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Solving the Initial Value Problem for the 3-Wave Interaction Equations in Multidimensions

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Abstract
Starting from the 3-wave interaction equations in 2+1 dimensions (i.e., two space dimensions and one time dimension), we complexify the independent variables, thus doubling the number of real variables, and hence we work in 4+2 dimensions: x_1, x_2, y_1, y_2 and t_1, t_2. In this paper we solve the initial value problem of the 3-wave interaction equations in 4+2 dimensions.

1 Introduction

The modern history of integrable systems begins in the late 1960s when Gardner, Greene, Kruskal and Miura solved the initial value problem of the Korteweg-de Vries (KdV) equation by what was later called the Inverse Scattering Transform [1]. Here "integrable" means that the system in question can be written as the compatibility condition of a set of linear equations, the so-called Lax pair [2,3]. For some years the KdV equation with its striking properties appeared to be a unique case, until Zakharov and Shabat showed that the Non-linear Schrödinger (NLS) equation can also be solved using the Inverse Scattering Transform [4].

The KdV and NLS equations are integrable evolution equations in one spatial dimension. There exist integrable generalizations of the aforementioned equations to two spatial dimensions: the Kadomtsev–Petviashvili (KP) [5] and Davey–Stewartson (DS) [6] equations, respectively. The KP equation, describing non-linear wave motion, has two forms (known as KPI and KPII) which differ in one particular sign appearing

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Dedicated to the memory of Stephan Ruscheweyh.

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in the equation. The choice of the sign depends on the relative magnitude of the gravitational forces and the surface tension. Progress towards solving these and other equations in 2+1 dimensions (i.e., two spatial dimensions and one time dimension) was made in the works of Zakharov and Manakov [7–9] and Segur [10]. In the early 1980s Ablowitz, one of the authors, and their collaborators solved both the initial value problem of the KPI equation using a non-local Riemann–Hilbert (RH) problem [11] and the analogous problem for the KPII equation (which cannot be written as a RH problem) using the d-bar formalism [12]. This formalism first appeared in the works of Beals and Coifman on problems with one spatial dimension [13,14]. By now the above methodologies have been used to solve the initial value problems of a wide variety of integrable evolution equations with two spatial dimensions [15,16].

One of the main current topics in the field of integrable systems concerns the existence of non-linear integrable evolution equations in more than two spatial dimensions. Several attempts were made to extend the inverse scattering method to non-linear evolution equations in three or more spatial dimensions [17–20] but although these works made important contributions to the solution of inverse problems, they did not result in the desired construction of integrable multidimensional non-linear evolution Partial Differential Equations (PDEs). The difficulty they encountered, known as the 'characterization problem', was that the associated scattering data must satisfy non-linear constraints which seem to be incompatible with the existence of local non-linear integrable evolution PDEs. These non-linear constraints arise in the above approaches because the corresponding eigenvalue equations involve *several* complex spectral variables.

The fact that integrable multidimensional non-linear evolution PDEs exist has been proven by one of the authors, who in 2006 derived equations of this type in four spatial dimensions, which however had the disadvantage of containing two time dimensions [21]. The Cauchy problem of these equations in 4+2 dimensions can be solved by means of a non-local d-bar problem. The main benefit of this approach is that the eigenvalue equations depend on *only one* complex spectral variable, whereas the second complex spectral variable only appears in the integration; thus the characterization problem does not appear here. Subsequently, the same author also identified a large class of integrable evolution equations in *any* number of spatial dimensions and 1 time dimension, by constructing a non-linear Fourier transform pair which can be used to solve the Cauchy problem of these equations [22,23]. This second class of equations, however, involves a *non-local* commutator. In this paper we will concentrate on the first class, i.e., the one with 2 time dimensions.

In general, the choice of the methodology (local RH, local d-bar, non-local RH, non-local d-bar) is closely connected to the dimensionality of the equations one wants to solve [3]. The initial condition q_0 of an integrable evolution equation in 1 and 2 space dimensions depends on 1 and 2 *real* spatial variables, respectively, hence the associated spectral function should also involve 1 and 2 *real* spectral variables. In the case of a local RH problem, the spectral function depends on 1 real spectral variable k, which describes the curve on which the "jump" of the RH problem occurs, so this formalism is suitable for solving equations in 1 spatial variable. In the case of a local d-bar problem, the associated spectral function depends on 2 real spectral variables (k_1, k_2) (or equivalently on $(k, \bar{k})$, where now $k = k_1 + ik_2$ and $\bar{k}$ is the complex conjugate of k) and, hence, this problem is used to solve equations in 2 spatial dimensions. Alternatively, sometimes the non-local RH formalism can be used for solving equations in 2 spatial dimensions (as mentioned above for the KPI equation); here the spectral function depends not only on the real variable k (specifying the curve along which the jump occurs) but also on the real variable λ, specifying the integration along a given curve. The non-local RH formalism can be generalized using a limiting procedure [9] to a non-local d-bar formalism. This is the formalism we have to employ when the initial data depend on 4 real spatial variables [21]. In this case, the spectral functions also depend on 4 real spectral variables $(k_1, k_2, \lambda_1, \lambda_2)$ (or equivalently on $(k, \bar{k}, \lambda, \bar{\lambda})$ with $k = k_1 + ik_2$, $\lambda = \lambda_1 + i\lambda_2$), where λ_1 and λ_2 define the integration on the complex λ-plane.

As a specific case study, we choose the N-wave interaction equations [24]:

$$q_{ij_t} = \alpha_{ij} q_{ij_x} + (C_i - J_i \alpha_{ij}) q_{ij_y} + \sum_{\substack{n=1 \\ n \neq i,j}}^{N} (\alpha_{in} - \alpha_{nj}) q_{in} q_{nj},$$
$$\text{for } i \neq j \text{ and } q_{ii} = 0, \quad i, j = 1, \ldots, N, \tag{1.1}$$

where $\alpha_{ij} = \frac{C_i - C_j}{J_i - J_j}$ (for $i \neq j$), $C_i, J_i \in \mathbb{R}$. In particular, we study the case $N = 3$ which is especially important because it is related to the three-wave resonant interaction equations [25]:

$$u_{i_t} + a_i u_{i_x} + b_i u_{i_y} = c_i \bar{u}_j \, \bar{u}_n, \tag{1.2}$$

where bar denotes complex conjugation, a_i, b_i, c_i are constants and $i, j, n = 1, 2, 3$ cyclically permuted. Indeed, Eq. (1.2) can be obtained from (1.1) by assuming that $q_{ij} = \sigma_{ij} \bar{q}_{ji}$ for $i > j$ and $\sigma_{32}\sigma_{21} = -\sigma_{31}$ (where the σ_{ij}'s are real normalizing constants) and taking $q_{12} = u_3, q_{23} = u_1, \bar{q}_{13} = u_2$. The three equations contained in Eq. (1.2) describe the non-linear interaction of wave packets and are found in numerous applications such as non-linear optics or internal waves in the ocean [26].

From now on, we will use the indices a and b rather than the more familiar i and j to avoid confusion with the imaginary number i, which appears frequently in the analysis that follows. By complexifying the independent variables of the N-wave interaction equations (1.1) for $N = 3$, we obtain the following system of non-linear integrable equations in 4+2 dimensions:

$$q_{ab_{\bar{t}}} = \alpha_{ab} q_{ab_{\bar{x}}} + (C_a - J_a \alpha_{ab}) q_{ab_{\bar{y}}} + (\alpha_{an} - \alpha_{nb}) q_{an} q_{nb}, \text{ for } a \neq b, \ n \neq a, b, \text{ and } q_{aa} = 0, \tag{1.3}$$

where $a, b, n = 1, 2, 3$ and

$$x = x_1 + ix_2, \ y = y_1 + iy_2, \ t = t_1 + it_2, \ x_1, x_2, y_1, y_2, t_1, t_2 \in \mathbb{R}. \tag{1.4}$$

The d-bar derivatives appearing in Eq. (1.3) are given by $\partial_{\bar{x}} = \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right)$ and analogously for $\partial_{\bar{y}}$ and $\partial_{\bar{t}}$ [27].

In this paper we apply various methods outlined in Refs. [21,28,29]. The spectral analysis of the time-independent part of the problem is presented in Sect. 2 via two different approaches. Subsequently, the full time-dependent problem in 4+2 dimensions is solved in Sect. 3. Finally, in Sect. 4 we discuss the question of reducing the problem to fewer dimensions by explicitly eliminating one of the two time variables.

2 Spectral Analysis of the Time-Independent Part of the Lax Pair

In this section we will derive non-linear Fourier transform pairs [30] tailor-made for solving the 3-wave interaction equations (1.3), by performing the spectral analysis of the following eigenvalue equation, which is the time-independent part of the Lax pair associated with the 3-wave interaction equations in $4+2$ dimensions:

$$\mu_{\overline{x}} - J\mu_{\overline{y}} - k[J,\mu] - Q\mu = 0. \tag{2.1}$$

The matrix μ is a 3×3 matrix valued function which depends on the six real variables $(x_1, x_2, y_1, y_2, k_1, k_2)$, the eigenvalue k is a complex spectral variable, and J and Q are defined by

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}, \qquad Q(x_1, x_2, y_1, y_2) = \begin{pmatrix} 0 & q_{12} & q_{13} \\ q_{21} & 0 & q_{23} \\ q_{31} & q_{32} & 0 \end{pmatrix},$$

where $J_1, J_2, J_3 \in \mathbb{R}\backslash\{0\}$, with $J_1 > J_2 > J_3$ and $q_{ab}(x_1, x_2, y_1, y_2)$, $a, b = 1, 2, 3$, are complex-valued functions which are sufficiently smooth and which decay rapidly enough for large values of the spatial variables. We observe that Eq. (2.1) can be written in component form as

$$\mu_{ab_{\overline{x}}} - J_a\mu_{ab_{\overline{y}}} - k\,(J_a - J_b)\,\mu_{ab} - (Q\mu)_{ab} = 0. \tag{2.2}$$

Let us introduce some notation:

$$\begin{aligned}
&x = x_1 + ix_2,\ \ y = y_1 + iy_2,\ \ \xi = \xi_1 + i\xi_2,\\
&\eta = \eta_1 + i\eta_2,\ \ k = k_1 + ik_2,\ \ \lambda = \lambda_1 + i\lambda_2,\\
&\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2,\ \ \mathrm{d}y = \mathrm{d}y_1\mathrm{d}y_2,\ \ \mathrm{d}\xi = \mathrm{d}\xi_1\mathrm{d}\xi_2,\\
&\mathrm{d}\eta = \mathrm{d}\eta_1\mathrm{d}\eta_2,\ \ \mathrm{d}k = \mathrm{d}k_1\mathrm{d}k_2 \text{ and } \mathrm{d}\lambda = \mathrm{d}\lambda_1\mathrm{d}\lambda_2,\\
&\text{where } x_1, x_2, y_1, y_2, \xi_1, \xi_2, \eta_1, \eta_2, k_1, k_2, \lambda_1, \lambda_2 \in \mathbb{R}.
\end{aligned}$$

Also, we shall generally write $f(k, \lambda, x, y)$ instead of $f(k_1, k_2, \lambda_1, \lambda_2, x_1, x_2, y_1, y_2)$.

At this point, it is important to emphasise that, although our approach is based on the complexification of the Lax pair of the 3-wave interaction equations, the resulting non-linear integrable system (1.3) is not the 3-wave interaction equations with 2+1 complex variables, but a genuine 4+2 dimensional system with six real variables. Namely, it does *not* depend on $x_1 + ix_2$, $y_1 + iy_2$, $t_1 + it_2$ but on $x_1, x_2, y_1, y_2, t_1, t_2$. This fact is not surprising. Indeed, in general if an equation involves a complex parameter, the solution depends on both the real and the imaginary parts of this parameter. For example, the time-independent part of the Lax pair Eq. (2.1) or equivalently Eq. (2.2) depends among other things on the complex variable $k = k_1 + ik_2$, but the associated eigenfunctions μ_{ab} depend on the real variables k_1, k_2. An explicit demonstration that the type of equations constructed here are genuinely 4+2 dimensional can be found in [31]; in this paper explicit solutions of the 4+2 Davey–Stewartson equation are analysed which clearly depend on 4 real spatial variables and on 2 real time variables.

We shall derive the non-linear Fourier transform pairs for the 4+2 dimensional 3-wave interaction equations in two ways.

2.1 First Method: Fourier Transform Formulation

In the above Eq. (2.2), the d-bar derivatives appear with respect to both complex spatial variables x and y. As an important first step towards solving the system Eq. (2.2), we will start by disentangling these derivatives. This can be achieved by introducing the following local coordinates $v_1^{(a)} (= v_1)$, $v_2^{(a)}$, $v_3^{(a)}$ and $v_4^{(a)}$:

$$\begin{aligned} &x_1 = v_1, \quad x_2 = v_2^{(a)} + v_4^{(a)}, \quad y_1 = v_3^{(a)} - J_a v_1, \quad y_2 = -J_a v_4^{(a)}, \\ &v_1 = x_1, \quad v_2^{(a)} = x_2 + \frac{1}{J_a} y_2, \quad v_3^{(a)} = y_1 + J_a x_1, \quad v_4^{(a)} = -\frac{1}{J_a} y_2, \end{aligned} \qquad \text{for } a = 1, 2, 3. \tag{2.3}$$

Hence,

$$\partial_{\overline{x}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial v_1} + J_a \frac{\partial}{\partial v_3^{(a)}} + i\frac{\partial}{\partial v_2^{(a)}}\right), \tag{2.4a}$$

$$\partial_{\overline{y}} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right) = \frac{1}{2}\left[\frac{\partial}{\partial v_3^{(a)}} + i\left(\frac{1}{J_a}\frac{\partial}{\partial v_2^{(a)}} - \frac{1}{J_a}\frac{\partial}{\partial v_4^{(a)}}\right)\right]. \tag{2.4b}$$

Defining the new complex variables

$$z^{(a)} := v_1 + i v_4^{(a)}, \text{ for } a = 1, 2, 3, \tag{2.5}$$

and using Eqs. (2.4a)-(2.4b), we find

$$\frac{\partial}{\partial \overline{z}^{(a)}} = \frac{1}{2}\left(\frac{\partial}{\partial v_1} + i\frac{\partial}{\partial v_4^{(a)}}\right) = \partial_{\overline{x}} - J_a \partial_{\overline{y}}. \tag{2.6}$$

Hence, Eq. (2.2) can be written in the desired disentangled form:

$$\frac{\partial \mu_{ab}}{\partial \overline{z}^{(a)}} - k\,(J_a - J_b)\,\mu_{ab} - (Q\mu)_{ab} = 0. \tag{2.7}$$

Taking into consideration the requirement of boundedness, we rewrite (2.7) in the form

$$\frac{\partial}{\partial \overline{z}^{(a)}}\left(\mu_{ab}\, e^{-k(J_a - J_b)\overline{z}^{(a)} + \overline{k}(J_a - J_b)z^{(a)}}\right) = e^{-k(J_a - J_b)\overline{z}^{(a)} + \overline{k}(J_a - J_b)z^{(a)}}\,(Q\mu)_{ab}\,. \tag{2.8}$$

In order to solve the direct problem we look for a solution of Eq. (2.8) such that $\mu \to I$ as the spatial variables tend to infinity, where I denotes the unit matrix. Using the Pompeiu formula [27] and the above boundary condition, we obtain

$$\mu_{ab} = \delta_{ab} - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{(Q\mu)_{ab}\ e^{-k(J_a-J_b)\left(\bar{\zeta}-\bar{z}^{(a)}\right)+\bar{k}(J_a-J_b)\left(\zeta-z^{(a)}\right)}}{\zeta - z^{(a)}}\mathrm{d}\zeta. \tag{2.9}$$

Differentiating Eq. (2.9) with respect to $\bar{k}$ we find

$$\begin{aligned}\frac{\partial \mu_{ab}}{\partial \bar{k}} = &-\frac{(J_a - J_b)}{\pi}\int_{\mathbb{R}^2} (Q\mu)_{ab}\ e^{-k(J_a-J_b)\left(\bar{\zeta}-\bar{z}^{(a)}\right)+\bar{k}(J_a-J_b)\left(\zeta-z^{(a)}\right)}\mathrm{d}\zeta \\ &- \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\left(Q\frac{\partial \mu}{\partial \bar{k}}\right)_{ab}\ e^{-k(J_a-J_b)\left(\bar{\zeta}-\bar{z}^{(a)}\right)+\bar{k}(J_a-J_b)\left(\zeta-z^{(a)}\right)}}{\zeta - z^{(a)}}\mathrm{d}\zeta,\end{aligned} \tag{2.10}$$

where

$$\mu_{ab} = \mu_{ab}\left(v_1, v_2^{(a)} + v_4^{(a)}, v_3^{(a)} - J_a v_1, -J_a v_4^{(a)}, k_1, k_2\right), \tag{2.11}$$

$$\begin{aligned}(Q\mu)_{ab} = \sum_{\substack{n=1\\ n\neq a}}^{3} q_{an}\left(\zeta_1, v_2^{(a)} + \zeta_2, v_3^{(a)} - J_a\zeta_1, -J_a\zeta_2\right)\mu_{nb}\left(\zeta_1, v_2^{(a)} + \zeta_2, v_3^{(a)}\right. \\ \left. -J_a\zeta_1, -J_a\zeta_2, k_1, k_2\right),\end{aligned} \tag{2.12}$$

$$\zeta = \zeta_1 + i\zeta_2 \text{ and } \mathrm{d}\zeta = \mathrm{d}\zeta_1\mathrm{d}\zeta_2. \tag{2.13}$$

Our aim is to complement the above equations for the μ_{ab}'s (which are written in terms of the q_{ab}'s) with an additional set of equations for the same μ_{ab}'s, but written in terms of appropriate "spectral functions" of the q_{ab}'s. In order to obtain this additional set of equations for the μ_{ab}'s, we shall construct a d-bar problem in which the d-bar derivatives $\frac{\partial \mu_{ab}}{\partial \bar{k}}$ are written in terms of the μ_{ab}'s alone (as opposed to Eq. (2.10), where the expressions on the right hand side contain both the μ_{ab}'s and the $\frac{\partial \mu_{ab}}{\partial \bar{k}}$'s). To achieve this, we need to relate Eqs. (2.9) and (2.10). In this connection, we first introduce appropriate notation for the functions appearing as forcing in (2.10),

$$\widetilde{f}_{ab}\left(v_2^{(a)}, v_3^{(a)}, k\right) := -\frac{(J_a - J_b)}{\pi}\int_{\mathbb{R}^2} (Q\mu)_{ab}\ e^{-k(J_a-J_b)\bar{\zeta}+\bar{k}(J_a-J_b)\zeta}\mathrm{d}\zeta_1\mathrm{d}\zeta_2, \tag{2.14}$$

where $(Q\mu)_{ab}$ is as in (2.12).

We can write $\widetilde{f}_{ab}\left(v_2^{(a)}, v_3^{(a)}, k\right)$ as the two dimensional Fourier transform of the function H_{ab}:

$$\widetilde{f}_{ab}\left(v_2^{(a)}, v_3^{(a)}, k\right) = \int_{\mathbb{R}^2} e^{\lambda\overline{w}^{(a)} - \bar{\lambda}w^{(a)}} H_{ab}(k, \lambda)\mathrm{d}\lambda, \tag{2.15}$$

where $w^{(a)} = v_3^{(a)} + i v_2^{(a)}$.
We next introduce the following new variables (the reasons for this particular choice will become clear later):

$$\lambda_1 = J_a \lambda_1' - J_b k_1, \qquad \lambda_2 = \lambda_2' - k_2. \tag{2.16}$$

Using the new variables, Eq. (2.15) takes the form

$$\begin{aligned}
&\widetilde{f}_{ab}\left(v_2^{(a)}, v_3^{(a)}, k\right)\\
&= |J_a| \int_{\mathbb{R}^2} e^{2i[-(J_a\lambda_1' - J_b k_1)v_2^{(a)} + (\lambda_2' - k_2)v_3^{(a)}]} H_{ab}(k_1, k_2, J_a\lambda_1' - J_b k_1, \lambda_2' - k_2)\mathrm{d}\lambda_1'\mathrm{d}\lambda_2'\\
&= |J_a| \int_{\mathbb{R}^2} e^{2i[-(J_a\lambda_1 - J_b k_1)v_2^{(a)} + (\lambda_2 - k_2)v_3^{(a)}]} H_{ab}(k_1, k_2, J_a\lambda_1 - J_b k_1, \lambda_2 - k_2)\mathrm{d}\lambda_1\mathrm{d}\lambda_2.
\end{aligned} \tag{2.17}$$

Denoting $f_{ab}(k, \lambda) := H_{ab}(k_1, k_2, J_a\lambda_1 - J_b k_1, \lambda_2 - k_2)$, by inverting equation (2.17) and taking into account equation (2.14), we obtain the following expression which can be considered as the non-linear Fourier transform for the problem under consideration:

$$f_{ab}(k, \lambda) = -\frac{(J_a - J_b)}{\pi^3} \int_{\mathbb{R}^4} \overline{E}_{ab}(k, \lambda, z^{(a)}, w^{(a)})\,(Q\mu)_{ab}\,\mathrm{d}z^{(a)}\mathrm{d}w^{(a)}, \tag{2.18}$$

where we have defined $E_{ab}(k, \lambda, z^{(a)}, w^{(a)})$ by

$$E_{ab}(k, \lambda, z^{(a)}, w^{(a)}) := e^{k(J_a - J_b)\overline{z}^{(a)} - \overline{k}(J_a - J_b)z^{(a)} + 2i[-(J_a\lambda_1 - J_b k_1)v_2^{(a)} + (\lambda_2 - k_2)v_3^{(a)}]}, \tag{2.19}$$

and where

$$\begin{aligned}
(Q\mu)_{ab} = \sum_{\substack{n=1\\ n\neq a}}^{3} q_{an}&\left(v_1, v_2^{(a)} + v_4^{(a)}, v_3^{(a)} - J_a v_1, -J_a v_4^{(a)}\right)\mu_{nb}\left(v_1, v_2^{(a)} + v_4^{(a)}, v_3^{(a)}\right.\\
&\left. - J_a v_1, -J_a v_4^{(a)}, k_1, k_2\right).
\end{aligned} \tag{2.20}$$

We next rewrite everything in the original coordinates and make the change of variables

$$\zeta_1 = \xi_1, \qquad -J_a \zeta_2 = \xi_2. \tag{2.21}$$

Equation (2.9) then becomes

$$\begin{aligned}
&\mu_{ab}(x, y, k)\\
&= \delta_{ab} - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{(Q\mu)_{ab}\ e^{-k(J_a - J_b)\left((\xi_1 - x_1) + \frac{i}{J_a}(\xi_2 - y_2)\right) + \overline{k}(J_a - J_b)\left((\xi_1 - x_1) - \frac{i}{J_a}(\xi_2 - y_2)\right)}}{|J_a|\,[(\xi_1 - x_1) + \frac{i}{J_a}(y_2 - \xi_2)]}\mathrm{d}\xi_1\mathrm{d}\xi_2.
\end{aligned} \tag{2.22}$$

Taking into account equation (2.17) and the definition of $f_{ab}(k,\lambda)$, Eq. (2.10) becomes

$$\begin{aligned}&\frac{\partial\mu_{ab}}{\partial\bar{k}}(x,y,k)\\&=|J_a|\int_{\mathbb{R}^2}\Bigg[e^{2i\left[-(J_a\lambda_1-J_bk_1)\left(x_2+\frac{1}{J_a}y_2\right)+(\lambda_2-k_2)(y_1+J_ax_1)+k_2(J_a-J_b)x_1+k_1\frac{(J_a-J_b)}{J_a}y_2\right]}\\&\times f_{ab}(k,\lambda)\Bigg]\mathrm{d}\lambda_1\mathrm{d}\lambda_2\\&-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\left(Q\frac{\partial\mu}{\partial\bar{k}}\right)_{ab}e^{-k(J_a-J_b)\left((\xi_1-x_1)+\frac{i}{J_a}(\xi_2-y_2)\right)+\bar{k}(J_a-J_b)\left((\xi_1-x_1)-\frac{i}{J_a}(\xi_2-y_2)\right)}}{|J_a|\left[(\xi_1-x_1)+\frac{i}{J_a}(y_2-\xi_2)\right]}\mathrm{d}\xi_1\mathrm{d}\xi_2,\end{aligned}\tag{2.23}$$

where in Eqs. (2.22), (2.23) we have

$$\begin{aligned}(Q\mu)_{ab}=&\left(Q\left(\xi_1,x_2+\frac{1}{J_a}(y_2-\xi_2),y_1+J_a(x_1-\xi_1),\xi_2\right)\mu\left(\xi_1,x_2\right.\right.\\&\left.\left.+\frac{1}{J_a}(y_2-\xi_2),y_1+J_a(x_1-\xi_1),\xi_2,k_1,k_2\right)\right)_{ab},\end{aligned}$$

and

$$\begin{aligned}\left(Q\frac{\partial\mu}{\partial\bar{k}}\right)_{ab}=&\left(Q\left(\xi_1,x_2+\frac{1}{J_a}(y_2-\xi_2),y_1+J_a(x_1-\xi_1),\xi_2\right)\frac{\partial\mu}{\partial\bar{k}}\left(\xi_1,x_2\right.\right.\\&\left.\left.+\frac{1}{J_a}(y_2-\xi_2),y_1+J_a(x_1-\xi_1),\xi_2,k_1,k_2\right)\right)_{ab}.\end{aligned}$$

Also, Eq. (2.18), written in the original coordinates, becomes

$$f_{ab}(k,\lambda)=-\frac{J_a-J_b}{|J_a|\,\pi^3}\int_{\mathbb{R}^4}\overline{E}_{ab}(k,\lambda,x,y)\,(Q(x,y)\mu(x,y,k))_{ab}\,\mathrm{d}x\mathrm{d}y,\tag{2.24}$$

with the function E_{ab} taking the form

$$\begin{aligned}E_{ab}(k,\lambda,x,y):&=\exp[(J_a\lambda-J_bk)\bar{x}-(J_a\bar{\lambda}-J_b\bar{k})x+(\lambda-k)\bar{y}-(\bar{\lambda}-\bar{k})y]\\&=\exp[2i\,((-J_a\lambda_1+J_bk_1)x_2+(J_a\lambda_2-J_bk_2)x_1\\&\quad+(k_1-\lambda_1)y_2+(\lambda_2-k_2)y_1)].\end{aligned}\tag{2.25}$$

This expression for E_{ab} in the original coordinates justifies the specific transformation (2.16) that we used to mix the spectral variables k and λ. The important thing to note here is the approximately symmetric role played by k and λ in this expression (2.25). This will allow us to replace k by λ in several steps of our analysis, which will prove to be very convenient.

We will now construct a d-bar problem via Eqs. (2.22) and (2.23). For concreteness, we concentrate on the second column of the matrix $\frac{\partial\mu}{\partial\bar{k}}$. Multiplying Eq. (2.23) for

$a = 1, 2, 3$ and $b = 2$ (second column) by the factor $e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}$, we find

$$e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}\frac{\partial\mu_{12}}{\partial\bar{k}}(x,y,k)$$
$$= |J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1x_2+J_1\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)} f_{12}(k,\lambda)\mathrm{d}\lambda$$
$$-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\left(q_{12}\frac{\partial\mu_{22}}{\partial\bar{k}}+q_{13}\frac{\partial\mu_{32}}{\partial\bar{k}}\right)e^{2i\left[-J_2k_1\left(x_2+\frac{1}{J_1}(y_2-\xi_2)\right)+J_2k_2\xi_1-k_1\xi_2+k_2(y_1+J_1(x_1-\xi_1))\right]}}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.26a}$$

$$e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}\frac{\partial\mu_{22}}{\partial\bar{k}}(x,y,k)$$
$$= -\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\left(q_{21}\frac{\partial\mu_{12}}{\partial\bar{k}}+q_{23}\frac{\partial\mu_{32}}{\partial\bar{k}}\right)e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}}{|J_2|\,[(\xi_1-x_1)+\frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.26b}$$

$$e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}\frac{\partial\mu_{32}}{\partial\bar{k}}(x,y,k)$$
$$= |J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)} f_{32}(k,\lambda)\mathrm{d}\lambda$$
$$-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\left(q_{31}\frac{\partial\mu_{12}}{\partial\bar{k}}+q_{32}\frac{\partial\mu_{22}}{\partial\bar{k}}\right)e^{2i\left[-J_2k_1\left(x_2+\frac{1}{J_3}(y_2-\xi_2)\right)+J_2k_2\xi_1-k_1\xi_2+k_2(y_1+J_3(x_1-\xi_1))\right]}}{|J_3|\,[(\xi_1-x_1)+\frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi. \tag{2.26c}$$

Multiplying Eq. (2.22) for $a = 1, 2, 3$ and $b = 1$ by $e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}$, we find

$$e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}\mu_{11}(x,y,k)$$
$$= e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}$$
$$-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{12}\mu_{21}+q_{13}\mu_{31})\,e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.27a}$$

$$e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}\mu_{21}(x,y,k)$$
$$= -\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{21}\mu_{11}+q_{23}\mu_{31})\;e^{2i\left[-J_1k_1\left(x_2+\frac{1}{J_2}(y_2-\xi_2)\right)+J_1k_2\xi_1-k_1\xi_2+k_2(y_1+J_2(x_1-\xi_1))\right]}}{|J_2|\,[(\xi_1-x_1)+\frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.27b}$$

$$e^{2i(-J_1k_1x_2+J_1k_2x_1-k_1y_2+k_2y_1)}\mu_{31}(x,y,k)$$
$$= -\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{31}\mu_{11}+q_{32}\mu_{21})\;e^{2i\left[-J_1k_1\left(x_2+\frac{1}{J_3}(y_2-\xi_2)\right)+J_1k_2\xi_1-k_1\xi_2+k_2(y_1+J_3(x_1-\xi_1))\right]}}{|J_3|\,[(\xi_1-x_1)+\frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi. \tag{2.27c}$$

In each of the above three equations we replace k by λ, multiply by $|J_1|\ f_{12}(k,\lambda)$ and integrate over dλ. Thus, we obtain the following three equations:

$$\begin{aligned}
&|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1 x_2+J_1\lambda_2 x_1-\lambda_1 y_2+\lambda_2 y_1)}\mu_{11}(x,y,\lambda) f_{12}(k,\lambda)\mathrm{d}\lambda\\
&=|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1 x_2+J_1\lambda_2 x_1-\lambda_1 y_2+\lambda_2 y_1)} f_{12}(k,\lambda)\mathrm{d}\lambda\\
&\quad-\frac{|J_1|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{12}\mu_{21}+q_{13}\mu_{31})\, e^{2i(-J_1\lambda_1 x_2+J_1\lambda_2 x_1-\lambda_1 y_2+\lambda_2 y_1)} f_{12}(k,\lambda)}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda,
\end{aligned} \tag{2.28a}$$

$$\begin{aligned}
&|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1 x_2+J_1\lambda_2 x_1-\lambda_1 y_2+\lambda_2 y_1)}\ \mu_{21}(x,y,\lambda) f_{12}(k,\lambda)\mathrm{d}\lambda\\
&=-\frac{|J_1|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{21}\mu_{11}+q_{23}\mu_{31})\ e^{2i\left[-J_1\lambda_1\left(x_2+\frac{1}{J_2}(y_2-\xi_2)\right)+J_1\lambda_2\xi_1-\lambda_1\xi_2+\lambda_2(y_1+J_2(x_1-\xi_1))\right]} f_{12}(k,\lambda)}{|J_2|\,[(\xi_1-x_1)+\frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda,
\end{aligned} \tag{2.28b}$$

$$\begin{aligned}
&|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1 x_2+J_1\lambda_2 x_1-\lambda_1 y_2+\lambda_2 y_1)}\ \mu_{31}(x,y,\lambda) f_{12}(k,\lambda)\mathrm{d}\lambda\\
&=-\frac{|J_1|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{31}\mu_{11}+q_{32}\mu_{21})\ e^{2i\left[-J_1\lambda_1\left(x_2+\frac{1}{J_3}(y_2-\xi_2)\right)+J_1\lambda_2\xi_1-\lambda_1\xi_2+\lambda_2(y_1+J_3(x_1-\xi_1))\right]} f_{12}(k,\lambda)}{|J_3|\,[(\xi_1-x_1)+\frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda.
\end{aligned} \tag{2.28c}$$

We apply a similar procedure to the third column of Eq. (2.22) (with $a = 1, 2, 3$ and $b = 3$): multiplying these three equations by $e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}$, we find

$$\begin{aligned}
&e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}\ \mu_{13}(x,y,k)\\
&\quad=-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{12}\mu_{23}+q_{13}\mu_{33})\ e^{2i\left[-J_3k_1\left(x_2+\frac{1}{J_1}(y_2-\xi_2)\right)+J_3k_2\xi_1-k_1\xi_2+k_2(y_1+J_1(x_1-\xi_1))\right]}}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi,
\end{aligned} \tag{2.29a}$$

$$\begin{aligned}
&e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}\mu_{23}(x,y,k)\\
&\quad=-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{21}\mu_{13}+q_{23}\mu_{33})\ e^{2i\left[-J_3k_1\left(x_2+\frac{1}{J_2}(y_2-\xi_2)\right)+J_3k_2\xi_1-k_1\xi_2+k_2(y_1+J_2(x_1-\xi_1))\right]}}{|J_2|\,[(\xi_1-x_1)+\frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi,
\end{aligned} \tag{2.29b}$$

$$\begin{aligned}
&e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}\mu_{33}(x,y,k)\\
&\quad=e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}\\
&\qquad-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{(q_{31}\mu_{13}+q_{32}\mu_{23})\, e^{2i(-J_3k_1x_2+J_3k_2x_1-k_1y_2+k_2y_1)}}{|J_3|\,[(\xi_1-x_1)+\frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi.
\end{aligned} \tag{2.29c}$$

In Eqs. (2.29a)-(2.29c) we replace k by λ and then multiply by $|J_3|\ f_{32}(k,\lambda)$ and integrate over dλ. In this way we obtain the following three equations:

$$|J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}\,\mu_{13}(x,y,\lambda)f_{32}(k,\lambda)\mathrm{d}\lambda$$
$$=-\frac{|J_3|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{12}\mu_{23}+q_{13}\mu_{33})\;e^{2i\left[-J_3\lambda_1\left(x_2+\frac{1}{J_1}(y_2-\xi_2)\right)+J_3\lambda_2\xi_1-\lambda_1\xi_2+\lambda_2(y_1+J_1(x_1-\xi_1))\right]}f_{32}(k,\lambda)}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda,\tag{2.30a}$$

$$|J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}\mu_{23}(x,y,\lambda)f_{32}(k,\lambda)\mathrm{d}\lambda$$
$$=-\frac{|J_3|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{21}\mu_{13}+q_{23}\mu_{33})\;e^{2i\left[-J_3\lambda_1\left(x_2+\frac{1}{J_2}(y_2-\xi_2)\right)+J_3\lambda_2\xi_1-\lambda_1\xi_2+\lambda_2(y_1+J_2(x_1-\xi_1))\right]}f_{32}(k,\lambda)}{|J_2|\,[(\xi_1-x_1)+\frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda,\tag{2.30b}$$

$$\begin{aligned}&|J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}\mu_{33}(x,y,\lambda)f_{32}(k,\lambda)\mathrm{d}\lambda\\&=|J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}f_{32}(k,\lambda)\mathrm{d}\lambda\\&\quad-\frac{|J_3|}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(q_{31}\mu_{13}+q_{32}\mu_{23})\,e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}f_{32}(k,\lambda)}{|J_3|\,[(\xi_1-x_1)+\frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi\mathrm{d}\lambda.\end{aligned}\tag{2.30c}$$

At this point, we use the similarity of the kernels of Eqs. (2.26), (2.28) and (2.30) to arrive at the desired d-bar problem. Introducing the notations

$$\begin{aligned}&\widetilde{M}_{a2}(x,y,k)\\&=e^{2i(-J_2k_1x_2+J_2k_2x_1-k_1y_2+k_2y_1)}\,\frac{\partial\mu_{a2}}{\partial\bar{k}}(x,y,k),\quad a=1,2,3,\end{aligned}\tag{2.31}$$

$$\begin{aligned}&M_{a1}(x,y,k)\\&=|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1x_2+J_1\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}\mu_{a1}(x,y,\lambda)f_{12}(k,\lambda)\mathrm{d}\lambda,\quad a=1,2,3,\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}&M_{a3}(x,y,k)\\&=|J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1x_2+J_3\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}\mu_{a3}(x,y,\lambda)f_{32}(k,\lambda)\mathrm{d}\lambda,\quad a=1,2,3,\end{aligned}\tag{2.33}$$

equations (2.26) can be rewritten as follows:

Component 12

$$\begin{aligned}\widetilde{M}_{12}(x,y,k)&=|J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1x_2+J_1\lambda_2x_1-\lambda_1y_2+\lambda_2y_1)}f_{12}(k,\lambda)\mathrm{d}\lambda\\&-\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{\left(q_{12}\widetilde{M}_{22}+q_{13}\widetilde{M}_{32}\right)}{|J_1|\,[(\xi_1-x_1)+\frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi,\end{aligned}\tag{2.34a}$$

Component 22 $\qquad \widetilde{M}_{22}(x,y,k) = -\dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}^2} \dfrac{\left(q_{21}\widetilde{M}_{12} + q_{23}\widetilde{M}_{32}\right)}{|J_2|\,[(\xi_1 - x_1) + \frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi,$

(2.34b)

Component 32

$$\widetilde{M}_{32}(x,y,k) = |J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1 x_2 + J_3\lambda_2 x_1 - \lambda_1 y_2 + \lambda_2 y_1)} f_{32}(k,\lambda)\mathrm{d}\lambda \\ -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\left(q_{31}\widetilde{M}_{12} + q_{32}\widetilde{M}_{22}\right)}{|J_3|[(\xi_1 - x_1) + \frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.34c}$$

where in the integrals of equations (2.34a)–(2.34c) the elements depend on the variables as follows (for $a = 1, 2, 3,\ b = 1, 2, 3$ with $a \neq b$)

$$q_{ab}\widetilde{M}_{b2} = q_{ab}\left(\xi_1, x_2 + \frac{y_2-\xi_2}{J_a}, y_1 + J_a(x_1-\xi_1), \xi_2\right) \\ \times \widetilde{M}_{b2}\left(\xi_1, x_2 + \frac{y_2-\xi_2}{J_a}, y_1 + J_a(x_1-\xi_1), \xi_2, k_1, k_2\right).$$

Adding Eq. (2.28a) to (2.30a), Eq. (2.28b) to (2.30b) and Eq. (2.28c) to (2.30c), and using the notations introduced in (2.32) and (2.33), we obtain the following three expressions:

Sum of 11 and 13

$$M_{11}(x,y,k) + M_{13}(x,y,k) = |J_1|\int_{\mathbb{R}^2} e^{2i(-J_1\lambda_1 x_2 + J_1\lambda_2 x_1 - \lambda_1 y_2 + \lambda_2 y_1)} f_{12}(k,\lambda)\mathrm{d}\lambda \\ -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{(q_{12}\,(M_{21}+M_{23}) + q_{13}\,(M_{31}+M_{33}))}{|J_1|\,[(\xi_1 - x_1) + \frac{i}{J_1}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.35a}$$

Sum of 21 and 23

$$M_{21}(x,y,k) + M_{23}(x,y,k) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{(q_{21}\,(M_{11}+M_{13}) + q_{23}\,(M_{31}+M_{33}))}{|J_2|\,[(\xi_1 - x_1) + \frac{i}{J_2}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.35b}$$

Sum of 31 and 33

$$M_{31}(x,y,k) + M_{33}(x,y,k) = |J_3|\int_{\mathbb{R}^2} e^{2i(-J_3\lambda_1 x_2 + J_3\lambda_2 x_1 - \lambda_1 y_2 + \lambda_2 y_1)} f_{32}(k,\lambda)\mathrm{d}\lambda \\ -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{(q_{31}\,(M_{11}+M_{13}) + q_{32}\,(M_{21}+M_{23}))}{|J_3|\,[(\xi_1 - x_1) + \frac{i}{J_3}(y_2-\xi_2)]}\mathrm{d}\xi, \tag{2.35c}$$

where in the integrals of equations (2.35a)–(2.35c) the elements depend on the variables as follows (for $a = 1, 2, 3, b = 1, 2, 3$ with $a \neq b$ and $j = 1, 3$)

$$q_{ab} M_{bj} = q_{ab}\left(\xi_1, x_2 + \frac{y_2 - \xi_2}{J_a}, y_1 + J_a(x_1 - \xi_1), \xi_2\right) \\ \times M_{bj}\left(\xi_1, x_2 + \frac{y_2 - \xi_2}{J_a}, y_1 + J_a(x_1 - \xi_1), \xi_2, k_1, k_2\right).$$

Comparing Eqs. (2.34a)-(2.34c) with Eqs. (2.35a)-(2.35c), we find

$$\widetilde{M}_{12} = M_{11} + M_{13}, \quad \widetilde{M}_{22} = M_{21} + M_{23}, \quad \widetilde{M}_{32} = M_{31} + M_{33},$$

which in the original notation yields the second column of the d-bar problem:

$$\frac{\partial \mu_{a2}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \sum_{\substack{n=1 \\ n \neq 2}}^{3} |J_n|\, \mu_{an}(x, y, \lambda) f_{n2}(k, \lambda) E_{n2}(k, \lambda, x, y) \mathrm{d}\lambda, \quad a = 1, 2, 3. \tag{2.36}$$

In a similar way, following the same procedure for the first and third columns of the matrix $\frac{\partial \mu}{\partial \bar{k}}$, we arrive at the full d-bar problem for the 3-wave interaction equations in $4 + 2$ dimensions:

$$\frac{\partial \mu_{ab}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n|\, \mu_{an}(x, y, \lambda) f_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}\lambda, \quad a, b = 1, 2, 3, \tag{2.37}$$

where the E_{nb}'s are defined in (2.25).
The above d-bar problem can be written in a more concise form as

$$\frac{\partial \mu}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \mu(x, y, \lambda) F(k, \lambda, x, y) \mathrm{d}\lambda, \tag{2.38}$$

where $F(k, \lambda, x, y)$ is the 3×3 off-diagonal matrix with its ab-th entry equal to

$$F_{ab}(k, \lambda, x, y) = |J_a|\, f_{ab}(k, \lambda) E_{ab}(k, \lambda, x, y), \quad \text{for } a \neq b.$$

Using the Pompeiu formula, and also using the condition that $\mu \to I + O\left(\frac{1}{k}\right)$ in the limit $k \to \infty$, Eq. (2.37) yields

$$\mu_{ab}(x, y, k) = \delta_{ab} + \frac{1}{\pi} \int_{\mathbb{R}^4} \sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n|\, \mu_{an}(x, y, \lambda) f_{nb}(k', \lambda) E_{nb}(k', \lambda, x, y) \frac{\mathrm{d}k' \mathrm{d}\lambda}{k - k'}. \tag{2.39}$$

In summary, Eq. (2.22) expresses μ_{ab} in terms of the q_{ab}'s, whereas Eq. (2.39) expresses μ_{ab} in terms of the f_{ab}'s. We can now obtain a relation between the q_{ab}'s and f_{ab}'s by noting that in the limit $k \to \infty$ Eqs. (2.39) and (2.2) imply, respectively, the following relations:

$$\mu_{ab}(x, y, k) \sim \delta_{ab} + \frac{1}{\pi k} \int_{\mathbb{R}^4} \sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n| \, \mu_{an}(x, y, \lambda) f_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}k \mathrm{d}\lambda + O\left(\frac{1}{k^2}\right) \tag{2.40}$$

and, for $a \neq b$,

$$\begin{aligned} \mu_{ab}(x, y, k) &\sim -\frac{1}{k(J_a - J_b)} q_{ab}(x, y) \mu_{bb}(x, y, k) + O\left(\frac{1}{k^2}\right) \\ &\sim -\frac{1}{k(J_a - J_b)} q_{ab}(x, y) + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \end{aligned} \tag{2.41}$$

The $O\left(\frac{1}{k}\right)$ terms of Eqs. (2.40) and (2.41) yield

$$q_{ab}(x, y) = -\frac{J_a - J_b}{\pi} \int_{\mathbb{R}^4} \sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n| \, \mu_{an}(x, y, \lambda) f_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}k \mathrm{d}\lambda. \tag{2.42}$$

These above expressions for the q_{ab}'s in terms of the f_{ab}'s, together with the associated expressions for the f_{ab}'s in terms of the q_{ab}'s given by (2.24), define the non-linear Fourier transform pair needed for the solution of the Cauchy problem of the 4+2 3-wave interaction equations.

2.2 Second Method: Green's Function Formulation

In order to solve the direct problem we look for a solution of Eq. (2.2) such that $\mu \to I$ as the spatial variables tend to infinity, where I denotes the unit matrix. A solution of Eq. (2.2) statisfying the above condition is given by the following equation

$$\mu_{ab}(x, y, k) = \delta_{ab} + \int_{\mathbb{R}^4} G_{ab}(x - x', y - y', k) \left(Q(x', y')\mu(x', y', k)\right)_{ab} \, \mathrm{d}x'\mathrm{d}y', \tag{2.43}$$

where we require that the Green's function $G(x, y, k)$ satisfies

$$\begin{aligned} &G_{ab_{\overline{x}}}(x, y, k) - J_a G_{ab_{\overline{y}}}(x, y, k) - k\,(J_a - J_b)\, G_{ab}(x, y, k) \\ &= \delta(x)\delta(y) = \frac{1}{\pi^4} \int_{\mathbb{R}^4} \mathrm{e}^{\rho x - \overline{\rho x} + \sigma y - \overline{\sigma y}} \mathrm{d}\rho \mathrm{d}\sigma, \end{aligned} \tag{2.44}$$

where $\rho = \rho_1 + i\rho_2$, $\sigma = \sigma_1 + i\sigma_2$, $\mathrm{d}\rho = \mathrm{d}\rho_1 \mathrm{d}\rho_2$, $\mathrm{d}\sigma = \mathrm{d}\sigma_1 \mathrm{d}\sigma_2$ and we have used the notation $\delta(x) = \delta(x_1)\delta(x_2)$, $\delta(y) = \delta(y_1)\delta(y_2)$. Hence, from Eq. (2.44) we have

$$G_{ab}(x, y, k) = \frac{1}{\pi^4} \int_{\mathbb{R}^4} \frac{\mathrm{e}^{\rho x - \overline{\rho x} + \sigma y - \overline{\sigma y}}}{-\overline{\rho} + J_a \overline{\sigma} - k(J_a - J_b)} \mathrm{d}\rho \mathrm{d}\sigma. \tag{2.45}$$

Our aim again is to complement the above equations for the μ_{ab}'s with another set of equations for the same μ_{ab}'s, but now written in terms of appropriate "spectral functions" of the q_{ab}'s, which we will denote by $\hat{q}_{ab}$. In order to obtain this additional set of equations for the μ_{ab}'s, we shall construct a d-bar problem. Differentiating Eq. (2.45) with respect to $\bar{k}$ we find, for $a \neq b$

$$\frac{\partial G_{ab}}{\partial \bar{k}}(x, y, k) = \frac{1}{\pi^4} \int_{\mathbb{R}^4} \frac{e^{\rho x - \overline{\rho x} + \sigma y - \overline{\sigma y}}}{-(J_a - J_b)} \frac{\partial}{\partial \bar{k}} \left(\frac{1}{-\frac{-\bar{\rho} + J_a \bar{\sigma}}{J_a - J_b} + k} \right) \mathrm{d}\rho \mathrm{d}\sigma. \tag{2.46}$$

Using the formula

$$\frac{\partial}{\partial \bar{z}} \left(\frac{1}{z - \zeta} \right) = \pi \delta(z - \zeta) = \pi \delta(z_1 - \zeta_1) \delta(z_2 - \zeta_2),$$

where $z = z_1 + i z_2$ and $\zeta = \zeta_1 + i \zeta_2$, with $z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{R}$, we obtain

$$\frac{\partial}{\partial \bar{k}} \left(\frac{1}{k - \frac{-\bar{\rho} + J_a \bar{\sigma}}{J_a - J_b}} \right) = \pi \delta \left(k_1 - \frac{-\rho_1 + J_a \sigma_1}{J_a - J_b} \right) \delta \left(k_2 - \frac{\rho_2 - J_a \sigma_2}{J_a - J_b} \right). \tag{2.47}$$

So combining Eqs. (2.46) and (2.47) we get

$$\begin{aligned}
&\frac{\partial G_{ab}}{\partial \bar{k}}(x, y, k) \\
&= \frac{1}{\pi^4} \int_{\mathbb{R}^4} \frac{e^{\rho x - \overline{\rho x} + \sigma y - \overline{\sigma y}}}{-(J_a - J_b)} \pi \delta \left(k_1 - \frac{-\rho_1 + J_a \sigma_1}{J_a - J_b} \right) \delta \left(k_2 - \frac{\rho_2 - J_a \sigma_2}{J_a - J_b} \right) \mathrm{d}\rho \mathrm{d}\sigma \\
&= \frac{1}{\pi^3} \int_{\mathbb{R}^4} \Bigg[\frac{e^{\rho x - \overline{\rho x} + \sigma y - \overline{\sigma y}}}{-(J_a - J_b)} \left(\frac{J_a - J_b}{J_a} \right)^2 \delta \left(\frac{J_a - J_b}{J_a} k_1 + \frac{1}{J_a} \rho_1 - \sigma_1 \right) \\
&\quad \times \delta \left(-\frac{J_a - J_b}{J_a} k_2 + \frac{1}{J_a} \rho_2 - \sigma_2 \right) \Bigg] \mathrm{d}\rho \mathrm{d}\sigma \\
&= -\frac{1}{\pi^3} \frac{J_a - J_b}{(J_a)^2} \int_{\mathbb{R}^2} e^{2i(\rho_1 x_2 + \rho_2 x_1 + \sigma_1 y_2 + \sigma_2 y_1)} \Bigg|_{\substack{\sigma_1 = \frac{1}{J_a}[(J_a - J_b)k_1 + \rho_1] \\ \sigma_2 = -\frac{1}{J_a}[(J_a - J_b)k_2 - \rho_2]}} \mathrm{d}\rho \\
&= -\frac{1}{\pi^3} \frac{J_a - J_b}{(J_a)^2} \int_{\mathbb{R}^2} e^{2i\left\{\rho_1 x_2 + \rho_2 x_1 + \frac{1}{J_a}[(J_a - J_b)k_1 + \rho_1] y_2 - \frac{1}{J_a}[(J_a - J_b)k_2 - \rho_2] y_1\right\}} \mathrm{d}\rho,
\end{aligned}$$

where in the second step we have used the identity $\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$, $\alpha \in \mathbb{R} \backslash \{0\}$.
With the change of variables

$$\rho_1 = -J_a \lambda_1 + J_b k_1, \quad \rho_2 = J_a \lambda_2 - J_b k_2,$$

we find

$$\frac{\partial G_{ab}}{\partial \bar{k}}(x,y,k) = -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^2} e^{2i[(-J_a\lambda_1+J_bk_1)x_2+(J_a\lambda_2-J_bk_2)x_1+(k_1-\lambda_1)y_2+(\lambda_2-k_2)y_1]}d\lambda$$
$$= -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^2} e^{\left[(J_a\lambda-J_bk)\bar{x}-(J_a\bar{\lambda}-J_b\bar{k})x+(\lambda-k)\bar{y}-(\bar{\lambda}-\bar{k})y\right]}d\lambda$$
$$= -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^2} E_{ab}(k,\lambda,x,y)d\lambda, \tag{2.48}$$

where E_{ab} is defined in (2.25). Equation (2.48) was found for $a \neq b$, but this equation is also valid for $a = b$. Indeed, since Eq. (2.45) does not depend on k or $\bar{k}$ when $a = b$, we have that $\frac{\partial G_{aa}}{\partial \bar{k}}(x,y,k) = 0$, $a = 1, 2, 3$, which is consistent with Eq. (2.48).

Differentiating Eq. (2.43) with respect to $\bar{k}$ we get

$$\begin{aligned}
&\frac{\partial \mu_{ab}}{\partial \bar{k}}(x,y,k)\\
&= \int_{\mathbb{R}^4} \frac{\partial G_{ab}}{\partial \bar{k}}(x-x', y-y', k)\left(Q(x',y')\mu(x',y',k)\right)_{ab}\, dx'dy'\\
&\quad + \int_{\mathbb{R}^4} G_{ab}(x-x', y-y', k)\left(Q(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k)\right)_{ab} dx'dy'\\
&\overset{(2.48)}{=} -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^4}\int_{\mathbb{R}^2} E_{ab}(k,\lambda,x-x',y-y')\left(Q(x',y')\mu(x',y',k)\right)_{ab} d\lambda\, dx'dy'\\
&\quad + \int_{\mathbb{R}^4} G_{ab}(x-x', y-y', k)\left(Q(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k)\right)_{ab} dx'dy'\\
&= -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^4} E_{ab}(k,\lambda,x,y)\overline{E}_{ab}(k,\lambda,x',y')\left(Q(x',y')\mu(x',y',k)\right)_{ab}\, dx'dy'd\lambda\\
&\quad + \int_{\mathbb{R}^4} G_{ab}(x-x', y-y', k)\left(Q(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k)\right)_{ab} dx'dy'.
\end{aligned}$$

By defining

$$\hat{q}_{ab}(k,\lambda) := -\frac{J_a - J_b}{\pi^3}\int_{\mathbb{R}^4} \overline{E}_{ab}(k,\lambda,x,y)\left(Q(x,y)\mu(x,y,k)\right)_{ab} dxdy, \tag{2.49}$$

we have

$$\frac{\partial \mu_{ab}}{\partial \bar{k}}(x,y,k) = \int_{\mathbb{R}^2} E_{ab}(k,\lambda,x,y)\hat{q}_{ab}(k,\lambda)d\lambda$$
$$+ \int_{\mathbb{R}^4} G_{ab}(x-x', y-y', k)\left(Q(x',y')\frac{\partial \mu}{\partial \bar{k}}(x',y',k)\right)_{ab} dx'dy'. \tag{2.50}$$

Now, we shall again construct a d-bar problem, this time via equations (2.43) and (2.50).

Multiplying equations (2.43), for $a = 1, 2, 3$ and $b = 1$ with k replaced by λ, by $\hat{q}_{12}(k,\lambda)E_{12}(k,\lambda,x,y)$ and integrating over $d\lambda$, we find

$$\int_{\mathbb{R}^2} \mu_{11}(x, y, \lambda)\hat{q}_{12}(k, \lambda)E_{12}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2} E_{12}(k, \lambda, x, y)\hat{q}_{12}(k, \lambda)\mathrm{d}\lambda$$
$$+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{11}(x - x', y - y', \lambda)E_{12}(k, \lambda, x, y)\hat{q}_{12}(k, \lambda)\left[q_{12}(x', y')\mu_{21}(x', y', \lambda)\right.$$
$$\left. + q_{13}(x', y')\mu_{31}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda, \tag{2.51a}$$

$$\int_{\mathbb{R}^2} \mu_{21}(x, y, \lambda)\hat{q}_{12}(k, \lambda)E_{12}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{21}(x - x', y - y', \lambda)E_{12}(k, \lambda, x, y)\hat{q}_{12}(k, \lambda)\left[q_{21}(x', y')\mu_{11}(x', y', \lambda)\right.$$
$$\left. + q_{23}(x', y')\mu_{31}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda, \tag{2.51b}$$

$$\int_{\mathbb{R}^2} \mu_{31}(x, y, \lambda)\hat{q}_{12}(k, \lambda)E_{12}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{31}(x - x', y - y', \lambda)E_{12}(k, \lambda, x, y)\hat{q}_{12}(k, \lambda)\left[q_{31}(x', y')\mu_{11}(x', y', \lambda)\right.$$
$$\left. + q_{32}(x', y')\mu_{21}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda. \tag{2.51c}$$

Now, multiplying (2.43), for $a = 1, 2, 3$ and $b = 3$ with k replaced by λ, by $\hat{q}_{32}(k, \lambda)E_{32}(k, \lambda, x, y)$ and integrating over $\mathrm{d}\lambda$, we find

$$\int_{\mathbb{R}^2} \mu_{13}(x, y, \lambda)\hat{q}_{32}(k, \lambda)E_{32}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{13}(x - x', y - y', \lambda)E_{32}(k, \lambda, x, y)\hat{q}_{32}(k, \lambda)\left[q_{12}(x', y')\mu_{23}(x', y', \lambda)\right.$$
$$\left. + q_{13}(x', y')\mu_{33}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda, \tag{2.52a}$$
$$\int_{\mathbb{R}^2} \mu_{23}(x, y, \lambda)\hat{q}_{32}(k, \lambda)E_{32}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{23}(x - x', y - y', \lambda)E_{32}(k, \lambda, x, y)\hat{q}_{32}(k, \lambda)\left[q_{21}(x', y')\mu_{13}(x', y', \lambda)\right.$$
$$\left. + q_{23}(x', y')\mu_{33}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda, \tag{2.52b}$$
$$\int_{\mathbb{R}^2} \mu_{33}(x, y, \lambda)\hat{q}_{32}(k, \lambda)E_{32}(k, \lambda, x, y)\mathrm{d}\lambda$$
$$= \int_{\mathbb{R}^2} E_{32}(k, \lambda, x, y)\hat{q}_{32}(k, \lambda)\mathrm{d}\lambda$$
$$+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^4} G_{33}(x - x', y - y', \lambda)E_{32}(k, \lambda, x, y)\hat{q}_{32}(k, \lambda)\left[q_{31}(x', y')\mu_{13}(x', y', \lambda)\right.$$
$$\left. + q_{32}(x', y')\mu_{23}(x', y', \lambda)\right]\mathrm{d}x'\mathrm{d}y'\mathrm{d}\lambda. \tag{2.52c}$$

Adding equation (2.51a) to (2.52a), equation (2.51b) to (2.52b) and equation (2.51c) to (2.52c), we obtain the following three expressions:

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left[\mu_{11}(x,y,\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)+\mu_{13}(x,y,\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\right]\mathrm{d}\lambda\\
&=\int_{\mathbb{R}^2}E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mathrm{d}\lambda\\
&\quad+\int_{\mathbb{R}^4}\Bigg[q_{12}(x',y')\left(\int_{\mathbb{R}^2}G_{11}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{21}(x',y',\lambda)\right.\\
&\quad\left.+G_{13}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{23}(x',y',\lambda)\mathrm{d}\lambda\right)\\
&\quad+q_{13}(x',y')\left(\int_{\mathbb{R}^2}G_{11}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{31}(x',y',\lambda)\right.\\
&\quad\left.+G_{13}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{33}(x',y',\lambda)\mathrm{d}\lambda\right)\Bigg]\mathrm{d}x'\mathrm{d}y',
\end{aligned}
\tag{2.53a}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left[\mu_{21}(x,y,\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)+\mu_{23}(x,y,\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\right]\mathrm{d}\lambda\\
&=\int_{\mathbb{R}^4}\Bigg[q_{21}(x',y')\left(\int_{\mathbb{R}^2}G_{21}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{11}(x',y',\lambda)\right.\\
&\quad\left.+G_{23}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{13}(x',y',\lambda)\mathrm{d}\lambda\right)\\
&\quad+q_{23}(x',y')\left(\int_{\mathbb{R}^2}G_{21}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{31}(x',y',\lambda)\right.\\
&\quad\left.+G_{23}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{33}(x',y',\lambda)\mathrm{d}\lambda\right)\Bigg]\mathrm{d}x'\mathrm{d}y',
\end{aligned}
\tag{2.53b}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left[\mu_{31}(x,y,\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)+\mu_{33}(x,y,\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\right]\mathrm{d}\lambda\\
&=\int_{\mathbb{R}^2}E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mathrm{d}\lambda\\
&\quad+\int_{\mathbb{R}^4}\Bigg[q_{31}(x',y')\left(\int_{\mathbb{R}^2}G_{31}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{11}(x',y',\lambda)\right.\\
&\quad\left.+G_{33}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{13}(x',y',\lambda)\mathrm{d}\lambda\right)\\
&\quad+q_{32}(x',y')\left(\int_{\mathbb{R}^2}G_{31}(x-x',y-y',\lambda)E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mu_{21}(x',y',\lambda)\right.\\
&\quad\left.+G_{33}(x-x',y-y',\lambda)E_{32}(k,\lambda,x,y)\hat{q}_{32}(k,\lambda)\mu_{23}(x',y',\lambda)\mathrm{d}\lambda\right)\Bigg]\mathrm{d}x'\mathrm{d}y'.
\end{aligned}
\tag{2.53c}
$$

We now consider the second column of the matrix $\frac{\partial\mu}{\partial\bar{k}}$, i.e., the equations obtained from (2.50) for $b=2$:

$$
\begin{aligned}
\frac{\partial\mu_{12}}{\partial\bar{k}}(x,y,k)&=\int_{\mathbb{R}^2}E_{12}(k,\lambda,x,y)\hat{q}_{12}(k,\lambda)\mathrm{d}\lambda\\
&\quad+\int_{\mathbb{R}^4}G_{12}(x-x',y-y',k)\left(q_{12}(x',y')\frac{\partial\mu_{22}}{\partial\bar{k}}(x',y',k)\right.\\
&\quad\left.+q_{13}(x',y')\frac{\partial\mu_{32}}{\partial\bar{k}}(x',y',k)\right)\mathrm{d}x'\mathrm{d}y',
\end{aligned}
\tag{2.54a}
$$

$$\frac{\partial \mu_{22}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^4} G_{22}(x - x', y - y', k) \bigg(q_{21}(x', y') \frac{\partial \mu_{12}}{\partial \bar{k}}(x', y', k) + q_{23}(x', y') \frac{\partial \mu_{32}}{\partial \bar{k}}(x', y', k) \bigg) \mathrm{d}x'\mathrm{d}y', \tag{2.54b}$$

$$\frac{\partial \mu_{32}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} E_{32}(k, \lambda, x, y) \hat{q}_{32}(k, \lambda) \mathrm{d}\lambda + \int_{\mathbb{R}^4} G_{32}(x - x', y - y', k) \bigg(q_{31}(x', y') \frac{\partial \mu_{12}}{\partial \bar{k}}(x', y', k) + q_{32}(x', y') \frac{\partial \mu_{22}}{\partial \bar{k}}(x', y', k) \bigg) \mathrm{d}x'\mathrm{d}y'. \tag{2.54c}$$

Comparing equations (2.53) to equations (2.54) we find

$$\frac{\partial \mu_{12}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \Big[\mu_{11}(x, y, \lambda) E_{12}(k, \lambda, x, y) \hat{q}_{12}(k, \lambda) + \mu_{13}(x, y, \lambda) E_{32}(k, \lambda, x, y) \hat{q}_{32}(k, \lambda) \Big] \mathrm{d}\lambda, \tag{2.55a}$$

$$\frac{\partial \mu_{22}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \Big[\mu_{21}(x, y, \lambda) E_{12}(k, \lambda, x, y) \hat{q}_{12}(k, \lambda) + \mu_{23}(x, y, \lambda) E_{32}(k, \lambda, x, y) \hat{q}_{32}(k, \lambda) \Big] \mathrm{d}\lambda, \tag{2.55b}$$

$$\frac{\partial \mu_{32}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \Big[\mu_{31}(x, y, \lambda) E_{12}(k, \lambda, x, y) \hat{q}_{12}(k, \lambda) + \mu_{33}(x, y, \lambda) E_{32}(k, \lambda, x, y) \hat{q}_{32}(k, \lambda) \Big] \mathrm{d}\lambda, \tag{2.55c}$$

provided that

$$G_{12}(x - x', y - y', k) E_{12}(k, \lambda, x', y') = G_{11}(x - x', y - y', \lambda) E_{12}(k, \lambda, x, y), \tag{2.56a}$$

$$G_{12}(x - x', y - y', k) E_{32}(k, \lambda, x', y') = G_{13}(x - x', y - y', \lambda) E_{32}(k, \lambda, x, y), \tag{2.56b}$$

$$G_{22}(x - x', y - y', k) E_{12}(k, \lambda, x', y') = G_{21}(x - x', y - y', \lambda) E_{12}(k, \lambda, x, y), \tag{2.56c}$$

$$G_{22}(x - x', y - y', k) E_{32}(k, \lambda, x', y') = G_{23}(x - x', y - y', \lambda) E_{32}(k, \lambda, x, y), \tag{2.56d}$$

$$G_{32}(x - x', y - y', k) E_{12}(k, \lambda, x', y') = G_{31}(x - x', y - y', \lambda) E_{12}(k, \lambda, x, y), \tag{2.56e}$$

$$G_{32}(x - x', y - y', k) E_{32}(k, \lambda, x', y') = G_{33}(x - x', y - y', \lambda) E_{32}(k, \lambda, x, y). \tag{2.56f}$$

Thus, as a consistency check, we will now verify equations (2.56a) and (2.56b) (the other equations can be verified in a similar way). Equivalently, we want to show that

$$G_{12}(x,y,k) = G_{11}(x,y,\lambda)E_{12}(k,\lambda,x,y), \tag{2.57a}$$

$$G_{12}(x,y,k) = G_{13}(x,y,\lambda)E_{32}(k,\lambda,x,y). \tag{2.57b}$$

These conditions hold, since

(i) $G_{12}(x,y,k)$

$$\overset{(2.45)}{=} \frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{\rho x-\overline{\rho x}+\sigma y-\overline{\sigma y}}}{-\overline{\rho}+J_1\overline{\sigma}-k(J_1-J_2)}\mathrm{d}\rho\mathrm{d}\sigma$$

$$= \frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{\rho x-\overline{\rho x}+\sigma y-\overline{\sigma y}}}{(-\rho_1+J_1\sigma_1-k_1(J_1-J_2))+i\,(\rho_2-J_1\sigma_2-k_2(J_1-J_2))}\mathrm{d}\rho\mathrm{d}\sigma \tag{2.58}$$

We replace the variables $(\rho_1,\rho_2,\sigma_1,\sigma_2)$ with $(\rho_1+J_2k_1-J_1\lambda_1,\ \rho_2-J_2k_2+J_1\lambda_2,\ \sigma_1+k_1-\lambda_1,\ \sigma_2-k_2+\lambda_2)$. Then equation (2.58) becomes

$$\begin{aligned} G_{12}(x,y,k) &= \frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{2i(\rho_1x_2+\rho_2x_1+\sigma_1y_2+\sigma_2y_1)}}{(-\rho_1+J_1\sigma_1)+i\,(\rho_2-J_1\sigma_2)}\mathrm{d}\rho\mathrm{d}\sigma\ E_{12}(k,\lambda,x,y) \\ &= G_{11}(x,y,\lambda)E_{12}(k,\lambda,x,y) \end{aligned}$$

which verifies equation (2.57a) and consequently equation (2.56a).

(ii) We now replace the variables $(\rho_1,\rho_2,\sigma_1,\sigma_2)$ with $(\rho_1+J_2k_1-J_3\lambda_1,\ \rho_2-J_2k_2+J_3\lambda_2,\ \sigma_1+k_1-\lambda_1,\ \sigma_2-k_2+\lambda_2)$ in equation (2.58). Then (2.58) becomes

$$\begin{aligned} &G_{12}(x,y,k) \\ &= \frac{1}{\pi^4}\int_{\mathbb{R}^4} \frac{e^{2i(\rho_1x_2+\rho_2x_1+\sigma_1y_2+\sigma_2y_1)}}{(-\rho_1+J_1\sigma_1-\lambda_1(J_1-J_3))+i\,(\rho_2-J_1\sigma_2-\lambda_2(J_1-J_3))}\mathrm{d}\rho\mathrm{d}\sigma \\ &\quad\times E_{32}(k,\lambda,x,y) \\ &= G_{13}(x,y,\lambda)E_{32}(k,\lambda,x,y) \end{aligned}$$

which verifies equation (2.57b) and consequently equation (2.56b).

Hence, equations (2.55) hold and yield the second column of the d-bar problem

$$\frac{\partial\mu_{a2}}{\partial\bar{k}}(x,y,k) = \int_{\mathbb{R}^2}\sum_{\substack{n=1\\n\neq 2}}^{3}\mu_{an}(x,y,\lambda)\hat{q}_{n2}(k,\lambda)E_{n2}(k,\lambda,x,y)\mathrm{d}\lambda,\ a=1,2,3. \tag{2.59}$$

In a similar way, we arrive at the full d-bar problem for the 3-wave interaction equations in $4+2$ dimensions:

$$\frac{\partial \mu_{ab}}{\partial \bar{k}}(x, y, k) = \int_{\mathbb{R}^2} \sum_{\substack{n=1 \\ n \neq b}}^{3} \mu_{an}(x, y, \lambda) \hat{q}_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}\lambda. \tag{2.60}$$

Comparing equations (2.24) and (2.49), we observe that $\hat{q}_{ab}(k, \lambda) = |J_a| \, f_{ab}(k, \lambda)$. Hence, the above equation is exactly the same as equation (2.37), i.e., the d-bar problem we found in Sect. 2.1. Thus, we find again

$$\begin{aligned} q_{ab}(x, y) &= -\frac{J_a - J_b}{\pi} \int_{\mathbb{R}^4} \sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n| \, \mu_{an}(x, y, \lambda) f_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}k \mathrm{d}\lambda \\ &= -\frac{J_a - J_b}{\pi} \int_{\mathbb{R}^4} \sum_{\substack{n=1 \\ n \neq b}}^{3} \mu_{an}(x, y, \lambda) \hat{q}_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \mathrm{d}k \mathrm{d}\lambda. \end{aligned} \tag{2.61}$$

Hence, equations (2.61) and (2.49) comprise again a non-linear Fourier transform pair tailor-made for the solution of the Cauchy problem of the 4+2 3-wave interaction equations. This completes the analysis of the time-independent part of the Lax pair.

3 The Time-dependent Problem

Let us use the non-linear Fourier transform pair q_{ab} and f_{ab}, given by equations (2.42) and (2.24) respectively. Suppose that $f_{ab}(k, \lambda)$, $a, b = 1, 2, 3$ are allowed to depend on the complex variable t, where $t = t_1 + i t_2$, with $t_1, t_2 \in \mathbb{R}$. To avoid confusion, let $h_{ab}(k, \lambda, t)$ denote these functions. Then $q_{ab}(x, y)$ for $a, b = 1, 2, 3$ will also depend on the time variable t and we denote these functions by $g_{ab}(x, y, t)$.
Let us consider the following non-local d-bar problem:

$$\frac{\partial \mu}{\partial \bar{k}}(x, y, t, k) = \int_{\mathbb{R}^2} \mu(x, y, t, \lambda) F(k, \lambda, x, y, t) \mathrm{d}\lambda, \tag{3.1a}$$

$$\mu(x, y, t, k) = I + \frac{\mu^{(1)}(x, y, t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \tag{3.1b}$$

where

$$\begin{aligned} &F_{ab}(k, \lambda, x, y, t) \\ &\quad = |J_a| \, h_{ab}(k, \lambda, t) E_{ab}(k, \lambda, x, y) \\ &\quad = |J_a| \, f_{ab}(k, \lambda) \exp[(J_a \lambda - J_b k)\bar{x} - (J_a \bar{\lambda} - J_b \bar{k})x + (\lambda - k)\bar{y} - (\bar{\lambda} - \bar{k})y \\ &\qquad + (C_a \lambda - C_b k)\,\bar{t} - \left(C_a \bar{\lambda} - C_b \bar{k}\right) t], \quad \text{for } \ a, b = 1, 2, 3 \text{ with } a \neq b, \\ &\qquad \text{and } F_{aa}(k, \lambda, x, y, t) = 0, \quad \text{for } \ a = 1, 2, 3, \end{aligned} \tag{3.2}$$

where $C_1, C_2, C_3 \in \mathbb{R}\backslash\{0\}$ (with $C_1 > C_2 > C_3$), E_{ab}'s are defined in (2.25), f_{ab}'s in (2.24) and

$$h_{ab}(k, \lambda, t) = f_{ab}(k, \lambda) e^{(C_a\lambda - C_b k)\bar{t} - (C_a\bar{\lambda} - C_b\bar{k})t}. \tag{3.3}$$

We assume that the d-bar problem characterized by (3.1) has a unique solution. Which means that if we find two operators L and M such that (i) $L\mu$ and $M\mu$ satisfy Eq. (3.1a) and (ii) $L\mu$ and $M\mu$ are of $O\left(\frac{1}{k}\right)$ as $k \to \infty$, then we have $L\mu = 0$ and $M\mu = 0$. The above argument is the main idea of the dressing method (see e.g. [9,32]). Using the dressing method we will show that $\mu(x, y, t, k)$ satisfies the following eigenvalue equations:

$$L\mu = \mu_{\bar{x}} - J\mu_{\bar{y}} - k[J, \mu] - \widetilde{Q}\mu = 0, \tag{3.4}$$
$$M\mu = \mu_{\bar{t}} - C\mu_{\bar{y}} - k[C, \mu] - A\mu = 0, \tag{3.5}$$

where $\widetilde{Q}$ and A are off-diagonal 3×3 matrices defined in terms of μ by

$$\widetilde{Q} = \left[\mu^{(1)}, J\right], \quad A = \left[\mu^{(1)}, C\right], \tag{3.6}$$

where C is the diagonal 3×3 matrix with entries $C_1,\ C_2,\ C_3$. We note that Eqs. (3.6), by eliminating $\mu^{(1)}$, give that the ab-th entry of the 3×3 matrix A is equal to $\alpha_{ab}\widetilde{Q}_{ab}$, where $\alpha_{ab} := \frac{C_a - C_b}{J_a - J_b}$ for $a \neq b$.
Let us now introduce the following operators:

$$D_{\bar{x}}\mu := \mu_{\bar{x}} + k\mu J, \tag{3.7a}$$
$$D_{\bar{y}}\mu := \mu_{\bar{y}} + k\mu, \tag{3.7b}$$
$$D_{\bar{t}}\mu := \mu_{\bar{t}} + k\mu C. \tag{3.7c}$$

We shall show that

$$\left(D_{\bar{x}} - JD_{\bar{y}} - \widetilde{Q}\right)\mu = 0. \tag{3.8}$$

We do this in two steps:

(i) First we show that $\left(D_{\bar{x}} - JD_{\bar{y}} - \widetilde{Q}\right)\mu$ satisfies (3.1a). We observe that the operators $D_{\bar{x}},\ D_{\bar{y}}$ and $D_{\bar{t}}$, commute with the operator $\frac{\partial}{\partial\bar{k}}$. Hence, also using (3.2), we find

$$\frac{\partial}{\partial\bar{k}}\left(\left(D_{\bar{x}} - JD_{\bar{y}} - \widetilde{Q}\right)\mu(k)\right) = \left(D_{\bar{x}} - JD_{\bar{y}} - \widetilde{Q}\right)\left(\frac{\partial\mu}{\partial\bar{k}}(k)\right)$$
$$= \left(D_{\bar{x}} - JD_{\bar{y}} - \widetilde{Q}\right)\left(\int_{\mathbb{R}^2}\mu(\lambda)F(k, \lambda)\mathrm{d}\lambda\right)$$
$$= \int_{\mathbb{R}^2}\mu_{\bar{x}}(\lambda)F(k, \lambda) + \mu(\lambda)F_{\bar{x}}(k, \lambda) + k\mu(\lambda)F(k, \lambda)J - J\mu_{\bar{y}}(\lambda)F(k, \lambda)$$

$$
\begin{aligned}
&- J\mu(\lambda)F_{\overline{y}}(k,\lambda) - kJ\mu(\lambda)F(k,\lambda) - \widetilde{Q}\mu(\lambda)F(k,\lambda)\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^2} \mu_{\overline{x}}(\lambda)F(k,\lambda) + \mu(\lambda)\left(\lambda J F(k,\lambda) - kF(k,\lambda)J\right) + k\mu(\lambda)F(k,\lambda)J \\
&\quad - J\mu_{\overline{y}}(\lambda)F(k,\lambda) + J\mu(\lambda)(k-\lambda)F(k,\lambda) - kJ\mu(\lambda)F(k,\lambda) - \widetilde{Q}\mu(\lambda)F(k,\lambda)\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^2} \left[\mu_{\overline{x}}(\lambda) - J\mu_{\overline{y}}(\lambda) + \lambda\mu(\lambda)J - \lambda J\mu(\lambda) - \widetilde{Q}\mu(\lambda)\right] F(k,\lambda)\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^2} \left(\left(D_{\overline{x}} - JD_{\overline{y}} - \widetilde{Q}\right)\mu(\lambda)\right) F(k,\lambda)\mathrm{d}\lambda, \qquad (3.9)
\end{aligned}
$$

where for convenience we have surpressed the dependence of μ and F on $\{x, y, t\}$.

(ii) The second step is to recognize that

$$\left(D_{\overline{x}} - JD_{\overline{y}} - \widetilde{Q}\right)\mu = O\left(\frac{1}{k}\right) \text{ as } k \to \infty. \qquad (3.10)$$

Indeed, by taking into account the asymptotic expansion (3.1b) of μ, we obtain

$$\left(D_{\overline{x}} - JD_{\overline{y}}\right)\mu = \mu_{\overline{x}} - J\mu_{\overline{y}} - k[J,\mu] = \left[\mu^{(1)}, J\right] + O\left(\frac{1}{k}\right), \quad k \to \infty. \qquad (3.11)$$

From the definition (3.6) of $\widetilde{Q}$ in terms of μ, we conclude that (3.10) holds.

Therefore, by virtue of (3.9) and (3.10), we have indeed $\left(D_{\overline{x}} - JD_{\overline{y}} - \widetilde{Q}\right)\mu = 0$.
As a consequence, $\mu(x, y, t, k)$ satisfies (3.4).
In a similar way we can show that

$$\left(D_{\overline{t}} - CD_{\overline{y}} - A\right)\mu = 0. \qquad (3.12)$$

After expanding the operators, equation (3.12) takes the form (3.5).

Let us denote the ab-th entry of the matrix $\widetilde{Q}$ by $g_{ab}(x, y, t)$. Then by the note we made under (3.6), we have that the ab-th entry of A is equal to $\alpha_{ab}g_{ab}(x, y, t)$, where $\alpha_{ab} := \frac{C_a - C_b}{J_a - J_b}$ $(a \neq b)$. In the Appendix we show that (3.4) and (3.5) provide a Lax pair for the following 3-wave system in 4+2, i.e., in four spatial and two temporal dimensions

$$
\begin{aligned}
g_{ab_{\overline{t}}} &= \alpha_{ab}g_{ab_{\overline{x}}} + (C_a - J_a\alpha_{ab})g_{ab_{\overline{y}}} + (\alpha_{an} - \alpha_{nb})g_{an}g_{nb}, \\
&\text{for } a \neq b \text{ and } n \neq a, b, \quad \text{and } g_{aa} = 0, \qquad (3.13)
\end{aligned}
$$

where $a, b, n = 1, 2, 3$.
Using the non-linear Fourier transform pairs in four dimensions f_{ab} and q_{ab}, which are defined in equations (2.24) and (2.42), respectively, we can solve the Cauchy problem of equations (3.13) supplemented with the initial condition

$$g_{ab}(x_1, x_2, y_1, y_2, 0, 0) = q_{ab}(x_1, x_2, y_1, y_2), \quad \forall a, b = 1, 2, 3. \qquad (3.14)$$

Indeed, suppose that the functions q_{ab} are Schwartz functions in four dimensions which satisfy appropriate small norm conditions such that equations (2.22) and (2.39) have unique solutions. We define the f_{ab}'s by equations (2.24), then the h_{ab}'s are given by equations (3.3). We define $g_{ab}(x, y, t)$ by equations (2.42) where the f_{ab}'s are replaced by the corresponding h_{ab}'s. Then, the functions g_{ab} satisfy equation (3.13) and condition (3.14) for all $a, b = 1, 2, 3$. The above discussion implies that the solution of the Cauchy problem of the 3-wave interaction equations in 4+2 dimensions is given by

$$g_{ab}(x, y, t) = -\frac{J_a - J_b}{\pi} \int_{\mathbb{R}^4} \left[\sum_{\substack{n=1 \\ n \neq b}}^{3} |J_n| \, \mu_{an}(x, y, t, \lambda) f_{nb}(k, \lambda) E_{nb}(k, \lambda, x, y) \right. \\ \left. \times e^{(C_n\lambda - C_b k)\bar{t} - \left(C_n\bar{\lambda} - C_b\bar{k}\right)t} \right] \mathrm{d}k\mathrm{d}\lambda, \tag{3.15}$$

where the E_{nb}'s are defined in (2.25) and where the μ_{an}'s are the entries of the time-dependent 3×3 matrix-valued function μ which satisfies the non-local d-bar problem (3.1). Hence, these time-dependent μ_{an}'s are given by Eq. (2.9) with the f_{ab}'s replaced by the corresponding h_{ab}'s of Eq. (3.3).

4 On the Reduction to Fewer Dimensions

Having solved the problem in 4+2 dimensions, one would like to be able to reduce this to 3+1 dimensions in order to comply with the physical world. This is not a simple matter, as we will discuss in this section. A logical first step is to eliminate one of the two time variables, and this step at least can be accomplished in a straightforward manner. We show this by eliminating the t_1 variable, focusing in first instance on the relatively simple case of the linear limit, i.e., for $q_{ab} \to \epsilon q_{ab} + O(\epsilon^2)$, with $\epsilon \to 0$ and $a, b = 1, 2, 3$. The t_2 variable may be eliminated in analogous fashion. The present analysis may be compared with that in [29], where the reduction of the 4+2 dimensional Davey–Stewartson system to 3+1 dimensions was discussed.
In the linear limit we have, as can be inferred from Eq. (2.39),

$$\mu_{aa} \to 1, \quad \mu_{ab} \to 0, \quad \text{for } a \neq b, \tag{4.1}$$

implying that in Eq. (3.15) we now have only to deal with a single term out of the sum-over-n, i.e. only for $n = a$. Noting that the exponent appearing in Eq. (3.15) is of the following form (with $n = a$):

$$(C_a\lambda - C_b k)\,\bar{t} - \left(C_a\bar{\lambda} - C_b\bar{k}\right) t = 2i\left[(C_a\lambda_2 - C_b k_2)\,t_1 - (C_a\lambda_1 - C_b k_1)\,t_2\right], \tag{4.2}$$

we see that the t_1 variable can be eliminated by imposing the condition

$$C_a\lambda_2 - C_b k_2 = 0, \text{ i.e., } \lambda_2 = \frac{C_b}{C_a}k_2. \tag{4.3}$$

The t_2 variable may be eliminated in an analogous manner by imposing $C_a\lambda_1 - C_b k_1 = 0$. In what follows, however, we focus on the elimination of t_1.

In the linear limit, the 3-wave interaction equations decouple, so we only have to consider one of the equations (e.g. the one for $a = 1$ and $b = 2$) for a dependent variable denoted by u: $u(x_1, x_2, y_1, y_2, t_2)$. For the choice $a = 1$ and $b = 2$, the condition (4.3) becomes

$$\lambda_2 = \frac{C_2}{C_1}k_2. \tag{4.4}$$

Making in equations (2.42) and (2.24) the substitutions

$$\mu_{11} = 1,\ \mu_{13} = 0,\ \mu_{22} = 1,\ \mu_{32} = 0,\ f_{12}(k,\lambda) = \widetilde{f}(k,\lambda),\ q_{12}(x,y) = u(x,y;0), \tag{4.5}$$

we obtain the Fourier transform pair in four dimensions:

$$u(x,y;0) = -\frac{|J_1|(J_1-J_2)}{\pi}\int_{\mathbb{R}^4}\Big[e^{2i[(-J_1\lambda_1+J_2k_1)x_2+(J_1\lambda_2-J_2k_2)x_1+(k_1-\lambda_1)y_2+(\lambda_2-k_2)y_1)]} \times \widetilde{f}(k,\lambda)\Big]\mathrm{d}k\mathrm{d}\lambda, \tag{4.6}$$

$$\widetilde{f}(k,\lambda) = -\frac{(J_1-J_2)}{|J_1|\,\pi^3}\int_{\mathbb{R}^4}\Big[e^{-2i[(-J_1\lambda_1+J_2k_1)x_2+(J_1\lambda_2-J_2k_2)x_1+(k_1-\lambda_1)y_2+(\lambda_2-k_2)y_1)]} \times u(x,y;0)\Big]\mathrm{d}x\mathrm{d}y. \tag{4.7}$$

In order to impose the reduction (4.4) in the above pair, we make in equation (4.6) the substitution

$$\widetilde{f}(k,\lambda) = f(k,\lambda)\,\delta\left(\lambda_2 - \frac{C_2}{C_1}k_2\right). \tag{4.8}$$

Then, (4.6) becomes

$$u(x,y;0) = -\frac{|J_1|(J_1-J_2)}{\pi}\int_{\mathbb{R}^3}\Big[e^{2i\left[(-J_1\lambda_1+J_2k_1)x_2+\left(J_1\frac{C_2}{C_1}-J_2\right)k_2x_1+(k_1-\lambda_1)y_2+\left(\frac{C_2}{C_1}-1\right)k_2y_1)\right]} \times f(k_1,k_2,\lambda_1)\Big]\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}\lambda_1, \tag{4.9}$$

where $f(k_1,k_2,\lambda_1)$ denotes $f\left(k_1,k_2,\lambda_1,\frac{C_2}{C_1}k_2\right)$. Introducing the notation

$$\Phi(k_1,k_2,\lambda_1;x_1) = f(k_1,k_2,\lambda_1)e^{2i\left(J_1\frac{C_2}{C_1}-J_2\right)k_2x_1}, \tag{4.10}$$

equation (4.9) can be rewritten in the form

$$
\begin{aligned}
u(x,y;0) = &-\frac{|J_1|(J_1-J_2)}{\pi}\int_{\mathbb{R}^3}\Bigg[\mathrm{e}^{2i\left[(-J_1\lambda_1+J_2k_1)x_2+(k_1-\lambda_1)y_2+\left(\frac{C_2}{C_1}-1\right)k_2y_1\right]} \\
&\times\Phi(k_1,k_2,\lambda_1;x_1)\Bigg]\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}\lambda_1 \\
= &-\frac{|J_1|(J_1-J_2)}{\pi}\int_{\mathbb{R}^3}\Bigg[\mathrm{e}^{2i\left[(J_2x_2+y_2)k_1+\left(\frac{C_2}{C_1}-1\right)y_1k_2+(-J_1x_2-y_2)\lambda_1\right]} \\
&\times\Phi(k_1,k_2,\lambda_1;x_1)\Bigg]\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}\lambda_1. \qquad (4.11)
\end{aligned}
$$

Defining

$$\overrightarrow{p} := (k_1,k_2,\lambda_1) \equiv (p_1,p_2,p_3) \qquad (4.12)$$

and

$$\overrightarrow{r} := \left(2\left(J_2x_2+y_2\right), 2\left(\frac{C_2}{C_1}-1\right)y_1, 2\left(-J_1x_2-y_2\right)\right) \equiv (r_1,r_2,r_3). \qquad (4.13)$$

So we can write $u(x,y;0)$ as a function of $r_1,\ r_2,\ r_3$ and x_1, which we will denote by $h(r_1,r_2,r_3;x_1)$.
Hence, we can write equation (4.11) as

$$u(x,y;0) \equiv h(r_1,r_2,r_3;x_1) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{e}^{i\overrightarrow{p}\cdot\overrightarrow{r}}\left(-8\pi^2|J_1|(J_1-J_2)\Phi(\overrightarrow{p};x_1)\right)\mathrm{d}^3\overrightarrow{p}. \qquad (4.14)$$

Employing the inverse Fourier transform in the variables (r_1,r_2,r_3), the above equation yields

$$-8\pi^2|J_1|(J_1-J_2)\Phi(\overrightarrow{p};x_1) = \int_{\mathbb{R}^3}\mathrm{e}^{-i\overrightarrow{p}\cdot\overrightarrow{r}}h(r_1,r_2,r_3;x_1)\mathrm{d}^3\overrightarrow{r}. \qquad (4.15)$$

We want to change variables from $\mathrm{d}^3\overrightarrow{r}$ to $\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2$, hence we compute the Jacobian

$$\frac{\partial(r_1,r_2,r_3)}{\partial(x_2,y_1,y_2)} = 8\left(\frac{C_2}{C_1}-1\right)(J_1-J_2). \qquad (4.16)$$

Thus,

$$
\begin{aligned}
&\Phi(\overrightarrow{p};x_1) \\
&= -\frac{\left|\frac{C_2}{C_1}-1\right|}{\pi^2|J_1|}\int_{\mathbb{R}^3}\mathrm{e}^{-2i\left[(J_2x_2+y_2)k_1+\left(\frac{C_2}{C_1}-1\right)y_1k_2+(-J_1x_2-y_2)\lambda_1\right]}u(x,y;0)\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2. \qquad (4.17)
\end{aligned}
$$

Hence, recalling the definition of Φ from (4.10), the above equation becomes

$$f(k_1,k_2,\lambda_1) = -\frac{\left|\frac{C_2}{C_1}-1\right|}{\pi^2|J_1|}\int_{\mathbb{R}^3}\left[\mathrm{e}^{-2i\left[(-J_1\lambda_1+J_2k_1)x_2+\left(J_1\frac{C_2}{C_1}-J_2\right)k_2x_1+(k_1-\lambda_1)y_2+\left(\frac{C_2}{C_1}-1\right)k_2y_1)\right]}\right.$$
$$\left.\times\, u(x,y;0)\right]\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2. \tag{4.18}$$

The reduction (4.4) imposes the constraint

$$\partial_{x_1}u = \frac{J_1C_2 - J_2C_1}{C_2 - C_1}\,\partial_{y_1}u. \tag{4.19}$$

The "reduced" Fourier transform pair (4.9) and (4.18) can be used for the solution of the t_1-independent linearized version of equation (1.3) for $a = 1$ and $b = 2$, i.e. of the equation

$$\frac{i}{2}u_{t_2} = \alpha_{12}u_{\overline{x}} + (C_1 - J_1\alpha_{12})u_{\overline{y}}, \tag{4.20}$$

supplemented with the constraint (4.19). Indeed, using in (3.15) the substitutions (4.5) and (4.8) we find the equation

$$u(x,y,t_2)$$
$$= -\frac{|J_1|(J_1-J_2)}{\pi}\int_{\mathbb{R}^3}\left[\mathrm{e}^{2i\left[(-J_1\lambda_1+J_2k_1)x_2+\left(J_1\frac{C_2}{C_1}-J_2\right)k_2x_1+(k_1-\lambda_1)y_2+\left(\frac{C_2}{C_1}-1\right)k_2y_1)\right]}\right.$$
$$\left.\times e^{-2i(C_1\lambda_1-C_2k_1)t_2}f(k_1,k_2,\lambda_1)\right]\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}\lambda_1. \tag{4.21}$$

In summary, there exist two different ways to solve the system of equations (4.19) and (4.20). One way is to solve equations (4.19) and (4.20) using the novel Fourier transform pair (4.9) and (4.18): consider the Cauchy problem of equations (4.19) and (4.20), where

$$u(x_1,x_2,y_1,y_2,0) = u_0(x_1,x_2,y_1,y_2), \qquad x_1,x_2,y_1,y_2\in\mathbb{R}, \tag{4.22}$$

with u_0 a scalar function. Then, equation (4.21) with f defined in terms of u_0 by equation (4.18) where $u(x_1,x_2,y_1,y_2,0)$ is replaced by u_0, provides the solution of the above Cauchy problem. Indeed, it is straightforward to verify that if u is defined by the right hand side of (4.21), this function satisfies (4.19) and (4.20). Furthermore, evaluating (4.21) at $t_2 = 0$, we find (4.22) in lieu of the validity of the Fourier transform pair (4.9) and (4.18). Hence, the above defined function $u(x_1,x_2,y_1,y_2,t_2)$ solves the Cauchy problem.

The second way of solving Eqs. (4.19) and (4.20) is to use (4.19) to eliminate one of the space variables from (4.20) and then to use the standard Fourier transform, or alternatively the method of characteristics. For example, eliminating ∂_{x_1} from (4.20) we find

$$iu_{t_2} = \alpha_{12}\left(\frac{J_1C_2 - J_2C_1}{C_2 - C_1}\,u_{y_1} + iu_{x_2}\right) + (C_1 - J_1\alpha_{12})\left(u_{y_1} + iu_{y_2}\right) \tag{4.23}$$

and recalling that $\alpha_{12} = \frac{C_1-C_2}{J_1-J_2}$, we see that Eq. (4.23) simplifies further to

$$u_{t_2} = \alpha_{12}u_{x_2} + (C_1 - J_1\alpha_{12})u_{y_2}, \tag{4.24}$$

which has the form of the 2+1 dimensional linearized 3-wave interaction equations. This shows that, in the linear limit, the elimination of t_1 has brought about a greater reduction than intended; instead of a problem in 3+1 dimensions we are left with a 2+1 dimensional equation. That is to say, the evolution involving the variables x_2, y_2 and t_2 is described by Eq. (4.24), while the relation involving the x_1 and y_1 variables is governed by Eq. (4.19). This separation can also be inferred directly from the way in which the general solution (4.21), which satisfies both (4.19) and (4.24), depends on the four spatial variables x_1, x_2, y_1, y_2 and the time variable t_2. Namely, the exponential expression can be separated into one part containing x_1 and y_1 that involves only the spectral variable k_2, and a second part containing x_2, y_2 and t_2 that involves the remaining spectral variables k_1 and λ_1. In the end, then, Eqs. (4.19) and (4.24) can be solved independently of each other.

It is readily checked that the elimination of the t_2 variable leads to a similar conclusion. So the method presented above exceeds its original aim, giving an over-reduced set of equations, at least in the linear limit. This over-reduction appears to be a peculiarity of the 3-wave interaction equations, since the same method applied to the linear limit of the 4+2 dimensional Davey–Stewartson system yields a 3+1 dimensional result [29].

The non-linear problem is technically more difficult. In this case, in contrast to the linear limit, the μ_{an}'s do not simplify and hence, in the function g_{ab} of Eq. (3.15), we have to deal with *both* terms of the sum-over-n ($n = 1, 2, 3,\ \ n \neq b$). To eliminate t_1 (or t_2) from the exponential appearing in this function, we thus have to impose *two* conditions for each value of b, instead of the single condition (4.3) we had in the linear limit. Furthermore, since the equations remain coupled in the non-linear case, all three values $b = 1, 2, 3$ must be considered, giving six conditions in total. The implementation of these conditions, and their implications for the desired reduction of the full non-linear problem to 3+1 dimensions, are currently under investigation.

As a final remark, we note that the original 2+1 dimensional problem is easily recovered from the 4+2 dimensional version. Indeed, to accomplish *this* reduction, it is sufficient to assume that the functions g_{ab} appearing in Eq. (3.13) are independent of x_2, y_2 and t_2. Then Eq. (3.13) reduces to a non-linear problem in 2+1 dimensions, equivalent to the original N-wave interaction equations (1.1) for $N = 3$.

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Appendix

Here we shall show that equations (3.4) and (3.5) indeed provide a Lax pair for (3.13). In order to see this explicitly, let us prove that the compatibility condition

$$\mu_{\overline{t}\overline{x}} = \mu_{\overline{x}\overline{t}} \tag{5.1}$$

yields equation (3.13). Differentiating equation (3.4) with respect to $\overline{t}$ we get

$$\mu_{\overline{x}\overline{t}} - J\mu_{\overline{y}\overline{t}} - k\left[J, \mu_{\overline{t}}\right] - \widetilde{Q}_{\overline{t}}\mu - \widetilde{Q}\mu_{\overline{t}} = 0. \tag{5.2}$$

Likewise differentiating equation (3.5) with respect to $\overline{x}$ we get

$$\mu_{\overline{t}\overline{x}} - C\mu_{\overline{y}\overline{x}} - k\left[C, \mu_{\overline{x}}\right] - A_{\overline{x}}\mu - A\mu_{\overline{x}} = 0. \tag{5.3}$$

Hence, combining equations (5.1), (5.2) and (5.3) we obtain

$$J\mu_{\overline{y}\overline{t}} + kJ\mu_{\overline{t}} - k\mu_{\overline{t}}J + \widetilde{Q}_{\overline{t}}\mu + \widetilde{Q}\mu_{\overline{t}} = C\mu_{\overline{y}\overline{x}} + kC\mu_{\overline{x}} - k\mu_{\overline{x}}C + A_{\overline{x}}\mu + A\mu_{\overline{x}}. \tag{5.4}$$

We will use the following relations:

- $\mu_{\overline{y}\overline{t}} = C\mu_{\overline{y}\overline{y}} + k\left[C, \mu_{\overline{y}}\right] + A_{\overline{y}}\mu + A\mu_{\overline{y}}$, this equation is found by differentiating equation (3.5) with respect to $\overline{y}$ and using the compatibility condition $\mu_{\overline{y}\overline{t}} = \mu_{\overline{t}\overline{y}}$.
- $\mu_{\overline{t}} = C\mu_{\overline{y}} + k[C, \mu] + A\mu$, which is equation (3.5).
- $\mu_{\overline{y}\overline{x}} = J\mu_{\overline{y}\overline{y}} + k\left[J, \mu_{\overline{y}}\right] + \widetilde{Q}_{\overline{y}}\mu + \widetilde{Q}\mu_{\overline{y}}$, this equation was found by differentiating equation (3.4) with respect to $\overline{y}$ and using the compatibility condition $\mu_{\overline{x}\overline{y}} = \mu_{\overline{y}\overline{x}}$.
- $\mu_{\overline{x}} = J\mu_{\overline{y}} + k\left[J, \mu\right] + \widetilde{Q}\mu$, which is equation (3.4).

Substituting the above in equation (5.4) we arrive at

$$\begin{aligned}
&J\left(C\mu_{\overline{y}\overline{y}} + k\left[C, \mu_{\overline{y}}\right] + A_{\overline{y}}\mu + A\mu_{\overline{y}}\right) + kJ\left(C\mu_{\overline{y}} + k[C, \mu] + A\mu\right)\\
&- k\left(C\mu_{\overline{y}} + k[C, \mu] + A\mu\right)J + \widetilde{Q}_{\overline{t}}\mu + \widetilde{Q}\left(C\mu_{\overline{y}} + k[C, \mu] + A\mu\right)\\
&= C\left(J\mu_{\overline{y}\overline{y}} + k\left[J, \mu_{\overline{y}}\right] + \widetilde{Q}_{\overline{y}}\mu + \widetilde{Q}\mu_{\overline{y}}\right)
\end{aligned}$$

$$+ kC\left(J\mu_{\overline{y}} + k[J,\mu] + \widetilde{Q}\mu\right) - k\left(J\mu_{\overline{y}} + k[J,\mu] + \widetilde{Q}\mu\right)C$$
$$+ A_{\overline{x}}\mu + A\left(J\mu_{\overline{y}} + k[J,\mu] + \widetilde{Q}\mu\right).$$

Thus,

$$\widetilde{Q}_{\overline{t}} = A_{\overline{x}} + C\widetilde{Q}_{\overline{y}} - JA_{\overline{y}} + [A, \widetilde{Q}] \tag{5.5}$$

where we have used that $[J, A] = [C, \widetilde{Q}]$, i.e., $(J_a - J_b)A_{ab} = (C_a - C_b)g_{ab}$ which is true because of the definitions of the matrix A and the coefficient $\alpha_{ab} := \frac{C_a - C_b}{J_a - J_b}$. We observe that the component form of equation (5.5) gives us the 3-wave system (3.13) and, hence, equations (3.4) and (3.5) indeed provide a Lax pair for (3.13).

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