GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION
SPONSORED PROJECT INITIATION

Date: *June 18, 1976*

Project Title: *Locally One Dimensional Numerical Methods for Multidimensional Free Surface Problems*

Project No: *G-37-605*

Project Director: *Dr. G. H. Meyer*

Sponsor: *U. S. Army Research Office, Research Triangle Park, N.C.*

Agreement Period: From *6/1/76* Until *5/31/78*

Type Agreement: *Grant No. DAAG29-76-G-0261*

Amount: *$15,500 - ARO*
5,493 - GIT(G-37-315)
$20,993 - Total (1st yr. funding)

Reports Required: *Progress Report*
Technical (when justified)
Final Report

Sponsor Contact Person (s):

Technical Matters

Technology Division
U. S. Army Research Office
PO Box 1211
Research Triangle Park, NC 27706

Contractual Matters
(thru OCA)

ONR Resident Representative
Campus

Defense Priority Rating:

Assigned to: *Mathematics* (School/Laboratory)

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CA-3 (3/76)

G-37-605

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

SPONSORED PROJECT TERMINATION

Date: September 6, 1979

Project Title: Locally One Dimensional Numercial Methods for Multidimensional Free Surface Problems

Project No: G-37-605

Project Director: Dr. Karlovitz

Sponsor: U. S. Army Research Office; Research Triangle Park, NC 27709

Effective Termination Date: 5/31/79 (Grant Expiration)

Clearance of Accounting Charges: --

Grant/Contract Closeout Actions Remaining:

TERMINATED

- x Final ~~Invoice~~ Fiscal and Closing Documents
- _ Final Fiscal Report
- _ Final Report of Inventions
- x Govt. Property Inventory & Related Certificate
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- _ Other ____________

Assigned to: Mathematics (School/Laboratory)

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CA–4 (1/79)

G-37-605

PROGRESS REPORT

1. ARO PROPOSAL NUMBER: P-13874-M

2. PERIOD COVERED BY REPORT: July - December 1976

3. TITLE OF PROPOSAL: Locally one-dimensional numerical methods for multi-dimensional free surface problems

4. CONTRACT OR GRANT NUMBER: DAAG29-76-G-0261

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Gunter H. Meyer

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   1) An application of the method of lines to multi-dimensional free boundary problems, accepted for publication by the Journal Inst. Math. Applcs.

   2) The method of lines for Poisson's equation with nonlinear boundary conditions, submitted to Mathematics of Computation.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   Luis Kramarz was supported during the Summer of 1976 in his Ph.D. thesis research.

13874M

DR. GUNTER H. MEYER
GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF MATHEMATICS
ATLANTA, GA 30332

BRIEF OUTLINE OF RESEARCH FINDINGS

My research deals with the numerical solution of free boundary and interface problems for multi-dimensional partial differential equations which describe melting and ablation, groundwater flow, electro-chemical machining, semi-conductors and optimal decision processes. My specific goal is to apply locally one dimensional (LOD) solution techniques because as a rule one dimensional transient and steady state problems are manageable from a numerical point of view (see, e.g., the review article in SIAM Review, January 1977).

So far two distinct LOD methods have been applied to diffusion and potential problems with a free surface which include, one phase Stefan problems. The first technique is the so-called method of fractional steps (similar to an alternating direction method) which allows a non-iterative solution of those diffusion problems where the free surface is single valued with respect to each independent variable. Extensive numerical experiments with Stefan type problems were successfully carried out. They are described in a report completed before the contract period and about to appear in the International Journal on Numerical Methods in Engineering.

The main draw-back of the fractional step method is the a-priori limitation on the shape of the free surface. To overcome this problem I am now working with the method of lines where all but one of the independent variables are discretized and the resulting system of ordinary differential equations is solved by a continuous analogue of the line SOR method. Along each line I employ invariant imbedding. Again, extensive numerical experiments indicate that the method works well for diverse free boundary problems. A report on this work (abstract attached) will appear in the Journal of the Institute of Mathematics and Its Applications.

An analysis of the convergence properties of this algorithm so far has been carried out only for Poisson's equation with a radiation type boundary condition on a fixed domain. Convergence of the SOR iteration and of the discretization has been shown. This report (abstract attached) has been submitted to Mathematics of Computation.

In summary, I have met no failures yet in applying LOD methods to free boundary problems. I hope to continue to demonstrate that this approach is widely applicable, efficient, simple and mathematically sound.

THE METHOD OF LINES FOR POISSON'S EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia
U.S.A.

This work was supported by the U. S. Army Research Office under Grant DA-AG29-76-G-0261.

Abstract

The method of lines is used to solve Poisson's equation on an irregular domain with nonlinear boundary conditions. The partial differential equation is approximated by a system of second order ordinary differential equations subject to multi-point boundary conditions. The system is solved with an SOR iteration which employs invariant imbedding for each one dimensional problem. It is shown for a model problem with radiative boundary conditions that the SOR iteration converges and that the method of lines solution converges to the solution of the original problem. An application to free boundary problems is noted and the results of some sample calculations are presented.

AMS(MOS) Classification: 65N5, 65N10, 65N15.

AN APPLICATION OF THE METHOD OF LINES TO MULTI-DIMENSIONAL FREE BOUNDARY PROBLEMS

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia
U.S.A.

This work was supported by the U. S. Army Research Office under Grant DA-AG29-76-G-0261.

SUMMARY

Multi-dimensional free boundary problems for elliptic and parabolic differential equations, among them one phase Stefan problems, are partially discretized with the method of lines. The multi-point free boundary problem for the resulting system of second order ordinary differential equations is solved iteratively with an SOR type iteration. At each step of this iteration a free boundary problem for a single ordinary differential equation must be solved for which an initial value technique is used. The method can effectively cope with quite general space and time dependent free surface conditions.

G-37-605

PROGRESS REPORT

1. ARO PROPOSAL NUMBER: P-13874-M

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4. CONTRACT OR GRANT NUMBER: DAAG29-76-G-0261

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Gunter H. Meyer

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:
   1) Direct and iterative one dimensional front tracking methods for the two dimensional Stefan problem, to appear in the 1977 Army Numerical Analysis Conference Proceedings
   2) Heat transfer in an expanding bar, submitted to Int. J. Heat Mass Transfer
   3) Luis Kramarz, Global approximations to solutions of initial value problems, submitted to Math. Comp.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   None

13874M

. GUNTER H. MEYER
ORGIA INSTITUTE OF TECHNOLOGY
HOOL OF MATHEMATICS
LANTA, GA 30332

BRIEF OUTLINE OF RESEARCH FINDINGS

So far, the broad conjecture that locally one-dimensional numerical methods can be used to solve multi-dimensional free surface problems for partial differential equations has proved to be correct. During the first year of the project a number of experimental computer codes based on either a direct fractional step or on an iterative method of lines algorithm have been developed. They allow a straightforward numerical solution of a variety of technical problems which in the past had the reputation of being difficult to solve because of the presence of an a priori unknown and frequently moving surface. For example, by simply changing the data functions the same method of lines program can be used to solve the one phase Stefan problem in heat transfer with change of phase, the Reynolds equation for the hydrodynamic lubrication of a finite journal bearing, and an ablation problem arising during the electrochemical machining of certain metals. In no case are problem dependent mathematical transformations required. Instead, the method applies directly to the model equations. Some of this work has been described at the 1977 Army Numerical Analysis Conference while newer applications are presently being collected in an appropriate report.

Work has now begun on extending these methods to more complicated systems such as Stefan like two phase problems and to more complicated geometries. An attempt is also made to compare the two quite dissimilar methods proposed so far with respect to speed and accuracy, and to classify the problems where one method is to be preferred over the other. As the report #1 listed above indicates the choice of the right method can make the difference between success and relative failure even if a priori requirements placed on the expected solution are met (although in this particular case only marginally). This report also shows the need for more sophisticated locally one dimensional methods leading, perhaps, to a combination of the above two methods.

Concurrently, some new techniques for strictly one dimensional problems are being examined with a view toward building more comprehensive codes for problems with nonlinear diffusion equations or more complicated boundary conditions. We have examined, for example, a problem with distributed boundary conditions arising when the heat transfer model incorporates thermal expansion. This work is described in the report #2 listed above.

Finally, during the past summer grant funds were used to support Luis Kramarz in his thesis research under Prof. W. J. Kammerer. This work concerned initial and boundary value problems for ordinary differential equations which possibly has applications for solving the locally one dimensional equations.

The research has now been completed. A portion of it is described in the report #3 listed above.

DIRECT AND ITERATIVE ONE DIMENSIONAL FRONT TRACKING METHODS FOR THE TWO DIMENSIONAL STEFAN PROBLEM

Gunter H. Meyer

Abstract

An application of the method of fractional steps and of the method of lines to general two dimensional Stefan type problems is described. The numerical aspects are discussed and the performance of both methods for a model problem is compared.

Research supported by the U. S. Army Research Office under Grant DA-AG29-76-G-0261

HEAT TRANSFER IN AN EXPANDING BAR

Gunter H. Meyer

Abstract

A time implicit finite difference method is presented for conductive heat transfer in a bar when the bar is allowed to undergo thermal expansion. The resulting model is a free boundary problems with distributed boundary conditions.

This work was supported by the U. S. Army Research Office under Grant DA-AG29-76-G-0261

GLOBAL APPROXIMATIONS TO SOLUTIONS OF INITIAL VALUE PROBLEMS

Luis Kramarz

Abstract

A wide class of implicit one-step methods for the construction of global approximations to solutions of initial value problems is studied. Approximations more general than piecewise polynomials can be constructed to exploit certain characteristics of the differential equations. Error bounds are given for the general class of methods but emphasis is placed on methods based on Hermite interpolation, for which higher rates of convergence are obtained for special choices of interpolation points. Computational examples are presented.

This research was supported in part by the U. S. Army Research Office under Grant DA-AG29-76-G-0261

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PROGRESS REPORT

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3. TITLE OF PROPOSAL: Locally one-dimensional numerical methods for multi-dimensional free surface problems

4. CONTRACT OR GRANT NUMBER: DAAG 29-76-G-0261

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Gunter H. Meyer

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   1) Direct and iterative one dimensional front tracking methods for the two dimensional Stefan problem, submitted to Numerical Heat Transfer (this report is a revised version of the 1977 ARO report listed in the last progress report)
   2) The numerical solution of multi-dimensional Stefan problems - A survey, to appear in the proceedings of the Gatlinburg Conference on Free Boundary Problems.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   F. Gerwig, M.S. in Applied Mathematics (to be awarded)

13874-M

Dr. Gunter H. Meyer
Georgia Institute of Technology
School of Mathematics
Atlanta, GA 30332

BRIEF OUTLINE OF RESEARCH FINDINGS

Progress was made in fine-tuning the currently existing computer programs for one-phase two-dimensional free surface problems. Such fine-tuning was deemed necessary because the discrepancies in the results for a typical ablation problem where both methods should have been equally applicable. An updated version of an earlier report (which was included in the proceedings of the 1977 Army Numerical Analysis Conference) reflects the improved performance of the fractional step program. This new report has been submitted to "Numerical Heat Transfer" (report #1 listed above).

At the same time computations for the 1977 Gatlinburg Conference showed that the fractional step method could not effectively handle extended horizontal and vertical free surfaces. An attempt to improve the performance by extending the Yanenko fractional step method to a full ADI method failed. As the conference paper (report #2 listed above) shows, severe oscillations arise which indicate that the delicate balancing of the horizontal and vertical sweeps on irregular domains is lost when the ADI method is used in the presence of free boundaries. A more implicit ADI method than the Peaceman-Rachford-Douglas method remains to be examined.

The computer studies for the conference also showed that the method of lines simply and accurately gave answers where ADI methods failed and where competing methods required special starting procedures. In particular, computed results reinforced claims in the literature that popular engineering predictions for the melting of prisms (e.g. Karman-Pohlhausen approximation) are not very accurate. The repeated success of the method of lines approach suggests that its numerical implementation and its theoretical foundations receive continued attention.

Research on one-dimensional problems also continues. During the summer, F. Gerwig, a student in our master's program, carried out some numerical experiments on solving nonlinear two point boundary value problems with a hybrid imbedding-quasilinearization method. The results of these experiments proved inconclusive so far. It is planned to continue work along these lines.

DIRECT AND ITERATIVE ONE DIMENSIONAL FRONT TRACKING METHODS FOR THE TWO DIMENSIONAL STEFAN PROBLEM

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

ABSTRACT. An application of the method of fractional steps and of the method of lines to general two dimensional Stefan type problems is described. The numerical aspects are discussed and the performance of both methods for a model problem is compared.

1. INTRODUCTION. Two and three dimensional free surface problems for partial differential equations lead to many of the same computational difficulties observed with nonlinear boundary value problems on fixed domains. The data management demands and long computing times required for the iterative solution of nonlinear problems place a premium on the efficiency of the numerical method itself and on its implementation on a given computer installation. If a general class of problems can be identified which must be solved repeatedly, then a careful design of a computer algorithm and its code (and the concomitant expense) is justified. If, however, nonstandard problems are to be solved then an ad-hoc method with modest programming demands may be attractive even if computing times are not minimized. In our view free surface problems may be considered nonstandard because of the variety of differential operators and

Research supported by the U.S. Army Research Office under Grant DA-AG29-76-G-0261

THE NUMERICAL SOLUTION OF MULTIDIMENSIONAL STEFAN PROBLEMS - A SURVEY

Gunter H. Meyer

It is the purpose of this paper to give a brief survey over commonly used and some recent methods for the numerical solution of multi-dimensional Stefan and Stefan type problems.

1. STATEMENT OF THE PROBLEM

We shall choose as our model the Stefan problem of conductive heat transfer with change of state. For definiteness, we shall think of a two phase system such as water and ice occupying a given domain $D \subset \mathbb{R}_2$ with prescribed temperature or flux on the boundary ∂D of D. The classical Stefan problem requires the determination of the temperature field over D. The freezing isotherm in D, denoted here by $\Gamma(t)$, is a free surface which is not known a priori but must be found as part of the solution of the problem. For ease of notation it is assumed that D consists of two regions D_1 and D_2 occupied by ice and water, respectively, and separated by $\Gamma(t)$.

In D_i the temperature u_i has to satisfy the usual heat equation

$$L_i u_i \equiv \nabla \cdot k_i \nabla u_i - c_i \rho_i \frac{\partial u_i}{\partial t} = f_i, \quad i = 1,2 \tag{1.1}$$

where the thermal properties and the source terms may be functions of space and time. In applications, these quantities may also depend on the temperature; however, linearization and iteration are frequently employed so that at least in each iteration equations of the form (1.1) occur.

Variations in the formulation of the Stefan problem are due

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PROGRESS REPORT

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6. AUTHOR(S) OF REPORT: Gunter H. Meyer

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   Numerical Simulation of Electro-Chemical Machining, to appear in the 1978 Army Numerical Analysis Conference Proceedings.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   None.

13874-M

Dr. Gunter H. Meyer
Georgia Institute of Technology
School of Mathematics
Atlanta, GA 30332

NUMERICAL SIMULATION OF ELECTRO-CHEMICAL MACHINING

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

ABSTRACT. Anode erosion in an electrolytic cell can be described by a free boundary problem for the potential equation. It is shown that the method of lines solution technique allows the numerical simulation of electro-chemical machining of anodes for a variety of tool and work piece shapes.

1. INTRODUCTION.

While electro-chemical machining has become a common metal forming process, its numerical simulation remains a difficult and largely unsolved problem. Yet full scale simulation can aid in choosing tool shapes and feed rates and thus have a direct influence on the design and utilization of electro-forming machinery. The lack of simulation appears to be due not so much to the difficulty of building a mathematical model for the process but to the difficulty of solving the model equations for realistic applications.

Ideally, one would like to determine the tool shape necessary to produce a specified work piece. Unfortunately, such prediction would have to be based on an initial value problem for the potential equation which is mathematically unstable. To date such formulation remains practically unsolvable. On the other hand, the prediction of the work piece shape for a given tool leads to a stable mathematical boundary value problem with a moving surface for which numerical methods are now becoming available.

It is the purpose of this note to indicate how a fairly simple numerical method for the solution of the potential equation can be used to predict the shape of the work piece under dynamic working conditions. The method is implicit in time and yields simultaneously the potential in the electrolyte and the anode surface. Implicit methods as a rule allow larger time steps and more irregular anode shapes than competing explicit methods where the anode is predicted and the field equation is solved in the predicted gap between the electrodes. Moreover, neither special scaling of the geometry is required to avoid the spurious

Research sponsored by the U. S. Army Research Office under Grant DA-AG29-76-G-0261.

BRIEF OUTLINE OF RESEARCH FINDINGS

Considerable time was spent on modifying an existing 2 dimensional method of lines code for Stefan like problems to handle the full 3 dimensional problem. We now have an operational 3-D code for one phase problems with Dirichlet and/or Neumann data. The program is not particularly efficient and will require more research into the space and time discretization and the iteration algorithm; however, to our knowledge it stands without competitor for full 3-D problems.

We continually are looking for new applications leading to free surface problems in order to focus our research on useful problems. Electro-chemical machining is one such application where mathematical models and simulation are slowly coming into their own. An application of the method of lines to the prediction of the work piece surface during machining of metal is described in the above report.

To deepen our understanding of the method of lines we are now trying to establish the convergence of the numerical method for certain free boundary problems. As model problem we have chosen Reynolds equation for a journal bearing. Preliminary results indicate that one can bracket the solution of the discretized free surface problem between monotone sequences. This research is very much in progress.

6-37-605

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7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:
   1. Invariant imbedding for elliptic and parabolic free boundary problems, submitted to the SIAM Journal on Numerical Analysis.
   2. The method of lines for the Reynolds equation in hydrodynamic lubrication, submitted to the SIAM Journal on Numerical Analysis

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   D. Dvorak, graduate student, Summer 78.

13874-M

Les Andrew Karlovitz
Georgia Institute of Technology
School of Mathematics
Atlanta, GA 30332

6-37-605

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   D. Dvorak, graduate student, Summer 78.

13874-M

Les Andrew Karlovitz
Georgia Institute of Technology
School of Mathematics
Atlanta, GA 30332

BRIEF OUTLINE OF RESEARCH FINDINGS

During the summer months numerous computer runs were carried out with a variety of one, two and three-dimensional free surface problems. It was established that the method of lines can effectively solve combined heat and mass transfer problems arising in binary alloy solidification as well as certain fluid flow problems in a porous medium. In particular it was found that our method, at least qualitatively, can reproduce the almost unstable solutions arising in Hele-Shaw flow with suction. The results of our numerical experiments were presented during a ten-day workshop in Durham, U.K., and have appeared in a Brunel University Technical Report (see [1] below). Present experimental work is concentrating on "unstable" problems.

Concurrently with these numerical experiments the theoretical analysis of the method of lines approach to free boundary problems has been completed for a model problem arising in hydrodynamic lubrication. It has been shown that the discretized free surface problem can be solved with a convergent SOR iteration and that the solution of the discretized problem converges to that of the continuous problem (see [2] below). An extension of this result to a more complicated class of problems arising in porous media flow is presently our main theoretical research objective.

During the summer quarter David Dvork, a graduate student, was supported to instal V. Peyrera's TPBVP solver PASVA 3 on file at Georgia Tech. This project was an object lesson in the importance of "portability".

The Method of Lines and Invariant Imbedding for Elliptic and Parabolic Free Boundary Problems

Gunter H. Meyer

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

A combination of the method of lines and invariant imbedding is suggested as a general purpose numerical algorithm for free boundary problems. Its effectiveness is illustrated by computing the solidification of a binary alloy in one dimension, electrochemical machining and Hele-Shaw flow in two dimensions, and a Stefan and ablation problem in three dimensions.

This work was supported by the U.S. Army Research Office under Grant DA AG29-76-G-0261. The manuscript was completed while the author was a Senior Visiting Fellow in the Institute of Computational Mathematics at Brunel University under Science Research Council Grant GR/A49256.

THE METHOD OF LINES FOR THE REYNOLDS EQUATION IN HYDRODYNAMIC LUBRICATION

Gunter H. Meyer

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332.

ABSTRACT

Reynolds equation for a hydrodynamic journal bearing is discretized with the method of lines. A continuous analog of the line SOR iteration is set up to solve the resulting free multipoint system of ordinary differential equations via a sequence of one dimensional free boundary problems. It is shown that each one dimensional problem can be solved with invariant imbedding, that the line SOR method converges, and that the solution of the by-lines discretization converges to the solution of a variational inequality derived from Reynolds equation.

This work was carried out while the author was Senior Visiting Fellow in the Institute of Computational Mathematics at Brunel University, England. The research was supported in part by the U.S. Army Research Office under Grant DA AG29-76-G0261 and by the Science Resarch Council of Great Britain under Grant GR/A49256.

G-37-605

GUNTER H. MEYER

Final report on work performed under Grant DAAG 29-76-G-0261 on "Locally one dimensional numerical methods for multidimensional free surface problems"

The applicability of locally one dimensional methods to multidimensional free boundary problems was examined. Such problems require the solution of an elliptic or parabolic field equation over a domain which is unknown a priori and which must be determined from over-prescribed boundary data. Our main objective was to identify a numerical method for such problems which is widely and easily applicable, but which at the same time has a solid mathematical foundation.

Initially, we examined the method of fractional steps which for a multidimensional diffusion equation leads to a sequence of loosely coupled one dimensional free boundary problems in alternating orthogonal directions. In a report prepared just prior to the grant period this method is introduced and used together with invariant imbedding to solve a variety of Stefan type free boundary problems. The method works well and allows comparable numerical resolution in all space directions. However, it does impose certain restrictions on the shape of the free boundary which serverely limit the applicability of the ADI method.

To overcome this limitation we began working with the method of lines based on a simple finite difference discretization of time and of all but one space variable. The original multidimensional equation is thus replaced by a system of one dimensional problems along parallel lines. This system is solved in line SOR fashion so that again a sequence of uncoupled one dimensional problems results. As above we apply invariant imbedding to each one dimensional problem. A description of this approach appears in [1]. A comparison with the fractional step method for a two dimensional ablation problem [2] shows that only relatively few lines compared to the number of such points along each line should be used and hence that the numerical resolution is not equally fine in all directions. Nonetheless, even in this severe test problem the method of lines performs as well as the ADI method. Because free surfaces of more general shape are admissible further work has concentrated on the method of lines/SOR/invariant imbedding.

The success of this method, even in the crude implementation employed as far, was good motivation to establish the mathematical validity of the algorithm. As a first step convergence of the iterative numerical method and the discretization was established for Poisson's equation subject to nonlinear boundary data on a fixed domain [3]. Recently, the arguments were extended to establish convergence for an elliptic free boundary problem arising in hydrodynamic lubrication [4]. As far as we are aware this is the first time that a multidimensional front tracking algorithm can be completely analyzed. Current work is concerned with convergence for the equations arising in fluid flow through a porous dam. This problem is characterized by considerably more complicated boundary conditions than present in the lubrication problem.

Although we are primarily concerned with multidimensional problem the one dimensional solver is at the heart of our method and we continue to gather experience with the invariant imbedding method. An application to a problem with distributed boundary conditions is described in [7].

During the grant period we particapted in two conferences dedicated to free and moving boundary problems. A survey over numerical methods for Stefan problems and a comparison with the method of lines was presented in [5], while an application of our methods to a variety of practical problems including heat and mass transfer, Hele-Shaw flow, and three dimensional ablation is described in [6].

Two additional publications were prepared for the 1977 and 1978 Army Numerical Analysis Conferences. A revision of the first report was published as [2] while the material of the second is contained in [6].

Colloquium talks on our work were presented during the past three years of several universities and laboratories in the U.S., Canada, Austria, Germany and Great Britain.

In summary, the basic premise that sequential one dimensional methods provide a useful tool for multidimensional free boundary problems has proved to be correct. It remains to establish firmly the limits of their applicability and to provide an analysis for a wider class of problems than was possible so far. First and foremost, however, the software aspects of our methods deserve attention in order to improve the efficiency of our methods.

During the grant period the following graduate student assisted in our project.

L. Kramarz, Ph.D. from Georgia Tech, 1977, [8]
F. Gerwig, M.S. from Georgia Tech, 1977
D. Dvorak, graduate student in mathematics

Papers and reports prepared during the grant period

1. G. H. Meyer, An application of the method of lines to multi-dimensional free boundary problems, J. Inst. Maths. Applcs. 20 (1977), 317-329.

2. , Direct and iterative one dimensional front tracking methods for the two dimensional Stefan problem, Numerical Heat Transfer 1 (1978), 351-369.

3. , The method of lines for Poisson's equation with non-linear or free boundary conditions, Numer. Math. 29 (1978), 329-344.

4. , The method of lines for the Reynolds equation in hydro-dynamic lubrication, Brunel Inst. of Comp. Maths. Tech. Report TR 21, 1978, submitted for publication.

5. , The numerical solution of multidimensional Stefan problems - a survey, in Moving Boundary Problems, D. G. Wilson, A. D. Solomon, P. T. Boggs, edts., Academic Press, N. Y. 1978.

6. , Invariant imbedding for elliptic and parabolic free boundary problems, Brunel Inst. of Comp. Maths. Tech Report TR 13, 1978, submitted for publication.

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8. L. Kramarz, Global approximations to solutions of initial value problems, Math. Comp. 32 (1978), 35-59.

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...T FISCAL REPORT (AR 70-5) ( ) INTERIM (X) FINAL	UNITED STATES OF AMERICA DEPARTMENT OF THE ARMY	DATE: December 20, 1979

...OM (GRANTEE) ADDRESS:	TO (GRANTOR) ADDRESS:
...orgia Institute of Technology ...lanta, Georgia 30332	U. S. Army Research Office P. O. Box 1211 Research Triangle Park, N. C. 27706
...PORT OF EXPENDITURES UNDER GRANT NO.: DAAG29-76-G-0261	REPORTING PERIOD FROM: 6/1/76 TO: 7/31/79

...TEM NO.	ITEM	GRANTOR	GRANTEE
.	AMOUNT OF EXPENDITURES:		
1.	Salaries and Wages	$26,012.77	$10,704.64
2.	Equipment Purchased with Grant Funds:	XXXXXXXXXXXX	XXXXXXXXXXXX
	Title Vested in Grantee		
	Title Retained by the Government		
3.	Supplies, Materials, and Expendable Equipment	1,054.87	
4.	Travel	1,122.26	
5.	Publication Costs (Total Page Costs, Reprints, etc.):	XXXXXXXXXXXX	XXXXXXXXXXXX
	Page Costs Only, if Available		
	Reprints, Direct Labor, Any Other Costs	430.00	
6.	Other (Specify) Computer	3,068.44	
7.	Total Direct Costs (Sum of Lines 1 through 6)	31,688.34	10,704.64
8.	Indirect Costs	16,313.20	6,638.11
9.	Total Costs Expended (Sum of Lines 7 and 8)	48,001.54	17,342.75
B.	AMOUNT OF GRANT	48,067.00	
C.	UNEXPENDED BALANCE	65.46	

Government Furnished Equipment (GFE):
The Grantee is required to identify, either on the reverse side or by separate attachment, other equipment furnished by the Government. The listing of the equipment by source of acquisition (GSA, DIPEC, DARO, etc.), unit cost, and by any other pertinent identifying data, is required. This listing should be forwarded to the Government Property Administrator for disposition purposes. Title to the equipment may vest with the Grantee. If no GFE was acquired under the grant, so state in the "Comments" section of this report.

COMMENTS: (CONTINUE ON REVERSE SIDE, IF NECESSARY)

I CERTIFY THAT THIS FISCAL REPORT IS CORRECT AND THAT THE EXPENDITURES INCLUDED HEREIN ARE DEEMED PROPERLY CHARGEABLE TO THE GRANT

...YPED NAME AND TITLE	SIGNATURE
David V. Welch, Manager, Grants & Contracts Accounting	1/10/80

SUGGESTED FISCAL REPORT FORMAT

1/10/80

Dr. G. H. Meyer Professor Date