}$ is determined by $a_{j}^{i}, 1 \leq j \leq 3$; and hence $p_{j}^{i}$ is recursively defined by $p_{j}^{s}$, $1 \leq j \leq 3$ and $1 \leq s \leq i-1$. Now

$$
\mathbb{E}\left(x_{j l}^{n-t}\right)=\frac{5}{9} \sum_{i=1}^{n-t} e\left(u_{i}, G_{i-1}\right)+x_{j l}^{0} \leq \frac{5}{9} m+e\left(V_{1}\right),
$$

and

$$
\mathbb{E}\left(x_{j}^{n-t}\right) \leq \frac{1}{9} \sum_{i=1}^{n-t} e\left(u_{i}, G_{i-1}\right)+x_{j}^{0} \leq \frac{1}{9} m+e\left(V_{1}\right) .
$$

Clearly, changing the color of $u_{i}$ (i.e., changing $Z_{i}$ ) affects $x_{j l}:=x_{j l}^{n-t}$ and $x_{j}:=x_{j}^{n-t}$ by at most $d\left(u_{i}\right)$. So by Lemma 1.4.1,

$$
\mathbb{P}\left(x_{j l}>\mathbb{E}\left(x_{j l}\right)+z\right)<\exp \left(-\frac{z^{2}}{8 m^{2-\alpha}}\right),
$$

and

$$
\mathbb{P}\left(x_{j}>\mathbb{E}\left(x_{j}\right)+z\right)<\exp \left(-\frac{z^{2}}{8 m^{2-\alpha}}\right) .
$$

Let $z=(8 \ln 6)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then for $1 \leq j \neq l \leq 3$,

$$
\mathbb{P}\left(x_{j l}>\mathbb{E}\left(x_{j l}\right)+z\right)<\frac{1}{6},
$$

and for $1 \leq j \leq 3$,

$$
\mathbb{P}\left(x_{j}>\mathbb{E}\left(x_{j}\right)+z\right)<\frac{1}{6} .
$$

So there exists a partition $V(G)=X_{1} \cup X_{2} \cup X_{3}$ such that for $1 \leq j \neq l \leq 3$,

$$
e\left(X_{j} \cup X_{l}\right) \leq \mathbb{E}\left(x_{j l}\right)+z \leq \frac{5}{9} m+o(m),
$$

and for $1 \leq j \leq 3$,

$$
e\left(X_{j}\right) \leq \mathbb{E}\left(x_{j}\right)+z \leq \frac{1}{9} m+o(m) .
$$

The $o(m)$ term in both expressions is

$$
\frac{1}{2} m^{2 \alpha}+(8 \ln 6)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}
$$

Picking $\alpha=\frac{2}{5}$ to minimize $\max \{2 \alpha, 1-\alpha / 2\}$, the $o(m)$ term becomes $O\left(m^{\frac{4}{5}}\right)$.
By the same argument as in the proof of Theorem 3.4.4, using Lemma 3.4.3 instead of Lemma 3.4.2, we have the following result.

Theorem 3.4.5. Let $G$ be a graph with $m$ edges. Then there is a partition $V_{1}, V_{2}, V_{3}, V_{4}$ of $V(G)$ such that for $1 \leq i \leq 4$,

$$
e\left(V_{i}\right) \leq \frac{1}{16} m+O\left(m^{4 / 5}\right),
$$

and for $1 \leq i \neq j \leq 4$,

$$
e\left(V_{i} \cup V_{j}\right) \leq \frac{2}{5} m+O\left(m^{4 / 5}\right)
$$

## CHAPTER IV

## 3-UNIFORM HYPERGRAPHS

### 4.1 The main result

Recall Conjecture 1.3.10 (Bollobás and Thomason, see [7, 9, 11, 12]) that any $r$-uniform hypergraph with $m$ edges has a $r$-partition $V_{1}, \ldots, V_{r}$ such that $d\left(V_{i}\right) \geq \frac{r}{2 r-1} m$. For large graphs, the bound $r /(2 r-1)$ may be improved. In this section, we prove the following result, which implies Theorem 1.3.11; hence Conjecture 1.3.10 holds for $r=3$ asymptotically.

Theorem 4.1.1. Every 3-uniform hypergraph with $m$ edges has a partition into sets $V_{1}, V_{2}, V_{3}$ such that for $i=1,2,3$,

$$
d\left(V_{i}\right) \geq 0.65 m-O\left(m^{6 / 7}\right) .
$$

Bollobás and Scott $[11,12]$ made a more general conjecture. For integers $r, k \geq 2$, every $r$-uniform hypergraph with $m$ edges has a vertex-partition into $k$ sets, each of which meets at least $(1+o(1))\left(1-(1-1 / k)^{r}\right) m$ edges. In particular, for $r=k=3$, the bound in this conjecture is $19 / 27 m+o(m)$, where $19 / 27 \approx 0.7037$. Although our method can be modified to make further improvement on the current bound of 0.65 , it is unlikely to yield a bound close to 19/27.

We organize this chapter as follows. In Section 4.2, we first state two lemmas, Lemmas 4.2.1 and 4.2.2, which assert that certain inequalities hold. We then use these two lemmas to prove Lemma 4.2.3 which, in turn, is used to prove Theorem 4.1.1. In Lemma 4.2.3, we need to bound three quantities simultaneously. In Section 4.3, we prove two lemmas that can be used to bound two quantities simultaneously. These lemmas will then be used in Section 4.4 to prove Lemmas 4.2.1 and 4.2.2.

### 4.2 Proof of Theorem 4.1.1

We need two lemmas which provide inequalities needed for our proof of Theorem 4.1.1. The meaning of the parameters in these lemmas will be clear from the proof of Lemma 4.2.3; each is related to the number of edges of a certain type. The first lemma tries to bound three quantities $f_{i}\left(p_{i}\right), i=1,2,3$, which will be proved in Section 4.4. It says that, under certain conditions, there exist $p_{i}$ such that either all three functions are bounded from above, or can be made equal. We use $\mathbf{R}^{+}$to denote the set of nonnegative reals.

Lemma 4.2.1. Let $b_{i j}, x_{i}, a_{i}, c \in \mathbf{R}^{+}, 1 \leq i \neq j \leq 3$, such that $b_{i j}=b_{j i}, b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\}$, and $b_{12}+b_{23}+b_{31}+x_{1}+x_{2}+x_{3}+a_{1}+a_{2}+a_{3}+c=1$. For any permutation ijk of $\{1,2,3\}$, let

$$
f_{i}:=\left(1-p_{i}\right)\left(b_{j k}+x_{j}+x_{k}\right)+\left(1-p_{i}\right)^{2}\left(a_{j}+a_{k}\right)+\left(1-p_{i}\right)^{3} c .
$$

Then there exists $p_{1}, p_{2}, p_{3} \in[0,1]$ with $p_{1}+p_{2}+p_{3}=1$ such that
(i) $f_{i} \leq 0.35$ for $i=1,2,3$, or
(ii) $f_{1}=f_{2}=f_{3}$ and $p_{i} \in(0,1)$ for $i=1,2,3$.

The second lemma (when combined with Lemma 4.2.1) deals with the case $c=0$ of Lemma 4.2.3, and will be proved in Section 4.4.

Lemma 4.2.2. Let $a_{i}, x_{i}, b_{i j} \in \mathbf{R}^{+}, 1 \leq i \neq j \leq 3$, such that $b_{i j}=b_{j i}, b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\}$ and $b_{12}+b_{23}+b_{31}+x_{1}+x_{2}+x_{3}+a_{1}+a_{2}+a_{3}=1$. For any permutation $i j k$ of $\{1,2,3\}$, let

$$
f_{k}:=\left(1-p_{k}\right)\left(b_{i j}+x_{i}+x_{j}\right)+\left(1-p_{k}\right)^{2}\left(a_{i}+a_{j}\right)
$$

Suppose there exist $p_{1}, p_{2}, p_{3} \in(0,1)$ such that $p_{1}+p_{2}+p_{3}=1$ and $f_{1}=f_{2}=f_{3}$. Thenfor such $p_{1}, p_{2}, p_{3}$, we have $f_{k} \leq 0.35$ for $k=1,2,3$.

We can now prove the main lemma by using Lemma 4.2.1 and Lemma 4.2.2.

Lemma 4.2.3. Let $b_{i j}, x_{i}, a_{i}, c \in \mathbf{R}^{+}, 1 \leq i \neq j \leq 3$, such that $b_{i j}=b_{j i}, b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\}$ and $b_{12}+b_{23}+b_{31}+x_{1}+x_{2}+x_{3}+a_{1}+a_{2}+a_{3}+c=1$. Then there exist $p_{1}, p_{2}, p_{3} \in[0,1]$ with $p_{1}+p_{2}+p_{3}=1$ such that for any $\{i, j, k\}=\{1,2,3\}$,

$$
f_{i}:=\left(1-p_{i}\right)\left(b_{j k}+x_{j}+x_{k}\right)+\left(1-p_{i}\right)^{2}\left(a_{j}+a_{k}\right)+\left(1-p_{i}\right)^{3} c \leq 0.35 .
$$

Proof. By Lemma 4.2.1, we may assume that there exist $p_{1}, p_{2}, p_{3} \in(0,1)$ with $p_{1}+p_{2}+$ $p_{3}=1$ such that $f_{1}=f_{2}=f_{3}$. Let $\mathscr{D}$ be the set of points

$$
\left(a_{1}, a_{2}, a_{3}, x_{1}, x_{2}, x_{3}, b_{12}, b_{23}, b_{31}, c, p_{1}, p_{2}, p_{3}\right) \in[0,1]^{13}
$$

satisfying

$$
\begin{aligned}
& b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\}, \\
& b_{12}+b_{23}+b_{31}+x_{1}+x_{2}+x_{3}+a_{1}+a_{2}+a_{3}+c=1, \\
& p_{1}+p_{2}+p_{3}=1, \\
& p_{i} \in[0,1] \text { for } i=1,2,3, \text { and } \\
& f_{1}=f_{2}=f_{3} .
\end{aligned}
$$

Note that $\mathscr{D} \neq \emptyset$ and $\mathscr{D}$ is a compact subset of $[0,1]^{13}$. So $f_{1}(\mathbf{v})$ has an absolute maximum over $\mathscr{D}$. Let $\mathscr{M}$ denote all $\mathbf{v} \in \mathscr{D}$ for which $f_{1}(\mathbf{v})$ is the maximum of $f_{1}$ over $\mathscr{D}$. It suffices to show that there is some $\mathbf{v} \in \mathscr{M}$ such that $f_{i}(\mathbf{v}) \leq 0.35$ for $i=1,2,3$. Let

$$
\mathbf{v}:=\left(a_{1}, a_{2}, a_{3}, x_{1}, x_{2}, x_{3}, b_{12}, b_{23}, b_{31}, c, p_{1}, p_{2}, p_{3}\right) \in \mathscr{M} .
$$

We claim that $\mathbf{v}$ may be chosen so that $c=0$. For, suppose $c \neq 0$. Define

$$
\mathbf{v}^{\prime}:=\left(a_{1}+p_{1} c, a_{2}+p_{2} c, a_{3}+p_{3} c, x_{1}, x_{2}, x_{3}, b_{12}, b_{23}, b_{31}, 0, p_{1}, p_{2}, p_{3}\right)
$$

It is easy to check that $\mathbf{v}^{\prime} \in \mathscr{D}$ and $f_{i}\left(\mathbf{v}^{\prime}\right)=f_{i}(\mathbf{v})$ for $i=1,2,3$. Since $\mathbf{v} \in \mathscr{M}$, we have $\mathbf{v}^{\prime} \in \mathscr{M}$. Now it follows from Lemma 4.2.2 that for any $i=1,2,3, f_{i}(\mathbf{v})=f_{i}\left(\mathbf{v}^{\prime}\right) \leq 0.35$.

We also need the following lemma, which is easy to prove. Let $G$ be a graph (multiple edges allowed) and let $w: E(G) \rightarrow \mathbf{R}^{+}$. Recall that for any $S \subseteq V(G)$, we write $w(S)=$
$\sum_{V(e) \subseteq S} w(e)$; for any $S, T \subseteq V(G)$ with $S \cap T=\emptyset$, we use $(S, T)$ to denote the set of edges st with $s \in S$ and $t \in T$; and we write $w(S, T)=\sum_{e \in(S, T)} w(e)$.

Lemma 4.2.4. Let $G$ be a graph and let $w: E(G) \rightarrow \mathbf{R}^{+}$, and let $V(G)=V_{1} \cup \ldots \cup V_{k}$ be a $k$-partition minimizing $\sum_{i=1}^{k} w\left(V_{i}\right)$. Then for any $1 \leq i \neq j \leq k$

$$
w\left(V_{i}, V_{j}\right) \geq \max \left\{2 w\left(V_{i}\right), 2 w\left(V_{j}\right)\right\} .
$$

Proof. For any $v \in V_{i}$ and for any $j \in\{1, \ldots, k\} \backslash\{i\}$, we have

$$
\sum_{\left\{u v \in E(G): u \in V_{i}-v\right\}} w(u v) \leq \sum_{\left\{u v \in E(G): u \in V_{j}\right\}} w(u v) .
$$

Summing over $v \in V_{i}$, we get $2 w\left(V_{i}\right) \leq w\left(V_{i}, V_{j}\right)$.

Proof of Theorem 4.1.1. We may assume that $G$ is connected; as otherwise, we may simply consider the individual components. Hence every vertex of $G$ has positive degree.

Let $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}$ such that $d\left(v_{1}\right) \geq d\left(v_{2}\right) \geq \ldots \geq d\left(v_{n}\right)$. Let $U_{1}:=\left\{v_{1}, \ldots, v_{t}\right\}$ and $U_{2}:=V(G) \backslash U_{1}$, with $t=\left\lfloor m^{\alpha}\right\rfloor$ and $0<\alpha<1 / 3$. Since $m \leq\binom{ n}{3}$ and $t<m^{1 / 3}$, we have $t \leq n-2$ for $n \geq 3$ (by a simple calculation). Moreover,

$$
m^{\alpha} d\left(v_{t+1}\right) \leq(1+t) d\left(v_{t+1}\right) \leq \sum_{i=1}^{t+1} d(v)<\sum_{v \in V(G)} d(v)=3 m
$$

so $d\left(v_{t+1}\right)<3 m^{1-\alpha}$. Hence

$$
\sum_{i=t+1}^{n} d\left(v_{i}\right)^{2}<3 m^{1-\alpha} \sum_{i=1}^{n} d\left(v_{i}\right)=9 m^{2-\alpha} .
$$

For any partition $U_{1}=X_{1} \cup X_{2} \cup X_{3}$ and for $1 \leq i \neq j \leq 3$, define

$$
\begin{aligned}
& x_{i}=\left|\left\{e \in E(G):\left|V(e) \cap X_{i}\right|=2,\left|V(e) \cap U_{2}\right|=1\right\}\right|, \\
& a_{i}=\left|\left\{e \in E(G):\left|V(e) \cap X_{i}\right|=1,\left|V(e) \cap U_{2}\right|=2\right\}\right|, \\
& b_{i j}=\left|\left\{e \in E(G):\left|V(e) \cap X_{i}\right|=\left|V(e) \cap X_{j}\right|=\left|V(e) \cap U_{2}\right|=1\right\}\right|, \\
& c=\left|\left\{e \in E(G):\left|V(e) \cap U_{2}\right|=3\right\}\right| .
\end{aligned}
$$

Then $m=e\left(U_{1}\right)+b_{12}+b_{23}+b_{13}+x_{1}+x_{2}+x_{3}+a_{1}+a_{2}+a_{3}+c$.
By Lemma 4.2.4, we may choose the partition $U_{1}=X_{1} \cup X_{2} \cup X_{3}$ such that for $1 \leq i \neq$ $j \leq 3$,

$$
b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\} .
$$

For $1 \leq i \leq 3$, assign color $i$ to the vertices in $X_{i}$. We extend the coloring to $U_{2}$ as follows: each vertex in $U_{2}$ is independently colored $i$ with probability $p_{i}$ for $1 \leq i \leq 3$, where $p_{1}+p_{2}+p_{3}=1$ and $p_{i}$ will be determined by an application of Lemma 4.2.3.

For $i=1,2,3$, let $V_{i}$ be the vertices with color $i$, and let

$$
y_{i}=\mid\left\{e \in E(G): V(e) \subseteq U_{1} \text { and } V(e) \cap X_{i} \neq \emptyset\right\} .
$$

Then, for any permutation $i j k$ of $\{1,2,3\}$,
$\mathbb{E}\left(d\left(V_{i}\right)\right)=b_{i j}+b_{i k}+x_{i}+a_{i}+p_{i}\left(b_{j k}+x_{j}+x_{k}\right)+\left(1-\left(1-p_{i}\right)^{2}\right)\left(a_{j}+a_{k}\right)+\left(1-\left(1-p_{i}\right)^{3}\right) c+y_{i}$. Thus

$$
f_{i}:=m-\mathbb{E}\left(d\left(V_{i}\right)\right)-e\left(U_{1}\right)+y_{i}=\left(1-p_{i}\right)\left(b_{j k}+x_{j}+x_{k}\right)+\left(1-p_{i}\right)^{2}\left(a_{j}+a_{k}\right)+\left(1-p_{i}\right)^{3} c,
$$

and

$$
\alpha:=m-e\left(U_{1}\right)=b_{12}+b_{23}+b_{31}+a_{1}+a_{2}+a_{3}+x_{1}+x_{2}+x_{3}+c .
$$

By applying Lemma 4.2.3 (with $b_{i j} / \alpha, a_{i} / \alpha, x_{i} / \alpha, c / \alpha$ as $b_{i j}, a_{i}, x_{i}, c$, respectively), there exist $p_{i} \in[0,1]$ with $p_{1}+p_{2}+p_{3}=1$ such that for $1 \leq i \leq 3, f_{i} / \alpha \leq 0.35$. So

$$
f_{i} \leq 0.35\left(m-e\left(U_{1}\right)\right)
$$

Hence

$$
\mathbb{E}\left(d\left(V_{i}\right)\right)=m-f_{i}-e\left(U_{1}\right)+y_{i} \geq 0.65 m-0.65 e\left(U_{1}\right)+y_{i} .
$$

Changing the color of any $v_{j}, t+1 \leq j \leq n$, affects $d\left(V_{i}\right)$ by at most $d\left(v_{j}\right)$. So by Lemma 1.4.1, we have for $i=1,2,3$,

$$
\mathbb{P}\left(d\left(V_{i}\right)<\mathbb{E}\left(d\left(V_{i}\right)\right)-z\right) \leq \exp \left(\frac{-z^{2}}{2 \sum_{j=t+1}^{n} d\left(v_{j}\right)^{2}}\right)<\exp \left(\frac{-z^{2}}{18 m^{2-\alpha}}\right) .
$$

Taking $z=\sqrt{18 \ln 3} m^{1-\alpha / 2}$, we have for $i=1,2,3$,

$$
\mathbb{P}\left(d\left(V_{i}\right)<\mathbb{E}\left(d\left(V_{i}\right)\right)-z\right)<1 / 3 .
$$

Therefore, there exists a partition $V(G)=V_{1} \cup V_{2} \cup V_{3}$ such that for $i=1,2,3$,

$$
d\left(V_{i}\right) \geq \mathbb{E}\left(d\left(V_{i}\right)\right)-z \geq 0.65 m-0.65 e\left(U_{1}\right)+y_{i}-z \geq 0.65 m-0.65 e\left(U_{1}\right)-z
$$

Since $\left|U_{1}\right|=t \leq m^{\alpha}, e\left(U_{1}\right)=O\left(m^{3 \alpha}\right)$. So

$$
0.65 e\left(U_{1}\right)+z=O\left(m^{3 \alpha}\right)+\sqrt{18 \ln 2} m^{1-\alpha / 2}
$$

Choosing $\alpha=\frac{2}{7}$ to minimize $\max \{3 \alpha, 1-\alpha / 2\}$, we have the desired bound.

### 4.3 Bounding two quantities

In this section, we prove two lemmas to be used in our proofs of Lemmas 4.2.1 and 4.2.2. The first is a slight variation of the main lemma in [9]. The difference is that here we relax the constraint $z \geq \max \{2 x, 2 y\}$ in [9] to $z \geq x+y$; as a consequence we have a weaker bound. Our proof mimics that in [9], where a more general result is proved.

Lemma 4.3.1. Let $a, b, x, y, z, e \in \mathbf{R}^{+}$such that $z \geq x+y$ and $a+b+x+y+z+e=1$. Then there exists $p \in(0,1)$ such that

$$
p^{2} a+p x+p^{3} e \leq 1 / 7, \text { and }(1-p)^{2} b+(1-p) y+(1-p)^{3} e \leq 1 / 7 .
$$

Proof. For convenience, let

$$
f_{1}:=p^{2} a+p x+p^{3} e, \text { and } f_{2}:=(1-p)^{2} b+(1-p) y+(1-p)^{3} e .
$$

Note that $f_{1}$ and $f_{2}$ are continuous functions of $p$ on $[0,1]$. We may assume that
(1) $a+x+e>0$ and $b+y+e>0$.

Otherwise, by symmetry, we may assume $a+x+e=0$. Then $a=x=e=0$ and $f_{1}=0<1 / 7$. Since $f_{2}$ is a continuous function of $p$, there exist $0<\epsilon<1$ such that
$\left|f_{2}(\epsilon)-f_{2}(1)\right|<1 / 7$. Thus, because $f_{2}(1)=0$, we have $f_{2}(\epsilon)<1 / 7$. So letting $p=\epsilon$, the assertion of the lemma holds. Thus we may assume (1).
$\operatorname{By}(1), f_{1}(1)=a+x+e>0$ and $f_{2}(0)=b+y+e>0$. Therefore, since $f_{1}(0)=$ $0=f_{2}(1)$ and because $f_{1}(p)$ (respectively, $f_{2}(p)$ ) is increasing (respectively, decreasing) and continuous on $[0,1]$, we have
(2) for any $a, b, x, y, z, e$ satisfying (1), there exists a unique $p \in(0,1)$ such that $f_{1}=f_{2}$.

We call $\mathbf{v}:=(a, b, x, y, z, e, p) \in[0,1]^{7}$ a satisfying point if $a, b, x, y, z, e, p \in \mathbf{R}^{+}, a+b+x+$ $y+z+e=1, z \geq x+y, p \in[0,1]$, and $f_{1}=f_{2}$. (In fact, $p \in(0,1)$ by (2).) Let $\mathscr{D}$ denote the set of all satisfying points. Note $\mathscr{D}$ is a compact subset of $[0,1]^{7}$. A point in $\mathscr{D}$ is said to be a maximal point if the value of $f_{1}$ at that point is the maximum of $f_{1}$ over $\mathscr{D}$. Let $\mathscr{M}$ be the set of maximal points, which is nonempty since $\mathscr{D} \neq \emptyset$ (by (1) and (2)) and $\mathscr{D}$ is compact.

It then suffices to show that $f_{1}(\mathbf{v}) \leq 1 / 7$ for any $\mathbf{v} \in \mathscr{M}$. We do so by looking for a special maximal point. First, we show that
(3) there exists $(a, b, x, y, z, e, p) \in \mathscr{M}$ such that $e=0, z=x+y$, and $a b=0$.

Let $\mathbf{v}:=(a, b, x, y, z, e, p) \in \mathscr{M}$. If $e>0$, then let $\mathbf{v}^{\prime}:=(a+p e, b+(1-p) e, x, y, z, 0, p)$. It is easy to check that $\mathbf{v}^{\prime} \in \mathscr{D}$ and $f_{i}\left(\mathbf{v}^{\prime}\right)=f_{i}(\mathbf{v})$ for $i=1,2$. Hence $\mathbf{v}^{\prime} \in \mathscr{M}$, since $\mathbf{v} \in \mathscr{M}$ and $f_{1}\left(\mathbf{v}^{\prime}\right)=f_{1}(\mathbf{v})$. So we may assume $e=0$.

We may assume $z=x+y$. For, otherwise, assume $z>x+y$. Let $\mathbf{v}^{\prime}:=(a+z-$ $\left.-x-y, b, x, y, x+y, 0, p^{\prime}\right)$ with $p^{\prime} \in[0,1]$, which satisfies (1). So by (2), we may choose $p^{\prime} \in(0,1)$ so that $f_{1}\left(\mathbf{v}^{\prime}\right)=f_{2}\left(\mathbf{v}^{\prime}\right)$; then $\mathbf{v}^{\prime} \in \mathscr{D}$. If $p^{\prime}<p$, then $f_{2}\left(\mathbf{v}^{\prime}\right)>f_{2}(\mathbf{v})$, contradicting the assumption that $\mathbf{v} \in \mathscr{M}$. So $p^{\prime} \geq p$. Then

$$
\begin{aligned}
f_{1}\left(\mathbf{v}^{\prime}\right)-f_{1}(\mathbf{v}) & \geq p^{2}(z-x-y)>0, \quad \text { and } \\
f_{2}\left(\mathbf{v}^{\prime}\right)-f_{2}(\mathbf{v}) & =b\left(\left(1-p^{\prime}\right)^{2}-(1-p)^{2}\right)+y\left(\left(1-p^{\prime}\right)-(1-p)\right) \\
& =-\left(p^{\prime}-p\right)\left(\left(2-p-p^{\prime}\right) b+y\right) \\
& \leq 0 .
\end{aligned}
$$

Hence $f_{1}\left(\mathbf{v}^{\prime}\right)>f_{1}(\mathbf{v})=f_{2}(\mathbf{v}) \geq f_{2}\left(\mathbf{v}^{\prime}\right)$, a contradiction.
Now suppose $a>0$ and $b>0$. Let $\varepsilon=\min \{p a,(1-p) b\}$, and let

$$
\mathbf{v}^{\prime}=\left(a^{\prime}, b^{\prime}, x^{\prime}, y^{\prime}, z^{\prime}, e^{\prime}, p^{\prime}\right):=\left(a-\frac{\varepsilon}{p}, b-\frac{\varepsilon}{1-p}, x+\varepsilon, y+\varepsilon, z+2 \varepsilon, 0, p\right) .
$$

It is easy to see that $e^{\prime}=0, z^{\prime}=x^{\prime}+y^{\prime}, a^{\prime} b^{\prime}=0$, and $f_{i}\left(\mathbf{v}^{\prime}\right)=f_{i}(\mathbf{v})$ for $i=1,2$ (and hence $\left.f_{1}\left(\mathbf{v}^{\prime}\right)=f_{2}\left(\mathbf{v}^{\prime}\right)\right)$. Since $a+b+x+y+z=1$,

$$
a^{\prime}+b^{\prime}+x^{\prime}+y^{\prime}+z^{\prime}=1+4 \varepsilon-\left(\frac{\varepsilon}{p}+\frac{\varepsilon}{1-p}\right) .
$$

Since $p(1-p) \leq 1 / 4$ (with equality iff $p=1 / 2$ ),

$$
4 \varepsilon \leq \frac{\varepsilon}{p}+\frac{\varepsilon}{1-p}
$$

So we have $a^{\prime}+b^{\prime}+x^{\prime}+y^{\prime}+z^{\prime} \leq 1$.
If $a^{\prime}+b^{\prime}+x^{\prime}+y^{\prime}+z^{\prime}=1$ then $p=1 / 2$ and $\mathbf{v}^{\prime} \in \mathscr{D}$. Since $f_{i}\left(\mathbf{v}^{\prime}\right)=f_{i}(\mathbf{v})$, we have $\mathbf{v}^{\prime} \in \mathscr{M}$; and hence (3) holds with $\mathbf{v}^{\prime}$. We may thus assume that $a^{\prime}+b^{\prime}+x^{\prime}+y^{\prime}+z^{\prime}<1$. Let

$$
\alpha=\frac{\varepsilon}{p}+\frac{\varepsilon}{1-p}-4 \varepsilon,
$$

and let

$$
\mathbf{v}^{\prime \prime}:=\left(a^{\prime \prime}, b^{\prime \prime}, x^{\prime \prime}, y^{\prime \prime}, z^{\prime \prime}, e^{\prime \prime}, p^{\prime \prime}\right)=\left(a^{\prime}+\alpha, b^{\prime}, x^{\prime}, y^{\prime}, z^{\prime}, 0, p^{\prime \prime}\right)
$$

with $p^{\prime \prime} \in[0,1]$.
Note that $e^{\prime \prime}=0, z^{\prime \prime}=x^{\prime \prime}+y^{\prime \prime}, a^{\prime \prime}+b^{\prime \prime}+x^{\prime \prime}+y^{\prime \prime}+z^{\prime \prime}=1$, and $\mathbf{v}^{\prime \prime}$ satisfies (1). So by (2), we may choose $p^{\prime \prime} \in(0,1)$ such that $f_{1}\left(\mathbf{v}^{\prime \prime}\right)=f_{2}\left(\mathbf{v}^{\prime \prime}\right)$, and hence $\mathbf{v}^{\prime \prime} \in \mathscr{D}$. If $p^{\prime \prime} \geq p^{\prime}$ then $f_{1}\left(\mathbf{v}^{\prime \prime}\right)>f_{1}\left(\mathbf{v}^{\prime}\right)=f_{1}(\mathbf{v})$ (since $a^{\prime \prime}>a^{\prime}$ and $f_{1}$ increases with $p$ ). If $p^{\prime \prime}<p^{\prime}$ then $f_{2}\left(\mathbf{v}^{\prime \prime}\right)>f_{2}\left(\mathbf{v}^{\prime}\right)=f_{2}(\mathbf{v})$ (since $f_{2}$ decreases with $p$ ). In either case, we obtain a contradiction to the assumption that $\mathbf{v} \in \mathscr{M}$. Thus, (3) holds.

Let $\mathscr{M}^{\prime}=\{(a, b, x, y, z, e, p) \in \mathscr{M}: a=b=e=0$ and $z=x+y\}$. We may assume that (4) $\mathscr{M}^{\prime}=\emptyset$.

For otherwise, let $\mathbf{v}=(0,0, x, y, x+y, 0, p) \in \mathscr{M}^{\prime}$. Then $f_{1}(\mathbf{v})=p x, f_{2}(\mathbf{v})=(1-p) y$, and $x+y=1 / 2$. Since $f_{1}(\mathbf{v})=f_{2}(\mathbf{v})$, we have $p x=(1-p)(1 / 2-x)$. Hence, $p=1-2 x$, and $f_{1}(\mathbf{v})=x(1-2 x)=1 / 8-2(1 / 4-x)^{2} \leq 1 / 8<1 / 7$. So the assertion of the lemma holds; and thus we may assume (4).

By (3) and (4), we may assume without losing generality that there exists $\mathbf{v}=(0, b, x, y, x+$ $y, 0, p) \in \mathscr{M}$ such that $b \neq 0$. Then $b+2(x+y)=1$, and hence $x=(1-b) / 2-y$. So

$$
f_{1}(\mathbf{v})=x p=(1-b) p / 2-y p, \text { and } f_{2}(\mathbf{v})=y(1-p)+b(1-p)^{2} .
$$

Since $\mathbf{v} \in \mathscr{M}, f_{1}(\mathbf{v})$ is the maximum value of $f_{1}$ over $\mathscr{D}$ subject to $g:=f_{1}-f_{2}=0$, where $f_{1}, f_{2}, g$ are considered as functions of $b, y, p$.

Case 1. $y \neq 0$.
Then $y \in(0,1)$ and $b \in(0,1)$; so $\mathbf{v}$ is a critical point of $f_{1}$ (as a function of $b, y$ ). Hence $\mathbf{v}$ must satisfy $\partial f_{1} / \partial b=\lambda \partial g / \partial b$ and $\partial f_{1} / \partial y=\lambda \partial g / \partial y$, where $\lambda$ is a Lagrange multiplier. Thus

$$
p=\lambda\left(p+2(1-p)^{2}\right), \text { and } p=\lambda(p+(1-p))=\lambda
$$

Since $p \in(0,1)$, we have $\lambda \neq 0$. So from the above equations we deduce that $(1-p)=$ $2(1-p)^{2}$. Again since $p \neq 1$, we have $p=1 / 2$. Let

$$
\mathbf{v}^{\prime}:=\left(a^{\prime}, b^{\prime}, x^{\prime}, y^{\prime}, z^{\prime}, e^{\prime}, p^{\prime}\right)=(0,0, x, y+b / 2, z+b / 2,0, p) .
$$

Then $a^{\prime}+b^{\prime}+x^{\prime}+y^{\prime}+z^{\prime}+e^{\prime}=1, z^{\prime}=x^{\prime}+y^{\prime}$, and $f_{1}\left(\mathbf{v}^{\prime}\right)=f_{1}(\mathbf{v})$. Since $p=1 / 2$,

$$
f_{2}\left(\mathbf{v}^{\prime}\right)=(1-p)(y+b / 2)=(1-p) y+(1-p) b / 2=(1-p) y+(1-p)^{2} b=f_{2}(\mathbf{v}) .
$$

This implies $\mathbf{v}^{\prime} \in \mathscr{M}^{\prime}$, contradicting (4).

Case 2. $y=0$.
Then $f_{1}(\mathbf{v})=(1-b) p / 2$ and $f_{2}(\mathbf{v})=b(1-p)^{2}$. By (1) and (2) and since $f_{1}(\mathbf{v})=f_{2}(\mathbf{v})$, we have $b \in(0,1)$ and $p \in(0,1)$. Since $f_{1}(\mathbf{v})$ is the maximum of $f_{1}$ over $\mathscr{D}$ subject to
$g:=f_{1}-f_{2}=0$ (considered as functions of $p$ and $b$ ), $\mathbf{v}$ satisfies $\partial f_{1} / \partial p=\lambda \partial g / \partial p$ and $\partial f_{1} / \partial b=\lambda \partial g / \partial b$ for some $\lambda$. Therefore,

$$
(1-b) / 2=\lambda((1-b) / 2+2 b(1-p)), \text { and } p / 2=\lambda\left(p / 2+(1-p)^{2}\right) .
$$

Since $p \in(0,1)$, we have $\lambda \neq 0$; so we derive from above that $b=(1-p) /(1+p)$. From $f_{1}(\mathbf{v})=f_{2}(\mathbf{v})$, we deduce $b=\frac{p}{p+2(1-p)^{2}}$. Hence

$$
\frac{p}{p+2(1-p)^{2}}=\frac{1-p}{1+p} .
$$

Simplifying this we get $p^{3}-2 p^{2}+3 p-1=0$. Since the function $p^{3}-2 p^{2}+3 p-1$ is always increasing and takes value 0.036125 when $p=9 / 20$, so $p<9 / 20$.

We now claim that $f_{1} \leq 1 / 7$. For otherwise, we have $f_{1}>1 / 7$, i.e.,

$$
\frac{(1-b) p}{2}=\frac{p^{2}}{1+p}>1 / 7 .
$$

But this gives $p>\frac{1+\sqrt{29}}{14}>9 / 20$, a contradiction. This proves Lemma 4.3.1.
In the next lemma we show that under certain conditions two functions can be made equal and bounded from above. The proof is similar to that of Lemma 4.3.1.

Lemma 4.3.2. Let $\mathscr{D}$ denote the set of all points $(a, b, x, y, e, p)$ such that $a, b, x, y, e \in \mathbf{R}^{+}$, $p \in[0.18,1], a+b+2(x+y+e)=1$, and $p^{2} a+p x+p^{3} e=(1.18-p)^{2} b+(1.18-$ $p) y+(1.18-p)^{3} e$. Suppose $\mathscr{D} \neq \emptyset$. Then for any $(a, b, x, y, e, p) \in \mathscr{D}, p^{2} a+p x+p^{3} e \leq$ $\left(1.18^{2} / 8\right)(1-0.82 e)$.

Proof. For convenience, let

$$
\begin{aligned}
& g_{1}(a, b, x, y, e, p):=p^{2} a+p x+p^{3} e, \text { and } \\
& g_{2}(a, b, x, y, e, p):=(1.18-p)^{2} b+(1.18-p) y+(1.18-p)^{3} e .
\end{aligned}
$$

A point $\mathbf{v}:=(a, b, x, y, e, p) \in \mathscr{D}$ is said to be maximal if $g_{1}(\mathbf{v})$ is the maximum of $g_{1}$ over $\mathscr{D}$. Let $\mathscr{M}$ denote the set of all maximal points. Since $\mathscr{D}$ is compact and $\mathscr{D} \neq \emptyset, \mathscr{M} \neq \emptyset$. Let $M:=g(\mathbf{v})$ for $\mathbf{v} \in \mathscr{M}$. We claim that
(1) for any $\mathbf{v}=(a, b, x, y, e, p) \in \mathscr{D}$, we have $e=0$ and $g_{1}(\mathbf{v}) \leq M(1-0.82 e)$.

It is clear that (1) holds when $e=0$. So assume $e \neq 0$. Let

$$
\mathbf{v}^{\prime}:=\left(a^{\prime}, b^{\prime}, x^{\prime}, y^{\prime}, e^{\prime}, p^{\prime}\right)=\left(\frac{a+p e}{1-0.82 e}, \frac{b+(1.18-p) e}{1-0.82 e}, \frac{x}{1-0.82 e}, \frac{y}{1-0.82 e}, 0, p\right) .
$$

Then $a^{\prime}+b^{\prime}+2\left(x^{\prime}+y^{\prime}+e^{\prime}\right)=1$, and $g_{1}\left(\mathbf{v}^{\prime}\right)=g_{1}(\mathbf{v}) /(1-0.82 e)=g_{2}(\mathbf{v}) /(1-0.82 e)=g_{2}\left(\mathbf{v}^{\prime}\right)$; so $\mathbf{v}^{\prime} \in \mathscr{D}$. Now $g_{1}(\mathbf{v})=g_{1}\left(\mathbf{v}^{\prime}\right)(1-0.82 e) \leq M(1-0.82 e)$, proving (1).

Therefore, it suffices to prove that $M \leq 1.18^{2} / 8$. Let $\mathscr{M}^{\prime}=\{(a, b, x, y, e, p) \in \mathscr{M}: x=$ $y=e=0\}$. We may assume
(2) $\mathscr{M}^{\prime}=\emptyset$.

For, suppose there exists some $\mathbf{v}=(a, b, x, y, e, p) \in \mathscr{M}^{\prime}$. Then $a+b=1$,

$$
g_{1}(\mathbf{v})=p^{2} a, \text { and } g_{2}(\mathbf{v})=(1.18-p)^{2} b
$$

Since $g_{1}(\mathbf{v})=g_{2}(\mathbf{v})$, we have

$$
b=\frac{p^{2}}{p^{2}+(1.18-p)^{2}} .
$$

Note that for any $s, t \in \mathbf{R}^{+}$, we have $2 \sqrt{s t} \leq s+t$ and $2 s t \leq s^{2}+t^{2}$; so $8 s^{2} t^{2} \leq(s+t)^{2}\left(s^{2}+t^{2}\right)$, which implies

$$
\frac{s^{2} t^{2}}{s^{2}+t^{2}} \leq \frac{1}{2}\left(\frac{s+t}{2}\right)^{2}
$$

Thus

$$
M=g_{2}(\mathbf{v})=\frac{p^{2}(1.18-p)^{2}}{p^{2}+(1.18-p)^{2}} \leq \frac{1}{2}\left(\frac{1.18}{2}\right)^{2}=\frac{1.18^{2}}{8}
$$

and the assertion of the lemma holds. So we may assume (2).

By (1) and (2), there exists $\mathbf{v}=(a, b, x, y, e, p) \in \mathscr{M}$ such that $e=0$, and $x \neq 0$ or $y \neq 0$.
We now show that $\mathbf{v}$ may be chosen so that
(3) $y=0$.

For, suppose $y \neq 0$. Since $a+b+2(x+y+e)=1$ and $e=0, x=(1-a-b-2 y) / 2$. So

$$
\begin{aligned}
& g_{1}(\mathbf{v})=p^{2} a+p \frac{1-a-b-2 y}{2}, \text { and } \\
& g_{2}(\mathbf{v})=(1.18-p)^{2} b+(1.18-p) y
\end{aligned}
$$

Suppose $b \neq 0$. Then since we assume $y \neq 0$ and because $\mathbf{v} \in \mathscr{M}, \mathbf{v}$ is a critical point of $g_{1}$ subject to $g:=g_{1}-g_{2}=0$, where $g_{1}, g_{2}, g$ are considered as functions of $b$ and $y$. By applying the method of Lagrange multipliers, we have $\partial g_{1} / \partial b=\lambda \partial g / \partial b$ and $\partial g_{1} / \partial y=\lambda \partial g / \partial y$. Hence

$$
-\frac{p}{2}=\lambda\left(-\frac{p}{2}-(1.18-p)^{2}\right), \text { and }-p=\lambda(-p-(1.18-p))
$$

Since $p \in[0.18,1], \lambda \neq 0$. Hence from the above expressions we deduce that $(1.18-p)^{2}=$ $(1.18-p) / 2$. So $p=0.68$, since $p \in[0.18,1]$. Let

$$
\mathbf{v}^{\prime}:=\left(a^{\prime}, b^{\prime}, x^{\prime}, y^{\prime}, e^{\prime}, p^{\prime}\right)=(a, b+2 y, x, 0,0, p)
$$

Then

$$
\begin{aligned}
& a^{\prime}+b^{\prime}+2\left(x^{\prime}+y^{\prime}+e^{\prime}\right)=a+b+2(x+y)=1, \\
& g_{1}\left(\mathbf{v}^{\prime}\right)=p^{2} a+p x=g_{1}(\mathbf{v}), \text { and } \\
& g_{2}\left(\mathbf{v}^{\prime}\right)=(1.18-p)^{2} b^{\prime}=(1.18-p)^{2} b+2(1.18-p)^{2} y=(1.18-p)^{2} b+(1.18-p) y=g_{2}(\mathbf{v}) .
\end{aligned}
$$

The last equality holds because $p=0.68$. So $g_{1}\left(\mathbf{v}^{\prime}\right)=g_{2}\left(\mathbf{v}^{\prime}\right)=g_{1}(\mathbf{v})$. This means that $\mathbf{v}^{\prime} \in \mathscr{M}$, with $e^{\prime}=0$ and $y^{\prime}=0$; and (3) holds by replacing $\mathbf{v}$ with $\mathbf{v}^{\prime}$.

Now suppose $a=0$ and $b=0$. Then $g_{1}(\mathbf{v})=p(1-2 y) / 2$ and $g_{2}(\mathbf{v})=(1.18-p) y$. So $g_{1}(\mathbf{v})=g_{2}(\mathbf{v})$ implies $y=p / 2.36$. Hence,

$$
M=g_{1}(\mathbf{v})=\frac{p}{2}-\frac{p^{2}}{2.36}=\frac{1.18}{8}-\frac{1}{2 \times 1.18}\left(p-\frac{1.18}{2}\right)^{2} \leq \frac{1.18}{8}<\frac{1.18^{2}}{8}
$$

and the assertion of the lemma holds.
So we may assume $a \neq 0$ and $b=0$. Then

$$
g_{1}(\mathbf{v})=p^{2} a+p(1-a-2 y) / 2, \text { and } g_{2}(\mathbf{v})=(1.18-p) y .
$$

Now $\mathbf{v}$ must be a critical point of $g_{1}$ subject to $g:=g_{1}-g_{2}=0$, where $g_{1}, g_{2}, g$ are considered as functions of $a$ and $y$. So there exists $\lambda$ (Lagrange multiplier) such that $\partial g_{1} / \partial a=\lambda \partial g / \partial a$ and $\partial g_{1} / \partial y=\lambda \partial g / \partial y$. This gives

$$
p^{2}-\frac{p}{2}=\lambda\left(p^{2}-\frac{p}{2}\right), \text { and }-p=\lambda(-p-(1.18-p))=-1.18 \lambda .
$$

Since $p \in[0.18,1], \lambda \neq 1$ (from the second equation) and $p=1 / 2$ (from the first equation). Hence, $g_{1}(\mathbf{v})=(1-2 y) / 4$ and $g_{2}(\mathbf{v})=0.68 y$. Since $g_{1}(\mathbf{v})=g_{2}(\mathbf{v})$, we have $(1-2 y) / 4=$ $0.68 y$, and so $y=1 / 4.72$. Hence $M=g_{2}(\mathbf{v})=0.68 / 4.72<1.18^{2} / 8$. This completes the proof of (3).

By (2) and (3), $x \neq 0$ and $\mathbf{v}=(a, b, x, 0,0, p)$. Hence $x=(1-a-b) / 2$,

$$
g_{1}(\mathbf{v})=p^{2} a+p \frac{1-a-b}{2}, \text { and } g_{2}(\mathbf{v})=(1.18-p)^{2} b .
$$

Note that when $b=0$, we have $M=g_{2}(\mathbf{v})=0<1.18^{2} / 8$. Hence, we may assume (4) $b \neq 0$.

We consider two cases: $a \neq 0$, and $a=0$.

Case 1. $a \neq 0$.
Then $\mathbf{v}$ is a critical point of $g_{1}$ subject to $g:=g_{1}-g_{2}=0$, all considered as functions of $a$ and $b$. So there exists $\lambda$ such that $\partial g_{1} / \partial a=\lambda \partial g / \partial a$ and $\partial g_{1} / \partial b=\lambda \partial g / \partial b$, which give

$$
p^{2}-\frac{p}{2}=\lambda\left(p^{2}-\frac{p}{2}\right), \text { and }-\frac{p}{2}=\lambda\left(-\frac{p}{2}-(1.18-p)^{2}\right) .
$$

Since $p \in[0.18,1]$, we have $\lambda \neq 1$ from the second equation; so $p^{2}-p / 2=0$ (from the first equation), which implies $p=1 / 2$. Define

$$
\mathbf{v}^{\prime}:=\left(a^{\prime}, b^{\prime}, x^{\prime}, y^{\prime}, e^{\prime}, p^{\prime}\right)=(a+2 x, b, 0,0,0, p) .
$$

Then $a^{\prime}+b^{\prime}+2\left(x^{\prime}+y^{\prime}+e^{\prime}\right)=a+b+2 x=1$ and $g_{2}(\mathbf{v})=g_{2}\left(\mathbf{v}^{\prime}\right)$. Also, because $p=1 / 2$, $g_{1}\left(\mathbf{v}^{\prime}\right)=p^{2} a^{\prime}=p^{2} a+2 p^{2} x=p^{2} a+p x=g_{1}(\mathbf{v})$. Therefore, $\mathbf{v}^{\prime} \in \mathscr{M}^{\prime}$, contradicting (2).

Case 2. $a=0$.
Then $g_{1}(\mathbf{v})=p(1-b) / 2$ and $g_{2}(\mathbf{v})=(1.18-p)^{2} b$. Since $g_{1}(\mathbf{v})=g_{2}(\mathbf{v})$, we have

$$
b=\frac{p / 2}{(1.18-p)^{2}+p / 2} .
$$

If $p=0.18$ then $b=0.18 / 2.18$; so $M=g_{2}(\mathbf{v})=b<1.18^{2} / 8$. If $p=1$ then $b=1 / 1.0648$;
so $M=g_{2}(v)=0.18^{2} b<1.18^{2} / 8$. Hence we may assume $p \in(0.18,1)$.
Since $b \neq 0$ (by (4)) and $p \in(0.18,1), \mathbf{v}$ is a critical point of $g_{1}$ subject to $g:=g_{1}-g_{2}=$ 0 , all considered as functions of $b$ and $p$. So there exists $\lambda$ such that $\partial g_{1} / \partial b=\lambda \partial g / \partial b$ and $\partial g_{1} / \partial p=\lambda \partial g / \partial p$, which gives

$$
-\frac{p}{2}=\lambda\left(-\frac{p}{2}-(1.18-p)^{2}\right) \text { and } \frac{1-b}{2}=\lambda\left(\frac{1-b}{2}+2 b(1.18-p)\right) .
$$

Since $p \in(0.18,1)$, we have $\lambda \neq 0$ (from the first equation). So

$$
\frac{p}{2}\left(\frac{1-b}{2}+2 b(1.18-p)\right)=\frac{1-b}{2}\left(\frac{p}{2}+(1.18-p)^{2}\right) .
$$

By a simple calculation, we derive

$$
b=\frac{1.18-p}{1.18+p} .
$$

Therefore, we have $(1.18-p)^{3}=p^{2}$.
Note that $h(p):=(1.18-p)^{3}-p^{2}$ is a decreasing function over $(0.18,1)$, and a simple calculation shows $h(0.53)=-0.006275<0$. So $p<0.53$. Also note that $g_{1}(\mathbf{v})=p^{2} /(1.18+p)$ is an increasing function over $(0.18,1)$. So

$$
g_{1}(\mathbf{v})=\frac{p^{2}}{1.18+p}<\frac{(0.53)^{2}}{1.18+0.53}<0.165<\frac{1.18^{2}}{8}
$$

This completes the proof of Lemma 4.3.2.

### 4.4 Proofs of Lemmas 4.2.1 and 4.2.2

Proof of Lemma 4.2.1. For any permutation $i j k$ of $\{1,2,3\}$, let

$$
\alpha_{i}:=b_{j k}+x_{j}+x_{k}, \beta_{i}:=a_{j}+a_{k}, \text { and } \gamma_{i}:=\alpha_{i}+\beta_{i}+c .
$$

Then for $i=1,2,3$,

$$
f_{i}\left(p_{i}\right)=\left(1-p_{i}\right) \alpha_{i}+\left(1-p_{i}\right)^{2} \beta_{i}+\left(1-p_{i}\right)^{3} c .
$$

By symmetry, we may assume that

$$
\gamma_{1} \leq \gamma_{2} \leq \gamma_{3} .
$$

We may further assume that
(1) $\gamma_{1} \geq 0.35$.

For, suppose $\gamma_{1}<0.35$. Let $p_{1}=0$; then $f_{1}=\gamma_{1}<0.35$. We wish to apply Lemma 4.3.1 to show that there exist $p_{2}, p_{3} \in(0,1)$ such that $p_{2}+p_{3}=1$ and $f_{2}=f_{3} \leq 0.35$. Let

$$
m=\alpha_{2}+\alpha_{3}+\beta_{2}+\beta_{3}+\left(\alpha_{2}+\alpha_{3}\right)+c .
$$

Let $x=\alpha_{2} / m, y=\alpha_{3} / m, a=\beta_{2} / m, b=\beta_{3} / m, z=\left(\alpha_{2}+\alpha_{3}\right) / m$, and $e=c / m$. Then $a+b+x+y+z+e=1$ and $z \geq x+y$. Thus by Lemma 4.3.1, there exist $p_{2}, p_{3} \in(0,1)$ such that $p_{2}+p_{3}=1$ and $f_{2} / m=f_{3} / m \leq 1 / 7$.

Note that

$$
m=2\left(b_{13}+x_{1}+x_{3}+b_{12}+x_{1}+x_{2}\right)+\left(a_{1}+a_{2}+a_{1}+a_{3}\right)+c \leq 2+2 x_{1} .
$$

Since $b_{i j} \geq \max \left\{2 x_{i}, 2 x_{j}\right\}$ for $1 \leq i \neq j \leq 3$, we have $5 x_{1} \leq x_{1}+b_{12}+b_{13} \leq 1$. Hence $x_{1} \leq 1 / 5$, and so $m \leq 12 / 5$. Therefore, $f_{2}=f_{3} \leq(12 / 5) / 7<0.35$; so (i) holds and we may assume (1).

We now write $f_{i}\left(p_{i}\right)$ for $f_{i}$, considering it as a function of $p_{i}$ over [ 0,1$]$ (while fixing the other parameters). Differentiating with respect to $p_{i}$, we have $f_{i}^{\prime}\left(p_{i}\right)=-\alpha_{i}-2\left(1-p_{i}\right) \beta_{i}-$ $3\left(1-p_{i}\right)^{2} c \leq 0$ and $f_{i}^{\prime \prime}\left(p_{i}\right)=2 \beta_{i}+6\left(1-p_{i}\right) c \geq 0$. Note from (1) that $f^{\prime}\left(p_{i}\right)<0$ with the possible exception when $p_{i}=1$. So
(2) each $f_{i}\left(p_{i}\right)$ is both decreasing and convex over $[0,1]$.

Because of (2), we approximate $f_{i}\left(p_{i}\right)$ (for each $i$ ) with the line $h_{i}\left(p_{i}\right)$ through the the points $\left(0, f_{i}(0)\right)$ and $\left(1, f_{i}(1)\right)$ in the Euclidean plane. Hence $h_{i}\left(p_{i}\right)=\left(1-p_{i}\right) \gamma_{i}$. It is also convenient to consider the reflection of $f_{3}\left(p_{3}\right)$ with respect to the line $p_{3}=1 / 2$, namely $f_{4}\left(p_{3}\right)=f_{3}\left(1-p_{3}\right)=p_{3} \alpha_{3}+p_{3}^{2} \beta_{3}+p_{3}^{3} c$. Let $h_{4}\left(p_{3}\right)=\gamma_{3} p_{3}$, which is the reflection of $h_{3}\left(p_{3}\right)$ with respect to the line $p_{3}=1 / 2$.

By (2) and by definition, we have
(3) $f_{4}\left(p_{3}\right)$ is convex and increasing over $[0,1]$; and for $i=1,2,3,4, f_{i}\left(p_{i}\right) \leq h_{i}\left(p_{i}\right)$ when $p_{i} \in[0,1]$.

For each $0 \leq \alpha \leq \gamma_{1}$ and for $i=1,2,3,4$, let $p_{i}(\alpha)$ denote the unique root of $f_{i}\left(p_{i}\right)=\alpha$ in $[0,1]$, and $q_{i}(\alpha)$ the unique root of $h_{i}\left(q_{i}\right)=\alpha$ in $[0,1]$. Note that from (2) and (3), we have
(4) for $\alpha \in\left[0, \gamma_{1}\right]$ and for $i=1,2,3, p_{i}(\alpha) \leq q_{i}(\alpha), p_{i}(\alpha)$ and $q_{i}(\alpha)$ decreases with $\alpha$; and $p_{4}(\alpha)$ and $q_{4}(\alpha)$ increases with $\alpha$.

Let $(a, b)$ be the point where $f_{2}$ and $f_{4}$ intersect, that is, $f_{2}(a)=f_{4}(a)=b$; so $p_{2}(b)=$ $p_{4}(b)=a$. Let $\left(a^{\prime}, b^{\prime}\right)$ be the point where $h_{2}$ and $h_{4}$ intersect, i.e., $h_{2}\left(a^{\prime}\right)=h_{4}\left(a^{\prime}\right)=b^{\prime}$. By (2) and (3), we have $b \leq b^{\prime}$. By solving $h_{2}\left(a^{\prime}\right)=h_{4}\left(a^{\prime}\right)=b^{\prime}$, we have

$$
a^{\prime}=\frac{\gamma_{2}}{\gamma_{2}+\gamma_{3}}, \text { and } b^{\prime}=\frac{\gamma_{2} \gamma_{3}}{\gamma_{2}+\gamma_{3}} .
$$

Since $h_{3}\left(1-a^{\prime}\right)=h_{4}\left(a^{\prime}\right)=b^{\prime}$ and by definition, we have $q_{3}\left(b^{\prime}\right)=1-q_{2}\left(b^{\prime}\right)$; and so $q_{2}\left(b^{\prime}\right)+q_{3}\left(b^{\prime}\right)=1$.

We may assume
(5) $b^{\prime}=\frac{\gamma_{2} \gamma_{3}}{\gamma_{2}+\gamma_{3}} \geq \gamma_{1}$.

For, suppose $b^{\prime}<\gamma_{1}$. Then $b<\gamma_{1}$; so $p_{i}(b)$ is defined for $i=1,2,3,4$. Since $f_{3}$ and $f_{4}$ are reflections through the line $p_{3}=1 / 2, p_{3}(b)+p_{4}(b)=1$. Since $p_{2}(b)=p_{4}(b)=a$ and $p_{1}(b)>0$, we have $p_{1}(b)+p_{2}(b)+p_{3}(b)=p_{1}(b)+1>1$. Also, $p_{1}\left(\gamma_{1}\right)=0$, and

$$
p_{2}\left(\gamma_{1}\right)+p_{3}\left(\gamma_{1}\right) \leq q_{2}\left(\gamma_{1}\right)+q_{3}\left(\gamma_{1}\right)<q_{2}\left(b^{\prime}\right)+q_{3}\left(b^{\prime}\right)=1 ; \text { so } p_{1}\left(\gamma_{1}\right)+p_{2}\left(\gamma_{1}\right)+p_{3}\left(\gamma_{1}\right)<1 \text {. }
$$

Since $p_{1}(\alpha)+p_{2}(\alpha)+p_{3}(\alpha)$ is a decreasing function of $\alpha$, there exists $\alpha \in\left(b, \gamma_{1}\right)$ (and hence by (4), $p_{i}(\alpha) \in(0,1)$ for $\left.i=1,2,3\right)$ such that $p_{1}(\alpha)+p_{2}(\alpha)+p_{3}(\alpha)=1$; so (ii) holds with $f_{i}\left(p_{i}\right)=\alpha$ for $i=1,2,3$.

We claim that
(6) $\gamma_{1} \leq 1 / 2,0.4 \leq \gamma_{2} \leq 1,0.7 \leq \gamma_{3} \leq 1, \gamma_{2}+\gamma_{3} \geq 1.4$, and $c-\sum_{1 \leq i<j \leq 3} b_{i j} \geq-0.25$.

By (5), $\frac{\gamma_{2} \gamma_{3}}{\gamma_{2}+\gamma_{3}} \geq \gamma_{1}$. So by Cauchy-Schwarz,

$$
\gamma_{2}+\gamma_{3} \geq \frac{4}{\frac{1}{\gamma_{2}}+\frac{1}{\gamma_{3}}} \geq 4 \gamma_{1} .
$$

Hence by (1), $\gamma_{2}+\gamma_{3} \geq 1.4$. Then $\gamma_{2} \geq 0.4$ and, since $\gamma_{3} \geq \gamma_{2}, \gamma_{3} \geq\left(\gamma_{2}+\gamma_{3}\right) / 2 \geq 0.7$. Since

$$
\gamma_{1}+\gamma_{2}+\gamma_{3}=2+c-\sum_{1 \leq i<j \leq 3} b_{i j},
$$

we have $5 \gamma_{1} \leq \gamma_{1}+\gamma_{2}+\gamma_{3}=2+c-\sum_{i<j} b_{i j}$, and so $\gamma_{1} \leq 2 / 5+\left(c-\sum_{i<j} b_{i j}\right) / 5$. Therefore, since $\gamma_{2}+\gamma_{3} \leq 2$,

$$
2+c-\sum_{i<j} b_{i j}=\gamma_{1}+\gamma_{2}+\gamma_{3} \leq 2+\frac{2}{5}+\frac{c-\sum_{i<j} b_{i j}}{5} .
$$

So $c-\sum_{i<j} b_{i j} \leq 1 / 2$, which in turn implies $5 \gamma_{1} \leq 2+c-\sum_{i<j} b_{i j} \leq 5 / 2$. Thus, $\gamma_{1} \leq \frac{1}{2}$. By (1), $1.75 \leq 5 \gamma_{1} \leq 2+c-\sum_{i<j} b_{i j}$, which implies $c-\sum_{i<j} b_{i j} \geq-0.25$.

We also claim that
(7) $x_{i} \leq 1.25 / 9$, for $i=1,2,3$.

Since $b_{i j} \geq 2 x_{i}$ and $b_{i j} \geq 2 x_{j}, c+5 x_{i} \leq 1$. By (6), $c-\sum b_{i j} \geq-0.25$; so $c-4 x_{i} \geq-0.25$. Hence $1-5 x_{i} \geq 4 x_{i}-0.25$, which gives (7).

We now prove that
(8) $f_{1}(0.18) \leq 0.35$.

This is true if $\gamma_{1} \leq 0.35 / 0.82$ as $f_{1}(0.18) \leq 0.82 \gamma_{1}$. So we may assume that $\gamma_{1}>0.35 / 0.82$. From the proof of (6) we see that $c \geq \sum_{i<j} b_{i j}+5 \gamma_{1}-2$. Then, since $b_{12} \geq 2 x_{2}, b_{13} \geq 2 x_{3}$ and $\alpha_{1}=b_{23}+x_{2}+x_{3}$, we have $c \geq \alpha_{1}+5 \gamma_{1}-2$. Also, $\gamma_{1} \geq \alpha_{1}+c$. So $\gamma_{1}-\alpha_{1} \geq \alpha_{1}+5 \gamma_{1}-2$. Therefore, $2 \gamma_{1}+\alpha_{1} \leq 1$. Hence, since $\gamma_{1}>0.35 / 0.82$, we have $\alpha_{1} \leq 1-0.7 / 0.82$ and $c \geq 5 \gamma_{1}-2 \geq 5 \times(0.35 / 0.82)-2=0.11 / 0.82$. This implies that $0.82 \alpha_{1}+0.82^{3} c<0.7\left(\alpha_{1}+c\right)$. Hence, since $0.82^{2}<0.7, f_{1}(0.18)<0.7 \gamma_{1} \leq 0.35$ (as $\gamma_{1} \leq 1 / 2$ by (6)). So we have (8).

Now let $p_{1}=0.18$; then by $(8), f_{1}\left(p_{1}\right) \leq 0.35$. We wish to apply Lemma 4.3.2 to prove the existence of $p_{2}$ and $p_{3}$ such that $p_{2}+p_{3}=1-p_{1}=0.82, f_{2}\left(p_{2}\right) \leq 0.35$ and $f_{3}\left(p_{3}\right) \leq 0.35$. Let $1-p_{2}=p$ and $1-p_{3}=1.18-p$. Let

$$
m=\beta_{2}+\beta_{3}+2\left(\alpha_{2}+\alpha_{3}+c\right),
$$

and let $a=\beta_{2} / m, b=\beta_{3} / m, x=\alpha_{2} / m, y=\alpha_{3} / m, e=c / m, g_{1}(p)=f_{2}(p) / m$, and $g_{2}(p)=f_{3}(p) / m$. Then $a+b+2(x+y+e)=1$,

$$
g_{1}(p)=p^{2} a+p x+p^{3} e, \text { and } g_{2}(p)=(1.18-p)^{2} b+(1.18-p) y+(1.18-p)^{3} e .
$$

Note that
$m=2 a_{1}+a_{2}+a_{3}+2\left(b_{12}+b_{13}+2 x_{1}+x_{2}+x_{3}+c\right)=2+2 x_{1}-\left(a_{2}+a_{3}+2 b_{23}\right) \leq 2+2 x_{1}$, and

$$
\begin{aligned}
m & =2+2 x_{1}-\left(a_{2}+a_{3}+2 b_{23}\right) \\
& =2+2 x_{1}-\gamma_{1}+x_{2}+x_{3}+c-b_{23} \\
& \leq 2+2 x_{1}-\gamma_{1}+c \quad\left(\text { since } b_{23} \geq \max \left\{2 x_{1}, 2 x_{3}\right\}\right) \\
& \leq 2+2(1.25 / 9)-0.35+c \quad(\text { by }(1) \text { and }(7)) .
\end{aligned}
$$

We claim that
(9) $\gamma_{2} / m>0.18$ and $\gamma_{3} / m>0.18$.

By (7), $m \leq 2+2(1.25 / 9)$; so by ( 6 ), $\gamma_{3} / m \geq 0.7 /(2+2.5 / 9)>0.18$. If $\gamma_{2} \geq 0.5$, then $\gamma_{2} / m \geq 0.5 /(2+2.5 / 9)>0.18$. So we may assume that $\gamma_{2}<0.5$. Then by (6), $\gamma_{3}>0.9$. Hence, $2 x_{1} \leq b_{13} \leq b_{13}+b_{23}+x_{3}+a_{3}=1-\gamma_{3}<0.1$. So $m \leq 2+2 x_{1}<2.1$ and, by (6), $\gamma_{2} / m \geq 0.4 / 2.1>0.18$. Thus, we have (9).

In order to apply Lemma 4.3.2, we need to show that there exists $p \in[0.18,1]$ such that $g_{1}(p)=g_{2}(p)$. To see this, consider $g_{1}, g_{2}$ as functions of $p$. By (9), we note that

$$
\begin{aligned}
& g_{1}(0.18) \leq 0.18(a+x+e) \leq 0.18, \text { and } \\
& g_{2}(0.18)=b+y+e=\gamma_{3} / m>0.18 .
\end{aligned}
$$

So $g_{1}(0.18)<g_{2}(0.18)$. Similarly, we can show $g_{1}(1)>0.18 \geq g_{2}(1)$. By $(2), g_{1}(p)$ is an increasing function, and $g_{2}(p)$ is a decreasing function. So there exists $p \in(0.18,1)$ such that $g_{1}(p)=g_{2}(p)$.

We can now apply Lemma 4.3.2. As a consequence, $g_{1}(p)=g_{2}(p) \leq\left(1.18^{2} / 8\right)(1-$ $0.82 e)$, so $f_{2}(p)=f_{3}(p) \leq\left(1.18^{2} / 8\right)(m-0.82 c)$. If $c \leq 0.35$ then, since $m \leq 2+2(1.25 / 9)-$ $0.35+c$,

$$
f_{2}(p)=f_{3}(p) \leq \frac{1.18^{2}}{8}(2+2.5 / 9-0.35+0.18 \times 0.35)<0.347<0.35 .
$$

So we may assume $c>0.35$. Then, since $m \leq 2+2 x_{1} \leq 2+2.5 / 9$ by (7),

$$
f_{2}(p)=f_{3}(p) \leq \frac{1.18^{2}}{8}(2+2.5 / 9-0.82 \times 0.35)<0.35 .
$$

Note that $p_{2}=1-p$ and $p_{3}=p-0.18$. Since $p \in(0.18,1)$, we have $p_{2}, p_{3} \in(0,1)$. Clearly, $p_{1}+p_{2}+p_{3}=1$. So (i) holds, which completes the proof of Lemma4.2.1.

In order to prove Lemma 4.2.2, we first deal with the special case when $b_{i j}=x_{i}+x_{j}$ for $1 \leq i<j \leq 3$.

Lemma 4.4.1. Let $b_{i}, y_{i} \in \mathbf{R}^{+}$for $i=1,2,3$ such that $\sum_{i=1}^{3}\left(3 y_{i}+b_{i}\right)=2$. Suppose there exist $q_{i} \in(0,1), i=1,2,3$, such that $q_{1}+q_{2}+q_{3}=2$ and $2 y_{1} q_{1}+b_{1} q_{1}^{2}=2 y_{2} q_{2}+b_{2} q_{2}^{2}=$ $2 y_{3} q_{3}+b_{3} q_{3}^{2}$. Then for $i=1,2,3,2 y_{i} q_{i}+b_{i} q_{i}^{2} \leq 0.35$.

Proof. For convenience, let $f_{i}:=2 y_{i} q_{i}+b_{i} q_{i}^{2}, i=1,2,3$. Let $\mathscr{D}$ denote the set of all points $\left(b_{1}, b_{2}, b_{3}, y_{1}, y_{2}, y_{3}, q_{1}, q_{2}, q_{3}\right)$ such that $b_{i}, y_{i} \in \mathbf{R}^{+}$and $q_{i} \in[0,1]$ for $i=1,2,3$,

$$
\begin{aligned}
& \sum_{i=1}^{3}\left(3 y_{i}+b_{i}\right)=2, \\
& q_{1}+q_{2}+q_{3}=2, \text { and } \\
& f_{1}=f_{2}=f_{3} .
\end{aligned}
$$

So $\mathscr{D}$ is a compact subset of $[0,2]^{3} \times[0,2 / 3]^{3} \times[0,1]^{3}$. Note that $\mathscr{D} \neq \emptyset$ by assumption of the lemma. Let

$$
\mathbf{v}:=\left(b_{1}, b_{2}, b_{3}, y_{1}, y_{2}, y_{3}, q_{1}, q_{2}, q_{3}\right) \in \mathscr{D}
$$

such that $f_{1}(\mathbf{v})$ is the maximum of $f_{1}$ over $\mathscr{D}$. It suffices to show that $f_{1}(\mathbf{v}) \leq 0.35$.
We may assume that $q_{i} \neq 0$ for $i=1,2,3$; as otherwise we have $f_{i}(\mathbf{v})=0<0.35$ for $i=1,2,3$. Thus, since $f_{1}=f_{2}=f_{3}$, we see that if $f_{i}=0$ for some $i \in\{1,2,3\}$ then $b_{i}=y_{i}=0$ for $i=1,2,3$, contradicting the condition that $\sum_{i=1}^{3}\left(3 y_{i}+b_{i}\right)=2$. Hence, we have
(1) for each $i \in\{1,2,3\}, q_{i}>0$, and $b_{i}>0$ or $y_{i}>0$.

We may assume that
(2) there exists some $i \in\{1,2,3\}$ such that $b_{i}>0$.

For, suppose $b_{i}=0$ for $i=1,2,3$. Then $f_{i}=2 y_{i} q_{i}$ and $y_{i}>0($ by (1)) for $i=1,2,3$, and $y_{1}+y_{2}+y_{3}=2 / 3$. Hence, by Cauchy-Schwarz,

$$
\frac{1}{y_{1}}+\frac{1}{y_{2}}+\frac{1}{y_{3}} \geq \frac{9}{y_{1}+y_{2}+y_{3}}=\frac{27}{2} .
$$

Setting $f_{1}=f_{2}=f_{3}=\alpha$, we have $q_{i}=\alpha / 2 y_{i}$ for $i=1,2,3$. Therefore, since $q_{1}+q_{2}+q_{3}=2$,

$$
\alpha=\frac{4}{\frac{1}{y_{1}}+\frac{1}{y_{2}}+\frac{1}{y_{3}}} \leq \frac{8}{27}<0.35 .
$$

We may also assume that
(3) there exists some $j \in\{1,2,3\}$ such that $y_{j}>0$.

For, otherwise, $y_{1}=y_{2}=y_{3}=0$. Then $f_{i}=b_{i} q_{i}^{2}$ and $b_{i}>0$ (by (1)) for $i=1,2,3$, and $b_{1}+b_{2}+b_{3}=2$. Setting $f_{1}=f_{2}=f_{2}=\alpha$, we have $q_{i}=\sqrt{\alpha / b_{i}}$. Since $q_{1}+q_{2}+q_{3}=2$, we have (by Cauchy-Schwarz),

$$
\alpha=\frac{4}{\left(\frac{1}{\sqrt{b_{1}}}+\frac{1}{\sqrt{b_{2}}}+\frac{1}{\sqrt{b_{3}}}\right)^{2}} \leq \frac{4}{81}\left(\sqrt{b_{1}}+\sqrt{b_{2}}+\sqrt{b_{3}}\right)^{2} \leq \frac{4}{9} \frac{b_{1}+b_{2}+b_{3}}{3}=\frac{8}{27}<0.35 .
$$

We may further assume that
(4) there exists some $i \in\{1,2,3\}$ such that $b_{i} y_{i} \neq 0$.

Otherwise, we have two cases (by symmetry): $y_{1}=y_{2}=b_{3}=0$, or $b_{1}=b_{2}=y_{3}=0$
First, assume $y_{1}=y_{2}=b_{3}=0$. Then, $b_{1}>0, b_{2}>0, y_{3}>0, b_{1}+b_{2}+3 y_{3}=2$,

$$
f_{1}=b_{1} q_{1}^{2}, f_{2}=b_{2} q_{2}^{2}, \text { and } f_{3}=2 y_{3} q_{3} .
$$

Setting $\alpha=f_{1}=f_{2}=f_{3}$ and using $q_{1}+q_{2}+q_{3}=2$, we have

$$
\frac{\sqrt{\alpha}}{\sqrt{b_{1}}}+\frac{\sqrt{\alpha}}{\sqrt{b_{2}}}+\frac{\alpha}{2 y_{3}}=2 .
$$

So

$$
\sqrt{\alpha}=\frac{4}{\sqrt{\left(1 / \sqrt{b_{1}}+1 / \sqrt{b_{2}}\right)^{2}+4 / y_{3}}+\left(1 / \sqrt{b_{1}}+1 / \sqrt{b_{2}}\right)} .
$$

Note that

$$
\left(\frac{1}{\sqrt{b_{1}}}+\frac{1}{\sqrt{b_{2}}}\right)^{2} \geq \frac{4}{\sqrt{b_{1} b_{2}}} \geq \frac{8}{b_{1}+b_{2}}=\frac{8}{2-3 y_{3}},
$$

so

$$
\sqrt{\alpha} \leq \frac{4}{\sqrt{\frac{8}{2-3 y_{3}}+\frac{4}{y_{3}}}+\sqrt{\frac{8}{2-3 y_{3}}}} .
$$

Let $f\left(y_{3}\right):=\sqrt{8 /\left(2-3 y_{3}\right)+4 / y_{3}}+\sqrt{8 /\left(2-3 y_{3}\right)}$. Note that $y_{3} \in(0,2 / 3)$, and

$$
f\left(y_{3}\right) \geq \begin{cases}\sqrt{4+20}+\sqrt{4}, & \text { if } y_{3} \in(0,1 / 5] ; \\ \sqrt{8 /(7 / 5)+16}+\sqrt{8 /(7 / 5)}, & \text { if } y_{3} \in(1 / 5,1 / 4] ; \\ \sqrt{8 /(5 / 4)+12}+\sqrt{8 /(5 / 4)}, & \text { if } y_{3} \in(1 / 4,1 / 3] ; \\ \sqrt{8+8}+\sqrt{8}, & \text { if } y_{3} \in(1 / 3,1 / 2] ; \\ \sqrt{16+6}+\sqrt{16}, & \text { if } y_{3} \in(1 / 2,2 / 3) .\end{cases}
$$

Therefore, $f\left(y_{3}\right) \geq 6.819$, and hence $\alpha \leq(4 / 6.819)^{2}<0.35$
Now assume $b_{1}=b_{2}=y_{3}=0$. Then $y_{1}>0, y_{2}>0, b_{3}>0,3\left(y_{1}+y_{2}\right)+b_{3}=2$,

$$
f_{1}=2 y_{1} q_{1}, f_{2}=2 y_{2} q_{2}, \text { and } f_{3}=b_{3} q_{3}^{2} .
$$

Again, setting $\alpha=f_{1}=f_{2}=f_{3}$ and using $q_{1}+q_{2}+q_{3}=2$, we have

$$
\frac{\alpha}{2 y_{1}}+\frac{\alpha}{2 y_{2}}+\frac{\sqrt{\alpha}}{\sqrt{b_{3}}}=2
$$

So

$$
\sqrt{\alpha}=\frac{4}{\sqrt{1 / b_{3}+4\left(1 / y_{1}+1 / y_{2}\right)}+1 / \sqrt{b_{3}}} .
$$

Note that $1 / y_{1}+1 / y_{2} \geq 4 /\left(y_{1}+y_{2}\right)=12 /\left(2-b_{3}\right)$. Hence

$$
\sqrt{\alpha} \leq \frac{4}{\sqrt{1 / b_{3}+48 /\left(2-b_{3}\right)}+1 / \sqrt{b_{3}}} .
$$

Let $g\left(b_{3}\right):=\sqrt{1 / b_{3}+48 /\left(2-b_{3}\right)}+1 / \sqrt{b_{3}}$. Note that $b_{3} \in(0,2)$, and

$$
g\left(b_{3}\right) \geq \begin{cases}\sqrt{3+48 /(2-0)}+\sqrt{3}, & \text { if } b_{3} \in(0,1 / 3] ; \\ \sqrt{2+48 /(2-1 / 3)}+\sqrt{2}, & \text { if } b_{3} \in(1 / 3,1 / 2] ; \\ \sqrt{3 / 2+48 /(2-1 / 2)}+\sqrt{3 / 2}, & \text { if } b_{3} \in(1 / 2,2 / 3] ; \\ \sqrt{2 / 3+48 /(2-2 / 3)}+\sqrt{2 / 3}, & \text { if } b_{3} \in(2 / 3,3 / 2] ; \\ \sqrt{1 / 2+48 /(2-3 / 2)}+\sqrt{1 / 2}, & \text { if } b_{3} \in(3 / 2,2) .\end{cases}
$$

Therefore, $g\left(b_{3}\right) \geq 6.87$, and hence $\alpha \leq(4 / 6.87)^{2}<0.35$.

By (4) and by symmetry, we may assume that
(5) $b_{3} y_{3} \neq 0$.

We may further assume that
(6) $b_{1} y_{1}=0$ and $b_{2} y_{2}=0$.

For, otherwise, by symmetry, assume $b_{2} y_{2}>0$. Then $\mathbf{v}$ is a solution to the following optimization problem:

Maximize $f_{1}$
subject to

$$
\begin{aligned}
& h_{1}:=f_{1}-f_{2}=0, \\
& h_{2}:=f_{1}-f_{3}=0, \\
& h_{3}:=3\left(y_{1}+y_{2}+y_{3}\right)+\left(b_{1}+b_{2}+b_{3}\right)-2=0, \\
& h_{4}:=q_{1}+q_{2}+q_{3}-2=0 .
\end{aligned}
$$

Applying the method of Lagrange multipliers, we have, for each $u \in\left\{y_{i}, b_{i}: i=2,3\right\}$,

$$
\partial f_{1} / \partial u=\lambda_{1} \partial h_{1} / \partial u+\lambda_{2} \partial h_{2} / \partial u+\lambda_{3} \partial h_{3} / \partial u+\lambda_{4} \partial h_{4} / \partial u .
$$

Thus,

$$
\begin{aligned}
& \text { for } u=y_{2}, \text { we have } 0=\lambda_{1}\left(-2 q_{2}\right)+3 \lambda_{3}, \\
& \text { for } u=y_{3}, \text { we have } 0=\lambda_{2}\left(-2 q_{3}\right)+3 \lambda_{3}, \\
& \text { for } u=b_{2} \text {, we have } 0=\lambda_{1}\left(-q_{2}^{2}\right)+\lambda_{3}, \\
& \text { for } u=b_{3} \text {, we have } 0=\lambda_{2}\left(-q_{3}^{2}\right)+\lambda_{3} .
\end{aligned}
$$

Clearly, if $\lambda_{i}=0$ for some $i \in\{1,2,3\}$ then $\lambda_{i}=0$ for all $i=1,2,3$ (since $q_{i}>0$ by (1)). In fact, $\lambda_{i} \neq 0$ for all $i=1,2,3$. To see this we notice that either $b_{1}>0$ or $y_{1}>0$, so $\mathbf{v}$ also satisfies $\partial f_{1} / \partial u=\lambda_{1} \partial h_{1} / \partial u+\lambda_{2} \partial h_{2} / \partial u+\lambda_{3} \partial h_{3} / \partial u+\lambda_{4} \partial h_{4} / \partial u$ for $u=b_{1}$ or $u=y_{1}$. For $u=b_{1}$, we have $q_{1}^{2}=\lambda_{1} q_{1}^{2}+\lambda_{2} q_{1}^{2}+\lambda_{3}$, and for $u=y_{1}$ we have $2 q_{1}=\lambda_{1} 2 q_{1}+\lambda_{2} 2 q_{1}+3 \lambda_{3}$. In either case, we see that $\lambda_{i} \neq 0$ (since $q_{1}>0$ ).

Now using the partial derivatives with respect to $b_{2}$ and $y_{2}$, we get $q_{2}=2 / 3$; and using the partial derivatives with respect to $b_{3}$ and $y_{3}$ we obtain $q_{3}=2 / 3$. So $q_{1}=2 / 3$ since $q_{1}+q_{2}+q_{3}=2$. Then for $i=1,2,3$,

$$
f_{i}=\frac{4}{3} y_{i}+\frac{4}{9} b_{i}=\frac{4}{9}\left(3 y_{i}+b_{i}\right) .
$$

Since $f_{1}=f_{2}=f_{3}$ and $\sum_{i=1}^{3}\left(3 y_{i}+b_{i}\right)=2$, we get $3 y_{i}+b_{i}=2 / 3$ for $i=1,2,3$, and hence $f_{i}=8 / 27<0.35$. This proves (6)

By (5) and (6), we have three cases to consider: $b_{1}=b_{2}=0 ; y_{1}=y_{2}=0 ; y_{1}=b_{2}=0$ or $b_{1}=y_{2}=0$. Let $h_{1}, h_{2}, h_{3}, h_{4}$ be defined as in the proof of (6).

Case 1. $b_{1}=b_{2}=0$.
Then $y_{1}>0, y_{2}>0, f_{1}=2 y_{1} q_{1}, f_{2}=2 y_{2} q_{2}, f_{3}=2 y_{3} q_{3}+b_{3} q_{3}^{2}$. Moreover, $\mathbf{v}$ is a critical point of $f_{1}$ subject to $h_{1}=h_{2}=h_{3}=h_{4}=0$, all considered as functions of $y_{1}, y_{2}, y_{3}, b_{3}$. Hence for $u \in\left\{y_{1}, y_{2}, y_{3}, b_{3}\right\}, \mathbf{v}$ satisfies

$$
\partial f_{1} / \partial u=\lambda_{1} \partial h_{1} / \partial u+\lambda_{2} \partial h_{2} / \partial u+\lambda_{3} \partial h_{3} / \partial u+\lambda_{4} \partial h_{4} / \partial u .
$$

So

$$
\begin{aligned}
& \text { for } u=y_{1} \text {, we have } 2 q_{1}=\lambda_{1}\left(2 q_{1}\right)+\lambda_{2}\left(2 q_{1}\right)+3 \lambda_{3}, \\
& \text { for } u=y_{2} \text {, we have } 0=\lambda_{1}\left(-2 q_{2}\right)+3 \lambda_{3}, \\
& \text { for } u=y_{3} \text {, we have } 0=\lambda_{2}\left(-2 q_{3}\right)+3 \lambda_{3}, \\
& \text { for } u=b_{3} \text {, we have } 0=\lambda_{2}\left(-q_{3}^{2}\right)+\lambda_{3} .
\end{aligned}
$$

Clearly, $\lambda_{i} \neq 0$ for $i=1,2,3$. So from the partial derivatives with respect to $b_{3}$ and $y_{3}$, we have $q_{3}=2 / 3$, and hence $q_{1}+q_{2}=4 / 3$. Set $\alpha:=2 y_{1} q_{1}=2 y_{2} q_{2}=4\left(3 y_{3}+b_{3}\right) / 9$. In particular, $\alpha=4\left(3 y_{3}+b_{3}\right) / 9=4\left(2-3\left(y_{1}+y_{2}\right)\right) / 9$, and so $y_{1}+y_{2}=2 / 3-3 \alpha / 4$. Using $q_{1}+q_{2}=4 / 3$ and Cauchy-Schwarz, we get

$$
\frac{4}{3}=\frac{\alpha}{2 y_{1}}+\frac{\alpha}{2 y_{2}} \geq \frac{2 \alpha}{y_{1}+y_{2}}=\frac{2 \alpha}{2 / 3-3 \alpha / 4} .
$$

This implies $\alpha \leq 8 / 27<0.35$.
Case 2. $y_{1}=y_{2}=0$.
Then $b_{1}>0, b_{2}>0, f_{1}=b_{1} q_{1}^{2}, f_{2}=b_{2} q_{2}^{2}$ and $f_{3}=2 y_{3} q_{3}+b_{3} q_{3}^{2}$. Now $\mathbf{v}$ is a critical point of $f_{1}$ subject to $h_{1}=h_{2}=h_{3}=h_{4}=0$, all considered as functions of $b_{1}, b_{2}, b_{3}, y_{3}$.

Hence for $u \in\left\{b_{1}, b_{2}, b_{3}, y_{3}\right\}, \mathbf{v}$ satisfies

$$
\partial f_{1} / \partial u=\lambda_{1} \partial h_{1} / \partial u+\lambda_{2} \partial h_{2} / \partial u+\lambda_{3} \partial h_{3} / \partial u+\lambda_{4} \partial h_{4} / \partial u .
$$

Thus,

$$
\begin{aligned}
& \text { for } u=b_{1}, \text { we have } q_{1}^{2}=\lambda_{1}\left(q_{1}^{2}\right)+\lambda_{2}\left(q_{1}^{2}\right)+\lambda_{3}, \\
& \text { for } u=b_{2}, \text { we have } 0=\lambda_{1}\left(-q_{2}^{2}\right)+\lambda_{3}, \\
& \text { for } u=b_{3} \text {, we have } 0=\lambda_{2}\left(-q_{3}^{2}\right)+\lambda_{3} \\
& \text { for } u=y_{3}, \text { we have } 0=\lambda_{2}\left(-2 q_{3}\right)+3 \lambda_{3} .
\end{aligned}
$$

Clearly, $\lambda_{i} \neq 0$ for $i=1,2,3$. So from the partial derivatives with respect to $b_{3}$ and $y_{3}$, we have $q_{3}=2 / 3$, and hence $q_{1}+q_{2}=4 / 3$. Setting $\alpha:=y_{1} q_{1}^{2}=y_{2} q_{2}^{2}=4\left(3 y_{3}+b_{3}\right) / 9$, we have $q_{i}=\sqrt{\alpha} / \sqrt{b_{i}}$ for $i=1,2,3 y_{3}+b_{3}=9 \alpha / 4$, and $b_{1}+b_{2}=2-9 \alpha / 4$. So

$$
\frac{4}{3}=\frac{\sqrt{\alpha}}{\sqrt{b_{1}}}+\frac{\sqrt{\alpha}}{\sqrt{b_{2}}} \geq \frac{2 \sqrt{\alpha}}{\sqrt{\sqrt{b_{1}} \sqrt{b_{2}}}} \geq \frac{2 \sqrt{2 \alpha}}{\sqrt{b_{1}+b_{2}}}=\frac{2 \sqrt{2 \alpha}}{\sqrt{2-9 \alpha / 4}} .
$$

This gives $\alpha \leq 8 / 27<0.35$.
Case 3. $y_{1}=b_{2}=0$, or $y_{2}=b_{1}=0$.
By symmetry, we may assume that $y_{1}=b_{2}=0$. Then $b_{1}>0, y_{2}>0, b_{1}+3 y_{2}+\left(3 y_{3}+\right.$ $\left.b_{3}\right)=2, f_{1}=b_{1} q_{1}^{2}, f_{2}=2 y_{2} q_{2}$, and $f_{3}=2 y_{3} q_{3}+b_{3} q_{3}^{2}$.

So $\mathbf{v}$ is a critical point of $f_{1}$ subject to $h_{1}=h_{2}=h_{3}=h_{4}=0$, all considered as functions of $b_{1}, y_{2}, b_{3}, y_{3}$. Hence $\mathbf{v}$ satisfies $\partial f_{1} / \partial u=\lambda_{1} \partial h_{1} / \partial u+\lambda_{2} \partial h_{2} / \partial u+\lambda_{3} \partial h_{3} / \partial u+\lambda_{4} \partial h_{4} / \partial u$ for $u \in\left\{b_{1}, y_{2}, b_{3}, y_{3}\right\}$. Thus,

$$
\begin{aligned}
& \text { for } u=b_{1} \text {, we have } q_{1}^{2}=\lambda_{1}\left(q_{1}^{2}\right)+\lambda_{2}\left(q_{1}^{2}\right)+\lambda_{3}, \\
& \text { for } u=y_{2} \text {, we have } 0=\lambda_{1}\left(-2 q_{2}\right)+3 \lambda_{3}, \\
& \text { for } u=b_{3} \text {, we have } 0=\lambda_{2}\left(-q_{3}^{2}\right)+\lambda_{3} \\
& \text { for } u=y_{3} \text {, we have } 0=\lambda_{2}\left(-2 q_{3}\right)+3 \lambda_{3} .
\end{aligned}
$$

Clearly, $\lambda_{i} \neq 0$ for $i=1,2,3$. So from the partial derivatives with respect to $b_{3}$ and $y_{3}$, we have $q_{3}=2 / 3$, and hence $q_{1}+q_{2}=4 / 3$.

Set $\alpha=f_{1}(\mathbf{v})=f_{2}(\mathbf{v})=f_{3}(\mathbf{v})$. Then

$$
2=b_{1}+3 y_{2}+\left(3 y_{3}+b_{3}\right)=\left(\frac{1}{q_{1}^{2}}+\frac{3}{2 q_{2}}+\frac{9}{4}\right) \alpha=\left(\frac{1}{q_{1}^{2}}+\frac{3}{2\left(4 / 3-q_{1}\right)}+\frac{9}{4}\right) \alpha .
$$

Let $h\left(q_{1}\right):=1 / q_{1}^{2}+3 /\left(2\left(4 / 3-q_{1}\right)\right)$. Note that $q_{1} \in(0,4 / 3)$ and

$$
h\left(q_{1}\right) \geq \begin{cases}4+3 /(2(4 / 3-0)), & \text { if } q_{1} \in(0,1 / 2] \\ 9 / 4+3 /(2(4 / 3-1 / 2)), & \text { if } q_{1} \in(1 / 2,2 / 3] \\ 25 / 16+3 /(2(4 / 3-2 / 3))), & \text { if } q_{1} \in(2 / 3,4 / 5] \\ 1+3 /(2(4 / 3-4 / 5)), & \text { if } q_{1} \in(4 / 5,1] \\ 9 / 16+3 /(2(4 / 3-1)), & \text { if } q_{1} \in(1,4 / 3)\end{cases}
$$

So $h\left(q_{1}\right) \geq 3.8125$, and hence $\alpha=2 /\left(h\left(q_{1}\right)+9 / 4\right) \leq 2 /(3.8125+9 / 4)<0.35$.
Proof of Lemma 4.2.2. For any permutation $i j k$ of $\{1,2,3\}$, and let $y_{k}=x_{i}+x_{j}$ and $b_{k}=a_{i}+a_{j}$. Then

$$
f_{k}=\left(1-p_{k}\right)\left(b_{i j}+y_{k}\right)+\left(1-p_{k}\right)^{2} b_{k} .
$$

Set $\alpha=f_{1}\left(p_{1}\right)=f_{2}\left(p_{2}\right)=f_{3}\left(p_{3}\right)$. Note that we may assume $\alpha>0$ (otherwise we are done); and hence $b_{i j}+y_{k}+b_{k}>0$ for $k=1,2,3$. Since $p_{k} \in(0,1), 1-p_{k} \in(0,1)$; and hence by solving $f_{k}\left(p_{k}\right)=\alpha$ we get

$$
1-p_{k}=\frac{2 \alpha}{\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}+\left(b_{i j}+y_{k}\right)} .
$$

We wish to show that $\alpha \leq 0.35$; so we consider the following optimization problem.

Maximize $\alpha$
Subject to

$$
\begin{aligned}
& g_{1}:=\sum_{k=1}^{3} \frac{2 \alpha}{\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}+\left(b_{i j}+y_{k}\right)}-2=0, \\
& g_{2}:=b_{12}+b_{13}+b_{23}+\frac{1}{2}\left(y_{1}+y_{2}+y_{3}+b_{1}+b_{2}+b_{3}\right)-1=0, \\
& b_{i j} \geq y_{k} \geq 0, \text { for }\{i, j, k\}=\{1,2,3\} .
\end{aligned}
$$

Here, $g_{1}, g_{2}$ are considered as functions of $\alpha, b_{i j}, b_{k}, y_{k}$. By the assumption of the lemma, the feasible region of this optimization problem is nonempty.

Claim 1. $\alpha$ is maximized only when $b_{i j}=y_{k}$ or $y_{k}=0$, for all $\{i, j, k\}=\{1,2,3\}$.
For, suppose $b_{i j}>y_{k}>0$ for some permutation $i j k$ of $\{1,2,3\}$. By applying the method of Lagrange multipliers, we have $\partial \alpha / \partial u=\lambda_{1} \partial g_{1} / \partial u+\lambda_{2} \partial g_{2} / \partial u$, where $u \in\left\{\alpha, b_{i j}, y_{k}\right\}$. So

$$
\begin{aligned}
& \text { for } u=b_{i j}, \quad 0=\lambda_{1} \frac{-2 \alpha\left(b_{i j}+y_{k}+\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}\right)}{\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}\left(\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}+\left(b_{i j}+y_{k}\right)\right)^{2}}+\lambda_{2}, \\
& \text { for } u=y_{k}, \quad 0=\lambda_{1} \frac{-2 \alpha\left(b_{i j}+y_{k}+\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}\right)}{\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}\left(\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}+\left(b_{i j}+y_{k}\right)\right)^{2}}+\frac{\lambda_{2}}{2}, \\
& \text { for } u=\alpha, \quad 1=\lambda_{1} \frac{\partial g_{1}}{\partial \alpha}+\lambda_{2} \frac{\partial g_{2}}{\partial \alpha} .
\end{aligned}
$$

The first two equations give $\lambda_{1}=\lambda_{2}=0$, which contradicts the third equation.
Therefore, the maximum of $\alpha$ is achieved when $b_{i j}=y_{k}$ for some permutation $i j k$ of $\{1,2,3\}$, or when $y_{k}=0$ for some $k \in\{1,2,3\}$; so Claim 1 follows.

Claim 2. We may assume that $\alpha$ is maximized when $b_{i j}>y_{k}$ for some $\{i, j, k\}=\{1,2,3\}$.
For, otherwise, the maximum of $\alpha$ is achieved when $b_{i j}=y_{k}$ for all permutations $i j k$ of $\{1,2,3\}$. Set $q_{k}=1-p_{k}$ for $k=1,2,3$; and so $f_{k}=2 y_{k} q_{k}+b_{k} q_{k}^{2}$ and $3\left(y_{1}+y_{2}+y_{3}\right)+b_{1}+$ $b_{2}+b_{3}=2$. We can now apply Lemma 4.4.1 and conclude that $f_{k} \leq 0.35$ for $k=1,2,3$. So Claim 2 holds.

From Claim 1 and Claim 2, we deduce
Claim 3. $\alpha$ is maximized when there exists a permutation $i j k$ of $\{1,2,3\}$ such that $b_{i j}>0$ and $y_{k}=0\left(\right.$ so $\left.x_{i}=x_{j}=0\right)$.

We consider three cases.
Case 1. $\alpha$ is maximized when $x_{k}=b_{i k}=b_{j k}=0$ and $b_{k}=0$.
Then $b_{i j}+a_{k}=1, f_{k}=\left(1-p_{k}\right) b_{i j}, f_{i}=\left(1-p_{i}\right)^{2} a_{k}$, and $f_{j}=\left(1-p_{j}\right)^{2} a_{k}$.
Since $f_{i}=f_{j}$, we have $p_{i}=p_{j}$. In particular, $p_{i} \in(0,1 / 2)$ as $p_{i}+p_{j}+p_{k}=1$. Since $b_{i j}=1-a_{k}$ and $f_{k}=f_{i}$, we have $2 p_{i}\left(1-a_{k}\right)=\left(1-p_{i}\right)^{2} a_{k}$. Therefore, $a_{k}=2 p_{i} /\left(1+p_{i}^{2}\right)$,
and so,

$$
\alpha=\frac{2 p_{i}\left(1-p_{i}\right)^{2}}{1+p_{i}^{2}}=\frac{4}{1+p_{i}^{2}}+2 p_{i}-4 .
$$

Differentiating with respect to $p_{i}$, we have $\alpha^{\prime}\left(p_{i}\right)=2-8 p_{i} /\left(1+p_{i}^{2}\right)^{2}$ and $\alpha^{\prime \prime}\left(p_{i}\right)<0$. Thus $\alpha\left(p_{i}\right)$ has maximum when $\alpha^{\prime}\left(p_{i}\right)=0$, i.e., when $\left(1+p_{i}^{2}\right)^{2}=4 p_{i}$. We now estimate $\alpha\left(p_{i}\right)$ subject to $\left(1+p_{i}^{2}\right)^{2}=4 p_{i}$. Considering the function $g(x):=\left(1+x^{2}\right)^{2}-4 x$ for $x \in(0,1 / 2)$, we see that $g^{\prime}(x)=4\left(1+x^{2}\right) x-4<0, g(0.3)<0$, and $g(0.29)>0$; so $g(x)=0$ implies that $x \in(0.29,0.3)$. Hence, $\left(1+p_{i}^{2}\right)^{2}=4 p_{i}$ implies $p_{i} \in(0.29,0.3)$. On the other hand, $\left(1+p_{i}^{2}\right)^{2}=4 p_{i}$ implies $\alpha\left(p_{i}\right)=2 / \sqrt{p_{i}}+2 p_{i}-4$. Since the function $h(t):=2 / \sqrt{t}+2 t-4$ is decreasing over $[0.29,0.3]$ (because $h^{\prime}=2-t^{-3 / 2}<0$ for $t \in[0.29,0.3]$ ), we have $\alpha \leq \alpha\left(p_{i}\right)=h\left(p_{i}\right) \leq h(0.29)=2 / \sqrt{0.29}+2(0.29)-4<0.35$.

Case 2. $\alpha$ is maximized when $x_{k}=b_{i k}=b_{j k}=0$ and $b_{k}>0$.
Then $b_{i j}+\left(b_{i}+b_{j}+b_{k}\right) / 2=1, f_{i}=\left(1-p_{i}\right)^{2} b_{i}, f_{j}=\left(1-p_{j}\right)^{2} b_{j}$, and $f_{k}=\left(1-p_{k}\right) b_{i j}+$ $\left(1-p_{k}\right)^{2} b_{k}$. From $\partial \alpha / \partial b_{k}=\lambda_{1} \partial g_{1} / \partial b_{k}+\lambda_{2} \partial g_{2} / \partial b_{k}$, we obtain

$$
0=\lambda_{1} \frac{-4 \alpha^{2}}{\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}\left(\sqrt{\left(b_{i j}+y_{k}\right)^{2}+4 b_{k} \alpha}+\left(b_{i j}+y_{k}\right)\right)^{2}}+\frac{\lambda_{2}}{2} .
$$

Using this and the partial derivatives with respect to $u \in\left\{\alpha, b_{i j}\right\}$ (as in the proof of Claim $1)$, we deduce that $\lambda_{1} \neq 0$ and $\lambda_{2} \neq 0$, and

$$
4 \alpha=b_{i j}+\sqrt{b_{i j}^{2}+4 b_{k} \alpha} .
$$

Therefore, $\alpha$ is maximized when $4 \alpha=b_{i j}+\sqrt{b_{i j}^{2}+4 b_{k} \alpha}$, that is $4 \alpha=b_{k}+2 b_{i j}$ which implies $p_{k}=1 / 2$ (since $f_{k}\left(p_{k}\right)$ is decreasing and $f_{k}\left(p_{k}\right)=\alpha$ has a unique solution).

Write $b_{k}^{\prime}:=b_{k}+2 b_{i j}$; then $f_{k}=\left(1-p_{k}\right)^{2} b_{k}^{\prime}$ (because $\left.p_{k}=1 / 2\right)$. Note that $\left(b_{k}^{\prime}+b_{i}+b_{j}\right) / 2=$ $b_{i j}+\left(b_{k}+b_{i}+b_{j}\right) / 2=1$. Since $\alpha=f_{1}=f_{2}=f_{3}$ and $\left(1-p_{i}\right)+\left(1-p_{j}\right)+\left(1-p_{k}\right)=2$, we have

$$
\frac{\sqrt{\alpha}}{\sqrt{b_{k}^{\prime}}}+\frac{\sqrt{\alpha}}{\sqrt{b_{i}}}+\frac{\sqrt{\alpha}}{\sqrt{b_{j}}}=2 .
$$

Applying Cauchy-Schwarz, we have

$$
\alpha=\left(\frac{2}{\frac{1}{\sqrt{b_{k}^{\prime}}}+\frac{1}{\sqrt{b_{i}}}+\frac{1}{\sqrt{b_{j}}}}\right)^{2} \leq 4\left(\frac{\sqrt{b_{k}^{\prime}}+\sqrt{b_{i}}+\sqrt{b_{j}}}{9}\right)^{2} \leq \frac{4}{9} \frac{b_{k}^{\prime}+b_{i}+b_{j}}{3}=\frac{8}{27}<0.35 .
$$

Case 3. $\alpha$ is maximized when (i) $x_{k}>0$, or (ii) $x_{k}=0$ and $b_{i k}>0$ or $b_{j k}>0$.
We claim that there exist $a_{m}^{\prime}, x_{m}^{\prime}, b_{m n}^{\prime} \in \mathbf{R}^{+}$, for any $1 \leq m \neq n \leq 3$, such that $b_{m n}^{\prime}=b_{n m}^{\prime}$,

$$
\begin{aligned}
& b_{m n}^{\prime} \geq \max \left\{2 x_{m}^{\prime}, 2 x_{n}^{\prime}\right\}, \\
& b_{12}^{\prime}+b_{23}^{\prime}+b_{31}^{\prime}+x_{1}^{\prime}+x_{2}^{\prime}+x_{3}^{\prime}+a_{1}^{\prime}+a_{2}^{\prime}+a_{3}^{\prime}=1, \\
& b_{m n}^{\prime}+x_{m}^{\prime}+x_{n}^{\prime} \geq b_{m n}+x_{m}+x_{n}, \\
& a_{m}^{\prime}+a_{n}^{\prime}=a_{m}+a_{n}, \text { and } \\
& b_{s t}^{\prime}+x_{s}^{\prime}+x_{t}^{\prime}>b_{s t}+x_{s}+x_{t} \text { for some } 1 \leq s \neq t \leq 3 .
\end{aligned}
$$

There are two cases to consider. First, suppose $x_{k}>0$. Then there exists $\delta>0$ such that $x_{k}^{\prime}=x_{k}-\delta>0$ and $b_{i j}^{\prime}=b_{i j}-2 \delta \geq 2 \delta$. Let $b_{i k}^{\prime}=b_{i k}+\delta, b_{j k}^{\prime}=b_{j k}$ and $x_{i}^{\prime}=x_{j}^{\prime}=\delta$. In particular, $x_{k}>\delta$; and so $b_{i k} \geq 2 x_{k} \geq 2 \delta$ and $b_{j k} \geq 2 x_{k} \geq 2 \delta$. It is easy to verify that the claim holds by setting $a_{i}^{\prime}=a_{i}, a_{j}^{\prime}=a_{j}$ and $a_{k}^{\prime}=a_{k}$. Now assume that $x_{k}=0$, and $b_{i k}>0$ or $b_{j k}>0$. We may assume $b_{i k}>0$; the case $b_{j k}>0$ is symmetric. Then there exists $\delta>0$ such that $b_{i k}^{\prime}=b_{i k}-\delta / 2 \geq \delta$ and $b_{i j}^{\prime}=b_{i j}-\delta / 2 \geq \delta$. Let $b_{j k}^{\prime}=b_{j k}+\delta / 2$ and $x_{i}^{\prime}=\delta / 2$. It is easy to verify that the claim holds by setting $x_{j}^{\prime}=x_{j}=0, x_{k}^{\prime}=x_{k}=0, a_{i}^{\prime}=a_{i}, a_{j}^{\prime}=a_{j}$ and $a_{k}^{\prime}=a_{k}$.

For every permutation $m n l$ of $\{1,2,3\}$, let

$$
f_{l}^{\prime}:=\left(1-p_{l}\right)\left(b_{m n}^{\prime}+x_{m}^{\prime}+x_{n}^{\prime}\right)+\left(1-p_{l}\right)^{2}\left(a_{m}^{\prime}+a_{n}^{\prime}\right) .
$$

For convenience of comparison, recall that

$$
\alpha:=f_{l}=\left(1-p_{l}\right)\left(b_{m n}+x_{m}+x_{n}\right)+\left(1-p_{l}\right)^{2}\left(a_{m}+a_{n}\right) .
$$

By Lemma 4.2.1, there exist $p_{i}^{\prime} \in[0,1]$ with $p_{1}^{\prime}+p_{2}^{\prime}+p_{3}^{\prime}=1$ such that $f_{l}^{\prime}\left(p_{l}^{\prime}\right) \leq 0.35$ for $l=1,2,3$, or $f_{1}^{\prime}\left(p_{1}^{\prime}\right)=f_{2}^{\prime}\left(p_{2}^{\prime}\right)=f_{3}^{\prime}\left(p_{3}^{\prime}\right)$ and $p_{i}^{\prime} \in(0,1)$. Since $p_{i} \in[0,1]$ and $p_{1}+p_{2}+p_{3}=1$, there exists some $l$ such that $1-p_{l} \leq 1-p_{l}^{\prime}$.

If $f_{i}^{\prime}\left(p_{i}^{\prime}\right) \leq 0.35$ for $i=1,2,3$ then, since $b_{m n}^{\prime}+x_{m}^{\prime}+x_{n}^{\prime} \geq b_{m n}+x_{m}+x_{n}$ and $a_{m}^{\prime}+a_{n}^{\prime}=a_{m}+a_{n}$ for all $\{m, n, l\}=\{1,2,3\}$, we have $f_{l}\left(p_{l}\right) \leq f_{l}^{\prime}\left(p_{l}^{\prime}\right) \leq 0.35$. Hence $\alpha \leq 0.35$.

We may thus assume $f_{1}^{\prime}\left(p_{1}^{\prime}\right)=f_{2}^{\prime}\left(p_{2}^{\prime}\right)=f_{3}^{\prime}\left(p_{3}^{\prime}\right)$. Suppose $1-p_{l}<1-p_{l}^{\prime}$. Then, since $b_{m n}^{\prime}+x_{m}^{\prime}+x_{n}^{\prime} \geq b_{m n}+x_{m}+x_{n}$ and $a_{m}^{\prime}+a_{n}^{\prime}=a_{m}+a_{n}$, and because $b_{m n}+x_{m}+x_{n}+a_{m}+a_{n}>0$ (see the beginning of the proof), we have $f_{l}\left(p_{l}\right)<f_{l}^{\prime}\left(p_{l}^{\prime}\right)$, contradicting the maximality of $\alpha$. So $1-p_{l}=1-p_{l}^{\prime}$. Then $\left(1-p_{m}^{\prime}\right)+\left(1-p_{n}^{\prime}\right)=\left(1-p_{m}\right)+\left(1-p_{n}\right)$. So we may assume that $1-p_{n} \leq 1-p_{n}^{\prime}$. By the same argument above for $1-p_{l}^{\prime}=1-p_{l}$, we derive the contradiction $f_{n}\left(p_{n}\right)<f_{n}^{\prime}\left(p_{n}^{\prime}\right)$ if $1-p_{n}<1-p_{n}^{\prime}$; and so we must have $1-p_{n}^{\prime}=1-p_{n}$. Hence we have $p_{i}^{\prime}=p_{i}$ for $i=1,2,3$. Recall that there exist $1 \leq s \neq t \leq 3$ such that $b_{s t}^{\prime}+x_{s}^{\prime}+x_{t}^{\prime}>b_{s t}+x_{s}+x_{t}$. Let $r \in\{1,2,3\} \backslash\{s, t\}$. Then $f_{r}\left(p_{r}\right)<f_{r}^{\prime}\left(p_{r}^{\prime}\right)$, again a contradiction to the maximality of $\alpha$. This proves Lemma 4.2.2.

## CHAPTER V

## CONCLUDING REMARKS

Theorem 1.3.2 implies Conjecture 1.3 .3 when the number of edges in the graph is sufficiently large; however to prove the entire conjecture is quite challenging. The error term in Theorem 2.3.5 is $O\left(m_{2}^{4 / 5}\right)$, but Bollobás and Scott ask in [12] whether it is possible to replace the error term by $O\left(\sqrt{m_{1}+m_{2}}\right)$ or $O\left(\sqrt{m_{2}}\right)$, which is still open.

For Problem 1.3.5, the general bound in Theorem 1.3.6 does not seem to be optimal. Also it is interesting to ask a general version of Problem 1.3.5: for any integer $r \in[3, k-1]$, find a $k$-partition $V_{1}, \ldots, V_{k}$ that minimizes $\max \left\{e\left(V_{i_{1}} \cup \ldots \cup V_{i_{r}}\right): 1 \leq i_{1}<i_{2}<\ldots<i_{r} \leq\right.$ $k\}$. In Chapter 3, we further show that Conjecture 1.3.7 holds for dense graphs as well as asymptotically for $k=3,4,5$; to the best of our knowledge, Conjecture 1.3 .7 is still standing in general.

Conjecture 1.3.10 is open for $r \geq 4$. In fact, Bollobás and Scott made an asymptotic version of Conjecture 1.3.10: for integers $r, k \geq 2$, every $r$-uniform hypergraph with $m$ edges has a vertex-partition into $k$ sets, each of which meets at least $(1+o(1))\left(1-(1-1 / k)^{r}\right) m$ edges. Note that, this bound is the expected number of edges meeting each set in a random $k$-partition. For $r=k=3$, the bound becomes $19 m / 27+o(m)$. One of the reasons why our proof does not give a closer bound to $19 / 27$ is that in Lemma 4.2.1, we can not get a smaller bound than 0.35 for $(i)$ of Lemma 4.2.1.

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