LINEAR INDEPENDENCE IN LINEAR SYSTEMS ON ELLIPTIC CURVES

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Abstract. Let E be an elliptic curve, with identity O, and let C be a cyclic subgroup of odd order N, over an algebraically closed field k with $\operatorname{char} k \nmid N$. For $P \in C$, let s_P be a rational function with divisor $N \cdot P - N \cdot O$. We ask whether the N functions s_P are linearly independent. For generic (E, C), we prove that the answer is yes. We bound the number of exceptional (E, C) when N is a prime by using the geometry of the universal generalized elliptic curve over $X_1(N)$. The problem can be recast in terms of sections of an arbitrary degree N line bundle on E.

1. Introduction

Fix $N \geq 1$ and an algebraically closed field k such that $\operatorname{char} k \nmid N$. Let E be an elliptic curve over k. Let $C \subset E$ be a cyclic subgroup of order N.

Let $\mathscr{L}$ be a degree N line bundle on E. Since $\operatorname{Pic}^0(E)$ is divisible, there exist points $P \in E$ such that $\mathscr{O}(N \cdot P) \simeq \mathscr{L}$, or equivalently, such that there exists a global section s_P of $\mathscr{L}$ whose divisor of zeros is $N \cdot P$. The set of such P is a coset $E[N]'$ of $E[N]$. Let $C' \subset E[N]'$ be a coset of C. Then $\#C' = N$. On the other hand, $\dim \Gamma(E, \mathscr{L}) = N$ by the Riemann-Roch theorem.

Question 1.1. Are the sections s_P for $P \in C'$ linearly independent in $\Gamma(E, \mathscr{L})$?

The answer is sometimes yes, sometimes no.

Example 1.2. Let $O \in E(k)$ be the identity. Let $\mathscr{L} = \mathscr{O}(N \cdot O)$ and $C' = C$. Then s_P is a rational function on E with divisor $(s_P) = N \cdot P - N \cdot O$. Question 1.1 asks whether the s_P for $P \in C$ are linearly independent, i.e., whether they form a basis of $\Gamma(E, \mathscr{O}(N \cdot O))$.

Proposition 1.3. *The answer to Question 1.1 depends only on (E, C), not on the choice of degree N line bundle $\mathscr{L}$ or coset C' or s_P for $P \in C'$. More precisely, the codimension of* $\operatorname{Span}\{s_P : P \in C'\}$ *in $\Gamma(E, \mathscr{L})$ depends only on (E, C).*

We will prove Proposition 1.3 in Section 3.

The pair (E, C) corresponds to a k-point on the classical modular curve $Y_0(N)$.

Theorem 1.4. *Let N be an odd positive integer such that* $\operatorname{char} k \nmid N$. *Then for all but finitely many $(E, C) \in Y_0(N)(k)$, Question 1.1 has an affirmative answer.*

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We next work towards a quantitative version of Theorem 1.4, at least for prime N. Let $c_{(E,C)}$ be the codimension in Proposition 1.3, and let $D = \sum_{(E,C)} c_{E,C}\,(E,C) \in \operatorname{Div} Y_0(N)$.

Theorem 1.5. *Let $N > 3$ be a prime with $\operatorname{char} k \nmid N$. There exist effective divisors D_1 and D_2 on $Y_0(N)$ such that $D = D_1 + 2D_2$ with*

$$\deg D_1 \le (N^2-1)/24$$
$$\deg D_2 \le (N-3)(N^2-1)/48.$$

Conjecture 1.6. If $\operatorname{char} k = 0$, then D_1 and D_2 are reduced and disjoint, and the inequalities in Theorem 1.5 are equalities.

Remark 1.7. Conjecture 1.6 is equivalent to the claim that for prime $N > 3$ and $\operatorname{char} k = 0$, there are exactly $(N^2-1)/24$ points $(E,C) \in Y_0(N)(k)$ with $c_{E,C} = 1$, exactly $(N-3)(N^2-1)/48$ points with $c_{E,C} = 2$, and no points with $c_{E,C} > 2$.

The primes $N > 3$ for which the genus of $X_0(N)$ is 0 are 5, 7, and 13; for these we checked that Conjecture 1.6 is true, using methods to be described in Section 10. There we will also show that Conjecture 1.6 sometimes fails when $\operatorname{char} k > 0$.

2. Notation

Let μ be the group of roots of unity in k. Fix a primitive Nth root of unity $\zeta \in k$.

If C is a finite abelian group with $\operatorname{char} k \nmid \#C$, and V is a k-representation of C, and $\chi\colon C \to k^\times$ is a character, define the χ-isotypic subspace

$$V^\chi := \{v \in V : cv = \chi(c)\,v \text{ for all } c \in C\}.$$

Let X be a regular finite-type k-scheme. Let $\operatorname{Div} X$ be its divisor group. Now suppose in addition that X is integral. Let $k(X)$ be its function field. If $f \in k(X)^\times$, let $(f) = (f)_X \in \operatorname{Div} X$ be its divisor. For each irreducible divisor Z on X, let v_Z be the associated valuation. A finite morphism of regular integral curves $\phi\colon X \to Y$ induces a homomorphism $\phi_*\colon \operatorname{Div} X \to \operatorname{Div} Y$ (sending each point to its image) compatible with the norm homomorphism $\phi_*\colon k(X)^\times \to k(Y)^\times$, and a homomorphism $\phi^*\colon \operatorname{Div} Y \to \operatorname{Div} X$ compatible with the homomorphism $\phi^*\colon k(Y)^\times \to k(X)^\times$ sending f to $f \circ \phi$.

3. Codimension is independent of choices

Proof of Proposition 1.3. Fix (E,C). Once $\mathscr{L}$ and C' are also fixed, each s_P is determined up to scaling by an element of $k^\times$, which does not change the span.

For each $Q \in E(k)$, let $\tau_Q\colon E \to E$ be the morphism sending x to $x+Q$. Pulling back by τ_Q shows that the codimension for $(\mathscr{L}, C')$ is the same as for $(\tau_Q^*\mathscr{L}, \tau_Q^{-1}(C'))$. If $Q \in E[N]$, then $\tau_Q^*\mathscr{L} \simeq \mathscr{L}$ but $\tau_Q^{-1}(C')$ can be any other coset of C' in $E[N]'$; thus the codimension is independent of C'. As Q ranges over $E(k)$, the line bundle $\tau_Q^*\mathscr{L}$ ranges over all degree N line bundles; thus the codimension is independent of $\mathscr{L}$ too. □

4. Normalized functions

If $f \in k(E)^\times$ has divisor supported on $E[N]$, then $[N]_*(f) = 0$, so $[N]_* f \in k^\times$. Multiplying f by a constant $a \in k^\times$ multiplies $[N]_* f$ by $a^{\deg [N]} = a^{N^2}$. Call $f \in k(E)^\times$ normalized if there exists $N \ge 1$ such that $[N]_* f \in \mu$. In that case, $[N']_* f \in \mu$ for all multiples N' of

N. Therefore the normalized functions form a subgroup of $k(E)^\times$. Given a principal divisor supported on torsion points, there exists a normalized function with that divisor, uniquely determined up to multiplication by a root of unity. In particular, a normalized constant rational function is an element of μ. If f is normalized and P is a torsion point on E, then $\tau_P^* f$ is normalized too.

5. Character-weighted combinations

From now on, we assume that N is odd. View C as a degree N divisor on E. Choose $\mathscr{L} := \mathscr{O}(C)$. The group C acts on $\mathscr{L}$: each P acts as τ_P^* on sections of $\mathscr{L}$. Since N is odd, $\mathscr{L} \simeq \mathscr{O}(N \cdot O)$. Choose $C' = C$. Choose sections s_P as in Section 1.

If we view s_O as a rational function on E, then $(s_O) = N \cdot O - C$. Assume that s_O is normalized. For $P \in C' = C$, we may assume that $s_P := \tau_{-P}^* s_O$. Then $\operatorname{Span}\{s_P : P \in C\}$ is the image of a kC-module homomorphism $kC \to \Gamma(E, \mathscr{L})$, so it decomposes as a direct sum of distinct characters. For each character $\chi\colon C \to k^\times$, the projection of $\operatorname{Span}\{s_P : P \in C\}$ onto $\Gamma(E, \mathscr{L})^\chi$ is spanned by

$$g_\chi := \left(\sum_{P \in C} \chi(P) \tau_{-P}^* \right) s_O = \sum_{P \in C} \chi(P)\, s_P.$$

Then $c_{E,C} = \#\{\chi : g_\chi = 0\}$.

Lemma 5.1. *We have $[-1]^* s_O = s_O$.*

Proof. The divisor (s_O) is fixed by $[-1]^*$, so s_O is an eigenvector of $[-1]^*$, with eigenvalue ± 1. Since $v_O(s_O)$ is even, the eigenvalue is 1. □

Lemma 5.2. *For each χ, we have $[-1]^* g_\chi = g_{\chi^{-1}}$.*

Proof. Apply

$$[-1]^* \left(\sum_{P \in C} \chi(P) \tau_{-P}^* \right) = \left(\sum_{P \in C} \chi(P) \tau_P^* \right) [-1]^* = \left(\sum_{Q \in C} \chi(-Q) \tau_{-Q}^* \right) [-1]^*$$

to s_O and use Lemma 5.1. □

Lemma 5.3. *We have $\prod_{P \in C} s_P \in \mu$.*

Proof. It is a normalized rational function whose divisor is 0. □

6. An almost canonical basis

Fix (E, C). Let $\phi\colon E \to E'$ be an isogeny with kernel C. Let $\hat{\phi}\colon E' \to E$ be the dual isogeny. The Weil pairing

$$e_\phi\colon \ker \phi \times \ker \hat{\phi} \to k^\times$$

is nondegenerate, so choosing $Q \in \ker \hat{\phi}$ is equivalent to choosing a character $\chi\colon C \to k^\times$, related via $\chi(P) = e_\phi(P, Q)$ for all $P \in C$. Let $C_\chi = \phi^* Q \in \operatorname{Div} E$. Let h_χ be a normalized function with $(h_\chi) = C_\chi - C$.

Lemma 6.1. *For $P \in C$, we have $\tau_P^* h_\chi = \chi(P)\, h_\chi$.*

Proof. This is the definition of $e_\phi(P, Q)$, which equals $\chi(P)$; see [Sil09, Exercise 3.15(a)]. □

Thus $0 \neq h_\chi \in \Gamma(E, \mathscr{L})^\chi$ for all χ, but $\bigoplus_\chi \Gamma(E, \mathscr{L})^\chi$ is N-dimensional, so $\Gamma(E, \mathscr{L})^\chi = k h_\chi$. In particular, $g_\chi / h_\chi \in k$. Now

$$c_{E,C} = \#\{\chi : g_\chi = 0\} = \#\{\chi : g_\chi/h_\chi = 0\}. \tag{1}$$

Lemma 6.2. *For each χ, we have $[-1]^* h_\chi \equiv h_{\chi^{-1}} \pmod{\mu}$.*

Proof. Compare divisors, and observe that both sides are normalized. □

Lemma 6.3. *For any χ, we have $g_\chi/h_\chi \equiv g_{\chi^{-1}}/h_{\chi^{-1}} \pmod{\mu}$.*

Proof. By Lemmas 5.2 and 6.2, $[-1]^*(g_\chi/h_\chi) \equiv g_{\chi^{-1}}/h_{\chi^{-1}} \pmod{\mu}$. On the other hand, g_χ/h_χ is constant on E, so $[-1]^*(g_\chi/h_\chi) = g_\chi/h_\chi$. □

7. The universal elliptic curve

Given an elliptic curve E over k and a point $P \in E(k)$ of exact order N, we define C as the subgroup generated by P. For $m \in \mathbb{Z}/N\mathbb{Z}$, let $\chi\colon C \to k^\times$ be the character such that $\chi(P) = \zeta^m$, and set $g_m := g_\chi$ and $h_m := h_\chi$. We may assume that $h_0 = 1$.

Suppose that $N > 3$ and $\operatorname{char} k \nmid N$. Then the moduli space $Y_1(N)$ parametrizing pairs (E, P) is a fine moduli space (it can be viewed as an étale quotient of the affine curve $Y(N)$ constructed by Igusa [Igu59], because a pair (E, P) consisting of an elliptic curve and a point of exact order $N > 3$ has no nontrivial automorphisms). Thus there is a universal elliptic curve $\mathscr{E} \to Y_1(N)$. The construction of s_O makes sense on $\mathscr{E}$, except that normalizing it may require taking an N^2th root of an invertible function on $Y_1(N)$. Thus s_O is a rational function not on the elliptic surface $\mathscr{E} \to Y_1(N)$, but on a pullback $\mathscr{E}' \to Y_1(N)'$ by some finite étale cover $Y_1(N)' \to Y_1(N)$. Then s_O^n for some $n \geq 1$ lies in $k(\mathscr{E})^\times$, and s_O itself may be identified with $\frac{1}{n} \otimes s_O^n \in \mathbb{Q} \otimes_{\mathbb{Z}} k(\mathscr{E})^\times$. Its divisor (s_O) is then an element of $\mathbb{Q} \otimes \operatorname{Div} \mathscr{E}$. Given $m \in \mathbb{Z}/NZ$, we may also define $g_m, h_m \in k(\mathscr{E}')^\times$ and consider them as elements of $\mathbb{Q} \otimes k(\mathscr{E})^\times$. Then g_m/h_m is a *regular* function on $Y_1(N)'$ and we may consider it an as element of $\mathbb{Q} \otimes k(Y_1(N))^\times$. Its divisor on $Y_1(N)$ lies in $\operatorname{Div} Y_1(N)$, not just $\mathbb{Q} \otimes \operatorname{Div} Y_1(N)$, since $Y_1(N)' \to Y_1(N)$ is finite étale.

8. The universal generalized elliptic curve

We continue to assume $N > 3$. Complete $Y_1(N)$ to a smooth projective curve $X_1(N)$ over k. One can recover from [DR73, IV.4.14 and VI.2.7] that $\mathscr{E} \to Y_1(N)$ can be completed to a "universal generalized elliptic curve" $\pi\colon \overline{\mathscr{E}} \to X_1(N)$. The following description of the cusps of $X_1(N)$ and the associated Tate curves is well-known; see [DR73, VII.2] and [FJ95, §3.1].

The cusps on $X_1(N)$ are in bijection with

$$\coprod_{d|N} \frac{(\mathbb{Z}/d\mathbb{Z})^\times \times (\mathbb{Z}/e\mathbb{Z})^\times}{\{\pm 1\}},$$

where $e = N/d$ in each term. The integer e equals the ramification index of $X_1(N) \to X(1)$ at the cusp, and is called the width of the cusp. The cusp represented by (d, a, b), where $0 \leq a < d$ and $0 \leq b < e$ and $\gcd(a, d) = \gcd(b, e) = 1$, has a uniformizer q and a punctured formal neighborhood $\operatorname{Spec} k((q))$ above which is the Tate curve analytically isomorphic to $\left(\mathbb{G}_m/q^{e\mathbb{Z}}, \zeta^a q^b\right) \in Y_1(N)(k((q)))$. This Tate curve specializes above the cusp itself to an e-gon consisting of irreducible components $Z_i \simeq \mathbb{P}^1$ indexed by $i \in \mathbb{Z}/e\mathbb{Z}$ such that $0 \in Z_i$ is attached to $\infty \in Z_{i+1}$ for all i. We choose the coordinate $t_i\colon Z_i \xrightarrow{\sim} \mathbb{P}^1$ for each i such that a point $a_i q^i + \sum_{j>i} a_j q^j \in \mathbb{G}_m/q^{e\mathbb{Z}}$ with $a_i \in k^\times$ specializes to $a_i \in \mathbb{G}_m \subseteq \mathbb{P}^1 \simeq Z_i \subset \pi^{-1}(y)$. Let $t = t_0$. For each cusp y, define $F_y := \pi^* y = \sum_i Z_i \in \operatorname{Div} \overline{\mathscr{E}}$.

9. Divisors

Given a rational function f on $\mathscr{E}$ whose divisor on $\mathscr{E}$ is known, the divisor of f on $\overline{\mathscr{E}}$ is determined up to addition of a linear combination of the F_y. We now explain how to compute it, modulo the ambiguity, following [SS91, §2]. Fix a cusp y of $X_1(N)$, and let q be a uniformizer at y, and let $Z_0, \ldots, Z_{e-1}$ be the components of $\pi^{-1}(y)$. The valuations $n_i := v_{Z_i}(f)$ can be simultaneously computed, modulo addition of a constant independent of i, by the relations $(f/q^{n_i}).Z_i = 0$ for all i, which amount to linear equations in the n_i. Let us make these equations explicit. In the case where the zeros and poles of f specialize to smooth points of $\pi^{-1}(y)$, let r_i be the number of them specializing to a point of Z_i, counted with multiplicity, with poles counted as negative. In the equation $(f/q^{n_i}).Z_i = 0$, only Z_{i+1}, Z_{i-1}, and the horizontal divisors in (f) meet Z_i, so the equation says

$$(n_{i+1} - n_i) + (n_{i-1} - n_i) + r_i = 0.$$

There is one such equation for each i. Solving this system of e equations yields all the n_i up to a common additive constant, since the solutions to the corresponding homogeneous system are the arithmetic progressions that are periodic modulo N, i.e., constant sequences. If in addition, f is normalized, then $\sum n_i = 0$; now the n_i are uniquely determined.

The above procedure can be applied also to any $f \in \mathbb{Q} \otimes k(\mathscr{E})^\times$, and in particular to the functions s_P, g_m, and h_m.

Lemma 9.1. *For $f = s_O$,*

(a) *At a cusp of $X_1(N)$ above $\infty \in X_0(N)$, we have $e = 1$, $n_0 = 0$, and $s_O|_{Z_0} = (1-t)^N/(1-t^N)$ in $\mathbb{Q} \otimes k(Z_0)^\times$.*

(b) *At a cusp of $X_1(N)$ above $0 \in X_0(N)$, we have $e = N$, $n_i = (N^2-1)/12 - i(N-i)/2$ for $0 \le i < N$, and $\left(q^{(N^2-1)/24} s_O\right)|_{Z_{(N-1)/2}}$ has a zero at ∞ and not at 0, while $\left(q^{(N^2-1)/24} s_O\right)|_{Z_{(N+1)/2}}$ has a zero at 0 and not at ∞.*

Proof.

(a) A cusp above ∞ has a punctured neighborhood above which is the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ with cyclic subgroup μ_N, specializing to a 1-gon. In fact, the relation $\prod_{R \in C} \tau_R^* s_O = 1$ in $\mathbb{Q} \otimes k(\mathscr{E})^\times$ from Lemma 5.3 implies $N n_0 = 0$, so $n_0 = 0$.

The order N zero of s_O specializes to 1, and the N poles of s_O specialize to the Nth roots of unity, so $s_O|_{Z_0}$ is a nonzero scalar times $(1-t)^N/(1-t^N)$.

Since s_O is normalized, $[N]_* s_0 \in \mu$. On the other hand, the morphism $[N]$ specializes to the Nth power map on $Z_0 \simeq \mathbb{P}^1$, which pushes $(1-t)^N/(1-t^N)$ forward to the norm $\prod_{\omega \in \mu_N} (1-\omega t)^N/(1-(\omega t)^N) = (1-t^N)^N/(1-t^N)^N = 1$. By the previous two sentences, the scalar of the previous paragraph is in μ.

(b) A cusp above 0 has a punctured neighborhood above which is the Tate curve $\mathbb{G}_m/q^{N\mathbb{Z}}$ with cyclic subgroup generated by q. The N zeros specialize to Z_0, but the N poles specialize to different Z_i, one pole per Z_i. Thus $r_0 = N-1$ and $r_i = -1$ for $i \neq 0$. On the other hand, $\prod_{R \in C} \tau_R^* s_O = 1$ implies $\sum n_i = 0$. Together these imply that

$n_i = (N^2-1)/12 - i(N-i)/2$ for $0 \le i < N$. The most negative of these are $n_{(N-1)/2}$ and $n_{(N+1)/2}$, which are both $-(N^2-1)/24$.

The divisor of $\left(q^{(N^2-1)/24} s_O\right)|_{Z_{(N-1)/2}}$ on $Z_{(N-1)/2} \simeq \mathbb{P}^1$ is

$$(n_{(N+1)/2} - n_{(N-1)/2})(0) + (n_{(N-3)/2} - n_{(N-1)/2})(\infty) - (1) = (\infty) - (1).$$

Similarly, the divisor of $\left(q^{(N^2-1)/24} s_O\right)|_{Z_{(N+1)/2}}$ on $Z_{(N+1)/2}$ is

$$(n_{(N+3)/2} - n_{(N+1)/2})(0) + (n_{(N-1)/2} - n_{(N+1)/2})(\infty) - (1) = (0) - (1). \qquad \square$$

Corollary 9.2.

(a) *At the cusp above $\infty \in X_0(N)$ given by $(\mathbb{G}_m/q^{\mathbb{Z}}, \zeta)$, we have $g_0|_{Z_0} = N$, and for $m \ne 0$ we have $g_m|_{Z_0} = (-1)^m N\binom{N}{m} t^m/(1-t^N)$, in $\mathbb{Q} \otimes k(\mathbb{Z}_0)^\times$.*

(b) *At a cusp above 0, for any $m, i \in \mathbb{Z}/N\mathbb{Z}$, we have $v_{Z_i}(g_m) = -(N^2-1)/24$.*

Proof.

(a) Up to a root of unity which may be ignored, $s_O|_{Z_0} = (1-t)^N/(1-t^N)$ by Lemma 9.1(a). Translation by P restricts to multiplication by ζ on Z_0, so

$$\begin{aligned} s_{jP}|_{Z_0} &= \tau_{-jP}^* s_O|_{Z_0} \\ &= (1-\zeta^{-j}t)^N/(1-(\zeta^{-j}t)^N) \\ &= \frac{1}{1-t^N} \sum_{i=0}^{N} \binom{N}{i} (-1)^i \zeta^{-ij} t^i \\ g_m|_{Z_0} &= \sum_{j=0}^{N-1} \zeta^{mj} \frac{1}{1-t^N} \sum_{i=0}^{N} \binom{N}{i} (-1)^i \zeta^{-ij} t^i \\ &= \frac{1}{1-t^N} \sum_{i=0}^{N} (-1)^i \binom{N}{i} t^i \sum_{j=0}^{N-1} \zeta^{(m-i)j} \\ &= \frac{1}{1-t^N} \sum_{i=0}^{N} (-1)^i \binom{N}{i} t^i \begin{cases} N, & \text{if } m-i \equiv 0 \pmod{N}; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

If $m = 0$, then only the terms with $i = 0$ or $i = N$ are nonzero, and the sum becomes $(1-t^N)N$. If $m \ne 0$, then only the term with $i = m$ is nonzero, and the sum becomes $(-1)^m \binom{N}{m} t^m N$.

(b) The translation action of C acts simply transitively on the set of components Z_i above the cusp. Thus the numbers $v_{Z_i}(s_{jP})$ for $j = 0, \ldots, N-1$ equal the numbers $v_{Z_{i'}}(s_O)$ for $i' = 0, \ldots, N-1$ in some order, which are described by Lemma 9.1(b). Hence in the sum $g_m = \sum_{j=0}^{N-1} \zeta^{mj} s_{jP}$ there are exactly two terms with the most negative valuation along Z_i, so $v_{Z_i}(\zeta^{mj} s_{jP}) = -(N^2-1)/24$ for $j = j_1$ and $j = j_2$, say. The last two claims in Lemma 9.1(b) imply that one of the functions $(q^{(N^2-1)/24} \zeta^{mj} s_{jP})|_{Z_i}$ for $j = j_1$ and $j = j_2$ has a zero at ∞ and not at 0, while the other has a zero at 0 and not at ∞, so their sum is nonzero on Z_i. Thus $v_{Z_i}(g_m) = -(N^2-1)/24$ too. $\square$

Proof of Theorem 1.4. We may work on the finite cover $Y_1(N)'$ of $Y_0(N)$ defined in Section 7. By Corollary 9.2(b), no g_m is identically zero. Hence each function g_m/h_m on $Y_1(N)'$ has only finitely many zeros. Equation (1) shows that outside the union of these zeros, $c_{E,C} = 0$; i.e., the f_P are linearly independent. □

Let $G := g_1 g_2 \cdots g_{N-1}$ and $H := h_1 h_2 \cdots h_{N-1}$ in $\mathbb{Q} \otimes k(\mathscr{E})^\times$. The divisor of H on $\mathscr{E}$ is $\mathscr{E}[N] - NC$.

Lemma 9.3. *For $f = H$,*

(a) *At a cusp of $X_1(N)$ above $\infty \in X_0(N)$, we have $e = 1$ and $n_0 = -(N^2-1)/12$.*
(b) *At a cusp of $X_1(N)$ above $0 \in X_0(N)$, we have $n_i = 0$ for all i.*

Proof. We work on the universal generalized elliptic curve over $X(N)$, whose degenerate fibers are all N-gons, so that the zeros and poles of H do not specialize to the singular points of fibers. As usual, let $Z_0, \ldots, Z_{N-1}$ be the components above a cusp; let $n'_i = v_{Z_i}(H)$. The normalization implies that the product of all translates of H by N-torsion points is in μ, so $\sum n_i = 0$.

(a) We have $r_0 = -N(N-1)$ and $r_i = N$ for $i \neq 0$. The r_i here are $-N$ times the r_i in the proof of Lemma 9.1(b), so the resulting n'_i are also multiplied by $-N$; that is, $n'_i = -N(N^2-1)/12 + Ni(N-i)/2$ for $0 \leq i < N$. Finally, $X(N) \to X_1(N)$ has ramification index N at cusps above ∞, so $n_0 = n'_0/N$.
(b) Each h_m has one zero and one pole specializing to each Z_i, so $r_i = 0$ for all i. Thus $n'_i = 0$ for all i, so $n_i = 0$ for all i. □

Lemma 9.4. *Let $N > 3$ be prime.*

(a) *The element $\mathfrak{g}_0 = g_0/h_0 \in \mathbb{Q} \otimes k(\mathscr{E})^\times$ lies in $\mathbb{Q} \otimes k(X_0(N))^\times$, its valuations at the cusps of $X_0(N)$ are $v_\infty(\mathfrak{g}_0) = 0$ and $v_0(\mathfrak{g}_0) = -(N^2-1)/24$, and its divisor on $Y_0(N)$ is effective and of degree $(N^2-1)/24$.*
(b) *The $G/H = \prod_{m=1}^{N-1}(g_m/h_m) \in \mathbb{Q} \otimes k(\mathscr{E})^\times$ lies in $\mathbb{Q} \otimes k(X_0(N))^\times$, with $v_\infty(G/H) \geq (N^2-1)/12$ and $v_0(G/H) = -(N-1)(N^2-1)/24$. The divisor of G/H on $Y_0(N)$ is of degree $\leq (N-3)(N^2-1)/24$, and it is twice an effective divisor on $Y_0(N)$.*

Proof. Each g_m/h_m is constant on each elliptic curve fiber, so g_m/h_m lies in $\mathbb{Q} \otimes k(X_1(N))^\times$. The Galois group of $X_1(N) \to X_0(N)$ fixes g_0/h_0 and permutes the g_m/h_m, so g_0/h_0 and G/H are in $\mathbb{Q} \otimes k(X_0(N))^\times$.

(a) The valuations $v_\infty(\mathfrak{g}_0)$ and $v_0(\mathfrak{g}_0)$ are determined by Corollary 9.2. On the other hand, (a power of) $\mathfrak{g}_0 = g_0/h_0$ is regular on $Y_0(N)$, and its divisor on the projective curve $X_0(N)$ has degree 0.
(b) The valuation of G/H along the component Z_0 above a cusp of $X_1(N)$ above ∞ is $\geq \left(\sum_{m=1}^{N-1} 0\right) - (-(N^2-1)/12) = (N^2-1)/12$, by Corollary 9.2(a) and Lemma 9.3(a); thus $v_\infty(G/H) \geq (N^2-1)/12$. The valuation of G/H along any component Z_i above a cusp above 0 is $\left(\sum_{m=1}^{N-1} -(N^2-1)/24\right) - 0 = -(N-1)(N^2-1)/24$ by Corollary 9.2(b) and Lemma 9.3(b); thus $v_0(G/H) = -(N-1)(N^2-1)/24$.

Since the divisor of G/H on $X_0(N)$ has degree 0, its divisor on $Y_0(N)$ has degree at most $-(N^2-1)/12 + (N-1)(N^2-1)/24 = (N-3)(N^2-1)/24$.

That it is twice an effective divisor can be checked on the étale cover $Y_1(N)'$ of Section 7. There, each g_m/h_m is regular, and Lemma 6.3 shows that $g_{-m}/h_{-m} = g_m/h_m$, so G/H is a square. □

Proof of Theorem 1.5. Let $D_{Y_1(N)}$ be the pullback of D under $Y_1(N) \to Y_0(N)$. Let $(g_m/h_m)_{\text{red}} \in \operatorname{Div} Y_1(N)$ be the reduced divisor whose support equals the divisor of g_m/h_m on $Y_1(N)$. Equation (1) says that $D_{Y_1(N)} = \sum_{m=0}^{N-1}(g_m/h_m)_{\text{red}}$. The divisors $D_{Y_1(N),1} := (g_0/h_0)_{\text{red}}$ and $D_{Y_1(N),2} = \sum_{m=1}^{(N-1)/2}(g_m/h_m)_{\text{red}} = \frac{1}{2}\sum_{m=1}^{N-1}(g_m/h_m)_{\text{red}}$ are invariant under the Galois group of $Y_1(N) \to Y_0(N)$, so they are pullbacks of divisors D_1 and D_2 on $Y_0(N)$. We have $D_{Y_1(N)} = D_{Y_1(N),1} + 2D_{Y_1(N),2}$, so $D = D_1 + 2D_2$.

The degree of D_1 is bounded by the degree of g_0/h_0 on $Y_0(N)$, which is $(N^2-1)/24$ by Lemma 9.4(a). Similarly, the degree of $2D_2$ is bounded by the degree of G/H on $Y_0(N)$, which is at most $(N-3)(N^2-1)/24$ by Lemma 9.4(b). □

10. Examples

Let $N > 3$ be prime. On the Tate curve over $k((q))$ analytically isomorphic to $\mathbb{G}_m/q^{\mathbb{Z}}$ we can write down a function with prescribed divisor in terms of theta functions in u and q, where u is the coordinate on $\mathbb{G}_m$. In this way, we express the elements s_P, g_m, and h_m in terms of u and q and we compute the q-expansions of the rational functions g_0/h_0 and G/H on $X_0(N)$.

Now suppose in addition that the genus of $X_0(N)$ is 0; that is, $N \in \{5, 7, 13\}$. Let $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$. Then the function $(N^{1/2}\eta(q^N)/\eta(q))^{24/(N-1)}$ is the q-expansion of a rational function x on $X_0(N)$ with $k(x) = k(X_0(N))$ such that x has a zero at the cusp ∞ and a pole at the cusp 0. Because of Lemma 9.4, this lets us compute g_0/h_0 and G/H as polynomials $f_1(x)$ and $x^{(N^2-1)/12}f_2(x)$ whose zeros with $x \neq 0$ give the points $(E, C) \in Y_0(N)$ with $c_{E,C} > 0$; call these points exceptional. Moreover, in these cases, using an expression for j in terms of x, we may take the $k(x)/k(j)$ norm and take numerators to obtain polynomials $F_1(j)$ and $F_2(j)$ (determined up to scalar multiple) whose zeros are the j-invariants of the E such that $c_{E,C} > 0$ for some $C \subset E$.

For $N \in \{5, 7, 13\}$, we found that the polynomials $f_1(x)$ and $f_2(x)$ are of degrees $(N^2-1)/24$ and $(N-3)(N^2-1)/48$ and have disjoint distinct roots in $\overline{\mathbb{Q}}$ (in fact, they are irreducible over $\mathbb{Q}$); this verifies Conjecture 1.6 for these values of N. In fact, $F_1(j)$ and $F_2(j)$ had the same properties.

Example 10.1. Let $N = 5$. Then

$$\begin{aligned} f_1(x) &= x + 5 \\ f_2(x) &= x + 10 \\ F_1(j) &= j - 1600 \\ F_2(j) &= 2j + 25. \end{aligned}$$

Each of f_1 and f_2 has a unique zero, and these zeros are distinct, and they avoid the cusps (where $x = 0$ and $x = \infty$), except in characteristic 2 (we always exclude characteristic 5). Thus in characteristics $\neq 2, 5$, we have $c_{E,C} = 0$ except for one (E, C) with $c_{E,C} = 1$ and one (E, C) with $c_{E,C} = 2$, so the conclusion of Conjecture 1.6 for $N = 5$ holds in characteristics $\neq 2, 5$. In characteristic 2, we have $c_{E,C} = 0$ except for one (E, C) with $c_{E,C} = 1$, so the conclusion of Conjecture 1.6 fails.

Moreover, in characteristics $\neq 2, 5$, the two exceptional (E, C) have j-invariants 1600 and $-25/2$, which are distinct except in characteristics 3 and 43. In characteristics 3 and 43, we find that $c_{E,C} = 0$ always except that the E with $j(E) = 1600 = -25/2$ has two exceptional subgroups C_1 and C_2, with $c_{E,C_1} = 1$ and $c_{E,C_2} = 2$.

Example 10.2. Let $N = 7$. Then

$$\begin{aligned}
f_1(x) &= x^2 + 7x + 7 \\
f_2(x) &= x^4 + 21x^3 + 168x^2 + 588x + 735 \\
F_1(j) &= j^2 - 1104j - 288000 \\
F_2(j) &= 15j^4 - 28857j^3 + 20163177j^2 - 5403404499j - 141176604743
\end{aligned}$$

and the constant terms, discriminants, and resultants factor as follows:

$$\begin{aligned}
f_1(0) &= 7 \\
f_2(0) &= 3 \cdot 5 \cdot 7^2 \\
\mathrm{Disc}(f_1) &= 3 \cdot 7 \\
\mathrm{Disc}(f_2) &= -3^3 \cdot 7^6 \\
\mathrm{Res}(f_1, f_2) &= 7^4 \\
\mathrm{Disc}(F_1) &= 2^8 \cdot 3^3 \cdot 7^3 \\
\mathrm{Disc}(F_2) &= -3 \cdot 7^{18} \cdot 43^2 \cdot 139^2 \cdot 421^2 \cdot 591751^2 \\
\mathrm{Res}(F_1, F_2) &= 5 \cdot 7^{12} \cdot 47 \cdot 3491 \cdot 5939 \cdot 244603.
\end{aligned}$$

The values of $f_1(0)$, $f_2(0)$, $\mathrm{Disc}(f_1)$, $\mathrm{Disc}(f_2)$ show that in all characteristics $\neq 3, 5, 7$, we have $c_{E,C} = 0$ except for two (E, C) with $c_{E,C} = 1$ and four with $c_{E,C} = 2$, so the conclusion of Conjecture 1.6 for $N = 7$ holds in characteristics $\neq 3, 5, 7$. In characteristic 3, we have $c_{E,C} = 0$ except that $c_{E,C} = 1$ for one (E, C) (corresponding to the double root $x = 1$ of f_1, where $j(E) = 0$). In characteristic 5, we have $c_{E,C} = 0$ except for two (E, C) with $c_{E,C} = 1$ and only *three* (E, C) with $c_{E,C} = 2$.

Moreover, excluding characteristic 7 as always, the exceptional (E, C) have distinct values of $j(E)$ except in characteristics 2, 43, 47, 139, 421, 3491, 5939, 244603, and 591751, for which there are exactly two exceptional (E, C) sharing the same $j(E)$. In characteristic 2, these two have $c_{E,C} = 1$ (since 2 divides $\mathrm{Disc}(F_1)$ but not $\mathrm{Disc}(f_1)$) In characteristics 43, 139, 421, and 591751, these two have $c_{E,C} = 2$ (since these primes divide $\mathrm{Disc}(F_2)$ but not $\mathrm{Disc}(f_2)$). In characteristics 47, 5939, and 244603, these two have c-values 1 and 2, respectively (since these primes divide $\mathrm{Res}(F_1, F_2)$ but not $\mathrm{Res}(f_1, f_2)$).

Example 10.3. Let $N = 13$. Then $\deg f_1 = \deg F_1 = 7$ and $\deg f_2 = \deg F_2 = 35$, and each of the four polynomials has distinct zeros in $\overline{\mathbb{Q}}$. The analysis is similar to that for $N = 5$ and $N = 7$, except that we were unable to factor $\mathrm{Disc}(F_2)$ completely.

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