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Determination of all abstract groups of a given order

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DETERMINATION OF ALL ABSTRACT GROUPS OF A GIVEN ORDER

by

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TABLE OF CONTENTS

TABLES

HISTORY OF ABSTRACT GROUPS

The theory of abstract groups of finite order may be said to date from the time of Cauchy. To him are due the first attempts at classification with a view to forming a theory from a number of isolated facts.

While the determination of all permutation groups of given degrees was started by J. A. Serret in 1850, the determination of all abstract groups of a given order was started about four years later by A. Cayley and was called Cayley's problem.

The prototype of the abstract group is the special substitution group which is ofter called a permutation group. A considerable number of other fundamental theorems of abstract group theory were stated long before a satisfactory general definition of the term abstract group was formulated. For instance, A. Cayley determined the five possible abstract groups of order 8 before this time. It can be seen that a number of fundamental advances in abstract group theory were made by men who seemed to have confined their attention to substitution groups while developing methods which apply also to abstract groups.

It is natural that the steps toward abstract theory of groups were taken haltingly by writers who seemed often to feel insecure as regards their position, since in the early history of abstract groups, little attention was being paid by the writers on groups to the postulational definitions of the term group. Men like S. Lie (1842-1899) and F. Klein (1849-1925) continued to use the term group without defining it except that they assumed that the product of two elements of a given group is contained therein and sometimes they assumed also the existence of the inverse of every element within the group. Even the work on finite abstract groups was done largely independently of postulates after it became known that every abstract group of finite order can be represented by one and only one regular permutation group. Abstract group theory, however, did receive more and more attention during the second half of the nineteenth century, and towards the end thereof, it became well established.

The golden age of the theory of finite groups came at the end of the last century and the first decade of the present. During this period the fundamental results of the theory were obtained, the fundamental directions of research were laid down, and the fundamental methods were created. Generally, through the work of its principal promoters, (Frobenius, Hölder, Burnside, Schur, Miller), the theory of finite groups acquired at this time all the essential features it has at the present day.

CHAPTER I

DETERMINATION OF ALL ABSTRACT GROUPS OF A GIVEN ORDER BY REGULAR PERMUTATION GROUPS

This paper is mainly devoted to the determination of all abstract groups of a given order by utilizing the properties of conjugate sets of subgroups and Sylow theorems. However, given here is an example of determining all abstract groups of order 4 by the use of permutation theory with the necessary developement first.

Def. A permutation of a set M is a 1-1 function from M onto M.

Def. The degree of a permutation group is the number of letters used in the group.

Def. A permutation group is called transitive when, by means of its permutations, a given symbol a_1 can be changed into every other symbol $a_2, a_3, \ldots, a_n$ operated on by the group.

Def. A transitive group whose order is equal to its degree is called a regular permutation group.

Theorem. Every group G of finite order n can be represented as a regular permutation group on n symbols, the latter being isomorphic with G. In fact, such a representation can be set up in two ways and the two representations are distinct when G is not an abelian group.

Pf. Let $s_1 = 1, s_2, s_3, \ldots, s_n$ be the n elements of the given group G. Then the n elements $s_1 s_i, s_2 s_i, \ldots, s_n s_i$ are all distinct and all belong to G, where it follows that they are the elements of G is some order. Then,
$\begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_1 s_i, s_2 s_i, \ldots, s_n s_i \end{pmatrix}$ is a permutation S_i performed on the n symbols representing the elements of G. For brevity we denote S_i by the symbol $S_i = \begin{pmatrix} s \\ ss_i \end{pmatrix}$. The permutation $S_i^{-1} S_j$ replaces s_i by s_j. Hence, the permutation group P, consisting of the permutations $S_1, S_2, \ldots, S_n$ is transitive. Since its order is equal to its degree, it is regular. If s_i is made to correspond to S_i, for every i, the G and P are isomorphic, since $s_i s_j$ corresponds with $S_i S_j$ as may be seen from the relations

$$S_i S_j = \begin{pmatrix} s \\ ss_i \end{pmatrix}\begin{pmatrix} s \\ ss_j \end{pmatrix} = \begin{pmatrix} s \\ ss_i \end{pmatrix}\begin{pmatrix} ss_i \\ ss_i s_j \end{pmatrix} = \begin{pmatrix} s \\ ss_i s_j \end{pmatrix}.$$

The process by which this representation of G has been obtained may be called post-multiplication, since in forming S_i we multiplied the elements of G on the right by s_i. If we use pre-multiplication and write

$$S'_i = \begin{pmatrix} s_1, s_2, \ldots, s_n \\ s_i^{-1} s_1, s_i^{-1} s_2, \ldots, s_i^{-1} s_n \end{pmatrix} = \begin{pmatrix} s \\ s_i^{-1} s \end{pmatrix},$$

then we have a permutation group P', consisting of the permutations $S'_1, S'_2, \ldots, S'_n$. Since s_i^{-1} is replaced by s_j^{-1} in the permutation $(S'_i)^{-1} S'_j$, it follows that this group P' is transitive and that also it is regular. Moreover, we have

$$S'_i S'_j = \begin{pmatrix} s \\ s_i^{-1} s \end{pmatrix}\begin{pmatrix} s \\ s_j^{-1} s \end{pmatrix} = \begin{pmatrix} s \\ s_i^{-1} s \end{pmatrix}\begin{pmatrix} s_i^{-1} s \\ s_j^{-1} s_i^{-1} s \end{pmatrix} = \begin{pmatrix} s \\ s_j^{-1} s_i^{-1} s \end{pmatrix} =$$

$$\begin{pmatrix} s \\ (s_i s_j)^{-1} s \end{pmatrix}$$

and this is the permutation corresponding to $s_i s_j$. Hence by making s_i and S_i' correspond for every i, the groups G and P' are isomorphic. Now, if $S_i = S_j'$, we have

$$\begin{pmatrix} s \\ ss_i \end{pmatrix} = \begin{pmatrix} s \\ s_j^{-1} s \end{pmatrix},$$

where $ss_i = s_j^{-1}s$ for each element $s \varepsilon G$. Taking s_1 for s, we have $s_i = s_j^{-1}$. Hence $ss_i = s_i s$, so that s_i is permutable with every element of the group. From this it follows that the two representations of G are distinct, except in the case when G is an abelian group. This completes the proof.

The following notation will be used in the example:

(abcd)all represents all possible permutations on the letters a,b,c,d;

(abcd)pos represents the subgroup of (abcd)all involving only even permutations;

(abcd)4 represents the subgroup of (abcd)all of order 4 and is non-cyclic;

(abcd)cyc represents the cyclic subgroup of (abcd)all; and

(abcd)s represents the permutation group of degree 4 and order 8.

Now, according to the definition, there are 5 transitive groups of degree 4.[1] These are:

(abcd)all =	1	abc	abcd	ac	ab·cd
		acb	adcb	ab	ac·bd

		abd	acbd	ad	ad·bc
		adb	adbc	bc	
		acd	abdc	bd	
		adc	acdb	cd	
		bcd			
		bdc			
(abcd)pos =	1	ab·cd	abc	acb	
		ac·bd	bdc	bcd	
		ad·bc	adb	abd	
			acd	adc	
(abcd)s =	1	ac	ab·cd	abcd	
		bd	ac·bd	adcb	
			ad·bc		
(abcd)4 =	1	ab·cd	ac·bd	ad·bc	
(abcd)cyc =	1	ac·bd	abcd	adcb	

Therefore, it follows from the definition of a regular permutation group and the theorem, that (abcd)4 and (abcd)cyc are isomorphic to the two abstract groups of order 4.

CHAPTER II

THEOREMS AND DEFINITIONS FOR THE DETERMINATION OF ALL ABSTRACT GROUPS OF A GIVEN ORDER BY FINITE GROUP PROPERTIES

The main purpose of this paper is to present arguments which take advantage of finite group properties in order to determine all abstract groups of a given order. Therefore, I will first present a body of definitions and theorems on finite groups and use standard notation except, perhaps, the use of $< >$ to denote "the group generated by."

Theorem 1. (Lagrange). If H is a subgroup of group G, then the order n of H is a factor of the order m of G.

Pf. Let $t_1 = 1, t_2, \ldots, t_n$ be the n distinct elements of H, and let $s_1 \varepsilon G$ such that $s_1 \not\varepsilon H$. Then, $A_1 = \{t_1 s_1, t_2 s_1, \ldots, t_n s_1\}$ is a set of distinct elements which are distinct from H, since if $t_p s_1 = t_q s_1$ where $1 \leq p < q \leq n$, then $t_p = t_q$ which is contrary to assumption. Also, if $t_p = t_q s_1$, then $s_1 = t_q^{-1} t_p$ and $s_1 \varepsilon H$ which is contrary to our assumption.

Consider set $G - (H \cup A_1)$. Then either $G - (H \cup A_1) = \emptyset$ or $G - (H \cup A_1)$ has more than n elements. If $G - (H \cup A_1)$ has fewer than n elements and $s_2 \varepsilon G - (H \cup A_1)$, then the set of elements $A_2 = \{t_1 s_2, t_2 s_2, \ldots, t_n s_2\}$ which is distinct from H and A_1 by previous arguments and the fact that $t_i s_2 = t_i s_1$ implies

$s_2 = s_1$. Hence, there exists some $s_1 \varepsilon G - (H \bigcup (\bigcup_{j=1}^{i-1} A_j))$ such that $G = H \bigcup (\bigcup_{j=1}^{i} A_j)$. Hence, the order n of H is a factor of the order m of G.

Theorem 2. If $s \varepsilon G$, the order n of s is a factor of the order m of G.

Pf. This folllows immediately from Theorem 1 and the fact that the order of s is also the order of the cyclic subgroup of G generated by s.

Theorem 3. If G is a group and H is a closed subset of G, then H is a subgroup of G.

Pf. Let $a_1, a_2, \ldots, a_n$ be the n distinct elements of $H \subset G$ where these elements are closed under the internal law of composition $\cdot$ of G. But $1 \varepsilon H$ since for every $a_j \varepsilon H$ there exists positive integer n_j such that $a_j^{n_j} = 1$. Also, $a_j^{n_j - 1} \cdot a_j = 1$ where $a_j^{n_j-1}, a_j \varepsilon H$. Consequently, H is a subgroup of G.

Theorem 4. The elements common to a family of groups γ, form a group G whose order is a factor of the order of every $P \varepsilon \gamma$.

Pf. Clearly, $\bigcap \gamma = G$ is a finite set. Suppose $s, t \varepsilon G$. Then, $s, t \varepsilon P$ for every $P \varepsilon \gamma$ which implies $st \varepsilon G$. Hence, $G \subset P$ is a subgroup of every $P \varepsilon \gamma$ by Theorem 3. Consequently, the order of G is a factor of the order of every $P \varepsilon \gamma$ by Theorem 1.

Def. 1. Let G be a group, let $a, b \varepsilon G$, and let H, K be subgroups of G. Then

(1) a and b are <u>permutable</u> if and only if $ab = ba$;

(2) a and H are <u>permutable</u> if and only if $aH = Ha$; and

(3) H and K are <u>permutable</u> if and only if every element of H is <u>permutable</u> with K and every element of K is <u>permutable</u> with H.

Def. 2. Let G be a group. Two elements $a, b \varepsilon G$ (two subgroups H, K of G) are conjugate in G if and only if there exists an inner automorphism α on G such that $a\alpha = b$ ($H\alpha = K$). The set of all distinct $a\alpha$ ($H\alpha$), for all inner automorphisms α of G is called a complete set of conjugate elements (subgroups).

Def. 3. In a group G, a subgroup H is called <u>self-conjugate</u> if and only if for every $a \varepsilon G$, $a^{-1}Ha = H$.

Theorem 5. The elements of a group G, which are permutable with a given element a, form a subgroup $H \subset G$. Also, the order of G divided by the order of H is the number of elements conjugate to a.

Pf. Let $H = \{t_1, t_2, \ldots, t_n\}$ denote all distinct elements of G permutable with a. Then for $t_1, t_2 \varepsilon H$, $t_1 a = at_1$ and $t_2 a = at_2$. Thus, $t_1 t_2 a = t_1(t_2 a) = (t_1 a)t_2 = at_1 t_2$ which implies that $t_1 t_2 \varepsilon H$ and that H is closed. Hence, H is a subgroup of G by Theorem

3. Suppose mn is the order of G. Then for $s_1 \varepsilon G$, $t_1s_1, t_2s_1, \ldots, t_ns_1$ all transform a into the same conjugate, a', since $(t_is_1)^{-1}a(t_is_1) = s_1^{-1}t_i^{-1}at_is_1 = s_1^{-1}(t_i^{-1}t_ia)s_1 = s_1^{-1}as_1$ for all $t_i \varepsilon H$. Also, the elements $t_1s_1, t_2s_1, \ldots, t_ns_1$ are the only elements of G which transform a into a', since for $s_2 \varepsilon G$, $s_2 \neq s_1$, $s_2^{-1}t_is_2 = t_i'$ implies $s_1s_2^{-1}t_is_2s_1^{-1} = s_1t_i's_1^{-1} = t_i$ and therefore $s_2s_1^{-1} \varepsilon H$. Consequently, since we have mn elements for G with every t_is_j distinct for $j = 1, 2, \ldots, m$, then we have m distinct sets of n elements each of which maps a into m distinct conjugates.

Theorem 6. The elements of a group G which are permutable with a subgroup H form a subgroup I, which is either identical with H or contains H as a self-conjugate subgroup. The order of G divided by the order of I is the number of subgroups conjugate to H.[2]

Theorem 7. The elements common to a complete set of conjugate subgroups form a self-conjugate subgroup.

Pf. Let $\{H_1, H_2, \ldots, H_n\}$ be a complete set of conjugate subgroups of group G. Also, let $I = \bigcap_{i=1}^{n} H_i$. Clearly, $I \neq \emptyset$. Suppose $t_1, t_2 \varepsilon I$. Then $t_1, t_2 \varepsilon H_i$ for every i which implies that $t_1t_2 \varepsilon I$. Hence, I is a subgroup of G by Theorem 3.

Also, I is a self-conjugate subgroup of G since the set of conjugate subgroups when transformed by any element of G is changed into itself where I is the common subgroup of the set.

Theorem 8. (Corollary). The elements permutable with each of a complete set of conjugate subgroups form a self-conjugate subgroup.

Pf. This follows immediately from Theorem 7, since the elements permutable with a subgroup $H \subset G$ form a subgroup $I \subset G$ by Theorem 5. Also, the elements permutable with every subgroup of the conjugate set to which H belongs are the elements common to every subgroup of the conjugate set to which I belongs.

Theorem 9. If $\{t_1, t_2, \ldots, t_n\}$ is a complete set of conjugate elements of G, then the group $< t_1, t_2, \ldots, t_n >$, if it does not coincide with G, is a self-conjugate subgroup of G, and it is the self-conjugate subgroup of smallest order which contains t_1.

Pf. Suppose $H = \{t_1, t_2, \ldots, t_n\}$ is a complete set of conjugate elements of group G. Let H_1 be any self-conjugate subgroup of G which contains t_1. But for every $s \varepsilon G$, $s^{-1}H_1 s = H_1$. Hence, $H_1 \supset H$.

Suppose H_2 is the group generated by H. Let $z \varepsilon H_2$. Then $z = x_1 x_2 \cdots x_k$ where $x_j = t_i$ or $x_j = t_i^{-1}$ for some i. Let $s \varepsilon G$. Then, $s^{-1}zs = s^{-1}x_1 s s^{-1} x_2 s \cdots$

$s^{-1}x_k s$. Now, $w = s^{-1}x_j s = s^{-1}t_j s = t_u$ for some u or $s^{-1}t_j s = s^{-1}t_j^{-1}s = (s^{-1}t_j s)^{-1} = t_v^{-1}$ for some v. Hence, $w\varepsilon H_2$ and $z\varepsilon H_2$. Consequently, H_2 is a self-conjugate subgroup which contains t_1 since H_2 is generated by H.

Theorem 10. If an element s of order n is permutable with a group G and if s^m is the lowest power of s which occurs in G, then m is a factor of n and n/m is a factor of the order of G.[3]

Theorem 11. If every element of G transforms H into itself and every element of H transforms G into itself, and if G and H have no common element except 1, then every element of G is permutable with every element of H.[4]

Def. 4. A group G is simple if and only if no proper subgroup is self-conjugate.

Theorem 12. (Corollary). If every element of G transforms H into itself and every element of H transforms G into itself, and if either G or H is a simple group, then G and H have no common elements except 1 and every element of G is permutable with every element of H.

Pf. Suppose H and G are groups such that $g^{-1}Hg = H$ for every $g\varepsilon G$ and $h^{-1}Gh = G$ for every $h\varepsilon H$ where H is a simple group. Consider the element $z = g^{-1}h^{-1}gh$. Then $(g^{-1}h^{-1}g)h\varepsilon H$ and $g^{-1}(h^{-1}gh)\varepsilon G$. Suppose $z \neq 1$. Then $G \subset H$ is a subgroup of G and also of H, and contains more than 1. If $G \subset H$ or $H \subset G$, we are done.

Suppose $G \not\subset H$ and $H \not\subset G$. Then $G \cap H$ is a proper subgroup of H. Clearly, $G \cap H$ is a self-conjugate subgroup of $\langle G, H \rangle$ since every element of $G \cap H$ is permutable with the elements of $G \cup H$. Contradiction. Hence, $G \cap H = 1$ and by Theorem 11, every element of G is permutable with every element of H.

Theorem 13. If p is a prime and if p^m is less than and divides the order of a group G, then G has at least one subgroup, distinct from itself, whose order is divisible by p^m.[5]

Theorem 14. (Corollary). If p^m divides the order of a group G, then the group has at least one subgroup of order p^m.

Pf. This follows immediately from Theorem 13 and the fact that if a group has a proper subgroup whose order is divisible by p^m, then this subgroup will have a proper subgroup whose order is divisible by p^m until this process terminates in a subgroup of G of order p^m.

Theorem 15. (Cauchy). If p, a prime, divides the order of a group, then the group has elements of order p.

Pf. This follows immediately from Theorem 14 and the fact that a cyclic subgroup of order p is generated by an element of order p.

Theorem 16. The number of elements of a group of order m, whose n^{th} powers belong to a given conjugate set is zero or a multiple of the highest common factor of m and n.[6]

Theorem 17. (Corollary). If n is a factor of m, the order of G, then the number of elements of G satisfying the relation $s^n = 1$ is a multiple of n.

Pf. This follows immediately from Theorem 16 if we consider the fact that all the elements of G belonging to the conjugate set of 1 is m where $s^n = 1$ and the fact that $(n,m) = n$.

Theorem 18. (Corollary). If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_j^{\alpha_j}$ is a factor of $m = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$ and if the number of elements of G, of order m, which satisfy the relation $s^n = 1$ is equal to n, then either $\alpha_1 = \beta_1$ or G must contain elements of order $p_1^{\alpha_1+1}$.[7]

Theorem 19. (Corollary). If a group of order mn, where $(m,n) = 1$ contains a self-conjugate subgroup of order n, then the group contains just n elements whose orders divide n.

Pf. Let G be a group of order mn where $(m,n) = 1$. Suppose G contains a self-conjugate subgroup of order n, H. Let $s \notin H$, $s \in G$ be an element whose order divides n. Then the group $H_1 = \langle H, s \rangle$ would have by Theorem 17 and Theorem 18, an order which is a multiple of n greater than 1 and which would be relatively prime to mn. But, H_1 is a subgroup of G and this would contradict Theorem 1. Hence, G contains just n elements whose orders divide n.

Theorem 20. (Corollary). If G has a self-conjugate subgroup H of order mn, where $(m,n) = 1$, and if H has a self-conjugate subgroup K of order n, then K is a self-conjugate subgroup of G.

Pf. This follows immediately from Theorem 19, since H contains just n elements whose orders divide n, (mainly those of K), and since every element of G, since it transforms H into itself, must transform K into itself.

Def. 5. If s and t are any two elements of a group, then the element $s^{-1}t^{-1}st$ is called a <u>commutator</u>.

Def. 6. The group generated by the commutators of a group G is called the <u>commutator subgroup</u> or the <u>derived group</u> of G.

Def. 7. The derived group H has itself a commutator subgroup or derived group, which may or may not coincide with H. Suppose now that starting with a given group G, of finite order, G_1 is the derived group of G, and actually distinct from it. G_2 is the derived group of G_1 and actually distinct from it and so on. Since the order of each of these groups is less than the preceding, the series must terminate. This may happen in one of two ways. We may either arrive at a group which is identical with its derived group, or we may arrive at an abelian group, whose derived group is $\{1\}$. In the case the derived series terminates in $\{1\}$, then G is solvable.

Theorem 21. The derived group of G is that self-conjugate subgroup H of smallest order such that the quotient group G/H is abelian.

Pf. See Burnside, section 39 and add the following argument to the last statement of the proof: Suppose $h \varepsilon H'$, then $sht = ths$. Thus, $sht = tt^{-1}h_1ts$ where $h_1 \varepsilon H'$. Hence, $sh = h_1tst^{-1}$ and $shs^{-1} = h_1tst^{-1}s^{-1}$ where $shs^{-1} \varepsilon H'$. Consequently, for $s,t \varepsilon G$, all elements of the form $tst^{-1}s^{-1}$, i.e., commutators are in H'.

Theorem 22. If K is a self-conjugate subgroup of G, and if G/K is a solvable group with i terms in its derived series, then K contains G_i, the i^{th} derived group of G and does not contain G_{i-1}.[8]

Theorem 23. If H is any self-conjugate subgroup of group G, and if K, K' are two self-conjugate subgroups of G contained in H, such that there is no self-conjugate subgroup of G contained in H and containing either K or K' except H, K and K' themselves, and if L is the greatest common subgroup of K and K', so that L is necessarily self-conjugate in G, then the groups H/K and K'/L are isomorphic as also the groups H/K' and K/L.[9]

Def. 8. If G_1, a self-conjugate subgroup of G is such that the group $< G_1, t_1, t_2, \ldots, t_k >$ coincides with G, when $\{t_1, t_2, \ldots, t_k\}$ is any complete set of conjugate elements not contained in G_1, then G_1 is said to be a <u>maximum self-conjugate subgroup</u> of G. If H is a subgroup of G, and if, for every element $s \varepsilon G$ which does not belong to H, the group $< H, s >$ coincides with G, H is said to be a <u>maximum subgroup</u> of G.

A <u>minimum self-conjugate subgroup</u> and <u>minimum subgroup</u> are defined similarly.

Theorem 24. (Corollary). If H coincides with G, and K and K' are maximum self-conjugate subgroups of G and L is the greatest group common to K and K', then G/K and K'/L are isomorphic, as also are G/K' and K/L.

Pf. This follows from Theorem 23 and the fact that G/K and G/K' are simple groups which implies that K/L and K'/L are simple groups such that L must be a maximum self-conjugate subgroup of both K and K'.

Def. 9. Let G_1 be a maximum self-conjugate subgroup of a given group G, G_2 a maximum self-conjugate subgroup of G_1 and so on. Since G is a group of finite order, we must, after a finite number of subgroups, arrive in this way at a subgroup G_{n-1}, whose only self-conjugate subgroup is that formed of the identity alone, so that G_{n-1} is a simple group. The series of groups obtained in this manner is called a <u>composition series</u> of G. The set of groups G/G_1, $G_1/G_2, \dots, G_{n-2}/G_{n-1}, G_{n-1}/G_n$ is called a set of <u>quotient groups</u> of G, and the orders of these groups are said to form a set of <u>composition-factors</u> of G.

Def. 10. Suppose that a series of groups, each contained in the previous one, $G, H_1, H_2, \dots, H_{n-1}, \{1\}$ are chosen so that each one is a self-conjugate subgroup of G, while there is no self-conjugate subgroup of G contained in any one group of the series and containing the next group. The series of groups obtained in this manner just described is called a <u>chief-composition series</u> of G.

Theorem 25. Any two composition series of a group consist of the same number of subgroups, and lead to two sets of quotient groups which, except as regards the sequence in which they occur, are identical with each other.[10]

Def. 11. The group G is the <u>direct product</u> (or <u>direct sum</u> if the law of composition is addition) of its subgroups $H_1, H_2, \ldots, H_n$ if and only if

(1) The subgroups $H_1, H_2, \ldots, H_n$ are self-conjugate subgroups of G;

(2) G is generated by the subgroups $H_1, H_2, \ldots, H_n$; and

(3) The common part of each H_i with the subgroup H_i', generated by all the H_j, $i \neq j$, is $\{1\}$.

Theorem 26. Any two chief composition series of a group consist of the same number of terms and lead to two sets of quotient groups, which, except as regards the sequence in which they occur, are identical with each other.

Pf. This theorem follows immediately by a repetition of the same arguments of Theorem 23.

Theorem 27. If between two consecutive terms H_r and H_{r+1} in the chief-composition series of a group there occur the groups $G_{r,1}, G_{r,2}, \ldots, G_{r,s-1}$ of a composition series, then (i) the quotient groups $H_r/G_{r,1}$, $G_{r,1}/G_{r,2}, \ldots, G_{r,s-1}/H_{r+1}$ are all isomorphic, and (ii) H_r/H_{r+1} is the direct product of s groups of the type $H_r/G_{r,1}$.[11]

Theorem 28. (Corollary). If the order of H_r/H_{r+1} is a power, p^s, of a prime, H_r/H_{r+1} must be an abelian group whose elements, except 1, are all of order p.

Pf. This theorem follows from arguments used in proof of Theorem 27.

Theorem 29. If H is a subgroup of G, each composition factor of H must be equal to or be a factor of some composition factor of G.[12]

Theorem 30. (Corollary). If all composition factors of group G are primes, so also are the composition factors of every subgroup of G.

Pf. This is an immediate consequence of Theorem 29.

Theorem 31. A solvable group, the composition factors of which may be taken in any order, is the direct product of groups whose orders are powers of primes.[13]

Theorem 32. An abelian group G of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are distinct primes, is the direct product of groups P_i formed of all elements of G whose orders divide $p_i^{\alpha_i}$ where P_i is of order $p_i^{\alpha_i}$.[14]

Theorem 33. The elements of an abelian group, whose order is a power of p, can always be represented in the form $q_1^{x_1}q_2^{x_2}\cdots q_s^{x_s}$, $\left(\begin{array}{l} x_i = 0,1,\ldots,p^{m_i}-1 \\ i = 1,2,\ldots,s \end{array}\right)$, where the elements $q_1, q_2, \ldots, q_s$ are related by $q_i^{p^{m_i}} = 1$, $q_iq_j = q_jq_i$, $(i,j = 1,2,\ldots,s)$ and by no others.[15]

Theorem 34. The number of distinct types of abelian groups of order p^n, where p is a prime, is equal to the number of partitions of m, and each type may be completely represented by the symbol $(m_1, m_2, \ldots, m_s)$ of the corresponding partition. If the numbers in the partition are written in descending order, a group of the type $(m_1, m_2, \ldots, m_s)$ will have a subgroup of the type $(n_1, n_2, \ldots, n_t)$ when the conditions $t = s$, $n_i = m_i$ $(i = 1, 2, \ldots, t)$ are satisfied, and the type of every subgroup must satisfy these conditions.[16]

Theorem 35. Every group whose order is the power of a prime contains self-conjugate elements other than 1 and no such group can be simple.[17]

Theorem 36. A group whose order is the power of a prime is necessarily distinct from its derived group, and its series of derived groups terminates with the one containing 1 only.[18]

Theorem 37. If G_s of order p^s is a subgroup of G, which is of order p^m, then G must contain a subgroup of order p^{s+t}, $t \nless 1$, within which G_s is self-conjugate. In particular, every subgroup of order p^{m-1} of G is a self-conjugate subgroup.[19]

Theorem 38. (Lemma 1). If a group G is of order p^m, then the number of subgroups of G of order p^{m-1} is r_{m-1} where $r_{m-1} \equiv 1 (\text{mod } p)$.[20]

Theorem 39. (Lemma 2). If a group G is of order p^m, then the number of subgroups of G of order p is r_1 where $r_1 \equiv 1 (\text{mod } p)$.[21]

Theorem 40. The number of subgroups of any given order p^s of a group G of order p^m is congruent to 1(mod p).

Pf. If now G_s is any subgroup of G of order p^s, and if G_{s+t} is the greatest subgroup of G in which G_s is contained self-conjugately, then every subgroup of G in which G_s is contained self-conjugately is contained in G_{s+t}. But every subgroup of order p^{s+1}, which contains G_s, contains G_s self-conjugately. Therefore every subgroup of order p^{s+1} which contains G_s is itself contained in G_{s+t}. Hence, by Lemma 1, the number of subgroups of G_{s+t}/G_s of order p is congruent to 1(mod p). Thus, the number of subgroups of G of order p^{s+1}, which contain G_s of order p^s, is congruent to 1(mof p).

Now, let r_s represent the total number of subgroups of order p^s contained in a_x subgroups of order p^{s+1}, and if any one of the subgroups of order p^{s+1} contains b_y subgroups of order p^s, then

$$\sum_{x=1}^{x=r_s} a_x = \sum_{y=1}^{y=r_{s+1}} b_y$$

for the numbers on either side of this equation are equal to the number of subgroups of order p^{s+1} when each of the latter is reckoned once for every subgroup of order p^s which it contains. It has, however, been shown that for all values of x and y

$a_x \equiv 1$, $b_y \equiv 1 (\text{mod } p)$ by Lemma 1.

Hence, $r_s \equiv r_{s+1} (\text{mod } p)$. Also, $r_1 \equiv 1$ and $r_{m-1} \equiv 1 (\text{mod } p)$ by Lemmas 1 and 2. Therefore, for all values of s, $r_s \equiv 1 (\text{mod } p)$.

Theorem 41. (Corollary). The number of self-conjugate subgroups of order p^s of a group of order p^m is congruent to $1 (\text{mod } p)$.

Pf. This follows immediately from Theorem 6 and Theorem 40, since the number of subgroups in any conjugate set is a power of p.

Theorem 42. If G, of order p^m, where p is an odd prime, contains only one subgroup of order p^s, then G must be cyclic.[22]

Theorem 43. If a group G, of order 2^m, has a single subgroup of 2^s, $(s > 1)$, it must be cyclic. If it has a single subgroup of order 2, it is either cyclic or of the type defined by $p^{2^{n-1}} = 1$, $q^2 = p^{2^{m-2}}$, $q^{-1}pq = p^{-1}$ $(m > 2)$.[23]

Def. 12. The set $N = \{x: x \varepsilon G, xa = ax\}$ of group G is a subset of G called the <u>normalizer</u> of $a \varepsilon G$.

Theorem 44. The normalizer N of $a \varepsilon G$ is a subgroup of G.

Pf. Since $a \varepsilon N$, N is nonempty. Let $x, y \varepsilon N$. Since

$ya = ay$, upon multiplying on the left by y^{-1}, we have $a = y^{-1}ay$ and then upon multiplying on the right by y^{-1}, we have $ay^{-1} = y^{-1}a$. Therefore $y \varepsilon N$ implies $y^{-1} \varepsilon N$. Also, $1 \varepsilon N$.

Now, $(xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy)$. Hence, $xy \varepsilon N$. Consequently, N is a subgroup of G.

Def. 13. (Sylow). Let G be a finite group of order n and let $p \varepsilon Z^+$, where p is a prime. Further, let p^m be the highest power of p which divides n. Then a subgroup H of G is a <u>Sylow subgroup</u> if and only if the order of H is p^m.

Theorem 45. Let G be a finite group of order n and p be a positive prime dividing n, then G has at least one Sylow subgroup of order p^m.[24]

Theorem 46. Let G be a group of order n and p be a positive prime such that p^m is the highest power of p dividing n. Then the Sylow subgroups of order p^m form a complete set of conjugate subgroups, and the number is congruent to 1(mod p).[25]

Theorem 47. There is only one Sylow subgroup H of order p^m of H if and only if H is a self-conjugate Sylow subgroup of G.

Pf. This follows immediately from Theorem 46, since the only subgroup of a conjugate set implies that the subgroup is self-congugate and since every Sylow subgroup of a given order belongs to a conjugate set of Sylow subgroups of the same order.

Theorem 48. If every Sylow subgroup of a group G is self-conjugate, then G is the direct product of its Sylow subgroups.

Pf. This follows immediately from Theorem 47 and Definition 11 since (1) and (3) are obvious and (2) follows from the fact that if $a \varepsilon G$, then $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ where $\{p_1, p_2, \ldots, p_n\}$ is the set of distinct primes dividing the order of G. But every Sylow subgroup of G of order $p_i^{\alpha_i}$ contains elements of order p_i. Hence, a is generated by powers of these elements from Sylow subgroups of G.

Theorem 49. Let p^{α} be the highest power of a prime p which divides the order G, and let H be a subgroup of G of order p^{α}. Let h be a subgroup common to H and some other subgroup of order p^{α}, such that no subgroup, which contains h and is of greatest order, is common to any two subgroups of order p^{α}. Then there must be some element of G, of order prime to p, which is permutable with h and not with H.[26]

CHAPTER III

DETERMINATION OF ABSTRACT GROUPS WHOSE ORDERS ARE THE POWERS OF A SINGLE PRIME

I. ($p \neq 2$). I shall now proceed to discuss, in application of the foregoing theorems, the various types of groups of order p^m, which contain self-conjugate cyclic subgroups of order p^{m-1} or p^{m-2}, etc. It is clear from Theorem 43 that the case $p = 2$ requires independent investigation. Hence, at the moment I will deal with the case where p is an odd prime in determining all non-abelian groups of order p^m.

(i) First consider a group G of order p^m, which contains an element q of order p^{m-1}. The cyclic subgroup $\langle q \rangle$ is self-conjugate and contains a single subgroup $\langle q^{p^{m-2}} \rangle$ of order p. By Theorem 42, since G is not cyclic, it must contain an element s, of order p, which does not occur in $\langle q \rangle$. Since $\langle q \rangle$ is self-conjugate and G is not abelian, s must transform q into one of its own powers. Hence, $s^{-1}qs = q^{\alpha}$, and since s^p is permutable with q, it follows that $\alpha = 1 + kp^{m-2}$.[27]

Since G is not abelian, $k \neq 0$. But it may have any value from 1 to $p - 1$. If now $kx \equiv 1 \pmod p$, then $s^{-x}qs^{x} = q^{1+p^{m-2}}$, and therefore writing t for s^x, the group is defined by

$$q^{p^{m-1}} = 1, \quad t^p = 1, \quad t^{-1}qt = q^{1+p^{m-2}}.$$

These relations are clearly self-consistent and they define a group of order p^m. There is therefore a single type of nonabelian group of order p^m which contains elements of order p^{m-1}, because for any such group, a pair of generating elements may be chosen which satisfy the above relations.

(ii) Suppose next that G, a group of order p^m, has a self-conjugate cyclic subgroup $\langle q \rangle$ of order p^{m-2}, and that no element of G is of higher order than p^{m-2}. Then three cases may be distinguished at once for separate discussion, according as q is selfconjugate, one of p conjugate elements or one of p^2 conjugate elements.

Taking the first case, there can be no element $s\varepsilon G$ such that s^{p^2} is the lowest power of $s\varepsilon\langle q \rangle$, for if there were, $\langle s,q \rangle$ would be abelian, and its order being p^m, it would necessarily coincide with G. Hence any element $s\not\varepsilon\langle q \rangle$ with q generates an abelian group of type (m - 2,1), and we may choose q and t as independent generators of this subgroup, the order of t being p. If now $r\varepsilon G$ and $r\not\varepsilon\langle t,q \rangle$, then $\langle r,q \rangle$ is again an abelian group of type (m - 2,1). If q and r are independent generators of this group, the latter cannot occur in $\langle t,q \rangle$. Now, since t is not self-conjugate, $r^{-1}tr = tq^{\beta}$, and since r^p or 1 is permutable with t, then $q^{p^{\beta}} = 1$, so that $\beta \equiv 0 (\text{mod } p^{m-3})$. Hence, $r^{-1}tr = tq^{kp^{m-3}}$, where k is not a multiple of p. If finally q^k be taken as a generating element in the place of q, the group is defined by

$$q^{p^{m-2}} = 1, \quad t^p = 1, \quad r^p = 1, \quad r^{-1}tr = tq^{p^{m-2}}, \quad qt = tq,$$
$$qr = rq.$$

There is therefore a single type of group of order p^m, which contains a self-conjugate element of order p^{m-2} and no element of order p^{m-1}.

Next, let q be one of p conjugate elements. These must be $q^{1+kp^{m-3}}$, $(k = 1,2,\dots,p)$.[28] If $G/\langle q \rangle$ is cyclic, let s be an element, the lowest power of which in $< q >$ is s^{p^2}. If s were permutable with q, G would be abelian. Hence, we may take $s^{-1}qs = q^{1+p^{m-3}}$ while $s^{p^2} = q^{kp^2}$. These relations give $(sq^x)^{p^2} = q^{(x+k)p^2}$. Hence, if $sq^{-k} = t$, the group is defined by

$$q^{p^{m-2}} = 1, \quad t^{p^2} = 1, \quad t^{-1}qt = q^{1+p^{m-3}},$$

and there is a single type.

If $G/< q >$ is non-cyclic, G must contain a subgroup of order p^{m-1} in which q is self-conjugate and another in which q is one of p conjugate elements. The former is an abelian group of type (m - 2, 1) of which q and r may be taken as independent generating elements. The latter is a group of the type considered in (i), (with m -1 for m), defined by

$$q^{p^{m-2}} = 1, \quad t^p = 1, \quad t^{-1}qt = q^{1+p^{m-3}}.$$

With this group r is permutable and therefore $r^{-1}tr = t^{\alpha}q^{\beta p^{m-3}}$, since the only elements of order p in $< q,t >$ are of this form by (i).

Now, $r^{-1}t^{-1}qtr = q^{1+p^{m-3}}$ or $t^{-\alpha}qt^{\alpha} = q^{1+p^{m-3}}$, and therefore $\alpha = 1$. Also, $q^{-1}tq = tq^{-p^{m-3}}$, hence $q^{-\beta}r^{-1}trq^{\beta} = t$, and rq^{β} is an element of order p^{m-2}, and by assuption the group has no self-conjugate element of order p^{m-2}. Hence, β must be a multiple of p and r is a self-conjugate element. Again there is one type defined by

$$q^{p^{m-2}} = 1, \quad t^p = 1, \quad r^p = 1, \quad t^{-1}qt = q^{1+p^{m-3}},$$

$$r^{-1}qr = q, \quad r^{-1}tr = t.$$

It is the direct product of $< r >$ and $< q,t >$.

Lastly, let q be one of p^2 conjugate element. There must be $q^{1+kp^{m-4}}$ $(k = 1,2,\ldots,p^2)$ such elements.[29] This case can only occur if $m > 4$. The order of an element which transforms q into $q^{1+p^{m-4}}$

must be equal to or a multiple of p^2. If there were no elements of order p^2 effecting the transformation, every element of the group not belonging to $< q >$ would be of order p^2 or greater, and the group would only have one subgroup of order p. Hence, there must be an element of order p^2 transforming q into $q^{1+p^{m-4}}$. Denoting this element by t, there is again a single type defined by

$$q^{p^{m-2}} = 1, \quad t^{p^2} = 1, \quad t^{-1}qt = q^{1+p^{m-4}} .$$

The logic of this process may be continued indefinitely until all possible non-abelian groups of order p^m, p an odd prime, are determined.

Examples:

1. Determination of all Non-Abelian Groups of Order p^2:

   If a group of order p^2 contains an element of order p^2, it is cyclic. If not, its $p^2 - 1$ elements other than 1, are all of order p. A subgroup of order p contains $p - 1$ elements of order p which enter in no other such subgroup. There must therefore be $p + 1$ subgroups of order p, and hence at least one of them is self-conjugate. If this is $< q >$ and if s is an element of order p which is not a power of q, $s^{-1}< q >s = < q >$. Hence, $s^{-1}qs = q^{\alpha}$, $s^{-p}qs^{p} = q^{\alpha^p}$, $\alpha^p \equiv 1 (\text{mod } p)$, $\alpha = 1$, and $qs = sq$. The group is therefore an abelian group generated by two permutable elements of order p. Hence, all groups of order p^2 are abelian and hence the only distinct types are those represented by (2) and (1,1).

2. Determination of all Non-Abelian Groups of Order p^3:

If a nonabelian group of order p^3 contains an element of order p^2, the subgroup it generates is self-conjugate. Hence by (i), there is a single type of group defined by

$$q^{p^2} = 1, \quad t^p = 1, \quad t^{-1}qt = q^{1+p}.$$

If there is no element of order p^2, then since there must be a self-conjugate element of order p, by (ii) there is again a single type of group defined by

$$q^p = 1, \quad t^p = 1, \quad r^p = 1, \quad r^{-1}tr = q, \quad t^{-1}qt = q.$$

These two types exhaust all the possibilities for non-abelian groups of order p^3.

3. Determination of all Non-Abelian Groups of Order p^4:

For non-abelian groups of order p^4, which contain elements of order p^3 there is a single type given by (i).

For non-abelian groups of order p^4, which contain a self-conjugate cyclic subgroup of order p^2 and no element of order p^3, there are three distinct types given by (ii).

It remains now to determine all distinct types of groups of order p^4, which contain no element of order p^3 and no self-conjugate cyclic subgroup of order p^2. This case is discussed in great detail in Burnside beginning with page 140, line 10 from the bottom of the page. We then obtain the following non-abelian groups

a) $q^{p^2} = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}qr = qs,$

$$sqs = q, \quad r^{-1}sr = s;$$

b) Three possible groups

$q^{p^2} = 1, \quad s^p = 1, \quad s^{-1}qs = q^{1+p}, \quad r^{-1}qr = qs,$

$r^{-1}sr = s, \quad r^p = q^{\alpha p}$ where $\alpha = 0,1$ or any non-residue (mod p);

c) For $p > 3$

$q^p = 1, \quad s^p = 1, \quad r^p = 1, \quad t^p = 1,$

$t^{-1}rt = rs, \quad t^{-1}st = sq, \quad t^{-1}qt = q,$

$r^{-1}sr = s, \quad r^{-1}qr = q, \quad s^{-1}qs = q$

or for $p = 3$

$q^9 = 1, \quad s^3 = 1, \quad r^3 = 1, \quad s^{-1}qs = q,$

$r^{-1}qr = qs, \; r\, r^{-1}sr = q^{-3}s$; and

d) $q^{p^2} = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}qr = q^{1+p},$

$q^{-1}sq = s, \quad r^{-1}sr = s.$

II. (p = 2). Let G be a non-abelian group of order 2^m containing an element q of order 2^{m-1}, $m > 3$.

Let us first suppose that G contains no element of order 2 except the single element of this order contained in $< q >$. Then G is a non-abelian group having only a single subgroup of order 2. It is therefore the last type defined in Theorem 43.

There remains the case in which G contains an element s of order 2 not contained in $< q >$. Since $< q >$ is self-conjugate in G, it follows that $s^{-1}qs = q^{\alpha}$, where α is some odd positive integer between 1 and 2^{m-1} exclusive of these bounds. Then $q = s^{-2}qs^2 = q^{\alpha^2}$. Hence $\alpha^2 \equiv 1 (\text{mod } 2^{m-1})$. Writing $\alpha = 1 + 2^{\beta}k$, where k is odd, we have

$$\alpha^2 - 1 = (1 + 2^{\beta}k)^2 - 1 = 2^{\beta+1}(k + k^2 \cdot 2^{\beta-1}) \equiv 0 (\text{mod } 2^{m-1}).$$

Hence, (1) $\beta = m - 2$, or (2) $\beta = 1$ and $k(1 + k) \equiv 0 (\text{mod } 2^{m-3})$. In case (1) we have $\alpha = 1 + 2^{m-2}$. In case (2), since k is odd, we must have $k = -1 + 2^{m-3}\delta$, and hence $\alpha = -1 + 2^{m-2}\delta$, where δ is an integer. Then the only possible values for δ are $\delta = 1$ and $\delta = 2$. The three cases thus obtained give rise to three distinct types of non-abelian groups

(1) $q^{2^{m-1}} = s^2 = 1, \quad sqs = q^{1+2^{m-2}}$;

(2) $q^{2^{m-1}} = s^2 = 1, \quad (sq)^2 = q^{2^{m-2}}$; and

(3) $q^{2^{m-1}} = s^2 = (sq)^2 = 1$.

The logic of this process may be continued indefinitely until all possible non-abelian groups of order 2^m are determined.

The reader will note from the complexities of previous arguments and the rapidly increasing length of cases in the following table that the determination of abstract groups of a given order is very difficult for orders which contains primes of high order.

Table of Groups of Order p^n, p an Odd Prime

a) Non-Abelian Groups of Order p^3

(i) $q^{p^2} = 1, \quad s^p = 1, \quad s^{-1}qs = q^{1+p}$, and

(ii) $q^p = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}sr = sq,$

$r^{-1}qr = q, \quad s^{-1}qs = q.$

b) Non-Abelian Groups of Order p^4

(i) $q^{p^3} = 1, \quad s^p = 1, \quad s^{-1}qs = q^{1+p^2}$;

(ii) $q^{p^2} = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}sr = sq^p,$

$s^{-1}qs = q, \quad r^{-1}qr = q$;

(iii) $q^{p^2} = 1, \quad s^{p^2} = 1, \quad s^{-1}qs = q^{1+p}$;

(iv) $q^{p^2} = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}qr = q^{1+p},$

$q^{-1}sq = s, \quad r^{-1}sr = s;$

(v) $q^{p^2} = 1, \quad s^p = 1, \quad r^p = 1, \quad r^{-1}qr = qs,$

$s^{-1}qs = q, \quad r^{-1}sr = s;$

(vi) Three possible groups

$q^{p^2} = 1, \quad s^p = 1, \quad s^{-1}qs = q^{1+p}, \quad r^{-1}qr = qs,$

$r^{-1}sr = s, \quad r^p = q^{\alpha p}$ where $\alpha = 0$ or $\alpha = 1$

or α = any non-residue(mod p); and

(vii) For $p > 3$

$q^p = 1, \quad s^p = 1, \quad s^{-1}qs = q, \quad r^p = 1, \quad t^p = 1,$

$t^{-1}rt = rs, \quad t^{-1}st = sq, \quad t^{-1}qt = q, \quad r^{-1}sr = s,$

$r^{-1}qr = q$ or

for $p = 3$

$q^9 = 1, \quad s^3 = 1, \quad r^3 = 1, \quad s^{-1}qs = q,$

$r^{-1}qr = qs, \quad r^{-1}sr = q^{-3}s.$

Table of Groups of Order 2^m

a) Non-Abelian Groups of Order 2^3

(i) $q^4 = 1, \quad s^2 = 1, \quad s^{-1}qs = q^3;$ and

(ii) $q^4 = 1, \quad s^4 = 1, \quad s^{-1}qs = q^{-1}, \quad s^2 = q^2.$

b) Non-Abelian Groups of Order 2^4

(i) $q^8 = 1, \quad s^2 = 1, \quad s^{-1}qs = q^5;$

(ii) $q^4 = 1, \quad s^2 = 1, \quad r^2 = 1, \quad r^{-1}sr = sq^2,$

$s^{-1}rs = q, \quad r^{-1}qr = q;$

(iii) $q^4 = 1, \quad s^4 = 1, \quad s^{-1}qs = q^3;$

(iv) $q^4 = 1, \quad s^2 = 1, \quad r^2 = 1, \quad r^{-1}qr = q^3,$
$q^{-1}sq = s, \quad r^{-1}sr = s;$

(v) $q^4 = 1, \quad s^2 = 1, \quad r^2 = 1, \quad r^{-1}qr = qs,$
$s^{-1}qs = q, \quad r^{-1}sr = s;$

(vi) $q^4 = 1, \quad s^4 = 1, \quad r^2 = 1, \quad s^{-1}qs = q^{-1},$
$s^2 = q^2, \quad r^{-1}sr = s, \quad r^{-1}qr = q;$

(vii) $q^8 = 1, \quad s^2 = 1, \quad s^{-1}qs = q^{-1};$

(viii) $q^8 = 1, \quad s^2 = 1, \quad s^{-1}qs = q^3;$ and

(ix) $q^8 = 1, \quad s^4 = 1, \quad s^{-1}qs = q^{-1}, \quad s^2 = q^4.$

CHAPTER IV

DETERMINATION OF ALL ABSTRACT GROUPS OF ORDER $n = p_1p_2\cdots p_n$ WHERE $p_1, p_2, \ldots, p_n$ ARE DISTINCT PRIMES

Consider first a group G of order pq where $p < q$ and p,q are distinct primes. Then a group of order pq must contains a subgroup of order p and a subgroup of order q. By Theorem 46, if the latter is not self-conjugate, it must be one of p conjugate subgroups, which contain $p(q - 1)$ distinct elements of order q. The remaining p elements must constitute a subgroup of order p, which is therefore either a self-conjugate subgroup of order p or one of order q.

Suppose $< t >$ is a self-conjugate subgroup of order p. Let s be an element of order q. Then

$$s^{-1}ts = t^{\alpha},$$

$$s^{-q}ts^{q} = t^{\alpha^q},$$

$\alpha^q \equiv 1 (\text{mod } p)$, and therefore $\alpha \equiv 1 (\text{mod } p)$. In this case s and t are permutable and G is cyclic.

Suppose there is no self-conjugate subgroup of order p. Then there is necessarily a self-conjugate subgroup $< s >$ of order q, and if t is an element of order p, then

$$t^{-1}st = s^{\beta},$$

$$t^{-p}st^{p} = s^{\beta^p},$$

$\beta^p \equiv 1 (\text{mod } q)$, and therefore $\beta \equiv 1 (\text{mod } q)$. Again the same case as before. But suppose $q \not\equiv 1 (\text{mod } p)$. This then would involve $\beta = 1$ and $< t >$ would be self-conjugate which would contradict our assumption. Hence, if the group is noncyclic, $q \equiv 1 (\text{mod } p)$

and $s^{-1}ts = s^{\beta}$ where β is a root other than unity of the congruence $\beta \equiv 1 (\text{mod } p)$. Between the groups defined by

$$s^p = 1, \quad t^q = 1, \quad s^{-1}ts = t^{\beta} \text{ and}$$

$$s'^p = 1, \quad t'^q = 1, \quad s'^{-1}t's' = t^{\beta^a},$$

an isomorphism is established by taking s' and s^a, t' and t as corresponding elements. Hence, when $q \equiv 1(\text{mod } p)$ there is a single type of non-abelian group of order pq.

Example: Groups G of order 1909 = 23·83, where 23 and 83 are distinct primes such that $83 \not\equiv 1(\text{mod } 23)$ gives us only one possible abstract group of order 1909.

What can we do if our group G has an order comprised of three or more distinct primes? Then we have no easy generalized rule to determine all distinct types of abstract groups, but must instead attack each order separately according to its unique properties as determined through finite group theory. As an example, I wish to determine all abstract groups of order 30 = 2·3·5.

From Theorem 46, a group of order 30 contains either 1 or 6 subgroups of order 5.

In the latter case these 6 subgroups would contain 24 distinct elements of order 5, leaving 6 elements of G to be determined. Among these 6 elements there must be at least one, t, of order 3, and at least one, u, of order 2. If t transforms u into itself, then t and u generate a cyclic group of order 6, which exactly supplies the 6 missing elements. This group contains only a single subgroup, < t >, of order 3, which is therefore the only subgroup of this order contained in G. Again if t does not transform u into itself, then it transforms the group $< u >$ into 3 conjugate groups of order 2. These contain 3 distinct elements of order 2, leaving only 3 powers of t. In this case also, then, the group G contains only one subgroup, $< t >$ of order 3.

Suppose that s, any element of order 5 contained in G, transforms t into itsef. Accordingly, s and t generate a cyclic group H of order 15. This group contains all the elements of order 3, 5, and 15 which occur in G since if $r \varepsilon G$, $r \not\varepsilon H$, then H, rH, r^2H will all be different. But this implies we have 45 distinct elements which is impossible. Thus, for this case we can have only one subgroup of G of order 5, $< s >$.

If the subgroup of order 5 is $< s >$, and if t is any element of order 3 contained in G, then t must transform s into one of its powers. Consequently, s and t generate a group of order 15. Also, $< s,t >$ contains all elements of order 3, 5, and 15 which occur in G. Hence, $H = < s,t >$ is self-conjugate.

We now have to distinguish two principal cases according as H of 15 is cyclic or not.

A. The Subgroup H is Cyclic; st = ts

Since G contains only one subgroup of order 3 as well as only one of order 5, we must have $u^{-1}su = s^{\sigma}$, $u^{-1}tu = t^{\mu}$.

There are four possible subcases, according as 1) $\sigma = 1$, $\mu = 1$; 2) $\sigma = 1$, $\mu \neq 1$; 3) $\sigma \neq 1$, $\mu = 1$; and 4) $\sigma \neq 1$, $\mu \neq 1$.

1) In this case s,t and u being all permutable, the element stu is of order 30 and G is cyclic.

2) The elements u and t generate a non-cyclic group of order

6 where 3 subgroups of order 2 are $< t^{-\alpha}ut^{\alpha} >$, $(\alpha = 0,1,2)$. The elements $t^{-\alpha}ut^{\alpha}$ are all permutable with s and each of them, taken with s, generates a cyclic group of order 10. These three groups have only the powers of s in common. Also, they contain 15 distinct elements, which with the elements of H, make up the entire group. Its generators are st = ts, su = us, and $u^{-1}tu = t^{\beta}$. It contains

1 self-congugate subgroup of order 5, $< s >$;

1 self-conjugate subgroup of order 3, $< t >$;

3 conjugate subgroups of order 2, $< t^{-\alpha}ut^{\alpha} >$ where (α = 0, 1, 2);

1 self-conjugate cyclic subgroup of order 15, $< s,t >$;

1 self-conjugate, non-cyclic subgroup of order 6, $< t,u>$; and

3 conjugate cyclic subgroups of order 10, $< s, t^{-\alpha}ut^{\alpha} >$, $(\alpha = 0,1,2)$.

3) This case differs from 2) only in the exchange of the roles of s and t.

4) The elements of the group can be written as follows (where σ = st);

$1, \sigma, \sigma^2, \ldots, \sigma^{14}$

$u, \sigma^{-1}u\sigma, \ldots, \sigma^{-14}u\sigma^{14}$.

All of these elements are different, since $\sigma^{-i}u^{\mu}\sigma^{i} = \sigma^{-j}u^{\beta}\sigma^{j}$ would require $\sigma^{-(i-j)}u^{\mu}\sigma^{i-j} = u^{\beta}$ where $\beta = \mu$. But then a power of σ being permutable with u, either s or t would be permutable with u, which is excluded. Hence, this group is not an analogue of the non-cyclic type of order 15. Its generating relations are $st = ts$, $u^{-1}su = s^{\gamma}$, $u^{-1}tu = t^{\mu}$. It contains

1 self-conjugate subgroup of order 5, $< s >$;

1 self-conjugate subgroup of order 3, $< t >$;

15 conjugate subgroups of order 2, $< (st)^{\alpha}u(st)^{\alpha} >$, $(\alpha = 0,1,2,\ldots,14)$;

1 self-conjugate cyclic subgroup of order 15, $< s,t >$;

3 conjugate non-cyclic subgroups of order 10, $< s, t^{-\alpha}ut^{\alpha} >$, $(\alpha = 0,1,2)$; and

5 conjugate non-cyclic subgroups of order 6, $< t, s^{-\alpha}us^{\alpha} >$, $(\alpha = 0,1,2,3,4,)$.

B. The Subgroup H is Non-Cyclic; $t^{-1}st = s^{\gamma}$, $\gamma \neq 1$

The group H now contains 1 subgroup of order 5 and 5 subgroups of order 3. Any element u of order 3 must transform at least one of these three subgroups, say $< t >$ into itself. We have then, as under A,

$u^{-1}su = s^{\mu}$, $u^{-1}tu = t^{\rho}$, $u^{-1}(t^{-1}st)u = u^{-1}(s^{\beta})u = s^{\beta\mu}$. But on the other hand, $u^{-1}(t^{-1}st)u = u^{-1}t^{-1}uu^{-1}su\cdot u^{-1}tu = t^{-\rho}s^{\mu}t^{\rho} = s^{\beta^{\rho}\mu}$. Hence we must have

$$\beta^{\rho}\mu \equiv \beta\mu \pmod 5,$$

$$\beta^{\rho-1} \equiv 1 \pmod 5.$$ But since $\rho - 1 < 3$ and $\beta \neq 1$, this is only possible if $\rho = 1$. We have therefore, $u^{-1}su = s^{\mu}$, $u^{-1}tu = t$, and there are two cases to be distinguished, according as 5) $\mu = 1$ or 6) $\mu \neq 1$.

5) Here t and u generate a cyclic group of order 6 which s transforms into 5 distinct conjugate groups of this order.

These groups have the powers of u in common. Also, they contain 20 distinct elements, which with the powers of su make up the entire group. It is readily seen that this group is of essentially the same form as 2) and 3) of A, the elements u, s, t here playing the same role as s, t, u in 2).

6) In this case as in 5) the elements t and u generate a cyclic group of order 6, which s transforms into 5 conjugate groups of this order. But in the present case these groups have no element in common except 1. They contain therefore 25 distinct elements, which with the powers of s, make up the entire group. The generating relation are $t^{-1}st = s^{\beta}$, $u^{-1}su = s^{\mu}$, $u^{-1}tu = t$. It contains

1 self-conjugate subgroup of order 5, $\langle s \rangle$;

5 conjugate subgroups of order 3, $\langle s^{-\alpha}ts^{\alpha} \rangle$, ($\alpha = 0,1, 2, 3,4$);

5 conjugate subgroups of order 3, $\langle s^{-\alpha}us^{\alpha} \rangle$, ($\alpha = 0,1, 2,3, 4$);

1 self-conjugate non-cyclic subgroup of order 15, $\langle s,t \rangle$;

1 self-conjugate non-cyclic subgroup of order 10, $\langle s,u \rangle$; and

5 conjugate cyclic subgroups of order 6, $\langle s^{-\alpha}ts^{\alpha},u \rangle$, ($\alpha = 0,1,2,3,4$).

Consequently, there are only four types of abstract groups of order 30 as expressed by cases 1), 2), 4), and 6).

CHAPTER V

DETERMINATION OF ALL ABSTRACT GROUPS WHOSE ORDER IS $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ WHERE $p_1, p_2, \ldots, p_n$ ARE DISTINCT PRIMES

Finally, we will determine all abstract groups of order 40 which will demonstrate some of the obstacles involved in determining all abstract groups of a given order.

A group of order 40 must contain either 1 or 5 subgroups of order 8 and 1 subgroup of order 5 by Theorem 46. If it has 1 subgroup of order 5 and 1 of order 8, the group must, since each of these subgroups is self-conjugate, be their direct product. But from our table of 2^3, including abelian groups, there are five distinct types of group of order 8. Hence, this gives us five distinct types of group of order 40.

If there are 5 subgroups of order 8, some 2 of them must have a common subgroup of order 4. Also, this common subgroup must be a self-conjugate subgroup of the group of order 40. Moreover, if in this case, a subgroup of order 8 is abelian, each element of the self-conjugate subgroup of order 4 must be a self-conjugate element of the group of order 40.[30]

(i) Suppose a group of order 8 is to be cyclic, and let q be an element that generates it. If $\langle q \rangle$ is self-conjugate and s is an element of order 5, then

$$s^{-1}qs = q^{\alpha},$$

$$s^{-5}qs^{5} = q^{\alpha^5},$$

$\alpha^5 \equiv 1 (\text{mod } 8)$, and therefore $\alpha \equiv 1 (\text{mod } 8)$

which means that q and s are permutable. This is one of the types already obtained. Hence, for a new type, $< q >$ cannot be self-conjugate, and q^2 must be a self-conjugate element. Therefore, s is one of two conjugate elements, while $< s >$ is self-conjugate. Hence, the only possible new type in this case is given by $q^{-1}sq = s^{\gamma}$ where $\gamma \neq 1$.

(ii) Next, let a group of order 8 be an abelian group defined by

$$q^4 = 1, \quad s^2 = 1, \quad qs = sq.$$

If this is self-conjugate, then by considerations similar to those of the preceding case, we infer that the group is the direct product of groups of orders 8 and 5. Hence, there is not in this case a new type.

If the group of order 8 is not self-conjugate, then the self-conjugate group of order 4 may be either $< q >$ or $< q^2, s >$. In either case, if t is an element of order 5, it must be one of five conjugate elements while $< t >$ is self-conjugate. Hence, there are two new types given by

$$t^5 = 1, \quad sts = t^{\gamma}, \quad q^{-1}tq = t, \text{ and}$$

$$t^5 = 1, \quad q^{-1}tq = t^{\gamma}, \quad sts = t \text{ where } \gamma \neq 1.$$

(iii) Let a group of order 8 be an abelian group defined by

$$q^2 = 1, \quad s^2 = 1, \quad t^2 = 1, \quad qs = sq, \quad st = ts,$$

$$qt = tq.$$

If it is self-conjugate, and if the group of order 40 is not the direct product of groups of orders 8 and 5, and element u of order 5 must transform the 7 elements of order 2 among themselves, and must, therefore, be permutable with one of them. Now, the relations $u^{-1}qu = q$, $u^{-1}su = qs$ are not self-consistent, because they give $u^{-2}su^{2} = s$. Hence, since the group of order 8 is generated by q, s and any other element of order 2 except qs, we may assume, without loss of generality, that $u^{-1}qu = q$, $u^{-1}su = t$, $u^{-1}tu = q^{x}s^{y}t^{z}$. These relations give

$$s = u^{-3}su^{3} = u^{-1}q^{x}s^{y}t^{z}u = q^{x(1+z)}s^{yz}t^{y+z^{2}}, \text{ and therefore}$$

$y = z = 1$. Now, if $u^{-1}tu = qsq$, and if $qs = s'$, $qt = t'$, then $u^{-1}s'u = t'$, $u^{-1}t'u = s't'$. Thus, the two alternatives $x = 0$ and $x = 1$ lead to isomorphic groups.

Hence, there is in this case a single type. It is the direct product of $< q >$ and $< u,s,t >$, where $u^{-1}su = t$, $u^{-1}tu = st$.

If the group of order 8 is not self-conjugate, the self-conjugate group of order 4 may be taken to be $< q,s >$, and u being an element of order 5, there is a single new type given by

$$u^{5} = 1, \quad tut = u^{\gamma}, \quad quq = u, \quad sus = u \text{ where } \gamma \neq 1.$$

(iv) Let a group of order 8 be a non-abelian group defined by

$$q^{4} = 1, \quad s^{4} = 1, \quad q^{2} = s^{2}, \quad s^{-1}qs = q^{-1}$$

and let t be an element of order 5. If the group of order 8 is self-conjugate, and the group of order 40 is not a direct product of groups of orders 8 and 5, t must transform the 3 subgroups of order 4, $< q >$, $< s >$ and $< qs >$ among themselves. Hence, in this case there is only one new type given by

$$t^5 = 1, \quad t^{-1}qt = s, \quad t^{-1}st = qs.$$

If the subgroup of order 8 is not self-conjugate, the self-conjugate subgroup of order 4 is cyclic, and each of its elements must be permutable with t. Hence, again we get a single new type given by

$$t^5 = 1, \quad q^{-1}tq = t, \quad s^{-1}ts = t^{\gamma}, \ \gamma \neq 1.$$

(v) Lastly, let a subgroup of order 8 be a non-abelian group defined by $q^4 = 1$, $s^2 = 1$, $sqs = q^{-1}$. This contains one cyclic and two non-cyclic subgroups of order 4. If it is self-conjugate, the group of order 40 must therefore be the direct product of groups of orders 8 and 5, and there is no new type.

If the subgroup of order 8 is not self-conjugate, and the self-conjugate subgroup of order 4 is the cyclic group $< q >$, then q must be permutable with an element t of order 5, and there is a single new type given by

$$t^5 = 1, \quad q^{-1}tq = t, \quad sts = t^{\gamma}, \ \gamma \neq 1.$$

If the self-conjugate subgroup of order 4 is not cyclic, it may be taken to be $< 1, q^2, s, q^2s >$. If t is permutable with each element of this subgroup, there is a single type given by

$$t^5 = 1, \quad q^{-1}tq = t^{\gamma}, \quad sts = t, \ \gamma \neq 1.$$

If t is not permutable with every element of the self-congugate subgroup, it must transform q^2, s, q^2s among themselves and we may take $t^{-1}q^2t = s$, $t^{-1}st = q^2s$.

Now, $< t, q^2, s >$ is self-conjugate, and therefore q must transform t into another element of order 5 contained in this subgroup. Hence, $q^{-1}tq = t^x q^{2y} s^z$. The only values of x, y,z which are consistent with the previous relation $q^2tq^2 = tq^2s$ are $x = 2$, $y = z = 1$ or $x = 2$, $y = 1$, $z = 0$. Either set of values lead to the same type defined by

$$q^4 = 1, \quad t^5 = 1, \quad (qt)^2 = 1.$$

Consequently, there are 14 types of abstract groups of order 40.

In this paper I have tried to give a line of attack on all abstract groups of a given order through the properties of finite groups. Undoubtedly, this method is fraught with numerous difficulties and complexities as witness the arguments already presented. However, new techniques, perhaps allied with computer technology, may scale such problems to reasonable proportions. I hope the reader has received some insight into this problem. As a valuable source of reference to anyone who may endeavour in this field, I have included the following table of the number of abstract groups of a given order for orders through 160.

Table of the Number of Abstract Groups of a Given Order, (through 160)

Order	Factors	Number of Groups
4	2^2	2
6	$2 \cdot 3$	2
8	2^3	5
9	3^2	2
10	$2 \cdot 5$	2
12	$2^2 \cdot 3$	5

(continued)

Order	Factors	Number of Groups
14	$2\cdot7$	2
16	2^4	114
18	$2\cdot3^2$	5
20	$2^2\cdot5$	5
21	$3\cdot7$	2
22	$2\cdot11$	2
24	$2^3\cdot3$	15
25	5^2	2
26	$2\cdot13$	2
27	3^3	5
28	$2^2\cdot7$	4
30	$2\cdot3\cdot5$	4
32	2^5	51
34	$2\cdot17$	2
36	$2^2\cdot3^2$	14
38	$2\cdot19$	2
39	$3\cdot13$	2
40	$2^3\cdot5$	14
42	$2\cdot3\cdot7$	6
44	$2^2\cdot11$	4
45	$3^2\cdot5$	2
46	$2\cdot23$	2
48	$2^4\cdot3$	52
49	7^2	2

(continued)

Order	Factors	Number of Groups
50	$2 \cdot 5^2$	5
52	$2^2 \cdot 13$	5
54	$2 \cdot 3^3$	15
55	$5 \cdot 11$	2
56	$2^3 \cdot 7$	13
57	$3 \cdot 19$	2
58	$2 \cdot 29$	2
60	$2^2 \cdot 3 \cdot 5$	13
62	$2 \cdot 31$	2
63	$3^2 \cdot 7$	4
64	2^6	294
66	$2 \cdot 3 \cdot 11$	4
68	$2^2 \cdot 17$	5
70	$2 \cdot 5 \cdot 7$	4
72	$2^3 \cdot 3^2$	50
74	$2 \cdot 37$	2
75	$3 \cdot 5^2$	3
76	$2^2 \cdot 19$	4
78	$2 \cdot 3 \cdot 13$	6
80	$2^4 \cdot 5$	52
81	3^4	15
82	$2 \cdot 41$	2
84	$2^2 \cdot 3 \cdot 7$	15
86	$2 \cdot 43$	2

(continued)

Order	Factors	Number of Groups
88	$2^3 \cdot 11$	12
90	$2 \cdot 3^2 \cdot 5$	10
92	$2^2 \cdot 23$	4
93	$3 \cdot 31$	2
94	$2 \cdot 47$	2
96	$2^5 \cdot 3$	230
98	$2 \cdot 7^2$	5
99	$3^2 \cdot 11$	2
100	$2^2 \cdot 5^2$	16
102	$2 \cdot 3 \cdot 17$	4
104	$2^3 \cdot 13$	14
105	$3 \cdot 5 \cdot 7$	2
106	$2 \cdot 53$	2
108	$2^2 \cdot 3^3$	45
110	$2 \cdot 5 \cdot 11$	6
111	$3 \cdot 37$	2
112	$2^4 \cdot 7$	43
114	$2 \cdot 3 \cdot 19$	6
116	$2^2 \cdot 29$	5
117	$3^2 \cdot 13$	4
118	$2 \cdot 59$	2
120	$2^3 \cdot 3 \cdot 5$	47
121	11^2	2
122	$2 \cdot 61$	2

(continued)

Order	Factors	Number of Groups
124	$2^2 \cdot 31$	4
125	5^3	5
126	$2 \cdot 3^2 \cdot 7$	16
128	2^7	not determined
129	$3 \cdot 43$	2
130	$2 \cdot 5 \cdot 13$	4
132	$2^2 \cdot 3 \cdot 11$	10
134	$2 \cdot 67$	2
135	$3^3 \cdot 5$	5
136	$2^3 \cdot 17$	15
138	$2 \cdot 3 \cdot 23$	4
140	$2^2 \cdot 5 \cdot 7$	11
142	$2 \cdot 71$	2
144	$2^4 \cdot 3^2$	197
146	$2 \cdot 73$	2
147	$3 \cdot 7^2$	6
148	$2^2 \cdot 37$	5
150	$2 \cdot 3 \cdot 5^2$	13
152	$2^3 \cdot 19$	12
153	$3^2 \cdot 17$	2
154	$2 \cdot 7 \cdot 11$	4
155	$5 \cdot 31$	2
156	$2^2 \cdot 3 \cdot 13$	18
158	$2 \cdot 79$	2

(continued)

Order	Factors	Number of Groups
160	$2^5 \cdot 5$	238

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Honor's Day, twice

Thesis Title (Include name of adviser)

Determination of All Abstract Groups of a Giver Order

Adviser: Andrew O. Lindstrum, Jr.

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