

Bed-slope-related diffusion of an erodible hump

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ABSTRACT: In certain conditions, the bed-slope has a substantial influence on sediment transport rates and hence morphological evolution. Approaches to account for such influence usually suffer from a high degree of empiricism and/or mathematical complexity. We propose a bed-slope-related diffusivity parameter, derived from a morphodynamic model previously validated against empirical data for bedload transport on horizontal and steep sloping channels. The proposed diffusivity parameter is easy to include in a conventional morphodynamic model via the modification of a typical bedload formula originally developed for sediment transport on a nearly-horizontal channel. A conventional model modified through this parameter proves to yield enhanced results in the case study of a submerged migrating hump/sandbar, by avoiding the generation of unrealistic high-frequency oscillations in the bed profile, yet permitting the expected steepening of its downstream face with time. Other models derived for sloping channels do not satisfy the latter condition. It is also shown that unrealistic oscillations can be avoided through numerical means; however, their use should be interpreted carefully from a phenomenological viewpoint.

1 INTRODUCTION

The earliest studies of sediment transport related to the fluvial environment, where bed-slopes tend to be small (in the order of $< 0.05^\circ$). Thus, pioneering works in the field were all based on steady unidirectional flows over mild bed-slopes. The tendency of considering these types of flows has continued until today and expanded to other environments, some of which can be very different hydrodynamically from typical rivers. Neglect of the potential effect of bed-slope on bedload transport is justifiable in practice for most cases concerned with mature river hydraulics, given the predominance of small bed-slopes in practice. However, in other hydraulic environments, such as beach shores and mountain streams, the bed-slope can be of such magnitude that its effect on bedload transport (and hence, on morphological evolution) may not be negligible.

In mountain streams, beds are usually composed of coarse sediments and have slopes that are sufficiently large to affect the overall flow behaviour, including the sediment transport rate (Bayazit 1983). For coarse-sediment beds, bedload is the primary

mode of transport, and occurs at small flow velocities relative to the threshold for sediment motion. It is under these conditions that extra-hydrodynamic factors, in particular the bed-slope, may play an important role in the transport process. This has indeed been confirmed through laboratory experiments (e.g. Smart 1984; Damgaard, Whitehouse, & Soulsby 1997; Dey & Debnath 2001). The present paper is thus solely concerned with bedload (rather than total or suspended) sediment transport.

One way of accounting for the bed-slope influence is by adding a slope-related diffusivity term to a sediment-transport formula, which typically translates into an additional calibration parameter (Johnson & Zyserman 2002). Other alternatives include semi-empirical models based on Bagnoldian ideas (e.g. Bagnold 1963; Bailard & Inman 1981; Kovacs & Parker 1994); formulae explicitly derived for sloping beds, which often imply a significant degree of empiricism or complexity (e.g. Smart 1984; Chiari, Friedl, & Rickenmann 2010; Parker, Seminara, & Solari 2003); and modification of the threshold of motion for sloping beds by inclusion of the gravity contribution of a resting particle.

In this paper, we outline the derivation of an alternative analytical slope-related diffusivity parameter, which is then incorporated into a conventional morphodynamic model and compared against a Bagnoldian model by means of the benchmark case of a migrating hump/sandbar. Inclusion of numerical diffusion is also discussed. Remarks are made regarding the phenomenological importance of including physical (rather than numerical) slope-related diffusion.

2 METHODOLOGY

Conventional Morphodynamic Models (CMMs) often consist of a coupling between a hydrodynamic model and an equation governing the morphological evolution. The coupling is achieved through sediment transport formulae, which are in turn functions of hydrodynamic and sediment-related parameters. A popular example for bedload transport in one-dimension is provided by the set of equations formed by combining the Shallow Water Equations (SWEs) and the Exner morphological equation:

$$\frac{\partial h}{\partial t} + \frac{\partial(Uh)}{\partial x} = 0 \quad (1)$$

$$\frac{\partial(Uh)}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \frac{1}{2}gh^2 \right) = -ghS_b - \frac{\tau}{\rho} \quad (2)$$

$$\frac{\partial z_b}{\partial t} + \xi \frac{\partial q_b}{\partial x} = 0, \quad (3)$$

where h is the total depth of the flow; U denotes the horizontal velocity vertically averaged over h ; S_b is the local bed-slope (with negative value in the down-sloping direction); τ represents the bed shear stress; ρ is the density of water; q_b is the volumetric bedload transport rate; z_b is the bed level from a datum; ξ relates to bed porosity; g is the gravitational acceleration; and x is horizontal distance and t is time.

The set of equations (1)–(3) requires estimation of τ and q_b for closure. The bed friction, τ , can be computed from a Chézy expression, as:

$$\tau = \rho c_f U |U|, \quad (4)$$

where c_f is the bed friction coefficient. Numerous options exist in the literature to estimate q_b . A particularly popular formula to compute bedload transport is that of Meyer-Peter & Müller (1948). For the sheet-flow regime, a transport rate proportional to a power of U (usually 3) is often employed.

The influence of bed-slope on bedload can be

included by adding a term to the sediment transport formula –originally derived for nearly horizontal channels– that promotes (inhibits) sediment motion for down(up)-sloping beds (see e.g. Johnson & Zyserman 2002; Watanabe 1988; Bailard & Inman 1981). Such a term is proportional to the bed-slope, and can be added as follows:

$$q_{b\beta} = q_{bh} + \varepsilon |q_{bh}| S_b, \quad (5)$$

where $q_{b\beta}$ and q_{bh} represent bedload transport on a sloping and a horizontal bed, respectively; and ε denotes the slope-related diffusivity.

The reason for ε being referred to as a diffusion term is that, under certain circumstances (see e.g. Hudson & Sweby 2003), it is sensible to write the Exner equation (eq. 3) as an advection-diffusion equation. To illustrate this, consider $q_b = q_{b\beta}$ and observe that $S_b \equiv \partial z_b / \partial x$; thus, in unidirectional flow, eq. (3) can be manipulated into:

$$\begin{aligned} \frac{\partial z_b}{\partial t} + \xi \frac{\partial}{\partial x} \left(q_{bh} + \varepsilon q_{bh} \frac{\partial z_b}{\partial x} \right) &= 0 \\ \Rightarrow \frac{\partial z_b}{\partial t} + \left(\xi \frac{\partial q_{bh}}{\partial z_b} \right) \frac{\partial z_b}{\partial x} &= - \frac{\partial}{\partial x} \left(\varepsilon \xi q_{bh} \frac{\partial z_b}{\partial x} \right), \end{aligned} \quad (6)$$

where ε is related to the diffusion term on the right-hand side of (6). Such diffusion is responsible for smoothing out perturbations to the bed elevation profile that would otherwise be present, as will be exemplified later. In practice, ε typically represents an additional tuning parameter within morphological models, which naturally increases their level of empiricism.

Maldonado-Villanueva (2015) proposed a two-layer-like, Shallow-Water-Equation-based model for sediment transport and morphological evolution in open channels. The model, which has the novelty that it requires no empirical formulae for sediment transport rates (although some empiricism is still necessary for closure of the model), was validated satisfactorily against empirical data for bedload transport rates in horizontal and inclined channels. Fig. 1 illustrates the comparison between the Quasi-2-Layer model proposed by Maldonado-Villanueva (Q2L model) against the laboratory data from Damgaard, Whitehouse, & Soulsby (1997) and the model (originally derived for arbitrary bed-slopes) by Bagnold (1963).

Fig. 1 depicts predicted and measured bedload transport rates versus the bed-slope angle, β , for three different values of the non-dimensional bed shear stress, $\theta \equiv \tau / [\rho g (s - 1) D]$ (where s is the sediment

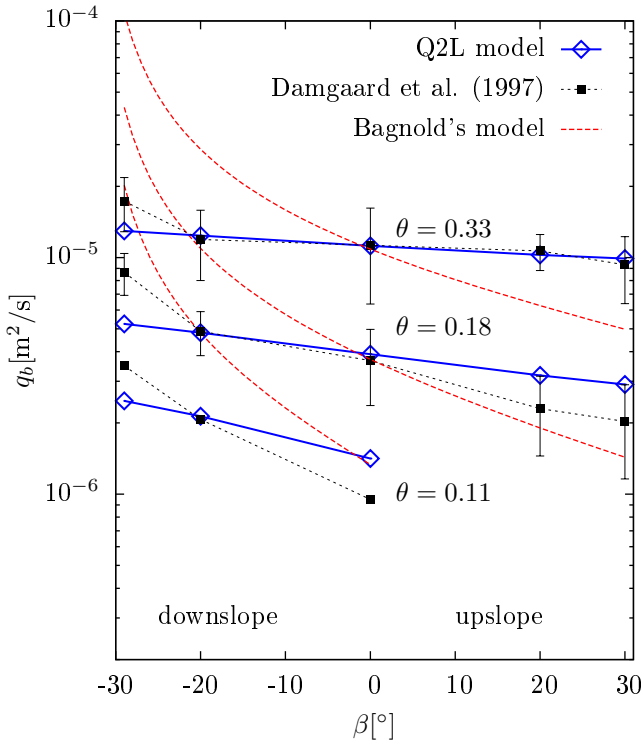


Figure 1: Bedload transport vs bed-slope angle for three different values of non-dimensional bed shear stress. Comparison between models by Maldonado-Villanueva (Q2L model) and Bagnold against empirical data reported by Damgaard et al. (1996)

relative density; and D is the sediment particle diameter). The model proposed by Maldonado-Villanueva (2015) is seen to agree well with empirical data reported by Damgaard, Whitehouse, & Soulsby (1997) for a wide range of bed-slopes and flow conditions, with agreement improving for larger values of bed shear stress. Potential explanations of the discrepancies found near $\pm\varphi$, where φ is the angle of repose (in the experiment by Damgaard, Whitehouse, & Soulsby 1997, $\varphi = 32.1^\circ$ is reported) may include the significant uncertainty involved in the estimation of the threshold of motion for steep-sloping channels (Chen, Ma, & Dey 2010). The model by Bagnold (1963) diverges significantly from the empirical data, especially for downsloping beds. However, Bagnold's model is particularly interesting because, unlike many others, it does not require additional tuning parameters to account for the effect of the bed-slope, and so a model based on Bagnold's ideas is further analysed in the following section.

After proving that the model could inherently account for the effect of bed-slope on bedload and morphological evolution, Maldonado-Villanueva (2015) then derived analytically an expression for the slope-related diffusivity, ε ; namely:

$$\varepsilon = \left(\frac{-\tau_{ch}}{\tau - \tau_{ch}} \right) \left(\frac{1}{\tan \varphi} \right), \quad (7)$$

where τ_{ch} denotes the critical bed shear stress for initiation of sediment motion (or dimensional

critical Shields parameter). The above expression has the advantage of not requiring any additional calibration parameter to tune the bed shear stress, as required within the hydrodynamic module of the model. Recall that the common practice is to treat ε as an additional tuning (often constant) parameter. Another strength of (7) is its physical meaningfulness – it predicts that the bed-slope influence vanishes at large bed shear stresses (note that $\varepsilon \rightarrow 0$ as $\tau \rightarrow \infty$), a phenomenon observed in experiments previously reported in the literature (see e.g. Damgaard, Whitehouse, & Soulsby 1997). Use of (7) is limited to values of the bed-slope of $S_b < 0.2$ (all negative values included), corresponding to angles of $\lesssim 11^\circ$. This range encompasses most of the applications to be expected in fluvial and coastal environments.

Another estimation of ε , which does not require additional calibration parameters, can be obtained from the Bagnoldian model of Bailard & Inman (1981). Manipulation of Bailard's work permits us to derive the following expression:

$$\varepsilon_{B\&I} = \frac{1}{\tan \varphi}. \quad (8)$$

It is worth noting that, although obtained following different approaches, both (7) and (8) predict a slope-related diffusivity inversely proportional to the angle of repose. However, $\varepsilon_{B\&I}$ is independent of the flow conditions, and so, unlike (7), it does not vanish at large flow velocities.

In the following section, the performance of the diffusivity parameter proposed in (7) is compared against that derived from Bagnoldian ideas (i.e. $\varepsilon_{B\&I}$) and against the case where no diffusion is included in (5) (i.e. $\varepsilon = 0$). The benchmark case employed is that of a completely submerged erodible hump (sandbar, in two dimensions) subject to a regular, nearly-uniform, unidirectional, subcritical current. The expected qualitative behaviour of the hump under these conditions is well known – it ought to migrate downstream and its downstream-face to steepen with time. Diffusion can also be introduced through numerical solution of the governing equations (see e.g. Johnson & Zyserman 2002) – one example which is also discussed below.

3 RESULTS

A submerged erodible hump (refer to Fig. 2) is located in an otherwise flat, horizontal channel of length, $l = 1000$ m. Bed friction is neglected within the hydrodynamic module ($\tau = 0$ in eq. 2), and the hump profile is described by:

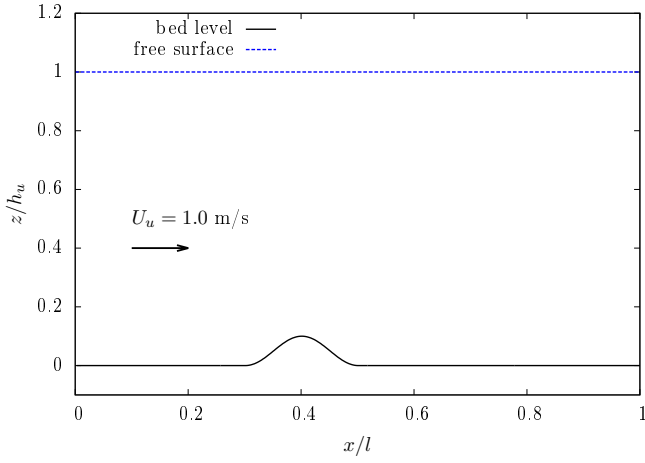


Figure 2: Sketch of the initial conditions for the case of a migrating submerged hump.

$$z_b(x, t_0) = \begin{cases} A \sin^2 \left[\frac{\pi(x-300)}{200} \right] & \text{if } 300 \leq x \leq 500\text{m} \\ 0 & \text{otherwise} \end{cases},$$

where $A = 1.0$ m is the maximum initial height of the hump. The initial total water depth at the upstream boundary, $h_u = 10$ m and the upstream steady flow velocity, $U_u = 1.0$ m/s.

The same case was studied by Hudson & Sweby (2003), who found an analytical solution based on the method of characteristics. Although this mathematical solution represents a sound verification tool for morphodynamic models following the structure described by (1)–(3), its predicted evolution of the hump does not necessarily match the physics of the phenomenon. The solution by Hudson & Sweby (2003) considers no diffusion, and so unrealistically steep slopes are permitted. The importance of slope-related diffusion from a phenomenological perspective is later discussed.

In the numerical simulations, the model is initially run with the bed fixed for 35×10^3 s until a steady state flow velocity is achieved, after which time the bedload transport is activated. A sediment particle diameter of 2.0 mm is assumed and $c_f = 0.012$ (required for estimation of τ in eq. 7) is considered. The formula proposed by Meyer-Peter & Müller (1948) is used to estimate the bedload transport rate.

The hydrodynamic part of the governing equations (eqs. 1 and 2) is solved by means of a Harten-Lax-van-Leer Contact (HLLC) Riemann solver with MUSCL-Hancock second-order time integration (van Leer 1979; Toro, Spruce, & Speares 1994). The Exner equation is discretised using a second-order central finite-difference scheme in space and second-order Adams-Bashforth time integration.

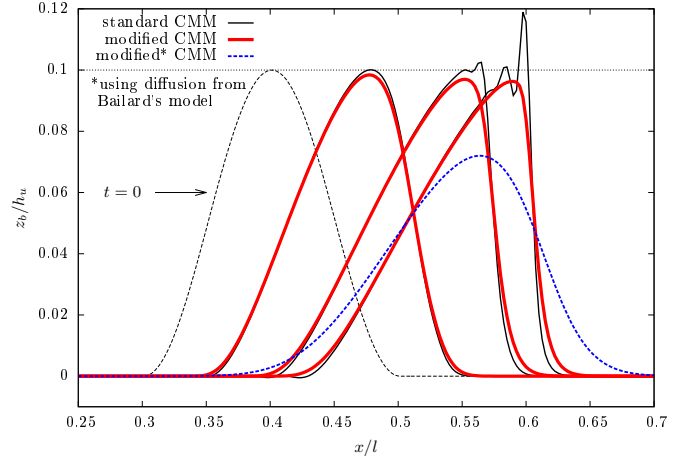


Figure 3: Evolution of the hump predicted by different approaches. Black thin line represents no modification of q_b ; red thick line denotes $q_{b\beta}$ modified through (7); dashed blue line represents $q_{b\beta}$ modified through $\varepsilon_{B\&I}$. Results shown (from left to right) at $t = 100 \times 10^3$, $t = 200 \times 10^3$ and $t = 250 \times 10^3$ s. Results for use of $\varepsilon_{B\&I}$ only shown for $t = 250 \times 10^3$ s (blue dashed line).

Fig. 3 depicts the evolution of the hump predicted by the governing equations (1)–(3), when i) no modification of q_b is considered (standard CMM); ii) q_b is modified through (5) and (7) (modified CMM); and iii) $\varepsilon_{B\&I}$ from Bailard & Inman (1981) is employed to estimate $q_{b\beta}$ (modified* CMM). When the CMM is not modified ($\varepsilon = 0$), the well-known oscillations in the bed level become very evident after 250×10^3 s of simulation, and eventually render the model unstable. When diffusivity from Bailard & Inman (1981) is invoked, high-frequency oscillations are prevented; however, the correct migration of the hump is not replicated – the hump attenuates, demonstrating that $\varepsilon_{B\&I}$ is over-diffusive. On the other hand, the modification of $q_{b\beta}$ through ε from (7) prevents development of oscillations in z_b , while predicting a realistic migration of the hump (steepening of its downstream face). The latter modification also allows us to run a much longer simulation, as it renders the model more stable (not shown here for brevity).

Similar results to those achieved through the inclusion of ε can be obtained by means of numerical techniques. For example, the Exner equation can be spatially discretised as the arithmetic mean of an upwind and a central finite difference scheme, such that

$$\frac{\partial q_b}{\partial x} = \frac{1}{2} \left[\frac{q_b(i+1) - q_b(i-1)}{2\Delta x} + \frac{3q_b(i) - 4q_b(i-1) - q_b(i-2)}{2\Delta x} \right], \quad (9)$$

where Δx is the length of a grid cell denoted by i . Without any modification to the bedload formula, the development of high-frequency oscillations can also

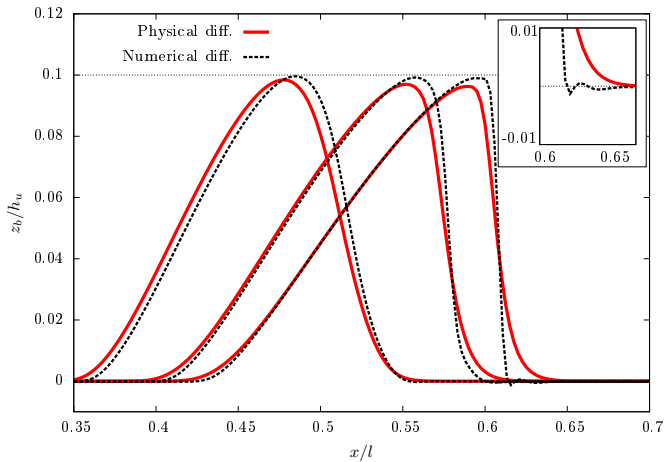


Figure 4: Comparison between hump evolution predicted with inclusion of physical diffusion ($q_{b\beta}$ modified through eq. 7) and numerical diffusion arising from discretisation of (3) described by eq. (9).

be delayed using this numerical technique. This is illustrated in Fig. 4, which compares results obtained using the above numerical technique against the use of ε from (7). The foregoing numerical technique retards the generation of oscillations (compare against the black thin curves in Fig. 3); these begin to develop at the upstream base of the hump towards the end of this simulation (see top-right panel in Fig. 4). Moreover, this approach yields significantly less diffusion than the use of (7).

It is important to note that, although numerical solvers can be used to avoid unrealistic oscillations in the bed level in some limiting cases, as has been demonstrated here, the primary goal of introducing a bed-slope-related diffusion term is not to avoid numerical oscillations. This term is included to ensure that the model represent more correctly observed physical phenomena such as the diffusion (smoothing effect) reported in the evolution of excavated holes in the surf zone (Moulton, Elgar, & Raubenheimer 2014). As such, the addition of ε is a more general approach—grounded in the physics of the phenomenon—that avoids development of numerical oscillations in the bed.

The two-dimensional version of the case analysed above can also be studied (i.e. a sandbar). The hump is projected 20 m in the direction orthogonal to x (i.e. y) and the flow is kept unidirectional (in the x -direction). The rest of the parameters are unchanged from above. Fig. 5 compares the four approaches previously considered. The findings are the same for the two-dimensional case. Lack of modification to the CMM promotes development of unrealistic oscillations in the bed level; use of $\varepsilon_{B\&I}$ shrinks the sandbar; and the diffusion parameter yields stable realistic results (which are also similar—although slightly more diffusive—to those achieved through numerical manipulation of the bed-update equation).

To model a fully two-dimensional flow (i.e. with velocity component in the y -direction) over a two-dimensional hump (where $\partial z_b/\partial y \neq 0$) it would be necessary to include the influence of the transverse bed-slope. Further research is recommended to address this problem.

4 CONCLUSIONS

In order to account for the influence of bed-slope on bedload (and thus, morphological evolution), we have proposed a physically meaningful diffusivity parameter that is easy to implement within conventional morphodynamic models based on the coupling described by (1)–(3). The parameter is derived from a morphodynamic model previously validated against empirical data for bedload transport rates in horizontal and steep sloping channels. The proposed expression for ε (eq. 7) does not require additional tuning variables and can enhance the performance of conventional morphodynamic models. This has been demonstrated through the benchmark cases of a submerged migrating hump (in one-dimension) and sandbar (in two-dimensions). The proposed ε prevents development of spurious oscillations in bed level, while reproducing realistic migration of the erodible hump including steepening of its downstream face; this is not achieved by implementing the model of Bailard & Inman (1981) which leads to over-diffusive results. High-frequency oscillations in z_b could also be avoided through mathematical means (Hudson & Sweby 2003) or numerical techniques (eq. 9; Johnson & Zyserman 2002). However, inclusion of slope-related diffusion has morphodynamic consequences, and so should not be used merely as a tool to stabilize numerical models. The present paper has focused on the effect of streamwise slope on bedload transport; extension of the approach to accommodate the transverse-slope effect is presently under investigation by the authors.

5 ACKNOWLEDGEMENTS

The first author was supported by scholarships awarded by the Mexican National Council for Science and Technology (CONACYT, Scholarship No. 310043) and the Mexican Ministry of Education (SEP). The second author is a recipient of a scholarship from the University of Edinburgh. All three authors would also like to acknowledge University College Cork, Ireland, where part of the work reported herein was undertaken.

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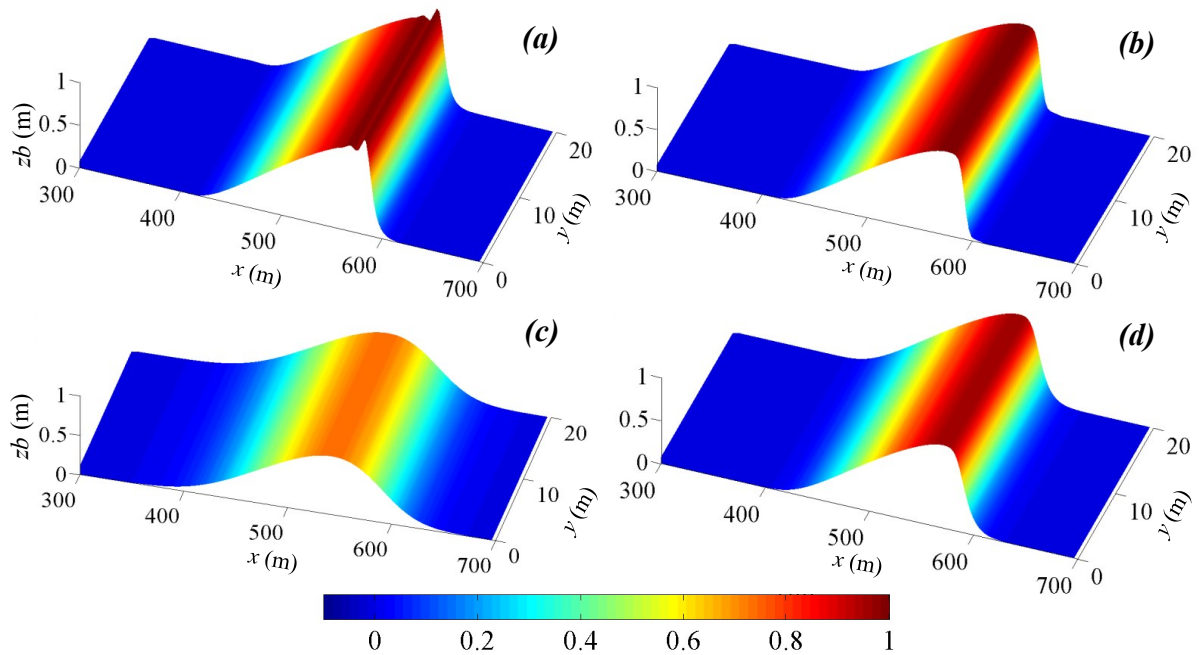


Figure 5: Evolution of a two-dimensional sandbar predicted by: a) CMM with no modification of q_b (i.e. $\varepsilon = 0$); b) discretisation of Exner equation according to (9); c) CMM with $q_{b\beta}$ modified through $\varepsilon_{B\&I}$; and d) CMM with $q_{b\beta}$ modified via the herein proposed expression for ε (eq. 7). Results shown at $t = 238 \times 10^3$ s.

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