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The divergence-free condition in axisymmetric MHD models

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ABSTRACT
Axisymmetric magnetohydrodynamic (MHD) models are useful in studies of magnetised winds and nonlinear Alfvén waves in solar and stellar atmospheres. We demonstrate that a condition often used in these models for the determination of a nearly vertical magnetic field is applicable to a radial field instead. A general divergence-free condition in curvilinear coordinates is self-consistently derived and used to obtain the correct condition for the variation of a nearly vertical magnetic field. The obtained general divergence-free condition along with the transfield equation complete the set of MHD equations in curvilinear coordinates for axisymmetric motions and could be useful in studies of magnetised stellar winds and nonlinear Alfvén waves.

Key words: Sun: corona – methods: analytical – waves – stars: winds, outflows – stars: magnetic field

1 INTRODUCTION

The evolution of Alfvén waves or twists in solar and stellar atmospheres is usually described by the magnetohydrodynamic (MHD) equations for axisymmetric motions. The 1.5 dimensional (1.5D) time-dependent equations characterising the evolution of the Alfvén waves or twists along a given magnetic field line were derived by Hollweg et al. (1982) using curvilinear orthogonal coordinates. An important effect captured by the 1.5D MHD model is nonlinear coupling and the transfer of energy from Alfvén waves to other MHD waves, which then dissipate rapidly since they steepen to form shocks (see, for example, Hollweg et al. 1982; Williams & Taroyan 2018). The 1.5D MHD model has been adopted by different authors to investigate the role of Alfvén waves in solar atmospheric heating and spicule dynamics (Hollweg 1992; Moriyasu et al. 2004), coronal rain (Antolin et al. 2010), wind acceleration (Kudoh & Shibata 1999), parametric decay instabilities (Shoda et al. 2018), and M dwarf stellar winds (Sakaue & Shibata 2021).

Since the seminal work by Weber & Davis (1967), many studies have examined different aspects of magnetic stellar winds and astrophysical jets. The launching mechanism, the propagation, and the extraction of angular momentum by winds and jets from rotating stellar objects are often described in curvilinear orthogonal coordinates (Okamoto 1975; Sakurai 1985; Heyvaerts & Norman 1989, 2003; Cui & Yuan 2020). The governing equations can be derived from Hollweg's equations in the time-independent limit. An additional transfield equation is used to provide the shape of field lines that are determined by the forces acting in the transverse direction.

An important condition for both time-dependent and time-independent models is the divergence free condition that allows the determination of the magnetic field. In stellar wind studies, the axisymmetric magnetic field is usually represented in terms of the field-stream function and the divergence-free condition is expressed in spherical or in cylindrical coordinates.

In studies of time-dependent axisymmetric motions or Alfvén waves, a divergence free condition in curvilinear coordinates is applied to field lines that remain close to the axis of symmetry. The condition was derived by Hollweg et al. (1982) and is based on general considerations of flux conservation. The aim of the present letter is the derivation of a self-consistent divergence-free condition in curvilinear coordinates. The consequences and applications of the obtained result to some special cases are discussed. The derived condition is different from those used in previous studies that were based on general considerations of flux conservation.

2 EQUATIONS

Consider the ideal MHD equations of mass continuity, momentum, and induction:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (1)$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} + \frac{\nabla p}{\rho} + \nabla \Phi + \frac{\nabla (B^2)}{8\pi\rho} = \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi\rho}, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

The last equation represents the divergence-free condition for the magnetic field. In cylindrical coordinates, (r, φ, z), the axisymmetric motions are characterised by the condition $\partial/\partial\varphi = 0$. To be consistent with Hollweg et al. (1982), we also assume time-independence of B_r and B_z: $\partial B_r/\partial t = \partial B_z/\partial t = 0$.

We also take the local curvilinear coordinates (a, φ, s), where s is the distance measured along the poloidal field line, a is the distance perpendicular to the poloidal field line, φ is the azimuthal angle measured around the rotation axis.

The magnetic field $\mathbf{B}$ may be decomposed either into cylindrical

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components, (B_r, B_φ, B_z) or into toroidal and poloidal components, $(0, B_\varphi, B_s)$, where B_s denotes the poloidal field and there is no component in the transverse a direction. The same applies to the velocity. Contopoulos (1996) considered a more general case where the poloidal flow is not parallel to the poloidal magnetic field.

We introduce the directional derivatives that relate the two coordinate systems:

$$\frac{\partial}{\partial s} = \frac{\partial r}{\partial s}\frac{\partial}{\partial r} + \frac{\partial z}{\partial s}\frac{\partial}{\partial z} \tag{5}$$

along the poloidal field, and

$$\frac{\partial}{\partial a} = \frac{\partial r}{\partial a}\frac{\partial}{\partial r} + \frac{\partial z}{\partial a}\frac{\partial}{\partial z} \tag{6}$$

in the transverse direction.

The equations of field lines, $r = r(a, s)$, $z = z(a, s)$, are obtained by solving:

$$\frac{\partial r}{\partial s} = -\frac{\partial z}{\partial a} = \frac{B_r}{B_s} = \sin\theta, \tag{7}$$

$$\frac{\partial z}{\partial s} = \frac{\partial r}{\partial a} = \frac{B_z}{B_s} = \cos\theta, \tag{8}$$

where θ denotes the angle between the poloidal field and the symmetry axis (Figure 1).

The adopted approach leads to an interchange of the dependent and the independent variables. A similar approach has been applied to static plasmas in a Cartesian geometry: Fiedler & Cally (1990) and Cally (1991) developed semi-inverse and fully inverse methods in which one or both Cartesian coordinates are dependent variables and are solved for as functions of a and s.

3 RESULTS

Using the relationships presented in the previous section we derive the governing equations for the axisymmetric motions:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{\rho}{B_s}V_s\right) = 0, \tag{9}$$

$$\frac{\partial}{\partial t}\left(\frac{r\rho V_\varphi}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{r\rho V_\varphi}{B_s}V_s\right) = \frac{1}{4\pi}\frac{\partial}{\partial s}\left(rB_\varphi\right), \tag{10}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho V_s}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{\rho V_s}{B_s}V_s\right) = \frac{\rho}{B_s}\frac{\partial \ln r}{\partial s}\left(V_\varphi^2 - \frac{B_\varphi^2}{4\pi\rho}\right) - \frac{1}{8\pi B_s}\frac{\partial B_\varphi^2}{\partial s} - \frac{1}{B_s}\frac{\partial p}{\partial s} - \frac{\rho}{B_s}\frac{\partial \Phi}{\partial s}, \tag{11}$$

$$\frac{\partial}{\partial t}\left(\frac{B_\varphi}{rB_s}\right) + \frac{\partial}{\partial s}\left(\frac{B_\varphi}{rB_s}V_s\right) = \frac{\partial}{\partial s}\left(\frac{V_\varphi}{r}\right). \tag{12}$$

Equations (9) - (12) have been derived by Hollweg et al. (1982).

Using the directional derivatives (5), (6), and the angle θ between the poloidal field and the symmetry axis we derive the following divergence-free condition:

$$\frac{\partial}{\partial s}\ln|B_s r| + \frac{\partial\theta}{\partial a} = 0. \tag{13}$$

An additional transfield equation can be cast in the following form:

$$\left(\rho V_s^2 - \frac{B_s^2}{4\pi}\right)\frac{\partial\theta}{\partial s} = \frac{\cos\theta}{r}\left(\rho V_\varphi^2 - \frac{B_\varphi^2}{4\pi}\right) - \frac{\partial}{\partial a}\left(\frac{B^2}{8\pi} + p\right) - \rho\frac{\partial\Phi}{\partial a}. \tag{14}$$

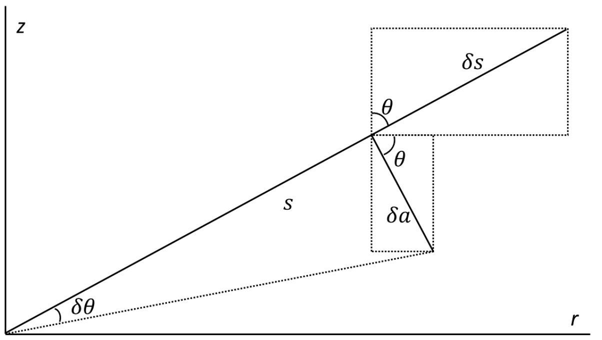

Figure 1. Distance s along a radial magnetic field and incremental increase along and across the poloidal field projected onto the radial, r, and the symmetry, z, axes.

The above equation expresses force balance in a direction perpendicular to the magnetic field and therefore represents a generalised Grad-Shafranov equation. It is equivalent to equation (21) in Okamoto (1975) and equation (7.41) in Mestel (2012). The transfield equation (14) determines the shape of the field lines as a result of the forces acting in the transverse direction. The field lines bend away or towards the symmetry axis depending on the sign of $\partial\theta/\partial s$.

The solenoidal condition (13) and the balance of forces in the transverse direction (14) complete the set of the governing equations for axisymmetric motions in curvilinear coordinates.

The azimuthal part of the energy density represents the sum of the azimuthal kinetic and magnetic energy densities. It is given by

$$W_\varphi = \frac{\rho V_\varphi^2}{2} + \frac{B_\varphi^2}{8\pi} \tag{15}$$

The set of governing equations (9) - (12) can be combined to derive the following equation:

$$\frac{\partial}{\partial t}\left(\frac{W_\varphi}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{F_W}{B_s}\right) = \frac{V_s}{B_s}\frac{\partial \ln r}{\partial s}\left(\frac{B_\varphi^2}{4\pi} - \rho V_\varphi^2\right) - \frac{B_\varphi^2}{8\pi}\frac{\partial}{\partial s}\left(\frac{V_s}{B_s}\right), \tag{16}$$

$$\text{where } F_W = V_s W_\varphi - \frac{B_s V_\varphi B_\varphi}{4\pi} \tag{17}$$

represents the azimuthal energy flux.

4 DISCUSSION

Equations (9) - (12) were derived by Hollweg et al. (1982) using curvilinear coordinates. For an axisymmetric field the divergence-free condition (4) takes the form

$$r h_\xi B_s = \text{constant along field lines}, \tag{18}$$

where r denotes distance from the axis of symmetry and h_ξ is an arbitrary curvilinear scale factor. It was argued that due to the conservation of magnetic flux the above condition (18) could be reduced to

$$B_s r^2 \approx \text{constant along field lines}, \tag{19}$$

for field lines close to the axis of symmetry. Condition (19) has been widely used in many studies of axisymmetric motions.

We note that the conservation of magnetic flux,

$$\int_S \mathbf{B}\cdot d\mathbf{S} = \text{constant along field lines,} \tag{20}$$

can be derived from the divergence-free condition by applying Gauss' theorem. Assuming nearly vertical field lines, B_s, close to the axis of symmetry, we have

$$\int_S \mathbf{B}\cdot d\mathbf{S} \approx \int_S B_s dS = \tilde{B}_s \pi r^2 = \text{constant along field lines,} \tag{21}$$

where $\tilde{B}_s$ is the mean magnetic field between 0 and r. However, the mean is not necessarily the same as the value of B_s at r because, in general, the strength of the magnetic field is variable across the field lines. Therefore, condition (19) cannot be derived from general considerations of flux conservation.

The divergence free condition (13) obtained in the present work is consistent with the remaining governing equations in curvilinear coordinates. It can be used to investigate axisymmetric motions with more general poloidal magnetic fields. We consider two applications: a radial field and a nearly vertical field.

As a first example we consider the case of a radial field shown in Figure 1 for which we have

$$s\,\delta\theta \approx \delta a, \tag{22}$$

where δ denotes incremental variation. For a radial field we also have

$$\frac{\delta r}{\delta s} = \frac{r}{s} \tag{23}$$

Therefore the derivative of θ is reduced to

$$\frac{\partial\theta}{\partial a} = \frac{1}{s} = \frac{1}{r}\frac{\partial r}{\partial s} \tag{24}$$

Equation (13) is reduced to

$$\frac{\partial}{\partial s}\ln|B_s r| + \frac{\partial}{\partial s}\ln r = 0. \tag{25}$$

or

$$B_s r^2 = \text{constant along field lines.} \tag{26}$$

Condition (26) was used by Cui & Yuan (2020) to study stellar winds along radial field lines. It is equivalent to the well known condition for $B_R R^2$ in spherical coordinates where R denotes radial distance.

Condition (26) was also used by Hollweg et al. (1982) and in subsequent studies of field lines in the neighbourhood of the symmetry axis. However, these studies did not assume a radial magnetic field and therefore condition (26) may not be compatible with the remaining governing equations that contain both B_s and r.

For field lines near the axis of symmetry that are nearly vertical the angle θ is small and the second term in equation (13) can be ignored. The resulting divergence-free condition is:

$$B_s r = \text{constant along field lines} \tag{27}$$

Condition (27) is applicable to arbitrary field lines that remain close to the symmetry axis. It is different from condition (26) which has been traditionally used in axisymmetric models. Replacing condition (27) with condition (26) should result in stronger variation of the poloidal field for a given expansion factor. It is important to note that both B_s and r figure in the governing equations and cannot be chosen independently.

Equation (27) can be used to derive an equation of energy for the neighbourhood of an arbitrary field line near the axis of symmetry. The azimuthal energy in a volume element dV containing a field line element ds is given by:

$$W_\varphi dV = \left(\frac{\rho V_\varphi^2}{2} + \frac{B_\varphi^2}{8\pi}\right) r d\varphi dr ds. \tag{28}$$

The total azimuthal energy in a thin axisymmetric shell of thickness dr is given by the integral:

$$2\pi dr \int W_\varphi r ds = 2\pi dr \int \left(\frac{\rho V_\varphi^2}{2} + \frac{B_\varphi^2}{8\pi}\right) r ds. \tag{29}$$

Integrating equation (16) and using the divergence-free condition (27), we obtain:

$$\frac{\partial}{\partial t}\int W_\varphi r ds + \int \frac{\partial}{\partial s}(F_W r)\, ds = \int V_s r \frac{\partial \ln r}{\partial s}\left(\frac{B_\varphi^2}{4\pi} - \rho V_\varphi^2\right) ds - \int \frac{B_\varphi^2}{8\pi}\frac{\partial}{\partial s}(V_s r)\, ds. \tag{30}$$

Equation (30) represents an equation for the temporal variation of the total azimuthal energy in a thin axisymmetric shell. The second term on the left-hand side of equation (30) represents the net azimuthal energy flux. The sources on the right-hand side of equation (30) represent the sum of the tension and centrifugal forces (first term) and the twist-flow coupling (second term).

It is worth noting that condition (27) separates two classes of field lines. More specifically, the divergence free condition (13) shows that for a diverging field ($\partial r/\partial s > 0$) of the form $B_s \sim r^{\alpha-1}$, where $\alpha < 0$, the field lines become less vertical and more inclined with distance a. The radial field with $\alpha = -1$ that we have considered above belongs in this category. On the contrary, for $\alpha > 0$, the field lines tend to become more vertical with distance a.

5 SUMMARY

Axisymmetric MHD models are commonly used to study the properties of magnetised stellar winds and the evolution of Alfvén waves in solar/stellar atmospheres. Both time-dependent and time-independent models rely on a divergence free condition to determine the poloidal magnetic field. Alternatively, it is represented in terms of a field-stream function, so the divergence-free condition is automatically satisfied. In both cases, the poloidal magnetic field is represented in spherical or cylindrical coordinates that are treated as independent variables.

We derive a general divergence-free condition for axisymmetric motions where the curvilinear coordinates s and a are treated as independent variables. It completes the set of governing equations in curvilinear coordinates.

As an application, we demonstrate that a well-known condition, previously thought to represent a nearly axial poloidal field, is consistent with a radial field. The correct condition for a nearly axial poloidal field is obtained. The derived divergence-free condition can be used in future studies to find more general solutions that are compatible with the remaining governing equations.

DATA AVAILABILITY

No new data were generated or analysed in support of this research.

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APPENDIX A: DERIVATION OF THE DIVERGENCE-FREE CONDITION

The radial component of the induction equation can be written in the following form:

$$\frac{\partial B_r}{\partial t} = \frac{(V_r B_\varphi - V_\varphi B_r)^{(\varphi)}}{r} - (V_z B_r - V_r B_z)^{(z)}, \tag{A1}$$

where the superscript denotes the partial derivative with respect to the enclosed variable. The z-component of the induction equation can be represented as:

$$\frac{\partial B_z}{\partial t} == \frac{(rV_z B_r - rV_r B_z)^{(r)} - (V_\varphi B_z - V_z B_\varphi)^{(\varphi)}}{r}. \tag{A2}$$

Using the adopted time-independence of B_r and B_z and the axisymmetric nature of the model ($\partial/\partial\varphi = 0$) we have:

$$\frac{\partial B_r}{\partial t} = (V_r B_z - V_z B_r)^{(z)} = 0, \quad \frac{\partial B_z}{\partial t} = \frac{(rV_z B_r - rV_r B_z)^{(r)}}{r} = 0, \tag{A3}$$

that are satisfied if:

$$V_r B_z = V_z B_r. \tag{A4}$$

We split the velocity and the magnetic field into toroidal, φ, and poloidal, s, components: $\mathbf{V} = (0, V_\varphi, V_s)$ and $\mathbf{B} = (0, B_\varphi, B_s)$, where

$$B_s^2 = B_r^2 + B_z^2, \quad V_s^2 = V_r^2 + V_z^2. \tag{A5}$$

The corresponding components in the direction transverse to the field are zero. Using equation A4 we have

$$\frac{V_s}{B_s} = \frac{V_z}{B_z}\sqrt{\frac{1+\frac{V_r^2}{V_z^2}}{1+\frac{B_r^2}{B_z^2}}} = \frac{V_z}{B_z}. \tag{A6}$$

We also have

$$V_s B_s = V_z B_z \sqrt{1+\frac{V_r^2}{V_z^2}}\sqrt{1+\frac{B_r^2}{B_z^2}} = V_z B_z\left(1+\frac{V_r^2}{V_z^2}\right) = V_z B_z + V_r B_r \tag{A7}$$

We introduce the directional derivatives:

$$\frac{\partial}{\partial s} = \frac{B_r}{B_s}\frac{\partial}{\partial r} + \frac{B_z}{B_s}\frac{\partial}{\partial z} \tag{A8}$$

along the magnetic field, and

$$\frac{\partial}{\partial a} = \frac{B_z}{B_s}\frac{\partial}{\partial r} - \frac{B_r}{B_s}\frac{\partial}{\partial z} \tag{A9}$$

transverse to the magnetic field. The equations of field lines can be found by solving:

$$\frac{\partial r}{\partial s} = -\frac{\partial z}{\partial a} = \frac{B_r}{B_s} = \sin\theta, \tag{A10}$$

$$\frac{\partial z}{\partial s} = \frac{\partial r}{\partial a} = \frac{B_z}{B_s} = \cos\theta. \tag{A11}$$

where we have introduced the angle θ between the poloidal field B_s and the symmetry axis $r = 0$. By combining (A8) and (A9), we obtain expressions for the partial derivatives with respect to r and z:

$$\frac{\partial}{\partial r} = \frac{B_r}{B_s}\frac{\partial}{\partial s} + \frac{B_z}{B_s}\frac{\partial}{\partial a} \tag{A12}$$

$$\frac{\partial}{\partial z} = \frac{B_z}{B_s}\frac{\partial}{\partial s} - \frac{B_r}{B_s}\frac{\partial}{\partial a} \tag{A13}$$

The divergence-free condition in cylindrical coordinates has the following form:

$$\nabla\cdot\mathbf{B} = \frac{(rB_r)^{(r)}}{r} + \frac{B_\varphi^{(\varphi)}}{r} + B_z^{(z)} = 0, \tag{A14}$$

The assumed axisymmetry ($\partial/\partial\varphi = 0$) reduces condition A14 to:

$$\frac{(rB_r)^{(r)}}{r} + B_z^{(z)} = 0, \quad \text{or} \quad B_r^{(r)} + B_z^{(z)} + \frac{B_r}{r} = 0, \tag{A15}$$

We rewrite this formula in the form:

$$\frac{rB_r^2}{B_z}\left(\frac{B_z B_r^{(r)}}{rB_r^2} - \frac{B_z^{(r)}}{rB_r} + \frac{B_z}{r^2 B_r}\right) + \frac{B_r B_z^{(r)}}{B_z} + B_z^{(z)} = 0 \tag{A16}$$

to obtain

$$-\frac{rB_r^2}{B_z}\left(\frac{B_z}{rB_r}\right)^{(r)} + \frac{B_r B_z^{(r)} + B_z^{(z)} B_z}{B_z} = 0. \tag{A17}$$

Alternatively, we have:

$$\frac{rB_r^2}{B_z}\left(\frac{B_z}{rB_r}\right)^{(r)} = \frac{B_s}{B_z}\frac{\partial B_z}{\partial s} \quad \text{or} \quad \frac{\partial}{\partial s}\ln|B_z| + \frac{B_r}{B_s}\frac{\partial}{\partial r}\ln\left|\frac{rB_r}{B_z}\right| = 0. \tag{A18}$$

The next step is to use the relationship (A12) and express all terms in equation (A18) via the directional derivatives:

$$\frac{\partial}{\partial s}\ln|B_z| + \frac{B_r^2}{B_s^2}\frac{\partial}{\partial s}\ln\left|\frac{rB_r}{B_z}\right| + \frac{B_r B_z}{B_s^2}\frac{\partial}{\partial a}\ln\left|\frac{rB_r}{B_z}\right| = 0. \tag{A19}$$

After some simple algebra, using expressions (A10) and (A11), we reduce equation (A19) to the following form:

$$\frac{1}{B_z}\frac{\partial B_z}{\partial s} + \frac{B_z^2}{B_s^2}\frac{\partial}{\partial a}\left(\frac{B_r}{B_z}\right) + \frac{B_r}{rB_s} + \frac{B_r B_z}{B_s^2}\frac{\partial}{\partial s}\left(\frac{B_r}{B_z}\right) = 0. \tag{A20}$$

Using the relationships (A10) and combining the first and last terms in condition (A20), we have:

$$\frac{B_z}{B_s^2}\frac{\partial B_z}{\partial s} + \frac{B_z^2}{B_s^2}\frac{\partial}{\partial a}\left(\frac{B_r}{B_z}\right) + \frac{\partial}{\partial s}\ln r + \frac{B_r}{B_s^2}\frac{\partial B_r}{\partial s} = 0. \tag{A21}$$

$$\text{or} \quad \frac{\partial}{\partial s}\ln|B_s r| + \frac{B_z^2}{B_s^2}\frac{\partial}{\partial a}\left(\frac{B_r}{B_z}\right) = 0. \tag{A22}$$

Finally, by using the relationships (A10), (A11), we can express the second term in the right hand side of (A22) in terms of the angle θ:

$$\frac{\partial}{\partial s}\ln|B_s r| + \frac{\partial \theta}{\partial a} = 0. \tag{A23}$$

Equation (A23) represents the divergence free condition for axisymmetric motions in curvilinear coordinates.

APPENDIX B: DERIVATION OF THE GOVERNING EQUATIONS

In the present section, we derive the conservation equations of mass, momentum and induction in terms of the introduced variables s and a and show their consistency with those derived by Hollweg et al. (1982). An additional transfield equation is derived.

From the mass conservation law (1) we have

$$\rho^{(t)} + \frac{(rV_r\rho)^{(r)}}{r} + (V_z\rho)^{(z)} = 0. \tag{B1}$$

Using equations (A4) and (A6) we have:

$$\rho^{(t)} + \frac{1}{r}\left(rB_r\rho\frac{V_s}{B_s}\right)^{(r)} + \left(B_z\rho\frac{V_s}{B_s}\right)^{(z)} = 0,$$

or, using the divergence-free condition (A14), we obtain

$$\rho^{(t)} + B_r\left(\rho\frac{V_s}{B_s}\right)^{(r)} + B_z\left(\rho\frac{V_s}{B_s}\right)^{(z)} = 0.$$

Finally, using the definition (A8) of the directional derivative and the time independence of B_s, we obtain the desired continuity equation (9):

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{V_s\rho}{B_s}\right) = 0. \tag{B2}$$

The φ- component of the equation of motion has the form:

$$V_\varphi^{(t)} + \frac{(rV_\varphi)^{(r)}V_r}{r} + V_\varphi^{(z)}V_z = \frac{(rB_\varphi)^{(r)}B_r + rB_\varphi^{(z)}B_z}{4r\pi\rho}$$

We multiply both sides by r and use the introduced directional derivative (A8) to obtain:

$$rV_\varphi^{(t)} + (rV_\varphi)^{(r)}V_r + rV_\varphi^{(z)}V_z = \frac{(rB_\varphi)^{(r)}B_r + rB_\varphi^{(z)}B_z}{4\pi\rho},$$

$$rV_\varphi^{(t)} + V_s\frac{\partial}{\partial s}(rV_\varphi) = \frac{B_s}{4\pi\rho}\frac{\partial}{\partial s}(rB_\varphi),$$

or, using the continuity equation, the equation of motion (10):

$$\frac{\partial}{\partial t}\left(\frac{r\rho V_\varphi}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{r\rho V_\varphi}{B_s}V_s\right) = \frac{1}{4\pi}\frac{\partial}{\partial s}(rB_\varphi). \tag{B3}$$

The φ component of the induction equation reads:

$$B_\varphi^{(t)} = (V_\varphi B_z - V_z B_\varphi)^{(z)} + (V_\varphi B_r - V_r B_\varphi)^{(r)}$$

$$\text{or } B_\varphi^{(t)} = (V_\varphi B_z - V_z B_\varphi)^{(z)} + \left[\frac{B_r}{B_z}(V_\varphi B_z - V_z B_\varphi)\right]^{(r)}$$

We use the definition of the partial derivative with respect to s to obtain:

$$B_\varphi^{(t)} = \frac{B_s}{B_z}\frac{\partial}{\partial s}(V_\varphi B_z - V_z B_\varphi) + \left(\frac{B_r}{B_z}\right)^{(r)}(V_\varphi B_z - V_z B_\varphi).$$

From equation (A18) we have:

$$\left(\frac{B_r}{B_z}\right)^{(r)} = -\frac{B_s}{rB_z^2}\frac{\partial}{\partial s}(rB_z).$$

We therefore obtain:

$$B_\varphi^{(t)} = \frac{B_s}{rB_z^2}\left(rB_z\frac{\partial}{\partial s}(V_\varphi B_z - V_z B_\varphi) - (V_\varphi B_z - V_z B_\varphi)\frac{\partial}{\partial s}(rB_z)\right),$$

$$B_\varphi^{(t)} = rB_s\frac{\partial}{\partial s}\left(\frac{V_\varphi B_z - V_z B_\varphi}{rB_z}\right)$$

$$\text{or } \frac{\partial}{\partial t}\left(\frac{B_\varphi}{rB_s}\right) + \frac{\partial}{\partial s}\left(\frac{B_\varphi}{rB_s}V_s\right) = \frac{\partial}{\partial s}\left(\frac{V_\varphi}{r}\right).$$

which is the induction equation (12) in curvilinear coordinates.

The z-component of the equation of motion reads:

$$V_z^{(t)} + V_z^{(z)}V_z + V_z^{(r)}V_r + \frac{p^{(z)}}{\rho} + \Phi^{(z)} + \frac{B^{(z)}B}{4\pi\rho} = \frac{1}{4\pi\rho}\left(B_z^{(z)}B_z + B_z^{(r)}B_r\right).$$

$$\text{or } V_z^{(t)} + \frac{V_r}{B_r}(V_z^{(z)}B_z + B_rV_z^{(r)}) + \frac{p^{(z)}}{\rho} + \Phi^{(z)} = -\frac{B^{(z)}B}{4\pi\rho} + \frac{1}{4\pi\rho}\left(B_z^{(z)}B_z + B_z^{(r)}B_r\right)$$

Using the definition of the partial derivative with respect to s we obtain:

$$\frac{\partial V_z}{\partial t} + V_s\frac{\partial V_z}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \Phi}{\partial z} + \frac{B}{4\pi\rho}\frac{\partial B}{\partial z} = \frac{B_s}{4\pi\rho}\frac{\partial B_z}{\partial s}. \tag{B4}$$

The radial component of the equation of motion can be represented in the following form:

$$V_r^{(t)} + V_r(V_r^{(r)} + V_r^{(z)}V_z/V_r) + \frac{p^{(r)}}{\rho} + \Phi^{(r)} = -\frac{B^{(r)}B - B_r^{(r)}B_r - B_r^{(z)}B_z}{4\pi\rho} + \frac{V_\varphi^2}{r} - \frac{B_\varphi^2}{4\pi\rho r}$$

Using expression (A8) we obtain:

$$\frac{\partial V_r}{\partial t} + V_s\frac{\partial V_r}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial \Phi}{\partial r} + \frac{B}{4\pi\rho}\frac{\partial B}{\partial r} - \frac{B_s}{4\pi\rho}\frac{\partial B_r}{\partial s} = \frac{V_\varphi^2}{r} - \frac{B_\varphi^2}{4\pi\rho r}. \tag{B5}$$

We multiply equation (B4) by $\rho B_z/B_s^2$, (B5) by $\rho B_r/B_s^2$, add the two together, and use the relationship (A7) to derive the equation of motion in the s-direction (11):

$$\frac{\partial}{\partial t}\left(\frac{\rho V_s}{B_s}\right) + \frac{\partial}{\partial s}\left(\frac{\rho V_s}{B_s}V_s\right) = \frac{\rho}{B_s}\frac{\partial \ln r}{\partial s}\left(V_\varphi^2 - \frac{B_\varphi^2}{4\pi\rho}\right) - \frac{1}{8\pi B_s}\frac{\partial B_\varphi^2}{\partial s} - \frac{1}{B_s}\frac{\partial p}{\partial s} - \frac{\rho}{B_s}\frac{\partial \Phi}{\partial s}, \tag{B6}$$

Finally, we multiply equation (B4) by $\rho B_r/B_s^2$, equation (B5) by

$\rho B_z/B_s^2$, and subtract the two from each other to derive the transfield equation (14):

$$\left(\rho V_s^2 - \frac{B_s^2}{4\pi}\right)\frac{\partial\theta}{\partial s} = \frac{\cos\theta}{r}\left(\rho V_\varphi^2 - \frac{B_\varphi^2}{4\pi}\right) - \frac{\partial}{\partial a}\left(\frac{B^2}{8\pi} + p\right) - \rho\frac{\partial\Phi}{\partial a}. \tag{B7}$$

Note that no time derivatives are present in equation (B7) due to the condition (A4) and the adopted time-independence of the magnetic field components B_r and B_z.

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