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RESEARCH ARTICLE

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Time delay control with sliding mode observer for a class of nonlinear systems: Performance and stability

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Abstract

Time delay control (TDC) is a type of disturbance observer (DO)-based control, where the disturbance estimation is performed by using the past information of control input and measurement signals. Despite its capability, there are concerns about its practical implementation. First, it requires acceleration measurements which are generally not available in many industrial systems. Second, input delays are introduced into the closed-loop system, but the relation between the size of the delay and the performance of TDC has not been studied. Finally, there is a lack of tools to analyze its performance in disturbance estimation and robust stability for a given set of control parameters. We construct Lyapunov-Krasovskii functionals for a class of nonlinear systems which leads to delay-dependent conditions in linear matrix inequalities (LMIs) for the ultimate boundedness of the closed-loop system. This provides a means for analyzing the trade-off between the accuracy of disturbance estimation and robust stability. To circumvent acceleration measurements, we construct a sliding mode (SM) observer where the resulting error dynamics turns into a neutral type delay system. The existence conditions of both the SM control and SM observer are provided via a single LMI. A simulation example considering the tracking control of an autonomous underwater vehicle at constant and varying speed with a comparison to a non-TDC shows the effectiveness of the proposed method.

KEYWORDS

linear matrix inequalities, nonlinearity-disturbance estimation, sliding mode controller and observer, time delay control

1 | INTRODUCTION

The technique of time delay control (TDC) has been originally developed to compensate for system uncertainties, for example, unmodeled dynamics, parameter variations, and the effect of disturbances. The technique utilizes the past information of the control input and measurement signals to estimate the effect of the nonlinearities and disturbances.^{1,2} While delays are considered to be undesirable in many systems, it might also have a stabilizing effect, see References 3-5. For introduction to the topic of time delay systems, please refer to References 6, 7, or 8. Another popular and effective robust control strategy is sliding mode control (SMC). It is well known for its inherent robust property against a class

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of unmeasurable disturbances and uncertainties.⁹ There are more recent advances in SMC. Gonzalez et al.¹⁰ considered finite-time convergence problem in variable gain super-twisting SMC for matched perturbations/uncertainties that are Lipschitz-continuous. An adaptive continuous higher order SMC was designed to mitigate the chattering effect.¹¹ SMC based on finite-time boundedness for a class of nonlinear systems was investigated in Reference 12. A dissipativity-based SMC of continuously switched stochastic systems was proposed in Reference 13. A more recent collection of SMC advances and applications can be found in Reference 14. Combining robust control strategies such as SMC with methods that give estimates of uncertainties and disturbances is an attractive proposition. Such a combination enables a reduction in the magnitude of discontinuous components in the control and thereby offers the possibility of mitigating the chattering in control. Such control strategy and its applications can be found in Reference 15-17.

TDC has been applied in experimental environment in many systems.¹⁸⁻²⁵ Despite its robustness, there are some critics about the usage of TDC.^{26,27} There is a lack of guidelines for how to select the TDC parameters for disturbance estimation and feedback controller gains. The delay is usually chosen as the smallest sample size available in digital control. But these delays introduced to the control inputs may cause stability issues and make the stability analysis quite complicated. To what size of the delay the closed-loop system can tolerate is yet to be investigated. SMC under input delay was investigated, see, for example, Reference 28, where ultimate bounded stability was derived. In addition, TDC requires acceleration measurements. This limits its application as the acceleration measurements are generally not available in many industrial systems. It is also difficult to construct the acceleration signal from the velocity signal by differentiation due to injection of noise with discrete time-derivation. Another problem as pointed out in Reference 29 is that in the presence of so-called hard nonlinearities, such as saturation or static friction, TDC reveals some problems commonly found in other methods, like PID control or disturbance observer (DO). An increase in the command input or the response speed leads to (or would lead to) large over-shoots, limit cycles, or even unstable responses on the outputs. A simple frequency domain analysis shows that TDC contains a natural integral action, which is generated from the time-delayed estimation of the uncertainties and disturbance. Owing to the integral action, therefore, when an actuator has a saturation element, a wind-up phenomenon occurs as the control input increases. Hence, the design of disturbance estimation has to be considered together with the design of the controller gains to prevent the wind-up phenomenon. While TDC has shown promising results in experimental studies, literature rigorously analyzing these aspects that delimit the capabilities of TDC is scarce. How a system would respond to larger size of delays and how the disturbance estimation parameters and controller gains are to be selected so that the natural integral action in TDC is prevented from destabilizing the system are still to be investigated.

While there has been a lack of theoretical results in TDC to explore its full capability and performance limits, there are on-going efforts in studying other type of disturbance estimation methods which originated from the same concept as TDC. A nonlinear disturbance observer (NDO), which circumvents the need to use delays and acceleration measurements, was proposed in Reference 30 to estimate constant disturbance torques caused by unknown friction in robotic manipulators. An additional variable is introduced to avoid the measurement of the acceleration signals, in the form of an either linear or nonlinear functions. However, the resulting error of disturbance estimation in NDO depends on the derivative of the disturbances, whereas the resulting error of disturbance estimation in TDC only depends on the difference of the disturbance over the delay duration. The bound on the derivative of the disturbances can much greater than the bound on the time difference of the disturbance. This restricts the NDO to account for a typical type of disturbances with some known properties. A harmonic disturbance was considered in Reference 31 with known frequency but unknown amplitude and phase rather than constant ones. Based on this, an enhanced version of DO is also provided. In Reference 32, a disturbance estimator which requires the full knowledge of the nonlinearities was designed, so that it is fully compensated in the disturbance estimation error. Hence, the disturbance error dynamics is free from the control input, exemplifying the separation principle, that is, separating the design of controller and DO into two tasks. First, a state feedback controller that stabilizes the system and meets other design specifications is designed. Then, a DO is obtained which minimizes the error between the disturbances and its estimates provided by the DO.

Another disturbance observation technique, which is originated from TDC is developed in Reference 26, where it was shown in frequency domain that a low-pass filter or an uncertainty and disturbance estimator (UDE) which does not use acceleration measurements can be designed and its performance was shown to be comparable to that of TDC. In Reference 27, a study was performed to provide uncertainty and disturbance estimation for linear uncertain systems. A detailed filter was designed to cover both the low and high frequency range for attenuating the disturbance estimation error. The control gain can be increased arbitrarily to attenuate the disturbance estimation error. A modified UDE was used in Reference 33 to compensate for model uncertainties and reject input disturbances for quadrotors with input/output delays. None of the methods, that is, DO and UDE, have considered the effect of the controller gain on the

bounding of the disturbance estimation error which is a function of the control inputs, meaning that larger controller gains designed for attenuating the disturbance estimation error could increase the estimation error and consequently violate the stability conditions. A comprehensive overview of disturbance-observer-based control (DOBC) can be found in Reference 34. Their limitations and further improvements can be summarized in the following two points. First, as a limitation, DOBC requires all of the states to be available, the low-pass filter, designed in frequency domain, still largely depends on tuning (under certain guidance). Second, as a question for further improvement, what is the limit of this approach? How to analyze the robust stability and performance for a designed DOBC strategy? For a prescribed level of the uncertainties and nonlinearities, how to develop a strategy that requires a minimum level of feedback and control bandwidth?

In this article, we attempt to tackle the above challenging questions by considering a sliding mode (SM) observer-based TDC control using only position and velocity measurements. The acceleration signal is estimated using a SM observer. In the literature of SM observer and controller for time delay systems, a SM observer for uncertain time delay systems was designed in Reference 35, where matched uncertainties and nonlinearities are considered. SMC for systems with delays, matched and mismatched model uncertainties, and external disturbances was performed in Reference 36. Readers are referred to Reference 37 for a comprehensive survey of SM observers. In this article, the system nonlinearities are assumed to be locally bounded by Lipschitz constants. We consider the effect of the controller and observer gains on the disturbance estimation error dynamics, and propose LMI conditions for minimizing the ultimate bound of the observer-based control system under either constant or varying delay. The resulting ultimate bound depends on the nonlinearities of the system which include external disturbances, controller and state observer inputs, and the generated reference speed signals. A scaling matrix Λ is introduced for tuning the trade-off between the accuracy of nonlinearity-disturbance estimation and robust stability depending on the size of the delay and its varying rate. It is shown that the resulting closed-loop system exhibits a delay system of a neutral type. For larger nonlinearities associated with larger speed variations, the scaling matrix needs to be reduced, implying a reduced estimation accuracy for increased stability. Hence, the proposed strategy prevents the problem of large overshoot and unstable responses in the output due to an increase in the command input, which commonly occurs with PID or DO-based controllers. The conditions for the existence SM for both the observer and the error neutral delay systems are provided in a single LMI. Based on the ultimate bound, a dynamical switching gain is designed to minimize the impact of the input-dependent TDC estimation error.

In Section 2, the generic model of the type of nonlinear system considered in this study is given and the problem to be solved is explained. The conditions for the existence of SM are given in Section 3. The closed-loop reachability condition is given in Section 4. The finite-time convergence conditions are given in Section 5. Simulation example and results are demonstrated in Section 6.

Notation: A standard notation is used throughout the article, \mathbb{R}^n denotes the *n*-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and P > 0 for $P \in \mathbb{R}^{n\times n}$ means that *P* is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by *. $\overline{\lambda}(P)$ and $\underline{\lambda}(P)$ denote the maximum and minimum eigen-value of the matrix *P*. The symbol $\|\cdot\|_{\infty}$ stands for essential supremum. Time dependent variables, such as x(t), are simply expressed as *x* wherever it does (would) not cause any confusion. col $\{\cdot\}$ denotes a column vector. Finally, $(X)^*$ stands for $X + X^T$.

2 | SYSTEM MODELING AND PROBLEM FORMULATION

Consider a nonlinear system governed by the following Euler-Lagrange dynamics

$$M(\eta)\ddot{\eta} + D(\eta,\dot{\eta})\dot{\eta} + d = F,$$
(1)

where $\eta \in \mathbb{R}^n$ is the generalized position in *n* axes, $M(\eta) \in \mathbb{R}^{n \times n}$ is a time varying positive definite inertia matrix, $D(\eta, \dot{\eta}) \in \mathbb{R}^{n \times n}$ represent the hydraulic damping for autonomous underwater vehicle (AUV),³⁸ the centrifugal and Coriolis force in space manipulators³⁹ or dry friction at each joint in exoskeleton robots.⁴⁰ $F \in \mathbb{R}^n$ is the generalized control inputs. The term $d \in \mathbb{R}^n$ represents any kind of disturbances such as external torques which are assumed to be bounded and $||d - d(t - L_t)|| \le \overline{d}$ is bounded by a positive constant $\overline{d} > 0$, where $L_t \triangleq L(t) \in (0, L^*]$ with $L^* > 0$ is a known time-varying delay with $\dot{L}_t \in [0, 1)$. Matrix $M(\eta)$ is always positive definite and is invertible. Let $M(\eta)$ be composed as $M(\eta) = M_d + M_b(\eta)$, where M_d is a diagonal matrix consisting of the constant parameters of the system. Let $\overline{M} = \text{diag}(\overline{m}_1, \dots, \overline{m}_n)$ be an user-defined matrix with $\overline{m}_i > 0$, $\forall i = 1, \dots, n$, then system (1) can be written as

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$$\overline{M}\ddot{\eta} + H = F,\tag{2}$$

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where $H \in \mathbb{R}^n$ lumps all the nonlinearities and disturbances and is given as

$$H = (M_d - \overline{M})\ddot{\eta} + \Phi, \tag{3}$$

where $\Phi = M_b(\eta)\ddot{\eta} + D(\eta,\dot{\eta})\dot{\eta} + d$.

Assumption 1. There exist known Lipschitz constants c_0 and c_1 such that

$$\|\Phi - \Phi(t - L_t)\| \le c_1 \|\ddot{\eta} - \ddot{\eta}(t - L_t)\| + c_0 \|\dot{\eta} - \dot{\eta}(t - L_t)\| + \overline{d}, \quad \forall \ \dot{\eta}, \ \ddot{\eta} \in \mathbb{R}^n.$$
(4)

When we only consider to design a local controller and observer, the assumption of global Lipschitz nonlinearity of Φ can be replaced by that Φ is a local Lipschitz function. All the results given in this note are then valid in a neighborhood around a nominal point. Lipschitz nonlinear systems have been investigated by many authors and some relative works can be found in References 41-43. Since $\Phi = 0$ when $\ddot{\eta}$, $\dot{\eta}$, d = 0, Assumption 1 implies that there exist constants c_1 , c_0 such that $\|\Phi\| \le c_1 \|\ddot{\eta}\| + c_0 \|\dot{\eta}\| + \|d\|$.

Let $x = col\{x_1, x_2\} \in \mathbb{R}^{2n}$, where $x_1 = \eta$, $x_2 = \dot{\eta}$, then (2) can be put into the state-space form

$$\dot{x} = Ax + B(F - H), \quad y = Cx, \tag{5}$$

where $A = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix}$, $B = \begin{pmatrix} 0_n \\ \overline{M}^{-1} \end{pmatrix}$, and $C = I_{2n}$ is an identity matrix. Consider the corresponding ideal reference model

$$\dot{x}_r = A_m x_r + B_1 u_r,\tag{6}$$

where $A_m = \begin{pmatrix} 0_n & I_n \\ A_{m_{21}} & A_{m_{22}} \end{pmatrix}$ and $A_{m_{21}}$, $A_{m_{22}} \in \mathbb{R}^{n \times n} < 0$ are diagonal matrices. Matrix $B_1 = \begin{pmatrix} 0_n \\ I_n \end{pmatrix}$ and $u_r \in \mathbb{R}^n$ is a command signal. Defining $G_1 = \overline{M} \begin{pmatrix} A_{m_{21}} & A_{m_{22}} \end{pmatrix}$ and $G_2 = \overline{M}$, it yields $BG_1 = A_m - A$ and $BG_2 = B_1$. Next, denoting the error between the ideal reference signals and the system measurements as $x_e = x_r - x$, one can obtain its derivative as

$$\dot{x}_e = A_m x_e + B(G_1 x + G_2 u_r - F + H).$$
(7)

In this article, we aim to construct the estimate of H and use this estimate in our control law to increase the robust performance of the closed-loop system subjected to external disturbances. However, the construction of this estimation requires the acceleration signals of system (5), which has the position and velocity signals available only. In the following, a SM observer will be designed to estimate the acceleration signals of the system.

Let $\hat{x}_e = \text{col}\{\hat{x}_{e_1}, \hat{x}_{e_2}\} \in \mathbb{R}^{2n}$ be the observer states, and $e = x_e - \hat{x}_e$ be the observation error. Our control law is defined as

$$F = \overline{M}v + \hat{H},\tag{8}$$

where $v = \overline{M}^{-1}G_1x + \overline{M}^{-1}G_2u_r + u$ and the expression of $u \in \mathbb{R}^n$ will be given later. $\hat{H} \in \mathbb{R}^n$ denotes the estimation of *H* in TDC and is given as

$$\hat{H} = \Lambda \left(F(t - L_t) - \overline{M} \left(\dot{x}_{r_2}(t - L_t) - \dot{\bar{x}}_{e_2}(t - L_t) \right) \right) = \Lambda H(t - L_t) - \Lambda \overline{M} \dot{e}_2(t - L_t), \tag{9}$$

where $\Lambda = \text{diag}\{a_1, \dots, a_n\}$ with $a_i \in [0, 1)$ is a positive diagonal matrix which governs the accuracy of the nonlinearity-disturbance estimation. It is assumed F(t), $\dot{x}_{r_2}(t)$, $\dot{x}_{e_2}(t) = 0$ for t < 0.

Remark 1. In practice, the smallest achievable L_t is the minimum sampling period in digital implementation. A digital control system behaves reasonably close to the continuous system if the sampling rate is larger than 30 times the bandwidth.⁴⁴ Hence, with L_t smaller than this level, H is assumed to be continuous and its effect can be estimated as \hat{H} in TDC. For sampled data control, Equation (9) becomes

$$\hat{H} = \Lambda \left(F_d(t_k) - \overline{M} \left(\dot{x}_{r_{2d}}(t_k) - \dot{\bar{x}}_{e_{2d}}(t_k) \right) \right), \ t_k \le t < t_{k+1}.$$
(10)

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Following the approach in References 45 and 46, the above equation can be formulated as a continuous-time system with a known delay as in (9), where $t \in [t_k, t_{k+1})$ and $L_t = t - t_k$. Sampling may be variable but subject to $t_{k+1} - t_k \le L^*$, $\forall k \ge 0$, that is, the time between any two sequential sampling instants is not greater than some pre-chosen $L^* > 0$. Then $L_t \in (0, L^*]$ with $\dot{L}_t = 1$ for $t \ne t_k$ is known with the known sampling instants t_k . For control design of a sampled-data system, one could refer to References 45 and 46. The sampled-data control will allow easier implementation of the disturbance estimation (9). Thus TDC observes the states and the inputs of the system one sample into the past at t_k , and determines the control action that should be commanded at time t.

Substituting (8) into (7) yields

$$\dot{x}_e = A_m x_e - B_1 u + B(H - \hat{H}).$$
(11)

Denoting $\epsilon = H - H(t - L_t)$, which is the disturbance and nonlinearity estimation errors, and substituting (9), (11) becomes

$$\dot{x}_e = A_m x_e - B_1 u + B\varepsilon + (I - \Lambda) \overline{M}^{-1} B_1 H(t - L_t) + \Lambda B_1 \dot{e}_2(t - L_t).$$

$$\tag{12}$$

We design a SM observer in the following form

$$\dot{\hat{x}}_e = A_m \hat{x}_e + B_f e - B_1 u + B_1 v_e, \tag{13}$$

where $B_f \in \mathbb{R}^{2n \times 2n}$ is the observer matrix to be constructed and $v_e \in \mathbb{R}^n$ is the observer control inputs to be designed. Then, denoting $D = \begin{pmatrix} 0_n & 0_n \\ 0_n & \Lambda \end{pmatrix}$, the observer output error dynamics can be written as follows

$$\dot{e} - D\dot{e}(t - L_t) = (A_m - B_f)e + B\epsilon + (I - \Lambda)\overline{M}^{-1}B_1h(t - L_t) - B_1v_e,$$
(14)

with initial condition $e(0) = e_0$, e(t) = 0, t < 0. Equation (14) is a time delay system of a neutral type. For more studies on this type of systems please refer to Reference 6 and references therein.

Denoting $\overline{\dot{\Phi}} = \overline{\Phi} - \Phi(t - L_t)$, $\overline{\dot{v}} = v - v(t - L_t)$, and $\overline{\ddot{e}}_2(t - L_t) = \dot{e}_2(t - L_t) - \dot{e}_2(t - 2L_t)$, then it can be shown that

$$\epsilon = \Lambda \left(I_n - M_d^{-1} \overline{M} \right) \epsilon(t - L_l) + \Lambda M_d^{-1} \overline{M} \overline{\Phi} + \Lambda \overline{M} \left(I_n - M_d^{-1} \overline{M} \right) \left(\overline{\dot{\nu}} - \overline{\ddot{e}}_2(t - L_l) \right) + (I - \Lambda) X, \tag{15}$$

where

$$X = \overline{M} \left(I_n - M_d^{-1} \overline{M} \right) v + M_d^{-1} \overline{M} \Phi - \Lambda \overline{M} \left(I_n - M_d^{-1} \overline{M} \right) \dot{e}_2(t - 2L_t) - \left(H(t - L_t) - \Lambda \left(I_n - M_d^{-1} \overline{M} \right) H(t - 2L_t) \right).$$
(16)

For the proof of (15), please see Appendix A.1.

Remark 2. Equation (15) can be validated by setting $\Lambda = 0$, then $\epsilon = X = \overline{M} \left(I_n - M_d^{-1} \overline{M} \right) \nu + M_d^{-1} \overline{M} \Phi - H(t - L_t)$ and $\hat{H} = 0$. Since by definition $\epsilon = H - H(t - L_t)$, the following holds:

$$H = \left(I_n - M_d^{-1}\overline{M}\right)F + M_d^{-1}\overline{M}\Phi.$$
(17)

Substituting H in (3), (17) becomes (2).

It is desirable to choose a larger \overline{m}_i in \overline{M} so that the effect of \overline{v} and $\overline{e}_2(t - L_t)$ on ϵ is reduced in (15). However, larger values of \overline{m}_i will increase the effect of $\overline{\Phi}$ on ϵ . Since \overline{v} depends on the control signal u and the reference model matrix G_1 and G_2 , and $\overline{e}_2(t - L_t)$ depends on the observer gain B_f , the choice of the controller and observer gains, as well as the reference signal parameters have a direct impact on TDC error ϵ . While we can reduce the values of the diagonal elements in Λ to minimize the dependence of ϵ on $\overline{\Phi}$, \overline{v} , and $\overline{e}_2(t - L_t)$, this causes ϵ to depend more on X and the accuracy of

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disturbance estimation (9) to be degraded. We aim to provide the delay-dependent LMI conditions to assess the trade-off between the performance of disturbance estimation and robust stability and provide the minimum control gains that preserves the performance of TDC.

Consider the following sliding surfaces:

$$\hat{S} = \{ \hat{x}_e \in \mathbb{R}^{2n} : B_1^T P_\mu \hat{x}_e = 0 \},$$
(18)

and

$$S_e = \{ e \in \mathbb{R}^{2n} : B_1^T P_\mu \left(e - De(t - L_t) \right) = 0 \},$$
(19)

where $P_{\mu} \in \mathbb{R}^{2n \times 2n} > 0$ is to be designed. It is desirable to design the estimation of nonlinearities and disturbance (9), the controller (8), the reference model parameters (6) and the observer (13) such that the closed-loop system is exponentially stable and the closed-loop system converges to the sliding surfaces (18) and (19) in finite time.

3 | SLIDING MANIFOLDS DESIGN

This section considers the stability of the closed-loop system once on the sliding surfaces (18) and (19). Let's define the control law as

$$u = K\hat{x}_e + v_u,\tag{20}$$

where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, $K_1 = A_{m_{21}} + \overline{K}_1$, $K_2 = A_{m_{22}} + \overline{K}_2$ and \overline{K}_1 , $\overline{K}_2 \in \mathbb{R}^{n \times n}$ are the control gains to be designed and $v_u = \rho \frac{\hat{s}}{\|\hat{s}\|}$ is the nonlinear control law. Let's define the observer matrix $B_f = \begin{pmatrix} B_{f_{11}} & I_n \\ B_{f_{21}} & \frac{1}{\mu}I_n \end{pmatrix}$, where $B_{f_{11}}$, $B_{f_{21}} \in \mathbb{R}^{n \times n}$ are to be designed. The observer control law is given in the form of

$$v_e = \rho_e \frac{s_e}{\|s_e\|}.$$
(21)

The switching gains ρ and ρ_e are some positive scalar functions of the outputs. We can write the observer system (13) and the error system (14) in the following form

$$\begin{aligned} \dot{x}_{e_1} &= \hat{x}_{e_2} + B_{f_{11}}e_1 + e_2, \\ \dot{x}_{e_2} &= -\overline{K}_1 \hat{x}_{e_1} - \overline{K}_2 \hat{x}_{e_2} + B_{f_{21}}e_1 + \frac{1}{\mu}e_2 - \nu_u + \nu_e, \\ \dot{e}_1 &= -B_{f_{11}}e_1, \\ \mu \left(\dot{e}_2 - \Lambda \dot{e}_2(t - L_t) \right) &= \mu (A_{m_{21}} - B_{f_{21}})e_1 + (\mu A_{m_{22}} - I)e_2 + \mu \overline{M}^{-1}\epsilon + \mu (I - \Lambda) \overline{M}^{-1} H(t - L_t) - \mu \nu_e, \end{aligned}$$
(22)

where ϵ is given in (15). Let P_{μ} be in the following structure

$$P_{\mu} = \begin{pmatrix} P_{11} & \delta_{1}\mu P_{22} \\ * & \delta_{2}\mu P_{22} \end{pmatrix}, \tag{23}$$

where P_{11} , $P_{22} > 0$ and μ , δ_1 , $\delta_2 > 0$ are user-defined parameters. A state transformation exists such that $\begin{pmatrix} \hat{x}_{e_1} \\ \hat{s} \end{pmatrix} = T\begin{pmatrix} \hat{x}_{e_1} \\ \hat{s}_{e_2} \end{pmatrix}$, and $\begin{pmatrix} \dot{e}_1 \\ \dot{s}_{e} \end{pmatrix} = T\begin{pmatrix} e_1 \\ e_2 - \Lambda e_2(t - L_l) \end{pmatrix}$, where $T = \begin{pmatrix} I_n & 0 \\ \mu \delta_1 P_{22} & \mu \delta_2 P_{22} \end{pmatrix}$. Hence system (22) can be rewritten as $\dot{x}_{e_1} = -\frac{\delta_1}{\delta_2} \hat{x}_{e_1} + \frac{1}{\delta_2 \mu} P_{22}^{-1} \hat{s} + B_{f_{11}} e_1 + e_2,$ $\dot{s} = \delta_1 \mu P_{22} \overline{K}_2 \hat{x}_{e_1} - P_{22} (\overline{K}_1 + \overline{K}_2) P_{22}^{-1} \hat{s} + \mu P_{22} (\delta_1 B_{f_{11}} + \delta_2 B_{f_{21}}) e_1 + (\delta_1 \mu + \delta_2) P_{22} e_2 + \delta_2 \mu P_{22} (v_e - v_u),$ (24)

$$\dot{e}_{1} = -B_{f_{11}}e_{1},$$

$$\dot{s}_{e} = \mu P_{22} \left(\delta_{1} \left(\frac{1}{\mu} I_{n} - B_{f_{11}} - A_{m_{22}} \right) + \delta_{2} \left(A_{m_{21}} - B_{f_{21}} \right) \right) e_{1} + P_{22} \left(A_{m_{22}} - \frac{1}{\mu} I_{n} \right) P_{22}^{-1} s_{e} + \delta_{2} \mu P_{22} \left(A_{m_{22}} - \frac{1}{\mu} I_{n} \right) \Lambda$$

$$\cdot e_{2}(t - L_{t}) + \delta_{2} \mu P_{22} \left(\overline{M}^{-1} \epsilon + (I - \Lambda) \overline{M}^{-1} H(t - L_{t}) - v_{e} \right).$$
(25)

Once the system trajectories are on the sliding surfaces (18) and (19), the dynamics of the systems (24) and (25) are governed by

$$\dot{\hat{x}}_{e_1} = -\frac{\delta_1}{\delta_2} \hat{x}_{e_1} + B_{f_{11}} e_1 + e_2,$$

$$\dot{e}_1 = -B_{f_{11}} e_1,$$

$$e_2 = \Lambda e_2 (t - L_t) - \frac{\delta_1}{\delta_2} e_1.$$
(26)

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Lemma 1. Given positive parameters L^* , $\overline{\alpha}$, δ_1 , δ_2 , positive diagonal matrix Λ , if there exist $n \times n$ matrices $\overline{P}_1 > 0$, $\overline{P}_2 \ge 0$ and matrices P_{f_1} , \overline{P}_3 , \overline{P}_4 such that the following LMI

$$\Omega = \begin{pmatrix} -2\frac{\delta_{1}}{\delta_{2}}\overline{P}_{1} + 2\overline{\alpha}\overline{P}_{1} & P_{f_{1}} & \overline{P}_{1} & 0 \\ * & (-P_{f_{1}})^{*} + 2\overline{\alpha}\overline{P}_{1} & -\frac{\delta_{1}}{\delta_{2}}\tilde{P}_{3} & -\frac{\delta_{1}}{\delta_{2}}\tilde{P}_{4} \\ * & * & \overline{P}_{2} + (-\overline{P}_{3}^{T})^{*} & \overline{P}_{3}^{T}\Lambda - \overline{P}_{4} \\ * & * & * & -e^{-2\overline{\alpha}L^{*}}\overline{P}_{2} + (\overline{P}_{4}^{T}\Lambda)^{*} \end{pmatrix} < 0$$
(27)

is feasible, then (26) is exponentially asymptotically stable with a decay rate $\overline{\alpha}$ for all $L \in [0, L^*]$. There exists $M_0 > 0$ such that the solution of (26) initialized by $\hat{x}_{e_1}(0) \in \mathbb{R}^n$, $e_1(0) \in \mathbb{R}^n$ satisfy the following inequality:

$$\|\hat{x}_{e_1}\|^2 + \|e_1\|^2 \le M_0 e^{-2\overline{\alpha}t} \left(\|\hat{x}_{e_1}(0)\|^2 + \|e_1(0)\|^2 \right), \quad \forall \ t \ge 0,$$
(28)

where $M_0 = \frac{1}{\underline{\lambda}(\overline{P}_1)}$. Moreover, the observer matrix $B_{f_{11}} = \overline{P}_1^{-1} P_{f_1}$.

Proof. Consider the following Lyapunov-Krasovskii functional

$$V = \hat{x}_{e_1}^T \overline{P}_1 \hat{x}_{e_1} + e_1^T \overline{P}_1 e_1 + \int_{t-L_t}^t e^{2\overline{\alpha}(s-t)} e_2^T(s) \overline{P}_2 e_2(s) \mathrm{d}s.$$
(29)

We define

$$W = \dot{V} + 2\alpha V = 2\hat{x}_{e_1}^T \overline{P}_1 \dot{\hat{x}}_{e_1} + 2e_1^T \overline{P}_1 \dot{e}_1 + 2\overline{\alpha} \hat{x}_{e_1} \overline{P}_1 \hat{x}_{e_1} + 2\overline{\alpha} e_1^T \overline{P}_1 e_1 + e_2^T \overline{P}_2 e_2 - e^{-2\overline{\alpha}L_t} e_2^T (t - L_t) \overline{P}_2 e_2 (t - L_t).$$
(30)

Then adding

$$0 = \left(e_2^T \tilde{P}_3^T + e_2^T (t - L_t) \tilde{P}_4^T\right) \left(-e_2 + \Lambda e_2 (t - L_t) - \frac{\delta_1}{\delta_2} e_1\right)$$
(31)

into (30), and defining $\overline{\xi}^T = (\hat{x}_{e_1}^T e_1^T e_2^T e_2^T (t - L_t))$, it follows W < 0 in (30) if $\overline{\xi}^T \Omega \overline{\xi} < 0$. W < 0 yields the solution of (26) to satisfy the bound $\underline{\lambda}(\overline{P}_1) \|\hat{x}_{e_1}\|^2 + \underline{\lambda}(\overline{P}_1) \|e_1\|^2 \le V \le e^{-2\overline{\alpha}t}V(0), t \ge 0$.

Remark 3. Lemma 1 provides conditions for the existence of SM for both the observer system and error system with neutral delay in a single LMI (27). Observer-based SMC control for systems with state delays has been studied in Reference 47, and for neutral delay systems has been studied in Reference 48. In those works, LMI conditions for the existence of SM with respect to the observer system were provided separately from the existence design for the observer error system. But the existence conditions related to the observer error dynamics were however missing. Lemma 1 shows that the existence of SM in observer-based SMC can be considered with respect to both the observer system and the error system.

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4 | REACHABILITY OF THE CLOSED LOOP SYSTEM

This section considers controller and observer design such that the closed-loop system (22) is exponentially attracted to an ultimate bound. Denoting $L_t = \mu \xi(t)$, where $0 \le \xi(t) \le \overline{L}$ and $\mu \overline{L} = L^*$, with $\mu \dot{\xi} \le \beta < 1$, then the following main result can be stated.

Theorem 1. Given positive tuning diagonal matrix Λ with its elements $a_i \in [0, 1)$ for i = 1, ..., n, positive $n \times n$ diagonal matrix \overline{M} , positive parameters μ , \overline{L} , $\mu \overline{L} = L^*$, and positive tuning scalars δ_1 , δ_2 , $\beta \in [0, 1)$, α , b, b_1 , b_2 , b_3 , b_4 , b_5 , b_6 , b_7 , positive scalars λ , c_0 , c_1 , and scalars $\hat{\delta}_2$, $\hat{\delta}_3$, $\hat{\delta}_4$, $\hat{\delta}_5$, if there exist a $2n \times 2n$ matrix $P_{\mu} > 0$ as in (23), and $n \times n$ matrices $S \ge 0$, $T \ge 0$, $R \ge 0$, $R_1 \ge 0$, $R_2 \ge 0$, $Q_1 \ge 0$, $P_{m_1} < 0$, $P_{m_2} < 0$, $n \times n$ matrices \hat{P}_2 , \hat{P}_3 , \hat{P}_4 , \hat{P}_5 , P_{k_1} , P_{k_2} , and P_{f_2} such that LMI $\Theta < 0$ with the following entries:

$$\begin{aligned} 1,1 &= (-\delta_{1}\mu P_{k_{1}})^{*} + 2aP_{11}, 1,2 &= P_{11} - \delta_{1}\mu P_{k_{1}} - \delta_{2}\mu P_{k_{1}}^{*} + 2a\delta_{1}\mu P_{22}, 1,3 &= P_{11}B_{j_{11}} + \delta_{1}\mu P_{j_{1}}, 1,4 &= P_{11} + \delta_{1}P_{22}, \\ 1,29 &= -L^{*}P_{k_{2}}^{T}, 2,2 &= 2\delta_{1}\mu P_{22} - (\delta_{2}\mu P_{k_{2}})^{*} + 2a\delta_{2}\mu P_{22}, 2,3 &= \delta_{1}\mu P_{22}B_{j_{11}} + \delta_{2}\mu P_{j_{2}}, 2,4 &= \delta_{1}\mu P_{22} + \delta_{2}P_{22}, 2,28 &= L^{*}Q_{1}, \\ 2,29 &= -L^{*}P_{k_{2}}^{T}, 3,3 &= (-P_{11}B_{j_{11}})^{*} + (\delta_{1}\mu P_{m_{1}} - \delta_{1}\mu P_{j_{2}})^{T} \Lambda - 2a\delta_{1}\mu P_{22}\Lambda, 3,7 &= \delta_{1}\mu P_{22} + \delta_{1}(\mu P_{m_{2}} - P_{22}) + \delta_{2}\mu(P_{m_{1}} - P_{j_{2}})^{T} \\ &+ 2a\delta_{1}\mu P_{22}, 3,5 &= \mu\delta_{1}B_{j_{11}}^{T}P_{22}\Lambda - \delta_{2}\mu(P_{m_{1}} - P_{j_{2}})^{T}\Lambda - 2a\delta_{1}\mu P_{22}\Lambda, 3,7 &= \delta_{1}\mu P_{22}\overline{M}^{-1}, 3,22 &= \delta_{1}\mu P_{22}(I - \Lambda)\overline{M}^{-1}, \\ 3,28 &= L^{*}B_{j_{11}}^{T}Q_{1}, 3,29 &= L^{*}P_{j_{1}}^{T}, 3,30 &= -L^{*}B_{j_{11}}^{T}Q_{2}, 3,31 &= \sqrt{2}L^{*}(P_{m_{1}} - P_{j_{2}})^{T}, 4,4 &= (\delta_{2}\mu P_{m_{2}})^{*} - 2\delta_{2}P_{22} + 2a\delta_{2}\mu P_{22} + T, \\ 4,5 &= -\delta_{2}(\mu P_{m_{2}} - P_{22})^{T}\Lambda - 2a\delta_{2}\mu P_{22}\Lambda, 4,7 &= \delta_{2}\mu P_{22}\overline{M}^{-1}, 4,22 &= \delta_{2}\mu P_{22}\overline{M}^{-1}, 4,28 &= L^{*}Q_{1}, 4,29 &= \overline{L}P_{22}, \\ 4,31 &= \sqrt{2}(L^{*}P_{m_{2}}^{T} - \overline{L}P_{22}), 5,5 &= -Te^{-2aL^{*}} + 2a\delta_{2}\mu \Lambda P_{22}\Lambda, 5,7 &= -\delta_{2}(\Lambda L - M_{1}^{-1}\overline{M}) - \delta_{3},7,9 &= \tilde{P}_{2}^{T}\Lambda\overline{M}(I - M_{1}^{-1}\overline{M}), \\ 7,11 &= \delta_{2}\mu\overline{M}^{-1}P_{22}, 6,31 &= \sqrt{2}\Lambda^{*}P_{22}, 7,7 &= S + (-\tilde{P}_{2}^{T})^{*}, 7,8 &= \tilde{P}_{2}^{T}\Lambda(I - M_{1}^{-1}\overline{M}) - \tilde{P}_{3}, 7,9 &= \tilde{P}_{2}^{T}\Lambda\overline{M}(I - M_{1}^{-1}\overline{M}), \\ 7,11 &= \delta_{2}\mu\overline{M}^{-1}P_{22}, 7,12 &= -\tilde{P}_{2}^{T}\Lambda\overline{M}(I - M_{1}^{-1}\overline{M}) - \delta_{3}\mu\overline{M}^{-1}P_{22}, 8,13 &= \tilde{P}_{3}^{T}\Lambda\overline{M}_{1}^{-1}\overline{M}, 8,14 &= \overline{M}^{-1}\tilde{P}_{4}, 8,15 &= \overline{M}^{-1}\tilde{P}_{5}, \\ 7,27 &= \tilde{P}_{2}^{T}(I - \Lambda), 7,31 &= \sqrt{2}L^{*}\overline{M}^{-1}\overline{M}, - \delta_{3}\mu\overline{M}^{-1}P_{22}, 8,13 &= \tilde{P}_{3}^{T}\Lambda\overline{M}_{1}^{-1}\overline{M}, 8,14 &= \overline{M}^{-1}\tilde{P}_{4}, 8,15 &= \overline{M}^{-1}\tilde{P}_{5}, \\ 8,27 &= \tilde{P}_{3}^{T}(I - \Lambda), 9, 9 &= R_{1} - 2\delta_{4}P_{22}, 9,10 &= -\delta_{5}P_{2}, 9,14 &= \delta_{4}P_{2}, 9,25 &= \delta_{4}P$$

and with the rest of the entries being zero, is feasible, then system (22) with the disturbance estimation error given in (15) is exponentially attracted by the ellipsoid

$$\begin{split} \lim_{t \to \infty} \sup_{t \to \infty} \left\| \left(\hat{x}_{e} \left(\begin{array}{c} e_{1} \\ e_{2} - \Lambda e_{2}(t - L_{t}) \end{array} \right) \right) \right\|^{2} &\leq \frac{1}{2\alpha \underline{\lambda}(P_{\mu})} \left(b \left\| H(t - L_{t}) \right\|_{\infty}^{2} + b_{1} \| \mathbf{X} \|_{\infty}^{2} + b_{2} \left\| \int_{t - L_{t}}^{t} \dot{x}_{r_{2}}(s) \mathrm{d}s \right\|_{\infty}^{2} + b_{3} \overline{d}^{2} \\ &+ b_{4} \| v_{u} - v_{u}(t - L_{t}) \|_{\infty}^{2} + b_{5} \| H(t - L_{t}) - H(t - 2L_{t}) \|_{\infty}^{2} + b_{6} \| v_{e} - v_{e}(t - L_{t}) \|_{\infty}^{2} + b_{7} \left\| \int_{t - L_{t}}^{t} \ddot{x}_{r_{2}}(s) \mathrm{d}s \right\|_{\infty}^{2} \right), \end{split}$$
(33)

with a decay rate α for all $\xi \in [0, \overline{L}]$, $\mu \dot{\xi} \leq \beta < 1$. Moreover, the following matrices can be obtained as $\overline{K}_1 = P_{22}^{-1}P_{k_1}$, $\overline{K}_2 = P_{22}^{-1}P_{k_2}$, $B_{f_{21}} = P_{22}^{-1}P_{f_2}$, and $A_{m_{21}} = P_{22}^{-1}P_{m_1}$, $A_{m_{22}} = P_{22}^{-1}P_{m_2}$.

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Proof. See Appendix A.2.

Corollary 1. If there exists Λ with $a_i \in [0, 1)$ for i = 1, ..., n arbitrarily close to 1 such that LMI (32) is feasible, then in the absence of switching controls, that is, $v_u = v_e = 0$, under constant disturbances $\overline{d} = 0$, and $u_r = 0$ in (6), system (22) is exponentially asymptotically stable with a decay rate α for all $\xi \in [0, \overline{L}]$, $\mu \dot{\xi} \leq \beta < 1$.

Proof. Suppose $\Lambda = I_n$. Then there is no scaling for nonlinearities-disturbance estimation in (9) and ϵ in (15) does not depend on X and \dot{e}_2 in (22) does not depend on H. The terms on the right-hand side of inequality (33) become $\frac{1}{2\alpha} \left(b_2 \| \int_{t-L_t}^t \dot{x}_{r_2}(s) ds \|_{\infty}^2 + b_3 \overline{d}^2 + b_4 \| v_u - v_u(t-L_t) \|_{\infty}^2 + b_6 \| v_e - v_e(t-L_t) \|_{\infty}^2 + b_7 \| \int_{t-L_t}^t \ddot{x}_{r_2}(s) ds \|_{\infty}^2 \right)$. Setting $v_u = v_e = 0$ and $u_r = 0$ and $\overline{d} = 0$ for constant disturbances, the right-hand side of inequality (33) becomes zero as $t \to \infty$.

Remark 4. In Reference 33, an UDE was considered for disturbance cancellation for systems with known input delays. A linear controller without switching parts v_u and v_e was considered. The uncertainties-disturbance estimation was constructed by delaying the control inputs and measurement signals and then the acceleration signals were estimated using a strictly proper low-pass filter. The disturbance estimation error was shown to be bounded by the system states and the disturbance estimation error itself. A controller was then designed to drive the system without external inputs, that is, $d = u_r = 0$, asymptotically to the origin. Theorem 1 in Reference 33 is an special case of Theorem 1 in this article, which is given by Corollary 1 with $a_i \in [0, 1)$ in Λ arbitrarily close to 1.

Remark 5. Both system (22) and (15) can be regarded as delay differential-algebraic equations, which have both delay and algebraic constraints. These types of systems often appear in various domains, including aircraft stabilization, chemical engineering systems, lossless transmission lines. In Reference 49, transforming such systems into a descriptor form has been considered and an LMI based stability criterion has been derived. The use of delay for stabilization, in our case for disturbance estimation, extends to other applications. For instance, a known delay is deliberately introduced for a SMC static control design.⁵⁰ Transformation of algebraic equations with time delay into neutral type system equations, for stability control using artificial delay has been considered in References 3 and 4.

It is shown in Reference 46 that a very small μ can be chosen to minimize the effect of the sampled-data measurement on observer error, effectively resulting in a singularly perturbed system with respect to the error dynamics. A high-gain observer design was proposed in Reference 51 based on *separation principle*, that is, designs of the controller and the observer are performed separately and then an output feedback controller is obtained by replacing the states by their estimates provided by the high-gain observer. It is well known that in the observer-based controller design, it is not possible to choose a very small μ for the system considered as the observer dynamics depends on the inverse of $\frac{1}{\mu}e_2$.

5 | **FINITE-TIME CONVERGENCE**

Since the closed loop system (22) is ultimately bounded by (33), and by the definition of ϵ in (15), definitions of $L\bar{v}$, \bar{e}_2 in (A10) and (A13), respectively, there exists a t_0 such that $\|\epsilon\| \le \overline{\delta}_{\epsilon_1} \|v_u - v_u(t - L_t)\| + \overline{\delta}_{\epsilon_2} \|v_e - v_e(t - L_t)\| + \overline{d}_e(t, \overline{d})$, where $\overline{\delta}_{\epsilon_1}$, $\overline{\delta}_{\epsilon_2}$, \overline{d}_e are some positive constants for all $t > t_0$. Also we have $\|H\| \le \overline{d}_H$, where \overline{d}_H is a positive constant for all $t > t_0$.

Corollary 2. Given positive constants $\overline{\delta}_{\epsilon_1}$, $\overline{\delta}_{\epsilon_2}$, $\overline{d}_{\epsilon} > 0$, and $\overline{d}_H > 0$, for any positive numbers $\gamma_1 > 0$, $\gamma_2 > 0$, then the following switching gains

$$\rho_e = \frac{\|\overline{M}^{-1}\|((\overline{\delta}_{\epsilon_1} + \overline{\delta}_{\epsilon_2})\rho_e(t - L_t) + \overline{\delta}_{\epsilon_1}(\Delta_u + \Delta_u(t - L_t)) + \overline{d}_e(t, \overline{d})) + \Delta_e}{1 - \|\overline{M}^{-1}\|(\overline{\delta}_{\epsilon_1} + \overline{\delta}_{\epsilon_2})}, \quad \rho = \rho_e + \Delta_u, \tag{34}$$

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where $\Delta_{u} = \left| \left| \overline{K}_{2} \hat{x}_{e_{1}} - \frac{1}{\delta_{2} \mu} (\overline{K}_{1} + \overline{K}_{2}) P_{22}^{-1} \hat{s} + \left(\frac{\delta_{1}}{\delta_{2}} B_{f_{11}} + B_{f_{21}} \right) e_{1} + \left(\frac{\delta_{1}}{\delta_{2}} + \frac{1}{\mu} \right) e_{2} \right| \right| + \gamma_{2} \quad and \quad \Delta_{e} = \left| \left| \left(\frac{\delta_{1}}{\delta_{2}} (\frac{1}{\mu} I_{n} - B_{f_{11}} - A_{m_{22}}) + (A_{m_{21}} - B_{f_{21}}) e_{1} + \frac{1}{\delta_{2} \mu} (A_{m_{22}} - \frac{1}{\mu} I_{n}) P_{22}^{-1} s_{e} + (A_{m_{22}} - \frac{1}{\mu} I_{n}) \Lambda e_{2} (t - L_{t}) \right| \right| + \left| (I - \Lambda) \overline{M}^{-1} \right| \left| \overline{d}_{H} + \gamma_{1}, \quad will \quad ensure \quad ideal \quad sliding motions are attained on (18) and (19) in finite time.$

Proof. Let $V_e = \frac{1}{2} s_e^T (\delta_2 \mu P_{22})^{-1} s_e$, then $\dot{V}_e = s_e^T (\delta_2 \mu P_{22})^{-1} \dot{s}_e$. Substituting \dot{s}_e in (25) and then ρ_e in (34) gives

$$\dot{V}_{e} \leq \|s_{e}\| \left(\left\| \left(\frac{\delta_{1}}{\delta_{2}} \left(\frac{1}{\mu} I_{n} - B_{f_{11}} - A_{m_{22}} \right) + (A_{m_{21}} - B_{f_{21}}) \right) e_{1} + \frac{1}{\delta_{2}\mu} \left(A_{m_{22}} - \frac{1}{\mu} I_{n} \right) P_{22}^{-1} s_{e} + \left(A_{m_{22}} - \frac{1}{\mu} I_{n} \right) \Lambda e_{2}(t - L_{t}) \right\| + \|\overline{M}^{-1}\| \|\varepsilon\| + \|(I - \Lambda)\overline{M}^{-1}\| \|H(t - L_{t})\| \right) - \rho_{e} \|s_{e}\|.$$

$$(35)$$

Rearranging (34) yields

$$\rho_{e} = \|\overline{M}^{-1}\|((\overline{\delta}_{\epsilon_{1}} + \overline{\delta}_{\epsilon_{2}})(\rho_{e} + \rho_{e}(t - L_{t})) + \overline{\delta}_{\epsilon_{1}}(\Delta_{u} + \Delta_{u}(t - L_{t})) + \overline{d}_{\epsilon}(t, \overline{d})) + \Delta_{e}$$

$$= \|\overline{M}^{-1}\|(\overline{\delta}_{\epsilon_{1}}(\rho + \rho(t - L_{t})) + \overline{\delta}_{\epsilon_{2}}(\rho_{e} + \rho_{e}(t - L_{t}) + \overline{d}_{\epsilon}(t, \overline{d}))) + \Delta_{e}.$$
(36)

Since $||H|| \le \overline{d}_H$, we also have $||H(t - L_t)|| \le \overline{d}_H$ for all $t > t_0$. Substituting (36) into (35), we have $\dot{V}_e < -\gamma_1 ||s_e||$. Next, let $V_s = \frac{1}{2} \hat{s}^T (\delta_2 \mu P_{22})^{-1} \hat{s}$, then substituting \dot{s} in (24)

$$\dot{V}_{s} \leq \|\hat{s}\| \left\| \overline{K}_{2} \hat{x}_{e_{1}} - \frac{1}{\delta_{2} \mu} (\overline{K}_{1} + \overline{K}_{2}) P_{22}^{-1} \hat{s} + \left(\frac{\delta_{1}}{\delta_{2}} B_{f_{11}} + B_{f_{21}} \right) e_{1} + \left(\frac{\delta_{1}}{\delta_{2}} + \frac{1}{\mu} \right) e_{2} \right\| + \hat{s}^{T} (v_{e} - v_{u}).$$

$$(37)$$

Since we have shown $\hat{s}(v_e - v_u) \le (\rho_e - \rho) \|\hat{s}\|$ in (A16), substituting ρ in (34) yields $\dot{V}_s < -\gamma_2 \|\hat{s}\|$. Thus sliding motions will be attained in finite time.

Remark 6. The switching gain ρ_e in (34) depends on $\overline{d_e}(t, \overline{d})$, where \overline{d} defines the bound on $||d - d(t - L_t)||$, this allows a smaller ρ_e to be chosen. It shows the advantage of using disturbance estimation (in our case, TDC) based SMC as in conventional SMC without using disturbance estimation technique, the switching gain needs to be large enough to attenuate the effect of the disturbance ||d||. Smaller switching gain is beneficial in reducing chattering when there is a delay in the control action.²⁸ Compared to TDC, in DO and UDE based control the filter gain design in disturbance estimation requires a priori knowledge of the upper bound of the disturbances ||d||, which is hard to estimate and its bound can be much larger for unknown disturbances.^{15,16,30,33} The clear advantage of TDC based disturbance estimation over DO and UDE is that the disturbance estimation error only depends on $||d - d(t - L_t)||$, whose upper bound is much smaller than ||d||.

5.1 | Design procedure

The following procedure provides guideline for selecting tuning parameters for LMIs (27) and (32) which minimize the ultimate bound (33).

- 1. Select the maximum delay size L^* . In digital control, the smallest delay achievable is a sample period. Select the set of parameters $\overline{\alpha}$, δ_1 , δ_2 and the diagonal matrix Λ such that LMI (27) is feasible. Larger values of Λ is preferred.
- 2. Determine the values of c_0 and c_1 such that the bound (4) holds. Using larger values for larger x_{r_2} , \dot{x}_{r_2} . They can be chosen zero for a constant speed tracking.
- 3. Select μ such that we can increase Λ . Ideally a smaller μ is preferable, but μ needs to be large enough as the feasibility of LMI (32) depends on a large enough μ .
- 4. Then choose a larger \overline{M} as close to M_d as possible.
- 5. Reduce α for larger values of Λ . Then reduce values for β , λ , b, b_1 , b_2 , b_3 , b_4 , b_5 , b_6 , b_7 such that LMI (32) is still feasible. Tuning parameters $\hat{\delta}_2$, $\hat{\delta}_3$, $\hat{\delta}_4$, and $\hat{\delta}_5$ can be chosen as small values to start with.

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6 | SIMULATION RESULTS

We consider the model of a 6-DOF model AUV which is assumed to be intrinsically stable in roll and pitch. Then the resulting 4-DOF AUV hydrodynamic model can be written as⁵²

$$m_{x}\dot{u} = m_{y}vr - k_{x}u - k_{x|x|}u|u| + m_{xy}\dot{v} + m_{xr}\dot{r} + d_{x} + F_{x},$$

$$m_{y}\dot{v} = -m_{x}ur - k_{y}v - k_{y|y|}v|v| + m_{yx}\dot{u} + m_{yr}\dot{r} + d_{y} + F_{y},$$

$$m_{z}\dot{w} = -k_{z}w - k_{z|z|}w|w| + m_{zx}\dot{u} + m_{zy}\dot{v} + d_{z} + F_{z} + W,$$

$$I_{r}\dot{r} = -(m_{y} - m_{x})uv - k_{r}r - k_{r|r|}r|r| + m_{rx}\dot{u} + m_{ry}\dot{v} + d_{r} + T_{r},$$

(38)

where u, v, win [m/s] are the linear velocities in the surge, sway, and heave, respectively, r [rad/s] is the angular velocity in the yaw. The surge and sway motions are usually coupled with the yaw motion. Mass values m_x , m_y , m_z , m_{xy} , m_{xr} , m_{yx} , m_{yr} , m_{zx} , m_{zy} , m_{rx} , m_{ry} , in [kg] stand for the masses that include both the rigid body mass and the added mass due to the surrounding fluid in the surge, sway, and heave; I_r [kg m²] is the moment of inertia in the yaw (including added mass and inertia); $k_x/k_{x|x|}$, $k_y/k_{y|y|}$, $k_z/k_{z|z|}$, and $k_r/k_{r|r|}$ are the linear/quadratic damping coefficients in the surge, sway, heave, and yaw, respectively. W [N] is the resultant weight accounting for the buoyancy force in heave, and finally F_x , F_y , F_z in [N] and T_r [N m] are the control inputs.

Considering underwater vehicle control in proximity to sub-sea structures, a vehicle is expected to respond quickly to locally generated flow disturbances while maintaining a stable position relative to a static or moving structure. The oscillation of the structure due to water flow generates local eddies and turbulence flow around the structure and the vehicle in close proximity to the structure. This makes the stable positioning of the vehicle relative to the moving structure rather challenging. In order to improve the robust performance under unavoidable and unknown disturbance, a SMC-based control is considered for its intrinsic robust property. The actuators of the AUV are thrusters whose rotational switching frequency (able to switch rotational direction every haft second) is relatively much faster compared to the reacting motion of the vehicle in the water. For sake of simplicity and space, we only show simulation results for surge, sway, and yaw motions and not for heave motion. The parameters of the physical model of the AUV that we consider are provided in Table 1.⁵³

To demonstrate the effectiveness of the method, we consider two cases. In case 1, we consider the vehicle to follow a constant speed reference. In case 2, we consider the vehicle to follow a variable speed reference.

For constant speed tracking, we can choose $c_0 = c_1 = 0$ in (4) as $x_2 - x_2(t - L_t) = 0$, $\dot{x}_2 - \dot{x}_2(t - L_t) = 0$. We have chosen a constant delay of 0.1 s. This represents the simplest case to investigate the best performance that we can obtain from the proposed control strategy. In LMI (27), choosing $\bar{\alpha} = 0.005$, $L^* = 0.1$ s, $\delta_1 = 0.1$,

| TABLE 1 Parameter values used for the Al |
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| Cyclops parameter | Value |
|--|---------------|
| Rigid body mass of Cyclops, m (kg) | 219.8 |
| Mass of Cyclops in surge, m_x (kg) | 391.5 |
| Linear drag coefficient in surge, $k_{\rm x}$ | 120 |
| Quadratic drag coefficient in surge, $k_{\mathrm{x} \mathrm{x} }$ | 229.4 |
| Mass of Cyclops in sway, m_y (kg) | 639.6 |
| Linear drag coefficient in sway, $k_{\rm y}$ | 131.8 |
| Quadratic drag coefficient in sway, $k_{y y }$ | 328.3 |
| Inertia of Cyclops in yaw, I_r (kg m ²) | 130 |
| Linear drag coefficient in yaw, $k_{\rm r}$ | 80 |
| Quadratic drag coefficient in yaw, $k_{\rm r r }$ | 280 |
| Other mass values, $m_{\rm xy}, m_{\rm xr}, m_{\rm yx}, m_{\rm yr}$ (kg) | 4, 7, 7, 29 |
| $m_{\rm zx}, m_{\rm zy}, m_{\rm rx}, m_{\rm ry}$ (kg) | 15, 25, 7, 20 |

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 $\delta_2 = 0.5$, and $\Lambda = \text{diag}\{0.84, 0.85, 0.85\}$, we obtain the observer matrix $B_{f_{11}} = \text{diag}\{0.68, 0.72, 0.72\}$. In LMI (32), choosing $\overline{M} = \text{diag}\{390, 639, 129\}$, $\beta = 0$, $\mu = 5$, $\hat{\delta}_2 = 0.01$, $\hat{\delta}_3 = 0.003$, $\hat{\delta}_4 = 0.1$, $\hat{\delta}_5 = 0.05$, $\alpha = 0.01$, $\lambda = 0.001$, $b = 0.7 \times 10^{-5}$, $b_1 = 0.0001$, $b_2 = 0.2 \times 10^{-8}$, $b_3 = 0.002$, $b_4 = 1$, $b_5 = 0.3 \times 10^{-6}$, $b_6 = 1$, $b_7 = 0.8$, we obtain the reference model, controller and observer matrices as $A_{m_{21}} = \text{diag}\{-0.018, -0.0027, -0.0024\}$, $A_{m_{22}} = \text{diag}\{-0.6, -0.445, -0.4\}$, $K_1 = \text{diag}\{0.448, 0.58, 0.77\}$, $K_2 = \text{diag}\{0.686, 1.49, 2.36\}$, $B_{f_{21}} = \{0.082, 0.06, -0.0064\}$. The Lyapunov matrix is $P_{22} = \text{diag}\{3.28, 2.77, 0.72\}$ and $\underline{\lambda}(P_{\mu}) = 0.094$. In the switching gain design in (34), we choose $\overline{\delta}_{\epsilon_1} = \overline{\delta}_{\epsilon_2} = 5$, $\overline{d}_{\epsilon} = 5$, $\overline{d}_{H} = 30$, $\gamma_1 = \gamma_2 = 0.001$.

For the varying speed tracking, we choose c_0 , c_1 such that $\|\Phi\| \le c_1 \|\dot{x}_2\| + c_0 \|x_2\| + \|d\|$, as implied by Assumption 1. By definition, we have $\Phi = M_b \dot{x}_2 + D(x_2)x_2 + d$, where $M_b = \begin{pmatrix} 0 & m_{xy} & m_{xr} \\ m_{yx} & 0 & m_{yr} \\ m_{rx} & m_{ry} & 0 \end{pmatrix}$ and $D(x_2)x_2 = D_k(x_2)x_2 + D_m(x_2)$ in system (38), with $D_k(x_2) = \text{diag}\{-k_{x|x|}|x_{2_1}|, -k_{y|y|}|x_{2_2}|, -k_{r|r|}|x_{2_3}\}$, $D_m(x_2) = \text{col}\{m_yx_{2_2}x_{2_3}, -m_xx_{2_1}x_{2_3}, (m_x - m_y)x_{2_1}x_{2_2}\}$ and $x_{2_1} = u$, $x_{2_2} = v$, $x_{2_3} = r$. We assume that the velocities $x_{2_1}, x_{2_2}, x_{2_3}$ are small such that the following bound $\|D(x_2)x_2\| \le c_0\|x_2\|$ holds, where $c_0 = \|\text{diag}\{-k_{x|x|}, -k_{y|y|}, -k_{r|r|}\}\| = 328$. We choose $c_1 = \|M_b\| = 30$. In LMI (27), we select $\Lambda = \text{diag}\{0.73, 0.77, 0.42\}$ and keep the other parameters the same as those in the case of constant speed tracking. We obtain $B_{f_{11}} = \text{diag}\{0.59, 0.67, 0.36\}$. In LMI (32), we choose $\beta = 0.3$, $\mu = 3$, $\hat{\delta}_2 = 0.001$, $\hat{\delta}_3 = 0.0003$, $\hat{\delta}_4 = 0.01$, $\hat{\delta}_5 = 0.0005$, $\lambda = 0.00002$, b = 0.7, $b_1 = 0.1$, $b_2 = 5$, $b_3 = 0.00003$, $b_4 = 10$, $b_5 = 0.004$, $b_6 = 0.9$, $b_7 = 1.5$ and keep \overline{M} , α the same as chosen in the case of constant speed. We obtain the reference model, controller and observer matrices as $A_{m_{21}} = \text{diag}\{-0.23, -0.01, -0.57\}$, $A_{m_{22}} = \text{diag}\{-2.1, -0.65, -3.75\}$, $K_1 = \text{diag}\{0.45, 0.85, 0.03\}$, $K_2 = \text{diag}\{-1.57, 1.06, -3.06\}$, $B_{f_{21}} = \text{diag}\{0.2, 0.08, 0.12\}$. The Lyapunov matrix is $P_{22} = \text{diag}\{10.6, 10.9, 8.2\}$ and $\underline{\lambda}(P_{\mu}) = 1.33$. For the switching gain design, we keep $\overline{\delta}_{e_1}, \overline{\delta}_{e_2}, \overline{d}_{e_1}, \overline{d}_{e_1}, \overline{d}_{e_2}, \overline{d}_{e_1}, \overline{d}_{e_1}, \overline{d}_{e_2}, \overline{d}_{e_1}, \overline{d}_{e_2}, \overline{d}_{e_2}, \overline{d}_{e_1}, \overline{d}_{e_1}, \gamma_1, \gamma_2$ the same as in the case of constant speed design.

In the simulation, we consider constant speed tracking in the first 35 s and variable speed tracking afterwards, as shown in Figure 1. The figure shows that the tracking performance was maintained in the presence of disturbances in surge, sway, and yaw. The delay L_t is constant during the constant speed tracking and becomes variable with a rate less than 0.3 afterwards (Figure 2). The TDC input \hat{H} is plotted in comparison to the actual nonlinearity and disturbance signals H in Figure 3. It can be seen that the estimation \hat{H} is in an approximate neighborhood of the actual H in surge and sway. In yaw, the estimation \hat{H} is quite close to the actual H during constant speed tracking but its estimation accuracy deteriorates during variable speed tracking as the scaling factor a_3 is reduced from 0.85 to 0.42.



FIGURE 1 Reference position tracking in the surge, sway, and yaw under disturbances



FIGURE 2 Delay L_t in TDC is constant and then become variable

Figure 4 shows that during constant speed tracking, as $c_0 = c_1 = 0$, we have $||\Phi - \Phi(t - L_t)|| \le \overline{d}$ in (4). Note that in the first 5 s, the condition $||\Phi - \Phi(t - L_t)|| \le \overline{d}$ does not hold since the vehicle's speed increases in the interval $t \in [0, 5]$ s, when its speed stays unchanged thereafter. In the meantime, $\overline{d} = 0$ in the first 5 s as seen in the zoom-in plots in Figure 1. During variable speed tracking, the zoom-in plots in Figure 4 show $||\Phi - \Phi(t - L_t)||$ is always bounded by the right-hand side of (4). Figures 5–7 show the sliding surface, control inputs, and the switching gain in TDC (blue line in the figures).

Below we demonstrate the effectiveness of TDC (with \hat{H}) in reducing the potential chattering caused by larger switching energy in SMC. We replace $\frac{\hat{s}}{\|\hat{s}\|}$, $\frac{s_e}{\|s_e\|}$ in v_u and v_e with $\frac{\hat{s}}{\|\hat{s}\|+\gamma_v}$ and $\frac{s_e}{\|s_e\|+\gamma_v}$, respectively, where $\gamma_v > 0$ is a small constant. This allows smoothing the discontinuity in the nonlinear switching control in SMC to obtain an arbitrarily close but continuous approximation of the discontinuous functions. This approximation is reasonable as the thrusters in an AUV change rotational speed and directions continuously.

In Figure 5, the sliding surface using TDC (with \hat{H}) and without using TDC (without \hat{H}) in surge, sway, and yaw are shown to be in the similar magnitudes. The control inputs are shown in Figure 6. It is seen in both cases that the same level of control inputs are present to keep the sliding surface \hat{s} at the same level. However, the result of not using TDC requires larger switching gain ρ , Figure 7. The value of ρ in non-TDC case is about 8 times larger than that in the TDC case. The smaller switching gain due to TDC (using \hat{H} to compensate for the effect of the actual nonlinear-disturbance H rather than using larger switching gain) reduces the risk of potential chattering due to the switching control action. In both of the controllers, TDC or non-TDC, the switching gain ρ is increased from constant speed tracking to variable speed tracking due to the decreased value of Λ in TDC and larger value of H in variable speed tracking. The effect of delay size on the scaling factor Λ can be seen in Figure 8. The data in the figure is obtained by setting $c_0 = c_1 = \beta = 0$ in the LMI (27) and (32). It is shown that the scaling factor Λ needs to be reduced for increasing delay size L^* , weakening the efficiency of TDC in compensating the nonlinearity-disturbances.

In this section, we have demonstrated that the effectiveness of using TDC in reducing the potential chattering in SMC, by compensating the nonlinearity-disturbance with its estimate from the past control and measurement information. In constant speed tracking, the estimation accuracy of nonlinearity-disturbance is higher compared to variable speed tracking. It is seen that in TDC, the level of compensation for *H* by \hat{H} is affected by the design parameter Λ . We have



FIGURE 3 TDC estimation and its comparison to the actual nonlinearity and disturbances in the surge, sway, and yaw



FIGURE 4 Bounding on the nonlinearity-disturbances $\|\Phi - \Phi(t - L_t)\|$ in Equation (4)



FIGURE 5 Sliding surfaces in the surge, sway, and yaw in TDC (with \hat{H}) with comparison to non-TDC (without \hat{H})

Time (s)

50

60

40

70

20

0 0

10



Control inputs in the surge, sway, and yaw with comparison to non-TDC FIGURE 6



FIGURE 7 Switching gain ρ used in TDC and non-TDC



FIGURE 8 Delay size L^* against scaling factor $a_{1,2,3}$ in Λ

shown that the maximum value we can achieve for $L^* = 0.1s$ is $\Lambda = \text{diag}\{0.84, 0.85, 0.85\}$, which is obtained at constant speed tracking with a constant delay. Note that Λ needs to be chosen by considering the size of the delay, controller gains, observer gains and the reference model parameters to maximize the TDC performance and to avoid potential instability.

7 | CONCLUSION

TDC is a simple but effective disturbance observation based control technique. However, it suffers from that it requires all system states to be available including the acceleration measurement which is not easily accessible in many physical systems. There is still a lack of analytical tools to find the limitations of this approach and to analyze the trade of between robust stability and performance for a designed TDC. In this article, a SM observer has been designed to circumvent the need for acceleration measurement that has been commonly assumed available in TDC. The resulting observer error system is shown to be a time delay system of neutral type. A tuning factor Λ is introduced for TDC-based nonlinearities-disturbance estimation, which governs the accuracy of the non-linearity-disturbance estimation and the robust performance of TDC. Delay-dependent linear matrix inequalities (LMIs) conditions are proposed for the design of TDC with the SM. The size of the delay, the controller gains, the observer gains, and the reference model parameters are determined from the LMIs which minimize the ultimate bound of the closed-loop system. It is shown with

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simulations that higher estimation accuracy can be achieved for a constant speed tracking than varying speed tracking. The advantage of using TDC is demonstrated with the simulation results as the substitution of the nonlinearity-disturbance estimation (TDC) in the control law compensate for the effect of the actual nonlinearities and disturbance and the resulting closed-loop system is constrained into a smaller neighborhood of the origin. As a consequence a smaller switching gain can be designed in SMC to induce a SM. The smaller switching gain reduces the potential undesirable chattering caused by the switching control action. It is shown that for larger delay size the scaling factor Λ needs to be reduced.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

This article describes entirely theoretical research. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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APPENDIX A

A.1 Proof of Equation (15)

Given the sliding surface (18) and sliding surface matrix (23), it follows

$$\hat{s} = B_1^T P_\mu \hat{x}_e = \delta_1 \mu P_{22} \hat{x}_{e_1} + \delta_2 \mu P_{22} \hat{x}_{e_2} = \delta_1 \mu P_{22} x_{e_1} - \delta_1 \mu P_{22} e_1 + \delta_2 \mu P_{22} x_{e_2} - \delta_2 \mu P_{22} e_2.$$
(A1)

From (12), we can write

$$\dot{x}_{e_2} = \left(A_{m_{21}}A_{m_{22}}\right)x_e - K\hat{x}_e - \rho \frac{\hat{s}}{\|\hat{s}\|} + \overline{M}^{-1}\epsilon + (I - \Lambda)\overline{M}^{-1}H(t - L_t) + \Lambda \dot{e}_2(t - L_t),\tag{A2}$$

then it follows

$$\dot{\hat{s}} = \delta_1 \mu P_{22} x_{e_2} - \delta_1 \mu P_{22} \dot{e}_1 - \delta_2 \mu P_{22} \dot{e}_2 + \delta_2 \mu P_{22} \bigg(\begin{pmatrix} A_{m_{21}} & A_{m_{22}} \end{pmatrix} x_e - K \hat{x}_e - \rho \frac{\hat{s}}{\|\hat{s}\|} + \overline{M}^{-1} \epsilon + (I - \Lambda) \overline{M}^{-1} H(t - L_t) + \Lambda \dot{e}_2(t - L_t) \bigg).$$
(A3)

Rearranging yields

$$-\rho \frac{\hat{s}}{\|\hat{s}\|} = \frac{1}{\delta_2 \mu} P_{22}^{-1} \dot{\hat{s}} - \frac{\delta_1}{\delta_2} x_{e_2} + \frac{\delta_1}{\delta_2} \dot{e}_1 + \dot{e}_2 - \left(A_{m_{21}} \quad A_{m_{22}}\right) x_e + K \dot{x}_e - \overline{M}^{-1} \epsilon - \Lambda \dot{e}_2 (t - L_t) - (I - \Lambda) \overline{M}^{-1} H (t - L_t).$$
(A4)

Next, substituting $\dot{\hat{s}} = \delta_1 \mu P_{22}(x_{e_2} - \dot{e}_1) + \delta_2 \mu P_{22}(\dot{x}_{e_2} - \dot{e}_2)$ into the above equation, we achieve

$$-\rho \frac{\hat{s}}{\|\hat{s}\|} = \dot{x}_{e_2} - \begin{pmatrix} A_{m_{21}} & A_{m_{22}} \end{pmatrix} x_e + K \hat{x}_e - \overline{M}^{-1} \epsilon - \Lambda \dot{e}_2 (t - L_t) - (I - \Lambda) \overline{M}^{-1} H (t - L_t).$$
(A5)

Given v in (8), we can write

$$\nu - \ddot{\eta} = \nu - \dot{x}_2 = \overline{M}^{-1} \epsilon + (I - \Lambda) \overline{M}^{-1} H(t - L_t) + \Lambda \dot{e}_2(t - L_t).$$
(A6)

Multiplying both sides of (A6) by $\overline{M}M_d$ and substituting $F = M_d \dot{\eta} + \Phi$, we have

$$\overline{M}M_d\nu + \overline{M}(\Phi - F) = M_d\varepsilon + M_d(I - \Lambda)H(t - L_t) + \overline{M}M_d\Lambda\dot{e}_2(t - L_t).$$
(A7)

From (3), (8), and (9), we can write

$$\overline{M}F = \overline{M}^2 v + \overline{M}\Lambda((M_d - \overline{M})\dot{\eta}(t - L_t) + \Phi(t - L_t) - \overline{M}\dot{e}_2(t - L_t)).$$
(A8)

Substituting (A8) into (A7) yields

$$\overline{M}M_{d}v + \overline{M}\Phi - \overline{M}^{2}v - \overline{M}\Lambda((M_{d} - \overline{M})\ddot{\eta}(t - L_{t}) + \Phi(t - L_{t}) - \overline{M}\dot{e}_{2}(t - L_{t}))$$

$$= M_{d}\epsilon + \overline{M}M_{d}\Lambda\dot{e}_{2}(t - L_{t}) + M_{d}(I - \Lambda)H(t - L_{t}).$$
(A9)

Then substituting $\ddot{\eta}$ from (A6) leads to (15).

A.2 Proof of Theorem 1

According to (8), (20), and using $x = x_r - \hat{x}_e - e$, it follows that

$$\bar{\psi} = \int_{t-L_t}^t \ddot{x}_{r_2}(s) ds - (A_{m_{21}} - K_1) \int_{t-L_t}^t \dot{x}_{e_1}(s) ds - A_{m_{21}} \int_{t-L_t}^t \dot{e}_1(s) ds - (A_{m_{22}} - K_2) \int_{t-L_t}^t \dot{x}_{e_2}(s) ds$$

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$$-A_{m_{22}} \int_{t-L_t}^t \dot{e}_2(s) \mathrm{d}s + v_u - v_u(t-L_t).$$
(A10)

Using (12), it can be shown that

$$\dot{x}_2 = v - \overline{M}^{-1} \epsilon - \overline{M}^{-1} (I - \Lambda) H(t - L_t) - \Lambda \dot{e}_2(t - L_t),$$
(A11)

and denoting $\overline{\ddot{x}}_2 = \dot{x}_2 - \dot{x}_2(t - L_t)$, we get

$$\bar{\ddot{x}}_2 = \bar{\dot{\nu}} - \overline{M}^{-1}(\epsilon - \epsilon(t - L_t)) - \overline{M}^{-1}(I - \Lambda)(H(t - L_t) - H(t - 2L_t)) - \Lambda \bar{\ddot{e}}_2(t - L_t).$$
(A12)

According to (22), we can write

$$\mu \overline{\dot{e}}_{2} = \mu \Lambda \overline{\dot{e}}_{2}(t - L_{t}) + \mu (A_{m_{21}} - B_{f_{21}}) \int_{t - L_{t}}^{t} \dot{e}_{1}(s) ds + (\mu A_{m_{22}} - I_{n}) \int_{t - L_{t}}^{t} \dot{e}_{2}(s) ds + \mu \overline{M}^{-1}(\epsilon - \epsilon(t - L_{t})) + \mu \overline{M}^{-1}(I - \Lambda)(H(t - L_{t}) - H(t - 2L_{t})) - \mu(v_{e} - v_{e}(t - L_{t})).$$
(A13)

Consider the Lyapunov–Krasovskii functionals $V = V_1 + V_2 + V_3 + V_4$ for the observer and the error dynamics in (22), as below:

$$V_{1} = \hat{x}_{e}^{T} P_{\mu} \hat{x}_{e} + \begin{pmatrix} e_{1}^{T} & e_{2}^{T} - e_{2}^{T} (t - L_{t}) \Lambda \end{pmatrix} P_{\mu} \begin{pmatrix} e_{1} \\ e_{2} - \Lambda e_{2} (t - L_{t}) \end{pmatrix},$$

$$V_{2} = \int_{t-L_{t}}^{t} e^{2\alpha(s-t)} e_{2}^{T}(s) T e_{2}(s) ds + \int_{t-L_{t}}^{t} e^{2\alpha(s-t)} e^{T}(s) S e(s) ds + \int_{t-L_{t}}^{t} e^{2\alpha(s-t)} \overline{\dot{e}}_{2}^{T}(s) R \overline{\dot{e}}_{2}(s) ds$$
$$+ \int_{t-L_{t}}^{t} e^{2\alpha(s-t)} \overline{\dot{v}}^{T}(s) R_{1} \overline{\dot{v}}(s) ds + \int_{t-L_{t}}^{t} e^{2\alpha(s-t)} \overline{\ddot{x}}_{2}^{T}(s) R_{2} \overline{\ddot{x}}_{2}(s) ds,$$

$$V_{3} = L^{*} \int_{-L^{*}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}_{e_{1}}^{T}(s) Q_{1} \dot{x}_{e_{1}}(s) ds + L^{*} \int_{-L^{*}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{e}_{1}^{T}(s) Q_{2} \dot{e}_{1}(s) ds + L^{*} \int_{-L^{*}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{x}_{e_{2}}^{T}(s) P_{22} \dot{x}_{e_{2}}(s) ds + L^{*} \int_{-L^{*}}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \dot{e}_{2}^{T}(s) P_{22} \dot{e}_{2}(s) ds,$$

$$V_4 = L^{*2} \int_{t-L_t}^t e^{2\alpha(s-t)} \dot{e}_2^T(s) P_{22} \dot{e}_2(s) \mathrm{d}s.$$
(A14)

Denoting $P_{k_1} = P_{22}\overline{K}_1$, $P_{k_2} = P_{22}\overline{K}_2$, $P_{m_1} = P_{22}A_{m_{21}}$, $P_{m_2} = P_{22}A_{m_{22}}$, and $P_{f_2} = P_{22}B_{f_{21}}$, and differentiating V_1 yields

$$\dot{V}_{1} + 2\alpha V_{1} = 2(\hat{x}_{e_{1}}^{T}P_{11} + \mu\delta_{1}\hat{x}_{e_{2}}^{T}P_{22})(\hat{x}_{e_{2}} + B_{f_{11}}e_{1} + e_{2}) + 2(\delta_{1}\hat{x}_{e_{1}}^{T} + \delta_{2}\hat{x}_{e_{2}}^{T})(-\mu P_{k_{1}}\hat{x}_{e_{1}} - \mu P_{k_{2}}\hat{x}_{e_{2}} + \mu P_{f_{2}}e_{1} + P_{22}e_{2} + \mu P_{22}(-\nu_{u} + \nu_{e})) + 2(e_{1}^{T}P_{11} + (e_{2}^{T} - e_{2}^{T}(t - L_{t})\Lambda)\mu\delta_{1}P_{22})(-B_{f_{11}}e_{1}) + 2(\delta_{1}e_{1}^{T} + \delta_{2}(e_{2}^{T} - e_{2}^{T}(t - L_{t})\Lambda)) \\ \cdot (\mu(P_{m_{1}} - P_{f_{2}})e_{1} + (\mu P_{m_{2}} - P_{22})e_{2} + \mu P_{22}\overline{M}^{-1}\epsilon + \mu P_{22}(I - \Lambda)\overline{M}^{-1}H(t - L_{t}) - \mu P_{22}\nu_{e}) + 2\alpha V_{1}.$$
(A15)

Note that since $\hat{s}_{\frac{\hat{s}_e}{\|\hat{s}_e\|}} \le \|\hat{s}\|$, it follows in (A15) that

$$(\delta_1 \hat{x}_{e_1} + \delta_2 \hat{x}_{e_2}) \mu P_{22}(-v_u + v_e) = \hat{s} \left(-\rho \frac{\hat{s}}{\|\hat{s}\|} + \rho_e \frac{s_e}{\|s_e\|} \right) = -\rho \|\hat{s}\| + \rho_e \hat{s} \frac{s_e}{\|s_e\|} \le -(\rho - \rho_e) \|\hat{s}\| < 0$$
(A16)

for $\rho > \rho_e > 0$, and

$$(\delta_1 e_1 + \delta_2 (e_2 - \Lambda e_2 (t - L_t)))(-\mu P_{22} v_e) = -s_e v_e = -\rho_e ||s_e|| < 0.$$
(A17)

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Then differentiating V_2 yields

$$\dot{V}_{2} + 2\alpha V_{2} = e_{2}^{T} T e_{2} - e^{-2\alpha L_{t}} e_{2}^{T} (t - L_{t}) T e_{2} (t - L_{t}) + \epsilon^{T} S \epsilon - e^{-2\alpha L_{t}} \epsilon^{T} (t - L_{t}) S \epsilon (t - L_{t}) + \overline{\ddot{e}}_{2}^{T} R \overline{\ddot{e}}_{2} - e^{-2\alpha L_{t}} \overline{\ddot{e}}_{2}^{T} (t - L_{t}) R \overline{\ddot{e}}_{2} (t - L_{t}) + \overline{\dot{v}}^{T} R_{1} \overline{\dot{v}} - e^{-2\alpha L_{t}} \overline{\ddot{v}}^{T} (t - L_{t}) R_{1} \overline{\dot{v}} (t - L_{t}) + \overline{\ddot{x}}_{2}^{T} R_{2} \overline{\ddot{x}}_{2} - e^{-2\alpha L_{t}} \overline{\ddot{x}}_{2}^{T} (t - L_{t}) R_{2} \overline{\ddot{x}}_{2} (t - L_{t}).$$
(A18)

In differentiation of $\dot{V}_3 + 2\alpha V_3$, we first consider the first term and denote $V_{3_1} = L^* \int_{-L^*}^0 \int_{t+\theta}^t e^{2\alpha(s-t)} \dot{\dot{x}}_{e_1}^T(s) Q_1 \dot{\dot{x}}_{e_1}(s) ds$, then it follows that

$$\dot{V}_{3_{1}} + 2\alpha V_{3_{1}} = L^{*2} \dot{x}_{e_{1}}^{T} Q_{1} \dot{x}_{e_{1}} - L^{*} e^{-2\alpha L^{*}} \left(\underbrace{\int_{t-L^{*}}^{t-L_{t}} \dot{x}_{e_{1}}^{T}(s) Q_{1} \dot{x}_{e_{1}}(s) ds}_{\text{neglected}} + \int_{t-L_{t}}^{t} \dot{x}_{e_{1}}^{T}(s) Q_{1} \dot{x}_{e_{1}}(s) ds \right).$$
(A19)

By Jensen's inequality⁶

$$-L^* e^{-2\alpha L^*} \int_{t-L_t}^t \dot{\hat{x}}_{e_1}^T(s) Q_1 \dot{\hat{x}}_{e_1}(s) \mathrm{d}s \le -e^{-2\alpha L^*} \int_{t-L_t}^t \dot{\hat{x}}_{e_1}^T(s) \mathrm{d}s Q_1 \int_{t-L_t}^t \dot{\hat{x}}_{e_1}(s) \mathrm{d}s.$$
(A20)

Differentiating the other terms in V_3 , Jensen's inequality can be applied to the similar terms in the same way. Using Schur complement for the following terms yields

$$L^{*^{2}} \dot{\hat{x}}_{e_{1}}^{T} Q_{1} \dot{\hat{x}}_{e_{1}} = \begin{pmatrix} \hat{x}_{e_{2}}^{T} & e_{1}^{T} & e_{2}^{T} \end{pmatrix} \begin{pmatrix} L^{*} Q_{1} \\ L^{*} B_{f_{11}}^{T} Q_{1} \\ L^{*} Q_{1} \end{pmatrix} Q_{1}^{-1} \begin{pmatrix} L^{*} Q_{1} & L^{*} Q_{1} B_{f_{11}} & L^{*} Q_{1} \end{pmatrix} \begin{pmatrix} \hat{x}_{e_{2}} \\ e_{1} \\ e_{2} \end{pmatrix},$$

$$L^{*^{2}} \dot{e}_{1}^{T} Q_{2} \dot{e}_{1} = e_{1}^{T} \begin{pmatrix} -L^{*} B_{f_{11}}^{T} Q_{2} \end{pmatrix} Q_{2}^{-1} \begin{pmatrix} -L^{*} Q_{2} B_{f_{11}} \end{pmatrix} e_{1}.$$
(A21)

Since we have shown that v_u and v_e terms in (A16) and (A17) are control signals rather than disturbance, we can analyze the linear part of \dot{x}_{e_2} and \dot{e}_2 when writing their Schur complement as follows:

$$L^{*2} \dot{\hat{x}}_{e_{2}}^{T} P_{22} \dot{\hat{x}}_{e_{2}} = \mu^{2} \overline{L}^{2} \dot{\hat{x}}_{e_{2}}^{T} P_{22} \dot{\hat{x}}_{e_{2}} = \left(\hat{x}_{e_{1}}^{T} \quad \hat{x}_{e_{2}}^{T} \quad e_{1}^{T} \quad e_{2}^{T} \right) \begin{pmatrix} -\mu \overline{L} P_{k_{1}}^{T} \\ -\mu \overline{L} P_{k_{2}}^{T} \\ \mu \overline{L} P_{f_{2}}^{T} \\ \overline{L} P_{22} \end{pmatrix} P_{22}^{-1} \left(-\mu \overline{L} P_{k_{1}} \quad -\mu \overline{L} P_{k_{2}} \quad \mu \overline{L} P_{f_{2}} \quad \overline{L} P_{22} \right) \begin{pmatrix} \hat{x}_{e_{1}} \\ \hat{x}_{e_{2}} \\ e_{1} \\ e_{2} \end{pmatrix}, \quad (A22)$$

and

$$\mu^{2}\overline{L}^{2}\dot{e}_{2}^{T}P_{22}\dot{e}_{2} = \begin{pmatrix} e_{1} \\ e_{2} \\ \mu\overline{L}\dot{e}_{2}(t-L_{t}) \\ \epsilon \\ H(t-L_{t}) \end{pmatrix}^{T} \begin{pmatrix} \mu\overline{L}(P_{m_{1}}-P_{f_{2}})^{T} \\ \mu\overline{L}P_{m_{2}}^{T}-\overline{L}P_{22} \\ \Lambda P_{22} \\ \mu\overline{L}\overline{M}^{-1}P_{22} \\ \mu\overline{L}(I-\Lambda)\overline{M}^{-1}P_{22} \end{pmatrix} P_{2}^{-1} \begin{pmatrix} \mu\overline{L}(P_{m_{1}}-P_{f_{2}})^{T} \\ \mu\overline{L}P_{m_{2}}^{T}-\overline{L}P_{22} \\ \Lambda P_{22} \\ \mu\overline{L}\overline{M}^{-1}P_{22} \\ \mu\overline{L}(I-\Lambda)\overline{M}^{-1}P_{22} \end{pmatrix} P_{2}^{-1} \begin{pmatrix} \mu\overline{L}(P_{m_{1}}-P_{f_{2}})^{T} \\ \mu\overline{L}P_{m_{2}}^{T}-\overline{L}P_{22} \\ \mu\overline{L}\overline{M}^{-1}P_{22} \\ \mu\overline{L}(I-\Lambda)\overline{M}^{-1}P_{22} \end{pmatrix}^{T} \begin{pmatrix} e_{1} \\ e_{2} \\ \mu\overline{L}\dot{e}_{2}(t-L_{t}) \\ \epsilon \\ H(t-L_{t}) \end{pmatrix}.$$
(A23)

Next, differentiating V_4 yields

$$\dot{V}_4 + 2\alpha V_4 \le L^{*2} \dot{e}_2^T P_{22} \dot{e}_2 - (1 - \beta) e^{-2\alpha L^*} L^{*2} \dot{e}_2^T (t - L_t) P_{22} \dot{e}_2 (t - L_t).$$
(A24)

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In (4), given that $x_2 = x_{r_2} - x_{e_2}$ and $x_{e_2} = \hat{x}_{e_2} + e_2$, we can write

$$c_0 \|x_2 - x_2(t - L_t)\| \le c_0 \left(\left\| \int_{t - L_t}^t \dot{x}_{r_2}(s) ds \right\| + \left\| \int_{t - L_t}^t \dot{\hat{x}}_{e_2}(s) ds \right\| + \left\| \int_{t - L_t}^t \dot{e}_2(s) ds \right\| \right).$$
(A25)

Then using Young's inequality it follows

$$\begin{aligned} c_{0}^{2} \left\| \int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s) ds \right\|^{2} &\leq 2c_{0}^{2} \left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s) ds \right)^{T} \left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s) ds \right), \\ c_{0}^{2} \left\| \int_{t-L_{t}}^{t} \dot{x}_{e_{2}}(s) ds \right\|^{2} &\leq 2c_{0}^{2} \left(\int_{t-L_{t}}^{t} \dot{x}_{e_{2}}(s) ds \right)^{T} \left(\int_{t-L_{t}}^{t} \dot{x}_{e_{2}}(s) ds \right), \\ c_{0}^{2} \left\| \int_{t-L_{t}}^{t} \dot{e}_{2}(s) ds \right\|^{2} &\leq 2c_{0}^{2} \left(\int_{t-L_{t}}^{t} \dot{e}_{2}(s) ds \right)^{T} \left(\int_{t-L_{t}}^{t} \dot{e}_{2}(s) ds \right). \end{aligned}$$
(A26)

Therefore, we can write

$$\bar{\Phi}^{T}\bar{\Phi} \leq c_{1}^{2}\bar{x}_{2}^{T}\bar{x}_{2} + c_{0}^{2}\left(2\left(\int_{t-L_{t}}^{t}\dot{x}_{r_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t}\dot{x}_{r_{2}}(s)ds\right) + 2\left(\int_{t-L_{t}}^{t}\dot{x}_{e_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t}\dot{x}_{e_{2}}(s)ds\right) + 2\left(\int_{t-L_{t}}^{t}\dot{e}_{2}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t}\dot{e}_{2}(s)ds\right) + 2\left(\int_{t-L_{t}}^{t}\dot{e}_{2}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t}\dot{e}_{2}(s)ds\right) + (d-d(t-L_{t}))^{T}(d-d(t-L_{t})).$$
(A27)

Using *S*-procedure,⁵⁴ the term $\lambda(-\bar{\Phi}^T\bar{\Phi} + \text{right-hand side of (A27)})$, where λ is any positive number, is added to the following

$$W = \dot{V} + 2\alpha V - bH^{T}(t - L_{t})H(t - L_{t}) - b_{1}X^{T}X - b_{2}\left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s)ds\right)$$

$$- b_{3}(d - d(t - L_{t}))^{T}(d - d(t - L_{t})) - b_{4}(v_{u} - v_{u}(t - L_{t}))^{T}(v_{u} - v_{u}(t - L_{t}))$$

$$- b_{5}(H(t - L_{t}) - H(t - 2L_{t}))^{T}(H(t - L_{t}) - H(t - 2L_{t})) - b_{6}(v_{e} - v_{e}(t - L_{t}))^{T}(v_{e} - v_{e}(t - L_{t}))$$

$$- b_{7}\left(\int_{t-L_{t}}^{t} \ddot{x}_{r_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t} \ddot{x}_{r_{2}}(s)ds\right) + \lambda c_{1}^{2}\ddot{x}_{2}^{T}\ddot{x}_{2} + \lambda\left(2c_{0}^{2}\left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t} \dot{x}_{r_{2}}(s)ds\right)$$

$$+ 2c_{0}^{2}\left(\int_{t-L_{t}}^{t} \dot{x}_{e_{2}}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t} \dot{x}_{e_{2}}(s)ds\right) + 2c_{0}^{2}\left(\int_{t-L_{t}}^{t} \dot{e}_{2}(s)ds\right)^{T}\left(\int_{t-L_{t}}^{t} \dot{e}_{2}(s)ds\right)$$

$$+ (d - d(t - L_{t}))^{T}(d - d(t - L_{t})) - \bar{\Phi}^{T}\bar{\Phi}\right).$$

(A28)

Then adding the right-hand side of the following terms

$$0 = 2(\epsilon^{T}\hat{P}_{2}^{T} + \epsilon^{T}(t - L_{t})\hat{P}_{3}^{T})(-\epsilon + \text{right-hand side of Equation (15)}),$$

$$0 = 2(\bar{\nu}^{T}\hat{\delta}_{4}P_{22} + \bar{\nu}^{T}(t - L_{t})\hat{\delta}_{5}P_{22})(-L\bar{\nu} + \text{right-hand side of Equation (A10)}),$$

$$0 = 2(\bar{x}_{2}^{T}\hat{P}_{4}^{T} + \bar{x}_{2}^{T}(t - L_{t})\hat{P}_{5}^{T})(-L\bar{x}_{2} + \text{right-hand side of Equation (A12)}),$$

$$0 = 2(\bar{e}_{2}^{T}\hat{\delta}_{2}P_{22} + \bar{e}_{2}^{T}(t - L_{t})\hat{\delta}_{3}P_{22})(-\mu L\bar{e}_{2} + \text{right-hand side of Equation (A13)}),$$

(A29)

to (A28) and defining $\zeta = \operatorname{col}\{\hat{x}_{e_1}, \hat{x}_{e_2}, e_1, e_2, e_2(t-L_t), \mu \overline{L} \dot{e}_2(t-L_t), \epsilon, \epsilon(t-L_t), \bar{v}, \bar{v}(t-L_t), \bar{e}_2, \bar{e}_2(t-L_t), \bar{\Phi}, \bar{x}_2, \bar{x}_2(t-L_t), \int_{t-L_t}^t \dot{x}_{e_1}(s) ds, \int_{t-L_t}^t \dot{x}_{e_2}(s) ds, \int_{t-L_t}^t \dot{e}_1(s) ds, \int_{t-L_t}^t \dot{e}_2(s) ds, \int_{t-L_t}^t \dot{x}_{e_2}(s) ds, \int_{t-L_t}^t \dot{x}_{e_2}(s) ds, H(t-L_t), H(t-L_t) - H(t-2L_t), v_e - v_e(t-L_t), v_u - v_u(t-L_t), d - d(t-L_t), X\}$, then W < 0 is satisfied if $\zeta^T \Theta \zeta < 0$, which is nothing but $\Theta < 0$, where Θ is explicitly given in (32).