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# Minimal Convex Extensions and Intersections of Primary $l$-Ideals in $f$-rings

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## INTRODUCTION

Let $n$ be a positive integer. An $f$-ring $A$ is said to satisfy the left $n$th-convexity property if for any $u, v \in A$ such that $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there exists a $w \in A$ such that $u = wv$. The right $n$th-convexity property is defined similarly and an $f$-ring is said to satisfy the $n$th-convexity property if it satisfies both the left and the right $n$th-convexity property. In this paper we study embedding a commutative semiprime $f$-ring into a commutative semiprime $f$-ring with a convexity property and apply these results to study intersections of primary ideals in commutative semiprime $f$-rings. Except where explicitly stated, all rings will be assumed to be commutative and semiprime.

Those $f$-rings which satisfy one or more of these convexity properties have been studied by several authors. In [GJ, 1D], L. Gillman and M. Jerison note that any $C(X)$, the $f$-ring of all real-valued continuous functions defined on a topological space $X$, satisfies the $n$th-convexity property for all $n \geqslant 2$, and in [GJ, 14.25], they give several properties that in $C(X)$ are equivalent to the 1st-convexity property. M. Henriksen proves some results about the ideal theory of an $f$-ring satisfying the 2nd-convexity property in [H], and S. Steinberg studies left quotient rings of $f$-rings satisfying the left 1st-convexity property in [S]. In [HP, Sects. 3, 4] C. Huijsmans and B. de Pagter use the 2nd-convexity property to prove some results about the ideal theory of uniformly complete archimedean $f$-algebras, and in [HP, Sect. 6; HP 1; HP 2; P] they give several properties that in archimedean $f$-algebras with identity element are equivalent to the 1st-convexity property. The author has looked at $f$-rings satisfying a convexity property in [L], giving several results concerning ideal theory and unitability of an $f$-ring satisfying a convexity property.



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Since $f$-rings which satisfy one of the convexity conditions have some nice properties, we consider how to start with an arbitrary $f$-ring and "get to" an $f$-ring satisfying a convexity property. Section II studies embedding an $f$-ring in an $f$-ring satisfying a convexity property, and finding a minimal such embedding for a commutative semiprime $f$-ring.

Section III gives an application showing how embedding an $f$-ring in a minimal $f$-ring satisfying a convexity property can be used in problems that do not originally mention a convexity property. There it is shown that in a commutative semiprime $f$-ring with identity element, an $l$-ideal $I$ satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I{:}\sqrt{I}$ is an intersection of primary $l$-ideals and a pseudoprime $l$-ideal $I$ satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I{:}\sqrt{I}$ is primary.

The problem of identifying $l$-ideals which are intersections of primary ideals in $C(X)$ has been studied by R. D. Williams in [W], and our results generalize some of that work.

## I. Preliminaries

By an ideal we will always mean a ring ideal. Suppose $A$ is a ring and $I$ an ideal of $A$. We will use the notation $I(a)$ for cosets of $I$. The ideal $I$ is called *semiprime* (*prime*) if whenever $J$ $(J_1, J_2)$ is an ideal such that $J_2 \subseteq I (J_1 J_2 \subseteq I)$, $J \subseteq I (J_1 \subseteq I$ or $J_2 \subseteq I)$. The ring $A$ is called semiprime (prime) if $\{0\}$ is a semiprime (prime) ideal.

An $f$-ring is a subdirect product of totally ordered rings. For background material on $f$-rings see [BKW]. A prime $f$-ring is a totally ordered domain and a semiprime $f$-ring is a subdirect product of totally ordered domains.

An ideal $I$ of an $f$-ring $A$ is said to be an $l$-ideal if $|x| \leqslant |y|$, $y \in I$ implies $x \in I$. Given a subset $S \subseteq A$ there is a smallest $l$-ideal containing $S$, and we will denote this by $\langle S\rangle$. It is well known that the sum of two $l$-ideals is again an $l$-ideal. It is also well known that the $l$-ideals containing a given prime $l$-ideal form a chain.

Recall that if $n$ is a positive integer, then an $f$-ring $A$ is said to satisfy the left $n$th-convexity property if for any $u, v \in A$ such that $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there exists a $w \in A$ such that $u = wv$. The right $n$th-convexity property is defined similarly and an $f$-ring satisfies the $n$th-convexity property if it satisfies both the right and the left $n$th-convexity property. If $n \geqslant 2$ and $A$ satisfies the (left) $n$th-convexity property, then we may assume that the element $w$ satisfies $0 \leqslant w \leqslant v^{n-1}$ (by replacing the element $w$, if necessary, by $(w \wedge v^{n-1}) \vee 0$). It is easily seen that

(1.1) An $f$-ring satisfying the 1st-convexity property also satisfies the $n$th-convexity property for all $n \geqslant 2$.

Let $A$ be a semiprime $f$-ring. The following appears in [L, 2.1]:

(1.2) If $n \geqslant 2$ and if whenever $u, v \in A$ with $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$, there is a $w \in A$ such that $0 \leqslant w \leqslant v^{n-1}$ and $u = wv$, then the element $w$ is unique.

The following is proved in [L, 2.3, 3.9] when $n \geqslant 1$ and $A$ satisfies the $n$th-convexity property.

(1.3) Any $l$-homomorphic image of $A$ satisfies the $n$th-convexity property.

(1.4) If $A$ has an identity element and if $0 \leqslant u \leqslant v$ and $u^{-1} \in A$, then $v^{-1} \in A$.

In [L, 4.4] the following is shown.

(1.5) Let $A$ be an $f$-ring satisfying the 2nd-convexity property. If $I, J$ are $l$-ideals of $A$, then $IJ$ is also an $l$-ideal in $A$.

## II

In this section, we discuss embedding an $f$-ring into an $f$-ring satisfying a convexity property.

2.1. Definition. Let $A$ be an $f$-ring. An $f$-ring $B$ is an $n$-convexity cover of $A$ if $A$ is embedded in $B$ and $B$ satisfies the $n$th-convexity property.

In the next theorem necessary and sufficient conditions for the existence of an $n$-convexity cover of a semiprime, but not necessarily commutative, $f$-ring are given. Recall that a (noncommutative) domain $R$ is a left Ore domain if for $a, b \in R$, there exist $a_1, b_1 \in R \setminus \{0\}$ such that $b_1 a = a_1 b$.

Theorem 2.2. *Let $n \geqslant 1$. If $A$ is a semiprime $f$-ring, then $A$ has a (semiprime) $n$-convexity cover if and only if $A$ can be embedded in a direct product of totally ordered division rings.*

*Proof.* Suppose $A$ has a semiprime $n$-convexity cover $B$. By (1.3), $B$ is a subdirect product of totally ordered domains which satisfy the left $n$th-convexity property. It follows that each of these totally ordered domains is a left Ore domain and hence is embeddable in a totally ordered division ring. ∎

In [J, II 6.1], D. Johnson gives an example of a totally ordered $l$-simple domain that cannot be embedded in a totally ordered division ring. So not every totally ordered domain has an $n$-convexity cover.

However, the last theorem does imply that every semiprime commutative $f$-ring has an $n$-convexity cover. Next we ask, for a semiprime commutative $f$-ring is there a minimal such cover, and if there is, does it enjoy a universal mapping property? To facilitate this discussion we make the following definitions.

2.3. DEFINITIONS. Let $n \geqslant 1$ and $A$ be an $f$-ring.

(1) An $n$-convexity cover $K_nA$ of $A$ with $e\colon A \to K_nA$ is a minimal $n$-convexity cover of $A$ if whenever $\phi\colon A \to B$ is an embedding into a semiprime $f$-ring $B$ which satisfies the $n$th-convexity property, there is an embedding $\bar{\phi}\colon K_nA \to B$ such that $\phi = \bar{\phi} \circ e$.

(2) Suppose $K_nA$ is a minimal $n$-convexity cover of $A$ with $e\colon A \to K_nA$. Then $K_nA$ satisfies the universal mapping property if whenever $\phi\colon A \to B$ is a homomorphism of $A$ into a semiprime $f$-ring $B$ satisfying the $n$th-convexity property, there is a homomorphism $\bar{\phi}\colon K_nA \to B$ such that $\bar{\phi} \circ e = \phi$.

For a commutative semiprime $f$-ring, we will always be able to find a minimal $n$-convexity cover if $n \geqslant 2$. If $n = 1$, the problem is not as easy.

THEOREM 2.4. *Let $n \geqslant 2$ and $A$ be a commutative semiprime $f$-ring. Then there is a unique (up to isomorphism) commutative semiprime $f$-ring $K_nA$ which is a minimal $n$-convexity cover of $A$, and which satisfies the universal mapping property. If $A$ is a subdirect product of the totally ordered domains $A_i$ and $\Pi Q(A_i)$ denotes the direct product of the quotient fields $Q(A_i)$, then $K_nA$ is isomorphic to a unique sub-$f$-ring of $\Pi Q(A_i)$. Moreover, if $A$ is a direct sum (direct product) of the $A_i$, then $K_nA$ is a direct sum (direct product) of the $K_n(A_i)$, the minimal convexity covers of the $A_i$.*

Portions of the proof will be separated out and stated in the following lemmas.

LEMMA 2.5. *Let $n \geqslant 2$ and $\{A_i : i \in I\}$ be a collection of $f$-rings contained in the semiprime $f$-ring $A$. If each $A_i$ satisfies the* nth-*convexity property, then $\bigcap \{A_i : i \in I\}$ satisfies the* nth-*convexity property.*

*Proof.* Suppose $0 \leqslant u \leqslant v^n$ and $v \geqslant 0$ in $\bigcap \{A_i : i \in I\}$. Then $0 \leqslant u \leqslant v^n$ and $v \geqslant 0$ in $A$ and in each $A_i$. By (1.2), there is a unique element $w \in A$ such that $0 \leqslant w \leqslant v^{n-1}$ and $u = wv$. Since each $A_i$ satisfies the $n$th-convexity property, $w \in \bigcap \{A_i : i \in I\}$. ∎

LEMMA 2.6. *Let $n \geqslant 2$ and let $B$ be an $n$-convexity cover of the $f$-ring $A$ with embedding $e\colon A \to B$. If $B$ is the convex sub-$l$-ring of $B$ generated by $e(A)$ then the following hold.*

(1) *For every $l$-ideal $I$ of $A$, $\langle e(I) \rangle \cap e(A) = e(I)$.*

(2) *For every semiprime $l$-ideal $I$ of $A$, $\sqrt{\langle e(I) \rangle} \cap e(A) = e(I)$, where $\sqrt{\langle e(I) \rangle}$ denotes the smallest semiprime $l$-ideal of $K_nA$ containing $e(I)$.*

*Moreover, if $B$ is a minimal $n$-convexity cover of $A$, or if $C$ is a semiprime $n$-convexity cover of $A$ and $B$ is the intersection of all the sub-$f$-rings of $C$*

*which satisfy the* $n$th-*convexity property and which contain* $e(A)$, *then* $B$ *satisfies the* $n$th-*convexity property and* $B$ *is the convex sub-l-ring of* $B$ *generated by* $e(A)$.

*Proof.* (1) The fact that $\langle e(I)\rangle \cap e(A) = e(I)$ follows easily from the hypothesis.

(2) Suppose that $a \in \sqrt{\langle e(I)\rangle} \cap e(A)$. Then $a^m \in \langle e(I)\rangle \cap e(A)$ for some $m$. But by (1), $\langle e(I)\rangle \cap e(A) = e(I)$, so $a^m \in e(I)$. Hence $a \in e(I)$.

Now suppose either that $B$ is a minimal $n$-convexity cover of $A$, or that $C$ is a semiprime $n$-convexity cover of $A$ and $B$ is the intersection of all the sub-$f$-rings of $C$ which satisfy the $n$th-convexity property and which contain $e(A)$. Let $B' = \{b \in B\colon |b| \leqslant e(a)$ for some $a \in A^+\}$. Then $B'$ is a sub-$f$-ring of $B$. Suppose $v \geqslant 0$ and $0 \leqslant u \leqslant v^n$ in $B'$. Then $v \leqslant e(a)$ for some $a \in A^+$. Also, there is a $w \in B$ such that $u = wv$ and $0 \leqslant w \leqslant v^{n-1}$. So $0 \leqslant w \leqslant v^{n-1} \leqslant e(a^{n-1})$, which implies $w \in B'$. Thus $B'$ satisfies the $n$th-convexity property. By hypothesis, $B$ either is embedded in or is contained in $B'$. ∎

LEMMA 2.7. *Let* $n \geqslant 2$ *and suppose* $A$ *is a sub-f-ring of an f-ring* $B$ *which satisfies the* $n$th-*convexity property. Suppose* $I \subseteq A$ *is a semiprime l-ideal in* $B$. *Then if* $A/I$ *satisfies the* $n$th-*convexity property,* $A$ *also satisfies the* $n$th-*convexity property.*

*Proof.* Suppose $0 \leqslant u \leqslant v^n$ and $v \geqslant 0$ in $A$. Then there is a $w \in B$ such that $u = wv$ and $0 \leqslant w \leqslant v^{n-1}$. Now $0 \leqslant I(u) \leqslant I(v^n)$ in $A/I$. So there is an element $w' \in A$ such that $I(u) = I(w'v)$ and $0 \leqslant I(w') \leqslant I(v^{n-1})$. Since $I$ is semiprime, $B/I$ is semiprime. In $B/I$, $I(u) = I(wv)$ with $0 \leqslant I(w) \leqslant I(v^{n-1})$ and at the same time, $I(u) = I(w'v)$ with $0 \leqslant I(w') \leqslant I(v^{n-1})$. By (1.2), $I(w) = I(w')$. That is, $w = w' + b$ for some $b \in I \subseteq A$. Therefore $w \in A$. ∎

We now give the proof of Theorem 2.4.

*Proof.* Let $\{I_i\colon i \in \Gamma\}$ denote the collection of all proper prime $l$-ideals in $A$. Then $A$ is a subdirect product of the totally ordered domains $A/I_i$. So there is an embedding $e\colon A \to \Pi Q(A/I_i)$ given by $[e(a)]_i = I_i(a)$. Note that $\Pi Q(A/I_i)$ is a semiprime $f$-ring satisfying the $n$th-convexity property. Let $K_n A$ be the intersection of all sub-$f$-rings of $\Pi Q(A/I_i)$ which contain $e(A)$ and which satisfy the $n$th-convexity property. By Lemma 2.5, $K_n A$ satisfies the $n$th-convexity property.

Now suppose that $\phi\colon A \to B$ embeds $A$ into a semiprime $f$-ring $B$ satisfying the $n$th-convexity property. Let $C$ be the intersection of all sub-$f$-rings of $B$ which contain $\phi(A)$ and which satisfy the $n$th-convexity property. By 2.6 $C$ satisfies the $n$th-convexity property, and $C$ is the convex sub-$l$-ring of $C$ generated by $\phi(A)$. There is no harm in assuming that $C = B$. Let

$\{J_j : j \in \Sigma\}$ denote the collection of all proper prime $l$-ideals in $B$. There is a natural embedding $e' : B \to \Pi Q(B/J_j)$ given by $[e'(b)]_j = J_j(b)$. Define a mapping $\psi : \Pi Q(A/I_i) \to \Pi Q(B/J_j)$ by the following. For each $j \in \Sigma$ there exists $k \in \Gamma$ with $J_j \cap \phi(A) = \phi(I_k)$ since $B$ is the convex sub-$l$-ring generated by $\phi(A)$. Then $\phi$ induces mappings $\phi_j : Q(A/I_k) \to Q(B/J_j)$, and the $\phi_j$ induce a mapping $\psi : \Pi Q(A/I_i) \to \Pi Q(B/J_j)$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[rrr, bend left, "\phi"] \arrow[dr] & \Pi Q(A/I_i) \arrow[r, "\psi"] \arrow[d] & \Pi Q(B/J_j) \arrow[d] & B \arrow[l, "e'"'] \arrow[dl] \\
 & Q(A/I_i) \arrow[r, "\phi_j"] & Q(B/J_j) &
\end{tikzcd}$$

We now have embeddings defined so that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, hook, "e"] \arrow[d, hook, "\phi"'] & \Pi Q(A/I_i) \arrow[d, hook, "\psi_j"] \\
B \arrow[r, hook, "e'"'] & \Pi Q(B/J_j)
\end{tikzcd}$$

But $e'(B) \supseteq e' \circ \phi(A) = \psi \circ e(A)$ and satisfies the $n$th convexity property. Therefore $\psi(K_n A) \subseteq e'(B)$. Thus there is an embedding of $K_n A$ into $B$. So $K_n A$ is a minimal $n$-convexity cover of $A$.

Next, we show that this minimal cover is unique up to isomorphism. Suppose that $C$ also is a minimal $n$-convexity cover of $A$ and let $e_1 : A \to C$ be an embedding. Then there is an embedding $\gamma : C \to K_n A$ such that $\gamma \circ e_1(A) = e(A)$. But $K_n A$ is a sub-$f$-ring of $\Pi Q(A/I_i)$, so we may consider $\gamma$ to map $C$ into $\Pi Q(A/I_i)$. Now $\gamma(C)$ satisfies the $n$th-convexity property and contains $e(A) = \gamma \circ e_1(A)$. So $\gamma(C) \subseteq K_n A \subseteq \gamma(C)$.

Next we show that $K_n A$ satisfies the universal mapping property. Suppose $\phi : A \to B$ is an $l$-homomorphism into a semiprime $f$-ring satisfying the $n$th-convexity property. Let $I = \ker \phi$ and $I^* = \sqrt{\langle e(I) \rangle}$ be the smallest semiprime $l$-ideal of $K_n A$ containing $e(I)$. By Lemma 2.6, $I^* \cap e(A) = e(I)$.

Note that $A/I$ is a semiprime commutative $f$-ring and so we may consider $K_n(A/I)$. We will show that $K_n(A/I) \cong (K_n A)/I^*$. Since $I^* \cap e(A) = e(I)$, $A/I$ is embedded in $(K_n A)/I^*$. So there are embeddings such that $A/I \to K_n(A/I) \to (K_n A)/I^*$. Thus there is a sub-$f$-ring $C \supseteq I^*$ of $K_n A$ such that $C/I^* \cong K_n(A/I)$ and $e(A) \subseteq C$. By Lemma 2.7, $C$ satisfies the

$n$th-convexity property. We have $e(A) \subseteq C \subseteq K_nA$ and $C$ satisfies the $n$th-convexity property. By our choice of $K_nA$, $C = K_nA$, and $K_n(A/I) \cong C/I^* \cong (K_nA)/I^*$.

Since $K_n(A/I)$ is a minimal $n$-convexity cover of $A/I$, there is an embedding $\gamma_2 \colon K_n(A/I) \to B$ such that the diagram commutes.

$$\begin{tikzcd} A/I \arrow[r, hook] \arrow[dr, hook] & K_n(A/I) \cong (K_nA)/I^* \arrow[d, hook', "\gamma_2"] \\ & B \end{tikzcd}$$

Let $\gamma_1 \colon K_nA \to (K_nA)/I^*$ be the natural $l$-homomorphism and $\gamma = \gamma_2 \circ \gamma_1$. Then $\gamma \colon K_nA \to B$ and the following diagrams commute.

$$\begin{tikzcd} A \arrow[r, hook, "e"] \arrow[d] & K_nA \arrow[d, "\gamma_1"] \\ A/I \arrow[d, hook] & (K_nA)/I^* \arrow[dl, hook', "\gamma_2"] \\ B & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook, "e"] \arrow[d, "\phi"'] & K_nA \arrow[dl, "\gamma"] \\ B & \end{tikzcd}$$

The proofs of the remaining assertions of the theorem are routine and omitted. ∎

*Remarks.* (1) In different terms, Theorem 2.4 states that $K_n$ is a functor which preserves monics in the category of commutative semiprime $f$-rings and $e \colon I \to K_n$ is a monic natural transformation.

(2) Theorem 2.4 is easily generalized to hold under the hypothesis that $A$ is a semiprime $f$-ring for which every prime $l$-homomorphic image is a left Ore domain.

This argument may not be used to obtain a minimal 1-convexity cover for arbitrary commutative semiprime $f$-rings since in that case, we may not apply Lemma 2.5. That is, we do not have a result stating that in a semiprime $f$-ring with the 1st-convexity property, the intersection of all sub-$f$-rings satisfying the 1st-convexity property also satisfies the 1st-convexity property. The reason we do not have such a result is that (1.2) does not hold for the 1st-convexity property. When $0 \leqslant u \leqslant v$ in an $f$-ring satisfying the 1st-convexity property, there is not necessarily a unique element $w$ such that $u = wv$ or even a unique element $w$ such that $0 \leqslant w \leqslant 1$ and $u = wv$ when an identity element is present.

The following theorem gives a condition under which a minimal 1-convexity cover exists and under which the minimal 1-convexity cover is the same as the minimal $n$-convexity cover for $n \geqslant 2$. Recall that given a commutative $f$-ring $A$ and a subset $S$ without zero divisors, there is a $f$-ring $A_s$, called the localization of $A$ at $S$, and an embedding $\lambda: A \to A_s$ such that (i) for every $s \in S$, $\lambda(s)$ is invertible in $A_s$, and (ii) for any $l$-homomorphism $\phi: A \to B$ mapping $A$ into an $f$-ring $B$ such that every $\phi(s)$ is invertible, there exists an $l$-homomorphism $\bar{\phi}: A_s \to B$ such that $\bar{\phi} \circ \lambda = \phi$.

THEOREM 2.8. *Let $A$ be a commutative $f$-ring with identity element in which every finitely generated ideal of $A$ is principal. If $S = \{s \in A: s \geqslant 1\}$, then $A_s$, the localization of $A$ at $S$, satisfies the* 1st-*convexity property. Thus, for $n \geqslant 1$, $A_s$ is a minimal $n$-convexity cover of $A$ which satisfies the universal mapping property.*

*Proof.* Suppose $A_s$ is the localization of $A$ at $S$, and $\lambda: A \to A_s$ is the embedding. Suppose $0 \leqslant u \leqslant v$ in $A_s$. There is an $x \in \lambda(S)$ such that $xu$, $xv \in \lambda(A)$. By hypothesis, $(xu, xv)_{\lambda(A)} = (d)_{\lambda(A)}$ for some $d \in \lambda(A)$. We may assume $xv \neq 0$, $d \neq 0$. So there are $p, q, r, s \in \lambda(A)$ such that $xu = pd$, $xv = qd$, and $rxu + sxv = d$. Let $I = \{a \in \lambda(A): ad = 0\}$. Then $I$ is a semiprime $l$-ideal of $\lambda(A)$. Now $(rp + sq) - 1 \in I$, so $I(rp + sq) = I(1)$. Also, $|p| - |p| \wedge |q| \in I$, so $I(|p|) \leqslant I(|q|)$. Thus $I(1) = I(rp + sq) \leqslant I((|r| + |s|)|q|)$. This implies there is an $i \in I$ such that $1 \leqslant (|r| + |s|)|q| + i$. Hence $((|r| + |s|)|q| + i)^{-1} \in A_s$. So in $A_s$, $xu = |p|\,|d| = |p|((|r| + |s|)|q| + i)^{-1}((|r| + |s|)|q| + i)|d| = |p|((|r| + |s|)|q| + i)^{-1}(|r| + |s|)|qd| = |p|((|r| + |s|)|q| + i)^{-i}(|r| + |s|)xv$. So there is an element $w \in A_s$ such that $xu = wxv$. But $x^{-1} \in A_s$, so $u = wv$. Therefore $A_s$ satisfies the 1st-convexity property.

If $n \geqslant 1$ and $\phi: A \to B$ is a homomorphism into an $f$-ring $B$ satisfying the $n$th-convexity property, then for every $s \in S$, $\phi(s)$ is invertible in $B$. Hence there exists a homomorphism $\bar{\phi}: A_s \to B$ such that $\bar{\phi} \circ \lambda = \phi$. ∎

## III

In this section, we give two results whose proofs use Lemma 2.6 but whose statements do not involve any of the convexity properties. This application will show how the $n$th-convexity property can be used in problems that do not originally mention it. In this section, we assume that $A$ has a identity element (in addition to the assumption that $A$ is commutative and semiprime).

An ideal $I$ in a ring $A$ is *primary* if $ab \in I$, and $a \notin I$ implies $b^n \in I$ for some positive integer $n$. Primary ideals in $C(X)$ have been studied by L. Gillman and C. Kohls in [GK] and by C. Kohls in [K]. The problem of identifying $l$-ideals which are intersections of primary ideals has been studied by

R. D. Williams in [W]. There he investigates necessary and sufficient conditions for an $l$-ideal of $C(X)$ to be an intersection of primary ideals. Recall that if $I$, $J$ are ideals of a ring then $I: J = \{a \in A: aJ \subseteq I\}$. We will generalize some of his results to show that in a commutative semiprime $f$-ring with identity element, if an $l$-ideal $I$ satisfies $I = \langle I \sqrt{I} \rangle$ or $I = I: \sqrt{I}$, then $I$ is an intersection of primary $l$-ideals. As a corollary, we show that if $I$ is a pseudoprime $l$-ideal satisfying $I = \langle I \sqrt{I} \rangle$ or $I = I: \sqrt{I}$, then $I$ is primary.

First, we need some facts concerning primary $l$-ideals in semiprime commutative $f$-rings satisfying a convexity property. An $f$-ring $A$ with identity is said to satisfy the *bounded inversion property* if $a \geqslant 1$ in $A$ implies $a^{-1} \in A$. By (1.4), an $f$-ring with identity element satisfying the $n$th-convexity property also satisfies the bounded inversion property. For $C(X)$, the result of the next lemma appears in [GK, 4.6]. The result holds in the more general context given next, and we omit the proof.

LEMMA 3.1. *Let $A$ be a commutative $f$-ring with identity element which satisfies the bounded inversion property. Let $P$ be a prime $l$-ideal of $A$. If $a$ is a positive nonunit of $A/P$, then*

$$mP|^{a} = \{b \in A/P: |b|^{m} < a^{m-1} \text{ for all } m \in \mathbf{N}\}$$

*and*

$$mP|_{a} = \{b \in A/P: |b|^{m} \leqslant a^{m+1} \text{ for some } m \in \mathbf{N}\}$$

*are primary $l$-ideals of $A/P$, and $a \in mP|^{a}$, $a \notin mP|_{a}$.*

Suppose $A$ is an $f$-ring satisfying the hypotheses of Lemma 3.1 and $P$ is a prime $l$-ideal of $A$. For each primary $l$-ideal $mP|^{a}$ (respectively $mP|_{a}$) of $A/P$, we may associate a primary $l$-ideal of $A$, namely $\{b \in A: P(b) \in mP|^{a}\}$ (respectively $\{b \in A: P(b) \in mP|_{a}\}$). We will denote these by $P|^{f}$ and $P|_{f}$, respectively, where $f \in A$ is an element such that $P(f) = a$.

Recall that a *pseudoprime* ideal $I$ is an ideal with the property that $xy = 0$ implies $x \in I$ or $y \in I$. Part (1) of the next lemma has been shown by H. Subramanian in [Su]. The result of Part (2) is shown to hold in a $C(X)$ by L. Gillman and C. Kohls in [GK]. However, their proof is valid for any semiprime $f$-ring with identity element.

LEMMA 3.2. *Let $A$ be a commutative semiprime $f$-ring with identity element.*

(1) *An $l$-ideal $I$ is pseudoprime if and only if it contains a prime $l$-ideal.*

(2) *An $l$-ideal $I$ is an intersection of pseudoprime $l$-ideals.*

We are now ready to give two results concerning $I\sqrt{I}$ and $I\colon\sqrt{I}$ in a commutative semiprime $f$-ring $A$ with identity element which satisfies the 2nd-convexity property. R. D. Williams has shown that $I\sqrt{I}$ is an intersection of primary $l$-ideals in [W, 2.8], and our first proof will mimic his.

THEOREM 3.3. *Suppose $A$ is a semiprime commutative $f$-ring with identity element satisfying the* 2nd-*convexity property and $I$ is an $l$-ideal of $A$. Then if $I=I\sqrt{I}$, it is an intersection of primary $l$-ideals.*

*Proof.* Let $f\in A\backslash I\sqrt{I}$. We will show there is a primary $l$-ideal that contains $I\sqrt{I}$ but not $f$. By 3.2, there is a pseudoprime $l$-ideal $Q$ containing $I=I\sqrt{I}$ but not $f$. Now let $P$ be a prime $l$-ideal contained in $Q$ (by 3.2), and let $M$ be the maximal $l$-ideal in which $P$ is contained. If $f\notin M$, then $M$ is a prime $l$-ideal containing $Q$, and hence $I\sqrt{I}$, but not $f$. Suppose now that $f\in M$. Then $P(|f|)$ is a nonunit of $A/P$. Now the $l$-ideals containing $P$ form a chain, and $f\in P|^{|f|}$ while $f\notin Q$. So $Q\subseteq P|^{|f|}$. Thus $I\subseteq P|^{|f|}$. We now show that $I\sqrt{I}\subseteq P|_{|f|}$. Suppose that $g\in I$, $h\in\sqrt{I}$. Then there is some $k\in\mathbf{N}$ such that $h^k\in I$. Also, since $I\subseteq P|^{|f|}$, $P(|g|^m)<P(|f|^{m-1})$ and $P(|h|^{km})<P(|f|^{m-1})$ for all $m\in\mathbf{N}$. Thus $P(|gh|^{k(k+2)})=P(|g|^{k(k+2)})$ $P(|h|^{k(k+2)})\leqslant P(|f|^{k(k+1)})P(|f|^{k+1})=P(|f|^{k(k+2)+1})$. So $gh\in P|_{|f|}$. ∎

THEOREM 3.4. *Let $n\geqslant 1$. Suppose $A$ is a semiprime commutative $f$-ring with identity element satisfying the* $n$th-*convexity property, and $I$ is an $l$-ideal of $A$. Then for any $x\in A\backslash(I\colon\sqrt{I})\colon\sqrt{I}$ there is a primary $l$-ideal which contains $I\colon\sqrt{I}$ but not $x$.*

*Proof.* Since $x\notin(I\colon\sqrt{I})\colon\sqrt{I}$, there is a $g\geqslant 0$ in $\sqrt{I}$ such that $xg\notin I\colon\sqrt{I}$. This implies that there is an $h\geqslant 0$ in $\sqrt{I}$ such that $xgh\notin I$. Let $f=g\vee h$. Then $f\in\sqrt{I}$ and $xf^2\notin I$. By 3.2, there is a pseudoprime $l$-ideal $Q$ containing $I\colon\sqrt{I}$ but not $xf$. Now let $P$ be a prime $l$-ideal contained in $Q$ and let $M$ be the maximal ideal containing $P$. If $x\notin M$, then $M$ is a prime $l$-ideal containing $Q$, and therefore containing $I\colon\sqrt{I}$, but not containing $x$. Suppose now that $x\in M$. The $l$-ideals containing $P$ form a chain, and $xf\in P|^{|x|f}$ while $xf\notin Q$. So $Q\subseteq P|^{|x|f}$. Thus $I\colon\sqrt{I}\subseteq P|^{|x|f}$.

Let $k$ be the smallest integer such that $f^k\in I$. Since $x\notin Q$, $x\notin P+I$. Since $A/P$ is totally ordered, $P(|x|)>P(f^k)$. So $P(|x|)^{k+1}>P(|x|f)^k$ and therefore, $x\notin P|^{|x|f}$. ∎

An $l$-ideal $I$ of an $f$-ring $A$ is *square dominated* if $I=\{a\in A\colon |a|\leqslant x^2$ for some $x\in A$ such that $x^2\in I\}$. A slight modification of this proof shows that if $A$ is a semiprime commutative $f$-ring satisfying the $n$th-convexity property with identity element, and $\sqrt{I}$ is a square dominated $l$-ideal of $A$, then $I\colon\sqrt{I}$ is an intersection of primary $l$-ideals.

We are now ready to prove our main result of this section.

THEOREM 3.5. *Let $A$ be a commutative semiprime $f$-ring with identity element and suppose $I$ is an $l$-ideal of $A$. Then if $I = \langle I\sqrt{I}\rangle$ or if $I = I : \sqrt{I}$, $I$ is an intersection of primary $l$-ideals.*

*Proof.* Let $B$ be a commutative semiprime minimal 2-convexity cover of $A$ with the identity element 1, and with the embedding $e: A \to B$. By 2.6, $B$ is the convex sub-$l$-ring of $B$ generated by $e(A)$. In $B$, let $J$ be the $l$-ideal generated by $e(I)$. By 2.6(1), $J \cap e(A) = e(I)$.

Suppose first that $I = \langle I\sqrt{I}\rangle$. Then $J = J\sqrt{J}$. By Theorem 3.3, $J$ is an intersection of primary $l$-ideals $Q_i$ in $B$. Now $Q_i \cap e(A)$ are primary $l$-ideals of $e(A)$ and so $e(I) = e(\langle I\sqrt{I}\rangle) = J \cap e(A) = (\bigcap Q_i) \cap e(A) = (Q_i \cap e(A))$. Therefore $I$ is an intersection of primary $l$-ideals.

Next, suppose that $I = I : \sqrt{I}$. Let $e(a) \in (J : \sqrt{J}) : \sqrt{J} \cap e(A)$. Then for any $b, c \in \sqrt{I}$, $e(a)e(b)e(c) \in J \cap e(A) = e(I)$. Thus, $e(a) \in e((I : \sqrt{I}) : \sqrt{I}) = e(I : \sqrt{I}) = e(I)$. We now have $(J : \sqrt{J}) : \sqrt{J} \cap e(A) \subseteq e(I)$. Clearly, the reverse inclusion also holds, and $(J : \sqrt{J}) : \sqrt{J} \cap e(A) = e(I)$.

For any $a \in B \backslash (J : \sqrt{J}) : \sqrt{J}$, there is a primary $l$-ideal $Q_i$ of $B$ which contains $J : \sqrt{J}$ but not $a$ by Theorem 3.4. Now $Q_i \cap e(A)$ are primary $l$-ideals of $e(A)$. So $e(I) = e(I : \sqrt{I}) = J \cap e(A) \subseteq J : \sqrt{J} \cap e(A) \subseteq (\bigcap Q_i) \cap e(A) = \bigcap (Q_i \cap e(A)) \subseteq (J : \sqrt{J}) : \sqrt{J} \cap e(A) = e(I)$. Thus $e(I) = \bigcap (Q_i \cap e(A))$ and $I$ is an intersection of primary $l$-ideals. ∎

COROLLARY 3.6. *Let $A$ be a commutative semiprime $f$-ring with identity element. If $I$ is a pseudoprime $l$-ideal that is an intersection of primary $l$-ideals, then $I$ is itself primary. Thus, if $I$ is a pseudoprime $l$-ideal satisfying $I = \langle I\sqrt{I}\rangle$ or $I = I : \sqrt{I}$, $I$ is a primary $l$-ideal.*

*Proof.* Suppose $I$ is a pseudoprime $l$-ideal that is an intersection of the primary $l$-ideals $Q_i$. Then $I$ contains a prime $l$-ideal and hence the set of all $l$-ideals containing $I$ form a chain. Now if $Q_i \supseteq \sqrt{I}$ for all $i$, then $\sqrt{I} \subseteq \bigcap Q_i = I$. Hence $I$ is semiprime and pseudoprime and therefore prime. We now may assume there is some $\alpha$ such that $Q_\alpha \subset \sqrt{I}$. Suppose that $ab \in I$ and $a \notin I$. There is some $\beta$ such that $a \notin Q_\beta \subseteq Q_\alpha \subseteq \sqrt{I}$. Since $ab \in Q_\beta$ and $a \notin Q_\beta$, $b \in \sqrt{Q_\beta} \subseteq \sqrt{I}$. Thus $I$ is primary. ∎

Finally, we give an example showing that an $l$-ideal $I$ with the property $I = \langle I\sqrt{I}\rangle$ or $I = I : \sqrt{I}$ is not the only type of ideal that is an intersection of primary $l$-ideals in a commutative semiprime $f$-ring with identity element. Another such example (which is not as simple) is given in [W, 2.11]. The $f$-ring described in this example was first given in [HP, 4.16].

EXAMPLE 3.7. In $C([0, 1])$, denote by $i$ the function $i(x) = x$, by $e$ the function $e(x) = 1$, and let $w = \sqrt{i}$. Let $\langle i\rangle$ denote the $l$-ideal of $C([0, 1])$

generated by $i$, and let $A = \{f \in C([0,1]) : f = ae + bw + g;\ g \in \langle i \rangle,\ a, b \in \mathbf{R}\}$. Give $A$ the inherited (componentwise) addition, multiplication, and ordering. Then it can be shown that $A$ is an $f$-ring. Also, $A$ is commutative, semiprime, and possesses an identity element.

Let $I = \{ae + bw + g \in A:\ a = b = 0\}$. Then $I$ is an $l$-ideal of $A$. Simple calculations show that $I$ is primary. Note that $\sqrt{I} = \{ae + bw + g \in A:\ a = 0\}$. Then $\langle I\sqrt{I}\rangle \subseteq \{ae + bw + g \in A:\ a = b = 0,\ g \leqslant ni^{3/2}\} \subset I$. Also, $1w \in A$ and $(1w)\sqrt{I} \subseteq I$. This implies $1w \in I : \sqrt{I}$ and yet $1w \notin I$. So $I \subset I : \sqrt{I}$. Thus $\langle I\sqrt{I}\rangle \subset I \subset I : \sqrt{I}$. Note also that $I$ is pseudoprime and so the converse to the corollary is also false.

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