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Adaptive Control for Nonlinear, Time Varying Systems

John Zelina
Embry-Riddle Aeronautical University, zelinaj@my.erau.edu

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By

John Zelina

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THESIS COMMITTEE

Richard J. Prazenica — Digitally signed by Richard J. Prazenica Date: 2021.04.30 11:18:37 -04'00'

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Troy A. Henderson — Digitally signed by Troy A. Henderson Date: 2021.04.30 10:28:01 -05'00'

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Marwan Al-Haik — Digitally signed by Marwan Al-Haik Date: 2021.04.30 15:00:06 -04'00'

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Dr. Marwan Al-Haik

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Date

Maj Mirmirani — Digitally signed by Maj Mirmirani Date: 2021.05.02 10:00:37 -04'00'

Dean of the College of Engineering,
Dr. Maj Mirmirani

5/2/2021

Date

Christopher Grant — Digitally signed by Christopher Grant Date: 2021.05.03 10:29:34 -04'00'

Associate Provost of Academic Support,
Dr. Christopher Grant

5/3/2021

Date

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ABSTRACT

It is common for aerospace systems to exhibit nonlinear, time varying dynamics. This thesis investigates the development of adaptive control laws to stabilize and control a class of nonlinear, time varying systems. Direct adaptive control architectures are implemented in order to compensate for time varying dynamics that could, for example, be caused by varying inertia resulting from fuel slosh or settling in a tank. The direct adaptive controller can also respond to external disturbances and unmodeled or nonlinear dynamics. Simulation results are presented for a prototype system that consists of two rotating tanks with time varying inertia due to the motion of fluid inside the tanks. This system is characterized by highly unstable rotational dynamics which are illustrated through simulation. An adaptive regulator is implemented to control the three-dimensional angular velocity to a desired operating point. It is shown that the adaptive controller provides improved performance compared to a baseline linear quadratic regulator designed using a simplified linear dynamics model of the plant. Finally, a direct model reference adaptive controller was implemented to enable the system to track trajectories generated by a reference model. The stability of this control law is investigated via Lyapunov analysis, and simulation results are provided showcasing overall controller performance in the presence of both internal and external disturbances and dynamical effects.

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1. Introduction

Chapter 1 will provide the necessary background for this thesis and provide a justification for the pursuit of this research. It is split into two sections. Section 1.1 will provide the motivation for pursuing this topic. Section 1.2 will provide a brief walkthrough of the thesis objectives, and provide a detailing of the structure of this thesis.

1.1. Motivation

When developing controls for simple systems such as Linear Time Invariant (LTI) systems, there is a myriad of techniques that can be deployed to produce acceptable levels of performance through a wide range of the system's operating envelope. However, it is common in aerospace applications to see systems exhibiting non-linear, time varying dynamics. Examples include the deployment of payloads or munitions, refueling scenarios, and the expenditure of fuel during flight. These types of scenarios are inertial changes that can be inimical to the desired translational and rotational dynamics. As one attempts to control more complex systems such as nonlinear time varying (NLTV) systems, the implementation of linear techniques will often result in undesired closed loop dynamics and inability to guarantee stability throughout the entire operational envelope. This is because when one applies linear techniques to a nonlinear system, the control law is tailored to a linearization point in the state space. Discretizing the operational envelope of such systems becomes more tedious as those systems become more complicated.

Development of highly dynamic plants, such as advanced unmanned aerial vehicles, hypersonic aircraft and space infrastructure applications creates systems with large performance envelopes whose characteristics may vary greatly throughout the use cycle.

Non-adaptive methods of control may prove inadequate to meet the demands of such systems; thus, more advanced methods of control are required in order to guarantee system stability and acceptable levels of performance for more complicated systems.

One such method of controlling these more complex systems is to create a simplified reference model, whose outputs are compared to the outputs of the plant. This provides a way to feed the controller a desired trajectory for the system to track. The comparison of the two is used to inform the control law and allows the controller gains to “adapt” to the often unknown and un-modeled time varying effects exhibited by the plant in real time. It also provides a way to cope with disturbances that may be inflicted upon the system. This approach is referred to as Model Reference Adaptive Control (MRAC).

1.2. Thesis Objectives

As will be discussed in later sections, there has been extensive research in the area of adaptive control for aerospace applications. However, much of this work falls into two distinct categories, linear systems with time varying effects, or nonlinear systems that are time invariant. The contributions that attempt to address both of these effects simultaneously are far sparser, and it is in this space that this document attempts to contribute.

In this thesis, the use of MRAC on a highly nonlinear, time varying system is explored. This will be accomplished by generating a candidate model system that exhibits both nonlinear and time varying dynamical behavior. This system will then be linearized in order to employ a linear quadratic regulator (LQR) as a baseline controller. A MRAC will then be generated to control this candidate system. Finally, simulations modeling possible operating regimes will be produced to demonstrate the overall effectiveness of MRAC for such systems.

This thesis will be presented in the following sections: Chapter 2 will contain a review of relevant literature including a sampling of knowledge on nonlinear and time varying systems, nonlinear control methods, and examples of the implementation of adaptive control. Chapter 3 will be a description of the methodology of achieving the goals described in the previous paragraph. This chapter will include the development of the candidate system, baseline controller, and adaptive controller. Chapter 4 will present simulation results that test the proposed adaptive controller on the candidate system. Finally, Chapter 5 will be a synthesis of the results and will interpret them in order to draw conclusions on this control strategy and provide paths for future research.

2. Review of the Relevant Literature

This chapter is intended as a primer to the concepts that are discussed later in this document as well as a sampling of related work in this area of research. The chapter is split into 3 sections. The first will discuss the concepts of nonlinear and time varying systems. The second section reviews some of the control methods that are commonly applied to these systems. Finally, Section 2.3 will discuss model reference adaptive control, the particular class of controller that is utilized in this thesis.

2.1. Nonlinear and Time Varying Systems

As stated in the introduction, few real world systems in aerospace exhibit linear, time invariant dynamical behavior. As this will prove to be an integral part of the work to follow, it is important to understand these concepts and their implications on system behavior and control. This section will be dedicated to providing a reference to these concepts.

Conceptually speaking, nonlinear systems are a group of systems that do not follow superposition. That is, if one were to scale the input to a given system, it would result in a proportional scaling of the output. A dynamical system is considered nonlinear if the time derivative of the states cannot be written as a linear combination of the states and inputs. Nonlinearities can take the form of coupled states, trigonometric functions and high order terms. Nonlinearities are common in real systems such as the equations of motion for fixed wing and rotor aircraft. A more complete description of nonlinear systems and their properties can be found in “Nonlinear Systems” by Khalil (2002).

Time varying systems are systems whose inherent parameters change with time. This includes qualities such as mass, moment of inertia and system form. Examples of time varying systems include plants that expend some form of fuel or change in shape thus affecting drag coefficient or other relevant parameters. In many cases, these changes are small enough that they can be ignored, such as the effect of changing mass in a car as it burns fuel. However, if these changes are drastic enough, they can have significant impacts on the dynamics of the system being considered.

2.2. Control Techniques for Nonlinear Systems

Outside of academic interest, there are relatively few systems of significance that happen to behave linearly. Almost all systems will exhibit nonlinearities, discontinuities, and other inconvenient effects. In the case of many systems however, linearization about some nominal operating point of interest is sufficient to capture the relevant dynamics and thus linear controls techniques may be utilized. If the system in question is highly nonlinear, or there are many operational conditions that the system must behave at, a simple linearized model of the system may not give an adequate description of the dynamics, and thus linear control techniques may prove ineffective. For such systems, there are many available nonlinear control methods. In this section, a subset of these methods including gain scheduling, sliding mode, $H\infty$, and model predictive control will be discussed in order to provide context of the nonlinear control landscape. While these will be discussed independently for the sake of structure, it is important to note that it is common to see some combination of these techniques in the literature.

One of the most common techniques for controlling nonlinear systems is gain scheduling. This is due to how conceptually simple this method is to implement, while still proving to be quite effective in real world applications. It requires two key parts: the first is a set of linearization points of interest, usually related to different operational points of the plant. The second is a logical operator that observes the system in order to determine which of these operational points the system is currently at, so the correct feedback gains will be used. As this is a popular method of control for nonlinear systems, in depth descriptions of its use are abundant in the literature (Leith, 2000). Gain scheduling is a common method of control for aerospace applications, but is limited in that it requires the discretization of a system's operational cycle and its ability to accommodate unmodeled dynamic effects such as external disturbances.

Another method of nonlinear control is sliding mode control. This is similar to gain scheduling as it targets discrete points in the state space of the system where it exhibits desirable dynamics. The control law changes based on which of these areas the plant is currently in where it attempts to force the system to "slide" along this subset of its dynamics. Like gain scheduling, this method also requires a system observer with some form of logical operator that switches the control law as the plant moves about the state space. As this method is so closely related to gain scheduling, it is common to see a hybrid of the two methods (Palm, 2001). Sliding mode control has also been used extensively in aerospace applications (Zou, 2017; Besnard, 2007). As discussed in the introduction to this document, the problem being addressed in this thesis is time varying as well. There are examples of sliding mode control for set NLTV systems in work conducted by Meza-Aguilar et al. (2019). Sliding mode control can be a very powerful tool, but still requires discretization of the plant's operational envelope. This can be very labor intensive in development, as it requires one to sort through the dynamics of the system at different conditions.

Another popular control strategy for nonlinear systems is H∞ control. This method takes the form of an optimization problem where the resulting control law is the solution that minimizes a user defined cost function. This method has been extensively explored for nonlinear and aerospace applications, as can be seen in work from Saat and Nguang (2014), as well as contributions from Raffo, Ortega, and Rubio (2010).

A final method of nonlinear control is nonlinear model predictive control (MPC). This method requires the projection of the plant dynamics into the future, and seeks to find the optimal control inputs over that time horizon to minimize a prescribed cost function. For nonlinear MPC, this is usually done by recursively generating numerical solutions to the optimization problem over the given projection time horizon. Aerospace examples of this method of control can be found throughout the literature including the work of Kang and Hedrick (2009), where nonlinear MPC is used to control a UAV to track a linear trajectory. As previously stated, many of these methods can be combined to form a control law, such as the work of Raffo et al. (2010) where MPC was used to make a quadrotor track a trajectory in conjunction with H∞ control to stabilize the plant's rotational dynamics. While there are mathematical simplifications that can be utilized when performing MPC, it still requires one to forecast future paths of the system, and thus can be computationally taxing for complex dynamical systems. This can make it difficult to implement as a real time control strategy.

2.3. Model Reference Adaptive Control

Model reference adaptive control (MRAC) refers to a class of adaptive controllers in which a reference plant, which usually represents a simplified model of the plant that is to be controlled, is used to generate trajectories for the plant to track. This can be accomplished in two different forms: direct MRAC and indirect MRAC. In the first case, the gains are left unrelated to the underlying parameters that are forcing a need for adaptation. This form of MRAC is simply concerned with mitigating tracking error. Indirect MRAC relates the adaptation of the gains to the unknown system parameters. Thus, indirect MRAC can be used for parameter ID if the system model is sufficiently accurate to the true system dynamics. In either case, the ultimate goal is to accurately track the desired trajectories provided by the reference model. In this thesis, direct model reference adaptive control is implemented, and thus will be the primary focus of this review.

Model reference adaptive control has received a significant amount of research interest especially in the era of unmanned systems. This is due to the resilience of the control architecture in its ability, as the name suggests, to adapt to unanticipated and unmodeled dynamical effects. This attribute provides opportunities for this controller to solve safety and reliability concerns for both civilian and military applications. One such example was demonstrated by Jourdan et al. (2010) in which MRAC was used for attitude control on a reduced scale model F-18 that was exposed to severe structural damage. These damages included missing or unresponsive control surfaces as well as up to 80% wing area detachment from one side. These theoretical results were extended to pitch and yaw axes, and flight tests were conducted with promising results. Similar tests were performed on a GT Twinstar using a MRAC for attitude control with a single layer neural network to update the control gains. In this case, 25% of the left wing was jettisoned mid-flight. These results were compared to a nominal test flight and shown to maintain similar performance (Chowdhary, 2013). In addition to structural damage conditions, publications can be found investigating the use of MRAC to accommodate actuator failures (Kutay, 2008; Idan, 2001). A derivative free update law has also been proposed, and is well suited to certain classes of plants with rapid dynamic changes (Yucelen, 2011; Yucelen, Calise, Nguyen, 2011). In all, there has been extensive research in the use of MRAC in applications where system survivability is at risk due to unforeseen failures.

In most of the previous examples, MRAC was developed from a linearized, time invariant model of the aircraft. However, research does exist for nonlinear plant models (Hovakimyan, 2002). In this case, the controller was implemented on both a Van der Pol oscillator and R-50 helicopter model. In another example, time varying effects are considered in the adaptive control formulation (Arabi, 2018). This paper also considered a nonlinear reference plant similar to a contribution by Yucelen et al. (2015), which allows more novel trajectories for highly maneuverable systems.

3. Methodology

In this chapter, the process of achieving the objectives set in Section 1.2 is discussed. The methodology chapter will be divided into 4 sections. The first will discuss the development of a candidate system that exhibits both time varying and nonlinear effects. This section also provides a mathematical model of the system dynamics, and graphically displays the system's behavior. Section 3.2 will discuss the development of the baseline LQR controller that will be used as a comparative tool to illustrate the need for adaptive control. Section 3.3 will show the development of the model reference adaptive controller. Finally Section 3.4 will provide a stability analysis of the proposed model reference adaptive controller.

3.1. Dynamical System Model

Consider a rotating system of changing moment of inertia as pictured Figure 3.1:

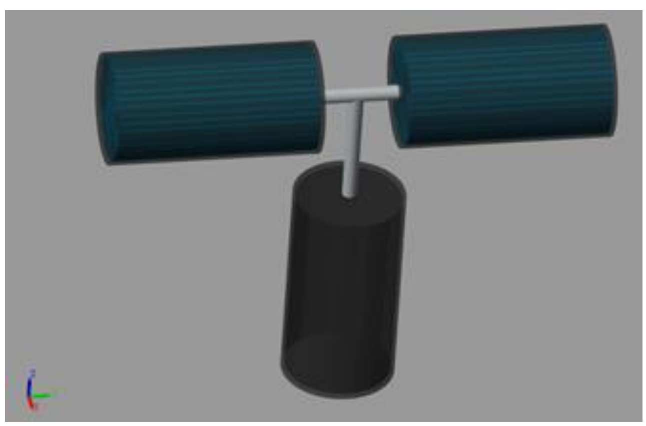

Figure 3.1 2-Tank System, Base Tank Translation

The system is intended to rotate about its z-axis. As it rotates, fluid in the upper tanks settles out to the ends, thus resulting in a time varying moment of inertia. A mathematical model of this system was generated using the definition of angular momentum for rigid bodies. Angular momentum about the center of mass, H_{cm}, is defined as follows:

$$H_{cm} = I_{cm}\omega$$

where I_{cm} is the positive definite 3 x 3 mass moment of inertia matrix computed with respect to body-fixed axes and ω is the three-dimensional angular velocity vector. Taking a time derivative of angular momentum results in:

$$\tau_{cm} = I_{cm}\dot{\omega} + \frac{{}^{B}d}{dt}(I_{cm})\omega + \omega \times I_{cm}\omega$$

Rearranging terms results in:

$$\dot{\omega} = I_{cm}^{-1}\left(\tau_{cm} - \frac{{}^{B}d}{dt}(I_{cm})\omega - \omega \times I_{cm}\omega\right) \quad (1)$$

where τ_{cm} is a control input torque that can be applied about all 3 axes, and ω, the 3 dimensional angular velocity vector, is equivalent to the state vector in this model. The derivative of the moment of inertia matrix is taken with respect to the body-fixed axes, and it is worth noting this term is not positive definite. It should be emphasized that the moment of inertia is time-varying due to the fluid motion in the tanks, which significantly affects the rotational stability of the system. Note that, in this model, only three states exist; for the purposes of this thesis, this will be sufficient to display results of the proposed control strategy. Angular position, therefore, will not be tracked.

A full 6 degree-of-freedom simulation was developed for a scale version of this two-tank rotational system under the following conditions. Figures 3.2 – 3.4 present results from a scenario in which the 2 12 inch long tanks contain fluid with density of 1000 kg/m^3. The system initially has zero angular velocity but is then subjected to a pulse torque input about the z-axis, which initiates rotational motion and induces fluid motion within the tanks. The results show that the uncontrolled rotational dynamics, as seen from the plots of angular velocity and Euler angles in Figures 3.2 and 3.3, show considerable instability. It should be noted that the angular velocity in Figure 3.2 is plotted with respect to inertial axes, which explains the periodic nature of the rotational rates in the x and y directions. The rotational instability also impacts the translational dynamics, as can be seen in the translational motion of the center of the base tank provided in Figure 3.4. Considering the inherent instability of this system and the underlying nonlinear, time-varying dynamics, the objective of this thesis is to design an adaptive control architecture to stabilize the rotational dynamics and control the angular velocity to track a desired trajectory.

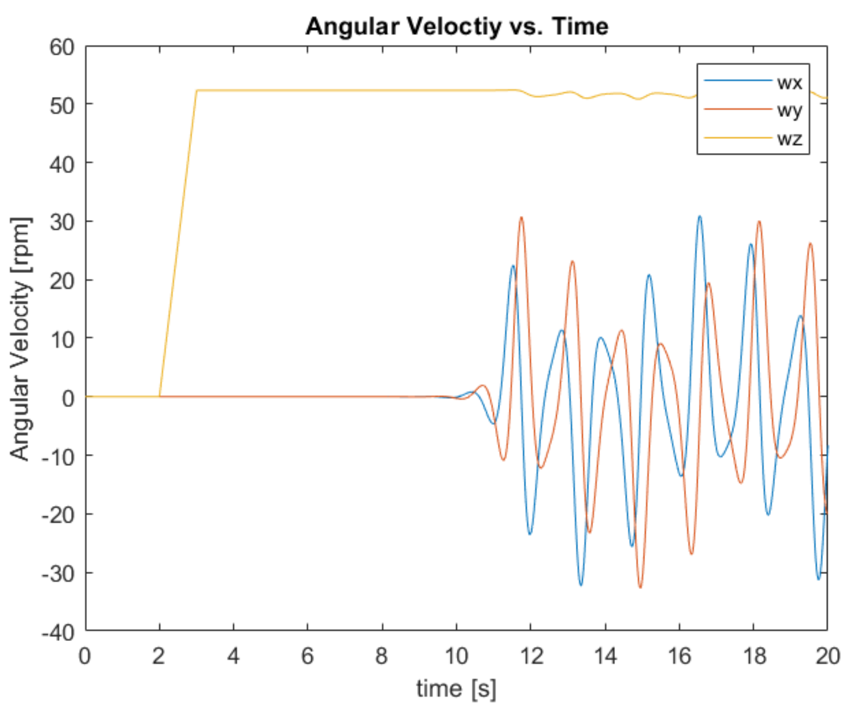

Figure 3.2 2-Tank System Angular Velocity (Inertial Axes)

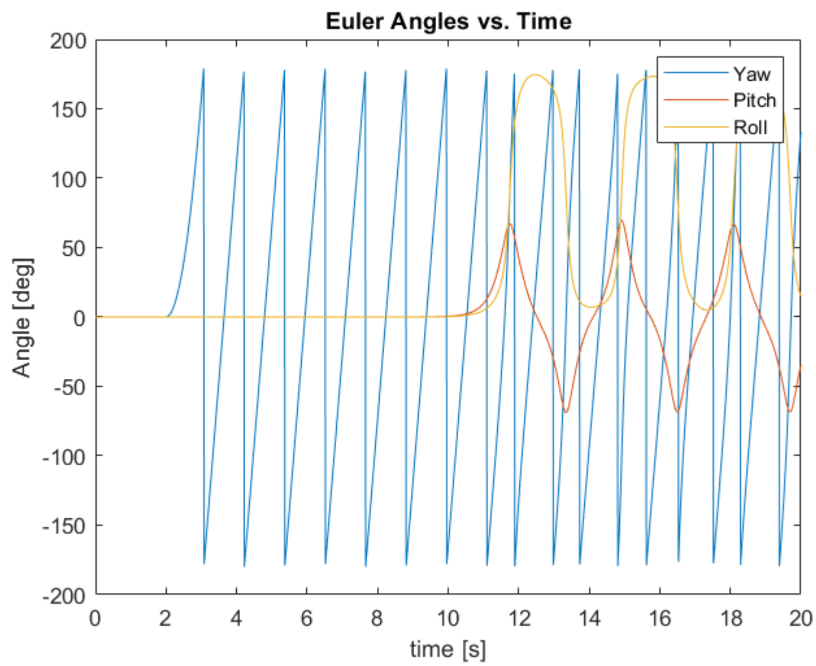

Figure 3.3 2-Tank System Euler Angles (3-2-1 Sequence)

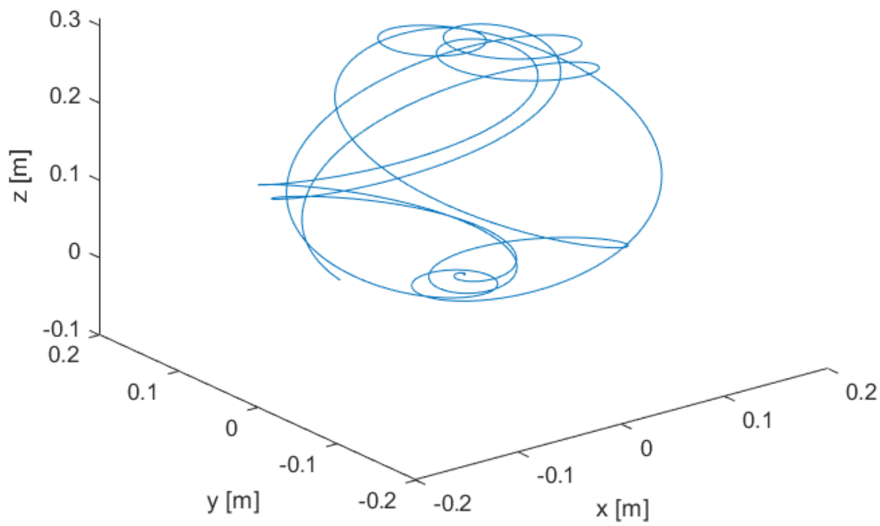

Figure 3.4 2-Tank System, Base Tank Translation

3.2. Baseline Controller Development

Using the rotational equation of motion derived previously, control laws can be constructed to mitigate undesired out-of-plane motion while maintaining the required rotational rate. Due to the time-varying nature of the described system, adaptive control techniques will likely be necessary to produce desired closed loop characteristics. However, a baseline LQR controller was developed using a linearized model of the system. This will later be used for comparison with the adaptive controller to verify its improved performance over the more simple techniques implemented in the baseline controller.

The classical linear quadratic regulator (LQR) addresses the following optimal control problem: Find the optimal control input $\underline{u}^*(t)$, $0 \le t \le T$, that minimizes the cost functional:

$$J = \tfrac{1}{2}\int_0^\infty \left\{ \underline{x}^T(t) Q \underline{x}(t) + \underline{u}^T(t) R \underline{u}(t) \right\} dt \tag{2}$$

where $Q \ge 0,\ R > 0$ (i.e., Q is a nxn symmetric, positive semi-definite matrix and R is a mxm symmetric, positive-definite matrix). These values act as weighting coefficients for the states and inputs within the cost function that is subject to the linear dynamical system model:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \qquad & \underline{x} \in \Re^n \\ \underline{x}(0) &= \underline{x}_0 \qquad & \underline{u} \in \Re^m \end{aligned} \tag{3}$$

where n is the number of states and m is the number of inputs, such that A is nxn and B is nxm. The cost function in Equation (2) is designed to drive the states to the origin while penalizing control effort. The optimal control law for the linear quadratic regulator is given by:

$$\underline{u}^*(t) = -K\underline{x}(t) \tag{4}$$

K is a mxn constant gain matrix computed as:

$$K = R^{-1}B^T S \tag{5}$$

where S is a nxn symmetric, positive-definite matrix obtained as the solution to the algebraic Riccati equation:

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \tag{6}$$

The classical LQR controller is only applicable to linear, time invariant systems. The candidate system meets neither of these criteria; thus some assumptions must be made. First, the time varying nature of the system will be addressed. This will be followed by a description of the linearization of the simplified time invariant system.

In order for LQR control to be applied to this system, a reduction in complexity is required. To this end, it is first assumed that the change in mass moment of inertia, defined in Equation (1) as $\frac{{}^B d}{dt}(I_{cm})$, is negligible and thus, the resulting equation of motion takes the form:

$$\dot{\omega} = I_{cm}^{-1}(\tau_{cm} - \omega \times I_{cm}\omega) \tag{7}$$

The LQR controller was designed using a linearized model of the nonlinear rotational dynamics Equation (7). The linearization was performed about a nominal equilibrium condition corresponding to zero angular velocity in the x and y directions, $\omega_x = \omega_y = 0$, and a constant angular velocity in the z direction, $\omega_z = \omega_*$, corresponding to a desired spin rate. All inputs are linearized about zero, such that $\tau_x = \tau_y = \tau_z = 0$. To complete this task, first the nonlinear equation is expanded:

$$\dot{\omega} = \begin{bmatrix} \frac{1}{I_x}(\omega_y\omega_z I_z - \omega_y\omega_z I_y) \\ \frac{1}{I_y}(-\omega_x\omega_z I_z + \omega_x\omega_z I_x) \\ \frac{1}{I_z}(\omega_x\omega_y I_y - \omega_x\omega_y I_x) \end{bmatrix} + \begin{bmatrix} \frac{1}{I_x}\tau_x \\ \frac{1}{I_y}\tau_y \\ \frac{1}{I_z}\tau_z \end{bmatrix}$$

A jacobian is then taken with respect to the states and inputs:

$$\frac{\partial\dot{\omega}}{\partial\omega} = -\begin{bmatrix} 0 & \frac{1}{I_x}(\omega_z I_z - \omega_z I_y) & \frac{1}{I_x}(\omega_y I_z - \omega_y I_y) \\ \frac{1}{I_y}(-\omega_z I_z + \omega_z I_x) & 0 & \frac{1}{I_y}(-\omega_x I_z + \omega_x I_x) \\ \frac{1}{I_z}(\omega_y I_y - \omega_y I_x) & \frac{1}{I_z}(\omega_x I_y - \omega_x I_x) & 0 \end{bmatrix}$$

$$\frac{\partial\dot{\omega}}{\partial\omega} = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

When the jacobian is evaluated about the linearization point, it yields a linear model of the form in Equation (3) with:

$$A = \begin{bmatrix} 0 & \frac{(I_y - I_z)}{I_x}\omega_* & 0 \\ \frac{(I_z - I_x)}{I_y}\omega_* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

such that the resulting linearized dynamical system takes the form:

$$\Delta\underline{\dot{x}}(t) = A\Delta x(t) + B\Delta u(t)$$

The states $\Delta x(t)$ and inputs $\Delta u(t)$ are now defined as deviations from their respective linearization points.

The LQR control law $\underline{u}^*(t) = -K\underline{x}(t)$ was then designed for the linear model using Equations (5) and (6), where the control inputs correspond to control torques that can be applied about all 3 axes, as would be the case with thrust control for example. The LQR controller was then implemented on the nonlinear rotational system as shown in Figure 3.5 below.

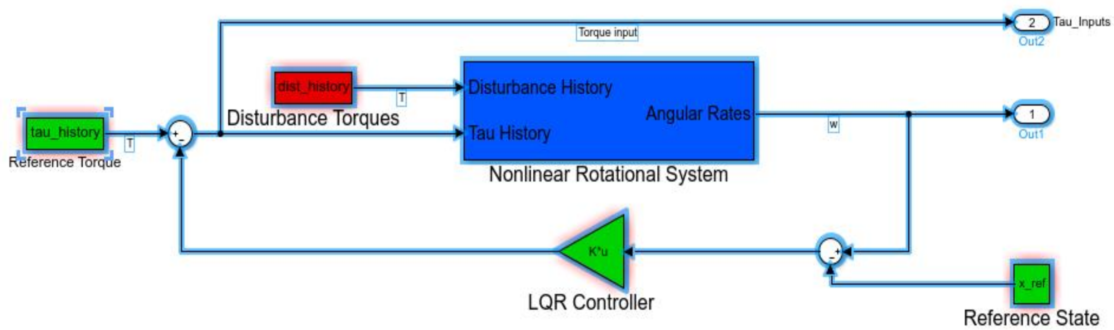

Figure 3.5: Block Diagram of State Feedback LQR controller

In general, it would be expected to see good performance of this control scheme when the states and inputs are relatively close to the linearization point. As the states and inputs deviate, the controller may struggle as it was formulated with a simplified model of the full dynamics of the system. It should also be noted that the linearization point is referred to as the reference state in the diagram, and must be subtracted from the true value of the states to produce a deviation from the linearization point, which is then fed to the control law.

3.3. Direct Model Reference Adaptive Control Development

The linear quadratic regulator controller is limited in its ability to compensate for external disturbances and time-varying dynamics. Therefore, to produce more desirable closed loop dynamics, adaptive control techniques must be utilized. The control design entails the design and implementation of a direct adaptive control system of the form depicted in Figure 3.6. This adaptive control architecture, which is based on implementations originally derived by Fuentes and Balas (2000) represents a form of direct model reference adaptive control that includes disturbance accommodation. In this implementation, the system is subjected to external disturbances, which are modeled as a linear combination of basis functions, which in turn can be used to construct a state space disturbance generator model. In this formulation, a linear reference model is used to provide reference trajectories for the adaptive control system to track. These reference trajectories correspond to desirable trajectories for system; for example, a first order response to some desired rotational rate.

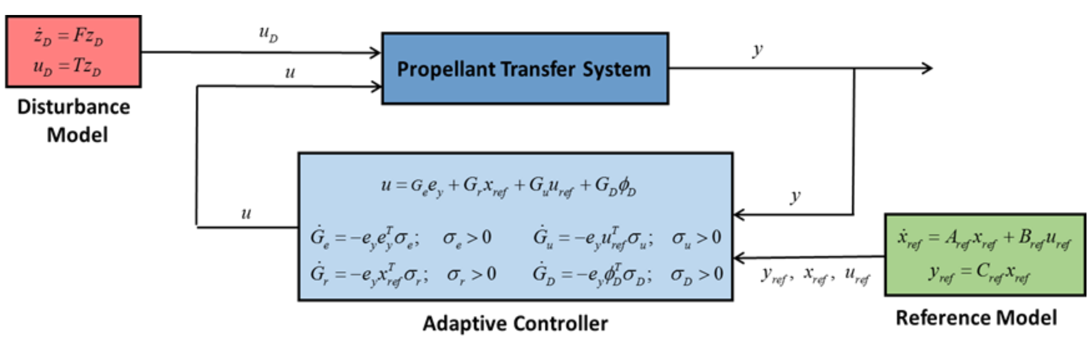

Figure 3.6: Adaptive Control Architecture

MRAC requires a reference model that the reference model produces bounded outputs, inputs and states, which can be accomplished by having a Hurwitz A_{ref}, or a control law within the reference model that bounds the outputs, inputs and states. In addition to this, the outputs of the reference model should correspond with the real outputs given by the plant. The direct adaptive control law includes four adaptive gains that are updated based on the output tracking error as well as the reference input, the reference states, and the disturbance basis functions. The adaptive control law, including the update laws for the adaptive gains, takes the form:

$$\begin{cases} u = G_e e_y + G_r x_{ref} + G_u u_{ref} + G_D \phi_D \\ \dot{G}_e = -e_y e_y^T \sigma_e; \quad \sigma_e > 0 \\ \dot{G}_r = -e_y x_{ref}^T \sigma_r; \quad \sigma_r > 0 \\ \dot{G}_u = -e_y u_{ref}^T \sigma_u; \quad \sigma_u > 0 \\ \dot{G}_D = -e_y \phi_D^T \sigma_D; \quad \sigma_D > 0 \end{cases} \tag{8}$$

In Equation (8), $x_{ref} \in \Re^n$ and $u_{ref} \in \Re^m$ are the reference states and control inputs that correspond to trajectories that are commanded by the guidance system, $\phi_D \in \Re^d$ represents a vector of disturbance basis functions of length d, and $e_y = y - y_{ref} \in \Re^p$ is the output tracking error, where p is the number of outputs. G_e, G_r, G_u and G_D are adaptive gains and $\sigma_e, \sigma_r, \sigma_u$ and σ_D are positive definite matrices that can be used to tune the controller response. It should be noted that in all results and analysis conducted in this thesis, the disturbance term is not included in the control law.

An adaptive regulator was first implemented in this analysis which includes a single adaptive gain matrix as an initial step towards mitigating the effects of time-varying moment of inertia. This implementation employed an adaptive control law of the form:

$$u = G_e e_y; \quad \dot{G}_e = -e_y e_y^T \sigma_e; \quad \sigma_e > 0 \tag{9}$$

The model reference adaptive control architecture does not require a model of the system to be controlled, although a model is employed to generate the reference trajectories. Therefore, the adaptive controller can be applied with little knowledge of the system or for systems that are modeled with a significant degree of error. However, it will be later shown that there are potential advantages to using a controlled linearized model of the system if it is possible to do so. The theoretical foundations of this adaptive control strategy have been developed and applied in numerous publications (Balas, 2000; Wen, 1989; Fuentes, 2014). There are also publications utilizing this form of control for aerospace systems specifically, including applications to aircraft and UAV flight control (Prabhakar, 2018).

3.4. Controller Stability Analysis

Lyapunov stability proofs for nonlinear time varying systems controlled by model reference adaptive controllers are extremely sparse in the current literature. The analysis where both nonlinear and time varying effects occur simultaneously is challenging, and is left as future work. In this thesis however, the system will be evaluated with these two effects occurring separately. In the first case where the system is linearized but allowed to vary with time; a Lyapunov analysis can guarantee a bounded result for the tracking error and gains. In the second case, the stability of the time invariant nonlinear system is evaluated. This second case also results in bounded tracking error and controller gains. Both stability proofs invoke Lyapunov's Theorem for uniform ultimate boundedness (UUB) which is commonly found in the literature (Khalil, 2002). The theorem states the following:

Given a Lyapunov function:

$$V(x,t) \quad \forall \left||x|\right| > R\,, t \in [0,\infty)$$

The solution to $\dot{x} = f(x,t)$ is uniformly bounded if:

$$\exists\ \Phi_1\left(\left||x|\right|\right) \in\ \mathrm{K}\,\mathfrak{R},\ \ \Phi_2\left(\left||x|\right|\right) \in\ \mathrm{K}\,\mathfrak{R}\ \ \text{class K functions}$$

such that:

$$1)\quad \Phi_1\left(\left||x|\right|\right) \le V(x,t) \le \Phi_2\left(\left||x|\right|\right)\ \ (decrecent)$$

$$2)\quad \dot{V}(x,t) \le 0$$

$$\forall \left||x|\right| > R\,, t \in [0,\infty)$$

The system is UUB if, in addition to the uniform boundedness conditions; it can also satisfy the following:

$$\exists\ \Phi_3\left(\left||x|\right|\right) \in\ \mathrm{K}\,\mathfrak{R}$$

such that:

$$\dot{V}(x,t) \le -\Phi_3\left(\left||x|\right|\right)$$

$$\forall \left||x|\right| > R\,, t \in [0,\infty)$$

3.4.1. Stability of the Time Varying Case

The first case will consider Equation (1). However, unlike the approach taken to produce the LQR architecture, the time varying dynamics will be preserved during linearization. This results in the following jacobians:

$$\frac{\partial\dot{\omega}}{\partial\omega} = -\begin{bmatrix} \frac{\dot{I}_x}{I_x} & \frac{1}{I_x}(\omega_z I_z - \omega_z I_y) & \frac{1}{I_x}(\omega_y I_z - \omega_y I_y) \\ \frac{1}{I_y}(-\omega_z I_z + \omega_z I_x) & \frac{\dot{I}_y}{I_y} & \frac{1}{I_y}(-\omega_x I_z + \omega_x I_x) \\ \frac{1}{I_z}(\omega_y I_y - \omega_y I_x) & \frac{1}{I_z}(\omega_x I_y - \omega_x I_x) & \frac{\dot{I}_z}{I_z} \end{bmatrix}$$

$$\frac{\partial \dot{\omega}}{\partial \omega} = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

The jacobians are evaluated at the following equilibrium point:

$$\omega_* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \tau_* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This results in the following time varying A and B matrices:

$$A = \begin{bmatrix} -\frac{\dot{I_x}(t)}{I_x(t)} & 0 & 0 \\ 0 & -\frac{\dot{I_y}(t)}{I_y(t)} & 0 \\ 0 & 0 & -\frac{\dot{I_z}(t)}{I_z(t)} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{I_x(t)} & 0 & 0 \\ 0 & \frac{1}{I_y(t)} & 0 \\ 0 & 0 & \frac{1}{I_z(t)} \end{bmatrix}$$

The linearized system now takes the form:

$$\Delta \dot{\underline{x}}(t) = A(t)\Delta \underline{x}(t) + B(t)\Delta \underline{u}(t)$$

$$\Delta \underline{y}(t) = C\Delta \underline{x}(t)$$

where C will be the identity matrix, corresponding to an output of the three states. This stability proof, which is found in the text written by Kaufman et al. (1998), is highly reliant on the system satisfying the condition of almost strictly passive (ASP); thus this condition must be proven first. A system of the form detailed above is considered ASP if it satisfies:

$$\dot{P}(t) + P(t)A_{cl}(t) + A_{cl}^T(t)P(t) = -Q(t) < 0 \tag{10}$$

$$P(t)B(t) = C^T(t) \tag{11}$$

where P and Q are positive definite and uniformly bounded and:

$$A_{cl}(t) = A(t) - B(t)K(t)C(t)$$

where $K(t)$ is a positive definite stabilizing feedback gain. From (11), and substituting B and C as defined prior:

$$P(t) = C^T(t)B^{-1}(t)$$

$$P(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} I_x(t) & 0 & 0 \\ 0 & I_y(t) & 0 \\ 0 & 0 & I_z(t) \end{bmatrix}$$

$$P(t) = \begin{bmatrix} I_x(t) & 0 & 0 \\ 0 & I_y(t) & 0 \\ 0 & 0 & I_z(t) \end{bmatrix}$$

Thus,

$$\dot{P}(t) = \begin{bmatrix} \dot{I}_x(t) & 0 & 0 \\ 0 & \dot{I}_y(t) & 0 \\ 0 & 0 & \dot{I}_z(t) \end{bmatrix}$$

To find a stabilizing gain, K is left generalized and used to solve for A_{cl} such that:

$$A_{cl}(t) = \begin{bmatrix} -\frac{\dot{I}_x(t) + K_{1,1}(t)}{I_x(t)} & -\frac{K_{1,2}(t)}{I_x(t)} & -\frac{K_{1,3}(t)}{I_x(t)} \\ -\frac{K_{2,1}(t)}{I_y(t)} & -\frac{\dot{I}_y(t) + K_{2,2}(t)}{I_y(t)} & -\frac{K_{2,3}(t)}{I_y(t)} \\ -\frac{K_{3,1}(t)}{I_z(t)} & -\frac{K_{3,2}(t)}{I_z(t)} & -\frac{\dot{I}_z(t) + K_{3,3}(t)}{I_z(t)} \end{bmatrix}$$

To prove the ASP condition, it must be shown that a stabilizing gain exists, even if it cannot be known in practice. To produce a stable A_{cl}, the following $K(t)$ matrix is chosen:

$$K(t) = \begin{bmatrix} \dot{I}_x(t)sgn(\dot{I}_x(t)) + \alpha & 0 & 0 \\ 0 & \dot{I}_y(t)sgn(\dot{I}_y(t)) + \beta & 0 \\ 0 & 0 & \dot{I}_z(t)sgn(\dot{I}_z(t)) + \gamma \end{bmatrix} \qquad \alpha, \beta, \gamma > 0$$

where α, β, γ are positive definite tuning parameters. With this choice in K values, the closed loop time varying system takes the form:

$$A_{cl}(t) = \begin{bmatrix} -\frac{\dot{I}_x(t) + \dot{I}_x(t)sgn(\dot{I}_x(t)) + \alpha}{I_x(t)} & 0 & 0 \\ 0 & -\frac{\dot{I}_y(t) + \dot{I}_y(t)sgn(\dot{I}_y(t)) + \beta}{I_y(t)} & 0 \\ 0 & 0 & -\frac{\dot{I}_z(t) + \dot{I}_z(t)sgn(\dot{I}_z(t)) + \gamma}{I_z(t)} \end{bmatrix}$$

This choice in K guarantees closed loop stability of A_{cl} such that the eigenvalues of A_{cl} will now satisfy:

$$\lambda_1 \leq -\frac{\alpha}{I_x(t)} < 0$$

$$\lambda_2 \leq -\frac{\beta}{I_y(t)} < 0$$

$$\lambda_3 \leq -\frac{\gamma}{I_z(t)} < 0$$

Finally, substituting $P(t)$, $P(\dot{t})$ and $A_{cl}(t)$ into Equation. (10):

$$-Q(t) = \begin{bmatrix} \dot{I}_x(t) & 0 & 0 \\ 0 & \dot{I}_y(t) & 0 \\ 0 & 0 & \dot{I}_z(t) \end{bmatrix} +$$

$$\begin{bmatrix} I_x(t) & 0 & 0 \\ 0 & I_y(t) & 0 \\ 0 & 0 & I_z(t) \end{bmatrix} \begin{bmatrix} -\frac{\dot{I}_x(t) + \dot{I}_x(t)sgn(\dot{I}_x(t)) + \alpha}{I_x(t)} & 0 & 0 \\ 0 & -\frac{\dot{I}_y(t) + \dot{I}_y(t)sgn(\dot{I}_y(t)) + \beta}{I_y(t)} & 0 \\ 0 & 0 & -\frac{\dot{I}_z(t) + \dot{I}_z(t)sgn(\dot{I}_z(t)) + \gamma}{I_z(t)} \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{\dot{I}_x(t) + \dot{I}_x(t)sgn(\dot{I}_x(t)) + \alpha}{I_x(t)} & 0 & 0 \\ 0 & -\frac{\dot{I}_y(t) + \dot{I}_y(t)sgn(\dot{I}_y(t)) + \beta}{I_y(t)} & 0 \\ 0 & 0 & -\frac{\dot{I}_z(t) + \dot{I}_z(t)sgn(\dot{I}_z(t)) + \gamma}{I_z(t)} \end{bmatrix} \begin{bmatrix} I_x(t) & 0 & 0 \\ 0 & I_y(t) & 0 \\ 0 & 0 & I_z(t) \end{bmatrix}$$

This equation reduces to:

$$Q(t) = \begin{bmatrix} I_x\dot{(t)} + 2I_x\dot{(t)}sgn(I_x\dot{(t)}) + 2\alpha & 0 & 0 \\ 0 & I_y\dot{(t)} + 2I_y\dot{(t)}sgn(I_y\dot{(t)}) + 2\beta & 0 \\ 0 & 0 & I_z\dot{(t)} + 2I_z\dot{(t)}sgn(I_z\dot{(t)}) + 2\gamma \end{bmatrix}$$

where:

$$Q(t) > 0, \quad -Q(t) < 0$$

Thus the time varying system is ASP.

This system is then provided a reference model that will provide trajectories to track that takes the following form:

$$\dot{x}_R(t) = A_R x_R(t) + B_R u_R(t)$$

This reference model is required to produce bounded trajectories and inputs.

$$y_R(t) = C_R x_R(t)$$

For simplicity, the linearized state space plant model will be written as:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = Cx(t)$$

An error term is defined for $C = C_R = I$ such that:

$$e_y(t) = e_x(t) = e(t) = y_R(t) - y(t) = x_R(t) - x(t) \qquad (12)$$

The time varying system is subjected to the following control law:

$$u = G_e e_y + G_r x_R + G_u u_R = G(t)r(t)$$

$$G(t) = [G_e \; G_r \; G_u]$$

$$r^T(t) = \left[e_y^T \; x_R^T \; u_R^T\right]$$

$$\dot{G(t)} = -e_y r^T(t)T$$

$$T = \begin{bmatrix} \sigma_e & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_u \end{bmatrix}$$

First, it must be shown that ideal trajectories exist, and perfect tracking of the reference model is possible. Ideal trajectories exist if:

$$y^*(t) = y_R(t)$$

such that:

$$Cx^*(t) = C_R x_R(t) \tag{13}$$

where the ideal state trajectory is defined as:

$$x^*(t) = X(t)x_R(t) + U(t)u_R(t)$$

$$u^*(t) = G_x^* x_R(t) + G_u^* u_R(t)$$

Substituting the ideal state trajectory into Equation (13) results in the first condition for perfect tracking such that:

$$CX(t)x_R(t) + CU(t)u_R(t) = C_R x_R(t) \tag{14}$$

where a solution exists when:

$$CX(t) = C_R, \quad U(t) = 0, \quad \dot{U}(t) = 0$$

In this case, $C = C_R = I$ thus:

$$X(t) = I, \quad \dot{X}(t) = 0$$

A second condition for perfect tracking is found by differentiating the ideal state equation such that:

$$\dot{x}^*(t) = \dot{X}(t)x_R(t) + X(t)\dot{x}_R(t) + \dot{U}(t)u_R(t) + U(t)\dot{u}_R(t)$$

Substituting $\dot{x}_R(t), x^*(t), u^*(t)$ with their respective definitions results in:

$$\dot{x}^*(t) = A(t)x^*(t) + B(t)u^*(t) - \left[A(t)X(t) + B(t)G_x^* - \dot{X}(t) - X(t)A_R\right]x_R(t)$$

$$- \left[A(t)U(t) + B(t)G_u^* - \dot{U}(t) - X(t)B_R\right]u_R(t) + U(t)\dot{u}_R(t)$$

where the first two terms are the definition of the state derivative. Thus all other terms must be zero such that:

$$-\left[A(t)X(t) + B(t)G_x^* - \dot{X}(t) - X(t)A_R\right]x_R(t)$$

$$-\left[A(t)U(t) + B(t)G_u^* - \dot{U}(t) - X(t)B_R\right]u_R(t) + U(t)\dot{u}_R(t) = 0 \tag{15}$$

Reducing terms by substituting $U(t)$, $\dot{U}(t)$, $X(t)$, $\dot{X}(t)$:

$$-[A(t) + B(t)G_x^* - A_R]x_R(t) - [B(t)G_u^* - B_R]u_R(t) = 0$$

As $x_R(t)$, and $u_R(t)$ are nonzero:

$$[A(t) + B(t)G_x^* - A_R] = 0$$

$$[B(t)G_u^* - B_R] = 0 \tag{16}$$

The second condition defined by Equation (16) cannot be met in this case as A and B are functions of time and the ideal gains are static. This suggests that perfect tracking is not possible, but bounded tracking can still be shown (Kaufman, 1998). However, Equation (14) can still be satisfied, with no requirements for G_x^* and G_u^* and definitions for $X(t)$ and $U(t)$ as well as their derivatives. With these terms defined, the derivative state error Equation (12) can be written as:

$$\dot{e}\,(t) = \dot{x}_R(t) - \dot{x}(t) =$$

$$A(t)[x^*(t) - x(t)] + B(t)G(t)r(t) - B(t)G_x^* x_R(t) - B(t)G_u^* u_R(t)$$

$$-\left[A(t)X(t) + B(t)G_x^* - \dot{X}(t) + X(t)A_R\right]x_R(t)$$

$$-\left[A(t)U(t) + B(t)G_u^* - \dot{U}(t) - X(t)B_R\right]u_R(t) + U(t)\dot{u}_R(t)$$

This can be written as:

$$\dot{e}\,(t) = A_{cl}(t)e\,(t) + B(t)\Delta G(t)r(t) - F(t) \tag{17}$$

where:

$$\Delta G(t) = G(t) - G^*(t)$$

$$G^*(t) = [G_e^*\ G_r^*\ \ G_u^*\ ]$$

$$F(t) = \left[A(t)X(t) + B(t)G_x^* - \dot{X}(t) - X(t)A_R\right]x_R(t)$$

$$+\left[A(t)U(t) + B(t)G_u^* - \dot{U}(t) - X(t)B_R\right]u_R(t) + U(t)\dot{u}_R(t)$$

Reducing terms by substituting $U(t)$, $\dot{U}(t)$, $X(t)$, $\dot{X}(t)$:

$$F(t) = [A(t) + B(t)G_x^* - A_R]x_R(t) + [B(t)G_u^* - B_R]u_R(t)$$

The residual term $F(t)$ is zero when the second condition for perfect tracking is satisfied. Note that in this case, $F(t)$ is bounded, as $x_R(t)$ and $u_R(t)$ are bounded by definition. $G_u^*, G_x^*,\ A_R$ and B_R are fixed. $A(t)$ and $B(t)$ are time varying due to the moment of inertia terms that appear in them. Both the moment of inertia and its derivative have physical bounds in any real system, and thus these terms are considered bounded. A Lyapunov stability analysis can now be conducted where:

$$V = e^T Pe + tr(\Delta G_e \sigma_e^{-1} \Delta G_e^T) + tr(\Delta G_u \sigma_u^{-1} \Delta G_u^T) + tr(\Delta G_r \sigma_r^{-1} \Delta G_r^T)$$

$$V(e, \Delta G_e, \Delta G_u, \Delta G_r) > 0, radially\ unbounded, decrecent$$

$$\dot{V} = e^T \dot{P} e + \dot{e}^T Pe + e^T P\dot{e} + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Substituting Equation (17) for $\dot{e}$ results in:

$$\dot{V} = e^T\left[\dot{P} + PA_{cl} + A_{cl}^T P\right]e + r^T \Delta G^T B^T Pe + e^T PB\Delta Gr - 2e^T PF + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Expanding the second and third term, and reducing the first term from Equation (10) yields:

$$\dot{V} = -e^T Qe + e^T PB\Delta G_e e + e^T PB\Delta G_u u_r + e^T PB\Delta G_r x_R + e^T \Delta G_e^T B^T Pe + u_R^T \Delta G_u^T B^T Pe + x_R^T \Delta G_r^T B^T Pe - 2e^T PF + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Note that:

$$e^T \Delta G_e^T B^T Pe, \qquad u_R^T \Delta G_u^T B^T Pe, \qquad x_R^T \Delta G_r^T B^T Pe$$

are scalars, which are therefore equal to their transpose, thus:

$$\dot{V} = -e^T Qe + 2e^T PB\Delta G_e e + 2e^T PB\Delta G_u u_r + 2e^T PB\Delta G_r x_R - 2e^T PF + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

where:

$$e^T PB\Delta G_e e = tr(\Delta G_e e e^T PB)$$

$$e^T PB\Delta G_u u_r = tr(\Delta G_u u_r e^T PB)$$

$$e^T PB\Delta G_r x_R = tr(\Delta G_r x_R e^T PB)$$

Thus:

$$\dot{V} = -e^T Qe + 2tr\left(\Delta G_e\left[ee^T PB + \sigma_e^{-1}\Delta\dot{G}_e^T\right]\right) + 2tr\left(\Delta G_u\left[u_r e^T PB + \sigma_u^{-1}\Delta\dot{G}_u^T\right]\right)$$
$$+ 2tr\left(\Delta G_r\left[x_R e^T PB + \sigma_r^{-1}\Delta\dot{G}_r^T\right]\right) - 2e^T PF$$

Using the prescribed update law from Equation (8) and the relationship provided in Equation (11) the trace terms are eliminated resulting in:

$$\dot{V} = -e^T Qe - 2e^T PF$$

Recall that all values in F are considered bounded in this system. This is because all time varying effects on the system have physical limits. While the first term is negative definite, the second term is undetermined. However, the quadratic nature of the first term will cause it to become dominant if the tracking error is too large. Due to the second term having some theoretical maximum value, the system is considered UUB. Therefore, one can conclude a bounded tracking result such that e, G_e, G_u, G_r are all bounded.

3.4.2. Stability of the Nonlinear Case

Next, the stability of the nonlinear case will be examined. The following equation of motion will be used to describe the system dynamics:

$$\dot{\omega} = I_{cm}^{-1}(\tau_{cm} - \omega \times I_{cm}\omega) \tag{18}$$

Note that this is simply Equation (1) with the time varying term omitted. This equation can be written in the following form:

$$\dot{x}(t) = A(x)x(t) + B(x)u(t)$$

$$\underline{y}(t) = C(x)x(t)$$

where:

$$A(x) = \begin{bmatrix} 0 & \frac{x_3 I_z}{I_x} & -\frac{x_2 I_y}{I_x} \\ -\frac{x_3 I_z}{I_y} & 0 & \frac{x_1 I_x}{I_y} \\ \frac{x_2 I_y}{I_z} & -\frac{x_1 I_x}{I_z} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}$$

As before, this proof follows what is provided in the adaptive control book written by Kaufman et al. (1998). It is again important to prove the system is ASP for the stability proof. A system of the form detailed above is considered strictly passive if it satisfies:

$$\dot{P}(x) + P(x)[A(x) - B(x)KC(x)] + [A(x) - B(x)KC(x)]^T P(x) = -Q(x) < 0 \quad (19)$$

$$P(x)B(x) = C^T(x) \quad (20)$$

Thus:

$$P(x) = C^T(x)B^{-1}(x)$$

$$P(x) = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} > 0\,, P^T = P$$

where the time invariant case dictates that:

$$\dot{P}(x) = 0$$

From Equation (19) and leaving K as a positive definite general matrix:

$$-Q(x) = P(x)\left(\begin{bmatrix} 0 & \frac{x_3 I_z}{I_x} & -\frac{x_2 I_y}{I_x} \\ -\frac{x_3 I_z}{I_y} & 0 & \frac{x_1 I_x}{I_y} \\ \frac{x_2 I_y}{I_z} & -\frac{x_1 I_x}{I_z} & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\right)$$

$$+\left(\begin{bmatrix} 0 & \frac{x_3 I_z}{I_x} & -\frac{x_2 I_y}{I_x} \\ -\frac{x_3 I_z}{I_y} & 0 & \frac{x_1 I_x}{I_y} \\ \frac{x_2 I_y}{I_z} & -\frac{x_1 I_x}{I_z} & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{I_x} & 0 & 0 \\ 0 & \frac{1}{I_y} & 0 \\ 0 & 0 & \frac{1}{I_z} \end{bmatrix}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}\right)^T P(x)$$

Reducing terms:

$$-Q(x) = P(x)\begin{bmatrix} -\frac{K_{11}}{I_x} & \frac{x_3 I_z - K_{12}}{I_x} & -\frac{x_2 I_y + K_{13}}{I_x} \\ -\frac{x_3 I_z + K_{21}}{I_y} & -\frac{K_{22}}{I_y} & \frac{x_1 I_x - K_{23}}{I_y} \\ \frac{x_2 I_y - K_{31}}{I_z} & -\frac{x_1 I_x + K_{32}}{I_z} & -\frac{K_{33}}{I_z} \end{bmatrix}$$

$$+\begin{bmatrix} -\frac{K_{11}}{I_x} & \frac{x_3 I_z - K_{12}}{I_x} & -\frac{x_2 I_y + K_{13}}{I_x} \\ -\frac{x_3 I_z + K_{21}}{I_y} & -\frac{K_{22}}{I_y} & \frac{x_1 I_x - K_{23}}{I_y} \\ \frac{x_2 I_y - K_{31}}{I_z} & -\frac{x_1 I_x + K_{32}}{I_z} & -\frac{K_{33}}{I_z} \end{bmatrix}^T P(x)$$

Further reducing gives:

$$-Q(x) = -\begin{bmatrix} 2K_{11} & K_{21} + K_{12} & K_{31} + K_{13} \\ K_{21} + K_{12} & 2K_{22} & K_{23} + K_{32} \\ K_{31} + K_{13} & K_{23} + K_{32} & 2K_{33} \end{bmatrix}$$

or equivalently:

$$Q(x) = \begin{bmatrix} 2K_{11} & K_{21} + K_{12} & K_{31} + K_{13} \\ K_{21} + K_{12} & 2K_{22} & K_{23} + K_{32} \\ K_{31} + K_{13} & K_{23} + K_{32} & 2K_{33} \end{bmatrix}$$

where $Q(x)$ will now need to be proven as positive definite. If K is defined as:

$$K(t) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \qquad \alpha, \beta, \gamma > 0$$

then:

$$Q(x) = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & 2\beta & 0 \\ 0 & 0 & 2\gamma \end{bmatrix}$$

where

$$Q(x) > 0, \quad -Q(x) < 0$$

Thus the nonlinear system is ASP. This system is then provided a reference model that will provide trajectories to track that takes the following form:

$$\underline{\dot{x}}_R(t) = A_R x_R(t) + B_R u_R(t)$$

$$\underline{y}_R(t) = C_R x_R(t)$$

The system is subjected to the following control law:

$$u = G_e e_y + G_r x_R + G_u u_R = G(t) r(t)$$

$$G(t) = [G_e \; G_r \; G_u]$$

$$r^T(t) = \left[e_y^T \; x_R^T \; u_R^T\right]$$

$$G(\dot{t}) = -e_y r^T(t) T$$

$$T = \begin{bmatrix} \sigma_e & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_u \end{bmatrix}$$

Just like the time varying case, ideal trajectories must be investigated. The existence of these trajectories is considered satisfying the following:

$$\dot{x}^*(t) = A^*(x^*)x^* + B^*(x^*)u^*(t) \tag{22}$$

$$y^*(t) = Cx^*$$

such that:

$$Cx^*(t) = C_R x_R(t) \tag{23}$$

The ideal state trajectory is defined as:

$$x^*(t) = Xx_R(t) + Uu_R(t)$$

$$u^*(t) = G_x^* x_R(t) + G_u^* u_R(t)$$

Substituting the ideal state trajectory into Equation (23) results in:

$$CXx_R(t) + CUu_R(t) = C_R x_R(t) \tag{24}$$

Solutions exist when:

$$CX = C_R, \quad U = 0$$

In this case, $C = C_R = I$, thus:

$$X = I$$

The plant is required to reach ideal states, thus an error term is formed:

$$e_x(t) = e_y(t) = e(t) = x^*(t) - x(t) = y_R(t) - y(t) \tag{25}$$

Differentiating Equation (25) results in:

$$\dot{e}\,(t) = \dot{x}^*(t) - \dot{x}(t) = A_{cl} e\,(t) + B\Delta G(t) r(t) + F(t) \tag{26}$$

where:

$$\Delta G(t) = G(t) - G^*(t)$$

$$G^*(t) = [G_e^* \; G_r^* \;\; G_u^* \,]$$

$$F(t) = \{[A^*(x^*) - A(x)]X + [B^* - B]G_x^*\}x_R(t)$$

$$+ \{[A^*(x^*) - A(x)]U + [B^* - B]G_u^*\}u_R(t)$$

Note that in the nonlinear case, the A matrix is a function of the states, but B is constant and only defined by the initial mass moment of inertia. Therefore, $F(t)$ reduces to:

$$F(t) = \{[A^*(x^*) - A(x)]X\}x_R(t) + \{[A^*(x^*) - A(x)]U\}u_R(t)$$

Finally, by substituting the values of X and U that satisfy Equation (24):

$$F(t) = \{A^*(x^*) - A(x)\}x_R(t)$$

A Lyapunov analysis can now be conducted. The following Lyapunov function is selected:

$$V = e^T P e + tr(\Delta G_e \sigma_e^{-1} \Delta G_e^T) + tr(\Delta G_u \sigma_u^{-1} \Delta G_u^T) + tr(\Delta G_r \sigma_r^{-1} \Delta G_r^T)$$

$$V(e, \Delta G_e, \Delta G_u, \Delta G_r) > 0, radially\ unbounded, decrecent$$

$$\dot{V} = e^T \dot{P} e + \dot{e}^T P e + e^T P \dot{e} + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Substituting Equation (26) for $\dot{e}$ results in:

$$\dot{V} = e^T\left[\dot{P} + PA_{cl} + A_{cl}^T P\right]e + r^T \Delta G^T B^T P e + e^T P B \Delta G r + 2e^T P F + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Expanding the second and third term, and reducing the first term from Equation (19) yields:

$$\dot{V} = -e^T Q e + e^T P B \Delta G_e e + e^T P B \Delta G_u u_r + e^T P B \Delta G_r x_R + e^T \Delta G_e^T B^T P e + u_R^T \Delta G_u^T B^T P e + x_R^T \Delta G_r^T B^T P e + 2e^T P F + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

Note that:

$$e^T \Delta G_e^T B^T P e, \qquad u_R^T \Delta G_u^T B^T P e, \qquad x_R^T \Delta G_r^T B^T P e$$

are scalars, which are equal to their transpose. Thus:

$$\dot{V} = -e^T Q e + 2e^T P B \Delta G_e e + 2e^T P B \Delta G_u u_r + 2e^T P B \Delta G_r x_R + 2e^T P F + 2tr\left(\Delta G_e \sigma_e^{-1} \Delta \dot{G}_e^T\right) + 2tr\left(\Delta G_u \sigma_u^{-1} \Delta \dot{G}_u^T\right) + 2tr\left(\Delta G_r \sigma_r^{-1} \Delta \dot{G}_r^T\right)$$

where:

$$e^T P B \Delta G_e e = tr(\Delta G_e e e^T P B)$$

$$e^T P B \Delta G_u u_r = tr(\Delta G_u u_r e^T P B)$$

$$e^T PB\Delta G_r x_R = tr(\Delta G_r x_R e^T PB)$$

Thus:

$$\dot{V} = -e^T Qe + 2tr\left(\Delta G_e\left[ee^T PB + \sigma_e^{-1}\Delta\dot{G}_e^T\right]\right) + 2tr\left(\Delta G_u\left[u_r e^T PB + \sigma_u^{-1}\Delta\dot{G}_u^T\right]\right)$$

$$+ 2tr\left(\Delta G_r\left[x_R e^T PB + \sigma_r^{-1}\Delta\dot{G}_r^T\right]\right) + 2e^T PF$$

Using the prescribed update law from Equation (8), the trace terms are eliminated resulting in:

$$\dot{V} = -e^T Qe + 2e^T PF$$

Similar to the time varying case discussed in the last section, it must now be shown that a bounded result will exist from this resulting expression for $\dot{V}$. That is to say, it can be shown that if the error term becomes too large, the negative definite first term will dominate resulting in a negative semi definite $\dot{V}$. To this end, the terms will be expanded as such:

$$-e^T Qe = -2\alpha e_1^2 - 2\beta e_2^2 - 2\gamma e_3^2$$

$$2e^T PF = 2e_1\left[x_{R,3}I_y(x_2 - x_2^*) - x_{R,2}I_z(x_3 - x_3^*)\right]$$

$$- 2e_2\left[x_{R,3}I_x(x_1 - x_1^*) - x_{R,1}I_z(x_3 - x_3^*)\right]$$

$$+ 2e_3\left[x_{R,2}I_x(x_1 - x_1^*) - x_{R,1}I_y(x_2 - x_2^*)\right]$$

where all values in I and x_R are either constant or bounded. Notice the expanded second term takes the form of the definition of the error terms. It can be re-written such that:

$$2e^T PF = 2\left[\left(-x_{R,3}I_y e_2 + x_{R,2}I_z e_3\right) \quad \left(x_{R,3}I_x e_1 - x_{R,1}I_z e_3\right) \quad \left(-x_{R,2}I_x e_1 + x_{R,1}I_y e_2\right)\right]e$$

where $\dot{V}$ is negative semi definite when:

$$[\alpha e_1 + \beta e_2 + \gamma e_3]e > \left[\left(-x_{R,3}I_y e_2 + x_{R,2}I_z e_3\right) \quad \left(x_{R,3}I_x e_1 - x_{R,1}I_z e_3\right) \quad \left(-x_{R,2}I_x e_1 + x_{R,1}I_y e_2\right)\right]e$$

$$\alpha e_1^2 + \beta e_2^2 + \gamma e_3^2 > I_x e_1\left(e_2 x_{R,3} - e_3 x_{R,2}\right) + I_y e_2\left(e_3 x_{R,1} - e_1 x_{R,3}\right) + I_z e_z\left(e_1 x_{R,2} - e_2 x_{R,1}\right)$$

The negative definite, quadratic first term will be dominant when:

$$\|e_1\| > \left\|\frac{I_x}{\alpha}(e_2 x_{R,3} - e_3 x_{R,2})\right\|, \quad \|e_2\| > \left\|\frac{I_y}{\beta}(e_3 x_{R,1} - e_1 x_{R,3})\right\|, \quad \|e_3\| > \left\|\frac{I_z}{\gamma}(e_1 x_{R,2} - e_2 x_{R,1})\right\|$$

where the values of $\alpha,\ \beta,\ \gamma$ can be arbitrarily large, positive definite scalars. Provided at the initialization of the controller, the error terms are finite; the arbitrarily large scalars can allow values of $\|e_1\|, \|e_2\|, \|e_3\|$ to be quite large and still result in a negative semi definite $\dot{V}$. Due to the second term not having a maximum value, the system is simply uniformly bounded. Thus, one can conclude a bounded tracking result such that e, G_e, G_u, G_r are all bounded.

4. Results

This chapter will present simulation results from MATLAB. It will display two distinct types of results. The first two simulations, 4.1 and 4.2, are used to justify using adaptive control for this type of system by comparing it to a baseline LQR controller. All results displayed after 4.2 show simulations using the MRAC developed in Section 3.3 , where a variety of reference trajectories will be tracked. Some of the following simulations provide quantifiable performance metrics in the form of error performance P_e and input performance P_u. These metrics were determined as follows:

$$P_e = ||e_x||_2 + ||e_y||_2 + ||e_z||_2 \quad where\ e\ is\ the\ error\ array\ from\ 0 \leq t \leq T$$

$$P_u = ||u_x||_2 + ||u_y||_2 + ||u_z||_2 \quad where\ u\ is\ the\ input\ array\ from\ 0 \leq t \leq T$$

For both cases, the optimal case would be $P_e = P_u = 0$, thus smaller values will be indicitive of better performance.

4.1. Adaptive Regulator with Slosh and Disturbance

The adaptive controller in Equation (9) and the baseline LQR controller were applied to the nonlinear rotational system model under the following conditions:

- Target angular velocity: ω_*.= [0 0 15]^T rpm
- Initial velocity= [0 0 0] ^{T} rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Persistent sinusoidal torque disturbances about all axes with frequencies of 0.5 Hz (30 rpm), 0.25 Hz (15 rpm) and 0.5 Hz (30 rpm), respectively.
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5\cos(0.2\pi t) & 0 \\ 0 & 0 & .5\sin(0.2\pi t) \end{bmatrix} \frac{kg*m^2}{s}$

- LQR $Q = R = I$, with closed loop eigenvalues: $\lambda_1, \lambda_2 = -.74 \pm .88i,\ \lambda_3 = -1.62$

These conditions are intended to simulate a sloshing motion in the tanks as the system is controlled from an initial angular velocity of zero to a desired spin rate of 15 rpm about the z-axis. The sloshing effect is modeled with orthogonal sine and cosine functions of equal magnitude and frequency on the y and z axis. Figures 4.1 – 4.3 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrix, respectively. The results demonstrate that, while both controllers result in stable rotational dynamics about the x and y axes, the adaptive regulator is better able to compensate for the time-varying nature of the system to control the angular velocity about the z-axis. The adaptive gains in Figure 4.3 demonstrate the adaptation in response to both the time-varying moment of inertia and external disturbances.

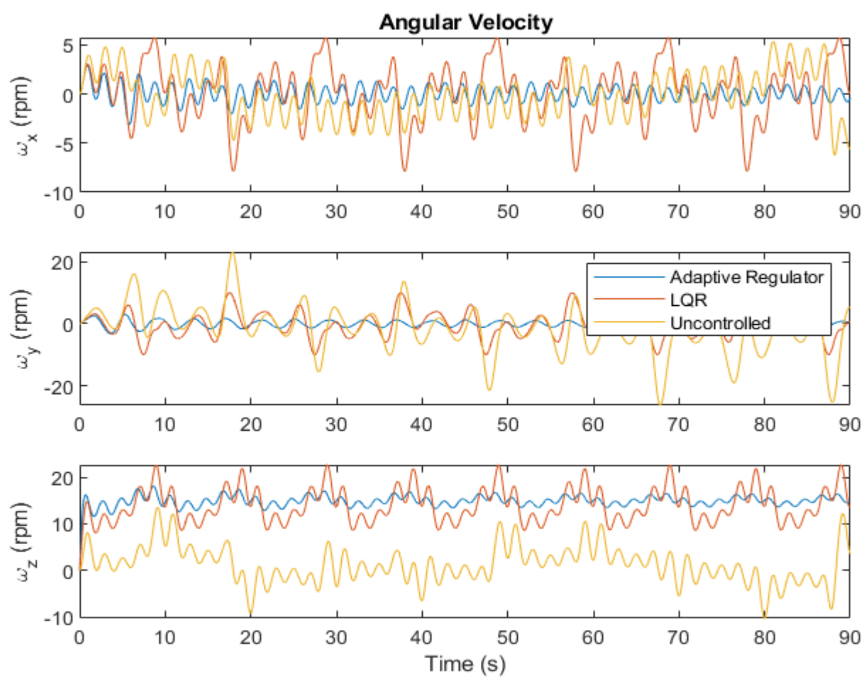

Figure 4.1 Angular Velocity, Simulation Case 1

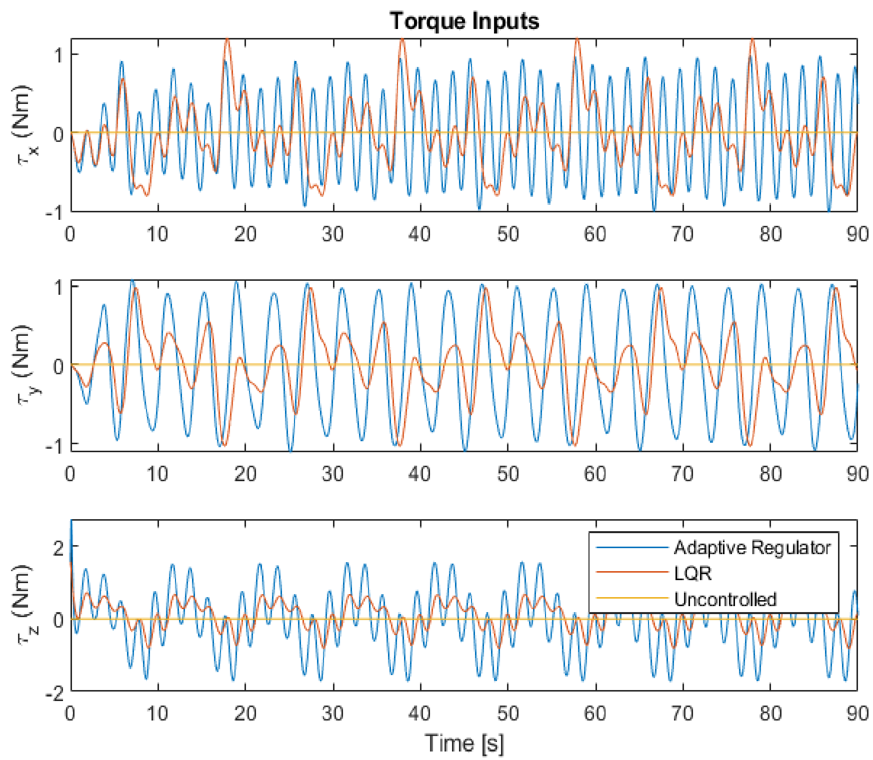

Figure 4.2 Torque Control Inputs, Simulation Case 1

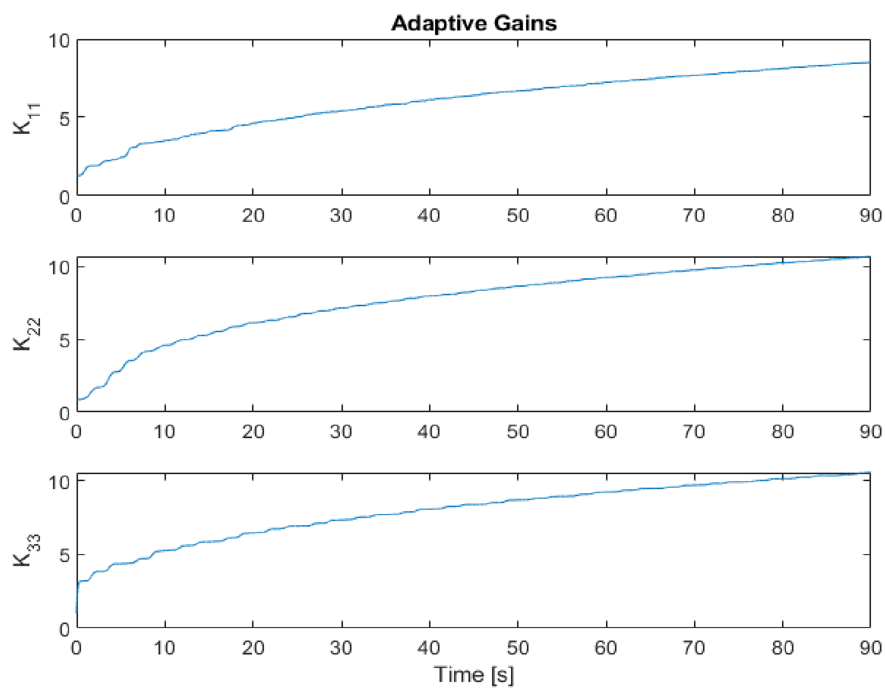

Figure 4.3 Adaptive Gains, Simulation Case 1

Table 4.1

Simulation 4.1 Performance Metrics

	$\boldsymbol{P_e}$	$\boldsymbol{P_u}$
LQR	108.16	198.71
Adaptive Regulator	28.92	117.51

4.2. Adaptive Regulator with Fluid Settling and Disturbance

This simulation is a representation of fluid moving to the ends of the tanks. However, the initial velocity is non-zero to demonstrate the uncontrolled system's reactions to this maneuver.

- Target angular velocity, ω_*= $[0\ 0\ 15]^T$ rpm
- Initial velocity= $[0\ 0\ 7]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \frac{kg*m^2}{s}$
- Persistent sinusoidal torque disturbances about all axes with frequencies of 0.5 Hz (30 rpm), 0.25 Hz (15 rpm) and 0.5 Hz (30 rpm), respectively.
- LQR $Q = R = I$, with closed loop eigenvalues: $\lambda_1, \lambda_2 = -.74 \pm .88i,\ \lambda_3 = -1.62$

These conditions are intended to simulate fluid settling in the tanks along with unmodeled external disturbances. Figures 4.4 – 4.6 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrix, respectively. The uncontrolled angular velocity is also plotted in this case to demonstrate that, in the absence of control, the system will decrease its spin rate as fluid moves towards the ends of the tanks and the moment of inertia about the axis of rotation increases.

As in the previous case, the results demonstrate that the adaptive regulator is better equipped to compensate for the time-varying nature of the system as well as the external disturbances. Specifically, the adaptive control provides better mitigation of undesired out-of-plane motion, and provides better tracking of the desired rotation rate about the z-axis. Note that, in the case of the LQR controller, as the fluid continues to settle further from the z-axis, the controller is unable to retain the rotation rate about z, and its angular velocity begins to approach zero like the uncontrolled system. It is also important to note that, in the case of the adaptive regulator, although no disturbance basis functions are provided in this example, the adaptive gain is still attempting to mitigate the effects of the external disturbance. This explains why the input performance, P_u, is higher for the adaptive regulator.

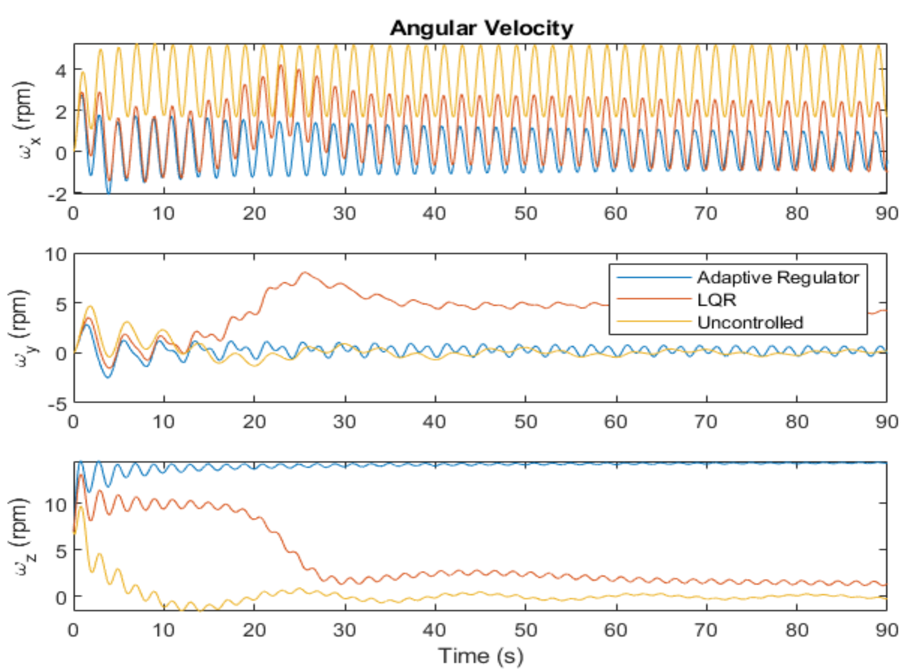

Figure 4.4 Angular Velocity, Simulation Case 2

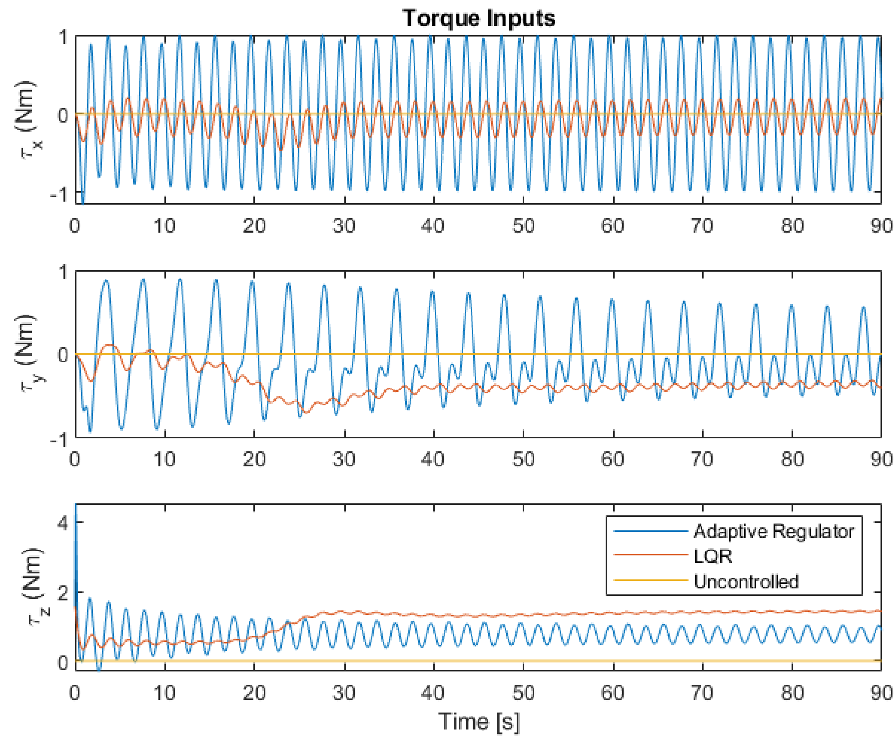

Figure 4.5 Torque Control Inputs, Simulation Case 2

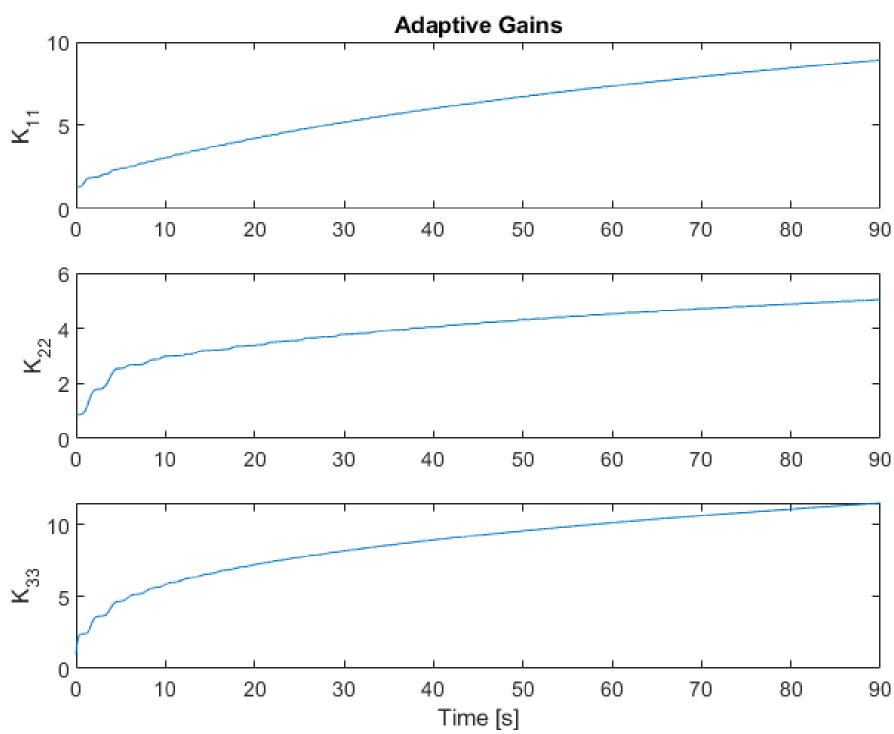

Figure 4.6 Adaptive Gains, Simulation Case 2

Table 4.2

Simulation 4.2 Performance Metrics

	P_e	P_u
LQR	42.19	43.25
Adaptive Regulator	9.35	54.43

The next section of results will display simulations with the MRAC controller implemented rather than an adaptive regulator. In these cases, the first three terms of the controller are active with the disturbance term omitted.

4.3. MRAC First Order Tracking with Fluid Settling

In this case, the model reference adaptive controller as described from Equation (8) will be tracking the output of the linearized, time invariant model controlled by the LQR controller discussed in Section 3.2 under the conditions described below. The only omission to the adaptive gains in Equation (8) is the optional disturbance accommodating term, which is zero for this, and all other MRAC simulation examples provided in this thesis. This simulation is meant to represent a fluid settling condition where fluid will flow from the axis of rotation as the system begins to spin in order to follow a reference trajectory.

- Initial velocity= $[0\ 0\ 0]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \frac{kg*m^2}{s}$
- Adaptive gain G_e is initialized as the gain matrix used by the LQR controller
- All remaining adaptive gains are initialized at zero

Figures 4.7 – 4.11 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrices G_e, G_r and G_u respectively. In this case, external disturbances are absent, so the oscillatory behavior is born from the internal disturbance of the time varying moment of inertia coupled with the nonlinear equation that is driving the system dynamics. The desired trajectory is simply to achieve an asymptotic first order response to a desired rotation rate about the z axis. Due to the reference trajectories being derived from a simplified dynamics model, much of the off axis effects are not present in the reference trajectories and thus provide relatively flat regulating trajectories for the x and y axis. The result is very little rotation about the x and y axes while a first order response is tracked in z. In addition to the production of desirable trajectories, the LQR controlled reference model also provides the initial gain, K, for G_e which helps generate such an effective response shortly after the initiation of the controller.

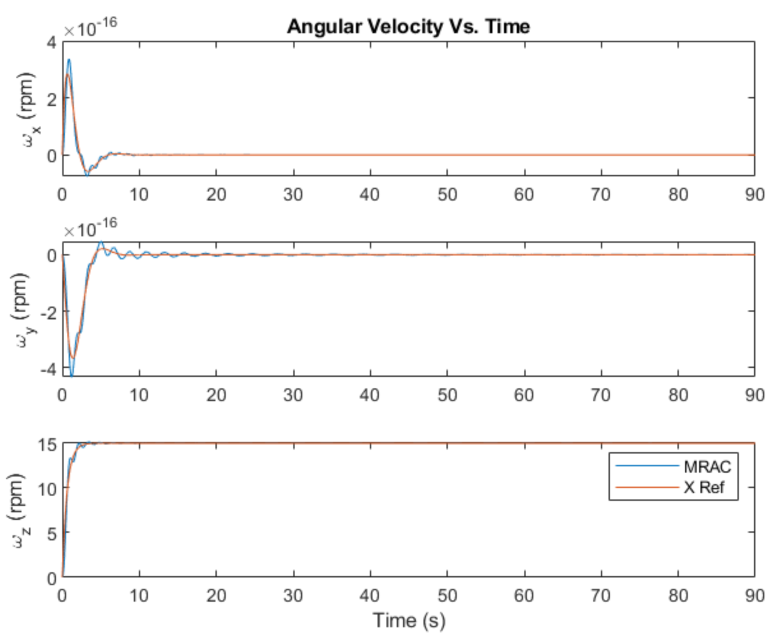

Figure 4.7 Angular Velocity, Simulation Case 3

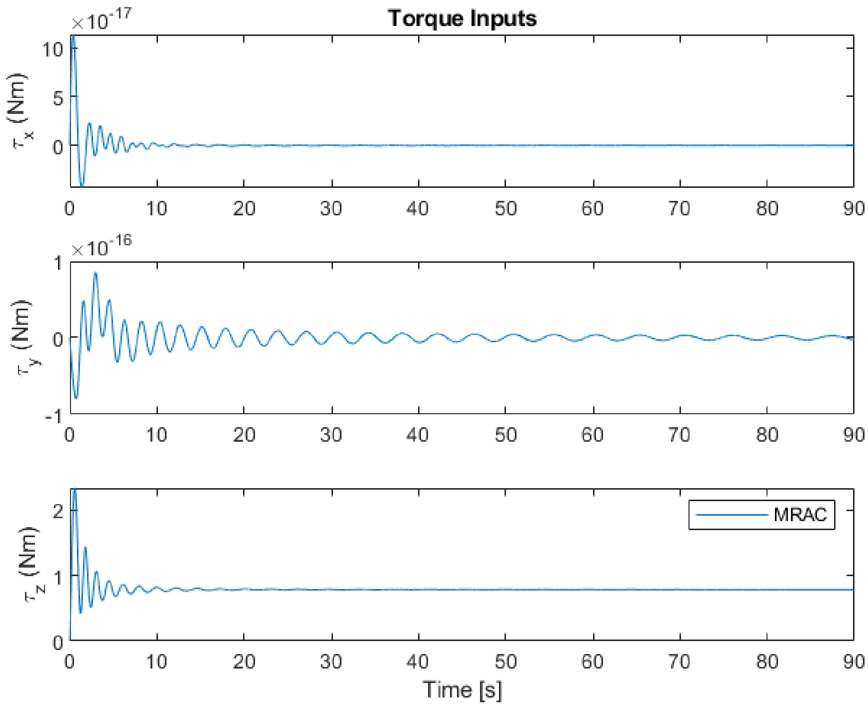

Figure 4.8 Torque Control Inputs, Simulation Case 3

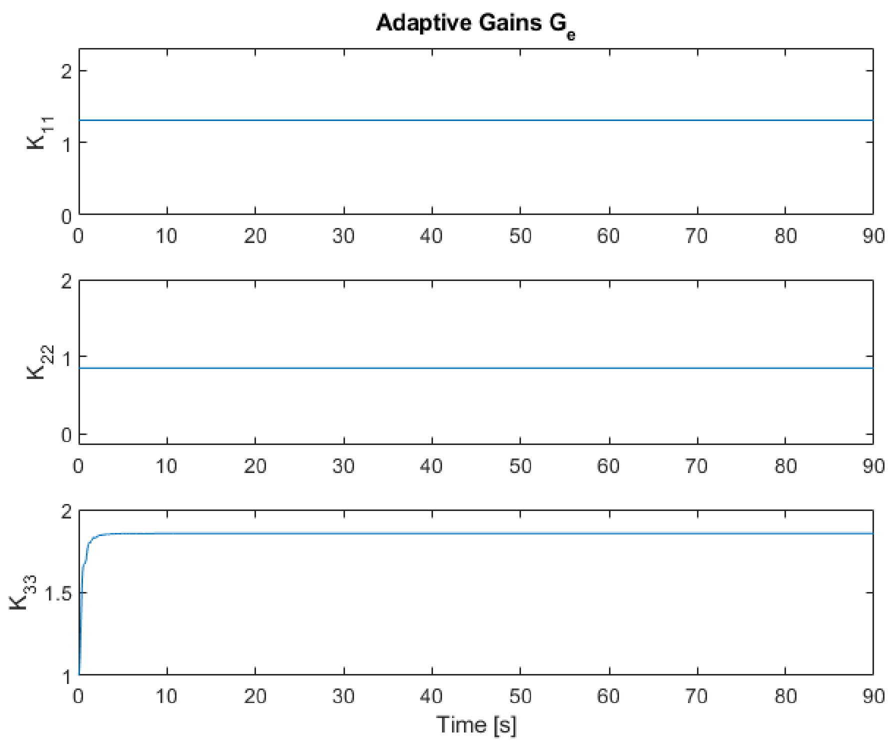

Figure 4.9 Adaptive Gains Ge, Simulation Case 3

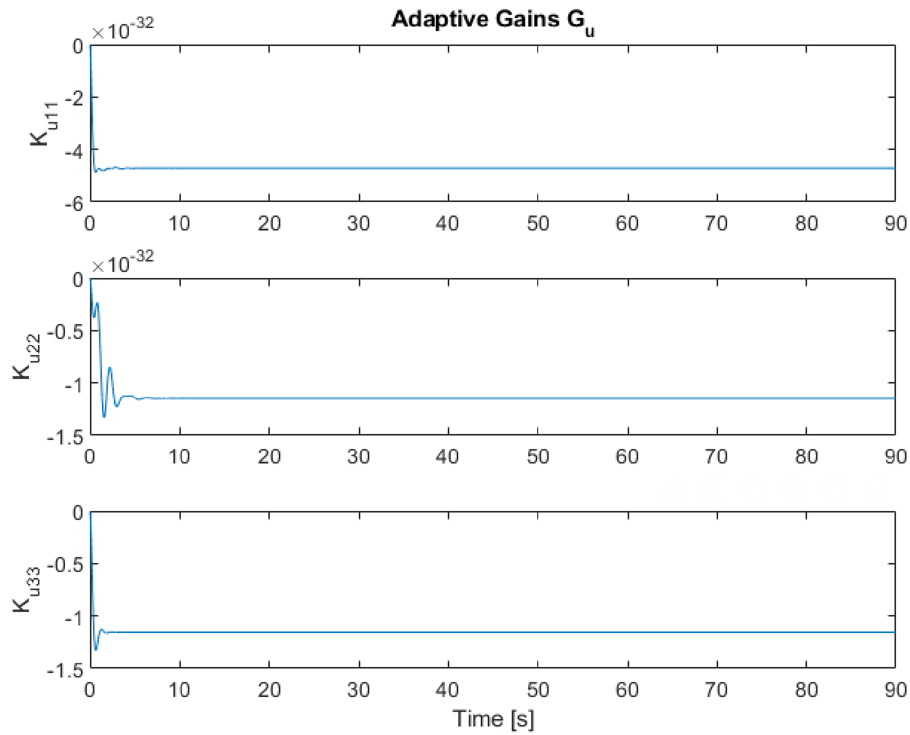

Figure 4.10 Adaptive Gains Gr, Simulation Case 3

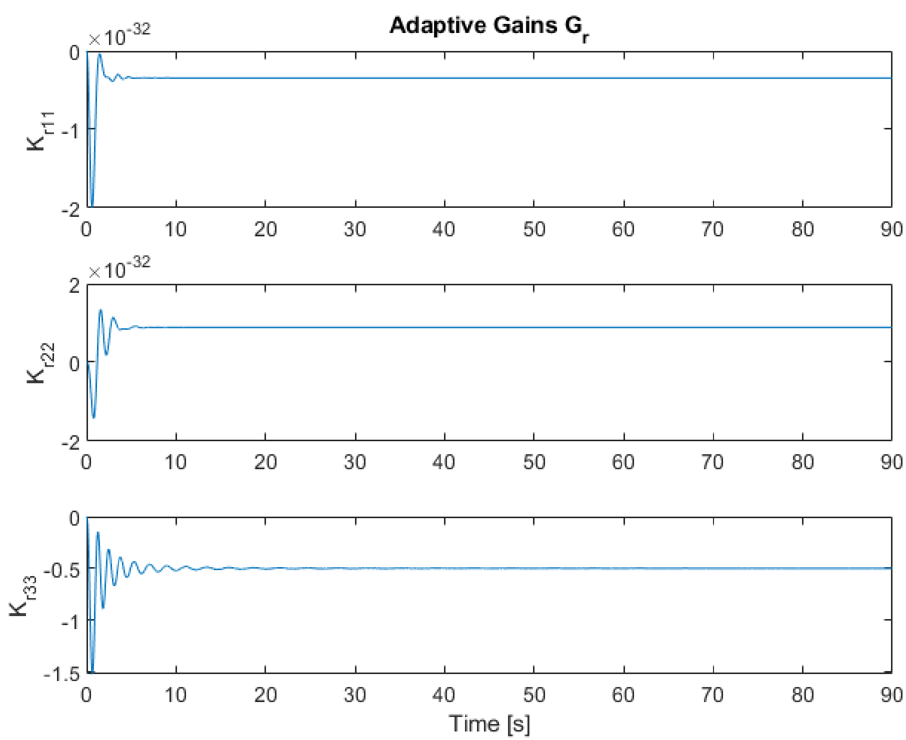

Figure 4.11 Adaptive Gains Gu, Simulation Case 3

4.4. MRAC First Order Tracking with Fluid Slosh

In this case, the controller will again be tracking the output of the linearized, time invariant model controlled by the LQR controller discussed in Section 3.2 under the conditions described below. This simulation is similar to the previous case, with the same reference trajectory being provided. However, this case is meant to represent a fluid slosh condition similar to simulation 4.1.

- Initial velocity= $[0\ 0\ 0]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5\cos(0.2\pi t) & 0 \\ 0 & 0 & .5\sin(0.2\pi t) \end{bmatrix} \frac{kg*m^2}{s}$
- Adaptive gain G_e is initialized as the gain matrix used by the LQR controller
- All remaining adaptive gains are initialized at zero

Figures 4.12 – 4.16 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrices G_e, G_r and G_u respectively. Note that as was the case in the previous simulation, no external disturbance has been added to the system. The MRAC is then only forced to attenuate internal effects while trying to track the reference trajectory.

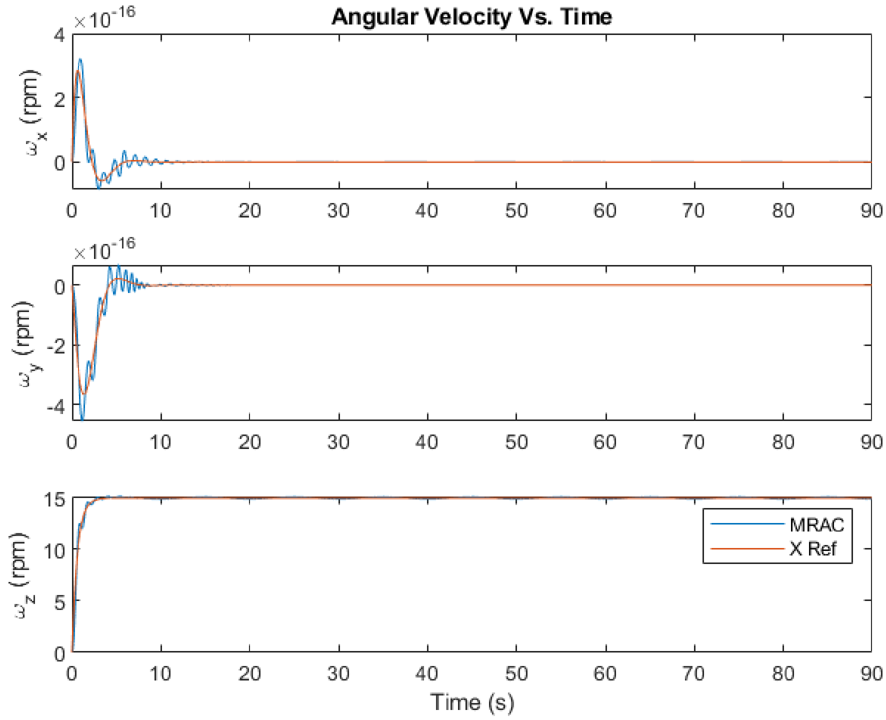

Figure 4.12 Angular Velocity, Simulation Case 4

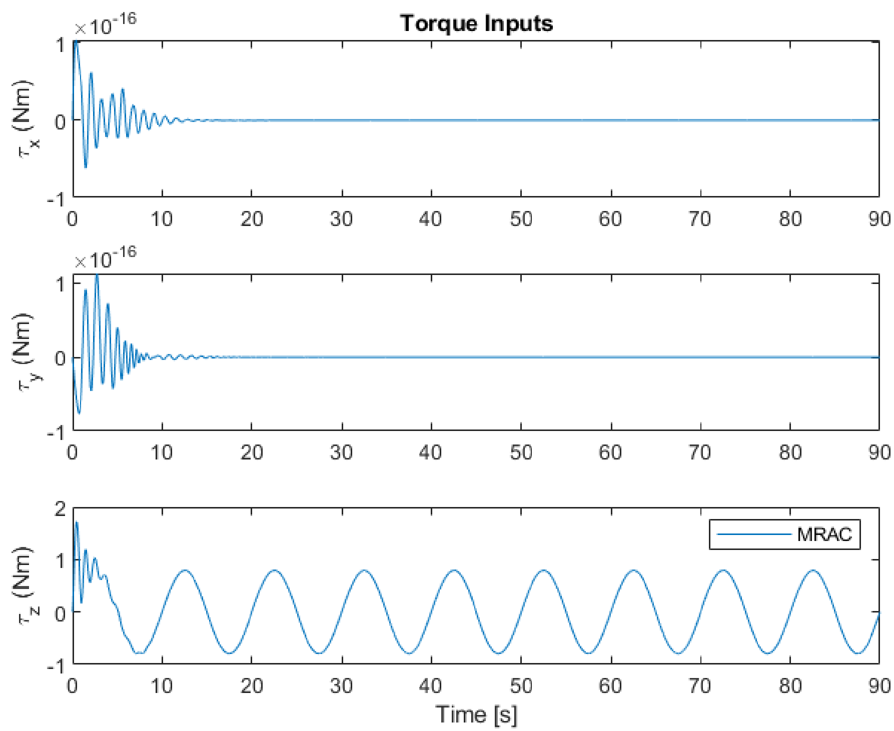

Figure 4.13 Torque Control Inputs, Simulation Case 4

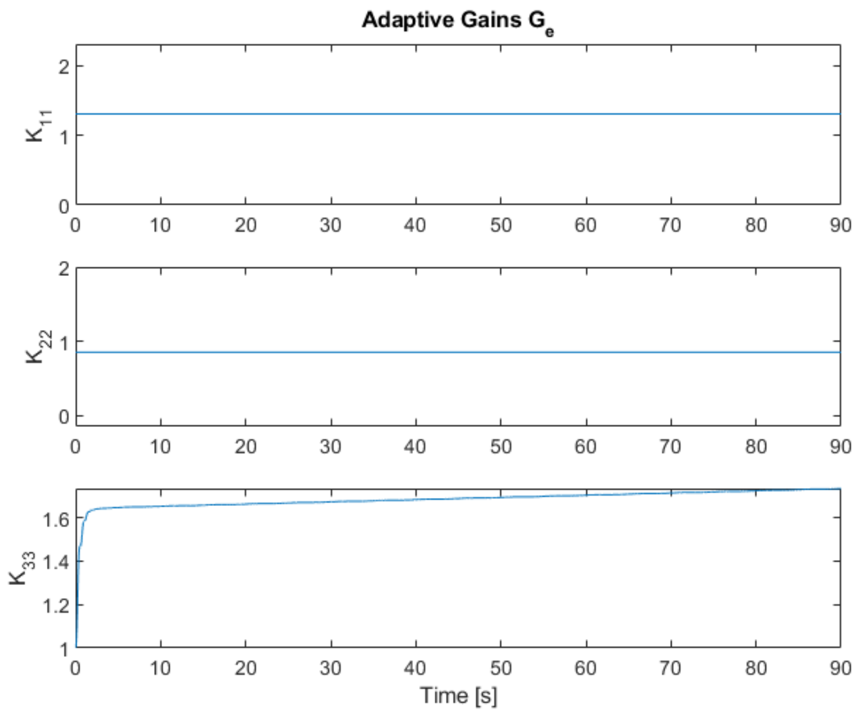

Figure 4.14 Adaptive Gains Ge, Simulation Case 4

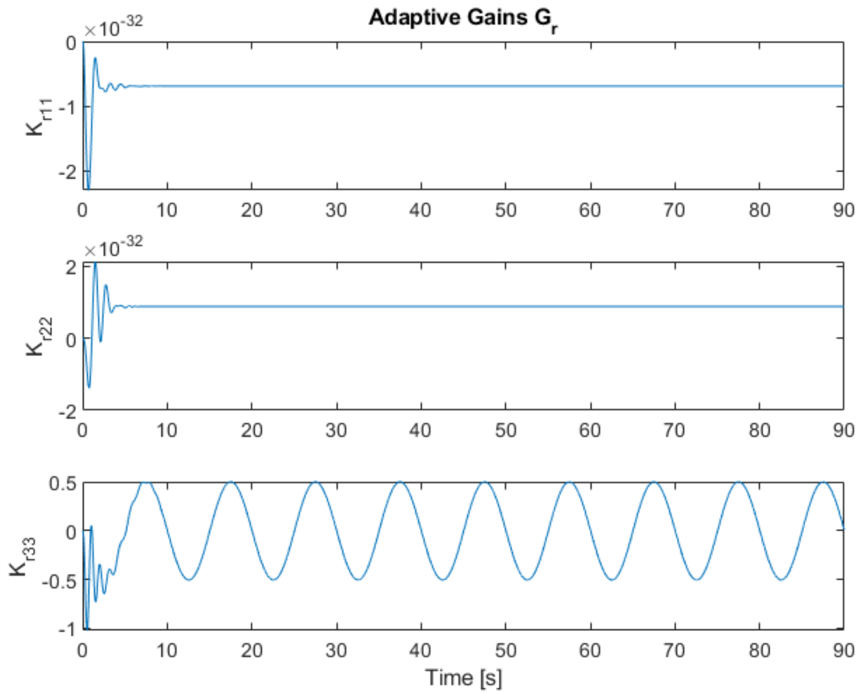

Figure 4.15 Adaptive Gains Gr, Simulation Case 4

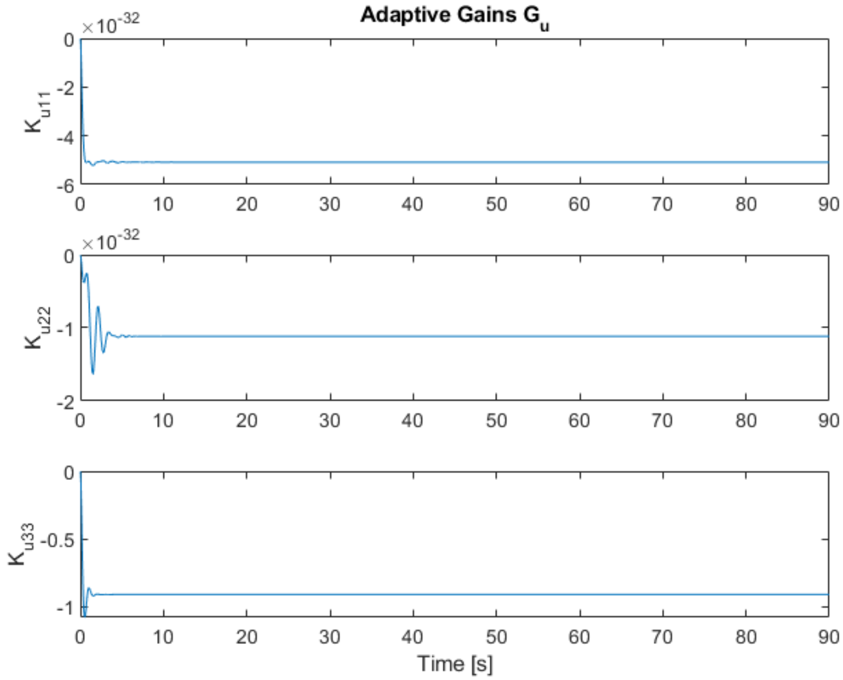

Figure 4.16 Adaptive Gains Gu, Simulation Case 4

4.5. MRAC First Order Tracking with Fluid Settling and Disturbance

This simulation is a representation of fluid moving to the ends of the tanks similar to Section 4.3. In this case, the controller will be tracking the output of the linearized model controlled by the LQR controller under the conditions described below.

- Initial velocity= $[0\ 0\ 0]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \frac{kg*m^2}{s}$
- Persistent sinusoidal torque disturbances about all axes with frequencies of 0.5 Hz (30 rpm), 0.25 Hz (15 rpm) and 0.5 Hz (30 rpm), respectively.
- Adaptive gain G_e is initialized as the gain matrix used by the LQR controller

- All remaining adaptive gains are initialized at zero

Figures 4.17 – 4.21 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrices G_e, G_r and G_u respectively. The results demonstrate the ability of the MRAC to force the nonlinear time-varying system to track the trajectory generated using the LQR controlled LTI model, as was used in simulations 4.4 and 4.5, while the system is experiencing both internal and external disturbances. Note that, although additional disturbances have been inflicted on the system, the MRAC is still able to track a reference trajectory and mitigate the effects of external disturbance. This is accomplished by continuing to adapt the gains to eliminate the external effects and is illustrated in Figures 4.19-4.21, which are much more active than the analogous disturbance free example in simulation 4.3.

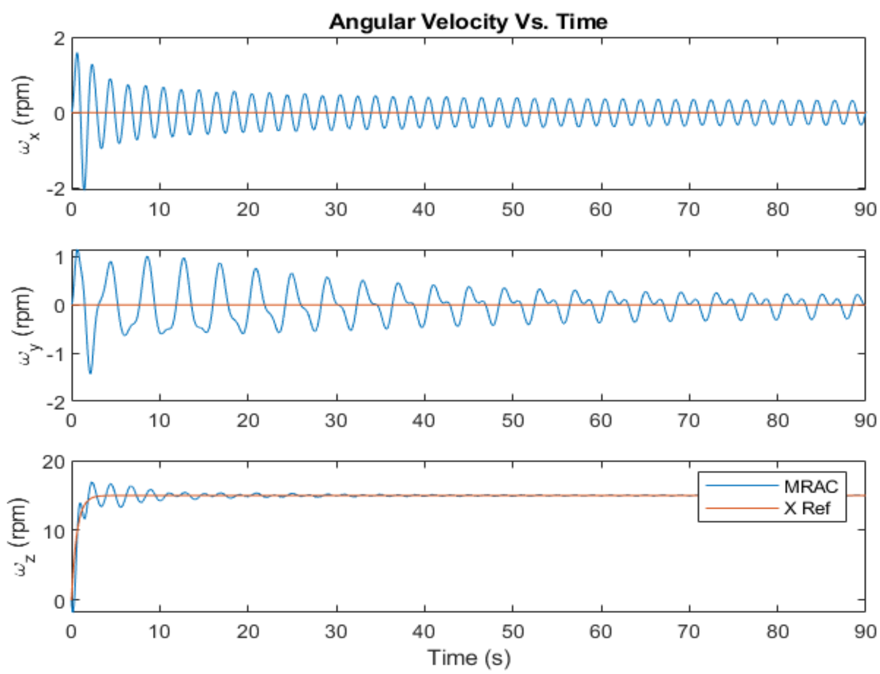

Figure 4.17 Angular Velocity, Simulation Case 5

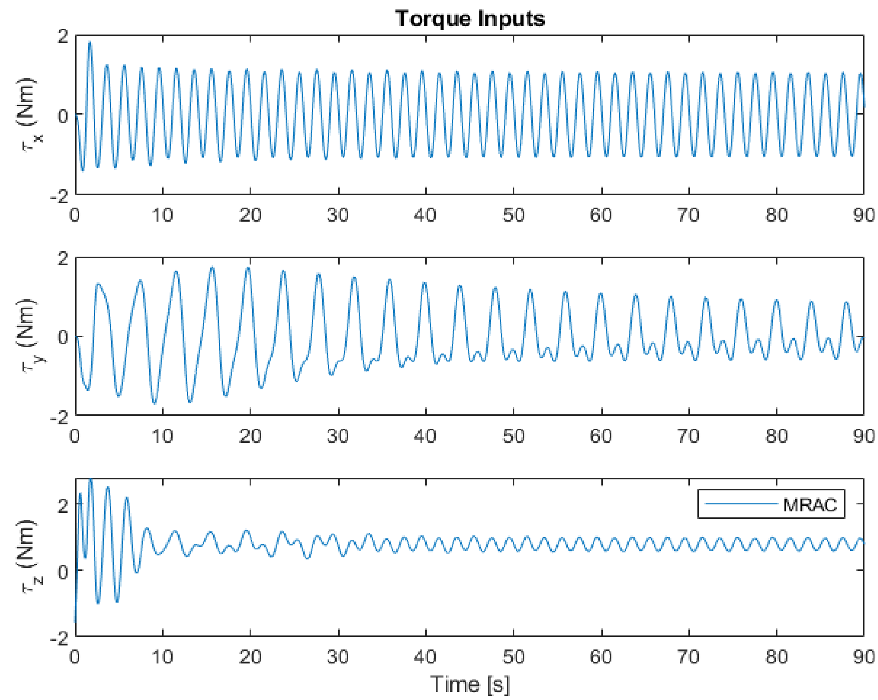

Figure 4.18 Torque Control Inputs, Simulation Case 5

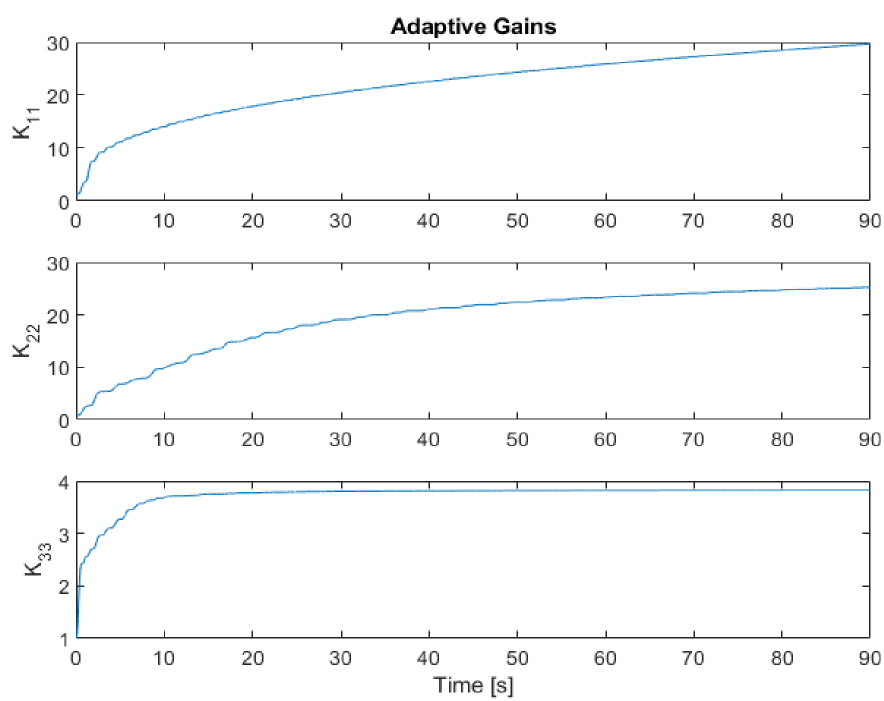

Figure 4.19 Adaptive Gains Ge, Simulation Case 5

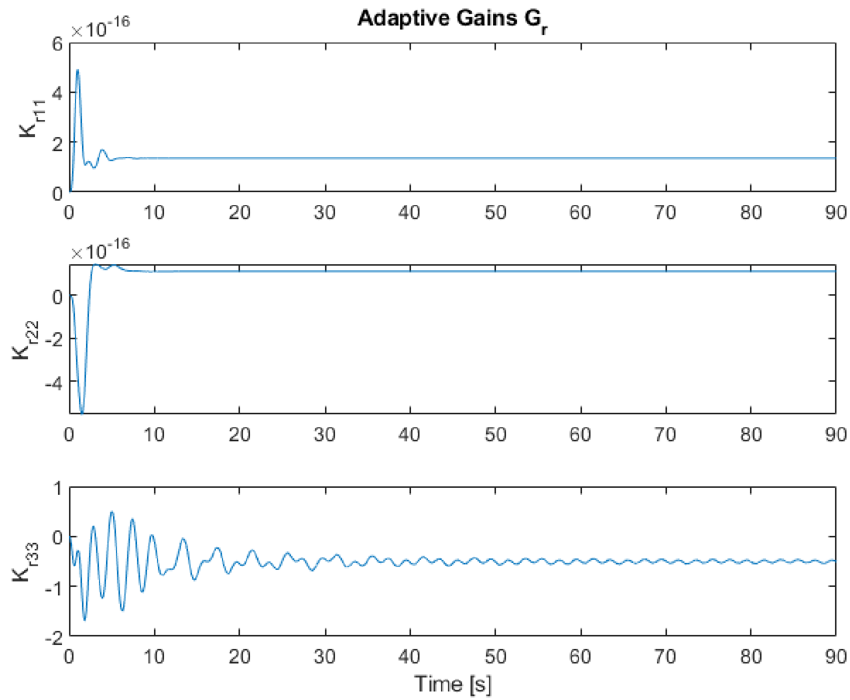

Figure 4.20 Adaptive Gains Gr, Simulation Case 5

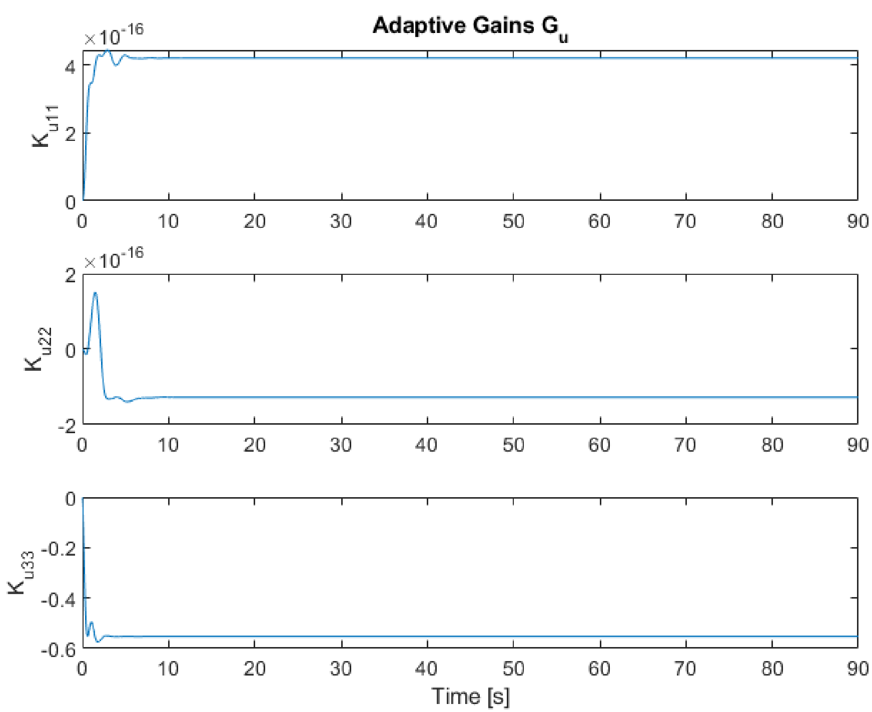

Figure 4.21 Adaptive Gains Gu, Simulation Case 5

4.6. MRAC Sinusoidal Tracking with Fluid Settling and Disturbance

This simulation is also a representation of fluid settling to the ends of the tanks. In this case, the system is trying to track the output from the LQR controlled linear system which is tracking a .1 Hz (3 rpm) sinusoid under the following conditions:

- Initial velocity= $[0\ 0\ 0]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix} \frac{kg*m^2}{s}$
- Persistent sinusoidal torque disturbances about all axes with frequencies of 0.5 Hz (30 rpm), 0.25 Hz (15 rpm) and 0.5 Hz (30 rpm), respectively.
- Adaptive gain G_e is initialized as the gain matrix used by the LQR controller
- All remaining adaptive gains are initialized at zero

Figures 4.22 – 4.26 provide the resulting angular velocity, control torque inputs about each axis, and the diagonal values of the adaptive gain matrices G_e, G_r and G_u respectively. In this case, the MRAC is trying to regulate all dynamics about the x and y axes and track a more novel trajectory in the form of a sinusoid about the z axis. This more dynamic trajectory coupled with time varying, nonlinear, and external disturbance effects, forces much more activity in all three adaptive gains. Ultimately, the result is small, bounded oscillations about x and y axes, with a slight lag in following the target trajectory about the z axis. It is also worth noting that, as can also be seen in example 4.5, the disturbances about all three axes are attenuated with time as the adaptive gains adjust.

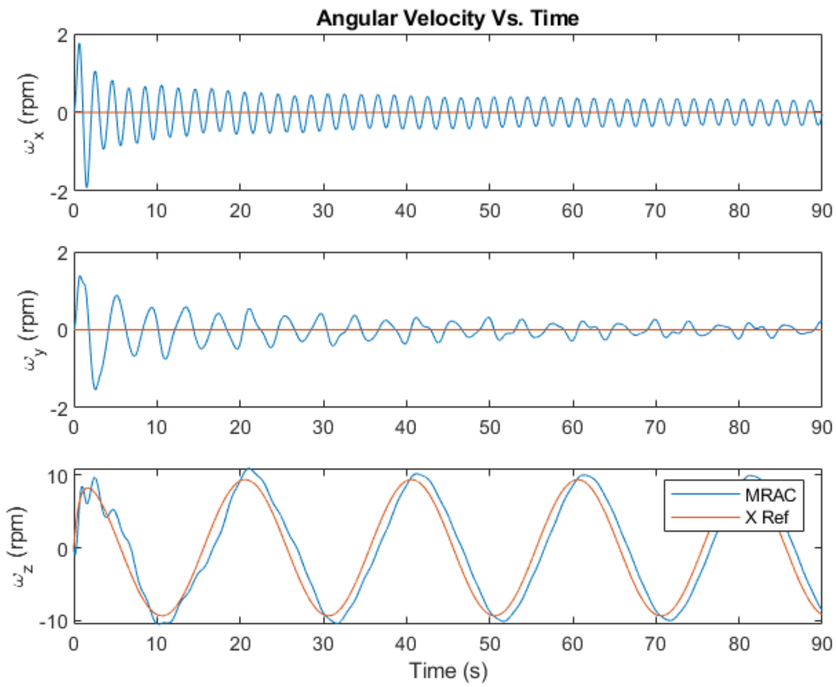

Figure 4.22 Angular Velocity, Simulation Case 6

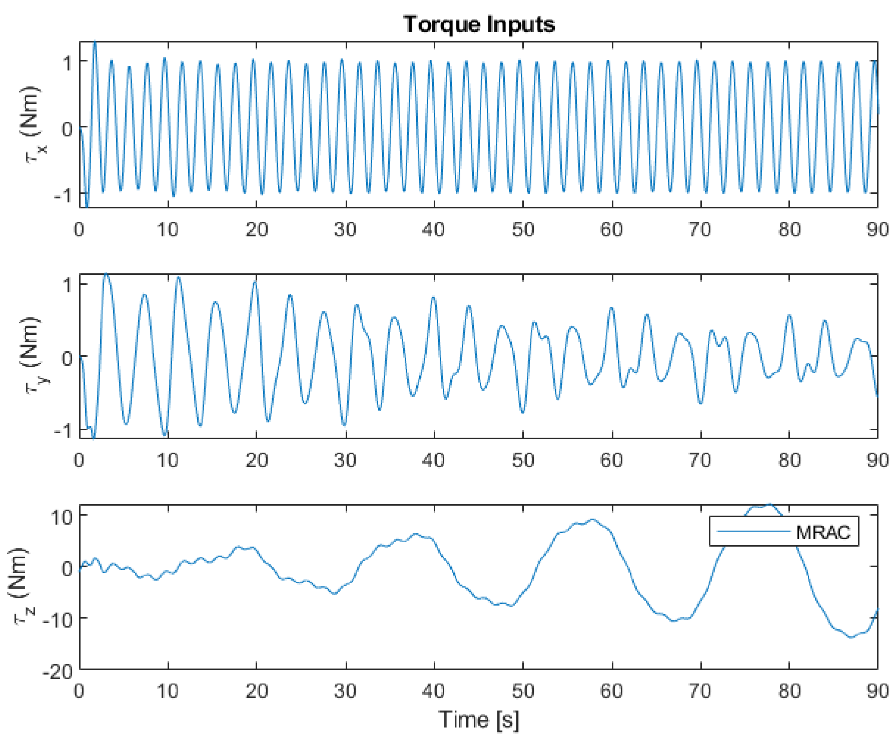

Figure 4.23 Torque Control Inputs, Simulation Case 6

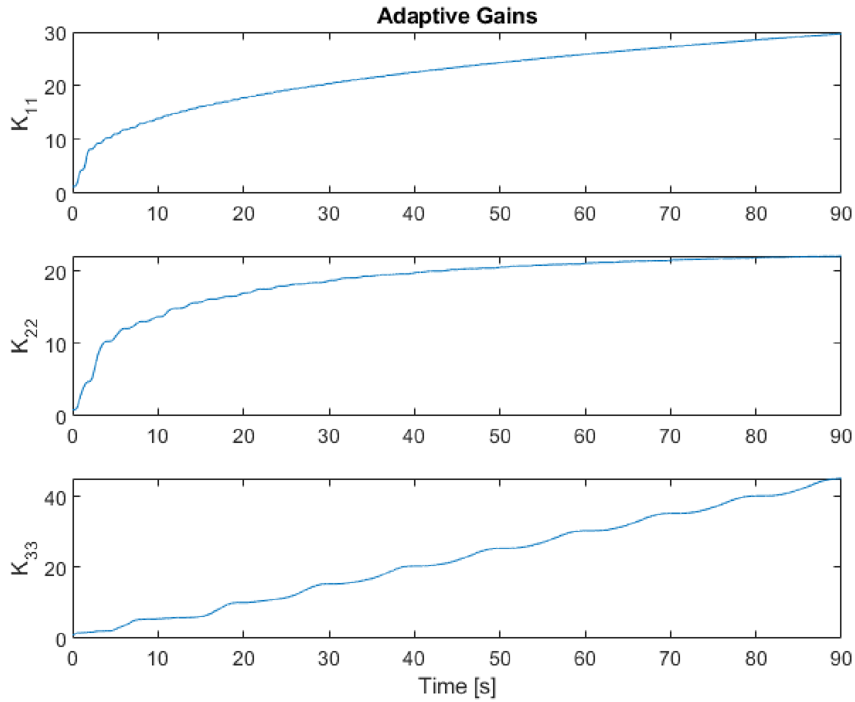

Figure 4.24 Adaptive Gains Ge, Simulation Case 6

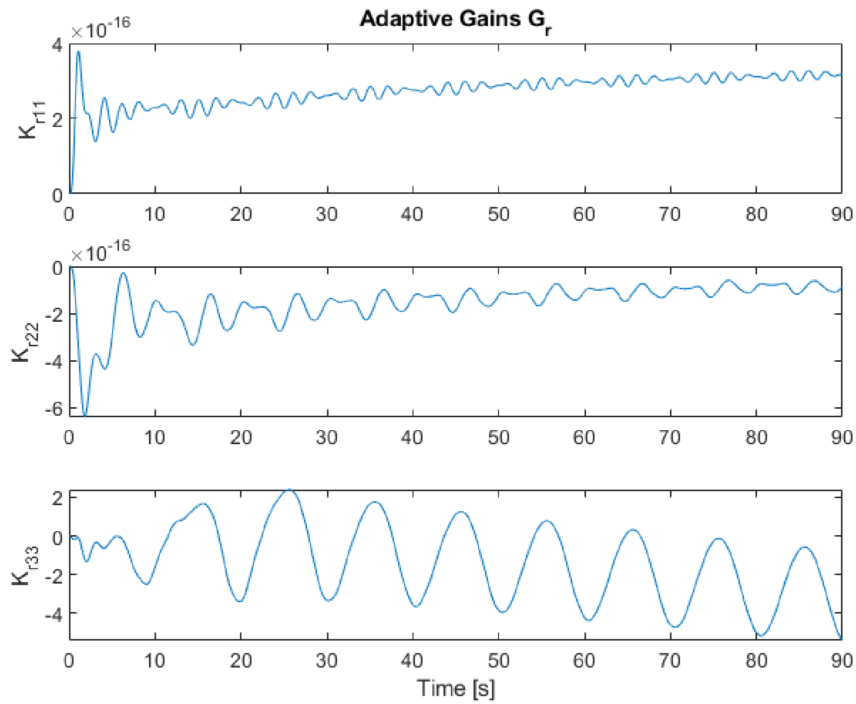

Figure 4.25 Adaptive Gains Gr, Simulation Case 6

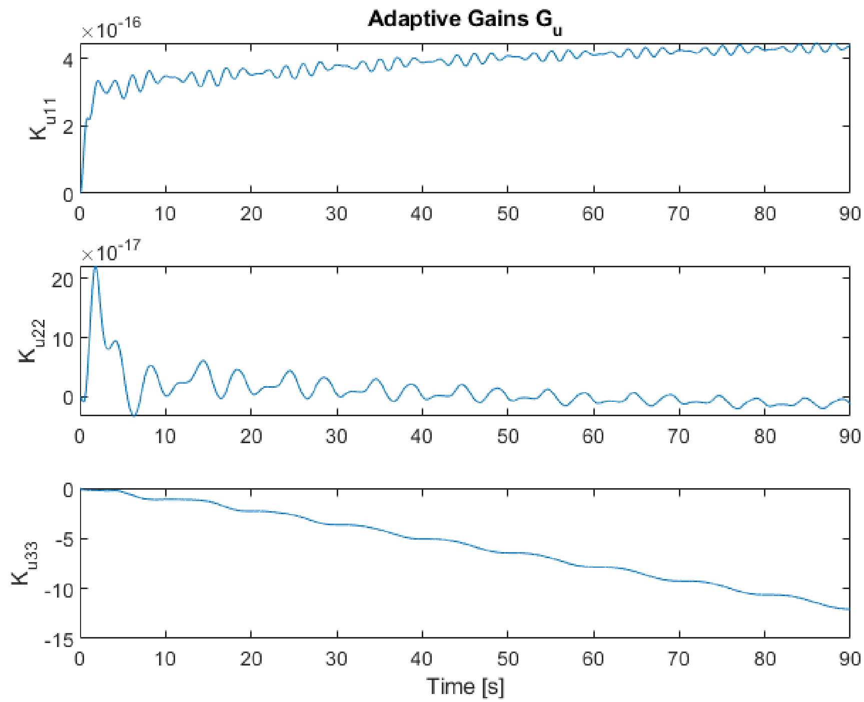

Figure 4.26 Adaptive Gains Gu, Simulation Case 6

4.7. MRAC Offset Sinusoidal Tracking with Fluid Settling, Slosh, Disturbance and Noise

This simulation is also a representation of fluid settling to the ends of the tanks. In this case, the system is trying to track the output from the LQR controlled linear system which is tracking a .1 Hz (3 rpm) sinusoid oscillating about 15 rpm under the following conditions:

- Initial velocity= $[2\ 5\ {-5}]^T$ rpm
- Initial inertia values: $I_0 = \begin{bmatrix} 1.74 & 0 & 0 \\ 0 & 1.1712 & 0 \\ 0 & 0 & 0.6171 \end{bmatrix} kg * m^2$
- Time varying inertia: $\frac{d}{dt}(I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & .5\cos(0.2\pi t) + .5 & 0 \\ 0 & 0 & .5\sin(0.2\pi t) + .5 \end{bmatrix} \frac{kg*m^2}{s}$
- Persistent sinusoidal torque disturbances about all axes with frequencies of 0.5 Hz (30 rpm), 0.25 Hz (15 rpm) and 0.5 Hz (30 rpm), respectively.

- Zero mean measurement noise with standard deviation: $\sigma = .15\ \frac{rad}{s}$
- Adaptive gain G_e is initialized as the gain matrix used by the LQR controller
- All remaining adaptive gains are initialized at zero

Figures 4.27 – 4.31 provide the resulting angular velocity, control torque inputs, and the diagonal values of the adaptive gain matrices G_e, G_r and G_u respectively. This simulation is meant to represent a fluid sloshing condition with a combination of all the effects discussed in prior examples to display the resilience of MRAC. In this case, there is a nonzero initial condition, external disturbance, internal slosh disturbance and a reference trajectory oscillating about 15 rpm. In addition to all of these effects, zero mean measurement noise was also introduced. Under all of these conditions, the MRAC is able to track the reference trajectory reasonably well, while the combination of nonlinear, time varying and external effects are well mitigated with small rotations about x and y.

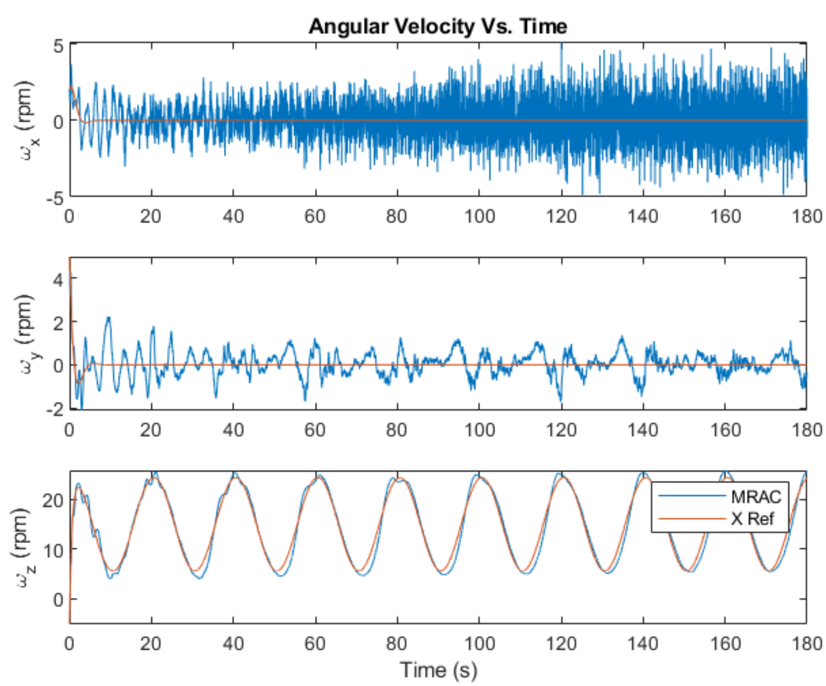

Figure 4.27 Angular Velocity, Simulation Case 7

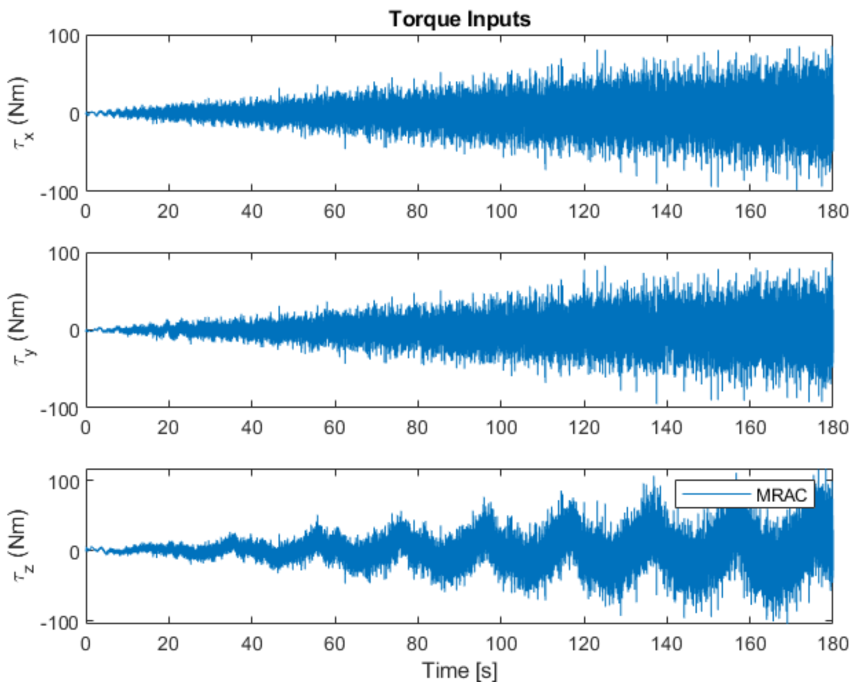

Figure 4.28 Torque Control Inputs, Simulation Case 7

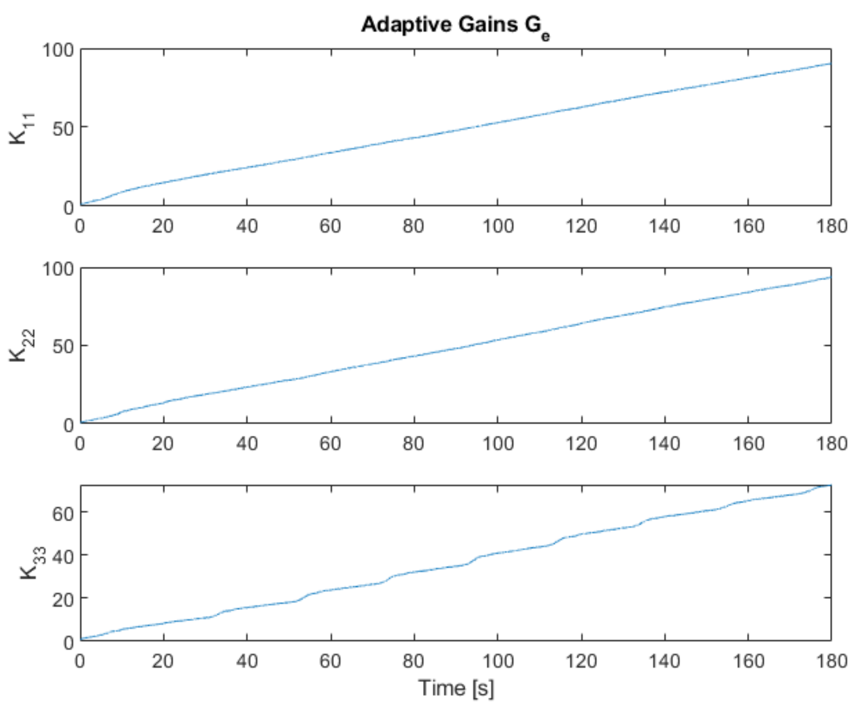

Figure 4.29 Adaptive Gains Ge, Simulation Case 7

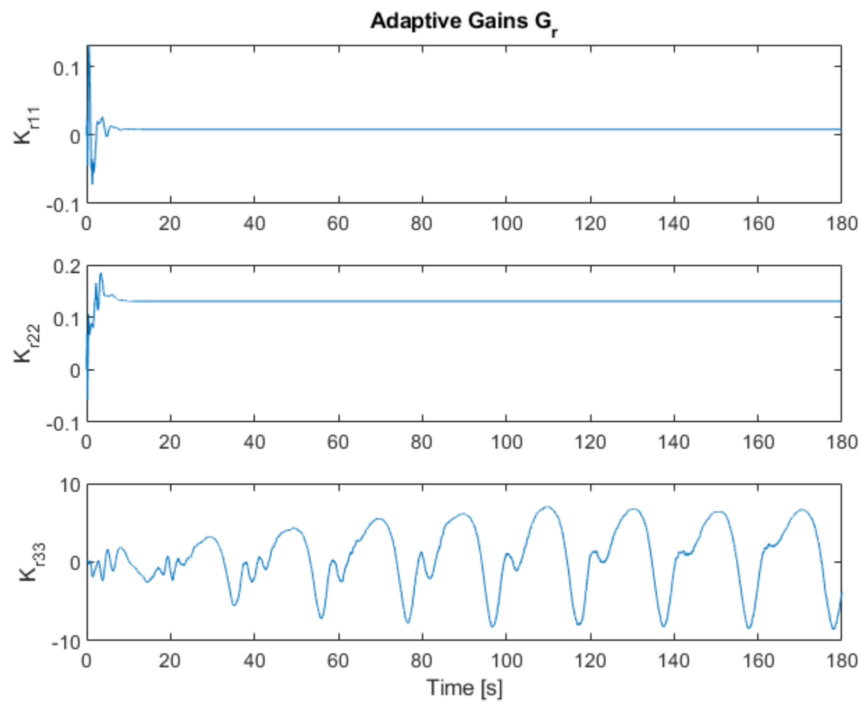

Figure 4.30 Adaptive Gains Gr, Simulation Case 7

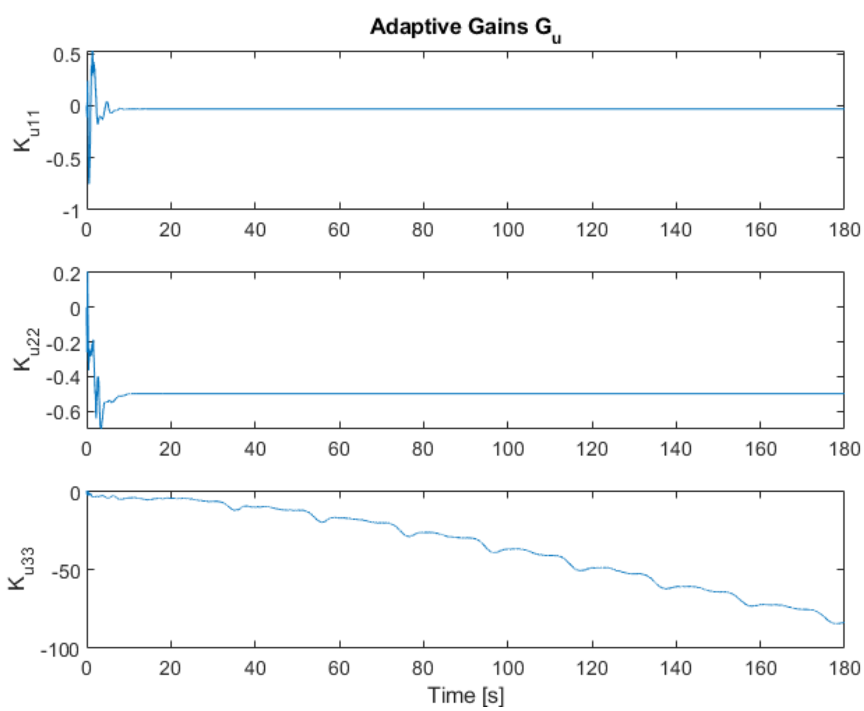

Figure 4.31 Adaptive Gains Gu, Simulation Case 7

4.8. Results Analysis

In this section, the results will be summarized so that conclusions on the use of MRAC for this class of system can be drawn. In sections 4.1 and 4.2, a pared down version of MRAC was applied to the plant along with an LQR which was generated from linearizing a time invariant model of the plant about a desired rotation rate about the z axis. These two cases demonstrate the need for more capable methods of control and show how MRAC can be used to reach a desired point in state space rather than track a trajectory. In these examples, the adaptive controller was quantifiably better at reaching the desired state compared to the LQR. However, the real benefit of MRAC is to use a simplified model of the plant in order to generate trajectories for the plant to track. To demonstrate this, further cases displayed results of the plant controlled using a full MRAC implementation.

In simulation 4.3 and 4.4, MRAC was applied to the nonlinear time varying plant, with trajectories provided by the LQR controlled, time invariant, linearized plant model. Both of these cases were void of external disturbances, but were subjected to internal time varying effects. Using reference trajectories rather than a desired reference state, as in simulations 4.1 and 4.2 reduces input spikes that are caused from large errors during initialization. To illustrate this, consider simulations 4.2 and 4.3. These cases are exposed to the same disturbances and initial conditions and are both trying to achieve a final state of $[0\ 0\ 15]^T$ rpm, but 4.2 is generating an error from its current state to this desired state, while 4.3 is generating an error based off the reference trajectory that achieves the desired state derived by a simplified model of the system. If one analyzes the input plots for the z axis of both systems, it can be seen that 4.2 initializes with an input spike while

4.3 does not. This is formed because of the large error seen at the controller's initiation. In other words, there is an increased potential of actuator saturation when a reference trajectory is not provided.

In addition to a lack of initial input spikes, using a reference model will provide a smooth trajectory to follow which may reduce more drastic oscillatory behavior when compared to providing a simple reference point. Another interesting aspect of using an LQR controlled linearized reference model is that these trajectories are inherently optimized to mitigate tracking error to the desired state as well as control effort based on Equation (2). Although this optimization is based on a simplified model of the plant dynamics, including some amount of state and input optimization may have potential benefits, although no empirical study was performed for this thesis.

Simulation 4.5 demonstrates the model reference adaptive controller's ability to compensate for both internal and external effects simultaneously while still tracking a first order response to the desired rotation rate about z. It is shown in these results that this combination of disturbances can be accommodated, with the desired steady state condition being reached, and respectable tracking exhibited through the transient phase of the first order response. Simulation 4.6 is simply a combination of all of these effects with a slightly more sophisticated reference trajectory and a nonzero initial condition. The only additional effect added was zero mean measurement noise. Measurement noise seems to affect the controller performance more than either internal or external disturbances. Although the states are bounded, the rotations about the x and y axes are not regulated as well as cases where measurement noise is absent. Measurement noise also forced the torque inputs to become more erratic than other cases, which may be hard to realize when actuator dynamics are included. If the standard deviation of the noise is increased, it will eventually cause the controller to fail. This may be solved by either through filtering, or by adding an additional term to the control law commonly referred to as a sigma or e-modification.

5. Conclusions and Recommendations

Modern aerospace systems often exhibit highly nonlinear, time varying dynamical effects. As the operational envelopes of these systems increase in complexity, the need for control methods that can adapt to both internal system changes and external effects become more imperative. While this can be sidestepped by discretizing the system into its main operating regimes, there are potential benefits to adaptive methods such as model reference adaptive control including reduced development time and resistance to failure in the event of unforeseen internal or external dynamical events. Implementation of such control methods can therefore result in safer systems with lower development costs. In this thesis, the control of a candidate system that exhibited both nonlinear and time varying dynamics was investigated. Using direct MRAC as the form of adaptive control architecture, simulations were produced to both demonstrate the increased performance of adaptive control compared to a linear quadratic regulator. After having demonstrated this, full MRAC was applied to the nonlinear, time varying system under various test conditions to illustrate its effectiveness for trajectory tracking under both internal and external effects.

Results from these simulations showed not only the advantages of using adaptive control over a linear quadratic regulator, but also the benefits of using model reference adaptive control when compared to an adaptive regulator. In all cases, the adaptive controller outperformed the LQR, and an analogous MRAC case was used to show how input spikes can be mitigated when compared to an adaptive regulator. It was also shown that the adaptive controller will attempt to compensate for external disturbances even if the disturbance accommodating term in the control law is not present. Activating this term, if it is assumed the form of the disturbance is known may yield even better tracking performance than what is displayed in this thesis, and is an area of research worth investigating in the future.

The simulation that was displayed in Section 4.7 demonstrated the effect of measurement noise on the system. It was discovered that this had a profound impact on the controller performance and that this noise can eventually cause the controller to fail. Future research should be invested in determining effective methods of adding controller robustness in the presence of measurement noise. Two potential approaches include filtering the output that is fed to the adaptive controller, or to mitigate the effects of noise within the controller by adding a so-called sigma or e-modification. Determining a solution to this particular failure mode of the controller is critical, as any real system will have measurement noise.

A full Lyapunov analysis of the nonlinear, time varying system was not attempted in this thesis; however nonlinear and time varying cases were separately investigated. An analysis of this kind where both effects occur simultaneously is very sparse in the current literature, and as such is worth investigating. Ultimately, although there is still much work to be done in this area of research, it was shown that the case can be made for MRAC as an effective control architecture for this class of nonlinear, time varying systems.

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