

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT

Closeout Notice Date 02/26/91

Project No. G-37-638 _____ Center No. R6512-0A0 _____

Project Director HARRELL E M II _____ School/Lab MATH _____

Sponsor NATL SCIENCE FOUNDATION/GENERAL _____

Contract/Grant No. DMS-8801309 _____ Contract Entity GTRC

Prime Contract No. _____

Title SPECTRAL AND VARIATIONAL PROBLEMS OF MATHEMATICAL PHYSICS _____

Effective Completion Date 901130 (Performance) 910228 (Reports)

Closeout Actions Required:	Y/N	Date Submitted
Final Invoice or Copy of Final Invoice	Y	
Final Report of Inventions and/or Subcontracts	Y	910222
Government Property Inventory & Related Certificate	N	_____
Classified Material Certificate	N	_____
Release and Assignment	N	_____
Other _____	N	_____

Comments ~~98A satisfies the requirement for the patent report; submitted with the final report.~~

Subproject Under Main Project No. _____

Continues Project No. _____

Distribution Required:

Project Director	Y
Administrative Network Representative	Y
GTRI Accounting/Grants and Contracts	Y
Procurement/Supply Services	Y
Research Property Management	Y
Research Security Services	N
Reports Coordinator (OCA)	Y
GTRC	Y
Project File	Y
Other _____	N
_____	N

NOTE: Final Patent Questionnaire sent to PDPI.

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I—PROJECT IDENTIFICATION INFORMATION

1. Institution and Address Georgia Institute of Technology Atlanta GA 30332-0160	2. NSF Program Modern Analysis	3. NSF Award Number DMS 8801306
	4. Award Period From 6/88 To 12/90	5. Cumulative Award Amount \$ 115,571.00

6. Project Title

SPECTRAL AND VARIATIONAL PROBLEMS OF MATHEMATICAL PHYSICS

PART II—SUMMARY OF COMPLETED PROJECT (FOR PUBLIC USE)

The proposal for this grant discussed several problems connected with spectral theory and variational analysis that arise in quantum physics. The main goal is to use modern, rigorous mathematics to derive new physical formulae and better understand physical theory. This was done in several papers, two Ph.D. theses, and many conferences, seminars, and less formal research efforts.

The areas where the most progress were made were:

Competing symmetries. With Carlen, Loss investigated how to exploit symmetries in order to analyze some nonlinear functionals of importance in physics and various areas of mathematics. Strikingly simple new proofs were discovered for some theorems about optimizers of these functionals.

Estimates of eigenvalue gaps. Small spectral gaps are known to arise in tunneling problems of quantum physics. With Ashbaugh and Svirsky, Harrell showed that the converse is also true to a certain extent.

Other topics. In addition to the topics described in the original proposal, Harrell and Loss investigated various new topics connected with laser dynamics and with numerical analysis. These are related to the proposed topics in their methods and in having applications to physics.

PART III—TECHNICAL INFORMATION (FOR PROGRAM MANAGEMENT USES)

1. ITEM (Check appropriate blocks)	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM	
				Check (✓)	Approx. Date
a. Abstracts of Theses		2			
b. Publication Citations		✓			
c. Data on Scientific Collaborators		✓			
d. Information on Inventions	✓				
e. Technical Description of Project and Results		✓			
f. Other (specify)					
2. Principal Investigator/Project Director Name (Typed) Evans M. Harrell and Michael Loss	3. Principal Investigator/Project Director Signature			4. Date 19 Feb 1991	

PART IV — FINAL PROJECT REPORT — SUMMARY DATA ON PROJECT PERSONNEL
(To be submitted to cognizant Program Officer upon completion of project)

The data requested below are important for the development of a statistical profile on the personnel supported by Federal grants. The information on this part is solicited in response to Public Law 99-383 and 42 USC 1885C. All information provided will be treated as confidential and will be safeguarded in accordance with the provisions of the Privacy Act of 1974. You should submit a single copy of this part with each final project report. However, submission of the requested information is not mandatory and is not a precondition of future award(s). Check the "Decline to Provide Information" box below if you do not wish to provide the information.

Please enter the numbers of individuals supported under this grant. *2 P's 2 grad students*
Do not enter information for individuals working less than 40 hours in any calendar year.

	Senior Staff		Post-Doctorals		Graduate Students		Under-Graduates		Other Participants	
	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.
A. Total, U.S. Citizens										
B. Total, Permanent Residents										
U.S. Citizens or Permanent Residents ² :										
American Indian or Alaskan Native . . .										
Asian										
Black, Not of Hispanic Origin										
Hispanic										
Pacific Islander										
White, Not of Hispanic Origin	2				2					
C. Total, Other Non-U.S. Citizens										
Specify Country										
1.										
2.										
3.										
D. Total, All participants (A + B + C)	2				2					
Disabled³										

Decline to Provide Information: Check box if you do not wish to provide this information (you are still required to return this page along with Parts I-III).

¹Category includes, for example, college and precollege teachers, conference and workshop participants.
²Use the category that best describes the ethnic/racial status for all U.S. Citizens and Non-citizens with Permanent Residency. (If more than one category applies, use the one category that most closely reflects the person's recognition in the community.)
³A person having a physical or mental impairment that substantially limits one or more major life activities; who has a record of such impairment; or who is regarded as having such impairment. (Disabled individuals also should be counted under the appropriate ethnic/racial group unless they are classified as "Other Non-U.S. Citizens.")

AMERICAN INDIAN OR ALASKAN NATIVE: A person having origins in any of the original peoples of North America, and who maintain cultural identification through tribal affiliation or community recognition.
ASIAN: A person having origins in any of the original peoples of East Asia, Southeast Asia and the Indian subcontinent. This area includes, for example, China, India, Indonesia, Japan, Korea and Vietnam.
BLACK, NOT OF HISPANIC ORIGIN: A person having origins in any of the black racial groups of Africa.
HISPANIC: A person of Mexican, Puerto Rican, Cuban, Central or South American or other Spanish culture or origin, regardless of race.
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THIS PART WILL BE PHYSICALLY SEPARATED FROM THE FINAL PROJECT REPORT AND USED AS A COMPUTER SOURCE DOCUMENT. DO NOT DUPLICATE IT ON THE REVERSE OF ANY OTHER PART OF THE FINAL REPORT.

The proposal for this grant discussed several problems connected with spectral theory and variational analysis that arise in quantum physics. Some of them progressed very satisfactorily, others are still being investigated, and, in addition, new related subjects of research arose and were investigated. Two students received Ph.D.'s under Harrell's direction, and a third expects to finish his dissertation in summer, 1991. Of the two students who finished already, one of them, D.T. Smith, was supported by research assistanceships with this grant. E.L. Green, who expects to finish in the summer, was also supported by research assistanceships. Our research was described in numerous articles, conference talks, and colloquia.

The five specific areas mentioned in the grant proposal were:

1. Variational problems and spectral bounds. The main progress in this area was a breakthrough by Loss and Carlen, who discovered a new, unified approach to find and characterize minimizers to many variational inequalities, called "competing symmetries." The idea of competing symmetries put forward in [1] turns out to be far more fruitful than originally anticipated. While the deeper structure does not seem to be fully understood a numerous collection of examples can be analyzed by this method. Here is an updated list of the inequalities in their sharp form obtained so far. Proofs of e) and g) have been outlined in [2].

- a) Hardy-Littlewood-Sobolev inequality
- b) the Sobolev inequality
- c) the hypercontractive estimate for the Mehler kernel
- d) the logarithmic Sobolev inequality
- e) Onofri's inequality
- f) a Mihlin Lebedev inequality
- g) Sobolev trace theorem due to Escobar
- h) a mixed Sobolev trace inequality due to Escobar
- i) a multilinear version of the Hardy-Littlewood-Sobolev inequality
- j) HLS trace theorem
- k) HLS trace theorems for higher codimension

There is more work in progress, some of the inequalities we conjecture are new. All this will be summarized in a coherent fashion in a paper jointly with Eric Carlen, which is in preparation. Another direction of research is to apply competing symmetries in the context of homogeneous groups. We tried, so far without success, to apply this method to inequalities on the Heisenberg group.

2. Semiclassical analysis. At the time of the proposal, Harrell had begun to work on inverse semiclassical problems, and had successfully shown that optimally eigenvalue gaps in a Schrödinger operator indicated that the potential was of double-well form, under assumptions of symmetry or in one-dimension. Together with Ashbaugh and Svirsky he has proved similar results in the fully n -dimensional case [3]. Secondly, parts of D. Smith's 1990 Ph.D. thesis [4] were concerned with semiclassical analysis in a discrete setting, for Jacobi matrices that arise in the theory of orthogonal polynomials.

3. Spectral theory of infinite banded matrices. This was the subject of D. Smith's 1990 Ph.D. thesis [4], under Harrell's direction. Smith proved several essentially optimal theorems characterizing the asymptotic decay of eigenvectors and resolvent matrices of Jacobi matrices arising in the theory of orthogonal polynomials. The coefficients diverge at

infinity, preventing them from falling under well-understood theorems, and necessitating novel analysis of a semiclassical nature.

4. Stability of matter. Loss has begun writing a book with one of the leading authorities in this field, Elliott Lieb [5]. To date there are over 200 pages, including many new proofs and results.

5. Skyrme model. No new publications resulted from this work in the period covered by the grant.

New topics.

Harrell directed G. James's Ph.D. thesis [6] and coauthored some papers on laser dynamics [7-9]. These papers consisted mainly of applications of classical dynamical systems theory to situations closely tied to experimental physics; the project was a collaboration with R. Roy, an experimental optical physicist. Of theoretical mathematical interest were a conservation theorem and as yet unresolved questions of metastability.

Loss worked with Estep and Rauch [10] on the theory of numerical analysis of hyperbolic equations with shocks. The goal is to control oscillations in the approximation near a discontinuity. The technical problem is to control some highly oscillatory singular integrals, which was done with a normal-form transformation.

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- [4] D.T. Smith, Exponential Decay of Resolvents of Banded Infinite Matrices and Asymptotics of Solutions of Linear Difference Equations, Georgia Institute of Technology Ph.D. thesis, 1990.
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- [6] G.E. James, Models of Intracavity Frequency Doubled Lasers, Georgia Institute of Technology Ph.D. thesis, 1990.
- [7] Instabilities and Chaos in Intracavity Doubled Lasers, *Proceedings of the Sixth Rochester Conference on Coherence and Quantum Optics*, to appear 1991 (with G. James and R. Roy). New York: Plenum, in press.
- [8] Instabilities and Chaos in Intracavity Doubled Lasers, II, *Phys. Rev. A* 41(1 March 1990)2778-2790 (with G. James and R. Roy).
- [9] On the Elimination of Chaos in an Intracavity Doubled Nd:YAG Laser, *Optics Letters*, to appear 1991 (with G. James, C. Bracicowski, R. Roy, and K. Wiesenfeld).
- [10] D. Estep, M. Loss, and J. Rauch, Boundedness of Dispersive Difference Schemes, *Math.Comp.*55(1990)55-87.

Harrell-2055
DMS 8801306

EXPONENTIAL DECAY OF RESOLVENTS OF BANDED MATRICES
AND ASYMPTOTICS OF SOLUTIONS OF LINEAR
DIFFERENCE EQUATIONS

A DISSERTATION
Presented to
The Academic Faculty

By

Dale T. Smith

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Georgia Institute of Technology
March, 1990

Thesis abstract of

D.T. Smith

Exponential Decay of Resolvents of Banded
Infinite Matrices and Asymptotics of Solutions
of Linear Difference Equations

SUMMARY

Georgia Institute of Technology
March, 1990

In this dissertation, we consider closed, m -banded infinite matrices J on the Banach spaces $\ell^p(\mathbb{Z}^+)$, where $1 \leq p \leq \infty$. Quantum Mechanics was originally formulated in terms of infinite matrices, and matrix mechanics has become an important part of Quantum Mechanics. Infinite matrices were also the origin of the modern theory of Functional Analysis. Today, infinite matrices are used in the theory of spline approximation, continued fractions, finite element approximations, orthogonal polynomials, Padé approximation, Functional Analysis, and in the analysis of eigenvalue problems in Quantum Mechanics.

We say $J = (c_{rs})$ is m -banded if m is a positive even integer, and

$$c_{rs} = 0 \text{ for } |r - s| > m/2.$$

The purpose of the dissertation is to find conditions on the elements c_{rs} of J which allow us to prove exponential bounds of the form

$$|(J - z)^{-1}(r,s)| \leq C_1 \exp(-C_2 |w(r) - w(s)|),$$

where C_1 and C_2 are positive constants, w is some positive sequence, and z is in the resolvent set of J . One of the two main results, in Chapter IV, uses the Combes-Thomas method as in Geronimo, Harrell, and Van Assche[65] and an idea of Simon[118] to prove

such bounds for a large class of infinite matrices. This method uses the theory of holomorphic families of linear operators of type (A) in the sense of Kato, and extends the result of Demko, Moss, and P. Smith[44] for $p = 2$ in two ways: first to a class of unbounded matrices; second to $p \neq 2$. The other main result (in Chapter V) is a comparison theorem for second-order difference equations, which allows us to bound solutions of the perturbed equation in terms of solutions of the comparison equation. These results seem to be similar to analogous results which can be derived by using a Green's function approach as in Geronimo[63]. Many examples of exponential decay are given in Chapter I, and in the discussion of the main results. We also discuss some known results in Chapter III as background and to compare them with the results proved in this dissertation. In Chapter VI, we discuss some incomplete results. First, we investigate the possibility of the existence of a Liouville-Green, or WKBJ, approximation for linear second-order difference equations. Second, we prove a discrete version of a result of Shnol for elliptic differential operators (for the differential operator case, see Glazman[67]), and discuss extending the results of Shnol and Agmon[2] to the difference equation case.

Harrell and Loss
DMS 8801306

NOTE:

G. James did not
receive funds from
8801306, but

finished his
dissertation with
Harrell in the
grant period.

MODELS OF INTRACAVITY FREQUENCY DOUBLED LASERS

A DISSERTATION
Presented to
The Academic Faculty

by

Glenn Edward James

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in Mathematics

Georgia Institute of Technology
March 1990

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SUMMARY

In many laser applications, the frequency of light produced by the laser is doubled by a crystal with nonlinear optical properties. The presence of such a crystal inside a laser cavity can produce large, irregular fluctuations in the output intensity. This thesis examines nonlinear systems of ordinary differential equations for the longitudinal mode intensities (physical observables) and gains of intracavity doubled lasers. A new system of equations is derived which models the frequency doubling of a general class-B laser with any number of intracavity birefringent elements, and the relevant features of the cavity configuration are reduced to two parameters. General results are also presented on the possible polarization states of the output beam; these polarizations are explicitly included in a dynamical system for the first time. This analysis is also the first to include the possibility of birefringence in the gain medium.

The complete range of behavior of the intensity output is characterized for one, two and three longitudinal modes in this general framework. A novel approach to the linearized stability analysis of the model leads to explicit stability criteria for the cavity parameters, and to several successful predictions of ways to stabilize the laser output. Several experimental laser configurations previously studied prove to be special cases of the general model; all the experimental results confirm the correspondence between the theory and experiment. Extensive numerical integrations also display a wide range of dynamical behavior consistent with experimental observations.

In the specific case of an intracavity doubled Nd:YAG laser, numerical results trace an intermittency route to chaos, with cross saturation as the control parameter. Samples of experimental output are closely matched by numerical integrations. An additional set of rate equations are developed which display dynamics seen in experiments but not in previous

numerical results. Still another set of rate equations validates our approximation of the lasing transition in Nd:YAG as a two-level system. The intracavity doubled Nd:YAG laser is found, in theory and experiment, to be a rich source of nonlinear dynamics.