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Dimension theory

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CHAPTER 0

INTRODUCTION

The notion of dimension takes on a different definition in the various fields of mathematics. Each of these several definitions, some of which will be discussed below, suffice to provide the areas of mathematics involved with a usable tool. These definitions of dimension, however, introduce many inconsistencies into notions derived from them, such as the notion that there are more points in the plane than in the line. With the development of point set topology came a definition of dimension which cleared up many of these inconsistencies and also holds for all of the areas of mathematics. The general concept of this definition is presented here with the particular development being given for spaces which are separable metrics.

0.1 SOME NOTIONS ON DIMENSION

The common notion of dimension is that the dimension is equal to the least number of components necessary to define a point in a space. The number of components necessary to define a point is found by taking the least number of vectors required to span the space (that is, every vector in the space can be expressed as a linear combination of these vectors). This definition of dimension is particularly useful when one is considering taking the derivative or integral of a well-behaved or continuous function. However, with this definition one gets the feeling that there are many more points in the plane than there are in the line. However, as will be shown later, Cantor's 1-1 correspondence between the points of a line and the points of a plane and also Peano's continuous map of an interval onto the square show this idea to be false. In a similar fashion, to the algebraist dimension means the maximum number of linearly independent vectors in a vector space. That is to say, a set of equations has dimension n if there exists n linearly independent equations.

In geometry the definition has an intuitive concept. The word 1-dimensional refers to those subsets which have length only, 2-dimensional to those which have length and depth only, and 3-dimensional to those which have length, depth and height. This notion of dimension has very serious shortcomings, in particular that one cannot go beyond 3 dimensions. The most serious shortcoming of these definitions is that they do not allow for the correspondence between Euclidean n-space and Euclidean m-space. With the advent of topology it was seen that none of these definitions of dimension would conform to the rigors of topology; hence a new and more general definition of dimension was necessary.

0.2 THE TOPOLOGICAL CONCEPT OF DIMENSION

About the turn of the century Poincaré gave his intuitive definition of dimension in an essay in the Journal, Revue de Métaphysique et de Morale, wherein he discusses dimension in the paper, "Why Space Has 3 Dimension". His definition of dimension is based upon the notion of "cuts". "... consider first of all a closed curve, that is, a continuum of 1 dimension. If on this curve we take any two points through which we shall not permit ourselves to pass, the curve will be cut into two parts and it will become impossible to go from one to the other still remaining on the curve but not passing through the excluded points. Let us consider, on the other hand, a closed surface which forms a continuum of two dimensions. It will be possible to take on this surface one, two, or any number of excluded points whatever. The surface will not be divided into two parts because of this; it will be possible to go from one point to another on this surface without encountering any obstacle because it will always be possible to go around the excluded points. But if we trace on the surface one or many closed curves and if we consider them as cuts which may not be crossed, the surface can then be cut into several parts...." Poincaré continued on using this idea of cutting to the 2 dimension closed surface and then to the 3 dimension surface. One will note that by this notion of dimension a space of dimension n is cut by a space of dimension n-1, but is not cut by a space of dimension n-2. Shortly after Poincaré came out with this paper Brouwer, Menger, and Urysohn, independent of one another, developed a topologically invariant definition of dimension based upon Poincaré's intuitive notion. As stated by Menger, the null set has dimension -1 and the dimension of a space is the least integer n $\ni$ every point has arbitrarily small neighborhoods with boundaries of dimension less than n.

0.3 INTRODUCTORY DEFINITIONS

In this paper the spaces which are in consideration will be separable metric spaces.

DEFINITION 0.1: TOPOLOGICAL SPACE Let X be a non-empty set and T a collection of subsets of X $\ni$:

1. $X \in T$ and $\emptyset \in T$.
2. If $A_1, A_2, \ldots, A_n \in T$ then $\bigcap_{i=1}^{n} A_i \in T$.
3. Let I be the indexing set. If for each $\alpha \in I$, $A_\alpha \in T$, then $\bigcup_{\alpha \in I} A_\alpha \in T$.

The couple (X,T) is called a topological space. The set X is the underlying set and T is the topology on the set X. The elements of T are the open sets.

<u>DEFINITION 0.2: METRIC SPACE</u> A space is metric if given any two points x, y $\exists$ a real number $d(x,y) \geq 0 \ni$

1. $d(x,y) = 0 \Longleftrightarrow x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) \leq d(x,z) + d(y,z)$ (This latter is the triangle axiom.)

<u>DEFINITION 0.3:</u> Let B be a collection of open sets of a metric space X, B is a <u>basis for the open</u> sets of X if:

1. Each set in B is an open set.
2. A subset V of X is open $\Longleftrightarrow$ V is the union of sets belonging to the collection B.

The subset of X consisting of those points $x \in X$, $\ni$ for $a \in X$ and $\delta > 0$ $d(a,x) < \delta$ is called a spherical neighborhood of a. These spherical neighborhoods about each point of the space form a basis for the space.

<u>DEFINITION 0.4: SEPARABLE SPACE</u> A topological space X is separable $\Longleftrightarrow \exists$ adenumerable subset $A \subset X$ which is dense in X. In a metric space this is equivalent to the existence of denumerable basis.

<u>DEFINITION 0.5: NORMAL SPACE</u> A space is normal $\Longleftrightarrow \forall$ disjoint pair of closed sets, A and B, $\exists$ disjoint open sets U and V such that $A \subset U$ and $B \subset V$. This definition can be stated as: a space is normal $\Longleftrightarrow$ disjoint closed sets have disjoint neighborhoods.

<u>DEFINITION 0.6: REGULAR SPACE</u> A topological space is regular $\Longleftrightarrow \forall$ point x and each neighborhood $U(x) \ni$ a closed neighborhood V of $x \in V \subset U$. Alternatively, the family of closed neighborhoods of each point is a base for the neighborhood system of the point, or $\forall$ point x and each closed set A if $x \notin A$ there are disjoint open sets U and $B \ni x \in U$ and $A \subset V$.

DEFINITION 0.7: Two subsets A and B are **separated** in a topological space X $\Longleftrightarrow$ $\bar{A} \cap B$ and $A \cap \bar{B}$ are both null.

DEFINITION 0.8: The boundary of a set X is equal to the intersection of the closure of X and the closure of the complement of X. The boundary of X is denoted by Bdry(X). If X is an open set then Bdry(X) = $\mathcal{C}_{\bar{x}}$ X.

DEFINITION 0.9: SEPARATION OF SUBSETS Let A_1, A_2, and B be mutually disjoint subsets of a space X, then A_1, A_2 are separated in X by B if $\exists$ A_1', A_2' $\ni$

$$A_1 \subset A_1', \; A_2 \subset A_2',$$

$$A_1 \cap A_2' = \emptyset, \; A_1' \cap A_2 = \emptyset, \text{ and } \mathcal{C}B = A_1' \cup A_2'$$

where A_1', A_2' are both open and/or closed in $\mathcal{C}$B. If A_1, A_2 are separated by the null set then A_1, A_2 are said to be separated in X_1. A_1, A_2 are separated if $\exists$ a set $A_1' \ni A_1 \subset A_1'$, $A_1' \cap A_2 = \emptyset$ and A_1' is both open and closed (i.e. has null boundary). For A_2' is then $\in A_1'$.

<u>DEFINITION 0.10:</u> A set X is connected if the only two subsets of X that are simultaneously open and closed are X itself and the null set. A set that is not connected is termed disconnected. A subset D of a space X is said to disconnect X if $\in$ D is disconnected.

<u>DEFINITION 0.11:</u> Let X be a topological space, then $\forall$ point $a \in X$, a is contained in a maximal connected subset of X called the <u>component of a</u>.

0.4 DIMENSION OF E_n

An absolutely necessary consequence of this definition of dimension is that dimension of $E_n = n$. Many of the proofs of this statement are developed through the use of combinatorial topological properties rather than those of point set topology. The proof of this statement as developed in this paper is an inductive one using point set topological properties and is one of the major consequences of the paper.

CHAPTER 1

DIMENSION OF GENERAL SPACES AND EUCLIDEAN SPACES

The dimension described herein is dependent upon the dimension of the boundaries of each point. The dimension of a set X is to be denoted as dim X = n.

1.1 DIMENSION n

DEFINITION 1.1: DIMENSION n

1. The space X has dimension -1 if and only if $X = \emptyset$.

2. A space X has dimension 0 at a point p if $\forall$ neighborhood N of p $\exists$ a neighborhood M of p $\ni M \subset N$, and $Bdry(M) = \emptyset$.

3. A space X has dimension $\leq n$ $(n \geq 0)$ at a point p if the boundaries of all of the neighborhoods of the point p have dimension $\leq n-1$.

4. The dimension of the space X (dim X) is $\leq$ n if X has dimension $\leq$ n at each of its points.

5. The space X has dimension n at a point p if it is true that X has dimension $\leq$ n at p, and it is false that X has dimension $\leq$ n-1 at p.

6. For the space X, dim X = n if X has dimension $\leq$ n and at least one of its points has dimension n.

7. X has dimension ∞ if dim X $\leq$ n is false for each n.

Since this definition of dimension is completely dependent upon the topological property of the existence of boundaries, then the dimension n of a set X or at a point $p \varepsilon X$ is topologically invariant. This property, however, is not invariant under continuous transformations, since it is possible to map E_1 onto E_2 (see Peano's space filling curve in the appendix) and it is also possible to map the unit interval into the unit square (see Cantor's mapping in the appendix). Since for a set X of dimension n $\exists$ a boundary of dimension n-1 for each point p in the set X, then for each point $p \epsilon X$ there exists an open set containing the point. Since $\forall$ point $p \epsilon X$ $\exists$ an open set containing the point, then the collection of all of these open sets is a basis for the topology of the set.

1.2 FOUR SIMPLE EXAMPLES

EXAMPLE I: A finite set, S, of points is 0-dimensional.

PROOF: For any neighborhood of a point $x \epsilon S$ $\exists$ $\epsilon > 0 \ni N(x, \epsilon)$ does not contain any points of the set. Hence the boundary is the null set (i.e. the intersection with a neighborhood of dim n would be null). Q.E.D.

The sets Q of rational numbers and H of irrational real numbers are sets of dimension 0.

<u>EXAMPLE II:</u> The set Q of rational real numbers is of dimension 0.

PROOF: Given any point $p \in Q$ and any neighborhood N of $p \exists$ irrational numbers $\rho, \sigma \in N \ni \rho < p < \sigma$, and the set M of rational numbers between ρ, σ is contained in N. In Q, M is open and has a null boundary. Therefore, Bdry(M) has dimension -1. Hence the dimension at each point is 0. Q.E.D.

<u>EXAMPLE III:</u> In a similar manner the set H of irrational numbers can be shown to be 0-dimensional.

<u>EXAMPLE IV:</u> The Euclidean line (E_1) and an interval (I_1) in the Euclidean line have dimension 1.

PROOF: Let $N = (a_1, a_2)$ be a neighborhood of any point in E_1. The dimension of the boundary of N is 0 since the boundary consists of the two points a_1 and a_2.

Since $\{a_1, a_2\}$ is a set consisting of a finite number of points, then dim $\{a_1, a_2\} = 0$. Hence the dimension at any point is 1. The same argument holds for an interval I_1. Q.E.D.

1.3 DIMENSION OF THE UNION OF SETS AND OF SUBSPACES

Example IV above indicates that the dimension of the union of sets is not necessarily the sum of the dimension of the sets. For in Example IV one can see that $E_1 = Q \cup H$, that is E_1 is the union of two 0-dimensional sets, yet the dim $E_1 = 1$. It is obvious that in determining the dimension of the union of sets one must consider the dimension of subspaces (such as Q,H used above).

<u>THEOREM 1.1:</u> A subspace of a space of dimension $\leq n$ has dimension $\leq n$.

PROOF: The statement is obvious for $n = -1$. Assume it now for $n-1$ and use induction. Let dim $X \leq n$, Y a subspace of X, and $p \in Y$. Let N be a neighborhood of p in Y, then $\exists$ a neighborhood M in X of p $\ni$ N = M$\cap$Y. Since dim X$\leq$ n, $\exists$ a set S open in X and satisfying pε S$\subset$M and dim Bdry(S)$\leq$ n-1. Let T = S$\cap$Y. Then T is open in Y, and pε T$\subset$N. Let B be the Bdry(S) in X, and C = Bdry(T) in Y. Hence C$\subset$B$\cap\bar{Y}$. Since by the hypothesis of the induction dim X$\leq$ n, then dim C$\leq$ n-1, because it is a boundary of a neighborhood of a point in X. Thus completing the induction. Q.E.D.

The necessary and sufficient conditions for the dimension of a subspace are:

<u>THEOREM 1.2:</u> A subspace Y of a space X has dimension $\leq$ n $\Longleftrightarrow$ $\forall$ point pε Y $\exists$ arbitrarily small neighborhoods in X, whose boundaries have intersections with Y of dimension $\leq$ n-1.

PROOF: $\Longleftarrow$ Let pε Y and V a neighborhood of p in Y. Then $\exists$ a neighborhood U of p in X $\ni$ V = U$\cap$Y. Hence $\exists$ a set S which is open in X $\ni$ pε S$\subset$U,

$\dim(Y \cap \text{Bdry}(S)) \leq n-1$. Let $T = S \cap Y$. Then T is open in Y, $p \in T \subset V$. Denote by B and C the boundaries of S in X and of T in Y giving $C \subset B \cap Y$. Therefore, by theorem 1.1 $\dim C \leq n-1$, and $\dim Y \leq n$.

$\Longrightarrow$ Suppose $\dim Y \leq n$. Let p be any point of Y and U a neighborhood of p in X. Then $V = U \cap Y$ is a neighborhood in Y of p. Hence $\exists$ a neighborhood T in Y of $p \ni p \in T \subset V$ and $\dim C \leq n-1$, where $C = \text{Bdry}(T)$ in Y. Neither of the disjoint open sets T and $\mathcal{C}_Y \bar{T}$ contains an accumulation point of the other, so that since X is completely normal (i.e. every subspace is normal), $\exists$ an open set $W \ni T \subset W$ and $\bar{W} \cap \mathcal{C}_Y \bar{T} = \emptyset$. Since W is an open set then $\text{Bdry}(W) = \mathcal{C}_{\bar{W}} W$ contains no points of $\mathcal{C}_Y \bar{T}$ and no points of T. This last statement is true since X is a normal space then $\exists$ open sets V_1 and $V_2 \ni \bar{W} \subset V_1$ and $\mathcal{C}_Y T \subset V_2$ and $V_1 \cap V_2 = \emptyset$. It follows that the intersection of Y with the boundary of W is contained in C and hence has dimension $\leq$ n-1 so that the necessary condition is satisfied. Q.E.D.

This latter theorem will now be used to prove the following theorem on the dimension of the union of sets.

<u>THEOREM 1.3:</u> Let A,B be two subspaces of a space X. Then $\dim(A \cup B) \leq 1 + \dim A + \dim B$.

PROOF: The proof is obvious for $\dim A = \dim B = -1$. Let $\dim A = n$ and use induction on dim B. Again it is obvious for $\dim B = -1$. Now assume $\dim B = m$ and that the theorem holds for:

$$\dim A \leq n \qquad \dim B \leq m-1.$$

Let $p \in (A \cup B)$ with $p \in A$, and $U \subset X$ a neighborhood of p. By theorem 1.2 $\exists$ an open set $V \ni p \in V \subset U$ and $\dim(W \cap A) \leq n-1$ where W is the boundary of V. Since $W \cap B$ is a subset of B then $\dim(W \cap B) \leq m$. By the hypothesis

$\dim(W \cap (A \cup B)) \leq m + n$. This proves by theorem 1.2 that $\dim(A \cup B) \leq m + n + 1$, which completes the induction.

1.4 DIMENSION OF A TOPOLOGICAL PRODUCT

THEOREM 1.4: Given $A \times B$ with at least one of the sets being non-empty, then

$$\dim(A \times B) \leq \dim A + \dim B.$$

PROOF: The proof is obvious for $\dim A = -1$ or for $\dim B = -1$. Assume $\dim A = n$ and $\dim B = m$ and use induction $\dim B = m$ where the theorem holds for $\dim A = n$ and $\dim B = m - 1$. Now for $p = (a,b)$ with $a \in A$ and $b \in B$, and U a neighborhood of a and V a neighborhood of b, $\exists$ neighborhoods $U \times V$ of the point p in $A \times B$. The boundary of $U \times V$ is:

$$\mathrm{Bdry}(U \times V) = (\bar{U} \times \mathrm{Bdry}(V)) \cup (\bar{V} \times \mathrm{Bdry}(U))$$

Now then $\dim \mathrm{Bdry}(U) \leq n-1$ and $\dim \mathrm{Bdry}(V) \leq m-1$. Since for $\mathrm{Bdry}(U \times V)$ each summond is closed then by the hypothesis the dimension $\leq m + n - 1$. By the sum theorem

$\dim(\text{Bdry}(U \times V)) \leq m + n - 1$, which proves that $\dim(A \times B) = m + n$.

1.5 DIMENSION OF THE UNION OF SETS

Theorem 1.3 for the dimension of the union of sets requires that the sets be disjoint. The problems that this restriction causes are cleared up in the special case of closed sets by the sum theorem for dimension n. In lemma 1.1 and theorem 1.5 below the notation Σ_n is used to denote the sum theorem for dimension n.

LEMMA 1.1: Any space of dimension $\leq n$ is the union of a subspace of dimension $\leq n-1$ and a subspace of dimension ≤ 0. Let Δ_n denote the condition of this lemma.

PROOF: Let X be a space of dimension $\leq n$. Then $\exists$ a basis for the open sets of X made up of sets with boundaries having dimension $\leq n-1$. Since X is separable $\exists$ a denumerable basis $\{U_i\}$, $i = 1, 2, \ldots$, the sets of which have boundaries $\{B_i\}$ with dimension $\leq$ n-1. From $\leq_{n-1}$ it follows that $B = \bigcup_{i=1}^{\infty} B_i$ has dimension $\leq$ n-1. Now then $\dim \mathcal{C}_X B \leq 0$ since the boundaries of the sets U_i are disjoint from $\mathcal{C}_X B$ and hence $\mathcal{C}_X B$ is both open and closed with a null boundary. $\triangle_n$ then follows from the equation $X = B \cup \mathcal{C}_X B$. Q.E.D.

<u>THEOREM 1.5:</u> A space which is the countable union of closed subsets of dimension $\leq$ n has dimension $\leq$ n.

PROOF: By induction. The case for $n = -1$, $\leq_{-1}$, is obvious. Now combine $\leq_{n-1}$ and $\triangle_n$ to prove $\leq_n$. Suppose $X = C_1 \cup \ldots \cup C_i \cup \ldots$, $\dim C_i \leq n$, each C_i closed. Now it is necessary to show that $\dim X \leq n$. Let

$$K_1 = C_1,$$

$$K_i = \mathcal{C}_{C_i} \bigcup_{j=1}^{i-1} C_j = C_i \cap (\mathcal{C}_X \bigcup_{j=1}^{i-1} C_j), \; i = 2, 3, \ldots$$

Then

(A) $X = \bigcup_{i=1}^{\infty} K_i$

(B) $K_i \cap K_j = \emptyset$ if $i \neq j$

(C) K_i is a countable sum of closed subsets in X denoted as F_σ

(D) $\dim K_i \leq n$

Statements (A) and (B) are obvious. Statement (C) is true since $\mathcal{C} \bigcup_{j=1}^{i-1} C_j$ is open and, therefore, as an open set in a metric space is an F_σ K_i as the intersection of this F_σ with the closed set C_i is true also as F_σ. Statement (D) is true since K_i is a subset of C_i. By (D) it is possible to apply Δ_n to each K_i, giving:

$$K_i = M_i \cup N_i$$

$$\dim M_i \leq n-1, \ \dim N_i \leq 0.$$

Denote $\cup M_i$ by M and $\cup N_i$ by N. From (A) $X = M \cup N$.

Each M_i is an F_σ in M. For

$$M_i = M_i \cap K_i = (M_1 \cup \dots \cup M_i \cup \dots) K_i = M \cap K_i,$$

since $M_i \subset K_i$ and $K_i \cap K_j = \emptyset$ for $i \neq j$ by (B). Hence M_i as the intersection of M with K_i, which is an F_σ by (C) is itself an F_σ in M. Therefore, $\mathcal{E}_{n-1}$ can be applied to conclude that dim $M \leq n-1$. Similarly each N_i is an F_σ and, therefore, dim $N \leq 0$ by $\mathcal{E}_0$. Thus $X = M \cup N$ with dim $M \leq n-1$ and dim $N \leq 0$. Then by theorem 1.3 dim $X \leq n$.

Q.E.D.

<u>COROLLARY 1.1:</u> The union of two subspaces each of which has dimension $\leq n$ and one of which is closed has dimension $\leq n$.

PROOF: Let A,B be n-dimensional and B closed. Let $D = A \cup B$. Now $\mathcal{C}_D B$ is an open set relative to $(A \cup B)$ which is the denumerable union of closed subsets. With this fact and the equation $A \cup B = B \cup \mathcal{C}_D B$ the proof follows immediately. Q.E.D.

An example of two spaces for corollary 1 would be the space Z of integers and the space Q of rational numbers. The set of integers are of dimension 0 as are the rational numbers; however, the integers are a subspace of the rational numbers. The integers are a closed set, hence $\dim(Z \cup Q) = \dim Z + \dim Q = 0$. The same conclusion also is obvious for the sets $Z, (\mathcal{C}_Q X)$. Note that in the former example $(Z \cup Q) = Q$ and hence the theorem could be stated to give $\dim(Z \cup Q) = \dim Q$ if $Z \subset Q$.

<u>COROLLARY 1.2:</u> The dimension of a non-empty space cannot be increased by the adjunction of a single point.

PROOF: This is an obvious consequence of corollary 1.1.

<u>COROLLARY 1.3:</u> If a space X' of dimension $\leq$ n is a subspace of a space X then every point of X has neighborhoods (in X) whose boundaries have intersections with X' of dimension $\leq$ n-1.

PROOF: For each point p of X, $(X' \cup p)$ has dimension $\leq$ n by corollary 1.2; the statement thus follows from theorem 1.1.

<u>COROLLARY 1.4:</u> If a space has dimension $\leq$ n it is the union of a subspace of dimension $\leq$ n-1 and a subspace of dimension $\leq$ 0.

PROOF: Let Y be a space of dimension $\leq$ n. Then $\exists$ a basis for the open sets of Y made up of sets whose boundaries have dimension $\leq$ n-1. Since Y is separable $\exists$ a countable basis (U_i), i = 1, 2,..., made up of sets whose boundaries (B_i) have dimension $\leq$ n-1. From the previous theorem for dimension n-1 $B = \bigcup_{i=1}^{\infty} B_i$ has dimension $\leq$ (n-1).

Thus $\dim(\mathcal{C}_Y B) \leq 0$ since obviously the boundaries of the sets U_i do not intersect $(\mathcal{C}_Y B)$ and hence the existence of a basis for the open sets of Y is satisfied (with n = 0 and Y replaced by $(\mathcal{C}_Y B)$). Thus proving the corollary.

If one were to continue in the above manner to remove a 0-dimensional space from the resulting (n-m)-dimensional space (where m = 0, 1,..., n-1), one would then reduce the (n-m)-dimensional space to one of dimension 0. The validity of this statement can be seen by continued application of corollary 1.4 and theorem 1.3.

CHAPTER 2

EUCLIDEAN SPACES

2.1 THE DIMENSION OF E

THEOREM 2.1: $\dim E_n = n$ The proof of this theorem is an inductive one. Recall that the dimension of a space X is dependent upon the dimension of the boundaries of the spherical neighborhoods about each of the points in the space. The dim $X = n$ if $\forall\ p \in X$ the dimension at p is $\leq n$ and it is false that dim $X \leq n-1$ at $p\ \forall\ p \in X$. In the proof to follow, cubical neighborhoods are used instead of spherical ones. A cubical neighborhood of a point p is the set of points in the cube about p with p as the center. Hence it is necessary to show that the topology defined by using cubical neighborhoods is equivalent to that using spherical neighborhoods. The proof of this last statement is offered below.

LEMMA 2.1: The topology of spherical neighborhoods is equivalent to that of cubical neighborhoods.

PROOF: The topology for the space X defined by spherical neighborhoods is:

Given $\epsilon > 0$, $\forall x \in X$, $\exists$ a neighborhood $N_1(x, \epsilon)$ where $N_1(x, \epsilon)$ is the set of points whose distance from x is less than ϵ (i.e. ϵ is the radius of a sphere about x). If $N_1(x, \epsilon)$ is given then the cubical neighborhood $N_2(x, \frac{\epsilon}{\sqrt{2}})$, where $\frac{\epsilon}{\sqrt{2}}$ is the length of an edge of the cube, is a subset of $N_1(x, \epsilon)$. Likewise given a neighborhood $N_2(x, \delta)$ in the cubical topology, where δ is the length of an edge of a cube, then $N_1(x, \frac{\delta}{2})$ is a subset of $N_2(x, \delta)$. Given a point $x \in X$, then $\forall$ spherical neighborhood $N_1(x)$ $\exists$ a cubical neighborhood contained in $N_1(x)$, and likewise each cubical neighborhood $N_2(x)$ contains a spherical neighborhood. Hence since the spherical topology is imbedded in the cubical topology and vice-versa, then the two topologies are equivalent. Q.E.D.

In the proof I_n is used to denote a subspace of E_n. For simplification of terminology the term "piece" is used to define one of the components of a boundary. The proof begins with the interval $I_1 \subset E_1$, which was proved earlier and is now proved in the sense of the inductive proof. The cases for I_2 and I_3 are then developed in detail so as to clarify the method of the induction.

Now to prove $E_n = n$ using cubical neighborhoods.

PROOF: Let $I_1 = (a_1, a_2)$ be an interval in E_1. The dimension of the boundary of I_1 is zero since the boundary consists of the two points a_1 and a_2. The dim $\{a_1, a_2\} = 0$ since $\{a_1, a_2\}$ contains a finite number of points. Hence dim $I_1 = 1$ since $\{a_1, a_2\}$ is the boundary of I_1.

Next consider the dimension of a square $I_2 \subset E_2$. It is necessary to show that the dimension of the boundary of the square is 1 and, therefore, the dim $I_2 = 2$.

A general closed square is:

$\{(x_1 x_2) \mid a_1 \leq x_1 \leq a_2;\ b_1 \leq x_2 \leq b_2 \text{ with } a_1 \neq a_2,\ b_1 \neq b_2\}$.

$a_2 - a_1 = b_2 - b_1$.

Choose some point $d = (a_2, c)$ on the boundary of the square with c defined by $b_1 < c < b_2$. Define a neighborhood about the point d in the following manner:

Let $\epsilon > 0$ and $\ni \epsilon < a_2 - a_1$, $\epsilon < b_2 - c$, and $\epsilon < c - b_1$.

A neighborhood of (a_2, c) is

$a_2 - \epsilon < x_1 < a_2 + \epsilon;\ c - \epsilon < x_2 < c + \epsilon$.

The boundary of this neighborhood is given by:

$x_1 = a_2 - \epsilon$ $\quad$ $x_1 = a_2 + \epsilon$

$c - \epsilon \leq x_2 \leq c + \epsilon$ $\quad$ $c - \epsilon \leq x_2 \leq c + \epsilon$

and $a_2 - \epsilon \leq x_1 \leq a_2 + \epsilon$ $\quad$ $a_2 - \epsilon x_1 \leq a_2 + \epsilon$

$x_2 = c - \epsilon$ $\quad$ $x_2 = c + \epsilon$.

Note that each piece of the boundary is defined by holding one of the components fixed and letting the other vary within the prescribed limits. The point (a_2, c) lies in the interval:

$$x_1 = a_2, b_1 \leq x_2 \leq b_2.$$

The neighborhood of the point in the piece of the boundary is the intersection of this piece with the neighborhood of the point. The relationship

$$x_1 = a_2, c - \epsilon \leq x_2 \leq c + \epsilon$$

defines this neighborhood which is an interval and hence is of dimension 1 as was shown above. A similar relationship holds for all four pieces of the boundary.

Since the dimension of the boundary is equal to 1 and it is false that the dimension of the boundary is less than 1 then it is false that the dimension of I_2 is less than two. Hence the dimension of any point on the boundary is 1. Hence dim $I_2 = 2$.

Let I_3 be defined by:

$x = (x_1, x_2, x_3) \in I_3$ where

$I_3 = \{(x_1, x_2, x_3) \mid a_{11} \leq x_1 \leq a_{12}, a_{21} \leq x_2 \leq a_{22},$

$a_{31} \leq x_3 \leq a_{32}, a_{11} \neq a_{12}, a_{21} \neq a_{22}, a_{31} \neq a_{32}\}$,

$a_{12} - a_{11} = a_{22} - a_{21} = a_{32} - a_{31}$.

Choose some point $d = (a_2, y_1, y_2)$ on the boundary of I_3 with y_1 and y_2 defined as:

$b_1 \leq y_1 \leq b_2$ and $c_1 \leq y_2 \leq c_2$.

Define a neighborhood about d in the following manner:

Let $\epsilon > 0$ and $\ni$

$\epsilon < a_2 - a_1, \epsilon < b_2 - y_1, \epsilon < y_1 - b_1, \epsilon < c_2 - y_2,$

and $\epsilon < y_2 - c_1$.

A neighborhood of (a_2, y_1, y_2) is:

$$a_2 - \epsilon < x_1 < a_2 + \epsilon$$

$$y_1 - \epsilon < x_2 < y_1 + \epsilon$$

$$y_2 - \epsilon < x_3 < y_2 + \epsilon$$

The boundary of this neighborhood is:

$$x_1 = a_2 - \epsilon$$

$$y_1 - \epsilon \leq x_2 \leq y_1 + \epsilon$$

$$y_2 - \epsilon \leq x_3 \leq y_2 + \epsilon$$

$$x_1 = a_2 + \epsilon$$

$$y_1 - \epsilon \leq x_2 \leq y_1 + \epsilon$$

$$y_2 - \epsilon \leq x_3 \leq y_2 + \epsilon$$

$$a_2 - \epsilon \leq x_1 \leq a_2 + \epsilon$$

$$x_2 = y_1 - \epsilon$$

$$y_2 - \epsilon \leq x_3 \leq y_2 + \epsilon$$

$$a_0 - \epsilon \leq x_1 \leq a_2 + c$$

$$x_2 = y_1 + \epsilon$$

$$y_2 - \epsilon \leq x_3 \leq y_2 + \epsilon$$

$$a_2 - \epsilon \leq x_1 \leq a_2 + \epsilon$$

$$y_1 - \epsilon \leq x_2 \leq y_1 + \epsilon$$

$$x_3 = y_2 - \epsilon$$

$$a_2 - \epsilon \leq x_1 \leq a_2 + \epsilon$$

$$y_1 - \epsilon \leq x_2 \leq y_1 + \epsilon$$

$$x_3 = y_2 + \epsilon.$$

The point (a_2, y_1, y_2) lies on the piece of the boundary defined by $x_1 = a_2, b_1 \leq x_2 \leq b_2$ and $c_1 \leq x_3 \leq c_2$. The neighborhood in the piece of the boundary so defined is seen to be the set $N((a_2, y_1, y_2), \epsilon)) = \{(x_1, x_2, x_3) \mid x_1 = a_2, y_1 - \epsilon \leq x_2 \leq y_1 + \epsilon, y_2 - \epsilon \leq x_3 \leq y_2 + \epsilon\}$. This is as shown above of dimension 2, hence the subspace I_3 is of dimension 3.

Now to complete the induction assume that $\dim I_{n-1} = n-1$. It is necessary to show that $\dim I_n = n$ holds in a similar manner as was shown for I_1, I_2 and I_3. It takes n relationships to define each of the curves and there are 2n curves. Hence there are a total of $2n^2$ relationships.

First define I_n as the subspace:

$I_n = \{(x_1, x_2, \ldots, x_n) \mid a_{11} \leq x_1 \leq a_{12}, a_{21} \leq x_2 \leq a_{22}, \ldots, a_{n1} \leq x_n \leq a_{n2}; a_{11} \neq a_{12}, a_{21} \neq a_{22}, \ldots, a_{n1} \neq a_{n2}\}$.

$a_{12} - a_{11} = a_{22} - a_{21} = \ldots = a_{n2} - a_{n1}$.

Since dim $I_{n-1} = n-1$ then for some point $d_n = (a_{12}, y_1, y_2, \ldots, y_{n-1})$ on the boundary of I_n where $y_1, \ldots, y_{n-1}$ are defined as:

$a_{21} \le y_1 \le a_{22}; \ldots; a_{n1} \le y_{n-1} \le a_{n2}$, then $\exists\ \epsilon > 0$ and $\exists\ \epsilon < a_{12} - a_{11}, \epsilon < a_{22} - y_1, \epsilon < y_1 - a_{21}, \epsilon < a_{32} - y_2, \epsilon < y_2 - a_{31}, \ldots, \epsilon < a_{n2} - y_{n-1}, \epsilon < y_{n-1} - a_{n1}$.

Then a neighborhood $N((a_2, y_1, y_2, \ldots, y_n), \epsilon)$ is:

$N = \{(x_1, x_2, \ldots, x_n) \mid a_{12} - \epsilon < x_1 < a_{12} + \epsilon, y_1 - \epsilon < x_2 < y_1 + \epsilon; \ldots, y_{n-1} - \epsilon < x_n < y_1 + \epsilon\}$.

The boundary of this neighborhood is:

$$\begin{array}{ccc}
x_1 = a_{12} - \epsilon & \quad & x_1 = a_{12} + \epsilon \\
y_1 - \epsilon \le x_2 \le y_1 + \epsilon & & y_1 - \epsilon \le x_2 \le y_1 + \epsilon \\
y_2 - \epsilon \le x_3 \le y_2 + \epsilon & & y_2 - \epsilon \le x_3 \le y_2 + \epsilon \\
\vdots & & \vdots \\
y_{n-1} - \epsilon \le x_n \le y_{n-1} + \epsilon & & y_{n-1} - \epsilon \le x_n \le y_{n-1} + \epsilon \\
a_{12} - \epsilon \le x_1 \le a_{11} + \epsilon & & a_{12} - \epsilon \le x_1 \le a_{11} + \epsilon \\
x_2 = y_1 - \epsilon & & x_2 = y_1 + \epsilon
\end{array}$$

$$
\begin{array}{llll}
y_2 - \epsilon \le x_3 \le y_2 + \epsilon & & y_2 - \epsilon \le x_3 \le y_2 + \epsilon \\
\vdots & & \vdots \\
y_{n-1} - \epsilon \le x_n \le y_{n-1} + \epsilon & & y_{n-1} - \epsilon \le x_n \le y_{n-1} + \epsilon \\
\vdots & & \vdots \\
a_{12} - \epsilon \le x_1 \le a_{12} + \epsilon & & a_{12} - \epsilon \le x_1 \le a_{12} + \epsilon \\
y_1 - \epsilon \le x_2 \le y_1 + \epsilon & & y_1 - \epsilon \le x_2 \le y_1 + \epsilon \\
\vdots & & \vdots \\
y_{n-2} - \epsilon \le x_{n-1} \le y_1 + \epsilon & & y_{n-2} - \epsilon \le x_{n-1} \le y_1 + \epsilon \\
x_n = y_{n-1} - \epsilon & & x_n = y_{n-1} - \epsilon
\end{array}
$$

The point $(a_2, y_1, y_2 \ldots y_n)$ lies on the piece of the boundary defined by $x_1 = a_{12}, y_1 - \epsilon \le x_2 \le y_1 + \epsilon$ $y_2 - \epsilon \le x_3 \le y_2 + \epsilon \ldots, y_{n-1} - \epsilon \le x_n \le y_{n-1} + \epsilon$. The neighborhood in the piece of the boundary so defined is seen to be the set $N((a_{12}, y_1, y_2, \ldots, y_{n-1}), \epsilon) =$ $\{(x_1, x_2, \ldots, x_n) \mid x_1 = a_{12}, y_1 - \epsilon < x_2 < y_1 + \epsilon,$ $y_2 - \epsilon < x_3 < y_2 + \epsilon, \ldots, y_{n-1} - \epsilon < x_n < y_{n-1} + \epsilon\}$ which has dimension n-1.

Therefore, the induction is complete showing that $\dim I_n = n$.

Let $x = (x_1, \dots, x_n) \in E_n$ then $\exists$ an open cube I_n containing the point x and is a neighborhood of x. Let $\epsilon > 0$ then $\exists$ a neighborhood $N((x_1, \dots, x_n), \epsilon)$ as defined above, which is of dimension n; hence since every point of E_n is of dimension n then $\dim E_n = n$.

2.2 THE DIMENSION OF SUBSETS IN E_n

THEOREM 2.2: A subset $U \subset E_n$ is n-dimensional $\Longleftrightarrow$ U contains a non-null subset which is open in E_n.

PROOF: Suppose U contains a non-empty open set. Then $\dim U = n$ since $\exists$ a point $p \in U$ and a $t > 0 \ni$ the spherical neighborhood $N(p,t)$ is entirely contained in U and is homeomorphic to E_n. This homeomorphism can be demonstrated by the function:

$$f(x) = (2x - 1)/ [x(x-1)], \text{ where } x \in (0,1)$$

which is mapped onto the set of real numbers. This function can be generalized to E_n, thus giving homeomorphism of U onto E_n.

Now suppose dim U = n then it is necessary to show that U contains a non-null open set. Let $V = \mathcal{C}U$ then if $V \subset E_n$ is dense in E_n then $\dim \mathcal{C}V \leq n-1$. If V is a subset of a separable space then V contains a denumerable dense subset A where $\dim \mathcal{C}A \leq n-1 \Rightarrow \dim \mathcal{C}V \leq n-1$. Therefore, V can be taken to be a denumerable set. This last statement is equivalent to saying that for all denumerable dense subsets $A, B \subset E_n$, $\exists$ a homeomorphism of E_n on itself which maps A onto B.

Now to prove this latter statement define similarly placed sequences as follows:

Let (x_1, x_2), (y_1, y_2) be ordered pairs of points of E_n. If no parallel to a coordinate hyperplane contains more than one point (x_1, x_2) or more than one point (y_1, y_2) then (x_1, x_2) and (y_1, y_2) are similarly placed if (x_1, x_2) and (y_1, y_2) are in the same quadrant of E_n.

Let the coordinate axes be in general position with respect to A,B then A,B may be rearranged into similarly placed sequences. For let A,B be ordered arbitrarily:

$A = a_1,\ldots,a_i,\ldots$ and $B = b_1,\ldots,b_i\ldots$. Define the sequences $C = c_1,\ldots,c_i,\ldots$ and $D = d_1,\ldots,d_i,\ldots$ which are similarly placed rearrangements of A,B. This selection is done by setting $c_1 = a_1$, $d_1 = b_1$ and $d_2 = b_2$, $c_2 = a_\sigma$ where σ = least integer $\ni (c_1,a_\sigma)$, (d_1,d_2) are similarly placed. The component c_2 exists since A is dense in E_n. Suppose that $c_1,\ldots,c_{2j}$ and $d_1,\ldots,d_{2j}$ have been so chosen that they are similarly placed. Denote by c_{2j+1} the first component a not yet included in the c_j's and d_{2j+1} the first $b \ni c_1,\ldots,c_{2j},c_{2j+1},d_1,\ldots,d_{2j},d_{2j+1}$ are similarly placed. Again d_{2j+1} exists because B is dense. In a like manner we obtain first d_{2j+2} and c_{2j+2} where c_{2j+2} exists since A is dense. With the induction thus complete it is clear that c and d are similarly placed and c and d are rearrangements of A and B.

Now we see that we can consider the countable dense sequences A, B are similarly placed. Next it is necessary to extend 1-1 correspondence $f : a_1 \rightleftarrows b_1$ of A,B to a homeomorphism of E_n on itself. Let $x = (x_1, \ldots, x_n)$ be an arbitrary point of E_n not in A. Let $y = f(x) \ni \forall$ K where K = 1,2,...,n, be the K^{th} coordinate y_k of y. The points of A fall into two disjoint classes according as their K^{th} coordinates are $\leq x_k$ or are $> x_k$. Associated with this decomposition of A is a decomposition of B yielding two disjoint classes due to the nature of the mapping f(A) = B. The decomposition of B induces a decomposition into two disjoint classes of the set A of all the K^{th} coordinates of elements of B. Since A and B are similarly placed the decomposition of A has the characteristic that each element of one class is $<$ each element of the other class. Since A is dense in the reals this cut defines a real number which we take to be y_k. Since f is a 1-1 mapping of a dense set into a dense set then f is a homeomorphism. Since A and B are dense sets in E_n and f is a homeomorphism of dense sets then f is a homeomorphism of E_n onto itself. Hence the theorem is proved. Q.E.D.

CHAPTER 3

SETS OF DIMENSION 0

THEOREM 3.1: Every finite or denumerable space which is non-empty has dimension 0.

PROOF: Let N be a neighborhood of any point p, and let $\delta > 0$ define a spherical neighborhood of radius δ about p which is a subset of N. Let $x_1, x_2, \ldots$ be an enumeration of X and the distance, $d(x_i, p)$, from x_i to p. Then $\exists$ a real number $\sigma > 0 \ni \sigma < \delta$ and $\sigma \neq d(x_i, p) \ \forall_i = 1, 2, \ldots$. The spherical neighborhood of radius σ about p is then a subset of N with a null boundary. Hence dim X = 0. Q.E.D.

THEOREM 3.2: A non-null space has dimension 0 if and only if any two disjoint closed sets in it are separated.

PROOF: Let U, V be any two closed disjoint subsets of a non-null space X, which are separated. Then $\forall$ point $p \in V \subset \complement U$, $\exists$ a set $V' \ni V \subset V'$, $V' \cap U = \emptyset$ and V' is both open and closed. Since V' is open it is a neighborhood of p, and since it is also closed it has a null boundary. Therefore, since $\forall$ point p $\exists$ a neighborhood with a null boundary then X is of dimension 0.

Let X be of dimension 0. It is necessary to show that $\forall$ two disjoint closed subsets U, V in X $\exists$ a separation of these sets in X. Let $p \in X$. Either $p \cap U = \emptyset$ or $p \cap V = \emptyset$. Since U, V are both closed then either $V \cap N(p) = \emptyset$ or $U \cap N(p) = \emptyset$. Now then $\exists$ neighborhoods N(p) which are both open and closed $\forall$ p, since dim X = 0. Since X is a separable metric it has a denumerable open basis and $\exists$ a sequence $N_1, N_2, \ldots$ of these neighborhoods $\ni \bigcup_{p \in} N(p) = X$. We shall now determine another sequence of sets $\{S_i\}$ where $N_1 = S_1$ and

$$S_i = \mathcal{C}_{N_i}\left(\bigcup_{k=1}^{i-1} N_k\right) = N_i \cap \mathcal{C}_X \bigcup_{k=1}^{i-1} N_k \text{ where } i = 2, 3, \ldots$$

Then $X = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ if $i \neq j$, S_i is open and either $S_i \cap U = \emptyset$ or $S_i \cap V = \emptyset$. The first statement is true since each succeeding set contains those points not contained in previous sets. $S_i \cap S_j$ is null since the boundary of the neighborhood of a point is null, hence S_i and S_j are separated and, therefore, are disjoint. Since S_i are neighborhoods of p then either $S_i \cap U = \emptyset$ or $S_i \cap V = \emptyset$ since S_i is a neighborhood of the point $p \varepsilon \mathcal{C}_X U$ or $p \varepsilon \mathcal{C}_X V$. S_i is open since $\bigcup_{k=1}^{i-1} N_k$ is closed and $\mathcal{C}_X \bigcup_{k=1}^{i-1} N_k$ is open, hence S_i is the intersection of this open set and the open set N_i and, therefore, is open.

Let $U' = \bigcup_i S_i$ for which $S_i \cap V = \emptyset$ and $V' = \bigcup_i S_i$ for which $S_i \cap U = \emptyset$. Then $X = U' \cup V'$ since

$X = \bigcup_{i=1}^{\infty} S_i$, $U' \cap V' = \emptyset$ since $S_i \cap S_j = \emptyset$ for $i \neq i$,

U', V' are open since S_i is open, and $U' \cap V = U \cap V' = \emptyset$ since either $S_i \cap U = \emptyset$ or $S_i \cap V = \emptyset$. Then $U \subset U'$ and $V \subset V'$. Hence U' and V' give the separation of the closed disjoint sets U, V. Q.E.D.

COMPACTNESS AND DIMENSION 0

DEFINITION 3.1: A space X is said to be compact $\Longleftrightarrow \exists$ a collection of open subsets $\{V_\alpha\}$ where $\bigcup_{\alpha \varepsilon I} V_\alpha = X$ and I is the indexing set, then $\exists$ a finite subcollection $\{U_k\} \ni X = \bigcup_{k=1}^{n} U_k$.

THEOREM 3.3: Let X be a compact space, F a closed subset of X, and $p \varepsilon X$. If p and each point of F can be separated then p and F can be separated.

PROOF: For each point $q \varepsilon F \exists$ two disjoint sets U_q and V_q, with $p \varepsilon U_q$, $q \varepsilon V_q$, which are both open and closed in $U_q \cup V_q$. Since F is a closed subspace of a compact space $\exists$ a finite number of points $q_1, \ldots,$ $q_k \ni \bigcup_{k=1}^{n} V_{q_k} \supset F$. Let $U = \bigcap_{i=1}^{k} U_{q_i}$, $V = \bigcup_{i=1}^{k} V_{q_i}$. Then $p \in U$, $F \subset V$, and U,V are disjoint and both open and closed. Therefore, p,F are separated. Q.E.D.

THEOREM 3.4: Let X be a 0-dimensional compact space, $p \in X$, and M(p) the set of all points which cannot be separated from p. Then M(p) is connected.

PROOF: First show M(p) is closed. A point $x \in \complement M(p)$ if $\exists$ U,V $\ni U \cap V = \emptyset$, $x \in U$, $p \in V$, and U,V are open in $U \cup V$. Note that each point $x \in \complement M(p)$ has a neighborhood $n(X) \subset \complement M(p)$, giving $\complement M(p)$ is an open set and so M(p) is closed. Assume M(p) is disconnected. Then $M(p) = S \cup T$, $S \neq \emptyset$, $T \neq \emptyset$, $S \cap T = \emptyset$, S and T closed in M(p), and $p \in S$, and so S,T are closed in X. Since X is normal then $\exists$ an open set $U \subset X \ni S \subset U$ and $U \cap T = \emptyset$, and $Bdry(U) \cap M(p) = Bdry(U) \cap (S \cup T) = \emptyset$. Thus $\forall\ x \in Bdry(U)$, x is separated from p. Since Bdry(U) is closed then by the above theorem $\exists$ a set V $\ni$ Bdry(U)$\subset$V, (p)$\cap$V = $\emptyset$, and V is both open and closed, for p ε S$\subset$U, and p ε $\mathcal{C}_u$(U$\cap$V). Now $\mathcal{C}_u(U\cap V) = \mathcal{C}_u(U\cap\bar{V}) = \mathcal{C}_u(\bar{U}\cap V)$, since $V = \bar{V}$ and Bdry(U)$\subset$V, which shows that $\mathcal{C}_u$(U$\cap$V) is both open and closed. But $(\mathcal{C}_u(U\cap V))\cap T = \emptyset$. Hence p is separated from the points of T$\subset$M(p), which contradicts the definition of M(p). Q.E.D.

<u>THEOREM 3.5:</u> The following four properties are equivalent for compact spaces X:

(0) X is totally disconnected.

(1) Any two distant points in X can be separated.

(2) Any point can be separated from any closed set not containing it (i.e. V is 0-dimensional).

(3) Any two disjoint closed sets can be separated.

PROOF: (0)$\Longrightarrow$(1). Assume X is totally disconnected. Consider $\forall$ $p \in X$ the set M(p). By the preceding theorem M(p) is connected but consists of a single point since X is totally disconnected. Hence any two points of X are separated. By theorem 3.3 it is obvious that (1)$\Longrightarrow$(2). (2)$\Longrightarrow$(3) for spaces with countable bases and since X is compact then the statement is true. (3)$\Longrightarrow$(0) since every metric space is completely normal. Q.E.D.

CHAPTER 4

SOME PROPERTIES OF n-DIMENSIONAL SPACES

4.1 SEPARATION OF SETS IN n-DIMENSIONAL SPACES

THEOREM 4.1: An equivalent condition that X has dimension $\leq n$ is that every point $p \in X$ can be separated by a closed set of dimension $\leq n-1$ from any closed set C not containing p.

PROOF: It is necessary to show that this theorem is equivalent to the definition of $\dim X \leq n$. Let $\dim X \leq n$ be true according to definition 1.1. Since X is a metric space then X is regular and $p \in \complement_X C$ $\exists$ a neighborhood N of $p \ni \bar{N} \subset \complement_X C$. Then $\exists$ a neighborhood M of p for which $M \subset \bar{N}$ and Bdry(M) is of dimension $\leq n-1$. Therefore, Bdry(M) separates p and C, thus showing $\dim X \leq n$ according to the conditions of the theorem.

Conversely, let N be a neighborhood of p then $\mathcal{C}_x N$ is closed and $p \notin \mathcal{C}_x N$ and can be separated from p by a closed set B of dimension $\leq n-1$. This means that $\mathcal{C}_x B = N' \cup M'$, $p \in N'$, $\mathcal{C}_x N \subset M'$, $N' \cap M' = \emptyset$ with N',V' open in $\mathcal{C}_x B$ in X. Now N' is a neighborhood of p contained in N since the boundary of N' is contained in B. Since a subspace of a space X of dimension $\leq n$ has dimension $\leq n$ then the Bdry(N') has dimension $\leq n-1$. Thus completing the proof. Q.E.D.

<u>THEOREM 4.2:</u> Let C_1 and C_2 be disjoint closed sets in a space X and A a subset of X of dimension $\leq n$. Then $\exists$ a closed set B separating C_1 and C_2 with dim $A \cap B \leq n-1$.

PROOF: The theorem is obvious for n = 0 and dim A = -1. For n = 0 and dim A = 0:

Since X is normal $\exists$ open sets $U_1, U_2 \ni C_1 \subset U_1$, $C_2 \subset U_2$ and $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. The disjoint sets $\bar{U}_1 \cap A$ and $U_2 \cap A$ are closed in A and can, therefore, be separated in A, which is true by the theorem on separation for spaces of dimension. Then $\exists$ disjoint sets $C_1^1, C_2^1 \ni A = C_1^1 \cup C_2^1$ and $\bar{U}_1 \cap A \subset C_1^1, U_2 \cap A \subset C_2^1$, with C_1^1 and C_2^1 both open and closed in A. Hence $C_1^1 \cap \bar{U}_2 \cup C_2^1 \cap U_1 = \emptyset$, $C_1^1 \cap C_2^1 \cup C_1^1 \cap C_2^1 = \emptyset$ all of which $\Longrightarrow$ $C_1^1 \; \bar{C}_2 \cup C_2^1 \; C_1 = \emptyset$. Since U_1, U_2 are open, $C_1^1 \cap U_2 \cup \bar{C}_2^1 \cap U_1 = \emptyset$ and hence $\bar{C}_1^1 \cap C_2 \cup \bar{C}_2^1 \cap C_1 = \emptyset$. Also since $\bar{C}_1 \cap \bar{C}_2 = C_1 \cap C_2 = \emptyset$ then neither $C_1 \cup C_1^1$ and $C_2 \cup C_2^1$ contains a cluster point of the other. Since X is completely normal then $\exists$ an open set $W \ni C_1 \cup C_1^1 \subset W$ and $W \cap (C_2 \cup C_2^1) = \emptyset$. The Bdry(B) $= \mathcal{C}_{\bar{W}} W$ separater C_1 and C_2 and is disjoint from $C_1^1 \cup C_2^1 = A$. Hence the separation is shown for $n = 0$ and $\dim A = 0$. For $n > 0$. Applying corollary 4 of the sum theorem to A, then $A = D \cup E$ with $\dim D \leq n$, $\dim E \leq 0$. Then using the case

$n = 0$ to obtain a separation of C_1, C_2 by a set B not meeting E. Hence $A \cap B \subset D$. But $\dim D \leq n-1$; $\therefore \dim A \cap B \leq n-1$. Thus proving the theorem. Q.E.D.

4.2 n-DIMENSIONAL COMPACT SPACES

THEOREM 4.3: For a compact space X the following three properties are equivalent:

(1) Any two distinct points can be separated by a closed set of dimension $\leq n-1$.

(2) Any point can be separated from a closed set not containing it by a closed set of dimension $\leq n-1$.

(3) Any two disjoint closed sets can be separated by a closed set of dimension $\leq n-1$.

PROOF: (3) $\Longrightarrow$ (2) since a set consisting of a single point is a closed set and similarly (2) $\Longrightarrow$ (1). By the previous theorem on separated sets it is seen that (2) $\Longrightarrow$ (3). Now to show that (1) $\Longrightarrow$ (2) let C be a closed subset of X and p a point of X. Then p can be separated from C by a closed set of dimension $\leq$ n-1 if p can be separated from each point of C by a closed set of dimension $\leq$ n-1. This latter statement is proved by considering that $\forall$ $q \in C$ $\exists$ an open set $U(q) \ni q \in U(q)$, $p \notin \bar{U}(q)$, dim Bdry(U(q)) $\leq$ n-1. C is compact and $\therefore$ $\exists$ a finite number $q_1, \ldots, q_k$ of the points $q \ni C \subset U = \bigcup_{i=1}^{k} U(q)$. Let B = Bdry(U), $B = \mathcal{C}_{\bar{U}} U$. Hence $B \subset \bigcup_{i=1}^{k}$ Bdry($U(q_i)$) giving dim B $\leq$ n-U by the sum theorem for dimension n-1. Since $p \notin \bar{U}$ then p is separated from C by the closed set B of dimension $\leq$ n-1. Thus (1) $\Longrightarrow$ (2). Q.E.D.

<u>THEOREM 4.4:</u> The statements (1), (2), and (3) are equivalent.

(1) X can be disconnected by a subset D of dimension $\leq$ m.

(2) X contains an open set U which is neither empty nor dense and dim Bdry(U) $\leq$ m.

(3) $X = C_1 \cup C_2$ and $\dim(C_1 \cap C_2) \leq m$ where C_1 and C_2 are closed proper subsets of X.

PROOF: (2) $\Longrightarrow$ (3). If U satisfies (2) then $C_1 = \bar{U}$, $C_2 = \complement U$ satisfy (3).

(3) $\Longrightarrow$ (1). If C_1, C_2 satisfy (3) then $D = C_1 \cap C_2$ satisfies (1).

(1) $\Longrightarrow$ (2). Since D disconnects X then

$$\complement D = U_1 \cup U_2,\ U_1 \neq \emptyset,\ U_2 \neq \emptyset, \text{ and}$$

$$U_1 \cap \bar{U}_2 \cup \bar{U}_1 \cap U_2 = \emptyset.$$

Either $X = \overline{\complement D} = \bar{U}_1 \cup \bar{U}_2$ or D contains a non-null open set U with $\bar{U} \subset D$. U satisfies the condition of (2) since $\text{Bdry}(U) \subset D$. Since $U = \complement \bar{U}_1$ then $U_2 \subset U \subset \bar{U}_2$ and $U \neq \emptyset$. U is not dense since $\bar{U}_2 \neq X$ because $U_1 \cap \bar{U}_2 \cup \bar{U}_1 \cap U_2 = \emptyset$. Since $\text{Bdry}(U) \subset \complement_{\bar{U}} U_2 \subset D$, then $\dim \text{Bdry}(U) \leq m$. Thus satisfying (2). Q.E.D.

<u>THEOREM 4.5:</u> E_n cannot be disconnected by a subset of dimension $\leq n-2$.

PROOF: Suppose false then E_n would contain a non-null open set which is not dense with boundary having dimension $\leq n-2$. This contradicts the fact that the dimension of such an open set is n.

4.3 INFINITE DIMENSIONAL SPACE

Now to complete the discussion of dimension it is necessary to establish the existence of a space which is of infinite dimension. Hilbert Space, E_ω, is a good example of such a space.

DEFINITION OF A HILBERT SPACE, E_ω: Space of sequences of real numbers $x_1, x_2 \ldots$ for which $\sum_{n=1}^{\infty} x_n^2$ is finite. E_ω has the metric $d(x,y) = \sqrt{\sum_{i=1}^{\infty} (a_i - b_i)^2}$.

First it is necessary to show that the Hilbert Space is a separable metric.

THEOREM 4.6: A Hilbert Space is a separable metric space.

PROOF: A Hilbert Space, E_ω, is separable if E_ω has denumerable dense sequences. Let S be the set of all sequences of rational numbers with all but a finite number of the elements = 0. Then obviously every point of S is an element of E_ω. Let $C = (c_1, c_2, c_3, \ldots) \in E_\omega$ and $\epsilon > 0$. Choose $n \ni \sum_{i=n+1}^{\infty} c_i^2 < \frac{\epsilon^2}{2}$ for the first n coordinates of C. For each $i = 1, 2, \ldots, n$ choose a rational number $r_i \ni | c_i - r_i | \; \frac{\epsilon}{\sqrt{2n}}$ and $r_i = 0$ for $i > n$. Then

$$\sqrt{\sum_{i=1}^{\infty} (c_i - r_i)^2} = \sqrt{\sum_{i=1}^{n} (c_i - r_i)^2 + \sum_{i=n+1}^{n} c_i^2} <$$

$$\sqrt{\frac{n\epsilon^2}{2n} + \frac{\epsilon^2}{2}} = \sqrt{\frac{2\epsilon^2}{2}} = \epsilon.$$

Then letting $p = (r_1, r_2, \ldots)$ we have $d(p, C) < \epsilon$. Thus $\forall$ point $C \in E_\omega$ $\exists$ a point $p \in S$ within ϵ of C. Since S is denumerable then it is shown to be a denumerable dense set. Hence E_ω is separable. Q.E.D.

Now that it has been shown that E_ω is separable metric we will show that $\dim E_\omega = \infty$.

<u>THEOREM 4.7:</u> $\dim E_\omega = \infty$.

PROOF: Let H_n be a subspace of E_ω which is homeomorphic to E_n and where the points $a \in H_n$ with $a = (a_1, a_2, \ldots a_n, \ldots) \ni a_{n+1} = a_{n+2} = a_{n+3} = \ldots = 0$. It is obvious that for n as large as you please you can find a subspace H_n of E_ω where $\dim H_n \leq n$. Hence by theorem 1.1 and definition 1.1-7 $\dim E_\omega = \infty$. Q.E.D.

APPENDIX

A.1 CANTOR'S MAPPING

DEFINITION: Given the family of sets $\{A_\eta \mid \eta \varepsilon \mathcal{A}\}$, where $\mathcal{A}$ is the indexing set, then the cartesian product $\prod_{\eta \varepsilon \mathcal{A}} A$ is the set of all maps $c: \mathcal{A} \to \bigcup_\eta A_\eta \ni \forall \eta \varepsilon \mathcal{A} : c(\eta) \varepsilon A_\eta$. For simplification of notation let $\{a_\eta\}$ indicate for an element $c \varepsilon \prod_{\eta \varepsilon \mathcal{A}} A_\eta \ni c(\eta) = a_\eta \forall \eta$ where a_η is the η th coordinate of $\{a_\eta\}$.

DEFINITION: The Cantor set for the closed interval $[0,1]$. Remove from $[0,1]$ the open interval (1/2, 2/3), thus removing all real numbers in $[0,1]$ that require $\eta_1 = 1$ in their triadic expansion (i.e. each element of $[0,1]$ can be represented by numbers of the form $.\eta_1 \eta_2 \eta_3 \ldots$ where $\eta_1 \varepsilon \{x \mid x = 1, x = 2, \text{ or } x = 3\}$). Now remove the middle third open intervals (1/9, 2/9), (7/9, 8/9) from the remaining intervals $[0,1/3]$, $[2/3,1]$.

In so doing all real numbers are removed which require $\eta_2 = 1$ in the triadic expansion. Proceed in an analogous fashion for the nth removal of sets to remove the union M_n of middle thirds of the 2^{n-1} intervals remaining. Thus the Cantor set is the set $C = \mathcal{C}_I \bigcup_{n=1}^{\infty} M_n$ where $I = [0,1]$.

Consider now the Cantor set C just described. C consists of all numbers in $I = [0,1]$ that do not require the use of the digit "1" in their triadic expansion. Let $A_i = \{0,2\} \forall i \in Z^+$; $\prod_{\eta \in \mathcal{A}} A_\eta$ is then the set of all sequences of 0's and 2's; $\{\{\eta_i\} \mid \eta_i = 0$ or $\eta_i = 2;$ $i = 1,2,\ldots\}$. Then mapping $f: \times_\eta A_\eta \rightarrow I \subset E_1$ where

$$f(\{\eta_i\}) = \sum_{i=1}^{\infty} \frac{\eta_i}{3^i}$$

and f is obviously one-to-one. Since f is 1-1 then $\forall$ $c \in C$ $\exists$ one and only one element $x \in I$ and conversely $\forall$ $x \in I$ one and only one element of $c \in C$. Now it remains to show that f maps all of C onto all of E^1. Since the expansion $A_i = \{0,2\}$ can be mapped 1-1 and onto the binary numbers then each $x \in I$ can be represented by binary numbers of the form $.\mu_1\mu_2\mu_3\ldots$; $i = 1,2,3,\ldots$; $\mu_i \in \{\{\mu_i\} \mid \mu_i = 0 \text{ or } 1\}$. Therefore, $\exists$ a one-to-one and onto mapping of $C \rightarrow I \subset E^1$. Example of 1-1 and onto mapping of the unit interval onto the unit square.

Each element $c \in C$ is a sequence of the form $.\eta_1\eta_2\eta_3\ldots$ where $\eta_i \in \{\{\eta_i\} \mid \eta_i = 0 \text{ or } \eta_i = 2;\ i = 1,2,3,\ldots\}$. By the above mapping $\forall z \in [0,1]$ $\exists$ one only one element $c \in C$, hence one and only one expansion $.\eta_1\eta_2\eta_3\eta_4\eta_5\ldots$. Now consider the mapping:

$x = .\eta_1\eta_3\eta_5\ldots;\ \ni \eta_i \in \{\{\eta_i\} \mid \eta_i = 0 \text{ or } 2;$
$i = 1,3,5,7,\ldots\}$ and $y = .\eta_2\eta_4\eta_6\ldots;\ \ni \eta_i \in \{\{\eta_i\} \mid$
$\eta_i = 0 \text{ or } 2;\ i = 2,4,6,8,\ldots\}$, where $x \in I$ and $y \in I$ and again each of these mappings are one-to-one and onto. Since the unit square $I_2 = I \times I$ then the combination of these mappings are a one-to-one and onto map of $C \rightarrow I_2$ and consequently of $I \rightarrow I_2$.

<u>DEFINITION: CARTESIAN PRODUCT TOPOLOGY</u> Let X_α be given $\forall \alpha \in \mathcal{A}$. The cartesian product $\Pi\{X_\alpha : \alpha \in \mathcal{A}\}$ is defined to be the set of all functions X on $\mathcal{A} \ni X_\alpha \in X_\alpha \forall \alpha \in \mathcal{A}$. The set X_α is the α-th coordinate set and the projection P_α of the product into the α-th coordinate set is defined by $P_\alpha(X) = X_\alpha$. Suppose that a topology $\mathcal{J}_\alpha$ is given $\forall$ coordinate set. The construction of the product topology is motivated by the requirement that each projection P_α is to be continuous. In order to attain continuity of the projection it is necessary and sufficient that each set of the form $P_\alpha^{-1}[U]$ be open where U is an open subset of X_α. The family of all sets of this form is a subbase for a topology; it is clearly the smallest topology $\ni$ projections are continuous.

Let A_n be the discrete space $\{0,2\}$ $\forall$ $n = 1,2,\ldots.$ In the preceding example it has been shown that if C is the Cantor set the map $\varphi: \Pi A_n \rightarrow C$ given by

$\{a_i\} \to \sum_1^\infty \frac{a_i}{3^i}$ is a one-to-one and onto mapping of the set $\prod A_n$ onto the set C.

The following is an example of a continuous mapping of the unit interval onto the unit n-dimensional cube (Peano's Curve). That is, $\forall\, n \leq \infty$, $\exists$ a continuous and onto mapping $f: I \to I_n$.

PROOF: First consider the subspace topology $f: \prod_n A_n \to C$ discussed above where C is the Cantor set and f is one-to-one and onto. Now show that f is a homeomorphism. Let $\langle a \rangle$ for $a \subset A$ denote $f^{-1}(a)$. For finitely many $a_1 \subset A_1, \ldots, a_n \subset A_n$, the subset $\langle a_1 \rangle \cap \ldots \cap \langle a_n \rangle = f_1^{-1}(a_1) \cap \ldots \cap f_n^{-1}(a_n)$ is denoted by $\langle a_1, \ldots, a_n \rangle$. The function f is continuous since $\forall\, \varepsilon > 0$ and given $c = \sum_i \frac{a_i}{3^i} \in C$ and the neighborhood $N(c; \epsilon) \cap C$, then a j can be chosen large enough that $\sum_j^\infty \frac{2}{3^j} < \epsilon$ (note $2 \in \{0,2\}$); hence $f(\langle a_1, \ldots, a_n \rangle) \subset N(c, \epsilon) \cap C$. The function f^{-1} is continuous since f is a continuous mapping of open sets. This latter statement is true since $\forall\ x = \{a_i\}$ and neighborhood $(a_{i1},\ldots,a_{ik})$ containing x, let $i = \max\{i_1,\ldots,i_k\}$ and $W = N(f(x);\frac{1}{3^{j+1}}) \cap C$; then $W \subset f(\langle a_{i1},\ldots,a_{ik}\rangle)$ where $(a_{i1},\ldots,a_{ik})$ is a subbasis set and is, therefore, open. Since f maps an open set into an open set and f is continuous then f^{-1} is continuous and hence is a homeomorph of $f:\prod_n A_n \to C$. In an analogous manner the mapping $g:\prod_n A_n \to I$ where g is defined as $\{a_i\} \to \sum_1^{\infty} \frac{a_i}{2^{i+1}}$ can be shown to be continuous and onto by use of diadic expansions. Also then g^{-1} is seen to be continuous and g is an open mapping.

Now it is necessary to show that $\forall\ n \leq \infty$, $\exists$ a continuous open mapping of the Cantor set onto I_n. As just shown, $g^{-1}:C \to \prod A_n$ is a homeomorphism.

<u>LEMMA:</u> Let $\mathcal{A}$ be an arbitrary denumerable indexing set. For each $\alpha \in \mathcal{A}$ let $f_\alpha : X_\alpha \to Y_\alpha$ and define $\prod_{\alpha\in\mathcal{A}} f_\alpha : \prod_\alpha X_\alpha \to, \prod_{\alpha\in\mathcal{A}} Y_\alpha$ as $\{X_\alpha\} \to \{f_\alpha(X_\alpha)\}$. If

each f_α is continuous then $\prod_{\alpha \in \mathcal{A}} f_\alpha$ is continuous and if each f_α is open and all but a finite number of the f_α are onto then $\prod_{\alpha \in \mathcal{A}} f_\alpha$ is also an open map. First $\prod_{\alpha \in \mathcal{A}}$ f_α is continuous since given V_α a subbasis open set $\prod_\alpha Y_\alpha$ then $\left[\prod f_\alpha\right]^{-1}(V_\alpha) = f_\alpha^{-1}(V_\alpha)$ is a subbasis open set since f_α is continuous. The product f_α is an open mapping since a basis $(U_{\alpha_1},\ldots,U_{\alpha_n})$ maps to $f_{\alpha_1}(U_{\alpha_1}) \times \ldots \times f_{\alpha_n}$ $(U_{\alpha_n}) \times \prod\{f_\beta(X_\beta) / \beta \neq \alpha_1,\ldots \alpha_n\}$, and since all but at most finitely many f_β are onto then there are only a finite number of sets $\ni f_\beta(X_\beta) = Y_\beta$ with the rest of the sets being open. Since the image of the basis $(U_{\alpha_1},\ldots,$ $U_{\alpha_n})$ is an open set and since f maps open sets into open sets then $\prod f_\alpha$ is open. Q.E.D.

<u>COROLLARY A.1:</u> An obvious consequence of this lemma is that for the two families of spaces $\{X_\alpha | \alpha \in \mathcal{A}\}$, $\{Y_\beta | \beta \in \mathcal{B}\}$, and the one-to-one mapping $f: \mathcal{A} \to \mathcal{B}$, then $\forall_\alpha$, which X_α is homeomorphic to $Y_{f(2)}$, $\prod_\alpha X_\alpha$ is homeomorphic

to $\prod_\beta Y_\beta$.

Now consider that given a fixed space Y and the denumerable indexing set $\mathcal{A}$, let $Y_\alpha = Y \ \forall \alpha \in \mathcal{A}$ and $Z = \prod_\alpha Y_\alpha$. Then each cartesian product $Z \times Z \times \ldots \times Z$ consisting of at most a denumerable number of factors Z is homeomorphic to Z. This statement is true since there are a denumerable number of factors Y in $Z \times \ldots \times Z$ and by the preceding corollary. Hence $\exists$ a homeomorphism $h: \prod A_k \rightarrow \prod A_k \times \ldots \times \prod A_k$. Using the mapping $f: \prod A_k \rightarrow I$ defined by $\{a_i\} \rightarrow \sum_1^\infty \frac{a_i}{2^{i+1}}$ which is a continuous open mapping then $\exists$,

$$\prod_1^n f: \prod A_k \times \ldots \times \prod A_k \rightarrow I \times \ldots \times I = I_n$$

$\ni$ that $(\prod_1^n f) \circ h \circ g^{-1}: C \rightarrow I^n$ is the desired map.

<u>PEANO'S CURVE:</u> Let $f: C \rightarrow I_n = I \times \ldots \times I$ be the continuous onto mapping just described and let $p_i \circ f: C \rightarrow I$ be the coordinate functions. Extend each $p_i \circ f: C \rightarrow I$ to a continuous function $f_i: I \rightarrow I$ by

defining f_i to be linear on each omitted interval. Now then let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be any family of space and $f: X \to \prod Y_\alpha$ a map. Then f is continuous $\Longleftrightarrow$ $p_\beta \circ f$ is continuous for each $\beta \in \mathcal{A}$ where p_β is a continuous open mapping. The proof of this latter statement:

Let f be continuous since p_β is continuous $\forall_\beta$ then $p_\beta \circ f$ is continuous. Conversely assume that $p_\beta \circ f$ is continuous, then $\forall$ subbasic set U_β in $\prod Y_\alpha$, $f^{-1}(U_\beta)$ is open because $f^{-1}(U_\beta) = f^{-1}\left[p_\beta^{-1}(U_\beta)\right] = (p_\beta \circ f)^{-1}(U_\beta)$ and so f is continuous.

As an immediate consequence of the preceding statement for $f: X \to \prod_\alpha Y_\alpha$ with X a fixed space and a map $f_\alpha: X \to Y_\alpha$ given by $X \to \{f_\alpha(X)\}$, f is continuous $\Longleftrightarrow$ each given f_α is continuous. Hence $F(t) = \{f_i(t)\}$ is a continuous map $I \to I^k$ and since $F \cap (C \times L) = f$, F(t) is onto.

Q.E.D.

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