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Inverse Dynamics and Fourier-Based Approach to Solve Optimal Control Problems for Multi-Link Mechanisms

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Abstract. This paper is concerned with the investigation of multi-link mechanism controlled motion between fixed boundary conditions and given constraints on the phase coordinates. The moduli of the controlling moments at the hinges between the links are bounded. A computational method to the mathematical modeling of the optimal control laws which govern multi-link mechanism reaching motion is presented. This method is based on Fourier and spline approximations of the independently variable functions and inverse-dynamics approach. The method proposed makes it possible to satisfy the boundary conditions and some constraints on the phase coordinates automatically and accurately. The efficiency of the proposed method is illustrated by the solution of the energy-optimal control problem of a plane mobile three-link manipulator with a load on the grip.

Key words: multi-link mechanism, optimal control law, manipulator, inverse-dynamics approach, Fourier and spline approximation.

1. Statement of the Problem

Consider a multi-link mechanism, whose motion is described by Lagrange equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{z}_i} - \frac{\partial T}{\partial z_i} = D_i(z)u + Q_i(z,\dot{z},t), \qquad \text{i=1,...,n} \tag{1}$$

Here $z=(z_1,..,z_n)$ are generalized coordinates for the mechanism, n is the number of its degrees of freedom, $u=(u_1,..,u_n)$ is the vector of control forces, and the dot denotes differentiation with respect to time.
The generalized forces consist of control forces u_i that are to be determined together with the terms $Q_i(z,\dot{z},t)$ which consist of all the remaining outer and inner forces.

The kinetic energy of the mechanism is given by the quadratic form

$$T(z,\dot{z}) = \frac{1}{2}\sum_{i,j} a_{ij}(z)\dot{z}_i\dot{z}_j \tag{2}$$

where the a_{ij} are elements of a symmetric positive-definite (n×n)- matrix A(z) and the summation is carried out over values of i and j from 1 to n.
Using (1) and (2) we can reduce the equations of motion to the form

$$A(z)\ddot{z} + B(z,\dot{z},t) = D(z)u \tag{3}$$

where $B(z,\dot{z},t)$ and *D(z)* are the known vector-valued function and (n×n) matrix, respectively.

Suppose that the initial time (t=0) and final time (t=τ) of the control process are specified together with the boundary conditions

$$z(0) = z_0, \dot{z}(0) = \dot{z}_0, \ddot{z}(0) = \ddot{z}_0 \tag{4}$$
$$z(\tau) = z_\tau, \dot{z}(\tau) = \dot{z}_\tau, \ddot{z}(\tau) = \ddot{z}_\tau \tag{5}$$

Let the phase state of the mechanism have to satisfy the following constraints over the time $t \in [0,\tau]$

$$g(z,\dot{z}) \le 0 \tag{6}$$

Since any physical solution for the control variables $u_i(t)$ of the mechanism must not exceed some upper bounds M_i, respectively, we assume that for all $t \in [0,\tau]$

$$| u_i(t) | \le M_i, \qquad i = 1,.., n \tag{7}$$

Consider the problem:
<u>Problem A.</u> It is required to determine the controlled process of the mechanism $\{z(t),u(t), t\in[0,\tau]\}$ which satisfy the equations of motion (3), the boundary conditions (4), (5), given constraints on the phase coordinates (6), given restrictions on the controlling stimuli (7) and which minimize the functional

$$\Phi = \int_0^{\tau} F(z,\dot{z},u)dt \tag{8}$$

2. Inverse Dynamics and Fourier-Based Approach

The necessary conditions of optimality in Problem **A** can be obtained by well-known methods of optimal control theory, e.g. using the Pontryagin's maximum principle [1].
But as usually the numerical solution of optimal control problems for multidimensional nonlinear systems based on the necessary conditions of optimality is very complicate.

On the other hand, the low-cost determination and using of the suboptimal controlled processes of multi-link mechanisms is often very attractive for practical application.

Central in approach proposed for solving problem A is the idea that any optimal control problem can be converted into a standard nonlinear programming problem by parameterizing each of the variable functions. We shall assume that the generalize coordinates z_i are the variable functions which could be represented as follows [2,3]:

$$z_i(t)=P_i(t)+Q_i(t), \quad \mathrm{i}=1,...,\mathrm{n} \tag{9}$$

$$P_i(t) = \sum_{j=0}^{5} p_{ij} t^j$$

$$Q_i(t) = \sum_{k=1}^{N_i} (a_{ik} \cos\frac{2\pi k}{\tau} t + b_{ik} \sin\frac{2\pi k}{\tau} t)$$

where N_i, i=1,..,n are given integer positive numbers.
The boundary conditions (4), (5) lead to the following expressions:

$$P_i(0) + Q_i(0) = z_{i0}, \qquad \dot{P}_i(0) + \dot{Q}_i(0) = \dot{z}_{i0},$$

$$\ddot{P}_i(0) + \ddot{Q}_i(0) = \ddot{z}_{i0}, \qquad P_i(\tau) + Q_i(\tau) = z_{i\tau},$$

(10)

$$\dot{P}_i(t) + \dot{Q}_i(t) = \dot{z}_{it}, \qquad \ddot{P}_i(t) + \ddot{Q}_i(t) = \ddot{z}_{it},$$

Considering the formula (10) it is possible to find the coefficients of the function $P_i(t)$ in analytical form:

$$p_{i0} = z_{i0} - \sum_{k=1}^{N_i} a_{ik}$$

$$p_{i1} = \dot{z}_{i0} - \frac{2\pi}{\tau}\sum_{k=1}^{N_i} kb_{ik}$$

$$p_{i2} = \frac{1}{2}\ddot{z}_{i0} + \frac{2\pi^2}{\tau^2}\sum_{k=1}^{N_i} k^2 a_{ik}$$

$$p_{i3} = [10(z_{i\tau} - z_{i0}) + 20\pi\sum_{k=1}^{N_i} kb_{ik} - 4\pi^2\sum_{k=1}^{N_i} k^2 a_{ik}]/\tau^3 - (6\dot{z}_{i0} + 4\dot{z}_{i\tau})/\tau^2 - (3\ddot{z}_{i0} - \ddot{z}_{i\tau})/2\tau$$

$$p_{i4} = [15(z_{i0} - z_{i\tau}) - 30\pi\sum_{k=1}^{N_i} kb_{ik} + 2\pi^2\sum_{k=1}^{N_i} k^2 a_{ik}]/\tau^4 + (8\dot{z}_{i0} + 7\dot{z}_{i\tau})/\tau^3 + (\frac{3}{2}\ddot{z}_{i0} - \ddot{z}_{i\tau})/\tau^2$$

$$p_{i5} = [6(z_{\tau 0} - z_{i0}) + 12\pi\sum_{k=1}^{N_i} kb_{ik}]\tau^5 - 3(\dot{z}_{i0} + \dot{z}_{i\tau})/\tau^4 - (\ddot{z}_{i0} - \ddot{z}_{i\tau})/2\tau^3$$

Suppose that the law of motion of the multi-link mechanism, i.e., vector-value function z(t) is given by formula (9), (10). Using the equation (3) the inverse dynamics problem can be solved. All above mention give possible to convert the problem A into the parameter optimization problem: $\Phi = F(\mathbf{C}) \rightarrow \min$, $f(\mathbf{C}) \leq 0$.
Here the functions F and f are determined by means of (3)-(10), $\mathbf{C}=(a_{ik},b_{ik};k=1,..,N_i, i=1,..., n)$ is a vector of the variable parameters.
To solve this parameter optimization problem, the computation algorithm based on Rosenbrok's method [4] has been devised.

3. The Energy-Suboptimal Controlled Motions of the Mobile Three-Link Manipulator

Consider the plane multibody system which comprises the four elements model the mobile three-link manipulator (Fig.1). The motion of the body P (mobile platform) is a translation in the sagittal plane OXY of a fixed rectangular Cartesian coordinate system OXYZ.
The translation of the platform is executed under the action of the control force **R(t)**.
It is assumed that control moments $p_\alpha(t), p_\beta(t)$ and $p_\gamma(t)$ act in the manipulator joints S,E, and W, respectively.
In addition to the weights of the platform and manipulator links the external load acts on the system which model by the force **F(t)** and the moment $\mu(t)$ (Fig.1).
Let involve the following notations: x and y are the Cartesian coordinates of the joint S; α,β,γ are angles that specify the position of the links of the manipulator; m_i,l_i,r_i and J_i are the mass, length, the distance to the center of mass and the moment of inertia of the links SE, EW and WH with respect to the center of mass, respectively; m_0 is the mass of mobile platform; $R_x(t),R_y(t)$,and $F_x(t),F_y(t)$ are the horizontal and vertical

components of the control force **R(t)** and load force **F(t)**, respectively; g is the acceleration due to gravity.

The motion of the mobile manipulator can be described in terms of generalized coordinates $z=(x,y,\alpha,\beta,\gamma)$ and controlling stimuli $u=(R_x,R_y,p_\alpha,p_\beta,p_\gamma)$ through the application of Lagrange's equations (1).

The kinetic and potential energies of the mobile manipulator are given by

$$T=\frac{1}{2}[M(\dot{x}^2+\dot{y}^2)+J_a\dot{\alpha}^2+J_b\dot{\beta}^2+J_c\dot{\gamma}^2]+K_a h(\alpha)+K_b h(\beta)+K_c h(\gamma)+$$
$$l_1 K_b\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)+K_c\dot{\gamma}[l_1\dot{\alpha}\cos(\alpha-\gamma)+l_2\dot{\beta}\cos(\beta-\gamma)] \qquad (11)$$

$$\Pi=g(My-K_a\cos\alpha-K_b\cos\beta-K_c\cos\gamma) \qquad (12)$$

where $J_a=J_1+m_1r_1^2+(m_2+m_3)l_1^2$ $J_b=J_2+m_2r_2^2+m_3l_2^2$, $J_c=J_3+m_3r_3^2$

$K_a=m_1r_1+(m_1+m_2)l_1$, $K_b=m_2r_2+m_3l_2$, $K_c=m_3r_3$, $M=m_0+m_1+m_2+m_3$

$h(\xi)=\dot{\xi}(\dot{x}\cos\xi+\dot{y}\sin\xi)$, $\quad \xi=(\alpha,\beta,\gamma)$

Using the formula (1),(11),(12) we can obtain the equations of controlled motion of the mobile manipulator in the form (3).

Let us describe some results of solution the **Problem A** for considered manipulator.
In the model the following parameters of the links have been considered :

$$\begin{aligned}&m_0=0,\ m_1=2.17\text{ kg},\ m_2=1.26\text{ kg},\ m_3=0.53\text{ kg}\\&l_1=0.32\text{ m},\ l_2=0.247\text{ m},\ l_3=1.184\text{ m}\\&r_1=0.147\text{ m},\ r_2=0.105\text{ m},\ r_3=0.184\text{ m}\\&J_1=0.014\text{ kgm}^2,\ J_2=0.007\text{ kgm}^2,\ J_3=0.006\text{ kgm}^2\end{aligned} \qquad (13)$$

The linear and mass-inertial parameters (13) correspond to respective parameters of the human upper extremity[5].

The boundary conditions were specified by formula (4),(5) with

$$\dot{z}_0=\ddot{z}_0=\dot{z}_\tau=\ddot{z}_\tau=0,\ x(0)=x(\tau)=0,\ y(0)=y(\tau)=l,\ l=l1+l2+l3 \qquad (14)$$

$$\alpha(0)=0,\ \alpha(\tau)=\pi/2$$

The constraints which were imposed on the phase coordinates are as follows

$$\alpha(t)=\beta(t)=\gamma(t)\ ,\quad t\in[0,\tau]$$
$$-\pi/3\le\alpha\le\pi\ ,\quad t\in[0,\tau] \qquad (15)$$

The objective functional was determined by formula (8) and

$$F=\frac{1}{\tau M^2 g^2 l^2}\left(p_\alpha^2(t)+p_\beta^2(t)+p_\gamma^2(t)\right) \qquad (16)$$

The functional (8),(16) under some conditions [6-8] can be used for evaluation of energy expenditure of manipulator's controlled motion.

The external load was given over the time $t \in [0,\tau]$ by the following way

$$F_x(t)=-Mg,\ \ F_y(t)=0,\ \ \mu(t)=0$$

Energetically optimal law of motion of three-link manipulator within the frame of above-mentioned conditions is determined by formula (9),(10),(14),(15) and the following values of free parameters :

$a_{\alpha1}$=3.099609 $a_{\alpha2}$ =0.181234 $a_{\alpha3}$=0.019103 $a_{\alpha4}$=0.001501
$a_{\alpha5}$ =0.002017 $a_{\alpha6}$=0.000703 $a_{\alpha7}$=-0.000124 $a_{\alpha8}$ =0
$a_{\alpha9}$= -0.000195 $a_{\alpha10}$=-0.000195 $a_{\alpha11}$ =-0.000351 $a_{\alpha12}$=0 $a_{\alpha13}$=0 $a_{\alpha14}$ =0

$b_{\alpha1}$=15.441740 $b_{\alpha2}$=0.161301 $b_{\alpha3}$=0.015997 $b_{\alpha4}$=0.001478
$b_{\alpha5}$=-0.001638 $b_{\alpha6}$=-0.002373 $b_{\alpha7}$=-0.001727 $b_{\alpha8}$=-0.001171
$b_{\alpha9}$=-0.000781 $b_{\alpha10}$=-0.000585 $b_{\alpha11}$=-0.000351 $b_{\alpha12}$=-0.000195
$b_{\alpha13}$=-0.000195 $b_{\alpha14}$=0

τ=1.202109 s

Figure 2 shows the phase picture of manipulator motion (α -shoulder angle in rad, angular velocity in rad/s ; dashed curve corresponds to energetically optimal controlled process, solid curve is the initial phase picture, i.e., phase picture of the motion before optimization procedure).

Figure 3 shows graphs of the joint torques for the obtained energetically optimal law of motion of the manipulator (solid curve corresponds to dimensionless torque $p_\alpha(t)/Mgl$, dashed to $p_\beta(t)/Mgl$ and centered to $p_\gamma(t)/Mgl$). For given external load the torques in the joints of the manipulator are the same order.

The dimensionless horizontal and vertical components of the reaction force at joint S for the obtained energetically optimal law of motion are shown in Fig.4 (solid curve corresponds to $R_x(t)/Mg$, dashed to $R_y(t)/Mg$). The extremal values of the reaction force components are approximately 2-4 times greater than the entire weight of the manipulator.

4.Discussion and Conclusion

In this paper the analysis of a controlled motion of multi-link mechanism is based on the solution of the optimal control problem for a dynamical system described by Lagrange equations of the second kind.

To solve the nonlinear optimal control problem under the given boundary conditions, the restrictions on the phase coordinates and on the controlling stimuli we proposed a computational method based on the special Fourier and spline approximation of the independently variable functions and inverse-dynamics approach.

Using the computational method proposed an algorithm and respective software to solve the energy-optimal problems for the mobile three-link manipulator have been created. In this paper some results of solution the problem of energy-optimal lifting the given mass from lowest state to the final position corresponding to the horizontal state of the erect manipulator arm with fixed platform. As numerical calculations have shown, for a given load there is energy-optimal motion of the erect manipulator arm when the load from initial to final state is transferred.

It can be seen from Fig.2 (dashed curve) that obtained optimal motion of the erect arm comprises the reverse motion of the system (the part of phase trajectory with negative angular velocity).

For comparison the phase picture of energy optimal motion of manipulator found with considering the constraints

$$\beta(t)=\gamma(t), \ -\pi/3 \le \alpha(t) \le \pi \tag{17}$$

instead of the constraints (15) is shown in Fig.2 (centered curve). The optimal law of motion obtained under the constraints (17) has also the part of phase trajectory with negative angular velocity of the link SE. Several numerical calculations have shown that for considered manipulator as a rule the obtaining energy-optimal motions comprise the reverse motion.

The proposed computational method belongs to suboptimal approaches because only finite number of Fourier series are used for modeling the optimal trajectory. But when the number of Fourier series increase Then the value of suboptimal functional converges to its minimal value (See Fig.5, solid curve corresponds to solution of the energy-optimal control problem for manipulator with constraints (15), centered curve to solution of the same problem with constraints (17)). As numerical calculations show that if we refuse from constraint $\alpha(t)=\beta(t)$, $t \in [0,\tau]$ the time for execution of the energy-optimal control motion is about 30% of time of optimal process under the constraints (15).

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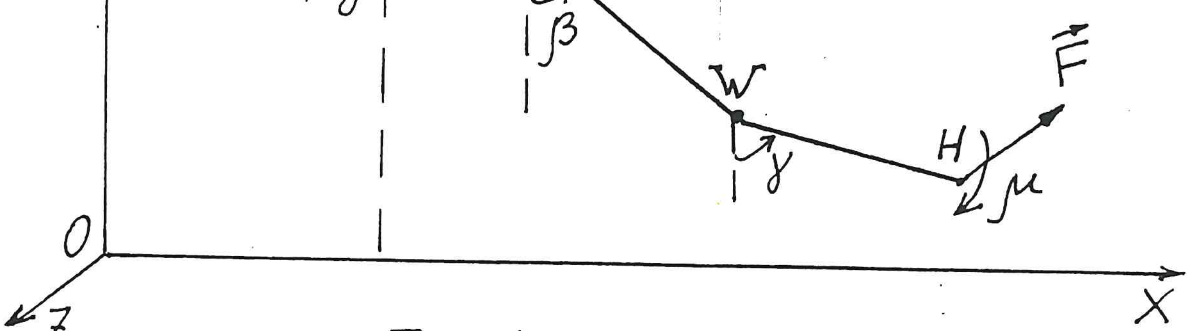

Fig.1

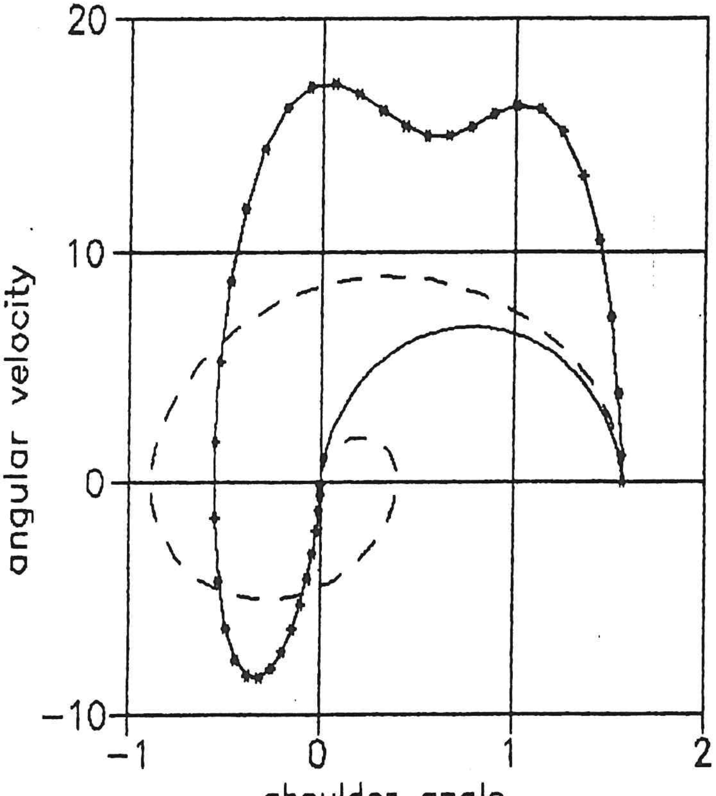

Fig.2

Fig.3

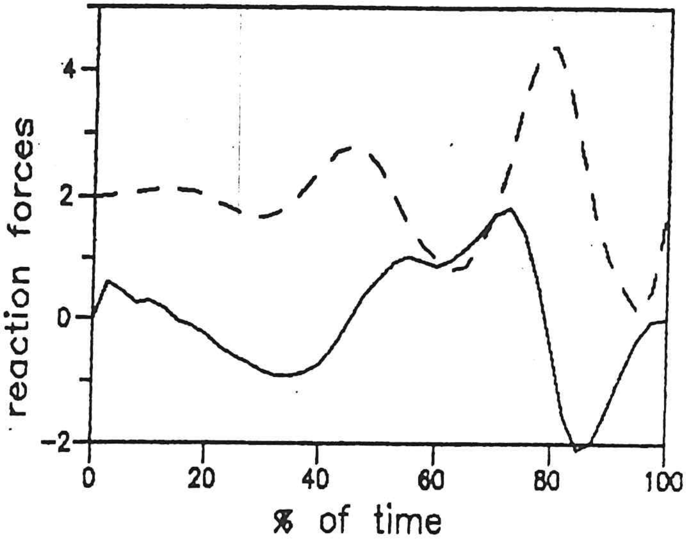

Fig.4

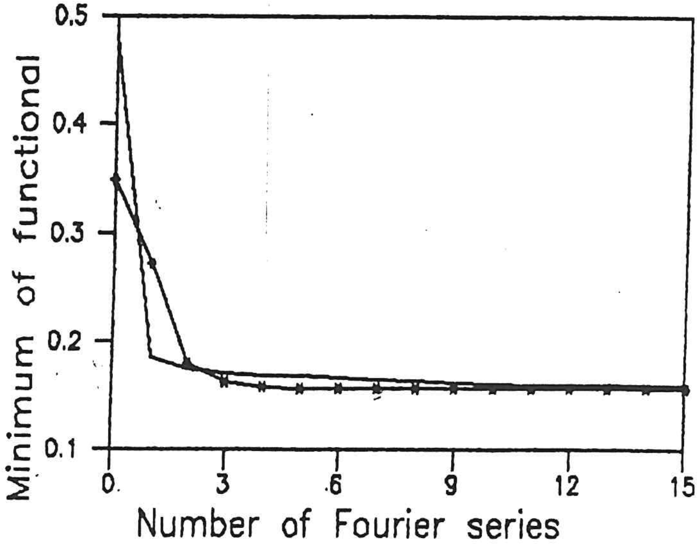

Fig.5