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# Wigner Measures and Coherent Quantum Control

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**Abstract**—We introduce Wigner measures for infinite-dimensional open quantum systems; important examples of such systems are encountered in quantum control theory. In addition, we propose an axiomatic definition of coherent quantum feedback.

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## 1. INTRODUCTION

The Wigner measure is a generalization of the notion of the Wigner function, which was introduced by Wigner (see [5] and references therein), and the representation of states of quantum systems in terms of Wigner measures is similar to the representation of states of classical Hamiltonian systems in terms of probability measures on the phase space. Describing the dynamics of open quantum systems in terms of Wigner measures allows one to apply methods similar to those used in describing the dynamics of open Hamiltonian systems. In particular, to pass to the description of a state of a subsystem of some larger quantum system, one can use the projection operation.

In the case where the dimension of the phase space is finite, the Wigner measure has density with respect to the Liouville measure on the phase space. This density coincides with the Wigner function introduced by Wigner; thus, the Wigner measure is a generalization of the Wigner function. Since (by Weil’s well-known theorem) there is no Liouville measure (i.e., Borel  $\sigma$ -additive  $\sigma$ -finite locally finite measure invariant under symplectic transformations) on an infinite-dimensional phase space, in this case one can either directly use the Wigner measure or introduce, instead of the Liouville measure, some “good” measure that, however, is not invariant under symplectic transformations; for example, if the phase space is linear, one can use the Gaussian measure. After that, one can again replace the Wigner measures by their densities with respect to the new measure, i.e., by Wigner functions. Moreover, if a Liouville measure does not exist, one can instead use a generalized measure invariant under the same symplectic transformations, which is naturally called a generalized Liouville measure; in this case, instead of the Wigner measure, one can consider its generalized density (cf. [10]) with respect to the generalized Liouville measure. This density is called a generalized Wigner function (see Definition 2 below).

Henceforth we consider in parallel the Wigner measures and their densities, including generalized ones, called Wigner functions and generalized Wigner functions, respectively.

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**Remark 1.** Integrals with respect to the generalized Liouville measure are analogous to the integrals from Feynman's first works (see [14] and references therein), in which the integrands are the exponentials of the classical action represented in the Lagrangian or Hamiltonian form. Moreover, one can assume that the integration in Feynman's studies is performed with respect to a translationally invariant generalized measure, which is an analog of the standard Lebesgue measure and is naturally called a generalized Lebesgue measure or a generalized Lebesgue–Feynman measure (in Feynman's studies, the integrals are defined as the limits of finitely multiple integrals). The classical action used there by Feynman contains a quadratic functional as an additive term. Mathematicians regard the exponential of the product of this functional and the imaginary unit as the generalized density of a generalized measure, which they call the Feynman measure. Then the integrals in Feynman's studies are viewed as the integrals of the exponential of the remaining part of the action with respect to this generalized measure. Thus, one can say that Feynman himself did not consider the integration with respect to the generalized measure called the Feynman measure by mathematicians; he used integration with respect to the generalized Lebesgue(–Feynman) measure. In this connection, we mention the title of the book [9], *Mathematical Feynman Path Integrals . . .*, which reflects the essence much more accurately than the possible title *Theory of Feynman Integrals . . .*

The paper is organized as follows. In Section 2, we consider the properties of Wigner measures and Wigner functions, including generalized ones; part of the results presented there can be viewed as a generalization of some results of [5]. In Section 3, we derive an equation describing the evolution of Wigner functions (including generalized ones) of quantum systems obtained by quantizing Hamiltonian systems with infinite-dimensional phase space; this equation is a consequence of a similar equation for the evolution of the Wigner measure (see [6]). Moreover, we consider the evolution of the Wigner measures and (generalized) Wigner functions of open quantum systems. Note that the Wigner measure is a signed cylindrical measure, and it would be interesting to study its properties; however, we do not address these issues in the present paper. Finally, Section 4 deals with quantum control theory (more information can be found in [2, 11, 1]). In particular, here we present an axiomatic definition of coherent quantum feedback. Apparently, this definition was first introduced in [4], which is a preliminary version of the present paper. Note, however, that the generalized Liouville measures and generalized Wigner functions used below do not appear in [4]. We consider the algebraic aspects of the theory and do not touch on assumptions of analytical character.

## 2. WIGNER MEASURES AND GENERALIZED WIGNER FUNCTIONS

Let  $E := Q \times P$  be the phase space of a Hamiltonian system, where  $Q$  and  $P$  are real locally convex spaces (LCSs) such that  $P = Q^*$  and  $Q = P^*$  (if  $X$  is an LCS, then  $X^*$  is its dual endowed with a locally convex topology consistent with the duality between  $X$  and  $X^*$ ); hence,  $E^* = P \times Q$ . Let, in addition,  $\langle \cdot, \cdot \rangle : P \times Q \rightarrow \mathbb{R}$  be the bilinear form providing the duality between  $P$  and  $Q$ . In this case, the linear mapping  $J : E \ni (q, p) \mapsto (p, q) \in E^*$  is an isomorphism, and we identify an element  $h \in E$  with  $Jh \in E^*$ . In particular, if  $h \in E$ , then  $\widehat{h}$  is a pseudodifferential operator in  $\mathcal{L}_2(Q, \mu)$  whose Weyl symbol<sup>1</sup> is the function  $Jh \in E^*$ . By  $\mu$  we denote a  $P$ -cylindrical (Gaussian) measure on  $Q$  whose Fourier transform  $\Phi_\mu : P \rightarrow \mathbb{R}$  is defined by  $\Phi_\mu(p) := \exp(-\langle p, B_\mu p \rangle / 2)$ , where  $B_\mu : P \rightarrow Q$  is a continuous linear mapping such that  $\langle p, B_\mu p \rangle > 0$  for  $p \neq 0$ . By  $\nu$  we denote a  $Q$ -cylindrical measure on  $P$  whose Fourier transform  $\Phi_\nu : Q \rightarrow \mathbb{R}$  is defined by  $\Phi_\nu(q) := \exp(-\langle B_\mu^* q, q \rangle / 2)$ . Henceforth, we assume that all LCSs are Hilbert spaces, although the main results can be extended to the general case. We identify the space  $Q$  with  $Q^*$  and the space  $P$  with  $P^*$ , so that  $B_\mu^* = B_\mu$  and  $B_\mu > 0$ ; note in addition that the measures  $\mu$  and  $\nu$  are  $\sigma$ -additive if the operator  $B_\mu$  is of trace class.

<sup>1</sup>The definition of a pseudodifferential operator  $\widehat{F}$  in  $\mathcal{L}_2(Q, \mu)$  with symbol  $F$  can be found in [5].

The Weyl operator  $\mathscr{W}(h)$  generated by an element  $h \in E$  is defined by the equality  $\mathscr{W}(h) := e^{-i\widehat{h}}$ . The Weyl function corresponding to a density operator  $T$  is a function  $\mathcal{W}_T: E \rightarrow \mathbb{R}$  defined as  $\mathcal{W}_T(h) := \text{tr}(T\mathscr{W}(h))$  (see [5]); it does not depend on  $\mu$ .

**Definition 1** (see [5]). The *Wigner measure* corresponding to a density operator  $T$  is an  $E^*$ -cylindrical measure  $W_T$  on  $E$  defined by

$$\int_{Q \times P} e^{i(\langle p_1, q_2 \rangle + \langle p_2, q_1 \rangle)} W_T(dq_1, dp_1) = \mathcal{W}_T(h)(q_2, p_2).$$

In other words,  $W_T$  is the (inverse) Fourier transform of the function  $\mathcal{W}_T(h)$ . Thus, the following equality holds:

$$W_T(dq, dp) = \int_Q \int_P \mathcal{W}_T(h)(q_2, p_2) F_{E \times E}(dq_2, dp_2, dq, dp),$$

where  $F_{E \times E}$  is the Hamiltonian Feynman measure on  $E \times E$  (see below).

The Feynman measure  $F_{\mathcal{K}}$  on a Hilbert space  $\mathcal{K}$  is a generalized measure (i.e., a distribution in the sense of the Sobolev–Schwartz theory) on  $\mathcal{K}$ ; in other words, this is a continuous (in an appropriate sense) linear functional on some space of test functions on  $\mathcal{K}$ . Just as a standard measure, the functional  $F_{\mathcal{K}}$  can be conveniently described in terms of its Fourier transform  $\widetilde{F}_{\mathcal{K}}: \mathcal{K} \ni z \mapsto F_{\mathcal{K}}(\varphi_z) \in \mathbb{C}$ , where  $\varphi_z: \mathcal{K} \rightarrow \mathbb{C}$  is defined as follows:  $\varphi_z(x) := e^{i\langle z, x \rangle}$ .

If  $\mathcal{K} = E = Q \times P$  and  $\widetilde{F}_{\mathcal{K}}(q, p) = e^{i\langle q, p \rangle}$ , then  $F_{\mathcal{K}}$  is called the Hamiltonian Feynman measure; it can be used to introduce the Fourier transform that acts on functions defined on infinite-dimensional spaces and maps them into measures. Here the structure of the Hilbert space matters little, and, like the Gaussian measure, the Feynman measure can be defined on any LCS; in particular, the Hamiltonian Feynman measure can be defined on any symplectic LCS (see [3, 12, 15] for more information).

**Proposition 1** (see [6]). *If  $G$  is the Weyl symbol of a pseudodifferential operator in  $\mathcal{L}_2(Q, \mu)$ , then  $\int_P \int_Q G(q, p) W_T(dq, dp) = \text{tr}(T\widehat{G})$ .*

This proposition can also be used as a definition (cf. [5, Definition 3]; however, it is assumed there that  $\dim Q = \dim P < \infty$ , and so only the Wigner functions rather than measures are considered).

**Definition 2.** The density  $\Phi_T$  of the Wigner measure  $W_T$  with respect to a measure  $\eta$  on  $Q \times P$  (if this density exists) is called the  $\eta$ -Wigner function. If  $\eta$  is a generalized Liouville measure, then the generalized density of the Wigner measure with respect to  $\eta$  is called a *generalized Wigner function*; it is denoted by the same symbol  $\Phi_T$ .

If  $\dim Q = \dim P < \infty$  and  $\eta$  is the Liouville measure on  $Q \times P$ , then the  $\eta$ -Wigner function is the classical Wigner function.

Next, we assume that  $\eta = \mu \otimes \nu$ ; however, when dealing with the  $(\mu \otimes \nu)$ -Wigner function, we will refer to it simply as the Wigner function for brevity.

**Corollary 1.** *If the hypotheses of Proposition 1 are satisfied, then*

$$\int_P \int_Q G(q, p) \Phi_T(q, p) \mu \otimes \nu(dq, dp) = \text{tr}(T\widehat{G}).$$

**Proposition 2.** *The following equality holds:*

$$\begin{aligned} \Phi_T(q, p) &:= e^{(\langle p_1, B_\mu^{-1} p_1 \rangle + \langle q_1, B_\mu^{-1} q_1 \rangle)/2} \\ &\times \int_{Q \times P} e^{-i(\langle p_1, q_2 \rangle + \langle p_2, q_1 \rangle)} \mathcal{W}_T(h)(q_2, p_2) e^{(\langle p_2, B_\mu^{-1} p_2 \rangle + \langle q_2, B_\mu^{-1} q_2 \rangle)/2} (\mu \otimes \nu)(dq_2, dp_2). \end{aligned}$$

The function  $(q, p) \mapsto e^{-\langle p, B_\mu^{-1} p \rangle + \langle q, B_\mu^{-1} q \rangle} / 2$  is the generalized density of the Gaussian measure  $\mu \otimes \nu$  (see [10] and references therein). The formulas just presented and similar ones can be obtained according to the following heuristic rule. First, one should write the relevant formulas for the case of  $\dim Q < \infty$  with the Gaussian measures replaced by their densities with respect to the Lebesgue (= Liouville) measures on the spaces  $Q$  and  $Q \times P$ ; to obtain these formulas, in turn, one should apply the standard isomorphisms of the spaces of square integrable functions with respect to the Lebesgue measure and of square integrable functions with respect to the Gaussian measures. Then, to pass to the infinite-dimensional case, one should replace the Gaussian densities with respect to the Lebesgue measures by generalized densities. Here one should keep in mind that the generalized densities of the Gaussian measures are defined only up to multiplication by positive numbers; therefore, only formulas invariant under multiplication of the Gaussian densities by positive numbers can be extended to the infinite-dimensional case by the method described above.

The following propositions can be viewed as definitions of the Wigner measures and functions; they are similar to those given in [5].

**Proposition 3.** *For any density operator  $T$  in  $\mathcal{L}_2(Q, \mu)$  and  $\varphi \in \mathcal{L}_2(Q, \mu)$ , the following equalities hold:*

$$(T\varphi)(q) = e^{\langle B_\mu^{-1} q, q \rangle / 4} \int_P \int_Q e^{-i\langle p, q_1 - q \rangle} \varphi(q_1) e^{-\langle B_\mu^{-1} q_1, q_1 \rangle / 4} W_T\left(\frac{d(q_1 + q)}{2}, dp\right),$$

$$(T\varphi)(q) = e^{\langle B_\mu^{-1} q, q \rangle / 4} \int_P \int_Q e^{-i\langle p, q_1 - q \rangle} \varphi(q_1) e^{\langle B_\mu^{-1} q_1, q_1 \rangle / 4} \Phi_T\left(\frac{q_1 + q}{2}, p\right) e^{\langle B_\mu^{-1} p, p \rangle / 2} (\mu \otimes \nu)(dq, dp).$$

In the first formula, the mapping  $q \mapsto W_T(d(q_1 + q)/2, dp)$  is a function while the mapping  $(dq_1, dp) \mapsto W_T(d(q_1 + q)/2, dp)$  is a measure. The function  $q \mapsto e^{-\langle B_\mu^{-1} q, q \rangle / 2}$  is the generalized density of the Gaussian measure  $\mu$ , and the function  $p \mapsto e^{-\langle B_\mu^{-1} p, p \rangle / 2}$  is the generalized density of the measure  $\nu$ .

For the density operator  $T$  in  $\mathcal{L}_2(Q, \mu)$ , define its integral kernel  $\rho_T^1$  by the equality

$$(T\varphi)(q) = e^{\langle B_\mu^{-1} q, q \rangle / 4} \int_Q e^{\langle B_\mu^{-1} q_1, q_1 \rangle / 4} \varphi(q_1) \rho_T^1(q, q_1) \mu(dq_1).$$

**Proposition 4.** *For any  $\varphi \in \mathcal{L}_2(Q, \mu)$ , the following equality holds:*

$$\Phi_T(q, p) = e^{\langle B_\mu q, q \rangle + \langle B_\mu p, p \rangle} / 2 \int_Q \rho_T^1\left(q - \frac{1}{2}r, q + \frac{1}{2}r\right) e^{i\langle r, p \rangle} e^{\langle B_\mu^{-1} r, r \rangle / 2} \mu(dr).$$

Let  $\rho_T^2$  be the integral kernel of  $T$  in  $\mathcal{L}_2(Q, \mu)$  defined by the equality

$$(T\varphi)(q) = e^{\langle B_\mu^{-1} q, q \rangle / 4} \int_Q \varphi(q_1) e^{-\langle B_\mu^{-1} q_1, q_1 \rangle / 4} \rho_T^2(q, dq_1).$$

Thus,  $\rho_T^2$  is a function of the first argument and is a measure with respect to the second argument.

It follows from Proposition 1 that

$$\rho_T^2(q, dq_1) = \int_P e^{-i\langle p, q_1 - q \rangle} W_T\left(\frac{d(q_1 + q)}{2}, dp\right).$$

Setting  $s - r = q$  and  $s + r = q_1$  and making the change of variables in the formula, we obtain

$$\rho_T^2(s - r, d(s + r)) = \int_P e^{-i\langle p, 2r \rangle} W_T(ds, dp), \quad \text{or} \quad \rho_T^2\left(q - \frac{r}{2}, d\left(q + \frac{r}{2}\right)\right) = \int_P e^{-\langle p, r \rangle} W_T(dq, dp),$$

which means that the “measure”  $dp \mapsto W_T(dq, dp)$  is the inverse Fourier transform of the function  $r \mapsto \rho_T^2(q - r/2, dq + r/2)$ . This implies the following proposition.

**Proposition 5.** *Let  $F_E$  be the Hamiltonian Feynman measure on  $E := Q \times P$ . Then*

$$W_T(dq, dp) = \int_Q \rho_T^2\left(q - \frac{r}{2}, d\left(q + \frac{r}{2}\right)\right) F_E(dr, dp);$$

here, to integrate with respect to the “measure”  $dq \mapsto W_T(dq, dp)$ , one should apply the so-called Kolmogorov integral.<sup>2</sup>

### 3. DYNAMICS OF WIGNER FUNCTIONS AND MEASURES

We continue to use the assumptions and notation of the previous section. For any  $t \in \mathbb{R}$ , let  $W_T(t)$  be the Wigner measure describing the state of a quantum system at time  $t$  (so in this section  $W_T(\cdot)$  denotes a function of a real argument whose values are Wigner measures, while in the previous section  $W_T$  denoted the Wigner measure). Then  $W_T(\cdot)$  satisfies the following equation [6]:

$$\dot{W}_T(t) = 2 \sin\left(\frac{1}{2} \mathcal{L}_{\mathcal{H}}^*(W_T(t))\right), \tag{3.1}$$

where  $\mathcal{H}$  is the space of  $E^*$ -cylindrical measures on  $E$  and  $\sin(a\mathcal{L}_{\mathcal{H}}^*)$  for every  $a \in \mathbb{R}$  is the linear operator acting in  $\mathcal{H}$  and adjoint to the operator  $\sin(a\mathcal{L}_{\mathcal{H}})$  that acts in the space of functions on  $E$  and is defined by

$$\sin(a\mathcal{L}_{\mathcal{H}}) := \sum_{n=1}^{\infty} \frac{a^{2n-1}}{(2n-1)!} \mathcal{L}_{\mathcal{H}}^{(2n-1)}.$$

Here  $\mathcal{L}_{\mathcal{H}}^{(n)}$  is defined as follows: for every function  $\Psi: E \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $\mathcal{L}_{\mathcal{H}}^{(n)}\Psi(x) := \{\Psi, \mathcal{H}\}^{(n)}(x)$ ,  $x \in E$ , where  $\{\Psi, \mathcal{H}\}^{(n)}(x) := \Psi^{(n)}(x)I^{\otimes n}\mathcal{H}^{(n)}(x)$ ,  $\Psi^{(n)}$  and  $\mathcal{H}^{(n)}$  are the  $n$ th derivatives of the functions  $\Psi$  and  $\mathcal{H}$ , respectively, and  $I^{\otimes n}$  is the  $n$ th tensor power of the operator  $I$  defining a symplectic structure on the phase space  $E$  (see [6]).

Equality (3.1) implies an equation describing the evolution of the  $\mu$ -Wigner function. To derive this equation, it suffices to notice that for any function  $\Phi: E \rightarrow \mathbb{R}$  the  $n$ th derivative of the product  $\Phi^n \mu$  can be calculated by the Leibnitz rule and that the derivatives of the Gaussian measure  $\mu$  can be calculated as follows. If  $h, h_1, h_2, \dots \in B_{\mu}^{1/2}Q$ , then

$$\mu' h = -\langle B_{\mu}^{-1} h, \cdot \rangle \mu, \quad \mu'' h_1 h_2 = -\langle B_{\mu}^{-1} h_1, h_2 \rangle \mu + \langle B_{\mu}^{-1} h_1, \cdot \rangle \langle B_{\mu}^{-1} h_2, \cdot \rangle \mu,$$

and so on. These equalities are versions of the Wick formulas. Here, for every  $k \in B_{\mu}^{1/2}Q$ , the expression  $\langle B_{\mu}^{-1} k, \cdot \rangle$  denotes a function defined  $\mu$ -almost everywhere on  $Q$  that has the following properties (see [13]):

- (1) its domain of definition is a measurable vector subspace of  $Q$  of full measure;
- (2) this function is linear on its domain of definition;
- (3) if  $x \in B_{\mu}^{1/2}Q$ , then  $\langle B_{\mu}^{-1} k, x \rangle = \langle B_{\mu}^{-1/2} k, B_{\mu}^{-1/2} x \rangle$

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<sup>2</sup>The Kolmogorov integral is the trace in the tensor product of the space of functions on  $Q$  and the space of measures on  $Q$ ;  $\rho_T^2$  is an element of this space (for the original definition, which involves neither the tensor product nor the trace, see [8]).

(such a function exists, and any two functions with properties (1)–(3) coincide  $\mu$ -almost everywhere; see [13]).

For any  $a > 0$ , the operator  $\sin(aL_{\mathcal{H}}^*)$  acting on functions on  $E$  is defined by the equality  $\sin(aL_{\mathcal{H}}^*)\varphi(\mu \otimes \nu) := (\sin a\mathcal{L}_{\mathcal{H}}^*)(\varphi\mu \otimes \nu)$ . For every  $t \in \mathbb{R}$ , we also introduce the  $\mu$ -Wigner function  $\Phi_T(t)$  that describes the state of a quantum system at time  $t$ .

**Theorem 1.** *The mapping  $\Phi_T(\cdot)$  taking values in the set of  $\mu$ -Wigner functions satisfies the equation*

$$\dot{\Phi}_T(t) = 2 \sin\left(\frac{1}{2}L_{\mathcal{H}}^*(\Phi_T(t))\right).$$

Let  $\rho_T^1$  and  $\rho_T^2$  be the integral kernels introduced above for the density operator  $T$  of a quantum system that is a quantum version of the classical Hamiltonian system with the phase space  $E_1 \times E_2$ ,  $E_1 = Q_1 \times P_1$ ,  $E_2 = Q_2 \times P_2$ . Then the integral kernels of the reduced density operator  $T_1$  acting in  $\mathcal{L}_2(Q_i, \mu_i)$ ,  $i = 1, 2$  (here and in what follows, we use natural generalizations of the above notation and assumptions), are given by the equalities

$$\begin{aligned} \rho_{T_1}^1(q_1^1, q_2^1) &= \int_{Q_2} \rho_T^1(q_1^1, q_2^1, q^1, q^2) e^{\langle B_{\mu_1 \otimes \mu_2}(q^1, q^2), (q^1, q^2) \rangle / 2} \mu_2(dq_2), \\ \rho_{T_1}^2(q^1, dq_2^1) &= \int_{Q_2} \rho_T^2(q^1, dq_2^1, q^2, dq^2); \end{aligned}$$

the latter integral is again a Kolmogorov integral. Therefore, Propositions 4 and 5 imply the following theorem.

**Theorem 2.** *Let  $W_T$  and  $\Phi_T$  be the Wigner measure and Wigner function of a quantum system with Hilbert space  $\mathcal{L}_2(Q_1 \times Q_2, \mu_1 \otimes \mu_2)$ . Then the Wigner measure  $W_{T_1}$  and Wigner function  $\Phi_{T_1}$  of its subsystem with Hilbert space  $\mathcal{L}_2(Q_1, \mu_1)$  are defined by the relations*

$$W_{T_1}(dq_1, dp_1) = \int_{Q_2 \times P_2} W_T(dq_1, dp_1, dq_2, dp_2)$$

and

$$\begin{aligned} \Phi_T(q_1, p_1) &= e^{\langle (B_{\mu_1}^{-1}q_1, q_1) + (B_{\mu_1}^{-1}p_1, p_1) \rangle / 2} \\ &\times \int_{Q_2 \times P_2} e^{\langle (B_{\mu_2}^{-1}q_2, q_2) + (B_{\mu_2}^{-1}p_2, p_2) \rangle / 2} \Phi_T(q_1, p_1, q_2, p_2) (\mu_2 \otimes \nu_2)(dq_2, dp_2). \end{aligned}$$

**Remark 2.** Analogs of Theorems 1 and 2 and the propositions of the previous section remain valid for the generalized Wigner function as well.

#### 4. MATHEMATICAL MODELS OF COHERENT QUANTUM FEEDBACK

Everywhere below, for a Hilbert space  $\mathcal{T}$ , we denote by  $\mathcal{L}^s(\mathcal{T})$  the set of all self-adjoint operators in  $\mathcal{T}$ .

So, let  $\mathcal{P}$ ,  $\mathcal{P}_1$ ,  $\mathcal{P}_2$  and  $\mathcal{C}$ ,  $\mathcal{C}_1$ ,  $\mathcal{C}_2$  be Hilbert spaces. We assume that  $\mathcal{P}$  is the Hilbert space of a controlled quantum system,  $\mathcal{C}$  is the Hilbert space of a controlling quantum system, and  $\mathcal{P}_j$  and  $\mathcal{C}_j$ ,  $j = 1, 2$ , are the Hilbert spaces of parts of the controlled and controlling systems, respectively. Let  $\mathcal{H} := \mathcal{P} \otimes \mathcal{C}$  be the Hilbert space of the combined quantum system, and let  $\hat{\mathcal{H}}_{\mathcal{P}} \in \mathcal{L}^s(\mathcal{P})$ ,  $\hat{\mathcal{H}}_{\mathcal{C}} \in \mathcal{L}^s(\mathcal{C})$ ,  $\hat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \in \mathcal{L}^s(\mathcal{P}_1 \otimes \mathcal{C}_1)$ , and  $\hat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2} \in \mathcal{L}^s(\mathcal{P}_2 \otimes \mathcal{C}_2)$ . Define

$$\hat{\mathcal{H}}_{\text{feedback}} := \hat{\mathcal{H}}_{\mathcal{P}} \otimes \text{Id}_{\mathcal{C}} + \text{Id}_{\mathcal{P}} \otimes \hat{\mathcal{H}}_{\mathcal{C}} + \hat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \text{Id}_{\mathcal{P}_2 \otimes \mathcal{C}_2} + \text{Id}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \hat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2} \in \mathcal{L}^s(\mathcal{H}),$$

where  $\text{Id}_{\mathcal{P}} \in \mathcal{L}^s(\mathcal{P})$ ,  $\text{Id}_{\mathcal{C}} \in \mathcal{L}^s(\mathcal{C})$ ,  $\text{Id}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \in \mathcal{L}^s(\mathcal{P}_1 \otimes \mathcal{C}_1)$ , and  $\text{Id}_{\mathcal{P}_2 \otimes \mathcal{C}_2} \in \mathcal{L}^s(\mathcal{P}_2 \otimes \mathcal{C}_2)$  are the identity operators in the corresponding spaces. The first term in  $\widehat{\mathcal{H}}_{\text{feedback}}$  describes the evolution of an isolated controlled quantum system, the second term describes the evolution of an isolated controlling quantum system, and the last two terms describe (coherent) quantum feedback. Note that the definition of  $\widehat{\mathcal{H}}_{\text{feedback}}$  is symmetric with respect to the controlled quantum system, controlling quantum system, and feedback.

A more general Hamiltonian  $\widehat{\mathcal{H}} := \widehat{\mathcal{H}}_{\mathcal{P}} \otimes \text{Id}_{\mathcal{C}} + \text{Id}_{\mathcal{P}} \otimes \widehat{\mathcal{H}}_{\mathcal{C}} + \widehat{\mathcal{K}}$ , where  $\widehat{\mathcal{K}} \in \mathcal{L}^s(\mathcal{P} \otimes \mathcal{C})$  (see [7]), can describe coherent quantum control both with and without feedback. In particular, in the case where  $\widehat{\mathcal{K}} = \widehat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \text{Id}_{\mathcal{P}_2 \otimes \mathcal{C}_2} + \text{Id}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \widehat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2}$ , we obtain the previous model. On the other hand, if  $\widehat{\mathcal{K}} := \widehat{\mathcal{K}}_1 \otimes \text{Id}_{\mathcal{P}_2 \otimes \mathcal{C}_2}$ , we obtain a model of (coherent) quantum control without feedback.

If the controlled and controlling quantum systems are obtained by quantizing Hamiltonian systems, we can assume that, in the natural notation,  $\mathcal{P}_j = \mathcal{L}_2(Q_{\mathcal{P}_j}, \mu_j)$ ,  $\mathcal{C}_j = \mathcal{L}_2(Q_{\mathcal{C}_j}, \nu_j)$ ,  $\mathcal{P} = \mathcal{L}_2(Q_{\mathcal{P}_1} \times Q_{\mathcal{P}_2}, \mu_1 \otimes \mu_2)$ , and  $\mathcal{C} = \mathcal{L}_2(Q_{\mathcal{C}_1} \times Q_{\mathcal{C}_2}, \nu_1 \otimes \nu_2)$ ,  $j = 1, 2$ . In this case, the Wigner function and Wigner measure of the combined quantum system (incorporating both the controlled and controlling parts) are defined on the space  $Q_{\mathcal{P}_1} \times Q_{\mathcal{P}_2} \times Q_{\mathcal{C}_1} \times Q_{\mathcal{C}_2}$ , and their evolution is described by the equations of Section 3. To obtain the dynamics of the Wigner measure, Wigner function, and generalized Wigner function of the controlled quantum system, one should apply Theorem 2 and Remark 2.

**Remark 3.** To determine the dynamics of the controlled quantum system, it is necessary to find the Hamiltonians  $\mathcal{K}_1$  and  $\mathcal{K}_2$  or  $\mathcal{K}$  (in appropriate classes of Hamiltonians). This problem is similar to the simpler problem of choosing a time-dependent Hamiltonian function  $\mathcal{K}_1(\cdot)$  on  $Q_{\mathcal{P}}$  such that the required dynamics in  $\mathcal{L}_2(Q_{\mathcal{P}}, \mu)$  corresponds to  $\mathcal{K}_1(\cdot)$  under the assumption that  $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_1 + \widehat{\mathcal{K}}_1(t)$ , where  $\widehat{\mathcal{H}}_1 \in \mathcal{L}^s(\mathcal{P})$  and  $\widehat{\mathcal{K}}_1(t) \in \mathcal{L}^s(\mathcal{P})$ . Although this model is not a particular case of any of the models described above, we expect that it can be obtained as the limit of an appropriate sequence of the above models.

**Remark 4.** We can extend our model by assuming that the controlled quantum system also interacts with another quantum system that perturbs the dynamics of the controlled system. Of course, we can also believe that this source of perturbations is a part of the controlled quantum system.

**Remark 5.** The approach presented in Sections 2 and 3 can be directly applied to quantum systems obtained by the Schrödinger quantization of classical Hamiltonian systems. To consider more general cases, including, for example, spin systems, we should extend our approach by applying the methods of superanalysis. We assume that all our results can be generalized to this case.

**Remark 6.** In our quantum model with (coherent) feedback, we can describe in more detail the internal dynamics of the controlled and controlling quantum systems. In particular, we can assume that

$$\begin{aligned} \widehat{\mathcal{H}} = & (\widehat{\mathcal{H}}_{\mathcal{P}_1} \otimes \text{Id}_{\mathcal{P}_2} + \text{Id}_{\mathcal{P}_1} \otimes \widehat{\mathcal{H}}_{\mathcal{P}_2}) \otimes \text{Id}_{\mathcal{C}} + \text{Id}_{\mathcal{P}} \otimes (\widehat{\mathcal{H}}_{\mathcal{C}_1} \otimes \text{Id}_{\mathcal{C}_2} + \text{Id}_{\mathcal{C}_1} \otimes \widehat{\mathcal{H}}_{\mathcal{C}_2}) \\ & + \widehat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{P}_2} \otimes \text{Id}_{\mathcal{C}_1 \otimes \mathcal{C}_2} + \text{Id}_{\mathcal{P}_1 \otimes \mathcal{P}_2} \otimes \widehat{\mathcal{K}}_{\mathcal{C}_1 \otimes \mathcal{C}_2} + \widehat{\mathcal{K}}_{\mathcal{P}_1 \otimes \mathcal{C}_1} \otimes \text{Id}_{\mathcal{P}_2 \otimes \mathcal{C}_2} + \widehat{\mathcal{K}}_{\mathcal{P}_2 \otimes \mathcal{C}_2} \otimes \text{Id}_{\mathcal{P}_1 \otimes \mathcal{C}_1}. \end{aligned}$$

Here again the parts of the Hamiltonian that describe the controlled and controlling quantum systems and the interaction between them appear symmetrically.

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