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Constructing illoyal algebra-valued models of set theory

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Abstract. An algebra-valued model of set theory is called *loyal to its algebra* if the model and its algebra have the same propositional logic; it is called *faithful* if all elements of the algebra are truth values of a sentence of the language of set theory in the model. We observe that non-trivial automorphisms of the algebra result in models that are not faithful and apply this to construct three classes of illoyal models: tail stretches, transposition twists, and maximal twists.

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1. Background

The construction of *algebra-valued models of set theory* starts from an algebra $\mathbb{A}$ and a model of set theory forming an $\mathbb{A}$-valued model of set theory. If the algebra $\mathbb{A}$ is a Boolean algebra, this construction results in *Boolean-valued models of set theory* which are closely connected to the theory of forcing and independence proofs in set theory [1]. If the algebra $\mathbb{A}$ is not a Boolean algebra, the construction gives rise to algebra-valued models of set theory whose logic is, in general, not classical logic. Examples of this are Heyting-valued models of intuitionistic set theory, lattice-valued models, orthomodular-valued models, and an algebra-valued model of paraconsistent set theory of Löwe and Tarafder [10, 25, 16, 14, 24].

The central idea of this construction is that the logic of the algebra $\mathbb{A}$ should be reflected in the resulting $\mathbb{A}$-valued model of set theory. E.g., if $\mathbb{H}$ is

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any finite Heyting algebra, then the logic of the $\mathbb{H}$-valued model is classical if and only if the logic of the algebra $\mathbb{H}$ is classical (i.e., $\mathbb{H}$ is a Boolean algebra; cf. Proposition 3.3).

But how closely does the logic of the algebra-valued model of set theory resemble the logic of the algebra it is constructed from? In this paper, we introduce the concepts of *loyalty* and *faithfulness* to describe the relationship between the logic of the algebra $\mathbb{A}$ and the logical phenomena witnessed in the $\mathbb{A}$-valued model of set theory: a model is called *loyal* to its algebra if the propositional logic in the model is the same as the logic of the algebra from which it was constructed and *faithful* if every element of the algebra is the truth value of a sentence in the model.

The classical construction of Boolean-valued models and also the mentioned construction of a model of paraconsistent set theory from [14] are all loyal (cf. Lemma 3.2 and Theorem 3.4). This raises the following natural questions:

(1) Are there models that are illoyal to their algebra?
(2) Can you characterise the class of algebras that only have loyal models?
(3) Can you characterise the class of logics that can hold in an algebra-valued model of set theory?

In this paper, we solve question (1) by giving constructions to produce illoyal models by stretching and twisting Boolean algebras. Our results can also be seen as a first step towards solving questions (2) and (3). (Note that question (3) depends on the precise requirements of being a "model of set theory", i.e., which axioms of set theory do you require to hold in such a model.)

Related work

Our two main notions of *loyalty* and *faithfulness* were introduced by Paßmann in a more general setting for classes of so-called *Heyting structures* in the sense of [9] (cf. [17, Definitions 2.39 and 2.40]). The concepts of loyalty and faithfulness also have proof-theoretic applications: *de Jongh's theorem* states that the propositional logic of Heyting arithmetic is **IPC**, the intuitionistic propositional calculus; using our terminology, this theorem can be proved by providing a loyal class of Kripke models of arithmetic (cf. [21,5]). Paßmann recently constructed a faithful class of models of set theory to prove that the propositional logic of IZF is **IPC** [18].

Outline of the paper

After we give the basic definitions in Section 2, we remind the reader of the construction of algebra-valued models of set theory in Section 3. In Section 4, we introduce our main technique: non-trivial automorphisms of an algebra $\mathbb{A}$ exclude values from being truth values of sentences in the $\mathbb{A}$-valued model of set theory (Corollary 4.3). Finally, in Section 5, we apply this technique to produce three classes of models: tail stretches (Section 5.2), transposition twists (Section 5.3), and maximal twists (Section 5.4).

2. Basic definitions

2.1. Algebras

As usual in logic, if Λ is a finite list of finitary logical connectives, a Λ-*algebra* $\mathbb{A}$ is an underlying set A with a finite list of finitary operations on A corresponding to the symbols in Λ. In this paper, we shall assume that

$$\{\wedge, \vee, \mathbf{0}, \mathbf{1}\} \subseteq \Lambda \subseteq \{\wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1}\}$$

and that $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded distributive lattice. As usual, we use the same notation for the syntactic logical connectives and the operations on $\mathbb{A}$ interpreting them. In the rare cases where proper marking of these symbols improves readability, we attach a subscript $\mathbb{A}$ to the algebra operations in $\mathbb{A}$, e.g., $\wedge_{\mathbb{A}}$, $\vee_{\mathbb{A}}$, $\bigwedge_{\mathbb{A}}$, or $\bigvee_{\mathbb{A}}$. We can define $\leq$ on $\mathbb{A}$ by $x \leq y$ if and only if $x \wedge y = x$. An element $a \in A$ is an *atom* if it is $\leq$-minimal in $A \backslash \{\mathbf{0}\}$; we write $\mathrm{At}(\mathbb{A})$ for the set of atoms in $\mathbb{A}$. If $\Lambda = \{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication algebra* and if $\Lambda = \{\wedge, \vee, \rightarrow \neg, \mathbf{0}, \mathbf{1}\}$, we call $\mathbb{A}$ an *implication-negation algebra.*

We call a Λ-algebra $\mathbb{A}$ with underlying set A *complete* if for every $X \subseteq A$, the $\leq$-supremum and $\leq$-infimum exist; in this case, we write $\bigvee X$ and $\bigwedge X$ for these elements of $\mathbb{A}$. A complete Λ-algebra $\mathbb{A}$ is called *atomic* if for every $a \in A$, there is an $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$.

2.2. Boolean algebras, complementation, and Heyting algebras

An algebra $\mathbb{B} = (B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ is called a *Boolean algebra* if for all $b \in B$, we have that $b \wedge \neg b = \mathbf{0}$ and $b \vee \neg b = \mathbf{1}$. As usual, we can define an implication by

$$x \rightarrow y := \neg x \vee y; \qquad (\#)$$

using this definition, we can consider Boolean algebras as implication algebras or implication-negation algebras. An implication algebra $(B, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ is called a *Boolean implication algebra* if there is a Boolean algebra $(B, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ such that $\rightarrow$ is defined by (#) from $\vee$ and $\neg$ or, equivalently, if the negation defined by $\neg_* x := x \rightarrow \mathbf{0}$ satisfies $\neg_* b \wedge b = \mathbf{0}$ and $\neg_* b \vee b = \mathbf{1}$.

On an atomic bounded distributive lattice $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, we have a canonical definition for a negation operation, the *complementation negation*: since $\mathbb{A}$ is atomic, every element $a \in A$ is uniquely represented by a set $X \subseteq \mathrm{At}(\mathbb{A})$ such that $a = \bigvee X$. Then we define the complementation negation by

$$\neg_{\mathrm{c}}(\bigvee X) := \bigvee \{t \in \mathrm{At}(\mathbb{A})\,;\, t \notin X\}.$$

In this situation, $(A, \wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1})$ is an atomic Boolean algebra. Moreover, if $(A, \wedge, \vee, \neg, \mathbf{0}, \mathbf{1})$ is an atomic Boolean algebra and $\neg_{\mathrm{c}}$ is the complementation negation of the atomic bounded distributive lattice $(A, \wedge, \vee, \mathbf{0}, \mathbf{1})$, then $\neg = \neg_{\mathrm{c}}$. Of course, for every set X, the power set algebra $(\wp(X), \cap, \cup, \varnothing, X)$ forms an atomic bounded distributive lattice and, with the set complementation operator, a Boolean algebra.

If $(H, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded distributive lattice, then an implication algebra $\mathbb{H} = (H, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ is called a *Heyting algebra* if and only if the

Law of Residuation holds, i.e., for all $a, b, c \in H$, we have that

$$c \wedge a \leq b \text{ if and only if } c \leq a \rightarrow b.$$

If $\mathbb{H}$ is a complete lattice, then this is equivalent to

$$a \rightarrow b = \bigvee\{x \in H\,;\, a \wedge x \leq b\} \qquad (\dagger)$$

and we say that $\mathbb{H}$ is a *complete Heyting algebra*. In a Heyting algebra $\mathbb{H}$, we can define a negation $\neg_{\mathbb{H}}$ by $\neg_{\mathbb{H}} x := x \rightarrow \mathbf{0}$. Note that Boolean implication algebras are Heyting algebras.

It is well known that the class of Heyting algebras forms a variety [13, p. 8] and that not every complete bounded distributive lattice can be turned into a Heyting algebra (e.g., the dual of the Heyting algebra of open subsets of $\mathbb{R}$; cf. [2, Proposition 51.2]).

A Heyting algebra is called *linear* if $(H, \leq)$ is a linear order; the formula $(p \rightarrow q) \vee (q \rightarrow p)$ characterises the variety of Heyting algebras generated by the linear Heyting algebras [20,7,11] (cf. also [19] for a discussion of Skolem's 1913 results).

We shall later use the following linear three element complete Heyting algebra $\mathbf{3} := (\{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}, \wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1})$ with $\mathbf{0} \leq \mathbf{1/2} \leq \mathbf{1}$. Then $\rightarrow$ is uniquely determined by (†):

$\rightarrow$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$.

2.3. Languages

Fix a set S of non-logical symbols, a countable set P of propositional variables, and a countable set V of first-order variables. We denote the set of well-formed propositional formulas with connectives Λ and propositional variables P by $\mathcal{L}_\Lambda$ and the set of well-formed first-order formulas with connectives Λ, variables in V and constant, relation and function symbols in S by $\mathcal{L}_{\Lambda,S}$. The subset of sentences of $\mathcal{L}_{\Lambda,S}$ will be denoted by $\mathrm{Sent}_{\Lambda,S}$. Note that both $\mathcal{L}_\Lambda$ and $\mathrm{Sent}_{\Lambda,S}$ have the structure of a Λ-algebra and that the Λ-algebra $\mathcal{L}_\Lambda$ is generated by closure under the connectives in Λ from the set P.

For arbitrary sets Λ of logical connectives and S of non-logical symbols, we define NFF_Λ to be the closure of P under the logical connectives other than $\neg$ and $\mathrm{NFF}_{\Lambda,S}$ to be the closure of the atomic formulae of $\mathcal{L}_{\Lambda,S}$ under the logical connectives other than $\neg$. These formulas are called the *negation-free* Λ*-formulas*. Clearly, if $\neg \notin \Lambda$, then $\mathcal{L}_\Lambda = \mathrm{NFF}_\Lambda$ and $\mathcal{L}_{\Lambda,S} = \mathrm{NFF}_{\Lambda,S}$.

2.4. Homomorphisms, assignments, and translations

For any two Λ-algebras $\mathbb{A}$ and $\mathbb{B}$, a map $f : A \rightarrow B$ is called a Λ-*homomorphism* if it preserves all operations in Λ; it is called a Λ-*isomorphism* if it is a bijective Λ-homomorphism; isomorphisms from $\mathbb{A}$ to $\mathbb{A}$ are called Λ-*automorphisms*.

If $\mathbb{A}$ and $\mathbb{B}$ are two complete Λ-algebras and $f{:}A{\to}B$ is a Λ-homomorphism, we call it *complete* if it preserves the operations $\bigvee$ and $\bigwedge$, i.e.,

$$f(\bigvee_{\mathbb{A}} X) = \bigvee_{\mathbb{B}}(\{f(x)\,;\,x \in X\}) \text{ and}$$
$$f(\bigwedge_{\mathbb{A}} X) = \bigwedge_{\mathbb{B}}(\{f(x)\,;\,x \in X\})$$

for $X \subseteq A$.

Since $\mathcal{L}_\Lambda$ is generated from P, we can think of any Λ-homomorphism defined on $\mathcal{L}_\Lambda$ as a function on P, homomorphically extended to all of $\mathcal{L}_\Lambda$. If $\mathbb{A}$ is a Λ-algebra with underlying set A, we say that Λ-homomorphisms $\iota : \mathcal{L}_\Lambda \to A$ are $\mathbb{A}$-*assignments*; if S is a set of non-logical symbols, we say that Λ-homomorphisms $T : \mathcal{L}_\Lambda \to \mathrm{Sent}_{\Lambda,S}$ are S-*translations*.

2.5. The propositional logic of an algebra

A set $D \subseteq A$ is called a *designated set* or *filter* if the following four conditions hold: (i) $\mathbf{1} \in D$, (ii) $\mathbf{0} \notin D$, (iii) if $x \in D$ and $x \leq y$, then $y \in D$, and (iv) for $x, y \in D$, we have $x \wedge y \in D$. For any designated set D, the *propositional logic of* $(\mathbb{A}, D)$ is defined as

$$\mathbf{L}(\mathbb{A}, D) := \{\varphi \in \mathcal{L}_\Lambda\,;\,\iota(\varphi) \in D \text{ for all } \mathbb{A}\text{-assignments } \iota\}.$$

Since the classical propositional calculus **CPC** is maximally consistent, we obtain that if $\mathbb{B}$ is a Boolean algebra and D is any designated set, then $\mathbf{L}(\mathbb{B}, D) = \mathbf{CPC}$ [3, Theorem 5.11].

2.6. Algebra-valued structures and their propositional logic

If $\mathbb{A}$ is a Λ-algebra and S is a set of non-logical symbols, then any Λ-homomorphism $[\![\cdot]\!] : \mathrm{Sent}_{\Lambda,S} \to A$ will be called an $\mathbb{A}$-*valued* S-*structure*. Note that if $S' \subseteq S$, $\Lambda' \subseteq \Lambda$, $\mathbb{A}$ is a Λ-algebra and $\mathbb{A}'$ its Λ'-reduct, and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $[\![\cdot]\!]\restriction\mathrm{Sent}_{\Lambda,S'}$ is an $\mathbb{A}$-valued S'-structure and $[\![\cdot]\!]\restriction\mathrm{Sent}_{\Lambda^*,S}$ is an $\mathbb{A}^*$-valued S-structure.

We define the *propositional logic of* $([\![\cdot]\!], D)$ as

$$\mathbf{L}([\![\cdot]\!], D) := \{\varphi \in \mathcal{L}_\Lambda\,;\,[\![T(\varphi)]\!] \in D \text{ for all } S\text{-translations } T\}.$$

Note that if T is an S-translation and $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure, then $\varphi \mapsto [\![T(\varphi)]\!]$ is an $\mathbb{A}$-assignment, so

$$\mathbf{L}(\mathbb{A}, D) \subseteq \mathbf{L}([\![\cdot]\!], D). \tag{‡}$$

Clearly, $\mathrm{ran}([\![\cdot]\!]) \subseteq A$ is closed under all operations in Λ (since $[\![\cdot]\!]$ is a homomorphism) and thus defines a sub-Λ-algebra $\mathbb{A}_{[\![\cdot]\!]}$ of $\mathbb{A}$. The $\mathbb{A}$-assignments that are of the form $\varphi \mapsto [\![T(\varphi)]\!]$ are exactly the $\mathbb{A}_{[\![\cdot]\!]}$-assignments, so we obtain

$$\mathbf{L}([\![\cdot]\!], D) = \mathbf{L}(\mathbb{A}_{[\![\cdot]\!]}, D \cap \mathbb{A}_{[\![\cdot]\!]}).$$

We should like to point out that the *propositional logic of the structure* $([\![\cdot]\!], D)$ as defined above treats all Λ, S-sentences as propositional atoms and thus cannot take their internal construction into account; this is in line with the usual definitions of propositional logics of first-order theories (cf., e.g., [5]). Note that ignoring the internal structure of sentences can result in a situation where a structure $([\![\cdot]\!], D)$ is non-classical, but satisfies $\mathbf{L}([\![\cdot]\!], D) = \mathbf{CPC}$. E.g., consider the Heyting algebra $\mathbb{H}$ with $H = \mathbb{Z} \cup \{\mathbf{0}, \mathbf{1}\}$ from Proposition 4.7

where we prove that $\mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}\}) = \mathbf{CPC}$. It is easy to see that the sentence $\varphi := \forall x \forall y (x \in y \vee x \notin y)$ (cf. the proof of Proposition 3.3) evaluates to $\mathbf{0}$ in $\mathbb{H}$, so $\mathbb{H}$ is non-classical. (This was pointed out by one of the referees.)

2.7. Loyalty and faithfulness

An $\mathbb{A}$-valued S-structure $[\![\cdot]\!]$ is called *loyal to* $(\mathbb{A}, D)$ if the converse of (‡) holds as well, i.e., if $\mathbf{L}(\mathbb{A}, D) = \mathbf{L}([\![\cdot]\!], D)$; it is called *faithful to* $\mathbb{A}$ if for every $a \in A$, there is a $\varphi \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\varphi]\!] = a$; equivalently, if $\mathbb{A}_{[\![\cdot]\!]} = \mathbb{A}$. (Cf. the paragraph on **Related Work** in Section 1 for the genesis of these notions.)

Lemma 2.1. *Let Λ be any set of propositional connectives, S be any set of non-logical symbols, $\mathbb{A}$ be a Λ-algebra, and $[\![\cdot]\!]$ be an $\mathbb{A}$-valued S-structure. Then, if $[\![\cdot]\!]$ is faithful to $\mathbb{A}$, then it is loyal to $(\mathbb{A}, D)$ for any designated set D.*

Proof. By (‡), we only need to prove one inclusion; if $\varphi \notin \mathbf{L}(\mathbb{A}, D)$, then let $p_1, \ldots, p_n$ be the propositional variables occurring in φ and let ι be an assignment such that $\iota(\varphi) \notin D$. By faithfulness, find sentences $\sigma_i \in \mathrm{Sent}_{\Lambda,S}$ such that $[\![\sigma_i]\!] = \iota(p_i)$ for $1 \leq i \leq n$. Let T be any translation such that $T(p_i) = \sigma_i$ for $1 \leq i \leq n$. Then $[\![T(\varphi)]\!] = \iota(\varphi) \notin D$, and hence T witnesses that $\varphi \notin \mathbf{L}([\![\cdot]\!], D)$. □

A proof of Lemma 2.1 in the more general setting for classes of Heyting structures can be found in [17, Proposition 2.50].

Note that faithfulness and loyalty depend on the choice of S. As mentioned above, if $S^* \subseteq S$ and $\Lambda^* \subseteq \Lambda$ then $\mathrm{Sent}_{\Lambda^*,S^*} \subseteq \mathrm{Sent}_{\Lambda,S}$ and thus we can easily see the following:

Observation 2.2. *Let $\mathbb{A}$ be a Λ-algebra, $\mathbb{A}^*$ its Λ^*-reduct, and $[\![\cdot]\!]$ be an $\mathbb{A}$-valued S-structure. If $[\![\cdot]\!]\restriction\mathrm{Sent}_{\Lambda^*,S^*}$ is faithful to $\mathbb{A}^*$, then $[\![\cdot]\!]$ is faithful to $\mathbb{A}$.*

However, the converse is not true in general: faithfulness cannot hold if the algebra $\mathbb{A}$ is bigger than the set $\mathrm{Sent}_{\Lambda,S}$, so for countable languages, no $\mathbb{A}$-valued S-structure can be faithful to an uncountable algebra $\mathbb{A}$. Thus, if $\mathbb{A}$ is an uncountable algebra, S an uncountable set of non-logical symbols, $[\![\cdot]\!]$ is an $\mathbb{A}$-valued S-structure that is faithful to $\mathbb{A}$, and S' is a countable subset of S, then $[\![\cdot]\!]\restriction\mathcal{L}_{\Lambda,S'}$ cannot be faithful to $\mathbb{A}$. The constructions in this paper will give another example that does not use a cardinality argument (cf. the remark after Theorem 5.10 at the end of this paper).

3. Algebra-valued models of set theory

In the following, we give an overview of general construction of an algebra-valued model of set theory following [14]. The original ideas go back to Boolean-valued models independently discovered by Solovay and by Vopěnka [28] and were further generalised to other classes of algebras [10, 22, 25, 26, 15, 16]. Details can be found in [1].

In the following, we shall use the phrase "*V is a model of set theory*" to mean that V is a transitive set such that $(V, \in) \models \mathsf{ZF}$. Of course, the existence

$$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y] \quad \text{(Extensionality)}$$
$$\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y)) \quad \text{(Pairing)}$$
$$\exists x[\exists y(\forall z(z \in y \rightarrow \mathbf{0}) \wedge y \in x) \wedge \forall w \in x \exists u \in x(w \in u)] \quad \text{(Infinity)}$$
$$\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(z \in x)) \quad \text{(Union)}$$
$$\forall x \exists y \forall z(z \in y \leftrightarrow \forall w \in z(w \in x)) \quad \text{(Power Set)}$$
$$\forall p_0 \cdots \forall p_n \forall x \exists y \forall z(z \in y \leftrightarrow z \in x \wedge \varphi(z, p_0, \ldots, p_n)) \quad (\text{Separation}_\varphi)$$
$$\forall p_0 \cdots \forall p_{n-1} \forall x[\forall y \in x \exists z \varphi(y, z, p_0, \ldots, p_{n-1}) \rightarrow \exists w \forall v \in x \exists u \in w\ \varphi(v, u, p_0, \ldots, p_{n-1})] \quad (\text{Collection}_\varphi)$$
$$\forall p_0 \cdots \forall p_n \forall x[\forall y \in x\ \varphi(y, p_0, \ldots, p_n) \rightarrow \varphi(x, p_0, \ldots, p_n)] \rightarrow \forall z \varphi(z, p_0, \ldots, p_n) \quad (\text{Set Induction}_\varphi)$$

FIGURE 1. The axioms of ZF formulated in $\mathcal{L}_{\{\wedge,\vee,\rightarrow,\mathbf{0},\mathbf{1}\},\{\in\}}$

of sets like this cannot be proved in ZF and requires some (mild) additional metamathematical assumptions. The choice of ZF as the set theory in our base model is not relevant for the constructions of this paper and one can generalise the results to models of weaker or alternative set theories; however, we shall not explore this route in this paper.

Since we are sometimes working in languages without negation, we need to formulate the axioms of ZF in a negation-free context given in Figure 1, following [14, Section 3]. Our negation-free axioms given are classically equivalent to what is usually called ZF, but not exactly the same axioms: e.g., we use Collection and Set Induction in lieu of Replacement and Foundation. Many authors call this axiom system IZF.

If V is a model of set theory and A is any set, then we construct a universe of *names* by transfinite recursion:

$$\begin{aligned} \mathrm{Name}_\alpha(V, A) &:= \{x\,;\, x \text{ is a function and } \mathrm{ran}(x) \subseteq A \text{ and} \\ &\qquad \text{there is } \xi < \alpha \text{ with } \mathrm{dom}(x) \subseteq \mathrm{Name}_\xi(V, A)\} \text{ and} \\ \mathrm{Name}(V, A) &:= \{x\,;\, \exists \alpha(x \in \mathrm{Name}_\alpha(V, A))\}. \end{aligned}$$

We let $S_{V,A}$ be the set of non-logical symbols consisting of the binary relation symbol $\in$ and a constant symbol for every name in $\mathrm{Name}(V, A)$ (as usual, we use the name itself as the constant symbol). The language $\mathcal{L}_{\Lambda,S_{V,A}}$ is usually called the *forcing language*.

If $\mathbb{A}$ is a Λ-algebra with underlying set A, we can now define a map $[\![\cdot]\!]^{\mathbb{A}}$ assigning to each $\varphi \in \mathcal{L}_{\Lambda,S_{V,A}}$ a truth value in $\mathbb{A}$ by recursion (the definition of $[\![u \in v]\!]^{\mathbb{A}}$ and $[\![u = v]\!]^{\mathbb{A}}$ is recursion on the hierarchy of names; the rest is a recursion on the complexity of φ):

$$
\begin{aligned}
[\![\mathbf{0}]\!]^{\mathbb{A}} &= \mathbf{0}, \\
[\![\mathbf{1}]\!]^{\mathbb{A}} &= \mathbf{1}, \\
[\![u \in v]\!]^{\mathbb{A}} &= \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!]^{\mathbb{A}}), \\
[\![u = v]\!]^{\mathbb{A}} &= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \to [\![x \in v]\!]^{\mathbb{A}}) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \to [\![y \in u]\!]^{\mathbb{A}}), \\
[\![\varphi \wedge \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \wedge [\![\psi]\!]^{\mathbb{A}}, \\
[\![\varphi \vee \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \vee [\![\psi]\!]^{\mathbb{A}}, \\
[\![\varphi \to \psi]\!]^{\mathbb{A}} &= [\![\varphi]\!]^{\mathbb{A}} \to [\![\psi]\!]^{\mathbb{A}}, \\
[\![\neg\varphi]\!]^{\mathbb{A}} &= \neg[\![\varphi]\!]^{\mathbb{A}}, \\
[\![\forall x \varphi(x)]\!]^{\mathbb{A}} &= \bigwedge_{u \in \mathrm{Name}(V,A)} [\![\varphi(u)]\!]^{\mathbb{A}}, \text{ and} \\
[\![\exists x \varphi(x)]\!]^{\mathbb{A}} &= \bigvee_{u \in \mathrm{Name}(V,A)} [\![\varphi(u)]\!]^{\mathbb{A}}.
\end{aligned}
$$

By construction, it is clear that $[\![\cdot]\!]^{\mathbb{A}}$ is an $\mathbb{A}$-valued $S_{V,A}$-structure and hence, by restricting it to $\mathrm{Sent}_{\Lambda,\{\in\}}$, we can consider it as an $\mathbb{A}$-valued $\{\in\}$-structure. Usually, it is the restriction to $\mathrm{Sent}_{\Lambda,\{\in\}}$ that set theorists are interested in: to reflect this shift of focus, we shall use the notation $[\![\cdot]\!]_{\mathbb{A}} := [\![\cdot]\!]^{\mathbb{A}} \restriction \mathrm{Sent}_{\Lambda,\{\in\}}$ and $[\![\cdot]\!]_{\mathbb{A}}^{\mathrm{Name}} := [\![\cdot]\!]^{\mathbb{A}}$.

The results for algebra-valued models of set theory were proved for Boolean algebras originally, then extended to Heyting algebras:

Theorem 3.1. *If V is a model of set theory, $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ is a Boolean algebra or Heyting algebra, and φ is any axiom of* ZF*, then $[\![\varphi]\!]_{\mathbb{B}} = \mathbf{1}$.*

Proof. Cf. [1, Theorem 1.33 and pp. 165–166]. □

Lemma 3.2. *Let $\mathbb{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ be a Heyting algebra and V be a model of set theory. Then $[\![\cdot]\!]_{\mathbb{H}}^{\mathrm{Name}}$ is faithful to $\mathbb{H}$ (and hence, loyal to $(\mathbb{H}, D)$ for every designated set D on $\mathbb{H}$ by Lemma 2.1).*

Proof. Consider $u := \varnothing \in \mathrm{Name}_1(V, H)$, $v := \{(\varnothing, a)\} \in \mathrm{Name}_2(V, H)$, and $\varphi := u \in v$ which is an element of $\mathrm{Sent}_{\Lambda, S_{V,H}}$. It is easy to check that $[\![\varphi]\!]_{\mathbb{H}}^{\mathrm{Name}} = a$. □

We can now prove the result for finite Heyting algebras mentioned in the introduction. The generalisation to infinite Heyting algebras is not true, as Proposition 4.7 will show. (Cf. [17, Corollary 5.15] for more on the logic of the class of all Heyting-valued models for a finite Heyting algebra.)

Proposition 3.3. *Let $\mathbb{H} = (H, \wedge, \vee, \to, \mathbf{0}, \mathbf{1})$ be a finite Heyting algebra and V be a model of set theory. Then $\mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}\}) = \mathbf{CPC}$ if and only if $\mathbb{H}$ is a Boolean algebra.*

Proof. To simplify notation, let $\neg a := \neg_{\mathbb{H}} a = a \to \mathbf{0}$. The direction "$\Leftarrow$" is clear.

For the direction "$\Rightarrow$", consider $h := \bigwedge\{a \vee \neg a\,;\, a \in H\}$. Since $\mathbb{H}$ is a Heyting algebra, we have the following equalities for all $a, b \in H$:

$$\neg\neg(a \vee \neg a) = \mathbf{1},$$
$$\neg(a \vee b) = \neg a \wedge \neg b \text{ (de Morgan for } \vee\text{), and}$$
$$\neg(a \wedge b) = \neg\neg(\neg a \vee \neg b) \text{ (weak de Morgan for } \wedge\text{).}$$

Using (weak) de Morgan, an induction shows for finite $A \subseteq H$ that

$$\neg\neg \bigwedge\{a\,;\, a \in A\} = \bigwedge\{\neg\neg a\,;\, a \in A\}.$$

Thus, since H is finite, we have that

$$\neg\neg h = \neg\neg \bigwedge\{a \vee \neg a\,;\, a \in H\}$$
$$= \bigwedge\{\neg\neg(a \vee \neg a)\,;\, a \in H\} = \mathbf{1}.$$

We now consider the sentence $\varphi := \forall x \forall y(x \in y \vee x \notin y)$. Clearly,

$$[\![\varphi]\!]_{\mathbb{H}} = \bigwedge\{[\![u \in v \vee u \notin v]\!]^{\mathbb{H}}\,;\, u, v \in \mathrm{Name}(V, H)\}$$
$$\geq \bigwedge\{a \vee \neg a\,;\, a \in H\} = h.$$

For $a \in H$, let $u_a := \{(\varnothing, a)\}$; then, $[\![u_{\mathbf{0}} \in u_a]\!]^{\mathbb{H}} = a$, and thus $[\![\varphi]\!]_{\mathbb{H}} \leq a \vee \neg a$, whence $[\![\varphi]\!]_{\mathbb{H}} = h$.

If $\mathbb{H}$ is not a Boolean algebra, then there is some a such that $a \vee \neg a \neq \mathbf{1}$, so $h \neq \mathbf{1}$, but then $\neg\neg p \to p \notin \mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{1\})$, as witnessed by φ. □

In order to formulate results for implication algebras, Löwe and Tarafder introduced NFF-ZF, the axiom system of all ZF-axioms where the two axiom schemata are restricted to instances of negation-free formulas [14, p. 197]. They furthermore introduced a three-element algebra $\mathbb{PS}_3$ [14, Figure 2 and Section 6] and proved the following result (for the sake of completeness, we give the definition of $\mathbb{PS}_3$ in Figure 2; for more on the algebra $\mathbb{PS}_3$, cf. [4]; for more on the set theory in the $\mathbb{PS}_3$-valued model, cf. [23]):

Theorem 3.4. *If V is a model of set theory and φ is any axiom of* NFF-ZF*, then $[\![\varphi]\!]_{\mathbb{PS}_3} = \mathbf{1}$. Furthermore, $[\![\cdot]\!]_{\mathbb{PS}_3}$ is faithful to $\mathbb{PS}_3$ and hence loyal to $(\mathbb{PS}_3, D)$ for every designated set D by Lemma 2.1.*

Proof. Cf. [14, Corollary 5.2] for the first claim. Löwe and Tarafder give a sentence $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$, $\varphi := \exists u, v, w(u = v \wedge w \in u \wedge w \notin v)$, such that $[\![\varphi]\!]_{\mathbb{PS}_3} = \mathbf{1/2}$ which establishes faithfulness [14, Theorem 6.2]. □

4. Automorphisms and algebra-valued models of set theory

Given a model of set theory V and any Λ-algebras $\mathbb{A}$ and $\mathbb{B}$ and a Λ-homomorphism $f : \mathbb{A} \to \mathbb{B}$, we can define a map

$$\widehat{f} : \mathrm{Name}(V, \mathbb{A}) \to \mathrm{Name}(V, \mathbb{B})$$

$\wedge$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

$\vee$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$\to$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$

$\neg$	
$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$

FIGURE 2. Connectives for $\mathbb{PS}_3$

by $\in$-recursion via

$$\operatorname{dom}(\widehat{f}(u)) := \{\widehat{f}(v)\,;\, v \in \operatorname{dom}(u)\} \text{ and}$$
$$\widehat{f}(u)(\widehat{f}(v)) := f(u(v)).$$

Proposition 4.1. *Suppose that V is a model of set theory, $\mathbb{A}$ and $\mathbb{B}$ are complete Λ-algebras and $f : \mathbb{A} \to \mathbb{B}$ is a complete Λ-isomorphism. Let $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$ with n free variables and $u_1, \ldots, u_n \in \operatorname{Name}(V, \mathbb{A})$. Then*

$$f([\![\varphi(u_1, \ldots, u_n)]\!]_{\mathbb{A}}) = [\![\varphi(\widehat{f}(u_1), \ldots, \widehat{f}(u_n))]\!]_{\mathbb{B}}.$$

Proof. For atomic formulas, this is easily proved by induction on the rank of the names involved. For non-atomic formulas, the claim follows by induction on the complexity of the formula (where the quantifier cases need the fact that f is a bijection). □

Corollary 4.2. *Suppose that V is a model of set theory, $\mathbb{A}$ and $\mathbb{B}$ are complete Λ-algebras and $f : \mathbb{A} \to \mathbb{B}$ is a complete Λ-isomorphism. Let $\varphi \in \operatorname{Sent}_{\Lambda,\{\in\}}$. Then*

$$f([\![\varphi]\!]_{\mathbb{A}}) = [\![\varphi]\!]_{\mathbb{B}}.$$

Corollary 4.3. *Suppose that V is a model of set theory, $\mathbb{A}$ is a complete Λ-algebra with underlying set A, $a \in A$, and that $f : \mathbb{A} \to \mathbb{A}$ is a complete Λ-automorphism with $f(a) \neq a$. Then there is no $\varphi \in \operatorname{Sent}_{\Lambda,\{\in\}}$ such that $[\![\varphi]\!]_{\mathbb{A}} = a$.*

Proof. By Corollary 4.2, if $[\![\varphi]\!]_{\mathbb{A}} = a$, then $f(a) = a$. □

Proposition 4.4. *If $\mathbb{A} = (A, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is an atomic bounded distributive lattice and $a \in A \backslash \{\mathbf{0}, \mathbf{1}\}$, then there is a $\{\wedge, \vee, \neg_{\mathrm{c}}, \mathbf{0}, \mathbf{1}\}$-automorphism f of $\mathbb{A}$ such that $f(a) \neq a$.*

Proof. Note that the assumptions imply that $A \neq \{\mathbf{0}, \mathbf{1}\}$ and hence $\operatorname{At}(\mathbb{A}) \neq \varnothing$. By atomicity, every permutation $\pi : \operatorname{At}(\mathbb{A}) \to \operatorname{At}(\mathbb{A})$ induces an automorphism of $\mathbb{A}$ preserving $\wedge$, $\vee$, $\neg_{\mathrm{c}}$, $\mathbf{0}$, and $\mathbf{1}$ by $f_\pi(\bigvee X) = \bigvee\{\pi(t)\,;\, t \in X\}$ for $X \subseteq \operatorname{At}(\mathbb{A})$. Let $a = \bigvee X_a$. Since $a \neq \mathbf{0}$, we have $X_a \neq \varnothing$; since $a \neq \mathbf{1}$, we have $X_a \neq \operatorname{At}(\mathbb{A})$. So, pick $t_0 \in X_a$ and $t_1 \in \operatorname{At}(\mathbb{A}) \backslash X_a$ and let π be the transposition that interchanges t_0 and t_1. Then

$$t_0 \leq \bigvee X_a = a, \text{ but}$$
$$t_0 \not\leq \bigvee\{\pi(t)\,;\, t \in X_a\} = f_\pi\left(\bigvee X_a\right) = f_\pi(a),$$

whence $a \neq f_\pi(a)$. □

Corollary 4.5. *If V is a model of set theory, $\mathbb{B}$ is an atomic Boolean (implication) algebra with more than two elements, and D is any designated set on $\mathbb{B}$, then $[\![\cdot]\!]_{\mathbb{B}}$ is loyal, but not faithful to $(\mathbb{B}, D)$.*

Proof. By Proposition 4.4, all elements except for $\mathbf{0}$ and $\mathbf{1}$ are moved by some automorphism of an atomic Boolean (implication) algebra and hence by Corollary 4.3, for each sentence $\varphi \in \mathcal{L}_{\Lambda,\{\in\}}$, we have that $[\![\varphi]\!]_{\mathbb{B}} \in \{\mathbf{0}, \mathbf{1}\}$. In particular, this means that $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}}, D) = \mathbf{L}(\{\mathbf{0}, \mathbf{1}\}, \{\mathbf{1}\}) = \mathbf{CPC} = \mathbf{L}(\mathbb{B}, D)$. □

Clearly, atomicity is not a necessary condition for the conclusion of Corollary 4.5: the Boolean algebra of infinite and co-infinite subsets of $\mathbb{N}$ is atomless and hence non-atomic, but every nontrivial element is moved by an automorphism, so Corollary 4.3 applies. We do not know whether this result extends to Boolean algebras without this property, e.g., rigid Boolean algebras (cf. [27, Section 2]):

Question 4.6. *Are there (necessarily countable) Boolean algebras $\mathbb{B}$ such that $[\![\cdot]\!]_{\mathbb{B}}$ is faithful to $\mathbb{B}$ for some designated set D?*

We can use our method of automorphisms to show that Proposition 3.3 does not generalise to infinite Heyting algebras:

Proposition 4.7. *There is an infinite complete Heyting algebra $\mathbb{H}$ that is not a Boolean algebra such that $\mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}\}) = \mathbf{CPC}$. Consequently, $[\![\cdot]\!]_{\mathbb{H}}$ is illoyal to $(\mathbb{H}, \{\mathbf{1}\})$.*

Proof. Let $\Lambda := \{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$ and let $H := \mathbb{Z} \cup \{\mathbf{0}, \mathbf{1}\}$ with the order where $\mathbf{0}$ is the smallest element, $\mathbf{1}$ is the largest element, and the elements of $\mathbb{Z}$ lie between them in their usual order. Then $\mathbb{H} = (H, \min, \max, \rightarrow, \mathbf{0}, \mathbf{1})$ with

$$a \rightarrow b := \begin{cases} \mathbf{1} & \text{if } a \leq b \text{ and} \\ b & \text{otherwise} \end{cases}$$

is a linear complete Heyting algebra with a nontrivial complete Λ-automorphism

$$\pi(a) := \begin{cases} a+1 & \text{if } a \in \mathbb{Z} \text{ and} \\ a & \text{if } a \in \{\mathbf{0}, \mathbf{1}\} \end{cases}$$

(cf. [6, Example 1.3.1]). By Corollary 4.3, for every $\varphi \in \mathrm{Sent}_{\Lambda,\{\in\}}$, $[\![\varphi]\!]_{\mathbb{H}} \in \{\mathbf{0}, \mathbf{1}\}$, so $\mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}\}) = \mathbf{CPC}$. □

5. Stretching and twisting the loyalty of Boolean algebras

5.1. What can be considered a negation?

In this section, we start from an atomic, complete Boolean algebra $\mathbb{B}$ and modify it to get an algebra $\mathbb{A}$ that gives rise to an illoyal $[\![\cdot]\!]_{\mathbb{A}}$. The first construction is the well-known construction of tail extensions of Boolean algebras to obtain a Heyting algebra. The other two constructions are *negation twists*: in these, we interpret $\mathbb{B}$ as a Boolean implication algebra via the definition

$a \to b := \neg a \vee b$, and then add a new, twisted negation to it that changes its logic.

So far, all negations we considered were the negations in Boolean algebras and Heyting algebras; now, we are going to modify these negations. Of course, not every unary function on an implication algebra is a sensible negation, and we need to argue that the modified negation operations in our examples meet the requirements of being a negation operation. In his survey of varieties of negation, Dunn lists Hazen's *subminimal negation* as the bottom of his *Kite of Negations*: only the rule of contraposition, i.e., $a \leq b$ implies $\neg b \leq \neg a$, is required [8]. In the following, we shall use this as a necessary requirement to be a reasonable candidate for negation. (Cf. also [12].)

5.2. Tail stretches

Let $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ be a Boolean algebra, and $\mathbf{1}^* \notin B$ be an additional element that we add to the top of $\mathbb{B}$ to form the *tail stretch* $\mathbb{H}$ as follows: $H := B \cup \{\mathbf{1}^*\}$, the complete lattice structure of $\mathbb{H}$ is the order sum of $\mathbb{B}$ and the one element lattice $\{\mathbf{1}^*\}$, and $\to^*$ is defined as follows:

$$a \to^* b := \begin{cases} a \to b & \text{if } a, b \in B \text{ such that } a \not\leq b, \\ \mathbf{1}^* & \text{if } a, b \in B \text{ with } a \leq b \text{ or if } b = \mathbf{1}^*, \\ b & \text{if } a = \mathbf{1}^*. \end{cases}$$

In $\mathbb{H}$, we use the (Heyting algebra) definition $\neg_{\mathbb{H}} h := h \to^* \mathbf{0}$ to define a negation; note that if $\mathbf{0} \neq b \in B$, $\neg_{\mathbb{H}} b = \neg b$, but $\neg_{\mathbb{H}} \mathbf{0} = \mathbf{1}^* \neq \mathbf{1} = \neg \mathbf{0}$.

Lemma 5.1. *The tail stretch* $\mathbb{H} = (H, \wedge, \vee, \to^*, \mathbf{0}, \mathbf{1}^*)$ *is a Heyting algebra with* $p \vee \neg p \notin \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\})$, *so in particular,* $\mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\}) \neq \mathbf{CPC}$.

Proof. If $b \neq \mathbf{0} \in B$, then by definition $b \to^* \mathbf{0} = \neg b$ where $\neg$ refers to the negation in $\mathbb{B}$. In particular, $b \vee \neg_{\mathbb{H}} b = b \vee \neg b = \mathbf{1} \neq \mathbf{1}^*$. □

Lemma 5.2. *If* $f : B \to B$ *is an automorphism of the Boolean algebra* $\mathbb{B}$, *then* $f^* : H \to H$ *defined by*

$$f^*(b) := \begin{cases} f(b) & \text{if } b \in B \text{ and} \\ \mathbf{1}^* & \text{if } b = \mathbf{1}^* \end{cases}$$

is an automorphism of $\mathbb{H}$.

Proof. Easy to check. □

Theorem 5.3. *Let* V *be a model of set theory,* $\mathbb{B}$ *an atomic Boolean algebra with more than two elements, and* $\mathbb{H}$ *be the tail stretch of* $\mathbb{B}$ *as defined above. Then the* $\mathbb{H}$*-valued model of set theory* $V^{\mathbb{H}}$ *is not faithful to* $\mathbb{H}$. *Furthermore, we have that*

$$(p \to q) \vee (q \to p) \in \mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}^*\}) \setminus \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\}).$$

Consequently, $V^{\mathbb{H}}$ *is illoyal to* $(\mathbb{H}, \{\mathbf{1}^*\})$.

Proof. Since $\mathbb{B}$ is atomic with more than two elements, each of the non-trivial elements of B is moved by an automorphism of $\mathbb{B}$ by Proposition 4.4. By Lemma 5.2, these remain automorphisms of $\mathbb{H}$. As a consequence, we can apply Corollary 4.2 to get that $\mathrm{ran}([\![\cdot]\!]_{\mathbb{H}}) \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{1}^*\}$ which is isomorphic to the linear Heyting algebra $\mathbf{3}$ and thus the range is a linear Heyting algebra. As mentioned, [11] proved that $(p \to q) \vee (q \to p)$ characterises the variety generated by the linear Heyting algebras, so $(p \to q) \vee (q \to p) \in \mathbf{L}([\![\cdot]\!]_{\mathbb{H}}, \{\mathbf{1}^*\})$. However, since $\mathbb{B}$ has more than two elements, we can pick imcomparable $a, b \in B$. Then $a \to^* b$ and $b \to^* a$ are both elements of B, and thus $(p \to q) \vee (q \to p) \notin \mathbf{L}(\mathbb{H}, \{\mathbf{1}^*\})$. □

We remark that $\mathrm{ran}([\![\cdot]\!]_{\mathbb{H}}) = \{\mathbf{0}, \mathbf{1}, \mathbf{1}^*\}$: one can show that the $[\![\cdot]\!]_{\mathbb{H}}$-value of the sentence formalising the statement "every subset of $\{\varnothing\}$ is either $\varnothing$ or $\{\varnothing\}$" is $\mathbf{1}$.

5.3. Transposition twists

Let $\mathbb{B} = (B, \wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1})$ be an atomic Boolean algebra, $a, b \in \mathrm{At}(\mathbb{B})$ with $a \neq b$, and let π be the transposition that transposes a and b. Since $\mathbb{B}$ is an atomic Boolean algebra, $\neg = \neg_{\mathrm{c}}$. Then f_π as defined in the proof of Proposition 4.4 is a $\{\wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1}\}$-automorphism of $\mathbb{B}$. We now define a twisted negation by

$$\neg_\pi\left(\bigvee X\right) := \bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}$$

and let the π-*twist* of $\mathbb{B}$ be $\mathbb{B}_\pi := (B, \wedge, \vee, \to, \neg_\pi, \mathbf{0}, \mathbf{1})$. (Note that we do not twist the implication $\to$ which remains the implication of the original Boolean algebra $\mathbb{B}$ defined by $x \to y := \neg_{\mathrm{c}} x \vee y$.) We observe that the twisted negation $\neg_\pi$ satisfies the rule of contraposition.

Lemma 5.4. *Let D be a designated set. If either $\neg_{\mathrm{c}} a = \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq a\}$ or $\neg_{\mathrm{c}} b = \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq b\}$ is not in D, then $\neg(p \wedge \neg p) \notin \mathbf{L}(\mathbb{B}_\pi, D)$. In particular, $\mathbf{L}(\mathbb{B}_\pi, D) \neq \mathbf{CPC}$.*

Proof. Without loss of generality, $\bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \neq b\} = \neg_{\mathrm{c}} b = \neg_\pi a \notin D$. Since $a \leq \neg_\pi a$, we have that $a = \neg_\pi a \wedge a$, and so $\neg_\pi(\neg_\pi a \wedge a) = \neg_\pi a \notin D$. □

Lemma 5.5. *There is an automorphism f of $\mathbb{B}_\pi$ such that $f(a) = b$. In particular, $[\![\cdot]\!]_{\mathbb{B}_\pi}$ is not faithful to $\mathbb{B}_\pi$.*

Proof. We know that f_π is an automorphism of $\mathbb{B}$. Since π is a transposition, we have that $\pi^2 = \mathrm{id}$ and $\pi = \pi^{-1}$; using this, we observe that f_π still preserves $\neg_\pi$:

$$\begin{aligned} f_\pi\left(\neg_\pi\left(\bigvee X\right)\right) &= f_\pi\left(\bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}\right) \\ &= \bigvee\{\pi(\pi(t)) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\} \\ &= \bigvee\{t \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\} \\ &= \neg_\pi\left(\bigvee\{\pi(t) \in \mathrm{At}(\mathbb{B})\,;\, t \notin X\}\right) \\ &= \neg_\pi\left(f_\pi\left(\bigvee X\right)\right). \end{aligned}$$

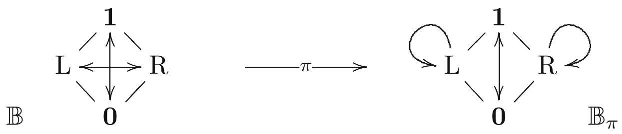

Figure 3. The four-element Boolean algebra and its transposition twist. Negations are indicated by arrows

Thus, f_π is an automorphism of $\mathbb{B}_\pi$; clearly, $f_\pi(a) = b$. The second claim follows from Corollary 4.3. □

Now let V be a model of set theory and $[\![\cdot]\!]_{\mathbb{B}_\pi}$ the $\mathbb{B}_\pi$-valued $\{\in\}$-structure derived from V and $\mathbb{B}$.

Lemma 5.6. *If $x \in \mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi})$, then $\neg_\pi x = \neg_\mathrm{c} x$.*

Proof. Let $x = \bigvee X$ for some $X \subseteq \mathrm{At}(\mathbb{B})$. By Corollary 4.3 and Lemma 5.5, if $x \in \mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi})$, then $f_\pi(x) = x$. This means that either both $a, b \in X$ or both $a, b \notin X$. In both cases, it is easily seen that $\neg_\pi x = \neg_\mathrm{c} x$. □

Theorem 5.7. *For any designated set D, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{CPC}$. In particular, if either $\neg_\mathrm{c} a$ or $\neg_\mathrm{c} b$ is not in D, then $[\![\cdot]\!]_{\mathbb{B}_\pi}$ is not loyal to $(\mathbb{B}_\pi, D)$.*

Proof. As mentioned in Section 2, if we let

$$\mathbb{C} := \mathbb{B}_{[\![\cdot]\!]_{\mathbb{B}_\pi}} = (\mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi}, \wedge, \vee, \rightarrow, \neg_\pi, \mathbf{0}, \mathbf{1}),$$

then $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{L}(\mathbb{C}, D)$. But Lemma 5.6 implies that

$$\mathbb{C} = (\mathrm{ran}([\![\cdot]\!]_{\mathbb{B}_\pi}, \wedge, \vee, \rightarrow, \neg_\mathrm{c}, \mathbf{0}, \mathbf{1}),$$

which is a Boolean algebra (as a subalgebra of $\mathbb{B}$). Thus, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D) = \mathbf{L}(\mathbb{C}, D)$ $= \mathbf{CPC}$. The second claim follows from Lemma 5.4. □

As the simplest possible special case, we can consider the Boolean algebra $\mathbb{B}$ generated by two atoms L and R; then, there is one nontrivial transposition $\pi(\mathrm{L}) = \mathrm{R}$ and all nontrivial elements of $\mathbb{B}$ are moved by the automorphism f_π. As a consequence of Corollary 4.3, all sentences will get either value $\mathbf{0}$ or value $\mathbf{1}$ under $[\![\cdot]\!]_{\mathbb{B}_\pi}$, and hence $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_\pi}, D)$ is classical (cf. Figure 3).

Note that the $\{\wedge, \vee, \rightarrow, \mathbf{0}, \mathbf{1}\}$-reduct of $\mathbb{B}_\pi$ is just the Boolean implication algebra underlying the Boolean algebra $\mathbb{B}$ that we started with. Thus, Observation 2.2 and Theorem 5.7 yield an alternative proof of Corollary 4.5.

5.4. Maximal twists

Again, let $\mathbb{B} = (B, \wedge, \vee, \rightarrow, \neg, \mathbf{0}, \mathbf{1})$ be an atomic Boolean algebra with more than two elements and define the maximal negation by

$$\neg_\mathrm{m} b := \begin{cases} \mathbf{1} & \text{if } b \neq \mathbf{1} \text{ and} \\ \mathbf{0} & \text{if } b = \mathbf{1} \end{cases}$$

for every $b \in B$. We let the *maximal twist* of $\mathbb{B}$ be $\mathbb{B}_\mathrm{m} := (B, \wedge, \vee, \rightarrow, \neg_\mathrm{m}, \mathbf{0}, \mathbf{1})$; once more observe that the maximal negation $\neg_\mathrm{m}$ satisfies the rule of contraposition.

Lemma 5.8. *Let D be a designated set. If there is some $\mathbf{0} \neq b \notin D$, then $(p \wedge \neg p) \rightarrow q \notin \mathbf{L}(\mathbb{B}_{\mathrm{m}}, D)$. In particular, $\mathbf{L}(\mathbb{B}_{\mathrm{m}}, D) \neq \mathbf{CPC}$.*

Proof. Let $c := \neg_{\mathrm{c}} b$. Note that the assumption $b \neq \mathbf{0}$ implies $c \neq \mathbf{1}$. In particular, $\neg_{\mathrm{m}} c = \mathbf{1}$, and thus $c \wedge \neg_{\mathrm{m}} c = c$. Also

$$\begin{aligned} c \rightarrow b &= \neg_{\mathrm{c}} b \rightarrow b \\ &= \neg_{\mathrm{c}} \neg_{\mathrm{c}} b \vee b \\ &= b \vee b = b. \end{aligned}$$

Thus, the assignment ι with $p \mapsto c$ and $q \mapsto b$ yields $\iota((p \wedge \neg p) \rightarrow q) = b \notin D$. □

Lemma 5.9. *For any $b \notin \{\mathbf{0}, \mathbf{1}\}$, there is an automorphism f of $\mathbb{B}_{\mathrm{m}}$ such that $f(b) \neq b$. In particular, $[\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}$ is not faithful to $\mathbb{B}_{\mathrm{m}}$.*

Proof. We claim that any automorphism f of $\mathbb{B}$ also preserves $\neg_{\mathrm{m}}$. Suppose f is an automorphism of $\mathbb{B}$. If $b = \mathbf{1}$, then clearly $f(\neg_{\mathrm{m}} \mathbf{1}) = f(\mathbf{0}) = \mathbf{0} = \neg_{\mathrm{m}} \mathbf{1} = \neg_{\mathrm{m}} f(\mathbf{1})$. Now let $b \neq \mathbf{1}$. Since f is bijective and $f(\mathbf{1}) = \mathbf{1}$, we have that $f(b) \neq \mathbf{1}$. So $f(\neg_{\mathrm{m}} b) = f(\mathbf{1}) = \mathbf{1} = \neg_{\mathrm{m}} f(b)$. The second claim follows from Corollary 4.3. □

Theorem 5.10. *For any designated set D, $\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}, D) = \mathbf{CPC}$. In particular, $[\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}$ is not loyal to $(\mathbb{B}_{\mathrm{m}}, D)$.*

Proof. Lemma 5.9 gives us that every nontrivial element of $\mathbb{B}$ is moved by an automorphism, so we can apply the argument from the proof of Corollary 4.5: since for each $\varphi \in \mathcal{L}_{\Lambda, \{\in\}}$, we have that $[\![\varphi]\!]_{\mathbb{B}_{\mathrm{m}}} \in \{\mathbf{0}, \mathbf{1}\}$, we get that

$$\mathbf{L}([\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}, D) = \mathbf{L}(\{\mathbf{0}, \mathbf{1}\}, \{\mathbf{1}\}) = \mathbf{CPC}.$$

The second claim follows from Lemma 5.8. □

As mentioned at the end of Sect. 2, our examples show that restricting the language can change faithful models into illoyal ones: for our twisted algebras $\mathbb{B}_{\pi}$ and $\mathbb{B}_{\mathrm{m}}$, the general faithfulness result Lemma 3.2 holds for $[\![\cdot]\!]^{\mathrm{Name}}_{\mathbb{B}_{\pi}}$ and $[\![\cdot]\!]^{\mathrm{Name}}_{\mathbb{B}_{\mathrm{m}}}$. However, Theorems 5.7 and 5.10 show that their restrictions $[\![\cdot]\!]_{\mathbb{B}_{\pi}}$ and $[\![\cdot]\!]_{\mathbb{B}_{\mathrm{m}}}$ are neither faithful nor loyal.

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