On the Varieties of Abstract Objects

DRAFT

James E. Davies

Word count: 6444 (7667 incl. abstract, footnotes & bibliography).

Keywords: abstract objects, metaphysics, philosophy of mathematics, philosophy of art.

Abstract

I reconcile the spatiotemporal location of repeatable artworks and impure sets with the non-location of natural numbers despite all three being varieties of abstract object. This is possible because while the identity conditions for all three can be given by abstraction principles, in the former two cases spatiotemporal location is a congruence for the relevant equivalence relation, whereas in the latter it is not. I then generalize this to other 'physical' properties like shape and mass.

Introduction

Here are three claims often taken to be individually true, but jointly inconsistent (e.g. Mag Uidhir [2012: 3-11]):

(1) Natural numbers are not located in space or time.

(2) Repeatable artworks are located in space and time: *Anna Karenina* came into existence after 1870, and is yet to reach the far ends of the cosmos.

(3) Both natural numbers and repeatable artworks are abstract objects.

These appear to be jointly inconsistent because (3) is taken to be inconsistent with (2). Supposedly, part of what it is to be an abstract object is being metaphysically incapable of spatiotemporally location (hereon simply *location*). My goal in this essay is to resolve this apparent inconsistency: the locatability of repeatable artworks does not impugn their abstractness.

Section 1 observes that both natural numbers and sets can have their identity conditions captured by *abstraction principles* – roughly (for now) statements individuating entities of one kind in terms of equivalence relations on other entities. Section 2 argues that attention to the equivalence relations featuring in the abstraction principles for numbers and sets reveals the consistency of regarding the latter as capable of location but not the former. Section 3 extends this argument to repeatable artworks by proposing an abstraction principle individuating such artworks socio-historically in terms of acts of creation (cf. Levinson [1980] and Thomasson [1999]), and running the reasoning from section 2. Section 4 steps back and makes general observations regarding which properties of concreta can also be had by which abstracta. I conclude by speculating that we have a *hierarchy* of abstract objects, with sets and repeatable artworks at a lower level than natural numbers.

1: Characterizing Abstract Objects

One standard contemporary view of abstract objects takes being metaphysically incapable of being spatiotemporally located as (at least partly) constitutive of what it is to be abstract.[1] The main alternative regards abstracts as inherently acausal. Noting that non-spatiotemporality plausibly entails acausality,[2] but not vice-versa,[3] suggests that non-spatiotemporality is more fundamental.

[1] Cf. chapter 2 of Cowling [2017] for an overview.
[2] E.g. Heck [2012: 200]: "If abstract objects are not even spatial, they presumably cannot cause anything to happen."
[3] E.g. Cowling [2017: p.82]: "Suppose, for example, that there could be entities of the sort that Forrest (1981) calls *epiphenomenalons* – "particles that than no more useless can be conceived of" – which are neither created nor destroyed and stand in no causal relations to other entities (including other epiphenomenalons). […] intuitively, these causally inactive entities would be concrete…"

Nevertheless this essay mainly concerns whether all abstracta are non-spatiotemporal; causality receives only brief discussion in section 4.

Now, characterizing abstracta as inherently non-spatiotemporal is a *negative* characterization, in terms of what abstracta are *not*. But we might prefer a characterization of abstract objects in terms of what they *are* rather than what they aren't. We could then ascertain the (non-) spatiotemporality of abstracta in light of this positive characterization.

Here is a positive characterization of one kind of abstract object: natural numbers. *Neo-Fregeans* about arithmetic take the identity conditions for natural numbers as given by an abstraction principle sometimes called *Hume's Principle*:[4]

$$\forall F \forall G\ [\#F = \#G \leftrightarrow F \text{ 1-1 } G].$$

Here "#" expresses a function from Fregean concepts (functions to truth-values) to objects (the natural numbers) and "F 1-1 G" abbreviates the (second-order) formula stating that there is a bijection between the entities falling under F and those falling under G, i.e. the Fs and Gs are *equinumerous*.[5]

Neo-Fregeans about arithmetic take Hume's Principle to be epistemically and metasemantically important in a right philosophical account of arithmetic because (i) it entails, in suitably strong logics, the Dedekind-Peano axioms for arithmetic; and (ii) they take it to be something like an analytic or conceptual truth.[6] However *our* concern is not with whether Hume's Principle plays any epistemic or metasemantic roles. Rather, all we require is that it capture the identity-conditions for natural numbers – i.e. that it be *true*.

[4] So named because Frege references Hume when giving the principle in *Grundlagen* §63

[5] $\exists R \forall x[(Fx \rightarrow \exists! y(Gy \,\&\, Rxy)) \,\&\, (Gx \rightarrow \exists! y(Fy \,\&\, Ryx))]$.

[6] Cf. e.g. Hale & Wright [2001: 4-14].

Sets have also been given a neo-Fregean treatment which takes them to be, like numbers, abstracts of concepts. Frege's own abstraction principle for sets[7] was his famously inconsistent Basic Law V; hence the hunt is still on for a consistent principle preserving the neo-Fregean payoffs.[8] The main avenue of investigation so far has been restricting Frege's law to prevent forming paradoxical sets. Every such restriction is a version of the following:

(RV): $\forall F_{Good(F)} \forall G_{Good(G)}[\{x: Fx\} = \{x: Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)]$.

Here "*Good*(*F*)" abbreviates a (to be determined) restriction of the prenex universal quantifiers to concepts *not* generating paradoxical sets. There is as yet little consensus on what this restriction should be. But note that every version of RV features the equivalence relation of *co-extensionality*, $\forall x(Fx \leftrightarrow Gx)$, on its right-hand side. Indeed, Hale (2000) argues that a necessary condition on an abstraction principle being a candidate principle *for sets* is that it feature co-extensionality (or a 'close relative'), on pain of changing the subject [Hale 2000: 332]. Moreover, as with numbers, whether RV plays epistemic or metasemantic roles is unimportant; what matters is its truth.

If this is correct, then we have at least one *positive* thing we can say about both sets and natural numbers – their identity conditions can be given by statements with the logical form:[9]

(AP): $\ulcorner \forall \alpha_{X(\alpha)} \forall \beta_{X(\beta)}[\S\alpha = \S\beta \leftrightarrow \Sigma(\alpha, \beta)] \urcorner$.

Here α and β range over a collection of Fregean concepts called the *base sort*; the subscript X(α) expresses a restriction of the subscripted quantifier; § a function from the base sort to objects; and

[7] Cognoscenti will be aware that Frege himself dealt with *extensions* or *value-ranges* of concepts rather than sets, and there is a historical debate about the extent to which these notions coincide. However rather than embrangling in such debates I indulge in anachronism by speaking as if Frege dealt directly with sets.
[8] There is another project – that of finding weakenings of second-order logic against which V is consistent – but I won't discuss it here; but see, e.g., Heck [1996].
[9] This is Davidson's notion of 'logical form', where the logical form of a sentence specifies the entailment relations it stands in: "to give the logical form of a sentence is to give its logical location in the totality of sentences, to describe it in a way that explicitly determines what sentences it entails and what sentences it is entailed by." [Davidson 1967/2000: 64]

Σ an equivalence relation on the base sort.[11] Given these conventions a more accurate statement of Hume's Principle is:

(HP): $\forall F_{ND(F)} \forall G_{ND(G)}[\#F = \#G \leftrightarrow F \text{ 1-1 } G]$.

The restriction to *ND* (Numerically Determinate) concepts rules out both *non-sortal* concepts like *is red* (and *is self-identical*) and 'indefinitely extensible' concepts like *is an ordinal number*, neither of which can be sensibly assigned a determinate number.[12] Both restrictions have been defended by at least one prominent neo-Fregean (cf. [Wright 1997/2001: 313-6]).

The next section shows that regarding both numbers and sets as having identity conditions articulated by statements of the form AP facilitates a unified explanation of why sets could be located by being co-located with their members, but numbers cannot be located. Section 3 extends this reasoning to repeatable artworks.

2: Natural Numbers, Sets, and Spatiotemporal Location

Let an assignment of spatiotemporal locations to entities of a kind *K* be *trivial* if it assigns no locations to any *K*s, assigns the same location(s) to all (located) *K*s, or assigns location(s) to *K*s in an *arbitrary* way. The idea then is that abstract *K*s can be located if and only if there is a nontrivial assignment of locations to *K*s.

[11] There are also abstraction principles whose base sorts are objects, rather than concepts, but I won't discuss them here. The technique of encoding restrictions to appropriate concepts is from Cook & Ebert [2005]. The absence of any restriction is elliptical for a limit-case restriction satisfied by all concepts (e.g. self-co-extensiveness).

[12] *Non-sortal* concepts are (at first blush) those for which it makes no sense to ask how many objects fall under them. *Indefinitely extensible* concepts are (at first blush) those such that whenever we have circumscribed the supposed totality of objects falling under one such concept, we can thereby generate another object falling under that concept but not in the given totality. For a concept *F* of either kind there appears to be no determinate answer to the question "how many *F*s are there?". Note that other restrictions of HP have been proposed, such as to *finite* concepts, e.g. Heck [1997].

It seems the only nontrivial way of locating a set is to co-locate it with the mereological fusion of its members.[14] Suppose the Eiffel Tower occupies a certain spatiotemporal region l_1 in central Paris and Big Ben another region l_2 in downtown London. Then the fusion of the Eiffel Tower and Big Ben occupies the (disconnected) spatiotemporal region comprised of both (and only) l_1 and l_2, which I'll call "$l_1 \cup l_2$". Seeing as the fusion of the Eiffel Tower and Big Ben occupies $l_1 \cup l_2$, the impure set {the Eiffel Tower, Big Ben} occupies $l_1 \cup l_2$ also. Thus (as I'll say) the fusion of Big Ben and the Eiffel Tower and {the Eiffel Tower, Big Ben} are *co-located*.[15]

Given RV, we can take a set as co-located with the fusion of the objects falling under the set-generating concept. Let '*fus*(*F*)' denote the fusion of the objects falling under *F*. Then our *located sets thesis* is:

(LST): $\forall F_{Good(F)}[\{x\text{: } Fx\}$ occupies $l_i \leftrightarrow fus(F)$ occupies $l_i]$.

The LST is our official statement that some (neo-Fregean) sets are spatiotemporally located.

Turning to natural numbers, parity of reasoning counsels that the only nontrivial assignment of locations to numbers is to take the number $\#F$ as co-located with the fusion of the *F*s. Call this the *located numbers thesis*:

(LNT): $\forall F_{ND(F)}[\#F$ occupies $l_i \leftrightarrow fus(F)$ occupies $l_i]$.[16]

---

[14] Cf. Cook [2013: 224]: "[I]f an impure set has any location whatsoever, then it must be co-located with its members. Any other location is as arbitrary as any other, and hence there is no conceivable reason for the set to have one location rather than another. Presumably, however, if an object has a location, then there must be some reason why it has *that* location." Note that the particular way of locating sets I'm about to give assumes both unrestricted composition and substantivalism. Nevertheless any way of assigning locations to *groups* of objects should do the trick (including pluralities; cf. footnote 16).

[15] More accurately, sets occupy the same region occupied by the fusions of the non-sets in their *transitive closure*: the sets of their members, members of members, etc. A set whose transitive closure includes at least one non-set is an *impure* set. Thus {Ø} occupies no region whilst {{Ø}, the Eiffel Tower} occupies l_1. Moreover, sets whose transitive closures contain co-located objects are themselves co-located; l_1 is occupied by all three of {{Ø}, the Eiffel Tower}, {the Eiffel Tower}, {{the Eiffel Tower}}, and many others. One special case of this is that every set is co-located with its own singleton (if either is located at all).

[16] These claims can also be made using pluralities without appeal to either Fregean concepts or fusions. E.g. we replace HP with $\forall xx \forall yy(\#xx = \#yy \leftrightarrow xx \text{ 1-1 } yy)$, and LNT with $\forall xx(\#xx$ occupies $l_i \leftrightarrow xx$ occupies $l_i)$. I've not gone this way for two reasons. First, plural logic is less universally accepted than singularist logic, functions (to truth values), and fusions. Second, the neo-Fregean research I'm leaning on has been almost exclusively carried out along

Here is why the LNT fails as a nontrivial assignment of locations to numbers. (This argument closely follows that in Cook [2013: 224]).

Take the concepts *identical to Luke or Leia* (hereon '*L*') and *identical to Han or Chewie* (hereon '*H*').[17] We thus have an instance of HP:

$\#L = \#H \leftrightarrow L$ 1-1 H.

The right-hand side of this instance of HP is itself true (in the fiction): the *L*s and *H*s are equinumerous. So the left-hand side is true also. We now reason as follows:

(1) $\#L$ occupies $l_i \leftrightarrow fus(L)$ occupies l_i. (Instance of LNT.)

(2) $\#H$ occupies $l_i \leftrightarrow fus(H)$ occupies l_i. (Instance of LNT.)

(3) $\#L = \#H$. (Consequence of HP.)

(4) $\#L$ occupies $l_i \leftrightarrow \#H$ occupies l_i. (3, laws of =.)

Therefore,

(5) $fus(L)$ occupies $l_i \leftrightarrow fus(H)$ occupies l_i. (1, 2, 4.)

Thus the fusion of Luke and Leia is co-located with the fusion of Han and Chewie. This is obviously false (in the fiction). Therefore the LNT fails: if HP gives the identity conditions for natural numbers, then such numbers cannot be assigned locations by the LNT. Given that the LNT describes the *only* nontrivial assignment, there is no such assignment. Natural numbers cannot be located.

Note as we're going past that this argument fails if we weaken the LNT from a biconditional to the following unidirectional conditional:

(LNT*): $fus(F)$ occupies $l_i \rightarrow \#F$ occupies l_i.

singularist lines, and I do not have space here to re-formulate it in pluralist terms. Nevertheless I conjecture (without proof) that the arguments I'm giving would go through in a pluralist setting.

[17] These are Cook's examples.

LNT* says if the fusion of the *F*s occupies l_i, then the number of *F*s occupies l_i too, but does not license the inference from #*F* occupying l_i to *fus*(*F*) occupying l_i. What distinguishes it from the LNT is that we could have *fus*(*L*) occupying l_1, *fus*(*H*) occupying l_2, *and* #*L* (= #*H*) occupying *both* l_1 and l_2, while *fus*(*L*) does not occupy l_2 nor does *fus*(*H*) occupy l_1.

If we replace (1) and (2) in the above inference with instances of LNT*, then 5 no longer follows from 1, 2, and 4:

(1*) *fus*(*L*) occupies $l_i \rightarrow$ #*L* occupies l_i. (Instance of LNT*.)

(2*) *fus*(*H*) occupies $l_i \rightarrow$ #*H* occupies l_i. (Instance of LNT*.)

(3) #*L* = #*H*. (Consequence of HP.)

(4) #*L* occupies $l_i \leftrightarrow$ #*H* occupies l_i. (3, laws of =.)

Therefore,

(5) *fus*(*L*) occupies $l_i \leftrightarrow$ *fus*(*H*) occupies l_i. (1*, 2*, 4.)

The move from (1*), (2*) and (4) to (5) is not valid – it infers the location of a fusion from a location of its number. This requires the *opposite* direction of the conditionals in (1*) and (2*) than those licensed by the LNT*.

LNT* not only provides a formal fix – it also embodies commitment to numbers being *multiply located* amongst the regions occupied by the fusions of the objects falling under the relevant concepts. So far this is neutral between numbers being *wholly* but not *exclusively* located in the distinct regions occupied by the relevant fusions (as some[18] say the colour red is wholly present in both my garden and the local fire station), versus having *parts* that are co-located with the fusions of the objects falling under the relevant concepts (just as my left hand is in one place, and my right another; but I am not bilocated).

[18] E.g. Armstrong [1989: 98-9].

While an interesting result, there is reason to deny that natural numbers are multi-located *à la* LNT*. An assignment of locations to abstract objects of kind *K* is nontrivial only if it doesn't co-locate all *K*'s. Let *a* be some entity occupying a location l_i. Then by LNT*, 1 is co-located with *a*, and so occupies l_i. Now *two* things occupy l_i (*a* and 1). And so by the LNT* 2 occupies l_i. And now *three* things occupy l_i – *a*, 1, and 2. Thus 3 occupies l_i. And so on… whence *every* natural number occupies l_i. The result generalizes; every number is co-located with every located entity (including every other number). Hence LNT* entails that all numbers are co-located, and is thus a trivial assignment of locations to numbers. Note that this argument can also be run for the LNT, so we now have two reasons that it fails as a nontrivial assignment of locations to numbers.

Turning now to whether there is a nontrivial assignment of locations to sets, one difference between numbers and sets is that HP features the equivalence relation of equinumerosity, whereas RV features the distinct relation of co-extensionality. Now, the LNT and LST both have abstract objects *inheriting* properties of fusions in the following way:

(PI)[19]: $\ulcorner\forall\alpha_{X(\alpha)}[\Phi(\S(\alpha)) \leftrightarrow \Psi\alpha]\urcorner$.

Suppose we have an abstraction principle $\ulcorner\forall\alpha_{X(\alpha)}\forall\beta_{X(\beta)}[\S\alpha = \S\beta \leftrightarrow \Sigma(\alpha, \beta)]\urcorner$ and a property-inheritance $\ulcorner\forall\alpha_{X(\alpha)}[\Phi(\S(\alpha)) \leftrightarrow \Psi\alpha]\urcorner$. This conjunction entails that Ψ expresses a *congruence* for Σ, i.e.:

$$\ulcorner\forall\alpha_{X(\alpha)}\forall\beta_{X(\beta)}[\Sigma(\alpha, \beta) \rightarrow (\Psi(\alpha) \leftrightarrow \Psi(\beta))]\urcorner.$$

This reveals a deeper explanation of the problem with the LNT. Spatiotemporal location is not a congruence for equinumerosity. That *F* 1-1 *G* and *fus*(*F*) occupies l_i does *not* entail that *fus*(*G*)

[19] 'PI' stands for 'property inheritance'. The idea that abstraction involves inheriting congruent properties has been discussed elsewhere, e.g. Linnebo [2012]; but (i) these discussions don't address the unity of apparently heterogenous abstracta for which I'm arguing, and (ii) in section 4 I'll propose another kind of property determination not discussed by Linnebo. There may also be important connections to Charles Parson's distinction between pure abstract and 'quasi-concrete' objects (which have 'intrinsic representations', cf. Parsons [2008: §7]). See footnote 38 below for (slightly) further discussion.

occupies l_i. And if location is not a congruence for equinumerosity, then one of HP or the LNT is not true. Given that HP is not up for grabs, we reject the LNT.

Things are different for sets. Assume $\forall x(Fx \leftrightarrow Gx)$. Now, if anything is F if and only if it is G, then the fusion of the Fs and the fusion of the Gs are *identical*. But if $fus(F) = fus(G)$ then *a fortiori* they are co-located. Location *is* a congruence for co-extensionality. Therefore the LST is consistent with RV.

If this is right, then there is a nontrivial assignment of locations to sets, but not to natural numbers. We now turn to repeatable artworks.

3: Repeatable Artworks

The repeatability of works of music and literature suggests they are not identical to any physical entity, and are thus abstract. However commonsense presumes that such repeatable artworks are created and destroyable [Levinson 1980: 7-9; Thomasson 1999: 9]. Now, it seems that if an entity *x* is created, there is a time before which *x* does not exist and after which it does. And, for *x* to be destroyed, there must be a time before which *x* exists and after which it does not. Moreover, some philosophers have claimed that repeatable artworks are spatially located, in the sense that *Anna Karenina* is present here on Earth, but not on Twin Earth where there is instead a qualitative duplicate [Burgess and Rosen 1997: 21-2]. If this is right, then repeatable artworks cannot be *non-spatiotemporal* abstracta. But if to be abstract is to lack location, then repeatable are not abstract – whence they must be concrete.

Amie Thomasson has argued that works of literature cannot be located [Thomasson 1999: 36-8]. Here is my generalization of her argument to repeatable artworks as a whole:

P1: If repeatable artworks have physical features like size, mass, or spatiotemporal location, they inherit those features from their copies (books, scores, performances, memories, etc.). (This is the only nontrivial assignment of such features.)

P2: Repeatable artworks don't change size or mass when their copies do. (Intuition.)

Therefore,

C1: Repeatable artworks ≠ (collections of) their copies. (P2, law of =.)

Therefore,

C2: Repeatable artworks are not located. (C1 and P1.)

But C2 does not follow from C1 and P1. This is because repeatable artworks being distinct from (collections or fusions of) their copies does not entail that repeatable artworks do not *inherit* location from their copies. The reasoning parallels that for sets. First we state the identity conditions for repeatable artworks in terms of an equivalence relation E on Fregean concepts. Next we state how works inherit the locations of their copies. And finally, we show that location is a congruence for E.

First we need an abstraction principle for repeatable artworks. According to Thomasson [1999] and Levinson [1980], works of literature and musical works (respectively) are individuated by their *origins*. Works w_1 and w_2 are identical if and only if they are the products of the same act of authorship or composition.[20] Two concrete entities are token copies of the same work if and only if they are each appropriately causally connected – via acts of copying of some kind – to the same act of creation.[21] For example, that the copy of *Anna Karenina* currently in my bookshelf is

[20] I'm assuming that distinct repeatable artworks can't both result from the very same act of creation. While this claim may require some defense – perhaps an author can write two novels using the very same token actions – I won't pause to give that defense here.

[21] See Levinson [1980: 25-26] and Thomasson [1999: 64-65] for discussions of what these appropriate causal relations could be. I set aside questions about individuating acts of creation themselves.

a copy of the same work of literature as some thing in my university's library results from both physical entities each standing in appropriate causal chains tracing back to Tolstoy's penning of the original manuscript.[22] A qualitative duplicate of my copy of *Anna Karenina* located on Twin Earth is not a copy of Tolstoy's greatest novel because it does not stand in the right causal connections to him. Likewise token performances, scores, recordings, or memories are copies of *Clair de Lune* if and only if appropriately causally connected to Debussy's original act of composition.

Here's how to capture this origin-determined identity condition on repeatable artworks using an abstraction principle. Let F be a Fregean concept under which fall all and only token copies of a given work (e.g. *is a copy of Anna Karenina*). Let w be a function from Fregean concepts to works, so that $w(F)$ is the work all and only copies of which fall under F. Let F and G be *co-created* if and only if the entities falling under F and those falling under G are appropriately causally connected to the *same* act of creation. Then we have:

(APW): $\forall F_{X(F)} \forall G_{X(G)}[w(F) = w(G) \leftrightarrow$ F and G are *co-created*].

That is, two works are identical if and only if all their copies stand in appropriate causal connections to the very same act of creation. This is the Thomasson-Levinson conception of works of literature and music described above.[25]

Here is how to locate repeatable artworks individuated by APW. Let *fus*(W) be the fusion of the entities falling under W (so that *fus*(*is a copy of Anna Karenina*) is the fusion of all copies of *Anna Karenina*). Then the claim that repeatable artworks inherit their locations from the fusions of their copies is the following *located works thesis*:

[22] I set aside questions about individuating acts of creation themselves.

[25] APW may not be very useful in explaining the identity conditions for repeatable artworks to someone who does not already grasp that notion. But just as with HP and RV, we are not asking APW to do any epistemic/metasemantic work – all we need is that it be true.

(LWT): $\forall F_{X(F)}[w(F)$ occupies $l_i \leftrightarrow fus(F)$ occupies $l_i]$.

The LWT says if one copy of *Anna Karenina* moves or ceases to exist, then *Anna Karenina* itself changes location. Thus that work was initially located only in Russia – because that's where all its copies were – but now envelopes the whole globe (but not Twin Earth). Likewise if all token scores, forthcoming performances, and people knowing how to play or remember hearing *Clair de Lune* move to the Southern Hemisphere, then so does that work itself.

It remains to show that location is a congruence for co-creation. If F and G are co-created, then any entity falling under F – i.e., any copy of $w(F)$ – also falls under G, and thus is a copy of $w(G)$, and vice-versa. Hence if F and G are co-created, then $fus(F) = fus(G)$. Therefore location is a congruence for co-creation.

Indeed, if the equivalence relation in the abstraction principle for repeatable artworks is *any* equivalence relation entailing that identical works have all and only the same copies – surely a desirable result – then the LWT describes a nontrivial assignment of locations to repeatable artworks.

Thus my generalizations of Thomasson's premises do not entail that there is *no* nontrivial assignment of locations to repeatable artworks in terms of those of their copies. We can take them to be co-located with the fusions of their copies. I make one brief observation before, in section 4, connecting this with our earlier discussion of numbers and sets.

Recall the commonsense presumption that repeatable artworks are created and destroyable. The LWT tells us exactly *when* this happens. A repeatable artwork is created when a fusion of its copies comes into existence.[26] That is, it is created when its first copy is; i.e., when the first

[26] Note that this is consistent with the concept (e.g.) *is a copy of Anna Karenina* not being created. If there are no copies of *Anna Karenina*, then that concept does not figure in any true instance of APW, whence '*w*(*is a copy of Anna Karenina*)' does not denote.

manuscript is authored, or the first act of storytelling enacted, or the first ‘memory’ of it occurs. (Note that any vagueness in these times will be inherited by the time of creation of the work itself. But we should view this as a feature of the view rather than a bug.) Similarly, a work is destroyed when there is no longer any fusion of its copies; which is to say, when its last copy is destroyed.[27] This coincides with Thomasson’s own conditions for creation and destruction of works of literature [Thomasson 1999:7-14]. Likewise for the creation of Levinson’s musical works (though he is ambivalent about whether musical works can be destroyed [2011: 261-3]).

However, there may seem to be a tension here: I’ve said that if a repeatable artwork lacks any location, then it doesn’t exist. Parity of reasoning might suggest the same for natural numbers and sets. By section 2, neither numbers nor pure sets are located. But we don’t want this to impugn the existence of those abstract mathematical objects.

Now, the difference here is that the concepts appearing in true instances of HP and RV can have both located *and* non-located entities fall under them. But the concepts featuring in true instances of APW can *only* have token copies of artworks – which are intelligently manufactured *concreta* – fall under them. Thus such concepts can only have *located* entities fall under them. Therefore if such a concept has no located entities falling under it, *nothing* falls under it, whence it stands in no co-creation relation to any concept (including itself). But then it figures in no true instance of APW, and there is no repeatable artwork corresponding to it.

The final section turns from location to other physical features like shape, mass, and possession of causal powers.

[27] There is a question here about whether works can be ‘resurrected’ – can all copies of a work be destroyed, and then later on a new copy of that *very same* work come into existence? I address this in section 3. I’m also setting aside any worry that if time travel is possible, then a work may have copies pre-dating its creation (but see Lewis [1976: 148-9]).

4: Abstract Entities, Mereological Fusions, and Physical Features

If what I've said so far is correct, then there is a nontrivial assignment of locations to impure sets and to repeatable artworks, but no such assignment for natural numbers. This is because (i) if location is not a congruence for an equivalence relation, then abstract objects individuated in terms of that relation cannot be nontrivially assigned locations; and (ii) location is a congruence for co-extensionality and co-creation, but not equinumerosity.

This result generalizes. We've seen that some equivalence relations E are such that when two concepts F and G stand in them, the fusions of the objects falling under them are *identical*:

(FI) $E(F, G) \rightarrow fus(F) = fus(G)$.

Given that for all (extensional) predicates Φ, $\ulcorner fus(F) = fus(G) \rightarrow [\Phi(fus(F)) \leftrightarrow \Phi(fus(G))]\urcorner$ – every (extensional) predicate expresses a congruence for *identity* – if R satisfies FI, then for every (extensional) predicate Φ, $\ulcorner E(F, G) \rightarrow [\Phi(fus(F)) \leftrightarrow \Phi(fus(G))]\urcorner$. But then, any extensional property of fusions can be nontrivially assigned to objects individuated in terms of R via a PI-assignment $\ulcorner \forall F_{X(F)}[\Phi(\S(F)) \leftrightarrow \Phi(fus(F)]\urcorner$.[28] Call this the *fusion congruence* result:

(FC): if $\ulcorner \forall \alpha_{X(\alpha)} \forall \beta_{X(\beta)}[\S\alpha = \S\beta \leftrightarrow \Sigma(\alpha, \beta)]\urcorner$ and $\ulcorner \Sigma(\alpha, \beta)] \rightarrow fus(\alpha) = fus(\beta)\urcorner$, then for any extensional predicate Φ, $\ulcorner \forall \alpha_{X(\alpha)}[\Phi(\S\alpha) \leftrightarrow \Phi(fus(\alpha))]\urcorner$ is nontrivial.

Thus when FI holds for an equivalence relation, then not just location but *any* extensional property of fusions can be nontrivially assigned by a PI-assignment. We can – *if we want to* – say that $\ulcorner \S F \urcorner$ weighs *n*kg if and only if *fus*(*F*) weighs *n*kg, or that $\ulcorner \S F \urcorner$ is *S*-shaped if and only if *fus*(*F*) is *S*-shaped – or even, that $\ulcorner \S F \urcorner$ has causal power *C* if and only if *fus*(*F*) has causal power *C*. (I discuss cases where such claims are intuitively false below.)

[28] Thanks to Dan Rabinoff for bringing this to my attention

On the other hand, when we don't have FI, then FC fails: not every PI-assignment ⌜$\forall\alpha_{X(\alpha)}[\Phi(\S\alpha) \leftrightarrow \Phi(fus(\alpha))]$⌝ is nontrivial. In particular, natural numbers we can have F 1-1 G even though $fus(F) \neq fus(G)$, and so we can have $fus(F)$ occupying l_i without $fus(G)$ doing so. Similarly, none of weight, shape, or possessing a given causal power are congruences for equinumerosity. Next I make three observations regarding FC.

The first observation is that while the FC means we can consistently nontrivially assign any extensional property of fusions to certain abstracta, this sometimes gives incorrect results. Intuitively, *Clair de Lune* is around 5 minutes long. Suppose we assign a duration to *Clair de Lune* in terms of the following *duration inheritance thesis*:

(DIT): $\forall F_{X(F)}[w(F)$ has duration $d \leftrightarrow fus(F)$ has duration $d]$.

Now, the duration of the fusion of copies of *Clair de Lune* far exceeds 5 minutes. Does this mean that *Clair de Lune* could be, appearances notwithstanding, many months long? This is not plausible. (Analogous reasoning holds for shapes of letter-types. Fusions of letter-tokens may occupy multifarious disconnected regions. So if we say letter-types inherit the shapes of the fusions of their tokens, the letter-type *S* is not '*S*'-shaped.)

Here is a response, inspired by Kit Fine's theory of arbitrary objects [Fine 1985[29]]. Fine says that an arbitrary object *a* – e.g. an arbitrary number or person – is Φ if and only if every object which *a* 'arbitrarily represents' is Φ. Thus an arbitrary natural number > 1 has a unique prime decomposition because every non-arbitrary natural number > 1 has a unique prime decomposition. We can transpose this to abstract objects as follows:

[29] See especially Fine [1985: 9-21]. (Thanks to an anonymous referee here.)

(PD): $\ulcorner\forall F_{X(F)}[\Phi(\S F) \leftrightarrow \forall x(Fx \rightarrow \Phi x)]\urcorner$.

Notice that PD captures the idea of an abstract object $\ulcorner\S F\urcorner$ *determining* a physical property of entities falling under *F*, rather than *inheriting* a property from the fusion of the *F*s. (Just as with PI-assignments, PD-assignments are permissible, not obligatory; there may be nontrivial PD-assignments that nevertheless seem intuitively false).

Applying this to durations of musical works yields the following *duration determination thesis*:

(DDT): $\forall F_{X(F)}[w(F)$ has duration $d \leftrightarrow \forall x(Fx \rightarrow x$ has duration $d)]$.

This entails that *Clair de Lune* is roughly 5 minutes long, provided that every copy of it is.[30] (Likewise, perhaps the letter-type *S* is '*S*'-shaped provided every token of is.[31]) So assigning durations to musical works via DDT rather than DIT nets the intuitively correct answer.

However this might look like an *ad hoc* postulation of a new kind of 'property correlation' just to get out of a bind. But it seems there is a genuine distinction here: while the *duration* of a copy of a musical work is determined by the length of the *work* it copies, the *location* of a musical work is determined by that *of the work's copies*. Some properties abstracta inherit from other entities (e.g. location), others (like duration and shape) abstracta determine other entities to have. The distinction between PI- and PD-assignments tracks that between these two 'directions of determination'. If that's right then we should expect there to be both PI-assignments and PD-

[30] Setting aside concerns about whether every copy of *Clair de Lune* even has a duration (maybe memories or scores of it don't).

[31] I.e. $\forall F_{X(F)}[\S(F)$ has shape $S \leftrightarrow \forall x(Fx \rightarrow x$ has shape $S)]$. See Wetzel [2009: 61-65] for an argument that shape is not in general common to all tokens of the same letter-type.

assignments. The move from DIT to DDT is not *ad hoc* (though I must leave for another time the question of which properties are determined and which inherited by which abstracta).

My second observation concerns abstracta having causal powers. Given that possessing a causal power is a congruence for identity of fusions, if *fus*(*F*) has a certain causal power, and FC holds for ⌜§*F*⌝s, then we can say that ⌜§*F*⌝ has the same power. So perhaps surprisingly, we can regard abstracta as having causal powers – *provided* we can regard fusions having powers of their own. While pursuing this line of thought further would take us too far afield, we have established a way of regarding an abstract object as located but *not* having any causal powers: the relevant fusion is located but has no powers (perhaps only its parts do).

On the other hand, we could use a PD-assignment to say that ⌜§*F*⌝ has every causal power had by every individual *F*. Again, investigating further would take us too far afield. But observe that if there are no powers common to every *F*, then ⌜§*F*⌝ will be acausal – and we have another way of regarding certain abstract as located by acausal.

Now to my third observation. The LST answers questions about the spatiotemporal locations of sets in terms of the locations of members. This entails, as a specific case, that the *temporal* location of a set whose members exist at different times is that of the fusion of those members. This gives us a response to Roy T. Cook's argument that there is no plausible assignment of temporal locations to impure sets whose members exist at different times [2013: 226-7].

Consider the set {Socrates, Frege}. What temporal location can it be sensibly assigned? It seems we have three options:

(i) {Socrates, Frege} occupies the intersection of the temporal locations occupied by Socrates and Frege.

(ii) {Socrates, Frege} occupies the union of the temporal locations occupied by Socrates and Frege.

(iii) {Socrates, Frege} occupies every temporal location.

The alternative is that {Socrates, Frege} occupies no temporal locations – it's 'abstract' in the standard platonist sense.

We can rule out option (iii) because its generalization temporally co-locates every impure set. (i) assigns {Socrates, Frege} no temporal location because there is no temporal region occupied by both Socrates and Frege,

This leaves (ii). Cook rejects (ii) for two reasons.[32] First, (ii) has {Socrates, Frege} occupying a discontinuous temporal region ‑ it occupies a region in the 5th century BCE, then occupies no region until a stretch of the mid 19th-early 20th centuries, and no region thereafter. Second, (ii) violates the 'spirit' of the extensionality of sets. The claim seems to be that during the 5th century BCE, given that Socrates exists but Frege does not, {Socrates, Frege} = {Socrates}. From the mid 19th to early 20th centuries, {Socrates, Frege} = {Frege}. And, we might infer, at all other times {Socrates, Frege} = Ø. Thus (ii) is out of the running. Having already disposed of (i) and (ii), we infer that impure sets of temporally non-overlapping individuals occupy no temporal locations.

My response is to press on Cook's argument against (ii). Recall that the LST reduces the question of the location of a set to that of the fusion of the entities falling under the concept from which the set is formed. So the atemporality of {Socrates, Frege} follows only if the fusion of Socrates and Frege is also atemporal. Now, Cook professes a 'natural thought' according to which

[32] Cook [2012: 226].

a set occupies l_i only of all its members do [2013: 226].[33] This has a mereological analogue: a fusion occupies a temporal location only if all of its parts do. Absent a reason think otherwise, the two stand and fall together. Now, it seems intuitively plausible that the fusion of Socrates and Frege exists, has one part in the 5th century BCE, another 23 centuries later, and as a whole occupies the union of those regions. What we say about the fusion, we can say about the set: {Socrates, Frege} occupies a discontinuous temporal region in virtue of having parts (members) occupying parts of that region. Insofar as we don't worry about fusions occupying discontinuous regions, we needn't worry for sets either.

Finally, Cook's misgivings about extensionality can also be relieved by giving up the 'natural thought' and saying that {Socrates, Frege} is never identical to any of {Socrates}, {Frege}, or Ø because it has members those other sets do not ‑ even if those members are temporally separated. Now, Parsons [2008: §7 f/n 57] observes that this requires an eternalist view of time. If this is right, then the same seems to be true of fusions of temporally separated entities – allowing that the fusion of Socrates and Frege is never identical to either individual requires eternalism. Hence Cook's argument against the LST requires rejecting eternalism. On the one hand, eternalism is a not unpopular view of time amongst contemporary philosophers. On the other, one may be content with the fusion of Socrates and Frege being in the past identical to Frege alone, and now identical to nothing. I leave the reader to pick their own poison. Either way, the thing to note is that the LST does not *by itself* decide the question, nor is it reduced to absurdity.

This might raise a question about pure sets. Pure sets are spatiotemporally located only if the empty set is. Given that the empty set has no members, the LST does not assign it a location.

---

[33] Compare also Parsons [2008: §7].

Thus the empty set exists *at no time*. Does it follow that neither the empty set nor any pure set exists? This would be the wrong result.

The best thing to say here is that, just as with numbers, that the empty set occupies no *time* need not entail that it does not exist *at all*. Pure sets (but not all sets) are atemporal. Provided the empty set exists, all pure sets exist; and if the empty set does not exist *in time*, then neither does any pure set. The question then is whether the empty set exists, given that none of its members exist (it has none). Fortunately, the existence of the empty set is a straightforward consequence of RV. Let *F* and *G* both be the concept of non-self-identity. Then we reason as follows:

(1) $\{x: x \neq x\} = \{x: x \neq x\} \leftrightarrow \forall x(x \neq x \leftrightarrow x \neq x)$	(RV with '$x \neq x$' for 'F' and 'G')
(2) $\forall x(x \neq x \leftrightarrow x \neq x)$	(truth of logic)
(3) $\{x: x \neq x\} = \{x: x \neq x\}$	(1, 2)
(4) $\exists x(x = \{x: x \neq x\})$	(3, $\exists$-introduction)

And so the empty set exists. That the LST assigns no location to the empty set is no threat to its existence.

And consider, finally, repeatable artworks. The general case is that the copies of a work exist at different times: the first act of storytelling might precede the authoring of the first manuscript, which itself may precede many printed copies and editions thereof. However we may not face exactly the same problem as we did with the set {Frege, Socrates}. This is because if we assume that for *c* to be a copy of *w* (hence fall under the concept *is a copy of w*), *c* must be appropriately causally connected to the creation of *w*, *and* that such causal chains are sustained by copies, then it is not possible for works to be 'resurrected': the destruction of all extant copies precludes the possibility of future copies.[37] From this it follows that a work exists if and only if at

[37] [acknowledgement redacted]

least one copy of it does too. And then there is no problem of temporally non-overlapping copies, for, for an object to *be* a copy, there must exist at all intervening times at least one copy. Thus repeatable artworks do not face the same problem of occupying discontinuous temporal regions as faced by impure sets of temporally separated individuals. (Note also that the identity of a repeatable artwork does not seem to be threatened by a change in its copies in the same way that the identity of a set is threatened by a change in its members.)

Conclusion

When the identity conditions for a given kind of abstract object *K* can be captured by an abstraction principle, whether *K*s can possess a given physical property *P* depends on whether that property is a congruence for the equivalence relation *E* featuring in that principle. If it is, then there is a nontrivial assignment of *P* to *K*s in terms of which concrete objects are *P*. But if *P* is not a congruence for *E*, then there is no such assignment. And so repeatable artworks (and impure sets) can be nontrivially assigned locations, whereas natural numbers cannot. So we now have an answer to our opening question of how both natural numbers and repeatable artworks could be abstract objects if being non-spatiotemporal is part of what it is to be abstract.

More generally, we have distinguished two kinds of abstract objects: those such that if ⌜$\S\alpha = \S\beta$⌝, then ⌜$\mathit{fus}(\alpha) = \mathit{fus}(\beta)$⌝, and that are not. We've seen that the former, but not the latter, can be nontrivially assigned *any* extensional property of fusions. This raises the prospect of something like a *hierarchy* of abstracta. We would have one 'level' at which abstracta bear some relation to fusions of concreta intimate enough for the potential inheritance of all properties possessed by those fusions; and another 'level', frustrating such universal inheritance and so perhaps further removed from concrete reality.[38] Further questions quickly suggest themselves: for instance, where do *propositions*, letter-types, and species occur in this hierarchy? Could propositions be at the same level as numbers, whereas letter-types and species are at the sets-and-artworks level? This hierarchy is intriguing and, I think, deserves further consideration.

Thanks to Philip Kremer, Sam Cowling, Chris Tillman, Wesley D. Cray, and Dan Rabinoff for discussions of and comments on the ideas in this essay.

[38] The idea that abstract objects comprise a hierarchy has been suggested before – in particular, in the appendix to chapter 2 of Hale [1987]. Hale's hierarchy is different from that described above, at least at first glance – his is in terms of the levels of the concepts comprising the base sorts for abstraction principles. A closer cousin may be Parsons' distinction between pure abstract and quasi-concrete objects (cf. footnote 19) – maybe the level of sets and repeatable artworks is that of Parsons' quasi-concreta. Unfortunately I must leave pursuing this for another time.

<u>*References*</u>

Armstrong, David M. 1989. *Universals: An Opinionated Introduction* Boulder, CO: Westview Press.

Burgess, John and Gideon Rosen 1997. *A Subject With No Object: Strategies for Nominalistic Interpretation of Mathematics* Oxford, UK: The Clarendon Press.

Cook, Roy T. and Philip A. Ebert 2005. Abstraction and Identity, *Dialectica* 59/2: 121-139.

Cook, Roy T. ed, 2007. *The Arché Papers on the Mathematics of Abstraction*

Cook, Roy T. 2013 Impure Sets are not Located: A Fregean Argument, *Thought* 1: 219-29.

Cowling, Sam 2017. *Abstract Entities* Oxford, UK: Routledge

Davidson, Donald 1967. The Logical Form of Action Sentences (with Criticism, Comment, and Defence), in *The Essential Davidson*, Donald Davidson 2006, Oxford, UK: The Clarendon Press: 37-71.

Davidson, Donald 2006. *The Essential Davidson* Oxford, UK: The Clarendon Press.

Fine, Kit 1985. *Reasoning with Arbitrary Objects* Oxford, UK: Basil Blackwell.

Hale, Bob 1987. *Abstract Objects* Oxford, UK: Basil Blackwell.

Hale, Bob 2000. Abstraction & Set Theory, in *The Arché Papers on the Mathematics of Abstraction*, ed. Roy T. Cook 2007, Dordrecht, NL: Springer: 331-52.

Hale, Bob and Crispin Wright 2001. *The Reason's Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics* Oxford, UK: Oxford University Press.

Heck, Richard G. 1996. The Consistency of Predicative Fragments of Frege's *Grundgesetze der Arithmetik*, *History and Philosophy of Logic* 17/1-2: 209-20.

Heck, Richard G. 1997. Finitude and Hume's Principle, in *Frege's Theorem*, Richard G, Heck 2011, Oxford: Oxford University Press: 237-66.

Heck, Richard G. 2011. *Frege's Theorem* Oxford, UK: Oxford University Press

Lewis, David 1976. Paradoxes of Time Travel, *American Philosophical Quarterly* 13/2: 145-52

Levinson, Jerrold 1980. What a Musical Work Is, *Journal of Philosophy*, 77/1: 5-28

Levinson, Jerrold 2011. *Music, Art, and Metaphysics* (second edition) Oxford, UK: Oxford University Press.

Linnebo, Øystein 2012. Reference by Abstraction, *Proceedings of the Aristotelian Society*, CXII: 45-71.

Mag Uidhir, Christy ed, 2012. *Art and Abstract Objects* Oxford, UK: Oxford University Press.

Parsons, Charles 2008. *Mathematical Thought and its Objects* Cambridge, UK: Cambridge University Press.

Thomasson, Amie 1999. *Fiction and Metaphysics* Cambridge, UK: Cambridge University Press.

Wetzel, Linda 2009. *Types and Tokens*. Cambridge, Mass: MIT Press.

Wright, Crispin 1997. Is Hume's Principle Analytic?, in *The Reason's Proper Study*, Bob Hale and Crispin Wright 2001, Oxford, UK: Oxford University Press: 307-32.