Homotopy Theory of Monoids and Group Completion

Nigel Burke

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

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Declaration

I hereby declare that this dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text

Nigel Burke
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Abstract

This thesis presents several complete and partial models for the homotopy theory of monoids and the derived functor of group completion. We show that there is a simplicial model structure on the category of reduced simplicial sets that is Quillen equivalent to the Quillen model structure of simplicial monoids. Using this Quillen equivalence we recover the fact that the derived functor of group completion is isomorphic to the homotopy type of loops on the classifying space of a monoid. We use the Street nerve to show that the derived functor of group completion of monoids in the category of ω-groupoids for the Gray tensor product is isomorphic to group completion for simplicial monoids in low degrees. Finally we exploit the connection of ω-groupoids with the theory of rewriting for presentations of monoids to calculate the second homotopy group of the classifying space BM of a monoid M in terms of a chosen presentation by generators and relations.

Contents

1 Introduction

The inspiration for the research in this thesis came from the fact proved by Barr and Beck in §1.2 of [BB69] that for a group G the derived functors of the abelianization functor $\mathrm{ab} : \mathbf{sGp} \to \mathbf{sAb}$ from simplicial groups to simplicial abelian groups correspond up to a shift of degree with the group homology of G

$$\mathbb{L}\,\mathrm{ab}_n(G) \cong H_{n+1}(BG, \mathbb{Z})$$

Around the same time I learned this I also learned about the group completion functor $L : \mathbf{Mon} \to \mathbf{Gp}$. Like the abelianization functor the group completion functor is a left adjoint to the inclusion of a reflective subcategory, namely monoids within the category of groups. Using the standard model structures on the categories of simplicial monoids and groups we can calculate the derived functors of group completion. This led me to ask whether there is an alternate interpretation of these derived functors of group completion in terms of another construction as is the case for abelianization of groups. The answer to this question was given by Dwyer and Kan in their paper [DK80] studying localization of categories. They showed that for a monoid M the derived functors of group completion correspond up to a degree shift with the homotopy groups of the classifying space of M

$$\mathbb{L}L_n(M) \cong \pi_{n+1}(BM) \tag{1}$$

This led me to ask the question:

Q1: Can this identification of homotopy groups as derived functors of group completion be used to do calculations of the homotopy groups of classifying spaces of discrete monoids?

This question is an ambitious one as by Theorem 1 of [McD79] all connected spaces have the homotopy type of the classifying space of a discrete monoid. My goal was to try to connect the data of a presentation of a monoid by generators and relations to its derived functor of group completion and so to the calculation of the homotopy groups of its classifying space. Some progress on this goal is realized in Section 5.4.3 with a formula for the second homotopy group $\pi_2(BM)$ of the classifying space based on the data of a presentation of a monoid.

In the course of investigating the homotopy theory of monoids and their group completions I found a lack of literature on models of ∞-monoids. Much work has been done in formalizing the notion of an $(\infty, 1)$-category, which is, as Riehl and Verity say in their lecture notes [RV19], a schematic notion for which many suitably equivalent models, in the form of model categories, exist. Some examples of these models are: categories enriched in simplicial sets, the Joyal model structure on the category of simplicial sets [Joy08], and the model structure for Segal categories on simplicial spaces [DKS89]. These all formalize the notion as defined in [RV19] of an $(\infty, 1)$-category as a category weakly enriched in ∞-groupoids, which are themselves models of spaces. My schematic approach to the question of models of ∞-monoids was to say that an ∞-monoid should be an $(\infty, 1)$-category with a single object. This led me to the second question:

Q2: Can models of ∞-monoids be made from models of $(\infty, 1)$-categories by restricting them to a single object?

In the case of categories enriched in simplicial sets it is simple to see that this works. The restriction to a single object is just the category of simplicial monoids, which has a well-known model structure transferred from the Kan model structure on $\mathbf{sSet}$. The only publication I found taking this approach to defining a model of ∞-monoids derived from another model of $(\infty, 1)$-categories was [Ber07]. In this paper Bergner shows that there is a model structure on reduced simplicial spaces that is the restriction of the model structure for Segal categories on simplicial spaces. As Bergner shows, for models of $(\infty, 1)$-categories not based on strict enrichment in spaces the restriction to the single object case is more difficult to construct. This is because for a given model category modelling $(\infty, 1)$-categories we are seeking the reflective subcategory not just of the category with the model structure but also of its homotopy category. Therefore, we need to show that restricting to the case of a single object interacts well with the model structure for $(\infty, 1)$-categories. This leads to the question of transfer of model structure along an adjunction, for which some general tools exist (see for example [Hes+17]) but a bespoke argument is almost always needed for the question of acyclicity as described in §2 of [Hes+17].

These are the two motivating questions that drove the research in this thesis. They are examined in reverse order, however, in the four chapters of this thesis. The first two chapters are devoted to constructing and comparing models of ∞-monoids via the approach of **Q2**. The final two chapters show how the data of a presentation of a monoid can be used to calculate the first derived functor of its group completion, giving a part of an answer to **Q1**. These two halves of the thesis are relatively independent, however the second half, dealing with **Q1**, benefits a little from having some of the models of ∞-monoids and their group completions defined and explored in the first half.

In Chapter 2, following the approach discussed above and inspired by Bergner, we prove the existence of a simplicial model structure on the category $\mathbf{sSet}_0$ of reduced simplicial sets, that is simplicial sets with a unique 0-simplex. This model structure is transferred from the Joyal model structure on simplicial sets of [Joy08]. The fibrant objects of this model structure are the quasi-monoids, which are reduced simplicial sets that have the right lifting property against all inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k < n$ and $n \geq 2$. These are the quasi-categories of [Joy08] that have a single object, which fits with the schematic approach to defining ∞-monoids given above. The model structure we prove is the following (see Theorem 2.37 below)

Theorem. *There exists a model structure on the category of reduced simplicial sets* $\mathbf{sSet}_0$ *such that*

- *the weak equivalences are the maps* $f : A \to B$ *such that* $(X^f)_* : (X^B)_* \to (X^A)_*$ *is a Kan weak equivalence for all quasi-monoids* X
- *the cofibrations are inclusions of reduced simplicial sets*
- *the fibrations are the maps with the right lifting property against all cofibrations that are weak equivalences*

This model structure is cofibrantly generated and the fibrant objects are the quasi-monoids.

The functor

$$(-^-)_* : \mathbf{sSet}_0^{\mathrm{op}} \times \mathbf{sSet}_0 \to \mathbf{sSet}$$

is the restriction to $\mathbf{sSet}_0$ of the pointed simplical mapping space functor for pointed simplicial sets. In Section 2.1 we describe this functor and show that it defines a simplicial enrichment of the category $\mathbf{sSet}_0$ forming part of the simplicial model structure on $\mathbf{sSet}_0$. We call this the quasi-monoid model structure after the fibrant objects and denote it by $(\mathbf{sSet}_0)_J$. This model structure has been proved in Lemma 3.2 of [CHL19], however we present a different proof based on the methods of [Joy08] and also show that it is a simplicial model structure. Our method makes use of a simplicial enrichment, tensoring, and cotensoring of the category $\mathbf{sSet}_0$ over simplicial sets, which we construct in Section 2.1.

The first step on the path to proving the quasi-monoid model structure is to show that $(-^{-})_*$ determines a mapping space for quasi-monoids, meaning that when X is a quasi-monoid $(X^A)_*$ is a Kan complex for any reduced simplicial set A. This mapping space can be used in place of the mapping space construction from [Joy08] Chapter 5 to apply to $\mathbf{sSet}_0$ the same reasoning used by Joyal to construct the quasi-category model structure on $\mathbf{sSet}$. This is the approach we take in Section 2.4 to prove several of the conditions for the quasi-monoid model structure. We deviate from Joyal at the end of this section to complete the proof, however. We will use the mapping space for $\mathbf{sSet}_0$ and the small object argument to construct a functor from $\mathbf{sSet}_0$ to the category graded groups that commutes with filtered colimits and detects whether a map in $\mathbf{sSet}_0$ belongs to the class of proposed weak equivalences of the quasi-monoid model structure. This functor then fits in to the context of [Bou75], where Bousfield showed that there is a model structure on $\mathbf{sSet}$ that has as its weak equivalences maps that induce isomorphisms on integer homology. Replacing the homology functor with the functor we have defined for $\mathbf{sSet}_0$ allows us to apply Bousfield's methods and show that the proposed quasi-monoid model structure exists and is cofibrantly generated.

We also show that the Kan model structure on $\mathbf{sSet}_0$, as described in [GJ99] §IV,is a left Bousfield localization of this model structure, recovering the reduced version of the corresponding fact for the Kan and Joyal model structures on $\mathbf{sSet}$. The localization of $(\mathbf{sSet}_0)_J$ is taken at a single map. This map adds the condition that 1-simplices of a quasi-monoid must be invertible, in a way that we will describe in Definition 2.9 and discuss in detail in Section 2.5. The Kan model structure on reduced simplicial sets is known (see for example [GJ99] §IV) to be Quillen equivalent to the model category of simplicial groups via the Kan loop group adjunction. Hence this localization of the model of ∞-monoids provided by $(\mathbf{sSet}_0)_J$ is a model for the (∞-)group completion of ∞-monoids. We will make this statement more precise in Chapter 3 when we compare the quasi-monoid model structure with that of simplicial monoids and show that they are Quillen equivalent.

The quasi-monoid model structure satisfies our definition as a model for ∞-monoids by being a model of the reflective subcategory of single object $(\infty, 1)$-categories in the homotopy category of a model of $(\infty, 1)$-categories, namely the Joyal model structure on simplicial sets. However, to show that this is a sensible way of modelling ∞-monoids we must show that it is equivalent to other models of ∞-monoids that are obtained through the same process. The only other such model I know of is that of Bergner in [Ber07]. The fibrant objects of Bergner's model structure are the Segal categories of the model structure from [JT07] that have a single object. Section 2.6 is devoted to showing that the quasi-monoid model structure is equivalent to the Bergner model structure on the category $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ of reduced simplicial spaces. This is done in Proposition 2.67 below, which is based on similar results for the Joyal model structure on $\mathbf{sSet}$ and the Segal category model structure on bisimplicial sets in [JT07]. This result shows that the diagonal functor

$$d : (\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B \to (\mathbf{sSet}_0)_J$$

forms the left and the right adjoint of two Quillen equivalences between the quasi-monoid model structure on reduced simplicial sets and the Bergner model structure on reduced simplicial spaces.

In the first chapter we followed our general scheme of constructing ∞-monoid models by restricting $(\infty, 1)$-category models to the single object case. We also showed that the quasi-monoid model structure we obtained from this approach was Quillen equivalent to the Bergner model structure for Segal monoids that motivated this approach. In this chapter we further support the identification of these model categories as models of ∞-monoids by showing that both are Quillen equivalent to the model structure for simplicial monoids. We also use this Quillen equivalence to show that the localization described in the previous chapter of $(\mathbf{sSet}_0)_J$ to obtain the Kan model structure on reduced simplicial sets is equivalent to the group completion functor for simplicial monoids on homotopy categories.

As we do for the Quillen equivalence between the quasi-monoid model structure and the Bergner model structure in Chapter 2 we will construct the Quillen equivalence with simplicial monoids by restricting a Quillen equivalence of the corresponding models for $(\infty, 1)$-categories. In this case we will use the homotopy coherent nerve and realization adjunction between categories enriched in simplicial sets and simplicial sets. This is a Quillen equivalence when **sSet** has the Joyal model structure by Theorem 2.2.5.1 of [Lur09]. In this chapter we show that this Quillen equivalence restricts to a Quillen equivalence between **sMon**, the category of simplicial monoids, and the quasi-monoid model structure.

We start by defining the cosimplicial simplicial monoid $\mathbb{C}^\bullet : \Delta \to \mathbf{sMon}$ that we call the homotopy coherent simplices. This cosimplicial simplicial monoid determines the homotopy coherent nerve-realization adjunction in the usual way. Rather than collapsing the corresponding notions for categories enriched in simplicial sets we construct these simplicial monoids as nerves of partially ordered monoids with respect to the partial order. To describe the realization of a reduced simplicial set by this cosimplicial simplicial monoid we make use of the terminology of beads and necklaces from [DS11a] and [DS11b]. This allows us to show that realization produces cofibrant simplicial monoids from reduced simplicial sets and also show that the homotopy coherent nerve-realization adjunction

$$\begin{tikzcd}
\mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_J \arrow[l, bend left, "\mathbb{C}"]
\end{tikzcd}$$

determined by $\mathbb{C}^\bullet$ is a Quillen adjunction.

Sections 3.3 to 3.6 contain the proof that the homotopy coherent nerve-realization adjunction is a Quillen equivalence. The method we use for this proof is the main new contribution of this chapter. We start from the same position as [Lur09]. We apply the arguments of §2.2 of [Lur09] for the proof of the $(\infty, 1)$-category Quillen equivalence to reduce the proof in the ∞-monoid case to:

1. constructing maps of simplicial sets $r_X : H(X) \to U\mathbb{C}(X)$ for X a reduced simplicial set, H a functor defined by Lurie that has the homotopy type of the mapping space $(X^{S^1})_*$ when X is a quasi-monoid, and $U : \mathbf{sMon} \to \mathbf{sSet}$ the forgetful functor

2. proving that r_X is a Kan weak equivalence when X is a quasi-monoid

3. proving that when $X = \mathbb{N}(M)$ for a Kan fibrant simplicial monoid M the composite map $r_{\mathbb{N}(M)}$

$$H(\mathbb{N}(M)) \xrightarrow{r_{\mathbb{N}(M)}} U\mathbb{C}\mathbb{N}(M) \xrightarrow{U\varepsilon_M} UM$$

is a Kan weak equivalence

The construction of the map for step 1 follows from Lurie by restricting to the single object or reduced simplicial set case. Step 2 is where we introduce new methods to this proof. Specifically, we define the map r_X by identifying the simplicial set $H(X)$ as the first stage of a filtration of the simplicial set $U\mathbb{C}(X)$ by simplicial subsets indexed by the natural numbers. This filtration is by a quantity we call the spine length, which is defined in Section 3.3. Spine length is a combined measure of the length of words in the free monoids of simplices of $\mathbb{C}(X)$ and the degeneracy of simplices that label the terms in these words. We show that the inclusions of each stage of this filtration in the next can be realized as pushouts of Kan acyclic cofibrations in **sSet** when X is a quasi-monoid and so r_X is a Kan weak equivalence in this case. Our construction of these colimits makes use of more of the technology of necklaces and beads from [DS11a] and [DS11b]. These papers offer an alternative proof of the Quillen equivalence of categories enriched in simplicial sets with the Joyal model structure on simplicial sets so the results in this chapter can be seen as a hybrid, connecting Lurie's approach with that of Dugger and Spivak.

In Section 3.7 we use the Quillen equivalence proved in the previous sections to show that the localization from Chapter 2 of $(\mathbf{sSet})_J$ that gives the Kan model structure on reduced simplicial sets is Quillen equivalent to the group completion adjunction for **sSet**. The results of the previous sections and Chapter 2 can be summarized in the following diagram where the Kan model structure on $\mathbf{sSet}_0$, is denoted $(\mathbf{sSet}_0)_K$

$$\begin{tikzcd}
\mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_J \arrow[l, bend left, "\mathbb{C}"] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{?} & (\mathbf{sSet}_0)_K \arrow[u, bend right, "\mathrm{Id}"']
\end{tikzcd}$$

The question mark in this diagram indicates our natural curiosity about what the localization adjunction for $(\mathbf{sSet}_0)_J$ on the right corresponds to for simplicial monoids under the horizontal Quillen equivalence. We answer this question by localizing **sMon** at the image under $\mathbb{C}$ of the localizing map of $\mathbf{sSet}_0$ for the vertical adjunction. This gives a model structure on **sMon** that is Quillen equivalent to the Kan model structure on $\mathbf{sSet}_0$. We show that this model structure has fibrant objects the Kan fibrant simplicial monoids whose monoid of connected components is a group. We also show that the group completion functor determines a Quillen equivalence of this model structure with the model structure on simplicial groups. Hence we obtain another Quillen equivalence between $(\mathbf{sSet}_0)_K$ and **sGp** by composing group completion with homotopy coherent realization $L\mathbb{C} \dashv \mathbb{N}I$. We can compare this Quillen equivalence to the Kan loop group adjunction to recover the result, which was mentioned above, of Dwyer and Kan in [DK80] that the derived functor of group completion of a Kan fibrant simplicial monoid is isomorphic to the homotopy type of the loop space of $\mathbb{N}(M)$.

The next chapter marks the beginning of our investigations of **Q1**. Our goal in these chapters is to calculate homotopy groups of the classifying space of a monoid M using the first non-trivial case of the identification (1) of homotopy groups of $\mathbb{N}(M) = BM$ with derived functors of group completion for a monoid M. This is the case $n = 1$ and it allows us to calculate $\pi_2(BM)$ as the first derived functor of group completion of M. We will calculate the first derived functor not in **sMon** but in another model category that captures the homotopy theory of 2-types of simplicial monoids. This is nevertheless sufficient for our purposes of calculating the first derived functor of a monoid The partial model of the homotopy theory of simplicial monoids we use is that of monoids in the category of strict ω-groupoids. This category was chosen because of the strong connection of the homotopy theory of monoids in this category to the theory of term rewriting for monoids. We will exploit this connection to show that data from a presentation of a monoid by term rewriting determines a resolution of the monoid and hence, by the definitions and identifications above, the second homotopy group of BM.

Chapter 4 is a recollection of the theory of strict ω-groupoids, which are the basis of the homotopy theory we will use to calculate the first derived functor of group completion. We define strict ω-groupoids, which are globular models of higher categories and record the folk model structure for the category $\omega\mathbf{Gpd}$ of strict ω-groupoids from [AM11]. This model structure, like many other results about strict ω-groupoids, was originally constructed for the category of crossed complexes in [BG89], which was shown to be equivalent to the category of strict ω-groupoids in [BH81b]. Some of the results of this chapter exist in the literature for crossed complexes but not for strict ω-groupoids, so in this chapter we gather these results and translate them where needed. Similarly, there exist results for strict ω-*categories* that we will specialize to strict ω-groupoids where needed. One example of this is the Gray tensor product for strict ω-groupoids, which we describe in Section 4.3. The construction we use was developed for strict ω-categories by Steiner in [Ste04] using the theory of augmented directed complexes, which are chain complexes of abelian groups with some additional structure. One key objective of this chapter is to recall the results of [AL20] which show that the folk model structure for strict ω-groupoids is monoidal with respect to the Gray tensor product. This opens the way to the study of Gray monoids, that is monoids for the Gray tensor product in $\omega\mathbf{Gpd}$, in the final chapter. The other key objective of Chapter 4 is to introduce the Street nerve of strict ω-groupoids. This is the construction that allows us to identify how the model structure for strict ω-groupoids models homotopy 2-types. This was shown in [MS93] for the case of 2-groupoids, but it is straightforward to apply their results to the case of strict ω-groupoids by localizing the model structure to ignore all homotopy information above degree 2.

In Chapter 5 we use the partial model of homotopy types given by strict ω-groupoids to obtain a partial model of homotopy types of monoids and their group completions. In the previous section we cited the results of [AL20] that show that $\omega\mathbf{Gpd}$ with the folk model structure is a monoidal model category with the Gray tensor product. It also satisfies the other technical lemmas that allow the transfer of the folk model structure on $\omega\mathbf{Gpd}$ to give a model structure on the category of monoids in strict ω-groupoids with the Gray tensor product. We call this the category **Gray** of Gray monoids. These Gray monoids are our partial model of the homotopy types of simplicial monoids, which we will use to calculate the first derived functor of group completion of a monoid M. To do this, we must translate the notion of group completion from simplicial monoids to Gray monoids and also show that Gray monoids do provide a partial model for the homotopy types of simplicial monoids as we have claimed. A key tool for both of these goals is the Street nerve that was defined in the previous chapter. We address both of these goals in the first two sections of this chapter.

In Section 5.2 we show that the Street nerve is lax monoidal for the Gray tensor product on $\omega\mathbf{Gpd}$ and the cartesian product on **sSet**. The proof of this fact follows immediately from the Alexander-Whitney map for chain complexes, which is in fact a map of Steiner's augmented directed complexes as defined in the previous chapter. Hence the theory of Steiner from [Ste04] as described in Chapter 4 immediately gives a corresponding Alexander-Whitney map of strict

ω-groupoids. This map was described for strict ω-categories in [Ver08], however the identification with the Alexander-Whitney map was not explicitly made there. This allows us to extend the Street nerve to an adjunction between categories of monoids in ω**Gpd** for the Gray tensor product and monoids in **sSet** for the cartesian product as in the following diagram

$$\begin{tikzcd}
\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
& \mathbf{sGp} \arrow[u, bend right, "I"']
\end{tikzcd}$$

As in Section 3.7 we are in a situation where we want to extend an adjunction along another adjunction, in this case we want to understand group completion for Gray monoids. The notion that fills in the lower left corner of this diagram is that of a Gray group, which we define in Section 5.1. A Gray group is a Gray monoid whose monoid of 0-cells is a group. We show in Section 5.2 that Gray groups are characterized by the condition that their Street nerve is a simplicial group. Hence we take the group completion of Gray monoids by adding inverses for the 0-cells.

To calculate derived functors of group completion we must first give a model structure on the category **GrayGp** of Gray groups. We show that there is a model structure on **GrayGp** that is transferred by the group completion adjunction from the Gray monoid model structure. Our method of proof involves construction a path Gray monoid for Gray monoids, which turns out to be a Gray group when the original Gray monoid is a Gray group. This approach is an adaptation of the arguments of §5 of [SS00] where a path space is constructed for monoids for the chain complex tensor product. This establishes the basic needs we outlined above to be able to speak about derived functors of group completion for Gray monoids. It remains to show that these derived functors of group completion agree with those for **sMon** in low degrees.

In Section 5.3 we use the same localization approach as in Section 4.8 to show that the Street nerve-realization adjunction

$$\begin{tikzcd}
\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"]
\end{tikzcd}$$

induces an equivalence when restricted to homotopy 2-types. In particular, we localize both **sMon** and **Gray** at maps that make the homotopy categories ignore all homotopy information above degree 2. It then follows by the results of the previous chapter and those of [SS03] on Quillen equivalences of categories of monoids that the Street nerve adjunction induces a Quillen equivalence between the localized model categories of monoids. We then show that this 2-types localization extends to the category of Gray groups and the Street nerve-realization adjunction is also a Quillen equivalence between the localized model categories of Gray groups and simplicial groups. Our method of proof here is the same as in Section 3.7. We localize the model category for 2-types of Gray monoids at a single map the gives a model structure for group-like 2-types of Gray monoids, which we can then show is Quillen equivalent to the model structure of 2-types of Gray groups. To complete this proof we us the results of [SS03] again, while making some slight modifications. This shows that, for calculating derived functors of group completion $\mathbb{L}L_n(M)$ of a monoid M if

$n \leq 2$ we can make the calculation in **sMon** or **Gray** and obtain the same result. We note that for Gray groups, this Quillen equivalence combined with the homotopy coherent nerve-realization from Chapter 3 recovers the result from [Ber99] that shows that groupoids enriched in the category of 2-groupoids with the Gray tensor product model all homotopy 3-types.

Although we have shown that we can calculate in **Gray** some low degree derived functors of group completion, and so low degree homotopy groups of monoid classifying spaces, we are not really any closer to an actual concrete calculation of these groups. Taking cofibrant replacements in **Gray** is not any easier than in **sMon** and the Gray tensor product introduces increased complexity in working with the cells of a Gray monoid. Our motivation for looking at strict ω-groupoids was their connection with the theory of term rewriting for monoids. This connection, however, is through *strict* monoids in ω**Gpd**, that is monoids with respect to the cartesian product of strict ω-groupoids rather than the Gray tensor product. In the last section of this chapter we show that there is a model structure on the category **Mon**(ω**Gpd**) of strict monoids in ω**Gpd** and recall how the data of presentation of a monoid by generators and relations can be used to construct a cofibrant replacement of a monoid in this model structure. Finally we show how this cofibrant replacement can be used to calculate the first derived functor of group completion for Gray monoids.

We show in Section 5.4.2 that the category **Mon**(ω**Gpd**) of strict monoids in ω**Gpd** is simultaneously

1. a reflective subcategory of **Gray**
2. isomorphic to the category $((\omega, 1)\mathbf{Cat})_{>0}$ of strict $(\omega, 1)$-categories that have a unique 0-cell

The category **Gp**(ω**Gpd**) of strict groups in ω**Gpd** has similar interpretations and allows us to define a group completion functor for strict monoids in ω**Gpd** that is the restriction of the Gray monoid group completion functor to strict monoids. The second perspective on these categories allows us to define model structures on **Mon**(ω**Gpd**) and **Gp**(ω**Gpd**) transferred from the corresponding folk model structures on $(\omega, 1)$**Cat** and ω**Gpd**. These model structures are also transferred from the model structures on **Gray** and **GrayGp** via their respective reflective subcategory inclusions and so make the group completion functor of strict monoids in ω**Gpd** a Quillen adjunction. Our final goal in this section is to show that the first derived functor of this group completion is isomorphic to that of group completion for Gray monoids, and so to that of simplicial monoids by the previous sections. We show that this holds at the end of Section 5.4.2 by showing that the calculation of the first derived functor collapses all differences between Gray monoids and strict monoids in ω**Gpd**, so we can make the calculation before or after reflection to the subcategory of strict monoids in ω**Gpd** and obtain the same result.

Finally it is in this context of strict monoids in ω**Gpd** (or equivalently strict $(\omega, 1)$-categories with a unique 0-cell) where we can apply the theory of term rewriting for monoids to do calculations of the derived functor of group completion. In term rewriting theory a presentation of a monoid M by generators and relations consists of an alphabet Σ and a set R of directed pairs of words from the free monoid Σ^* on the alphabet. The monoid M presented by this data is isomorphic to the quotient of Σ^* that is obtained by identifying words of the free monoid using the relations of R. This data can also be used to freely generate a $(2, 1)$-category with a unique object or equivalently a monoid in the category of groupoids. In Section 5.4 we describe this construction and also show how it can be extended to construct a cofibrant replacement of M in the model category **Mon**(ω**Gpd**) of monoids for the cartesian product in ω**Gpd**. This cofibrant replacement is called a polygraphic resolution in the literature on term rewriting (see for example [GM12] §2.3.3). Such a cofibrant replacement can then be used to calculate the derived functors of the group completion functor

$$L : \mathbf{Mon}(\omega\mathbf{Gpd}) \to \mathbf{Gp}(\omega\mathbf{Gpd})$$

for strict monoids. By the previous arguments the first derived functor of this group completion is isomorphic to that for group completion of simplicial monoids. The first derived functor only depends on the n-cells of a cofibrant replacement of M in $\mathbf{Mon}(\omega\mathbf{Gpd})$ for $n \leq 2$ or equivalently on n-cells for $n \leq 3$ for a cofibrant replacement of M in $((\omega, 1)\mathbf{Cat})_{>0}$. The data required to produce a 3-truncated cofibrant replacement of M in $((\omega, 1)\mathbf{Cat})_{>0}$ is called a coherent presentation in [GMM13]. It adds to the alphabet Σ and relations R a set P called a homotopy basis that consists of generating 2-cells of the $(2, 1)$-category generated by Σ and R. From this data and the coherent presentation of the monoid M we can therefore calculate the first derived functor of group completion and so the homotopy group $\pi_2(BM)$.

In the final section of this chapter we give a formula for the second homotopy group of the classifying space of a monoid in terms of a construction that has already been studied in term rewriting theory and homotopy theory. The construction we employ is the group of identities among relations defined in [BH82]. This is an abelian group determined by the data of the presentation of a group by generators and relations. We apply this construction to a presentation (Σ, R) of a monoid M by considering the group of identities among relations for the group presentation of LM obtained from the same sets of generators and relations. We denote this group by $N(R)$. Elements of the homotopy basis of a coherent presentation (Σ, R, P) of M correspond to elements of the group of identities among relations, so we can define a subgroup $N(P) \trianglelefteq N(R)$ that is generated by the elements of P. Given a coherent presentation (Σ, R, P) of a monoid M the formula in Theorem 5.26 gives the second homotopy group of BM as the quotient of the group of identities among relations by the subgroup generated by the homotopy basis.

Theorem. *Let M be a discrete monoid with coherent presentation (Σ, R, P). The second homotopy group $\pi_2(BM)$ of the classifying space of M is the quotient*

$$\pi_2(BM) = N(R)/N(P)$$

where $N(R)$ is the group of identities among relations for the presentation (Σ, R) and $N(P)$ is the subgroup of identities among relations determined by the set P.

This description follows automatically from the earlier proof that we can calculate the first derived functor of group completion in $\mathbf{Mon}(\omega\mathbf{Gpd})$. We call this a Hopf formula for the second homotopy group because of aesthetic similarities to the Hopf formula ([Hop41]) for the second group homology group. Calculations of coherent presentations and groups of identities among relations can be difficult, as we would expect, but there exists a large body of research on cases when these coherent presentations have nice properties that allow calculations. At the end of this thesis we apply some of these results to calculate $\pi_2(BM)$ in some simple examples. Much more work could be done to apply this formula to calculate second homotopy groups in cases where the hard work of constructing coherent presentations and the group of identities among relations has already been done.

2 ∞-Monoids

In this chapter we will describe and compare two models of ∞-monoids. These are both obtained using the main strategy of this part of the thesis, that of restricting previously described models of $(\infty, 1)$-categories to a single object. The first of these models is based on Joyal's model structure for **sSet** as defined in [Joy08]. This model structure is defined so that the fibrant objects are quasi-categories. These are simplicial sets X such that all lifting problems

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^n \arrow[r] & *
\end{tikzcd}$$

with $0 < k < n$ have solutions. These spaces generalize the defining property of nerves of categories amongst simplicial sets, which is that all such lifting problems have unique solutions. The 0-simplices of a quasi-monoid represent its objects, so we can restrict to the case of monoids by restricting the ambient category to the full subcategory of reduced simplicial sets $\mathbf{sSet}_0$, which all have a unique 0-simplex. In this chapter we will show that there is a simplicial model structure on $\mathbf{sSet}_0$ that is the restriction of the Joyal model structure. This model structure on $\mathbf{sSet}_0$ has as its fibrant objects the quasi-monoids, which will be the quasi-categories in $\mathbf{sSet}_0$ or equivalently quasi-categories with a unique object. We will call this the quasi-monoid model structure on $\mathbf{sSet}_0$.

Our approach to proving the quasi-monoid model structure will be a hybrid, using results about mapping spaces derived from Joyal's work on quasi-categories [Joy08] but applying them in the framework used by Bousfield in [Bou75] to show the existence of a model structure on **sSet** where the weak equivalences are homology equivalences. Using this approach was inspired by the proof by Goerss and Jardine in [GJ99] that there exists a model structure on reduced simplicial sets that is Quillen equivalent to the model structure of simplicial groups defined by Quillen in [Qui67].

Theorem 2.1 ([GJ99] Proposition 6.2). *There exists a model structure on* $\mathbf{sSet}_0$ *where a map* $f : X \to Y$ *is a*

- *weak equivalence if it is a weak equivalence of simplicial sets*
- *cofibration if it is a monomorphism*
- *fibration if it has the right lifting property against all acyclic cofibrations*

Furthermore, acyclic fibrations of this model structure are acyclic fibrations of the Kan model structure on simplicial sets.

We will also show that this Kan model structure on $\mathbf{sSet}_0$ is obtained by localizing the quasi-monoid model structure at a single map. This is similar to the unreduced case, where the Kan model structure on **sSet** can be obtained by localizing the Joyal model structure.

The second model of ∞-monoids in this chapter is Bergner's model structure for reduced Segal categories from [Ber07]. Bergner's approach in this paper of restricting an $(\infty, 1)$-category model structure to the unique object case was what inspired the investigation of the quasi-monoid model structure in this chapter. We will show that Bergner's model structure is closely related to the quasi-monoid model structure for $\mathbf{sSet}_0$. In particular, these model structures are Quillen equivalent. We will do this by adapting to the unique object case the results of [JT07], which show the equivalence of the Joyal model structure on **sSet** and the Segal category model structure on bisimplicial sets.

This chapter could be viewed as a proof of concept for the Bergner method of restricting to a unique object as a way to generate models of ∞-monoids from models of $(\infty, 1)$-categories. It shows that this approach is feasible in at least one other case than that in [Ber07] and also highlights some of the difficulties that can arise when making this restriction. The case of the quasi-monoid model structure is intriguing since the restriction to a unique object enhances the enrichment of the model structure: while the Joyal model structure on **sSet** is only enriched in itself, the quasi-monoid model structure on $\mathbf{sSet}_0$ is enriched in **sSet** with the Kan model structure. It is possible that this approach could be productively applied to other models of $(\infty, 1)$-categories to produce new models of ∞-monoids.

We begin in the first section by describing the enrichment of $\mathbf{sSet}_0$ in **sSet** and its powering and copowering over **sSet**. These will form part of a simplicial model structure and we will use their properties in the future to identify weak equivalences and prove the lifting properties needed for the definition of a quasi-monoid model structure on $\mathbf{sSet}_0$.

2.1 Simplicial Enrichment of $\mathbf{sSet}_0$

We want to show that the model structure for quasi-monoids we will prove in this chapter is a simplicial model category. To do this we will first need to show that $\mathbf{sSet}_0$ is enriched in simplicial sets and powered and copowered over **sSet**. In this section we will construct three functors which we denote by

$$\begin{aligned} (-) \;{}_*\!\times (-) &: \mathbf{sSet}_0 \times \mathbf{sSet} \to \mathbf{sSet}_0 \\ (-^-)_* &: {\mathbf{sSet}_0}^{\mathrm{op}} \times \mathbf{sSet}_0 \to \mathbf{sSet} \\ E_0(-^-) &: \mathbf{sSet}^{\mathrm{op}} \times \mathbf{sSet}_0 \to \mathbf{sSet}_0 \end{aligned}$$

for copowering, simplicial enrichment, and powering over **sSet** respectively. We will show that these functors form an adjunction of two variables in the sense of Definition 4.1.12 of [Hov99]. This means that there are natural isomorphisms

$$\mathrm{Hom}_{\mathbf{sSet}_0}(A, E_0(B^K)) \cong \mathrm{Hom}_{\mathbf{sSet}_0}(K \;{}_*\!\times A, B) \cong \mathrm{Hom}_{\mathbf{sSet}}(K, (A^B)_*)$$

The copowering of $\mathbf{sSet}_0$ over **sSet** is given by what we will call the half reduced product.

Definition 2.2. *The **half reduced product** of a simplicial set K with a reduced simplicial set $A \in$ **$sSet_0$** is the simplicial set obtained as the pushout*

$$\begin{tikzcd} * \times K \arrow[r, "!"] \arrow[d, hook, "* \times K"'] & * \arrow[d] \\ A \times K \arrow[r] & A \;{}_*\!\times K \end{tikzcd}$$

The simplicial enrichment of $\mathbf{sSet}_0$ is given by the standard simplicial enrichment of pointed simplicial sets. We first observe that a reduced simplicial set is necessarily uniquely pointed, as there is a unique 0-simplex. Hence for any reduced simplicial sets A and B the simplicial set of maps $A \to B$ is given by the unpointed simplicial set $U(B^A)_*$ whose set of n-simplices consists of pointed maps of simplicial sets

$$\Delta^n_+ \wedge A \to B$$

where $\wedge$ is the smash product of pointed simplicial sets and Δ^n_+ is the standard simplicial n-simplex with a freely adjoined basepoint. The forgetful functor U indicates that we are forgetting the pointed structure of $(B^A)_*$ and viewing it only as a simplicial set. Since we will not make any use of the pointed structure of this simplicial set in future we will suppress the forgetful functor U and simply refer to $(B^A)_*$ as the simplicial set of maps for reduced simplicial sets A and B.

The powering of $\mathbf{sSet}_0$ over $\mathbf{sSet}$ is accomplished by the construction of the 0^{th} Eilenberg subcomplex (see Definition 8.3 of [May67]) of the function complex A^K at the map $K \to * \to A$. This is done by taking the pullback

$$\begin{tikzcd}
E_0(A^K) \arrow[r] \arrow[d] & * \arrow[d] \\
A^K \arrow[r] & \operatorname{cosk}_0(A^K)
\end{tikzcd}$$

where cosk_0 is the right adjoint of the 0-skeleton functor.

Proposition 2.3. *The following data define an adjunction of two variables*

$$\begin{aligned}
(-)\ {}_*\!\times (-) &: \boldsymbol{sSet}_0 \times \boldsymbol{sSet} \to \boldsymbol{sSet}_0 \\
(-^-)_* &: \boldsymbol{sSet}_0{}^{\text{op}} \times \boldsymbol{sSet}_0 \to \boldsymbol{sSet} \\
E_0(-^-) &: \boldsymbol{sSet}^{\text{op}} \times \boldsymbol{sSet}_0 \to \boldsymbol{sSet}_0
\end{aligned}$$

Proof. We will first describe the isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}_0}(K\ {}_*\!\times A, B) \cong \mathrm{Hom}_{\mathbf{sSet}}(K, (A^B)_*)$$

A map of simplicial sets

$$K \to (B^A)_*$$

corresponds by the definition of the simplicial enrichment to a map $A \times K \to X$ such that the following diagram commutes

$$\begin{tikzcd}
* \times K \arrow[r] \arrow[d, hook] & * \arrow[d] \\
A \times K \arrow[r] & B
\end{tikzcd}$$

This clearly corresponds to a unique map $A\ {}_*\!\times K \to X$ by Definition 2.2.

We now describe the isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}_0}(A, E_0(B^K)) \cong \mathrm{Hom}_{\mathbf{sSet}_0}(K\ {}_*\!\times A, B)$$

A map of reduced simplicial sets $B \to E_0(A^K)$ corresponds to a map of simplicial sets $B \times K \to A$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{sk}_0(B) \times K \arrow[r, equal] \arrow[d, hook] & * \times K \arrow[r] & * \arrow[d] \\
B \times K \arrow[rr] & & X
\end{tikzcd}$$

This corresponds to a unique map $B \mathbin{{}_*\!\times} K \to A$ of reduced simplicial sets by Definition 2.2. □

As mentioned above, we want quasi-monoids to play the role of fibrant objects in our model structure on $\mathbf{sSet}_0$. In the next section we will define these reduced simplicial sets and study the horn lifting problems that they allow solutions for.

2.2 Horn Lifting Problems in $\mathbf{sSet}_0$

In this section we will consider lifting problems of simplicial sets of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\
\Delta^n \arrow[r] & B
\end{tikzcd} \tag{2}$$

for $f : A \to B$ a map of reduced simplicial sets. We will define conditions under which solutions exist for these problems and in later sections we will use these to construct a simplicial model structure on $\mathbf{sSet}_0$.

First we note that this problem (2) is defined in $\mathbf{sSet}$ rather than in $\mathbf{sSet}_0$, where we are hoping to define a model structure. We will fix this by defining reduced simplicial sets corresponding to the standard simplicial simplices. We will do this by making use of the left adjoint $\kappa : \mathbf{sSet} \to \mathbf{sSet}_0$ to the inclusion of reduced simplicial sets into the category of simplicial sets. This functor collapses all 0-simplices of a simplicial set to a single 0-simplex without affecting higher simplices. Composing this functor with the standard cosimplicial object of simplicial simplices $\Delta^\bullet : \Delta \to \mathbf{sSet}$ gives a reduced cosimplicial object of simplicial simplices which we denote by

$$S^\bullet = \kappa \circ \Delta^\bullet : \Delta \to \mathbf{sSet}_0$$

The coface and codegeneracy maps for this cosimplicial object will be denoted by the same notation as for $\Delta^\bullet$, namely $d^i : S^n \to S^{n+1}$ and $s^i : S^n \to S^{n-1}$, when no confusion can occur.

By the definition of $S^\bullet$ using the left adjoint κ there is a bijection for all reduced simplicial sets A between the set of maps $S^n \to A$ in $\mathbf{sSet}_0$ and the set of n-simplices of A

$$\mathrm{Hom}_{\mathbf{sSet}_0}(S^n, A) \cong A_n$$

that follows from the corresponding property for Δ^n.

The reduced inclusions of the boundaries and the horns for the reduced standard simplicial simplices will be denoted respectively by

$$\partial S^n \subseteq S^n \qquad \lambda^n_k \subseteq S^n$$

for $n \geq 0$ and $0 \leq k \leq n$. These are the inclusions obtained by collapsing to a single 0-simplex all 0-simplices of the maps

$$\partial\Delta^n \subseteq \Delta^n \qquad \Lambda^n_k \subseteq \Delta^n$$

Hence, in this section we will be considering the lifting problems in $\mathbf{sSet}_0$

$$\begin{tikzcd} \lambda^n_k \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ S^n \arrow[r] & B \end{tikzcd} \tag{3}$$

We will divide the horns $\lambda^n_k \subseteq S^n$ into two types: the inner and outer horns.

Definition 2.4. *An **inner horn** is a horn inclusion $\lambda^n_k \hookrightarrow S^n$ for $n \geq 2$ and $0 < k < n$. The remaining horns $\lambda^n_0 \hookrightarrow S^n$ and $\lambda^n_n \hookrightarrow S^n$ for $n \geq 0$ are called **outer horns**.*

We can extend the definition of horns to include other subsets of the the standard reduced simplicial simplex S^n. We will call these generalized horns as in [Joy08] §2.2.1.

Definition 2.5. *Let $T \subseteq [n]$ be a non-empty subset. Then we define the **generalized horn***

$$\lambda^n_T = \bigcup_{i \notin T} d^i(S^n)$$

Note this definition is quite compatible with our notation for the horns λ^n_k given above before Definition 2.4 since when $T = \{k\} \subseteq [n]$ we have

$$\lambda^n_{\{k\}} = \lambda^n_k$$

A map $f : A \to B$ between reduced simplicial sets such that all lifting problems (3) have solutions when $\lambda^n_k \subseteq S^n$ is an inner horn is called an **inner fibration**. Any monomorphism of reduced simplicial sets that has the left lifting property against all inner fibrations is called **inner anodyne**. This defines a weakly saturated class of morphisms $\mathcal{IA}$ in $\mathbf{sSet}_0$.

A map is a **right fibration** if it is an inner fibration and also has the right lifting property against all horn inclusions $\lambda^n_n \hookrightarrow S^n$ for $n \geq 1$. A map is **right anodyne** when it belongs to the saturated class of maps having the left lifting property against all right fibrations. Dually, there are also **left fibrations** and **left anodyne** maps.

The next Lemma from [Joy08] identifies those generalized horns as defined in Definition 2.5 that are also inner, left, or right anodyne.

Lemma 2.6. *A generalized horn $\lambda^n_T \subseteq S^n$ is inner anodyne if there exist $a < b < c \in [n]$ such that $a, c \notin T$ and $b \in T$.*

Proof. The proof of Proposition 2.12 (*iv*) of [Joy08] can be adapted to the reduced case as follows. We use induction on the size of the set $\emptyset \neq T \subseteq [n]$. If $T = \{k\}$ then clearly $0 < k < n$ so the horn $\lambda^n_{\{k\}} = \lambda^n_k \subseteq S^n$ is clearly inner anodyne. For the induction step, let T contain at least two elements. There exist $a < b < c \in [n]$ such that $a, c \notin T$ and $b \in T$ by assumption. Let $i \in T$ be different from b. In the unreduced case Joyal shows that the square

$$\begin{tikzcd}
\Lambda^{n-1}_{(d^i)^{-1}T} \arrow[r, hook, "d^i"] \arrow[d, hook] & \Lambda^n_T \arrow[d, hook] \\
\Delta^{n-1} \arrow[r, hook, "d^i"'] & \Lambda^n_{T\setminus\{i\}}
\end{tikzcd}$$

is a pushout, where $(d^i)^{-1}T \subseteq [n-1]$ is the preimage in $[n-1]$ of the subset $T \subseteq [n]$. The functor κ is a left adjoint so it preserves this pushout. The preimage of T under d^i satisfies the property of the hypothesis for $(d^i)^{-1}(a) < (d^i)^{-1}(b) < (d^i)^{-1}(c)$ so by the induction hypothesis the horn inclusion $\lambda^{n-1}_{(d^i)^{-1}T} \subseteq S^{n-1}$ is inner anodyne. Hence its pushout $\lambda^n_T \subseteq \lambda^n_{T\setminus\{i\}}$ along d^i is inner anodyne. The horn inclusion $\lambda^n_{T\setminus\{i\}} \subseteq S^n$ is inner anodyne by induction hypothesis, so the result holds. □

We can now state the definition of quasi-monoids in terms of a lifting problem within $\mathbf{sSet}_0$.

Definition 2.7. *A **quasi-monoid** is a reduced simplicial set X such that all lifting problems*

$$\begin{tikzcd}
\lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\
S^n \arrow[r] & *
\end{tikzcd}$$

for $0 < k < n$ have solutions.

Equivalently, a quasi-monoid is a reduced simplicial set such that the unique map $X \to *$ is an inner fibration. Hence the map $X \to *$ has the right lifting property against all inner anodyne maps, including all generalized horns λ^n_k

A quasi-monoid is sent by the inclusion functor $i : \mathbf{sSet}_0 \to \mathbf{sSet}$ to a quasi-category, as defined by Joyal [Joy08] Definition 1.5, that has only one object. This holds since κ is left adjoint to the inclusion of $\mathbf{sSet}_0$ in $\mathbf{sSet}$, so the lifting property for quasi-monoids in $\mathbf{sSet}_0$ corresponds to the quasi-category lifting property for simplicial sets. As in [Joy08] we will emphasize the connection of this definition with monoids. This comes from the nerve or classifying space functor

$$B : \mathbf{Mon} \to \mathbf{sSet}_0$$

The nerve of a monoid is a quasi-monoid. All nerves are 2-coskeletal, as for $n > 2$ the face maps act on an n-simplex $(a_1, a_2, \cdots, a_n) \in BM$ by

$$d_i(a_1, a_2, \cdots, a_n) = \begin{cases} (a_2, , \cdots, a_n) & i = 0 \\ (a_1, a_2, \cdots, a_i a_{i+1}, \cdots, a_n) & 0 < i < n \\ (a_1, a_2, \cdots, \cdots, a_{n-1}) & i = n \end{cases}$$

so an n-simplex is uniquely determined by its faces. For $n > 2$ the 2-skeletons of λ^n_k and S^n are isomorphic, so it only remains to show that the lifting problem

$$\begin{tikzcd}
\lambda^2_1 \arrow[r] \arrow[d, hook] & BM \arrow[d] \\
S^2 \arrow[r] & *
\end{tikzcd}$$

can be solved. The top horizontal map identifies $a, b \in M$ and so a solution to the lifting problem is given by the 2-simplex $(a, b) \in BM_2$.

In fact, as mentioned at the start of this chapter, the nerve functor embeds the category of monoids as a full subcategory of reduced simplicial sets. Thinking of monoids as their nerves and reduced simplicial sets as generalized nerves, 1-simplices or edges of a reduced simplicial set correspond to arrows or elements of the quasi-monoid. This perspective can be formalized by the left adjoint to the nerve B

$$\tau : \mathbf{sSet}_0 \to \mathbf{Mon}$$

This functor sends a reduced simplicial set A to its **homotopy monoid**, which is the monoid $\tau(A)$ generated by the 1-simplices of A subject to the relations

$$[d_1(\alpha)] = [d_2(\alpha)][d_0(\alpha)]$$

for all 2-simplices α of A. This construction is defined for all reduced simplicial sets, but in the case A is a quasi-monoid, every word in the monoid $\tau(A)$ can be represented by the class of a single edge. A word $[x][y] \in \tau(A)$ corresponds to a horn $\lambda_1^2 \to A$, which has a filler $\alpha : S^2 \to X$, so $[x][y] = [d_1(\alpha)]$ in $\tau(A)$.

Returning to the general case of a quasi-monoid X, a solution to the lifting problem

$$\begin{tikzcd}
\lambda_1^2 \arrow[r, "\langle x,y \rangle"] \arrow[d, hook] & X \arrow[d] \\
S^2 \arrow[r] & *
\end{tikzcd}$$

therefore can be seen as identifying a composite 1-simplex for the 1-simplices $x, y \in X_1$ determined by the map $\lambda_1^2 \to X$ as well as a 2-simplex $\alpha : S^2 \to X$ that witnesses the composite

$$\begin{tikzcd}
& * \arrow[dr, "y"] & \\
* \arrow[ur, "x"] \arrow[rr, dashed, "xy"'] & \alpha & *
\end{tikzcd}$$

In the next section we will examine when certain outer horn lifting problems for a quasi-monoid can have solutions.

2.2.1 Special Outer Horns

In this section we will describe conditions on outer horn lifting problems for quasi-monoids that guarantee that these lifting problems have solutions. Recall that the outer horns are those inclusions

$$\lambda_0^n \subseteq S^n \qquad \lambda_n^n \subseteq S^n$$

for $n \geq 1$. We will show that solving lifting problems against outer horns is related to the existence of inverses in the homotopy monoid. This is most easily seen in the case $n = 2$. Solving the lifting problem for a quasi-monoid X against an outer horn in the diagram

$$\begin{tikzcd}
\lambda_0^2 \arrow[r, "\langle x,y \rangle"] \arrow[d, hook] & X \arrow[d] \\
S^2 \arrow[r] & *
\end{tikzcd}$$

corresponds in the homotopy monoid by the discussion above to the existence of $z \in X_1$ such that $[x][z] = [y]$ in $\tau(X)$. Hence if X has the right lifting property against all outer horns then $\tau(X)$ is a group. This motivates our investigation in this section of the connection between the existence of solutions to outer lifting problems against quasi-monoids and invertible 1-simplices of X.

The notion of invertibility we will define for 1-simplices of a quasi-monoid X will be based on the existence of a solution to an extension problem for the reduced simplicial set map $x : S^1 \to X$ corresponding to a 1-simplex $x \in X_1$. The extension will be along the inclusion of S^1 into the reduced simplicial set

$$E^1 = \kappa(\operatorname{cosk}_0([1]))$$

For $r \geq 1$ the r-simplices of E^1 are length r sequences of 0 and 1. These include the non-decreasing sequences that can be identified with r-simplices of S^1, so there is an inclusion

$$01 : S^1 \hookrightarrow E^1$$

determined by the 1-simplex 01 which is common to both reduced simplicial sets.

The extra simplices of the space E^1 beyond those of S^1 allow us to identify an inverse for an edge in the homotopy monoid as well as all higher simplices that witness the edge's invertibility. In particular, the 2-simplices

$$\begin{tikzcd}[column sep=small] & * \arrow[dr, "01"] & & & & * \arrow[dr, "10"] & \\ * \arrow[ur, "10"] \arrow[rr, "11"'] & 101 & * & & * \arrow[ur, "01"] \arrow[rr, "00"'] & 010 & * \end{tikzcd}$$

witness that 10 is a left and right inverse respectively for the class of the 1-simplex $01 \in E^1$ in $\tau(E^1)$. Hence a map of reduced simplicial sets $\gamma : E^1 \to X$ identifies an invertible element of the homotopy monoid of X.

We will also define subspaces of E^1 that witness the existence of a right inverse and a left inverse respectively of an edge in the homotopy monoid.

Definition 2.8. *The subspace $R^1 \subseteq E^1$ is the image of the reduced simplicial set map*

$$010 : S^2 \to E^1$$

Dually, the space $L^1 \subseteq E^1$ is the image of the reduced simplicial set map

$$101 : S^2 \to E^1$$

The inclusion $01 : S^1 \hookrightarrow E^1$ factors through the inclusions $L^1 \subseteq E^1$ and $R^1 \subseteq E^1$, so there are inclusions $01 : S^1 \hookrightarrow R^1$ and $01 : S^1 \hookrightarrow L^1$ that are the identity below degree 2. The simplex 010 is a witness for 10 as the right inverse of 10 in the homotopy monoid. Dually, the simplex 101 witnesses 10 as the left inverse of 10 in the homotopy monoid. Using these spaces we make the following definitions.

Definition 2.9. *An edge $S^1 \to X$ of a quasi-monoid is* ***right invertible*** *(respectively* ***left invertible****) when there exists an extension $R^1 \to X$ (respectively $L^1 \to X$). An edge $S^1 \to X$ of a quasi-monoid is* ***invertible*** *when there exists an extension to $E^1 \to X$.*

Note that clearly invertible edges are both left and right invertible, but considering R^1 and L^1 themselves shows that not all right or left invertible edges are invertible, as in the following example.

Example 2.10. The map $01 : S^1 \hookrightarrow R^1$ has no extension along the map $S^1 \hookrightarrow E^1$. If $\alpha : E^1 \to R^1$ extends $01 : S^1 \to R^1$ then $d_0(d_0(\alpha(0101))) = 01$ and $d_2(d_3(\alpha(0101))) = 01$ in R^1, but this implies $\alpha(0101) = 0101$, which is not possible as this simplex does not belong to R^1.

We will show in Section 2.5 that these definitions of invertibility behave as we would hope in a quasi-monoid. In particular in Proposition 2.46 we will show that an edge of a quasi-monoid is invertible in the sense of Definition 2.9 if and only if its image in the homotopy monoid is invertible as an element of a monoid. This means, in particular, that in quasi-monoids an edge is invertible if and only if it is left and right invertible.

Let $n \geq 2$ and $f : X \to Y$ be an inner fibration between quasi-monoids. We say a lifting problem is a **special outer horn** if it is of the form of either of the squares

$$\begin{tikzcd}
\lambda_0^n \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(a)"] & X \arrow[d, "f"] \\
S^n \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_n^n \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "(b)"] & X \arrow[d, "f"] \\
S^n \arrow[r] & Y
\end{tikzcd}$$

where the edge 01 of the problem (a) (respectively the edge $n-1\ n$ of the problem (b)) is sent to a right invertible edge in X (respectively a left invertible edge). The importance of these diagrams is that these lifting problem can be solved.

Proposition 2.11. *Special outer horns have the left lifting property against all inner fibrations between quasi-monoids.*

The corresponding result for quasi-categories, of which this is a special case, has been proved in [Joy08] Theorem 4.13 using the join and slice of a simplicial set. In the rest of this section we will present a different proof that avoids introducing these definitions. We will show that solutions exist for lifting problems of type (a), with the other case following by duality. We will begin by defining a space that will help us construct solutions in our quasi-monoids.

Definition 2.12. *Let $n \geq 1$ and let E^n be the reduced simplicial set obtained by collapsing all 0-simplices of $\operatorname{cosk}_0([n])$. The subspace*

$$Q^n \subseteq E^n$$

consists of all sequences of integers from $[n]$ that are non-decreasing, except that 0 is allowed to follow 1.

For example, the sequence $0101010123 \cdots n$ belongs to Q^n. The space Q^n contains the n-simplex $012 \cdots n$ so there is an inclusion

$$012 \cdots n : S^n \hookrightarrow Q^n$$

corresponding to this n-simplex. The maps $\delta^{n+1} : [n] \hookrightarrow [n+1]$ determine inclusions $E^n \hookrightarrow E^{n+1}$ that restrict to an inclusion $Q^n \hookrightarrow Q^{n+1}$. This inclusion makes the diagram following diagram commute for $n \geq 1$

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, hook, "d^{n+1}"'] & Q^n \arrow[d, hook] \\
S^{n+1} \arrow[r, hook] & Q^{n+1}
\end{tikzcd}$$

Finally, we observe that since R^1 is generated by the 2-simplex 010, which is a valid 2-simplex of Q^n for all $n \geq 1$, we have an inclusion $R^1 \hookrightarrow Q^n$.

The role of Q^n is to give a factorization of a special outer lifting problem through an inclusion of a subspace of Q^n that we can then show is inner anodyne and so has a solution. To construct this factorization for a special outer lifting problem of type (a) we start by considering the inclusion of the edge 01 into S^n for $n \geq 2$ and take the pushout attaching the cells of R^1

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, hook, "01"'] & R^1 \arrow[d, hook] \arrow[ddr, hook, bend left] & \\
S^n \arrow[r, hook] \arrow[drr, hook, bend right] & S^n \vee_{S^1} R^1 \arrow[dr, hook, "q^n"] & \\
 & & Q^n
\end{tikzcd}$$

We will show that the map q^n arising from the inclusions of S^n and R^1 in Q^n is an inner anodyne map.

Lemma 2.13. *The comparison map q^n is an inner anodyne map.*

Proof. The space $S^n \vee_{S^1} R^1$ consists of the sequences 01, 10, and 010 from R^1 and all non-decreasing sequences from $[n]$. Let Q^n_k consist of the subspace of Q^n generated by the non-degenerate simplices of Q^n specified by sequences of length k where every number changes by exactly 1 at each step in the sequence. For example, 1012 is a generating 3-simplex of Q^n_4 but its face 102 is not a 2-simplex of Q^n_3. Note that $Q^n_2 \subseteq S^n \vee_{S^1} R^1$ since 01 and 10 belong to R^1. Also, Q^n is the colimit of

$$S^n \vee_{S^1} E^1 \hookrightarrow (S^n \vee_{S^1} E^1) \vee Q^n_3 \hookrightarrow (S^n \vee_{S^1} E^1) \vee Q^n_4 \hookrightarrow \cdots$$

where the maps in the colimit are the pushouts

$$\begin{tikzcd}
Q^n_m \arrow[r, hook] \arrow[d, hook] & (S^n \vee_{S^1} E^1) \vee Q^n_m \arrow[d, hook] \\
Q^n_{m+1} \arrow[r, hook] & (S^n \vee_{S^1} E^1) \vee Q^n_{m+1}
\end{tikzcd}$$

We claim that the inclusion $Q^n_m \subseteq Q^n_{m+1}$ is inner anodyne for $m \geq 2$. The generating m-simplices of Q^n_{m+1} that are not in Q^n_m are the length $m+1$ sequences of the form $e_t\delta$ for $\max\{1, m-n\} \leq t \leq m-1$ where e_t is the unique length t sequence of alternating 0s and 1s that ends with a 1 and $\delta = 012\cdots m-t$. The $0 \leq i < t$ faces all belong to Q^n_m, as they delete integers in the sequence e_t, producing degeneracies of the generating $m-1$-simplices of Q^n_m. The only face for $t \leq i \leq m+1$ that belongs to Q^n_m is the $m+1$ face, as all others are length m sequences where a number changes by 2 at some step. So this cell is attached as the pushout of a generalized horn $\lambda^{m+1}_{\{t,t+1,\cdots,m\}} \hookrightarrow S^{m+1}$, which is inner anodyne by Lemma 2.6. □

We can now prove Proposition 2.11.

Proof of Prop. 2.11. Given a left special outer horn lifting problem, the the image of the 1-simplex $01 \in \lambda^n_0$ in X extends to a map $R^1 \to X$. Consider the extended diagram built around the lifting problem for $T \subseteq \{0, 1, 2 \cdots, n-1\} \subseteq [n]$

$$\begin{tikzcd}
\lambda^n_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
S^n \arrow[r] & Y
\end{tikzcd} \tag{4}$$

made by extending the 1-simplex $S^1 \to X$ in the image of $01 : S^1 \hookrightarrow S^n \hookrightarrow \lambda^n_0$ along the inclusion $01 : S^1 \hookrightarrow R^1$ and taking pushouts as indicated

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d, hook, "01"'] & R^1 \arrow[d, hook] \arrow[rrdd, bend left] & & \\
S^{n-1} \arrow[r, hook] \arrow[d, hook, "d^n"'] & S^{n-1} \vee_{S^1} R^1 \arrow[r, hook, "q_{n-1}"] \arrow[d, hook] \arrow[rrd, dotted] & Q^{n-1} \arrow[d, hook] \arrow[rd, dashed] & \\
\lambda^n_0 \arrow[rrr] \arrow[rd, hook] \arrow[dd, hook] & & & X \arrow[dd, "f"] \\
& \lambda^n_0 \vee_{S^1} R^1 \arrow[r, hook] \arrow[d, hook] \arrow[rru, dotted] & \lambda^n_0 \vee_{S^{n-1}} Q^{n-1} \arrow[d, hook] \arrow[ru, dashed] & \\
S^n \arrow[rrr] \arrow[rd, hook] & & & Y \arrow[d] \\
& S^n \vee_{S^1} R^1 \arrow[r, hook, "q_n"'] \arrow[rru, dotted] & Q^n \arrow[r] \arrow[ru, dashed] & *
\end{tikzcd} \tag{5}$$

The dotted arrows in this diagram are the universal maps determined by the pushouts from the original lifting problem. The map $S^n \vee_{S^1} R^1 \dashrightarrow Y$ factors through the inner anodyne map $q_n : S^n \vee_{S^1} R^1 \hookrightarrow Q^n$ via the lowest dashed arrow of (5) because $Y \to *$ is an inner fibration, so a lift exists for the diagram

$$\begin{tikzcd}
S^{n-1} \vee_{S^1} R^1 \arrow[r, dotted] \arrow[d, hook, "q_n"'] & Y \arrow[d] \\
Q^n \arrow[r] \arrow[ru, dashed] & *
\end{tikzcd} \tag{6}$$

by Lemma 2.13. The top dashed arrow in (5) is the lift of the following square where the bottom horizontal arrow is the composite of the inclusion $Q^{n-1} \hookrightarrow Q^n$ with the lift $Q^n \dashrightarrow Y$ of the previous diagram (6) and the top horizontal arrow is the top dotted arrow of (5)

$$\begin{tikzcd}
S^{n-1} \vee_{S^1} R^1 \arrow[r, dotted] \arrow[d, hook, "q_{n-1}"'] & X \arrow[d, "f"] \\
Q^{n-1} \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

A solution to this lifting problem exists because f is an inner fibration and q_{n-1} is inner anodyne by Lemma 2.13.

The middle dashed arrow of (5) is the comparison map for the central pushout square arising from the dotted and dashed arrows with target X from its three corners. Hence the original lifting problem factors through the inclusion $\lambda^n_0 \vee_{S^1} Q^{n-1} \subseteq Q^n$ as shown

$$\begin{tikzcd}
\lambda^n_0 \arrow[r, hook] \arrow[d, hook] & \lambda^n_0 \vee_{S^1} Q^{n-1} \arrow[r, dashed] \arrow[d, hook] & X \arrow[d, "f"] \\
S^n \arrow[r, hook] & Q^n \arrow[r, dashed] & Y
\end{tikzcd}$$

We will solve the lifting problem of the right hand square, which will give a solution to the outer square lifting problem. We will do this by factoring the inclusion $\lambda^n_0 \vee_{S^{n-1}} Q^{n-1} \subseteq Q^n$ as a sequence of subspaces

$$\lambda^n_0 \vee_{S^{n-1}} Q^{n-1} = V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq Q^n$$

such that the inclusions $V_i \subseteq V_{i-1}$ are inner anodyne and the map $S^n \hookrightarrow Q^n$ factors through V_1. $V_n = \lambda^n_0 \vee_{S^{n-1}} Q^{n-1}$ is the subspace of Q^n that is generated by λ^n_0 and the non-degenerate simplices of Q^n that are represented by sequences of the form $e_t d^n$ where e_t is the unique length $t \geq 1$ sequence of alternating 0s and 1s that ends in 1 and $d^n = 012 \cdots n-1 \in S^n_{n-1}$. We construct V_k for $1 \leq k < n$ by attaching the $n+1$ simplex $01d^k \in Q^n_{n+1}$ to V_{k+1}, where

$$d^k = d_k(012 \cdots n) \in S^n_{n-1}$$

The faces of $01d^k$ are given by

$$d_j(01d^k) = \begin{cases} 1d^k & \text{if } j = 0 \\ 0d^k & \text{if } j = 1 \\ 01d_{j-2}(d^k) & \text{if } 2 \leq j \leq n \end{cases}$$

Note that $0d^k = s_0(d^k)$ so this face is present in V_{k+1} because it is present in λ^n_0. The faces $01d_{j-2}(d^k)$ are non-degenerate sequences of length n. The only such sequences starting 01 in V_{k+1} have their remaining part a face of d^l for $l > k$. Since $d_{j-2}(d^k) = d_{j-2}d_k(01 \cdots n)$, the only faces present in V_{k+1} are those for $k < j \leq n$. Hence the attachment is done by the generalized horn $\lambda^{n+1}_{\{0,2,\cdots,k\}} \subseteq S^{n+1}$, which is inner anodyne by Lemma 2.6. Finally, we have

$$01d^1 = 01023 \cdots n \in V_1$$

and so $012 \cdots n = d_2(01d^1) \in V_1$. □

2.3 Mapping Spaces for Quasi-Monoids

In this section we will show that when X is a quasi-monoid the space $(X^A)_*$ defined in Section 2.1 is a Kan complex. This is a key step towards constructing a simplicial model structure for quasi-monoids on $\mathbf{sSet}_0$, as the space $(X^A)_*$ will be a homotopy function complex for $\mathbf{sSet}_0$ in the sense of §17 of [Hir03]. In the next section we will use the homotopy properties of the spaces $(X^A)_*$ to define weak equivalences for this quasi-monoid model structure and show that they are weak equivalences of a simplicial model structure.

To show that $(X^A)_*$ is a Kan complex when X is a quasi monoid we will make use of the adjoint structure for the functor of two variables $(-^-)_*$. In particular, by Proposition 2.3 there is a correspondence between lifting problems in $\mathbf{sSet}$ and $\mathbf{sSet}_0$ respectively

$$\begin{tikzcd}
K \arrow[r] \arrow[d, hook, "i"'] & (X^B)_* \arrow[d, "(f^j)_*"] \\
L \arrow[r] & (X^A)_* \times_{(X^B)_*} (Y^B)_*
\end{tikzcd}
\qquad
\begin{tikzcd}
(B \,{}_*{\times}\, K) \cup_{A \,{}_*{\times}\, K} (A \,{}_*{\times}\, L) \arrow[r] \arrow[d, hook, "j\,{}_*{\times}\, i"'] & X \arrow[d, "f"] \\
B \,{}_*{\times}\, L \arrow[r] & Y
\end{tikzcd} \tag{7}$$

Kan complexes and quasi-monoids are defined by their lifting properties against horn inclusions and reduced horn inclusions respectively. Hence we can prove that for a map f the map of simplicial sets

$$(f^j)_* : (X^B)_* \to (X^A)_* \times_{(X^B)_*} (Y^B)_*$$

is a Kan fibration for any inclusion of reduced simplicial sets $A \hookrightarrow B$ by showing that for all $m, n \geq 0$ and all $0 \leq k \leq n$ the pushout product map of reduced simplicial sets

$$\begin{tikzcd}
(S^m \,{}_*{\times}\, \Lambda^n_k) \cup_{\partial S^m \,{}_*{\times}\, \Lambda^n_k} (\partial S^m \,{}_*{\times}\, \Delta^n) \arrow[d, hook] \\
S^m \,{}_*{\times}\, \Delta^n
\end{tikzcd} \tag{8}$$

has the left lifting property against f.

Our main goal for this section is to show this is the case when $f : X \to Y$ is an inner fibration between quasi-monoids. To do this we will borrow and adapt to the reduced case the terminology and tools of Appendix H of [Joy08], starting with the notion of prisms.

Definition 2.14. *A* ***reduced prism*** *is a reduced simplicial set of the form*

$$S^m \,{}_*{\times}\, \Delta^n$$

for $m, n \geq 0$.

In Appendix H of [Joy08] the spaces $\Delta^m \times \Delta^n$ are called prisms, which motivates our definition of reduced prisms. Theorem H.0.20 of [Joy08] shows that the inclusion in a prism $\Delta^m \times \Delta^n$ of the pushout cartesian product of an inner horn with the inclusion $\partial\Delta^n \subseteq \Delta^n$ is inner anodyne. We can use this result to show that when the map (8) arises from an inner horn $\Lambda^n_k \subseteq \Delta^n$ it is inner anodyne.

Proposition 2.15. *Let $m, n \geq 0$ and $0 < k < n$. The map*

$$(S^m \,{}_*{\times}\, \Lambda^n_k) \cup_{\partial S^m \,{}_*{\times}\, \Lambda^n_k} (\partial S^m \,{}_*{\times}\, \Delta^n) \hookrightarrow S^m \,{}_*{\times}\, \Delta^n \tag{9}$$

is inner anodyne.

Proof. For $0 < k < n$ and $0 < m$ the map (9) is the pushout of the corresponding unreduced prism inclusion along the unit of the adjunction $\kappa \dashv i$ for the inclusion of $\mathbf{sSet}_0$ in $\mathbf{sSet}$. That is, the following square is a pushout in $\mathbf{sSet}$

$$\begin{tikzcd}
(\Delta^m \times \Lambda^n_k) \cup (\partial\Delta^m \times \Delta^n) \arrow[r] \arrow[d, hook] & (S^m \,{}_*\!\times \Lambda^n_k) \cup (\partial S^m \,{}_*\!\times \Delta^n) \arrow[d, hook] \\
\Delta^m \times \Delta^n \arrow[r] & S^m \,{}_*\!\times \Delta^n
\end{tikzcd} \tag{10}$$

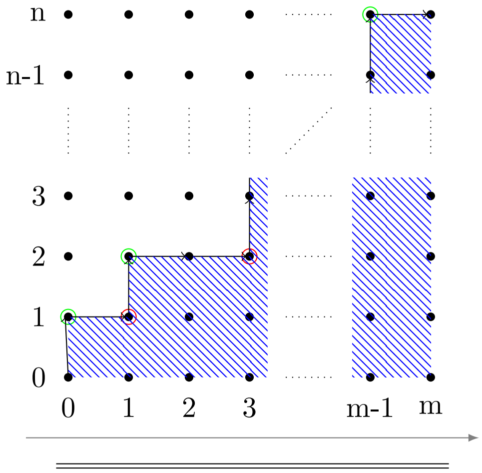
By Theorem H.0.20 of [Joy08] the map of simplicial sets

$$(\Delta^m \times \Lambda^n_k) \cup (\partial\Delta^m \times \Delta^n) \hookrightarrow \Delta^m \times \Delta^n$$

is inner anodyne. Any lifting problem against a map (9) can be extended to the left by the pushout square (10). Hence the maps (9) have the left lifting property against all inner fibrations, so they are inner anodyne. The case of $m = 0$ is trivial. □

The only cases for the maps (8) left to consider are the inclusions where the unreduced horn is an outer horn. As noted above, inner fibrations only have the right lifting property against outer horns in restricted circumstances, such as that of a special outer horn. In the rest of this section we will show that the outer horns occurring in the reduced prism inclusions (8) corresponding to outer horns $\Lambda^n_k \subseteq \Delta^n$ are all special outer horns and so the lifting problems can be solved. We will show this over the next few results, which rely on the following notation and constructions from [Joy08].

Consider the vertices of the prism $\Delta^m \times \Delta^n$, represented as a grid.



(11)

The simplicial set $\Delta^m \times \Delta^n$ is generated by $\binom{m+n}{n}$ $m+n$-simplices, which correspond to length $m+n$ paths in this grid. An example of such a path, which Joyal calls a **maximal chain**, is drawn with arrows on the grid (11). We identify two types of corners that occur in these chains. A **lower corner** of a maximal chain is a position (i, j) such that the chain

$$\begin{tikzcd}
 & (i,j+1) \\
(i-1,j) \arrow[r] & (i,j) \arrow[u]
\end{tikzcd} \tag{12}$$

is included in the maximal chain. The lower corners of the maximal chain in the diagram (11) are circled in red. Dually, an **upper corner** of a maximal chain is a position (i, j) such that the chain

$$\begin{tikzcd}
\text{(i,j)} \arrow[r] & \text{(i+1,j)} \\
\text{(i,j-1)} \arrow[u] &
\end{tikzcd} \tag{13}$$

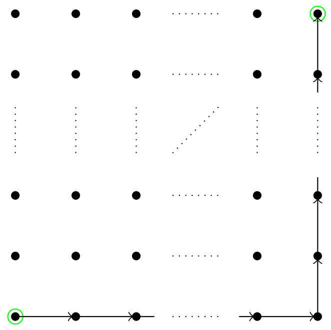
is included in the maximal chain. The upper corners of the maximal chain in the diagram (11) are circled in green.

Joyal defines a new partial order called the **transverse order** on the vertices by saying $(i, j) \preccurlyeq (k, l)$ when $i \geq k$ and $j \leq l$. The top element of $[m] \times [n]$ for this order is the corner $(0, n)$ in the top left and the bottom element is the corner $(m, 0)$ in the bottom right. A **transverse section** of the lattice $[m] \times [n]$ is a subset that is downward closed for this transverse order and contains the vertices $(0, 0)$ and (m, n). Given a maximal chain A, there is a corresponding transverse section called the **shadow** of A, which we denote $S_A \subseteq [m] \times [n]$. The shadow of A is the down-closed subset of $[m] \times [n]$ for the transverse order that contains the vertices of A. The shadow of the chain identified in the diagram (11) above consists of all vertices touching the blue shaded region. Note that in the transverse order, for every $(i, j) \in A$ there exists an upper corner u of A such that $(i, j) \preccurlyeq u$.

The map that sends a maximal chain A to the transverse section S_A is a bijection. This claim is made in Appendix H of [Joy08], we include a proof here for completeness.

Proposition 2.16. *The shadow is a bijection between the set* $\mathrm{Tr}(m, n)$ *of transverse sections of* $[m] \times [n]$ *and maximal chains of this space.*

Proof. Let T be a transverse section and let $M = \{m_x\}_{x \in I} \subseteq T$ be the set of maximal elements for the transverse order in T. Since T is down-closed, it consists of all elements $(i, j) \in [m] \times [n]$ such that $(i, j) \preccurlyeq m_x$ for some maximal element of T. Consider the unique chain A_T whose upper corners are those elements of M that are not $(0, 0)$ and (m, n). There is a unique such chain, as for a pair of upper corners there is a unique path joining them that doesn't create any intermediate upper corners, namely the path between the green circled upper corners below



The shadow of a maximal chain consists of all elements (i, j) such that $(i, j) \preccurlyeq u$ for some upper corner of the chain. The upper corners of A_T are exactly the maximal elements of T, so $S_{A_T} = T$. □

Finally, we define the simplicial subset of $\Delta^n \times \Delta^m$ that is spanned by the vertices of the shadow S_A. This is denoted $C(S_A)$ and it is equal to the union of all $m+n$-simplices whose maximal chains are contained in S_A.

The role of this partial order is to give a way of systematically extending a map defined on a subspace of $\Delta^m \times \Delta^n$ to the whole space. This is done by extending it to each $m+n$-simplex arising from a maximal chain in a way that agrees with the already established extension on the rest of the transverse section given by its shadow. The transverse sections form a finite inf-lattice, where the partial order is by inclusion. The meet is the intersection, which gives a transverse section because $(0,0)$ and (m,n) belong to all transverse sections and the intersection of down-closed subsets is down-closed. Joyal proves the following general result about extending maps that we will use.

Lemma 2.17 ([Joy08] H.0.16 and H.0.17)**.** *Let $\mathcal{A}$ be a class of morphisms in* ***sSet****. Suppose $E \subseteq X$ is a subspace of a simplicial set X and $\epsilon : \mathcal{L} \to \mathcal{P}(X)$ is an inf-preserving map of posets from a finite inf-lattice $\mathcal{L}$ to the poset of subspaces of X. Suppose for all $a \in \mathcal{L}$ the inclusion*

$$\bigcup_{b<a} \epsilon(b) \cup E \subseteq \epsilon(a) \cup E$$

has the left lifting property against all maps in $\mathcal{A}$. Then $E \subseteq X$ has the left lifting property against all maps in $\mathcal{A}$.

The simplices of $S^n \,{}_{*}\!\times \Delta^m$ consist of the simplices of $\Delta^m \times \Delta^n$ except for all the simplices of

$$\Delta^m \times [n] = \Delta^m \times (\Delta^n)_0 \subseteq \Delta^m \times \Delta^n$$

which are all identified to the degeneracy of the unique basepoint. Therefore we can unambiguously describe simplices in $S^n \,{}_{*}\!\times \Delta^m$ as ordered sequences of pairs $(i,j) \in [m] \times [n]$ as long as not all of the second coordinates are the same. Looking at the grid (11), this means that all horizontal sequences of arrows are the degeneracy of the unique basepoint, as indicated by the $=$ along the bottom axis. Hence the projection map to the quotient $\rho : \Delta^m \times \Delta^n \to S^n \,{}_{*}\!\times \Delta^m$ preserves intersections of subspaces $X, Y \subseteq \Delta^n \times \Delta^m$

$$\rho(X \cap Y) = \rho(X) \cap \rho(Y)$$

The inclusion of the left in the right always holds. The only way it could be a proper inclusion is if there were simplices $u_X \in X \setminus Y$ and $u_Y \in Y \setminus X$ such that $\rho(u_X) = \rho(u_Y)$ but $\rho(z) \neq \rho(u_X)$ for all $z \in X \cap Y$. This is impossible as $\rho(u_X) = \rho(u_Y)$ implies both are sent to the basepoint and the basepoint must belong to $\rho(X \cap Y)$.

This analysis allows us to adapt the proof of Theorem H.0.20 and Proposition H.0.21 in [Joy08] to prove the following half of the remaining cases for the lifting properties of maps (8).

Lemma 2.18. *The inclusion $(S^n \,{}_{*}\!\times \Lambda^m_m) \cup (\partial S^n \,{}_{*}\!\times \Delta^m) \subseteq S^n \,{}_{*}\!\times \Delta^m$ has the left lifting property against all inner fibrations between quasi-monoids.*

Proof. Let $\mathcal{L} = \mathrm{Tr}(m,n)$ be the poset of transverse sections of $[m] \times [n]$. Define a map

$$c : \mathrm{Tr}(m,n) \to \mathcal{P}(S^n \,{}_{*}\!\times \Delta^m)$$

sending a transverse section T to the image in $S^n \,{}_{*}{\times}\, \Delta^m$ of the simplicial set in $\Delta^m \times \Delta^n$ spanned by the vertices. In other words, $c(T) = \rho(C(T))$. The infimum of two transverse sections T, T' is their intersection and the discussion above shows that

$$c(T \cap T') = c(T) \cap c(T')$$

so c is an inf-preserving map of posets.

We now apply Lemma 2.17 to the subspace $(S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m) \subseteq S^n \,{}_{*}{\times}\, \Delta^m$ and the function $c : \mathrm{Tr}(m,n) \to \mathcal{P}(S^n \,{}_{*}{\times}\, \Delta^m)$. We denote

$$\dot{c}(S) = \bigcup_{T \lneq S} c(T) \subsetneq c(S)$$

So to show

$$(S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m)) \subseteq S^n \,{}_{*}{\times}\, \Delta^m$$

has the left lifting property against all inner fibrations between quasi-monoids it is sufficient to show that the inclusions

$$((S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m)) \cup \dot{c}(S) \subseteq ((S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m)) \cup c(S)$$

do for all transverse sections $S \in \mathrm{Tr}(m,n)$.

Every transverse section is the shadow of a unique maximal chain, so $S = S_A$ for a maximal chain A. The only $m+n$-simplex missing from $\dot{c}(S_A)$ is the simplex of $S^n \,{}_{*}{\times}\, \Delta^m$ corresponding to the maximal chain A, which we denote S^{m+n}_A. Hence $c(S_A) = \dot{c}(S_A) \cup S^{m+n}_A$ and the following square is a pushout in $\mathbf{sSet}_0$

$$\begin{tikzcd}
S^{m+n}_A \cap (\dot{c}(S_A) \cup (S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m))) \arrow[r, hook] \arrow[d, hook] & \dot{c}(S_A) \cup ((S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m)) \arrow[d, hook] \\
S^{m+n}_A \arrow[r, hook] & c(S_A) \cup ((S^n \,{}_{*}{\times}\, \Lambda^m_m) \cup (\partial S^n \,{}_{*}{\times}\, \Delta^m))
\end{tikzcd} \tag{14}$$

Joyal shows in Lemma H.0.19 of [Joy08] that

$$\Delta^{m+n}_A \cap \left(\left(\bigcup_{T \lneq S_A} C(S_A) \right) \cup (\Lambda^m_m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \right) = \Lambda^{m+n}_{\mathrm{lc(A)} \cup \{t\}} \subseteq \Delta^{n+m}_A$$

where Δ^{m+n}_A is the $n+m$-simplex spanned by the maximal chain A, $\mathrm{lc}(A)$ is the set of lower corners of A and t is the least element in the column $\{m\} \times [n]$ with respect to the standard partial order on $[m] \times [n]$. The image of this inclusion under ρ is the left hand inclusion in (14), since ρ preserves intersections. Therefore the square (14) is the pushout of a generalized horn $\lambda^{n+m}_{\mathrm{lc}(A) \cup \{t\}} \subseteq S^{n+m}$ for each choice of maximal chain A in $[m] \times [n]$. We will show that all these inclusions have the left lifting property against inner fibrations between quasi-monoids.

Consider the two possible cases for the last edge of A terminating at (m,n). We will show that in each case the lifting problem can be solved.

1. If the edge is $$\begin{tikzcd}(m,n) \\ (m,n-1) \arrow[u]\end{tikzcd}$$ then there exists a lower corner (m, j) for some $j < n$ and the highest element of A in the column $\{m\} \times [n]$ is not (m, n). In particular, $(0, 0)$, which is never a lower corner, and (m, n) do not belong to $\mathrm{lc}(A) \cup \{t\}$ but there exists some lower corner $(0, 0) < (m, j) < (m, n)$ belonging to this set. Hence $\lambda^{n+m}_{\mathrm{lc}(A)\cup\{t\}}$ is inner anodyne.

2. If the edge is $\begin{tikzcd}(m-1,n) \arrow[r] & (m,n)\end{tikzcd}$ then the top element of $\{m\} \times [n]$ is (m, n), but again $(0, 0) \notin \mathrm{lc}(A) \cup \{(m, n)\}$ so $\mathrm{lc}(A) \cup \{(m, n)\} \subseteq [1, \cdots , m + n]$ in S^{m+n}_A and the inclusion is right anodyne. But the last edge is $(m-1, n) \leq (m, n)$ so it is the identity in $S^n {}_* \!\times \Delta^m$ and so this inclusion has the left lifting property against all inner fibrations between quasi-monoids by Proposition 2.11.

□

The case of $S^n {}_* \!\times \Lambda^m_0$ follows from this by duality, so we have shown that all the maps (8) have the left lifting property against inner fibrations between quasi-monoids. Applying the correspondences of the adjunction of two variables defined in Proposition 2.3, we have the following two immediate corollaries that we promised at the start of this section.

Corollary 2.19. *The simplicial set $(X^A)_*$ is a Kan complex for every $X, A \in$ **sSet**$_0$ with X a quasi-monoid.*

Corollary 2.20. *For any inner fibration between quasi-monoids $f : X \to Y$ and any inclusion $A \subseteq B$ of reduced simplicial sets the map*

$$(X^B)_* \to (X^A)_* \times_{(Y^A)_*} (Y^B)_*$$

is a Kan fibration between Kan complexes.

2.4 The Quasi-Monoid Simplicial Model Structure

We will define a model structure on **sSet**$_0$ such that the cofibrations are the inclusions and the fibrant objects are the quasi-monoids. As mentioned at the start of this chapter, our approach is a hybrid of Joyal's methods for the mapping spaces $(-^-)_*$ defined in the previous section and the framework used by Bousfield in [Bou75] for the model structure on **sSet** for homology equivalences. In this section we will define the classes of proposed weak equivalences, cofibrations, and fibrations for the model structure and show that they have the required lifting and factorization properties. We will follow [Joy08] Chapters 4, 5, and 6 in making heavy use of the adjunction of two variables and the mapping space $(-^-)_*$ that we described in the previous section. These methods will give characterizations of the classes of maps for the model structure in terms of lifting properties. Then we will use the approach of Bousfield by showing that there exists a functor from the arrow category of **sSet**$_0$ to the category of graded groups that commutes with filtered colimits and exactly characterizes when a map belongs to the class of proposed weak equivalences for the quasi-monoid model structure. This allows us to replace the homology functor in the proofs in [Bou75] and use the same arguments to show that **sSet**$_0$ has a cofibrantly generated model structure. Finally, at the end of this section we will show that this model structure is the transferred model structure from the Joyal model structure on simplicial sets (Theorem 6.2 of [Joy08]) via the adjunction $\kappa \dashv i$ with the inclusion functor of $\mathbf{sSet}_0$ in $\mathbf{sSet}$.

We begin by defining $\mathcal{W}$, the proposed class of weak equivalences. The model structure we are proposing will be a simplicial model structure, so the mapping spaces $(-^-)_*$ will allow us to identify weak equivalences using Corollary 9.7.5 of [Hir03]. Anticipating these results we define $\mathcal{W}$ as follows.

Definition 2.21. *Let $\mathcal{W}$ be the class of maps $f : A \to B$ in $\mathbf{sSet}_0$ such that for all quasi-monoids X the map $f^* : (X^B)_* \to (X^A)_*$ is a Kan weak equivalence.*

It is clear that the 2 out of 3 property holds for maps in $\mathcal{W}$ by the corresponding property for Kan weak equivalences of $\mathbf{sSet}$.

We will now build up the classes of fibrations and cofibrations that will make up the model structure for quasi-monoids on $\mathbf{sSet}_0$. The class of cofibrations $\mathcal{C}$ will be the monomorphisms of reduced simplicial sets, as is the case for the Kan model structure on reduced simplicial sets of Theorem 2.1. The results of the previous section give the following characterization of maps in $\mathcal{C} \cap \mathcal{W}$.

Proposition 2.22. *The class $\mathcal{C} \cap \mathcal{W}$ consists of the monomorphisms $A \hookrightarrow B$ of reduced simplicial sets such that $(X^B)_* \to (X^A)_*$ is an acyclic fibration for all quasi-monoids X.*

Proof. By Corollary 2.20 applied to $A \subseteq B$ and $X \to *$, an inner fibration between quasi-monoids, $(X^B)_* \to (X^A)_*$ is a Kan fibration. Hence it is a Kan weak equivalence if and only if it is an acyclic fibration. $\square$

We will want quasi-monoids to be fibrant objects of this model structure, so we will need the inner horns $\lambda_k^n \hookrightarrow S^n$ to belong to $\mathcal{C} \cap \mathcal{W}$. We can show that they do by using the adjoint properties of the functor $(-^-)_*$ and the results of [Joy08] Appendix H for unreduced prisms.

Proposition 2.23. *Inner anodyne maps between reduced simplicial sets belong to $\mathcal{C} \cap \mathcal{W}$.*

Proof. By Proposition 2.22 we must show that for any inner anodyne map $A \hookrightarrow B$ and for any quasi-monoid X the map

$$(X^B)_* \to (X^A)_*$$

is an acyclic fibration of simplicial sets. By the adjointness of Proposition 2.3 the following lifting problems are equivalent.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & (X^B)_* \arrow[d] \\
\Delta^n \arrow[r] & (X^A)_*
\end{tikzcd}$$

$$\begin{tikzcd}
(B \mathbin{{}_*\times} \partial\Delta^n) \cup_{A \mathbin{{}_*\times} \partial\Delta^n} (A \mathbin{{}_*\times} \Delta^n) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
B \mathbin{{}_*\times} \Delta^n \arrow[r] & *
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & E_0(X^{\Delta^n}) \arrow[d] \\
B \arrow[r] & E_0(X^{\partial\Delta^n})
\end{tikzcd}$$

Hence it is sufficient to prove that for all $n \geq 0$ and all quasi-monoids X the map of reduced simplicial sets $E_0(X^{\partial\Delta^n}) \to E_0(X^{\Delta^n})$ is an inner fibration. Hence it is sufficient to prove that $(X^{S^m})_* \to (X^{\lambda_k^m})$ is an acyclic fibration for all $m \geq 2$, $0 < k < m$.

To do this, by the adjointness of lifting problems described in (7) we must show that for all $n \geq 0$

$$(S^m \mathbin{{}_*\times} \partial\Delta^n) \cup_{\lambda_k^m \mathbin{{}_*\times} \partial\Delta^n} (\lambda_k^m \mathbin{{}_*\times} \Delta^n) \hookrightarrow S^m \mathbin{{}_*\times} \Delta^n$$

is an inner anodyne map. For $m, n \geq 0$ and $0 < k < m$ the following squares are pushouts

$$\begin{tikzcd}
(\Lambda^m_k \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \arrow[r] \arrow[d, hook] & (\lambda^m_k {}_*\times \Delta^n) \cup (S^m {}_*\times \partial\Delta^n) \arrow[d, hook] \\
\Delta^m \times \Delta^n \arrow[r] & S^m {}_*\times \Delta^n
\end{tikzcd}$$

Now by Theorem H.0.20 of [Joy08] for $0 < k < m$ the map of simplicial sets

$$(\Lambda^m_k \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \hookrightarrow \Delta^m \times \Delta^n$$

is inner anodyne, so we are done.

□

We now begin to identify some of the fibrations of our proposed model structure. By standard model structure properties these will be the maps that have the right lifting property against all maps in $\mathcal{C} \cap \mathcal{W}$.

Proposition 2.24. *Let $f : X \to Y$ be an inner fibration between quasi-monoids. Then f has the right lifting property against all maps of $\boldsymbol{sSet}_0$ that belong to $\mathcal{C} \cap \mathcal{W}$.*

Proof. Let $A \hookrightarrow B$ belong to $\mathcal{C} \cap \mathcal{W}$. Consider the pullback

$$\begin{tikzcd}
(X^B)_* \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (X^A)_* \times_{(Y^A)_*} (Y^B)_* \arrow[r] \arrow[d] & (X^A)_* \arrow[d] \\
& (Y^B)_* \arrow[r] & (Y^A)_*
\end{tikzcd}$$

The maps $(X^B)_* \to (X^A)_*$ and $(Y^B)_* \to (Y^A)_*$ are acyclic fibrations by Proposition 2.22, so the comparison map

$$(X^B)_* \to (X^A)_* \times_{(Y^A)_*} (Y^B)_* \tag{15}$$

is a Kan weak equivalence. By Corollary 2.20 it is a Kan fibration, so it is an acyclic fibration. The left lifting property for (15) implies f has the right lifting property against

$$(A {}_*\times \Delta^n) \cup_{A_* \times \partial\Delta^n} (B {}_*\times \partial\Delta^n) \hookrightarrow B {}_*\times \Delta^n$$

for all $n \geq 0$. Taking $n = 0$ gives the result. □

In the proposed model structure, therefore, inner fibrations between quasi-monoids must be fibrations. Let $\mathcal{F}$ be the class of maps that have the right lifting property against all maps in $\mathcal{W} \cap \mathcal{C}$, which are the proposed fibrations of this model structure we are constructing. Before showing that there is a model structure with these classes of maps, we first recognize that the maps in $\mathcal{F} \cap \mathcal{W}$ are acyclic fibrations of the Kan model structure of Theorem 2.1. That is, these are maps of reduced simplicial sets that are acyclic fibrations in the Kan model structure on **sSet**, so they have the right lifting property against all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$. This will allow us to refer to acyclic fibrations of reduced simplicial sets without confusion as to which of the model structures we are referring to.

To do this, we will define the following notion of homotopy equivalence for $\mathbf{sSet}_0$.

Definition 2.25. *We say a map of reduced simplicial sets $f : X \to Y$ is a* ***reduced homotopy equivalence*** *when there exists a map $g : Y \to X$ and maps H and G making the following diagrams commute*

$$\begin{tikzcd}
X \arrow[d, hook] \arrow[dr, equal] & \\
X \,{}_{*}{\times}\, \Delta^1 \arrow[r, "H"'] & X \\
X \arrow[u, hook'] \arrow[r, "f"'] & Y \arrow[u, "g"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[d, hook] \arrow[dr, equal] & \\
Y \,{}_{*}{\times}\, \Delta^1 \arrow[r, "G"'] & Y \\
Y \arrow[u, hook'] \arrow[r, "g"'] & X \arrow[u, "f"]
\end{tikzcd}$$

We will eventually show that the quasi-monoid model structure on $\mathbf{sSet}_0$ is a simplicial model structure, so that $A \,{}_{*}{\times}\, \Delta^1$ is a cylinder object for a reduced simplicial set in this model structure. This definition, therefore, is describing a left homotopy equivalence, so we expect that such a structure makes f into a weak equivalence of the quasi-monoid model structure, which it does.

Proposition 2.26. *Let $f : A \to B$ be a reduced homotopy equivalence in $\boldsymbol{sSet}_0$. Then f belongs to $\mathcal{W}$.*

Proof. By Definition 2.21 we must show that for all quasi-monoids X the map of simplicial sets

$$(X^f)_* : (X^B)_* \to (X^A)_*$$

is a Kan weak equivalence. In fact, we will show that this map is a Kan weak equivalence for any reduced simplicial set, not only for a quasi-monoid X.

Let $g : B \to A$, $H : A \,{}_{*}{\times}\, \Delta^1 \to A$, and $G : B \,{}_{*}{\times}\, \Delta^1 \to B$ be the required maps as in Definition 2.25. The homotopies H and G are 1-simplices of $(A^A)_*$ and $(B^B)_*$ respectively. We will define a map of simplicial sets inspired by the map in Lemma 19.5 of [Rez17]

$$\gamma : (B^A)_* \to \underline{\mathrm{Hom}}_{\mathbf{sSet}_*}((X^B)_*, (X^A)_*)$$

where we are temporarily recalling the pointed structure of the simplicial sets $(X^A)_*$ which we have been forgetting ever since constructing the mapping space for $\mathbf{sSet}_0$ in Section 2.1. The space $\underline{\mathrm{Hom}}_{\mathbf{sSet}_*}((X^B)_*, (X^A)_*)$ is the mapping space for the pointed simplicial sets $(X^B)_*$ and $(X^A)_*$ where n-simplices are the maps

$$\Delta^n_+ \wedge (X^B)_* \to (X^A)_*$$

with Δ^n_+ the standard simplicial n-simplex with a freely adjoined basepoint.

We define the map γ to take an n-simplex $x : A \,{}_{*}{\times}\, \Delta^n \to B$ of $(B^A)_*$ and send it to the map

$$\gamma(x) : \Delta^n_+ \wedge (X^B)_* \to (X^A)_*$$

defined on t-simplices by

$$\gamma(x)(\theta : [t] \to [n] \in \Delta^n_t, y : B \,{}_{*}{\times}\, \Delta^t \to X \in ((X^B)_*)_t) = x_{\theta,y} : A \,{}_{*}{\times}\, \Delta^t \to X$$

where $x_{\theta,y}$ is defined on s-simplices by

$$x_{\theta,y}(\rho : [s] \to [t] \in \Delta^t_s, u \in A_s) = y(\rho : [s] \to [t], x(u \in A_s, \theta \circ \rho : [s] \to [n]))$$

This respects simplicial operations and the basepoint.

For any reduced simplicial set X the map γ sends H and G to homotopies between the simplicial sets $(X^A)_*$ and $(X^B)_*$. These homotopies show that $f^* : (X^B)_* \to (X^A)_*$ and $g^* : (X^A)_* \to (X^B)_*$ are homotopy inverses of each other, hence they are weak equivalences of reduced simplicial sets, and so of simplicial sets in the Kan model structure on **sSet**. ☐

We can now show that all Kan acyclic fibrations between reduced simplicial sets belong to $\mathcal{F} \cap \mathcal{W}$.

Corollary 2.27. *A Kan acyclic fibration between reduced simplicial sets is in* $\mathcal{F} \cap \mathcal{W}$.

Proof. First we observe that a Kan acyclic fibration belongs to $\mathcal{F}$ because it has the right lifting property against all maps in $\mathcal{C}$ and so in particular against all maps in $\mathcal{C} \cap \mathcal{W}$. We will show that an acyclic fibration belongs to $\mathcal{W}$ by showing that it is part of a reduced homotopy equivalence, so by Proposition 2.26 it belongs to $\mathcal{W}$. This is the standard proof for acyclic fibrations in a model category of cofibrant objects, see for example Proposition 1.3.2 of [Cis06].

All reduced simplicial sets are cofibrant in the reduced Kan model structure so an acyclic fibration $p : A \to B$ has a retract $j : B \hookrightarrow A$. This retract is a reduced homotopy inverse since p is an acyclic fibration so there exists a lift for the diagram

$$\begin{tikzcd}
(B \,{}_*\!\times \Delta^1) \cup_{A_* \times \partial\Delta^1} (A \,{}_*\!\times \partial\Delta^1) \arrow[rr] \arrow[d, hook] & & A \arrow[d, "p"] \\
A \,{}_*\!\times \Delta^1 \arrow[r] \arrow[rru, "H"'] & A \arrow[r, "p"'] & B
\end{tikzcd}$$

which determines one homotopy of Definition 2.25 of a reduced homotopy equivalence. The other homotopy, $G : B \,{}_*\!\times \Delta^1 \to B$ is trivial as j is a retract for p. ☐

We have shown that Kan acyclic fibrations all belong to $\mathcal{F} \cap \mathcal{W}$. We will now show that $\mathcal{F} \cap \mathcal{W}$ is exactly the class of Kan acyclic fibrations of reduced simplicial sets.

Proposition 2.28. *Let* $f : X \to Y$ *be a map with the right lifting property against all maps in* $\mathcal{W} \cap \mathcal{C}$. *Then* $f \in \mathcal{W}$ *if and only if* f *is a Kan acyclic fibration.*

Proof. We showed acyclic fibrations are in $\mathcal{W}$ in Corollary 2.27. For the converse, factor $f : A \to B$ in $\mathbf{sSet}_0$ as a cofibration $u : A \hookrightarrow E$ followed by a Kan acyclic fibration $p : E \to B$ using the Kan model structure on $\mathbf{sSet}_0$ of Theorem 2.1

$$\begin{tikzcd}
A \arrow[r, hook, "u"] \arrow[dr, "f"'] & E \arrow[d, "p"] \\
 & B
\end{tikzcd}$$

The Kan acyclic fibration is in $\mathcal{W}$ by Corollary 2.27, so by 2 out of 3 for $\mathcal{W}$ since f belongs to $\mathcal{W}$ by assumption the cofibration does too. As f belongs to $\mathcal{F}$ it has the left lifting property against u, which belongs to $\mathcal{C} \cap \mathcal{W}$, so there is a solution to the lifting problem

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, hook, "u"'] & A \arrow[d, "f"] \\
E \arrow[r, "p"'] \arrow[ur, dashed] & B
\end{tikzcd}$$

This makes f a retract of the acyclic fibration p so it is an acyclic fibration. $\square$

This identification of $\mathcal{F} \cap \mathcal{W}$ as the same as the acyclic fibrations of the model structure of Theorem 2.1 allows us to simply lift the existence of a weak factorization system $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ straight from the reduced Kan model structure on $\mathbf{sSet}_0$. We now have to show the weak factorization system $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$. We will start with the following special case for maps between quasi-monoids. Recall from Proposition 2.24 that inner fibrations between quasi-monoids belong to $\mathcal{F}$, so this is an instance of this factorization.

Proposition 2.29. *Let $f : X \to Y$ be a map between quasi-monoids. There exists a factorization*

$$\begin{tikzcd}
X \arrow[r, hook, "\sim", "i_X"'] \arrow[rd, "f"'] & P(f) \arrow[d, "p_f"] \\
& Y
\end{tikzcd}$$

where $P(f)$ is a quasi-monoid, $i_X \in \mathcal{W} \cap \mathcal{C}$ and p_f is an inner fibration.

Proof. Consider the diagram

$$\begin{tikzcd}
X \arrow[rrrd, bend left, equal] \arrow[rd, dashed, "i_X"] \arrow[rdd, bend right] & & & \\
& P(f) \arrow[r] \arrow[d, "\pi"] & X \times Y \arrow[r, two heads] \arrow[d] & X \arrow[d, "f"] \\
& E_0(Y^{\Delta^1}) \arrow[r] \arrow[rr, bend right, two heads, "E_0(Y^{d^1})"'] & E_0(Y^{\partial\Delta^1}) \arrow[r, two heads] & Y
\end{tikzcd}$$

The outer square is defined to to be the pullback of f along $E_0(Y^{d^1})$. The right square is a pullback because $E_0(Y^{\partial\Delta^1}) = Y \times Y$, so the left square is also a pullback. The composite of the bottom two maps is the map

$$E_0(Y^{d^1}) : E_0(Y^{\Delta^1}) \to E_0(Y^{\Delta^0}) = Y \tag{16}$$

By the correspondence of lifting problems

$$\begin{tikzcd}
\partial S^m \arrow[r] \arrow[d, hook] & E_0(Y^{\Delta^1}) \arrow[d] & & \Delta^0 \arrow[r] \arrow[d, hook, "d^1"'] & (Y^{S^m})_* \arrow[d] \\
S^m \arrow[r] & E_0(Y^{\Delta^0}) & & \Delta^1 \arrow[r] & (Y^{\partial S^m})_*
\end{tikzcd}$$

the lifting problem on the right has a solution by Corollary 2.20 since Y is a quasi-monoid and $d^1 : \Delta^0 \hookrightarrow \Delta^1$ is an anodyne map. Hence the map (16) is an acyclic fibration of reduced simplicial sets. Since the total square is a pullback the the top map is an acyclic fibration and hence in $\mathcal{W}$ by Corollary 2.27. The left hand map $X \to E_0(Y^{\Delta})$ is the composite

$$X \xrightarrow{f} Y = E_0(Y^{\Delta^0}) \xrightarrow{(s^0)^*} E_0(Y^{\Delta^1})$$

where $s^0 : \Delta^1 \to \Delta^0$ is the unique map, which is a retract of $d^1 : \Delta^0 \to \Delta^1$. The comparison map i_X is a retract of a map in $\mathcal{W}$, so it is in $\mathcal{W} \cap \mathcal{C}$.

Now consider the map

$$p_f : P(f) \xrightarrow{\pi} E_0(Y^{\Delta^1}) \xrightarrow{(d^0)^*} Y$$

Since $\pi \circ i_X = (s^0)^* \circ f$ and s^0 is also a retract of d^0 we have $p_f \circ i_X = f$. Now consider the diagram

$$\begin{tikzcd}
P(f) \arrow[r] \arrow[d, "\pi"] \arrow[ddr, bend left, "p_f"] & X \times Y \arrow[d] \arrow[dd, bend left=60, "\mathrm{pr}_1"] \\
E_0(Y^{\Delta^1}) \arrow[r] \arrow[dr, "(d^0)^*"'] & Y \times Y \arrow[d, "\mathrm{pr}_1"] \\
 & Y
\end{tikzcd} \tag{17}$$

The square is a pullback and $E_0(Y^{\Delta^1}) \to E_0(Y^{\partial\Delta^1}) = Y \times Y$ is an inner fibration by the correspondence of lifting problems

$$\begin{tikzcd}
\lambda^m_k \arrow[r] \arrow[d, hook] & E_0(Y^{\Delta^1}) \arrow[d] \\
S^m \arrow[r] & E_0(Y^{\partial\Delta^1})
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^m \,{}_*\!\times \partial\Delta^1) \cup_{\lambda^m_k \,{}_*\!\times \partial\Delta^1} (\lambda^m_k \,{}_*\!\times \Delta^1) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\
S^m \,{}_*\!\times \Delta^1 \arrow[r] & *
\end{tikzcd}$$

where the right lifting problem can be solved since Y is a quasi-monoid and by the proof of Proposition 2.23 for all $n \geq 0$, $m \geq 2$, and $0 < k < m$

$$(S^m \,{}_*\!\times \partial\Delta^n) \cup_{\lambda^m_k \,{}_*\!\times \partial\Delta^n} (\lambda^m_k \,{}_*\!\times \Delta^n) \hookrightarrow S^m \,{}_*\!\times \Delta^n$$

is an inner anodyne map. So the top map of the pullback square in (17) is an inner fibration. The product projection pr_1 is an inner fibration, so p_f is an inner fibration. □

This factorization allows us to replace any map between quasi-monoids with an inner fibration up to a map in $\mathcal{W}$. To replace any map between reduced simplicial sets by a map between quasi-monoids we will use the small object argument with the set of inner horn inclusions $h^k_n : \lambda^k_n \hookrightarrow S^n$ with $2 \leq n$ and $0 < k < n$ in $\mathbf{sSet}_0$.

Proposition 2.30 (Small Object Argument)**.** *There exists a functor $R : \boldsymbol{sSet}_0 \to \boldsymbol{sSet}_0$ and a natural transformation $\rho : \mathrm{Id} \to R$ such that for any reduced simplicial set X $R(X)$ is a quasi-monoid X and the components $\rho_X : X \to R(X)$ are inner anodyne. This functor preserves filtered colimits.*

Corollary 2.31. *We can replace any map by a map in $\mathcal{F}$, up to maps in $\mathcal{W} \cap \mathcal{C}$.*

Proof. Apply the functor R to the map $f : X \to Y$. Then factor $R(f)$ as a map in $\mathcal{W} \cap \mathcal{C}$ followed by an inner fibration using Proposition 2.29

$$\begin{tikzcd}
X \arrow[r, hook, "\rho_X"] \arrow[d, "f"'] & R(X) \arrow[r, "i_{R(X)}"] \arrow[d, "R(f)"'] & P(R(f)) \arrow[dl, "p_{R(f)}"] \\
Y \arrow[r, hook, "\rho_Y"] & R(Y) &
\end{tikzcd}$$

The maps ρ_X and ρ_Y are inner anodyne by Proposition 2.30 so by Proposition 2.23 they belong to $\mathcal{W} \cap \mathcal{C}$. Hence the top row maps are in $\mathcal{W} \cap \mathcal{C}$, so their composition is. The map $p_{R(Y)}$ is an inner fibration between quasi-monoids, so it has the right lifting property against all maps in $\mathcal{W} \cap \mathcal{C}$ by Proposition 2.24. $\square$

We will now define a functor that will give us another characterization of maps in $\mathcal{W}$. We will use this functor and a modified version of the approach in §11 of [Bou75] to complete our proof of the model structure for quasi-monoids on $\mathbf{sSet}_0$. The functor $h_\bullet$ we define here will replace the homology functor used in §11 of [Bou75] to prove the existence of a homology model structure for **sSet**. This functor $h_\bullet$ acts on objects of the arrow category by

$$h_\bullet(f : A \to B) = \pi_\bullet \left((\mathrm{fib}(p_{R(f)})^{S_1})_* \right)$$

where $\mathrm{fib}(p_{R(f)})$ is the fiber at the basepoint of the map $p_{R(f)}$ obtained in Corollary 2.31. The key fact needed to adapt the work of Bousfield is that this functor commutes with filtered colimits.

Proposition 2.32. *The functor $h_\bullet : \boldsymbol{sSet}_0^{\Delta^1} \to \mathrm{Gr}(\boldsymbol{Gp})$ from the arrow category of reduced simplicial sets to graded groups commutes with filtered colimits.*

Proof. The functor $h_\bullet$ sends a map $f : X \to Y$ to $\pi_\bullet \left((\mathrm{fib}(p_{R(f)})^{S_1})_* \right)$ The functor R commutes with filtered colimits by Proposition 2.30. The functor $E_0(-^{\Delta^1})$ and all the finite limits in the path space construction commute with filtered colimits. Taking the fiber is a pullback, so it commutes with filtered colimits. The map $p_{R(f)}$ is an inner fibration, so its fiber is a quasi-monoid. The functor $(-^{S^1})_*$ commutes with filtered colimits because S^1 is a finite simplicial set. The space $(\mathrm{fib}(p_{R(f)})^{S^1})_*$ is a Kan complex by Corollary 2.19, so its n^{th} homotopy group is the quotient of a subset of the n-simplices, which commutes with filtered colimits. $\square$

The role of this functor is in identifying maps in $\mathcal{W}$. The following result is based on methods of §7 of [DS11a]. In this section the authors define maps of simplicial sets called *DK-equivalences* and show how these are related to other notions of equivalences for simplicial sets. We observe that the second condition in the definition of DK-equivalences (Definition 7.1 [DS11a]) is the same as our condition for maps of $\mathbf{sSet}_0$ to belong to $\mathcal{W}$. Furthermore, by Lemma 7.2 an inner fibration between quasi-monoids is a DK-equivalence exactly when it belongs to $\mathcal{W}$, since all maps between reduced simplicial sets are surjective on 0-simplices. Therefore, we can adapt the proof of Propositions 7.5 and 7.6 of [DS11a] to show that a map of reduced simplicial sets that is an inner fibration between quasi-monoids and belongs to $\mathcal{W}$ is a Kan acyclic fibration.

Lemma 2.33. *Let $f : X \to Y$ be an inner fibration between quasi-monoids. Then f is a Kan acyclic fibration if and only if $(f^{S^1})_* : (X^{S^1})_* \to (Y^{S^1})_*$ is a Kan acyclic fibration.*

Proof. If f is a Kan acyclic fibration then it has the right lifting property against maps

$$(* \ {}_*\!\times \Delta^n) \cup (S^1 \ {}_*\!\times \partial\Delta^n) \hookrightarrow S^1 \ {}_*\!\times \Delta^n$$

for all $n \geq 0$, so $(f^{S^1})_*$ is an acyclic fibration.

Suppose $(f^{S^1})_*$ is a Kan acyclic fibration. We must show that f has the right lifting property against $\partial S^n \hookrightarrow S^n$ for all $n \geq 0$. The case of $n = 0$ follows immediately from the fact that X and

Y are reduced simplicial sets, so they have unique 0-simplices. For $n \geq 1$ we will show that the pullback product map

$$(X^{S^n})_* \to (Y^{S^n})_* \times_{(Y^{\partial S^n})_*} (X^{\partial S^n})_*$$

is an acyclic fibration, and in particular surjective. Hence there are solutions to the equivalent lifting problems

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & (X^{S^n})_* \arrow[d] \\
\Delta^0 \arrow[r] & (X^{\partial S^n}) \times_{(Y^{\partial S^n})_*} (Y^{S^n})_*
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n {}_*\!\times \emptyset) \cup_{(\partial S^n {}_*\!\times \emptyset)} (\partial S^n {}_*\!\times \Delta^0) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
S^n {}_*\!\times \Delta^0 \arrow[r] & Y
\end{tikzcd}$$

so f has the right lifting property against the remaining generating cofibrations $\partial S^n \hookrightarrow S^n$ for $n \geq 1$.

Consider maps

$$\bigvee_{i=0}^{n-1} S^1 \hookrightarrow S^n$$

including the reduced 1-simplices along the spine of the reduced n-simplex S^n for all $n \geq 1$. We will call these maps **spine inclusions**. These are inner anodyne by Proposition 2.13 of [Joy08]. In the square

$$\begin{tikzcd}
(X^{S^n})_* \arrow[r] \arrow[d, "\wr"] & (Y^{S^n})_* \arrow[d, "\wr"] \\
(X^{\vee_{i=0}^{n-1} S^1})_* \arrow[r] \arrow[d, equal, "\wr"] & (Y^{\vee_{i=0}^{n-1} S^1})_* \arrow[d, equal, "\wr"] \\
\prod_{i=0}^{n-1} (X^{S^1})_* \arrow[r] & \prod_{i=0}^{n-1} (Y^{S^1})_*
\end{tikzcd}$$

the vertical maps are acyclic fibrations by Propositions 2.22 and 2.23. The bottom horizontal map is the product of Kan weak equivalences between Kan complexes, so it is a Kan weak equivalence. Hence the top horizontal map is a Kan weak equivalence. In the pullback square

$$\begin{tikzcd}
(X^{S^n})_* \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& (Y^{S^n})_* \times_{(Y^{\partial S^n})_*} (X^{\partial S^n})_* \arrow[r] \arrow[d] & (X^{\partial S^n})_* \arrow[d] \\
& (Y^{S^n})_* \arrow[r] & (Y^{\partial S^n})_*
\end{tikzcd}$$

the comparison map and the bottom map are Kan fibrations by Corollary 2.20. We have just showed that the left map $(X^{S^n})_* \to (Y^{S^n})_*$ is a Kan weak equivalence, so to show the comparison map is an acyclic fibration for all $n \geq 1$ it is sufficient to show that $(X^{\partial S^n})_* \to (Y^{\partial S^n})_*$ is a Kan weak equivalence for all $n \geq 1$. Observe for $n \geq 1$ ∂S^n is covered by the subspaces $S^{n-1} \cong \mathrm{Im}(d^1) \subseteq S^n$ and λ_1^n, whose intersection is isomorphic to ∂S^{n-1}. Therefore there is a pushout of simplicial sets

$$\begin{tikzcd}
\partial S^{n-1} \arrow[r, hook] \arrow[d, hook] & S^{n-1} \arrow[d, hook, "d^1"] \\
\lambda_1^n \arrow[r, hook] & \partial S^n
\end{tikzcd}$$

that attaches the missing face of λ_n^1 needed to make ∂S^n along its boundary ∂S^{n-1}. This corresponds to a pullback of mapping spaces

$$\begin{tikzcd}
(X^{\partial S^n})_* \arrow[r] \arrow[d] & (X^{S^{n-1}})_* \arrow[d] \\
(X^{\lambda_1^n})_* \arrow[r] & (X^{\partial S^{n-1}})_*
\end{tikzcd}$$

where all maps are Kan fibrations. This square is homotopy cartesian, so it is sufficient to show that all three maps

$$(X^{\partial S^{n-1}})_* \to (Y^{\partial S^{n-1}})_*$$
$$(X^{\lambda_1^n})_* \to (Y^{\lambda_1^n})_*$$
$$(X^{S^{n-1}})_* \to (Y^{S^{n-1}})_*$$

are Kan weak equivalences. The last is by earlier, the middle is by the same discussion using the inner anodyne map $\vee_{i=0}^{n-1} S^1 \hookrightarrow \lambda_1^n$, and the first is by the inductive hypothesis. □

This argument shows that for maps in $\mathcal{W}$ between quasi-monoids, the maps $(X^{S^n})_* \to (Y^{S^n})_*$ and $(X^{\partial S^n})_* \to (Y^{\partial S^n})_*$ for all $n \geq 0$ are Kan weak equivalences. All simplicial sets are built as unions of n-simplices attached along their boundaries by pushouts, which correspond to pullbacks of products of mapping spaces since $(X^-)_*$ is contravariant. When X is a quasi-monoid, this functor sends inclusions to Kan fibrations. So the limits defining the construction of a mapping space $(X^A)_*$ by building up mapping spaces of skeleta of A are all homotopy limits. The Kan weak equivalences above, therefore, are preserved by the limits and we find that for any map in $\mathcal{W}$ between quasi-monoids $(X^A)_* \to (Y^A)_*$ is a weak equivalence for any reduced simplicial set A. This makes sense, since we will define a model structure on $\mathbf{sSet}_0$ where quasi-monoids are fibrant objects and maps in $\mathcal{W}$ are weak equivalences, with $(-^A)_*$ a right Quillen functor. This observation simply shows that $(-^A)_*$ preserves weak equivalences between fibrant objects.

Returning to the model structure, we can conclude the following characterization of maps in $\mathcal{W}$.

Proposition 2.34. *A map f in $\boldsymbol{sSet}_0$ belongs to $\mathcal{W}$ if and only if $h_\bullet(f) = 0$.*

Proof. We will show that this condition on $h_\bullet$ is equivalent to the map $({p_{R(f)}}^{S^1})_*$ being a Kan acyclic fibration. By Lemma 2.33 and Proposition 2.28, since $p_{R(f)}$ is an inner fibration between quasi-monoids this is equivalent to $p_{R(f)}$ belonging to $\mathcal{W}$. By Corollary 2.31 this is equivalent to f being in $\mathcal{W}$, so this will give the result.

Since $p_{R(f)}$ is an inner fibration between quasi-monoids $({p_{R(f)}}^{S^1})_*$ is a Kan fibration by Corollary 2.20. The functor $(-^{S^1})_*$ is a right adjoint so it preserves pullbacks and in particular fibers. Hence

$(\mathrm{fib}(p_{R(f)})^{S_1})_* = \mathrm{fib}({p_{R(f)}}^{S^1})_*$, so by the homotopy long exact sequence of a fibration $({p_{R(f)}}^{S^1})_*$ is an acyclic Kan fibration if and only if $h_\bullet(f) = 0$. □

We will adopt the notation of [Bou75] when $h_\bullet$ is applied to an inclusion $i : A \subseteq B$ of reduced simplicial sets, so we will write $h_\bullet(i) = h_\bullet(B, A)$. The following argument comes straight from [Bou75] Lemma 11.2.

Proposition 2.35. *Let $A \subseteq B$ be an inclusion of reduced simplicial sets that belongs to $\mathcal{W}$. There exists a subcomplex $C \subseteq B$ with $|C| \leq \omega$, $C \not\subseteq A$, and such that $C \cap A \subseteq C$ belongs to $\mathcal{W}$.*

Proof. Construct C inductively, starting with any space $C_1 \subseteq B$ with $|C_1| \leq \omega$ and $C_1 \cap A \neq \emptyset$. Since $h_\bullet$ commutes with filtered colimits, we have

$$h_\bullet(A, B) = \mathrm{colim}_{i \in P}\, h_\bullet(F_i, F_i \cap A)$$

where P is the poset of finite subspaces $F_i \subseteq B$. Suppose C_i exists for $i \geq 1$ satisfying the same conditions as C_1. For any $x \in h_\bullet(C_1, C_1 \cap A)$, there exists a finite subspace $F_{i_x} \subseteq B$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbb{Z}_n \arrow[r, "0"] \arrow[d, "x"'] & h_\bullet(F_{i_x}, F_{i_x} \cap A) \arrow[d, "\lambda_{i_x}"] \\
h_\bullet(C_i, C_i \cap A) \arrow[r] & h_\bullet(B, A)
\end{tikzcd}$$

where $\mathbb{Z}_n$ is the graded group with n^{th} group $\mathbb{Z}$ and all others trivial and λ_{i_x} is the map to the colimit. This holds because $\mathbb{Z}_n$ is a compact object of graded groups and $h_\bullet(B, A) = 0$ so any element of a graded group $h_\bullet(F_i, F_i \cap A)$ is eventually sent to 0 in a large enough finite subspace of B.

Now let

$$C_{i+1} = C_i \cup \left(\bigcup_{x \in h_\bullet(C_i, C_i \cap A)} F_{i_x} \right)$$

The construction of $h_\bullet$ shows that $|h_\bullet(C_i, C_i \cap A)| \leq \omega$ and the spaces F_{i_x} are finite, so $|C_{i+1}| \leq \omega$. Taking

$$C = \bigcup_{i \geq 1} C_i$$

we have a directed colimit

$$\begin{tikzcd}
h(C_1, C_1 \cap A) \arrow[r] \arrow[rrrrrd, bend right=10] & h(C_2, C_2 \cap A) \arrow[r] \arrow[rrrrd, bend right=10] & \cdots \arrow[r] & h(C_i, C_i \cap A) \arrow[r] \arrow[rrd, bend right] & & \cdots \\
& & & & & h(C, C \cap A)
\end{tikzcd} \tag{18}$$

where by construction of C_{i+1}, for all $x \in h_\bullet(C_i, C_i \cap A)$ the diagram

$$\begin{tikzcd}
\mathbb{Z}_n \arrow[rr, "0"] \arrow[d, "x"'] & & h_\bullet(F_{i_x}, F_{i_x} \cap A) \arrow[d, "\lambda_{i_x}"] \arrow[dl, "\lambda'_{i_x}"] \\
h_\bullet(C_i, C_i \cap A) \arrow[r] & h_\bullet(C_{i+1}, C_{i+1} \cap A) \arrow[r] & h_\bullet(B, A)
\end{tikzcd}$$

commutes. So the maps in the colimit (18) are all zero and $h_\bullet(C, C \cap A) = 0$. $\square$

This corollary is the adaptation of Lemma 11.3 of [Bou75] to the proposed quasi-monoid model structure that follows immediately from the adaptation of Lemma 11.2 to this case in the previous proposition.

Corollary 2.36. *A map $f : X \to Y$ belongs to $\mathcal{F}$ if and only if it has the right lifting property against all maps $A \hookrightarrow B$ in $\mathcal{C} \cap \mathcal{W}$ such that $|B| \leq \omega$.*

Proof. Suppose f has the right lifting property given above. Let $g : C \hookrightarrow D$ be a map in in $\mathcal{C} \cap \mathcal{W}$ and let

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, hook, "g"'] & X \arrow[d, "f"] \\
D \arrow[r, "v"'] & Y
\end{tikzcd} \tag{19}$$

be a lifting problem in $\mathbf{sSet}_0$. We will show that this lifting problem has a solution.

We define the poset $\mathcal{P}$ of subspaces lying between C and D. An element of this poset is therefore a reduced simplicial set $Z \subseteq D$ a monomorphism $g_Z : C \hookrightarrow Z$ such that $g|_Z = g_Z$. The poset order is by inclusion of subspaces of D. We take the subposet $\mathcal{P}' \subseteq \mathcal{P}$ consisting of those elements $Z \subseteq D$ for which the monomorphism $g_Z : C \hookrightarrow Z$ belongs to $\mathcal{C} \cap \mathcal{W}$ and there exists a solution h_Z to the factorization of the lifting problem (19) as follows

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, hook, "g_Z"'] & X \arrow[dd, "f"] \\
Z \arrow[ur, dashed, "h_Z"'] \arrow[d, hook] & \\
D \arrow[r, "v"'] & Y
\end{tikzcd}$$

This subposet is non-empty as it contains the subspace

$$C \underset{\cong}{\overset{g}{\hookrightarrow}} g(C) \subseteq D$$

of D. It is chain complete, so by Zorn's lemma there exists a maximal subspace $Z_0 \subseteq D$ in $\mathcal{P}'$. We will show that this maximal subspace is D, so that the lifting problem (19) has a solution.

If $Z_0 \neq D$ then by Proposition 2.35 applied to $Z_0 \subsetneq D$ exists a subspace $A \subseteq D$ with $|A| \leq \omega$, $A \not\subseteq Z_0$, and such that $A \cap Z_0 \subseteq A$ belongs to $\mathcal{W}$. By the assumption on f, therefore, and the definition of $\mathcal{P}'$ we can extend the solution to the lifting problem

$$\begin{tikzcd}
 & C \arrow[r, "u"] \arrow[d, hook, "g_{Z_0}"'] & X \arrow[ddd, "f"] \\
A \cap Z_0 \arrow[r, hook] \arrow[d, hook] & Z_0 \arrow[ur, dashed, "h_{Z_0}"] \arrow[d, hook] & \\
A \arrow[r, hook] & A \cup Z_0 \arrow[uur, dashed] \arrow[d, hook] & \\
 & D \arrow[r, "v"'] & Y
\end{tikzcd}$$

This contradicts maximality, however, so $Z_0 = D$ and the original lifting problem (19) has a solution. □

A choice of isomorphism representatives of maps in $\mathcal{C} \cap \mathcal{W}$ such that the domain has cardinality at most ω form a set of generating acyclic cofibrations, so we have shown the existence of a model structure.

Theorem 2.37. *There exists a model structure on the category of reduced simplicial sets* $\boldsymbol{sSet}_0$ *such that*

- *the weak equivalences are the maps* $f : A \to B$ *such that* $(X^f)_* : (X^B)_* \to (X^A)_*$ *is a Kan weak equivalence for all quasi-monoids* X
- *the cofibrations are inclusions of reduced simplicial sets*
- *the fibrations are the maps with the right lifting property against all cofibrations that are weak equivalences*

This model structure is cofibrantly generated and the fibrant objects are the quasi-monoids.

That the fibrant objects are exactly the quasi-monoids follows from Proposition 2.22 and Corollary 2.23. A map is a weak equivalence of the model structure of Theorem 2.37 if and only if $(X^B)_* \to (X^A)_*$ is an acyclic fibration for all quasi-monoids X, so quasi-monoids are fibrant objects. Corollary 2.23 gives that inner anodyne maps are weak equivalences of this model structure, so all fibrant objects are quasi-monoids. From this characterization, we will call the model structure of Theorem 2.37 the **quasi-monoid model structure**. To distinguish the two model structures on the category $\mathbf{sSet}_0$ we will denote the Kan model structure of Theorem 2.1 by $(\mathbf{sSet}_0)_K$ and the quasi-monoid model structure of Theorem 2.37 by $(\mathbf{sSet}_0)_J$.

The quasi-monoid model structure could equally well be called the reduced Joyal model structure because of the following result.

Proposition 2.38. *A map between reduced simplicial sets is a weak equivalence of the quasi-monoid model structure if and only if it is a Joyal weak equivalence.*

Proof. First observe that inner anodyne maps are Joyal equivalences and weak equivalences of the quasi-monoid model structure. The proof of Proposition 2.34 shows that a map f is a weak equivalence of the quasi-monoid model structure if and only if $p_{R(f)}$ is an acyclic fibration, where there is the following factorization

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[d, "f"'] & P(R(f)) \arrow[d, "p_{R(f)}"] \\
Y \arrow[r, "\sim"] & R(Y)
\end{tikzcd}$$

An acyclic fibration is a Joyal equivalence and a weak equivalence of the quasi-monoid model structure, so by 2 out of 3 for both classes of weak equivalences they are the same. □

Because of this we will refer to weak equivalences of the model structure of Theorem 2.37 as **reduced Joyal equivalences**. This also shows that the quasi-monoid model structure on $\mathbf{sSet}_0$ is the model structure transferred via the adjunction

$$\begin{tikzcd}
(\mathbf{sSet}_0)_J \arrow[r, bend left, "i"] \arrow[r, phantom, "\top"] & (\mathbf{sSet})_J \arrow[l, bend left, "\kappa"]
\end{tikzcd}$$

with the Joyal model structure on simplicial sets. We have just shown in Proposition 2.38 that the inclusion functor i creates weak equivalences for the quasi-monoid model structure, so to identify this as the transferred model structure it remains to show that i creates fibrations of the quasi-monoid model structure.

Proposition 2.39. *A map of reduced simplicial sets belongs to the class $\mathcal{F}$ of fibrations for the quasi-monoid model structure if and only if it is sent to a fibration of the Joyal model structure by the inclusion functor $i : \boldsymbol{sSet}_0 \to \boldsymbol{sSet}$.*

Proof. Let $f : A \to B$ be a map of reduced simplicial sets such that $i(f)$ is a fibration of the Joyal model structure. We will show that any lifting problem

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, hook, "g"'] & A \arrow[d, "f"] \\ D \arrow[r, "v"'] & B \end{tikzcd}$$

in $\mathbf{sSet}_0$ has a solution when $g : C \hookrightarrow D$ is an acyclic cofibration for the quasi-monoid model structure. To do this we show that the lifting problem in **sSet** obtained by applying the functor i to this problem has a solution. Since $\mathbf{sSet}_0$ is a reflective subcategory of **sSet** this then implies that the original lifting probem has a solution in $\mathbf{sSet}_0$.

An acyclic cofibration of the quasi-monoid model structure is an acyclic cofibration of the Joyal model structure on **sSet** by Proposition 2.38 so $i(g)$ is an acyclic cofibration in the Joyal model structure on **sSet**. Hence the lifting problem

$$\begin{tikzcd} i(C) \arrow[r, "i(u)"] \arrow[d, hook, "i(g)"'] & i(A) \arrow[d, "i(f)"] \\ i(D) \arrow[r, "i(v)"'] & i(B) \end{tikzcd}$$

can be solved since $i(f)$ is a fibration of the Joyal model category on simplicial sets by hypothesis.

It only remains to show that i preserves fibrations. By [JT07] Lemma 7.14 we can show this by proving that i preserves fibrations between fibrant objects. By Proposition 2.23 a fibration between fibrant objects in the quasi-monoid model structure is an inner fibration between quasi-monoids. Hence under i by Definition 4.1 and Theorem 6.12 of [Joy08] a fibration between fibrant objects of the quasi-monoid model structure is sent to a fibration of the Joyal model structure. □

Finally, as we promised, we can now show that the quasi-monoid model structure is a simplicial model structure. We proved that $\mathbf{sSet}_0$ is enriched in **sSet** and powered and copowered over **sSet** in Section 2.1, so it only remains to show that the adjunction of two variables defined in Proposition 2.3 is a Quillen adjunction.

Proposition 2.40. *The quasi-monoid model structure is a simplicial model structure.*

Proof. We will show that the functor $(-) \;{}_*\!\times (-) : \mathbf{sSet}_0 \times \mathbf{sSet} \to \mathbf{sSet}_0$ is a Quillen bifunctor. Let $f : A \hookrightarrow B$ and $g : K \hookrightarrow L$ be cofibrations in $\mathbf{sSet}_0$ and **sSet** respectively. Then $f \;{}_*\!\times g : A \;{}_*\!\times L \cup B \;{}_*\!\times K \hookrightarrow B \;{}_*\!\times L$ is a cofibration. The maps (8) have the left lifting property against all fibrations between fibrant objects in the quasi-monoid model structure, so by Lemma 7.14 of

[JT07] these maps are reduced Joyal equivalences. Hence $f\ {}_*\!\times\ g$ is a reduced Joyal equivalence when $K \hookrightarrow L$ is a Kan weak equivalence. The adjunction and Proposition 2.22 give that $f\ {}_*\!\times\ g$ is a reduced Joyal equivalence when $A \hookrightarrow B$ is. □

2.5 Kan Complexes and Quasi-monoids

In this section we will show that the Kan model structure on reduced simplicial sets $(\mathbf{sSet}_0)_K$ can be obtained from the quasi-monoid model structure as a consequence of simply requiring that one more map be a weak equivalence. The framework we will use to describe a new model structure obtained by adding more weak equivalences to an existing model structure is that of left Bousfield localization.

We will use the characterization of left Bousfield localizations given in [Hir03] Definition 3.3.1 to show that $(\mathbf{sSet}_0)_K$ is a left Bousfield localization of $(\mathbf{sSet}_0)_J$. Since we have shown in Proposition 2.40 that $(\mathbf{sSet}_0)_J$ is a simplicial model category with mapping space functor $(-^-)_*$, to show that $(\mathbf{sSet}_0)_K$ is the left Bousfield localization of $(\mathbf{sSet}_0)_J$ at a set H of morphisms of $\mathbf{sSet}_0$ we must show

1. the cofibrations of $(\mathbf{sSet}_0)_K$ are the same as the cofibrations of $(\mathbf{sSet}_0)_J$
2. a reduced simplicial set X is a reduced Kan complex if and only if it is a quasi-monoid and for all maps $f : A \to B$ in H the map of simplicial sets
$$(X^f)_* : (X^B)_* \to (X^A)_*$$
is a Kan weak equivalence
3. a map $f : A \to B$ of $\mathbf{sSet}_0$ is Kan weak equivalence if and only if for all H-local reduced simplicial sets X the simplicial set map
$$(X^f) : (X^B)_* \to (X^A)_*$$
is a Kan weak equivalence

The localizing set that works will be the single map

$$H = \{01 : S^1 \hookrightarrow R^1\}$$

defined for the purpose of identifying right invertible edges in Definition 2.9. By Theorems 2.1 and 2.37 the cofibrations of $(\mathbf{sSet}_0)_K$ are the same as the cofibrations of $(\mathbf{sSet}_0)_J$, so the first condition holds. We will now show that the second of these conditions holds for the set $H = \{01 : S^1 \hookrightarrow R^1\}$

Proposition 2.41. *Let X be a quasi-monoid. The following are equivalent*

(1) $(X^{R^1})_* \to (X^{S^1})_*$ is an weak equivalence

(2) $X \to *$ has the right lifting property against $S^1 \hookrightarrow R^1$

(3) X is a Kan complex

Proof. (1) $\implies$ (2) Corollary 2.20 implies that $(X^{R^1})_* \to (X^{S^1})_*$ is a Kan fibration of simplicial sets for a quasi-monoid X, so if it is a weak equivalence it is an acyclic fibration. An acyclic fibration is surjective, so the result holds.

(2) $\implies$ (3) Suppose $X \to *$ has the right lifting property against $S^1 \hookrightarrow R^1$. Then every edge of X is right invertible by Definition 2.9, so in the homotopy monoid $\tau(X)$ every class has a right inverse. This implies $\tau(X)$ is a group. If $[x] \in \tau(X)$ there exists $[y]$ such that $[x][y] = e$. But $[y]$ must also have a right inverse in $\tau(X)$, say $[z]$. But then $[z] = [x][y][z] = [x]$, so $[y][x] = e$. Since X is a quasi-monoid, this means there exists $\alpha \in X_2$ such that $d_0(\alpha) = x$, $d_1(\alpha) = s_0(*)$, and $d_2(\alpha) = y$. Equivalently, x is left invertible. Hence all edges of X are left and right invertible, so by Proposition 2.11, therefore, X has the right lifting property against all horns and so is a Kan complex.

(3) $\implies$ (1) Suppose X is a Kan complex. The square

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r] \arrow[d, hook] & S^1 \arrow[d, hook] \\
\Delta^2 \arrow[r, "010"] & R^1
\end{tikzcd}$$

is a pushout so $S^1 \hookrightarrow R^1$ is a Kan weak equivalence since the horn inclusion is a Kan acyclic cofibration. Hence the pushout product maps

$$(S^1 \;{}_*\!\times \Delta^n) \cup_{S^1 {}_*\times \partial\Delta^n} (R^1 \;{}_*\!\times \partial\Delta^n) \hookrightarrow R^1 \;{}_*\!\times \Delta^n$$

are Kan weak equivalences for all $n \geq 0$. So since X is a Kan complex $(X^{R^1})_* \to (X^{S^1})_*$ is an acyclic fibration.

□

Now we show the final condition holds and so the model structure of Theorem 2.1 is a left Bousfield localization of the quasi-monoid model structure.

Proposition 2.42. *A map of reduced simplicial sets $A \to B$ is a Kan weak equivalence if and only if for all reduced Kan complexes X the map $(X^B)_* \to (X^A)_*$ is a Kan weak equivalence.*

Proof. Factor $A \to B$ as a cofibration $i : A \hookrightarrow P$ followed by an acyclic fibration $p : P \to B$ in the reduced Kan model structure. The proof of Corollary 2.27 shows that p is a reduced homotopy equivalence in the sense of Definition 2.25. Hence by Proposition 2.26 p induces a homotopy equivalence $(X^B)_* \to (X^P)_*$ for any reduced Kan complex X, so it is a local equivalence. It is therefore sufficient to show that cofibrations are Kan weak equivalences if and only if they are local equivalences. Hence we will suppose $A \hookrightarrow B$ is a cofibration. Since a Kan complex is a quasi-monoid Corollary 2.20 gives that $(X^B)_* \to (X^A)_*$ is a Kan weak equivalence if and only if it is a Kan acyclic fibration. Hence a cofibration $A \hookrightarrow B$ is a local equivalence if and only if $(X^B)_* \to (X^A)_*$ is an acyclic fibration.

Let $A \hookrightarrow B$ be a local equivalence. Then for any Kan fibration between Kan complexes $f : X \to Y$ the map $(X^B)_* \to (X^A)_* \times_{(Y^A)_*} (Y^B)_*$ is an acyclic fibration, so $A \hookrightarrow B$ has the left lifting property against all such maps f. A map with the left lifting property against all fibrations between fibrant objects is an acyclic cofibration by Lemma 7.14 of [JT07].

Conversely, if $A \hookrightarrow B$ is a Kan weak equivalence then since **sSet** with the Kan model structure is a proper model category, the maps

$$(A \;{}_*\!\times \Delta^n) \cup_{A_* \times \partial\Delta^n} (B \;{}_*\!\times \partial\Delta^n) \hookrightarrow B \;{}_*\!\times \Delta^n$$

are Kan weak equivalences for all $n \geq 0$ so $(X^B)_* \to (X^A)_*$ is an acyclic fibration. $\square$

Hence the Kan model structure on $\mathbf{sSet}_0$ is the left Bousfield localization of the quasi-monoid model structure on $\mathbf{sSet}_0$ by the map $S^1 \hookrightarrow R^1$. The adjunction

$$\begin{tikzcd} (\mathbf{sSet}_0)_J \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "\mathrm{Id}"] \end{tikzcd} \tag{20}$$

is a Quillen adjunction. The localization of an object $A \in (\mathbf{sSet}_0)_J$ is its Kan fibrant replacement, so the left derived functor of localization is simply the homotopy type of a reduced simplicial set.

We will finish this section by returning to our definition of invertibility for 1-simplices of reduced simplicial sets from Definition 2.9. We motivated the definition by showing that invertibility of an edge implies that the class of the edge in the homotopy monoid is invertible as an element of the monoid. We will now show that when X is a quasi-monoid, these two notions of invertibility for 1-simplices are the same.

Recall the adjunction $\tau \dashv B$ where τ is the homotopy monoid functor and B is the nerve of a discrete monoid. Using the unit map of this adjunction we can define a functor that select subspaces of a reduced simplicial set A whose simplices have edges all corresponding to classes of $\tau(A)$ that are invertible.

Definition 2.43. *There is a functor $k : \boldsymbol{sSet}_0 \to \boldsymbol{sSet}_0$ defined by the pullback*

$$\begin{tikzcd} k(A) \arrow[r, hook] \arrow[d] & A \arrow[d, "\eta_A"] \\ Bg(\tau(A)) \arrow[r, hook] & B\tau(A) \end{tikzcd}$$

where $g\tau(A) \leq \tau(A)$ is the submonoid of $\tau(A)$ consisting of all invertible elements.

Applying the functor k to a quasi-monoid gives a reduced Kan complex.

Proposition 2.44. *If X is a quasi-monoid then $k(X)$ is a reduced Kan complex.*

Proof. We will show that all edges of $k(X)$ are left and right invertible in the sense of Definition 2.9. Hence all outer horn lifting problems for the map $X \to *$ will be special outer horns, so they can be solved by Proposition 2.11.

Let $x \in k(X)_1$. Then there exists a class $[y]$ such that $[x][y] = e$ in $g(\tau(X))$ where $y \in k(X)$. Now X is a quasi-monoid, so there exists a simplex $\alpha \in X_2$ such that $d_2(\alpha) = x$, $d_0(\alpha) = y$, and $d_1(\alpha) = s_0(*)$. Hence $\alpha : R^1 \to X$ defines an extension of $x : S^1 \to X$ and x is right invertible and α factors through $k(X) \hookrightarrow X$ because $[x]$, $[y]$, and e all belong to $g(\tau(X))$. The same argument applied to the identity $[y][x] = e$ in $g(\tau(X))$ gives that x is left invertible. $\square$

Recall that in Definition 2.9 we defined an edge $x : S^1 \to A$ of a reduced simplicial set A as invertible if there exists a map $\alpha : E^1 \to A$ such that $\alpha(01) = x$. As mentioned in the motivating discussion before that definition, if x is invertible in the sense of Definition 2.9 then the corresponding class in $\tau(A)$ is an invertible element of the monoid. For an arbitrary reduced simplicial set A it is not necessarily the case, however, that all invertible classes of $\tau(A)$ arise from invertible edges.

Example 2.45. Consider the 2-skeleton $\mathrm{sk}_2(E^1)$ of the reduced simplicial set E^1. This space has non-degenerate edges 01 and 10 and non-degenerate 2-simplices 010 and 101 but all other simplices degenerate. The homotopy monoid of this reduced simplicial set is $\mathbb{Z}$, the free group on one generator, so all classes of edges have inverses. However the map $01 : S^1 \to \mathrm{sk}_2(E^1)$ has no extension along the map $S^1 \hookrightarrow E^1$. If $\alpha : E^1 \to \mathrm{sk}_2(E^1)$ extends $01 : S^1 \to \mathrm{sk}_2(E^1)$ then $d_0(d_0(\alpha(0101))) = 01$ and $d_2(d_3(\alpha(0101))) = 01$ in E^1_2, but this implies $\alpha(0101) = 0101$, which is not possible as this simplex does not belong to $\mathrm{sk}_2(E^1)$.

We will show that these two conditions on edges of a reduced simplicial set are equivalent when the reduced simplicial set is a quasi-monoid.

Proposition 2.46. *Let X be a quasi-monoid. An edge $x \in X_1$ is invertible if and only if its class $[x] \in \tau(X)$ is an invertible element of the homotopy monoid.*

Proof. We have already shown that invertible elements have invertible classes. Suppose then that $[x] \in \tau(X)$ has an inverse in the homotopy monoid. Then the edge $x : S^1 \to X$ factors through the inclusion $k(X) \hookrightarrow X$. Now we observe that $S^1 \hookrightarrow E^1$ can be obtained as the colimit of inclusions

$$\begin{tikzcd}
S^1 \arrow[r, hook] & Z^{(1)} \arrow[r, hook] & Z^{(2)} \arrow[r, hook] & \cdots \arrow[r, hook] & Z^{(n)} \arrow[r, hook] & \cdots
\end{tikzcd}$$

where the reduced simplicial sets $Z^{(n)}$ are obtained by attaching the unique n-simplex of $z_n \in (E^1)_n$ that starts with a 0, for example the simplex $z_2 = 010$. All faces of z_n simplex belong to $Z^{(n-1)}$ except the d_0 face, as

$$d_j z_n = \; s_{j-1}z_{n-2} \quad 1 \leq j \leq n-1 z_{n-1} \quad j = n$$

and $Z^{(n-1)}$ contains z_m for all $1 \leq m \leq n-1$. Hence there are pushouts

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "z_n"] \\
Z^{(n-1)} \arrow[r, hook] & Z^{(n)}
\end{tikzcd}$$

and so $S^1 \hookrightarrow E^1$ is a Kan weak equivalence. Since $k(X)$ is Kan by Proposition 2.44 there is an extension of x along the Kan weak equivalence $S^1 \hookrightarrow E^1$, so x is invertible. □

As mentioned in Subsection 2.2.1 after Example 2.10, this also shows that in a quasi-monoid an edge is invertible if and only if it is left and right invertible in the sense of Definition 2.9.

2.6 Segal Monoids

We now turn to the second model of ∞-monoids that was proposed at the start of this chapter, the Segal monoids. These are the Segal categories with a unique object, just as quasi-monoids were the quasi-categories with a unique object. Segal categories are bisimplicial sets satisfying certain lifting properties, defined by Dwyer, Kan, and Smith in [DKS89]. Restricting to a unique object, in this case therefore means considering the category $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathbf{sSet}_0$. The model structure on this category that we study in this section, and indeed our approach of restricting model structures to the single object case that motivated the results of the previous sections, come from Bergner in [Ber07].

In this section we will show that we can relate Bergner's model structure to the quasi-monoid model structure defined in Section 2.4 by adapting the methods of [JT07] to the single object case. In this paper Joyal and Tierney constructed a model category whose fibrant objects are Segal categories and that is Quillen equivalent to the Joyal model structure on **sSet**. Our goal will be to show that the Bergner model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ has Segal monoids as its fibrant objects and is Quillen equivalent to the quasi-monoid model structure on $\mathbf{sSet}_0$.

2.6.1 Reduced Simplicial Spaces

Our starting point is the category $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ of simplicial objects in reduced simplicial sets or reduced simplicial spaces. In this section we will describe this category $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ and an adjunction of two variables that allows us to assemble a reduced simplicial space out of a reduced simplicial set and an ordinary simplicial set. We will also describe how $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ is enriched, tensored and cotensored in simplicial sets.

Reduced simplicial spaces are functors $X : \Delta^{\mathrm{op}} \to \mathbf{sSet}_0$. These are in particular bisimplicial sets, but we have to pay careful attention to the roles of rows and columns as the normal symmetry of bisimplicial sets has been broken. We record the convention for rows and columns we have chosen in the following definition.

Definition 2.47. *A* ***reduced simplicial space*** $X : \Delta^{\mathrm{op}} \to \boldsymbol{sSet}_0$ *is a simplicial object in* $\boldsymbol{sSet}_0$*. The* ***rows*** *of* X *are the reduced simplicial sets*

$$X_{\bullet,n} \in \boldsymbol{sSet}_0$$

for $n \geq 0$*. The* ***columns*** *are the simplicial sets*

$$X_{m,\bullet} \in \boldsymbol{sSet}$$

for $m \geq 0$*. In particular* $X_{0,\bullet} = \Delta^0$*.*

We will denote the unique 0-simplex of the reduced simplicial sets making up the rows of X as

$$X_{0,n} = \{*\}$$

for $n \geq 0$.

We now describe a construction that allows us to build reduced simplicial spaces out of a reduced simplicial set and a simplicial set. The **half reduced box product** for $A \in \mathbf{sSet}_0$ and $K \in \mathbf{sSet}$ is the reduced simplicial space $A_*\Box K$ defined by the functor

$$A_*\Box K : [n] \in \Delta^{op} \mapsto \bigvee_{\alpha \in K_n} A \in \mathbf{sSet}_0$$

Equivalently, by the definition of the coproduct in reduced simplicial sets, the square

$$\begin{tikzcd}
*\Box K \arrow[r, hook] \arrow[d] & A\Box K \arrow[d] \\
* \arrow[r] & A_*\Box K
\end{tikzcd}$$

is a pushout. From this it is clear that the diagonal of the bisimplicial set $A_*\square K$ recovers the half-reduced product of the previous section

$$d(A_*\square K) = A\ {}_*\!\times K$$

This functor $-{}_*\square-$ is based on the box product $-\square- : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$, which is an adjunction of two variables (see for example [JT07] §2). The functor

$$-{}_*\square- : \mathbf{sSet}_0^{\mathrm{op}} \times \mathbf{sSet} \to \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$$

is also an adjunction of two variables, with right adjoints for $A_*\square-$ and $-{}_*\square K$. For $A \in \mathbf{sSet}_0$ and $Z \in \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ define the simplicial set $Z/^*A$ by

$$(Z/^*A)_n = \mathrm{Hom}_{\mathbf{sSet}_0}(A, Z_{\bullet,n})$$

Recall that the rows of Z are reduced simplicial sets so this definition makes sense. For $K \in \mathbf{sSet}$ and $Z \in \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ define the reduced simplicial set $K\backslash^*Z$ by the end

$$K\backslash^*Z = \int_{[n]\in\Delta} Z_{\bullet,n}^{K_n}$$

Note that zero-simplices of $X_{\bullet,n}^{K_n}$ are simplicial maps $K_n \to X_{\bullet,n}$, but K_n is a discrete simplicial set and $X_{0,n} = *$ so $(K\backslash^*Z)_0 = *$.

Bisimplicial maps $S^m{}_*\square\Delta^n \to Z$ are in bijective correspondence with n,m-bisimplices of Z. Hence we have

$$Z/^*S^m = Z_{m,\bullet}$$

$$\Delta^n\backslash^*Z = Z_{\bullet,n}$$

so these right adjoints select columns and rows of Z respectively.

The Segal monoid model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ will be simplicial, so we must define an enrichment of $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ in simplicial sets that is powered and copowered. For $Z \in \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ and $K \in \mathbf{sSet}$ define $X\ {}_*\!\otimes K$ as the pushout in $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$

$$\begin{tikzcd}
* \times (*\square K) \arrow[r] \arrow[d] & X \times (*\square K) \arrow[d] \\
* \arrow[r] & X\ {}_*\!\otimes K
\end{tikzcd}$$

The bisimplicial set $*\square K$ has all columns equal to K and all rows constant.

With this operation we can define a simplicial set of maps $(Z^Y)_*$ for $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ as having n-simplices the maps $Y\ {}_*\!\otimes \Delta^n \to Z$. The powering of a reduced bisimplicial set Z with a simplicial set K is the reduced bisimplicial set $E_0(Z^K)$ whose m,n-bisimplices are the maps

$$S^m{}_*\square(\Delta^n \times K) \to Z$$

This follows from the observation that there is an isomorphism natural in all variables

$$(A_*\square K)\ {}_*\!\otimes L \cong A_*\square(K \times L)$$

2.6.2 The Segal Monoid Model Structure

In this section we will define Segal monoids in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ and recall the model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ defined in [Ber07]. We will also show that these Segal monoids are exactly the fibrant objects of the Bergner model structure and that this model structure can be obtained as a left Bousfield localization of the Reedy model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ when $\mathbf{sSet}_0$ has the quasi-monoid model structure of Theorem 2.37.

We will now give our definition of Segal monoids as reduced simplicial spaces satisfying lifting properties in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$.

Definition 2.48. *A* ***Segal monoid*** *is a reduced bisimplicial set Z such that the map $Z \to *$ has the right lifting property for $0 < l < m$ and $0 \leq k \leq n$ against all maps*

$$(\lambda_l^m{}_*\Box\Delta^n) \cup_{\lambda_l^m{}_*\Box\partial\Delta^n} (S^m{}_*\Box\partial\Delta^n) \hookrightarrow S^m{}_*\Box\Delta^n \tag{21}$$

$$(\partial S^m{}_*\Box\Delta^n) \cup_{\partial S^m{}_*\Box\Lambda_k^n} (S^m{}_*\Box\Lambda_k^n) \hookrightarrow S^m{}_*\Box\Delta^n \tag{22}$$

This definition is based on the results of [JT07], which show that Segal spaces can be characterized by their right lifting properties against the unreduced box product versions of the maps (21) and (22). From this identification it is clear that a Segal monoid is a Segal category with a single object, which is the same position we started from when we were considering quasi-monoids as a subset of quasi-categories.

Recall the identification of $Z/^*S^m$ with the m^{th} row of Z and $\Delta^n\backslash^*Z$ with the n^{th} column of Z. As was shown in [JT07] for Segal categories the following shows that a Segal monoid has rows that are quasi-monoids and columns that are Kan complexes.

Proposition 2.49. *Let Z be a Segal monoid. For all $A \in$ **sSet**$_0$ and $K \in$ **sSet** $Z/^*A$ is a Kan complex and $K\backslash^*Z$ is a quasi-monoid.*

Proof. By Definition 2.48 and the adjointness of the box product, $Z/^*S^m \to Z/^*\lambda_k^m$ has the right lifting property against $\partial\Delta^n \hookrightarrow \Delta^n$ for all $m, n \geq 0$. Hence it is an acyclic fibration and has the right lifting property against $\emptyset \hookrightarrow K$ for any $K \in \mathbf{sSet}$. Adjointness gives a lift for the square

$$\begin{tikzcd}
\lambda_k^n \arrow[r] \arrow[d, hook] & K\backslash^*Z \arrow[d] \\
S^n \arrow[r] & *
\end{tikzcd}$$

So $K\backslash^*Z$ is a quasi-monoid. The same argument starting from $\Delta^m\backslash^*Z \to \Lambda_k^m\backslash^*Z$ shows that $Z/^*A$ is a Kan complex. □

We will show that these Segal monoids are the fibrant objects of Bergner's model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ defined in the discussion following the proof of Theorem 3.9 in [Ber07]. This model structure is obtained by localizing another model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ at the reduced spine inclusions.

Theorem 2.50 ([Ber07] Prop. 3.8, Theorem 3.9, and Prop. 3.10)**.** *There exists a model structure $(\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C$ on reduced simplicial spaces that is cofibrantly generated with*

- *generating cofibrations the maps* (26)

- *generating acyclic cofibrations the maps*
$$(\partial S^m{}_*\Box\Delta^n)\cup_{\partial S^m{}_*\Box\partial\Delta^n}(S^m{}_*\Box\Lambda^n_k)\hookrightarrow S^m{}_*\Box\Delta^n \tag{23}$$
for all $0\leq k\leq n$

The weak equivalences of $(\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C$ *are the maps between reduced simplicial spaces that are column-wise Kan weak equivalences.*

There is another model structure $(\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_B$ *on reduced simplicial spaces that is obtained by localizing* $(\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C$ *at the maps of* $\boldsymbol{sSet}_0$ *for* $l\geq 2$
$$\varphi_l:\bigvee_l S^1\hookrightarrow S^l \tag{24}$$
viewed as constant reduced simplicial spaces.

The maps (24) are the spine inclusions described in the proof of Lemma 2.33. These maps include the wedge of circles into S^n as the 1-simplices $i\to i+1$ along the spine of the maximal n-simplex of S^n.

The localization gives a Quillen adjunction

$$\begin{tikzcd}(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C \arrow[l, bend left, "\mathrm{Id}"]\end{tikzcd}$$

Hence, as all objects of $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ are cofibrant, column-wise Quillen weak equivalences are weak equivalences in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ as well.

As mention above, the model structure $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ will be our Segal monoid model structure, so we will show that the fibrant objects of this model structure are the Segal monoids of Definition 2.48. In [Ber07] the fibrant objects of the model structure $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ are called **reduced Segal categories**. These are reduced simplicial spaces X with solutions to all lifting problems for $0\leq k\leq n$

$$\begin{tikzcd}(\partial S^m{}_*\Box\Delta^n)\cup(S^m{}_*\Box\Lambda^n_k) \arrow[r] \arrow[d, hook] & X \arrow[d] \\ S^m{}_*\Box\Delta^n \arrow[r] & *\end{tikzcd} \tag{25}$$

and such that for all $l\geq 2$ $X/^*\varphi_l$ is a weak equivalence of simplicial sets for all maps (24).

We will show that Bergner's reduced Segal categories are the Segal monoids using the arguments of Joyal and Tierney in [JT07]. They show exactly this result for unreduced Segal spaces, so we will adapt their methods, starting with the following lemma on inner anodyne maps. We recall here the simplicial spine inclusions that were mentioned in their reduced from in the proof of Lemma 2.33 and were used to localize the model structure $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ in Theorem 2.50.

For $n\geq 2$ the spine inclusion of the standard simplicial n-simplex Δ^n is the inclusion of the spine of the n 1-simplices $i\to i+1$ for $0\leq i\leq n-1$ into Δ^n. We denote the space of these 1-simplices joined end to end by I_n, which is a space isomorphic to the colimit of the diagram

$$\begin{tikzcd}[column sep=small] & \Delta^0 \arrow[dl, hook', "d^0"'] \arrow[dr, hook, "d^1"] & & \Delta^0 \arrow[dl, hook', "d^0"'] \arrow[dr, hook, "d^1"] & & {} \arrow[dl, hook'] & \cdots & {} \arrow[dr, hook] & & \Delta^0 \arrow[dl, hook', "d^0"'] \arrow[dr, hook, "d^1"] & \\ \Delta^1 & & \Delta^1 & & \Delta^1 & & & & \Delta^1 & & \Delta^1\end{tikzcd}$$

We denote the inclusion map by

$$I_n \hookrightarrow \Delta^n$$

Lemma 2.51 ([JT07] Lemma 3.5)**.** *Let $\mathcal{A} \subseteq$ **sSet** be a saturated class such that for all composable u, v if $uv \in \mathcal{A}$ and $v \in \mathcal{A}$ then $u \in \mathcal{A}$. If $\mathcal{A}$ contains all the spine inclusions $I_n \subseteq \Delta^n$ for $n \geq 2$ then it contains all inner anodyne maps.*

Proposition 2.52. *X is a Segal monoid if and only if it is a reduced Segal category.*

Proof. Let X be a Segal monoid. Then the lifting problems (25) all have solutions by definition, so we only need to show that $X/^*\varphi_l$ is a weak equivalence for all $l \geq 2$. The lifting property (25) implies that $X/^*u$ is a Kan fibration of simplicial sets for all inclusions u, so $X/^*\varphi_l$ is a weak equivalence if and only if it is a trivial fibration. But φ_l is inner anodyne for all $l \geq 2$ so by Definition 2.48 of Segal monoids $X/^*\varphi_l$ is an acyclic fibration.

Let X be a reduced Segal category and let $\mathcal{A}$ be the class of all inclusions f of simplicial sets such that $X/^* \operatorname{red}(f)$ is a weak equivalence. By the definition of reduced Segal categories $\mathcal{A}$ contains $I_n \subseteq \Delta^n$ for all $n \geq 2$. We claim that $\mathcal{A}$ is saturated and has the desired cancellation property. The cancellation property is clear using the two out of three property of weak equivalences. The first lifting property of (25) is equivalent to requiring $X/^* \operatorname{red}(f)$ be a Kan fibration of simplicial sets, so for it to be a weak equivalence is equivalent to $X/^* \operatorname{red}(f)$ having the right lifting property against all maps $\delta^n : \partial\Delta^n \hookrightarrow \Delta^n$. Equivalently, $\operatorname{red}(f)$ must have the left lifting property against all maps $\delta^n \backslash^* X$ for $n \geq 0$. The class of maps f which have this property is closed under saturation because red is a left adjoint and preserves pushouts. Hence by Lemma 2.51, X is a Segal monoid.

□

Hence we can call Bergner's model structure $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ of Theorem 2.50 the **Segal monoid model structure** on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$, since the fibrant objects of this model structure are the Segal monoids. We will call weak equivalences of this model structure **Segal monoid weak equivalences**.

With the definition of the tensor product $- \,{}_*\!\otimes -$ of Subsection 2.6.1 for $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$, the model category $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ is a simplicial model category.

Proposition 2.53. *$(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ is a simplicial model category.*

Proof. We note that

$$\begin{tikzcd}
((\partial S^n {}_*\square \Delta^m) \cup (S^n {}_*\square \partial\Delta^m)) \,{}_*\!\otimes K \arrow[dd, hook] & & (\partial S^n {}_*\square (\Delta^m \times K)) \cup (S^n {}_*\square (\partial\Delta^m \times K)) \arrow[dd, hook] \\
 & = & \\
(S^n {}_*\square \Delta^m) \,{}_*\!\otimes K & & S^n {}_*\square (\Delta^m \times K)
\end{tikzcd}$$

Let $- {}_*\widehat{\otimes} -$ denote the Leibniz product of maps with the tensor product. Then for a generating cofibration i of the form (26) and an anodyne map $j : K \hookrightarrow L$, a map $f : Y \to Z$ of reduced simplicial spaces has a solution to the lifting problem against $i {}_*\widehat{\otimes} j$ if and only if the problem

$$\begin{tikzcd}
\partial\Delta^m \times L \cup_{\partial\Delta^m \times K} \Delta^m \times K \arrow[r] \arrow[d, hook] & X/^*S^n \arrow[d] \\
\Delta^m \times L \arrow[r] & X/^*\partial S^n \times_{Y/^*\partial S^n} Y/^*S^n
\end{tikzcd}$$

has a solution, which it does for f a fibration in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ by the Kan simplicial model structure on $\mathbf{sSet}$. The case of a generating acyclic cofibration of $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ and a cofibration of $\mathbf{sSet}$ is the same but with $\partial\Delta^m$ replaced by Λ^m_k. $\square$

This is the source of the simplicial model structure we will prove for the Segal monoid model structure in Corollary 2.66 in the next subsection. It also allows us to describe Bergner's model structure of Theorem 2.50 as follows, since it is the localization of $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ with local objects the Segal monoids by Proposition 2.52.

Theorem 2.54 ([Ber07])**.** *There is a simplicial model structure on* $\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}}$ *with cofibrations the inclusions and fibrant objects the Segal monoids. Weak equivalences are the maps* $X \to Y$ *such that for all Segal monoids* Z *the maps*

$$(Z^Y)_* \to (Z^X)_*$$

are Quillen weak equivalences of simplicial sets.

From the results of Section 2.4 we can define another model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$, namely the Reedy model structure induced by the quasi-monoid model structure. We will denote this model structure by $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_R$ and show that the Segal monoid model structure $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ can be obtained by left Bousfield localization of the Reedy model structure as well.

Recall that the box product was defined by the tensoring of $\mathbf{sSet}_0$ over $\mathbf{sSet}$ by coproducts. [RV14] show in §6 and 7 that when a model category $\mathcal{M}$ is cofibrantly generated, the latching maps defining the Reedy model structure on $\mathcal{M}^{\Delta^{\mathrm{op}}}$ are cofibrations or acyclic cofibrations exactly when they belong to the saturated class generated by tensoring the generating cofibrations or acyclic cofibrations of $\mathcal{M}$ with the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$. Hence the Reedy model structure on $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ has the following properties.

Proposition 2.55 ([RV14] Proposition 7.7)**.** *The Reedy model structure on* $\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}}$ *induced by the quasi-monoid model structure is cofibrantly generated, with generating cofibrations the maps*

$$(\partial S^m{}_*\square\Delta^n) \cup_{\partial S^m{}_*\square\partial\Delta^n} (S^m{}_*\square\partial\Delta^n) \hookrightarrow S^m{}_*\square\Delta^n \tag{26}$$

for all $m, n \geq 0$ *and generating acyclic cofibrations the maps*

$$(A_*\square\Delta^n) \cup_{A_*\square\partial\Delta^n} (B_*\square\partial\Delta^n) \hookrightarrow B_*\square\Delta^n \tag{27}$$

for generating acyclic cofibrations $A \hookrightarrow B$ *of the quasi-monoid model structure.*

Fibrant objects in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_R$ satisfy one of the conditions of Definition 2.48 for Segal monoids, namely they have the right lifting property against the maps (21), since the inner horn inclusions $\lambda^n_k \hookrightarrow S^n$ are reduced Joyal equivalences by Proposition 2.23.

We will now show that the identity functor $\mathrm{Id} : \left(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}\right)_R \to (\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ is also left Quillen.

Proposition 2.56. *The adjunction*

$$\begin{tikzcd}
(\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_B \arrow[r, bend left, "Id"] \arrow[r, phantom, "\top"] & (\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_R \arrow[l, bend left, "Id"]
\end{tikzcd}$$

is Quillen.

Proof. It is sufficient to show that the generating cofibrations (26) and generating acyclic cofibrations (27) of the Reedy model structure are cofibrations and acyclic cofibrations in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$. The cofibrations of $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ are inclusions so the maps (26) and (27) remain cofibrations.

It only remains to show that for any reduced Joyal acyclic cofibration $A \hookrightarrow B$ and any inclusion of simplicial sets $K \subseteq L$ the map

$$(A_*\Box L) \cup_{A_*\Box K} (B_*\Box K) \hookrightarrow B_*\Box L$$

is an acyclic cofibration in the Segal monoid model structure. We will use [JT07] Lemma 7.14, which allows us to show that a map is an acyclic cofibration in a model structure by showing that it has the left lifting property against fibrations between fibrant objects.

Let $f : Y \to Z$ be a fibration between Segal monoids in the Segal monoid model structure. By adjointness we can consider the lifting problem

$$\begin{tikzcd}
K \arrow[r] \arrow[d, hook] & Y/^*B \arrow[d] \\
L \arrow[r] & Y/^*B \times_{Z/^*A} Z/^*A
\end{tikzcd}$$

where $K \hookrightarrow L$ is an inclusion of simplicial sets and $A \hookrightarrow B$ is a reduced Joyal acyclic cofibration. Hence we must show that the map on the right is an acyclic fibration of simplicial sets. Consider the pullback defining the map on the right

$$\begin{tikzcd}
Y/^*B \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y/^*B \times_{Z/^*A} Z/^*A \arrow[r] \arrow[d] & Y/^*A \arrow[d] \\
& Z/^*B \arrow[r] & Z/^*A
\end{tikzcd} \tag{28}$$

Since $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_B$ is defined by localizing $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ for any inclusion $u : A \hookrightarrow B$ of reduced simplicial sets and any horn $\Lambda^n_k \hookrightarrow \Delta^n$ in $\mathbf{sSet}$ the map

$$u_*\widehat{\Box}f : (A_*\Box\Delta^n) \cup_{A_*\Box\Lambda^n_k} (B_*\Box\Lambda^n_k) \hookrightarrow B_*\Box\Delta^n$$

obtained as the comparison map from the pushout is an acyclic cofibration in the Segal monoid model structure. Hence by adjointness since $Y \to Z$ is a fibration in the Segal monoid model structure the map

$$Y/^*B \to Y/^*A \times_{Z/^*A} Z/^*B$$

is a Kan fibration of simplicial sets.

Since Y and Z are Segal monoids and $A \hookrightarrow B$ is an inclusion of reduced simplicial sets the definition of Segal monoids in Definition 2.48 and Proposition 2.49 imply that for any inclusion of simplicial sets $\partial\Delta^n \hookrightarrow \Delta^n$ the map

$$\Delta^n\backslash^*Z \to \partial\Delta^n\backslash^*Z$$

is an inner fibration between quasi-monoids. Hence it is a fibration in the quasi-monoid model structure on $\mathbf{sSet}_0$, so for any reduced Joyal acyclic cofibration $A \hookrightarrow B$, $Z/^*B \to Z/^*A$ has the right lifting property against all inclusions $\partial\Delta^n \subseteq \Delta^n$ of simplicial sets and so is an acyclic fibration. Hence in (28) the maps $Y/^*B \to Y/^*A$ and $Z/^*B \to Z/^*A$ are acyclic fibrations, which are preserved by pullbacks, so the comparison map of (28) is an acyclic fibration of simplicial sets. The box product adjunction shows that $(A_*\Box L) \cup (B_*\Box K) \hookrightarrow B_*\Box L$ has the left lifting property against any fibration between fibrant objects in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$. Hence it is an acyclic cofibration in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ by [JT07] Lemma 7.14.

□

Corollary 2.57. *A row-wise reduced Joyal equivalence between reduced simplicial spaces is a weak equivalence of the Segal monoid model structure.*

This identifies the weak equivalences of the Reedy model structure, that is the row-wise reduced Joyal equivalences of reduced simplicial spaces, as Segal monoid weak equivalences. In particular, for any map of reduced simplicial spaces $f : Y \to Z$, if for all $n \geq 0$ the maps $f_{\bullet,n} : Y_{\bullet,n} \to Z_{\bullet,n}$ are reduced Joyal equivalences then f is a weak equivalence in the Segal monoid model structure. In fact, for a map between Segal monoids, we only have to check the zeroth row.

Proposition 2.58. *Let $f : Y \to Z$ be a map between Segal monoids. Then f is a weak equivalence in the Segal monoid model structure if and only if $f_{\bullet,0} : Y_{\bullet,0} \to Z_{\bullet,0}$ is a reduced Joyal equivalence.*

Proof. Suppose f is a weak equivalence. Then the right adjoint of the localization preserves weak equivalences between fibrant objects, so f is a weak equivalence of the Reedy model structure and so a levelwise equivalence. In particular, $f_{\bullet,0} : Y_{\bullet,0} \to Z_{\bullet,0}$ is a reduced Joyal equivalence.

Now suppose $f_{\bullet,0} : Y_{\bullet,0} \to Z_{\bullet,0}$ is a reduced Joyal equivalence. Since Y and Z are Segal monoids they have the right lifting property against maps

$$(\partial S^m{}_*\Box\Delta^n) \cup_{\partial S^m{}_*\Box\partial\Delta^n} (S^m{}_*\Box\Delta^0) \hookrightarrow S^m{}_*\Box\Delta^n$$

for all $n, m \geq 0$ Equivalently, $\Delta^n\backslash^* Z \to \Delta^0\backslash^* Z$ has the right lifting property against $\partial S^m \hookrightarrow S^m$, and similarly for Y. But $\Delta^n\backslash^* Z = Z_{\bullet,n}$ so the unique map $\Delta^n \to \Delta^0$ gives a section $Z_{\bullet,0} \to Z_{\bullet,n}$ for an acyclic fibration, hence is a reduced Joyal equivalence. The vertical maps in the following square are acyclic cofibrations in the quasi-monoid model structure

$$\begin{tikzcd}
Y_{\bullet,0} \arrow[r, "f_{\bullet,0}"] \arrow[d, hook] & Z_{\bullet,0} \arrow[d, hook] \\
Y_{\bullet,n} \arrow[r, "f_{\bullet,n}"'] & Z_{\bullet,n}
\end{tikzcd}$$

The map f is a row-wise equivalence since $f_{\bullet,0}$ is a reduced Joyal equivalence, hence it is a weak equivalence in the Segal monoid model structure. □

In the next section we will further analyze the weak equivalences of the Segal monoid model structure and show that this model structure is Quillen equivalent to the quasi-monoid model structure of Theorem 2.37.

2.6.3 Quillen Equivalences of the Segal Monoid and Quasi-Monoid Model Structures

We will show in this section that the Segal monoid model structure is Quillen equivalent to the quasi-monoid model structure in two different ways. We will follow the approach of §5 of [JT07], which proves similar results for the case of Segal categories and the Joyal model structure on **sSet**. The adjunctions we will show are Quillen equivalences both involve the diagonal functor.

Recall that the diagonal of a bisimplicial set X is the simplicial set with n-simplices $X_{n,n}$. The diagonal of a reduced simplicial space is a reduced simplicial set, so taking the diagonal is a functor

$$d : \mathbf{sSet}_0^{\Delta^{\mathrm{op}}} \to \mathbf{sSet}_0$$

The diagonal has a left and a right adjoint $d^* \dashv d \dashv d_*$. These functors are defined by

$$d^*(S^n) = S^n{}_*\square\Delta^n$$

$$d_*(A)_{n,m} = \mathrm{Hom}_{\mathbf{sSet}_0}(S^n{}_*\!\times \Delta^m, A)$$

where in the definition of d^* we are using the fact that all simplicial sets are colimits of their simplices and the left adjoint d^* must preserve colimits. We will show that both of these adjunctions are Quillen equivalences, and furthermore a map of reduced simplicial spaces $f : X \to Y$ is a weak equivalence in the Segal monoid model structure if and only if the diagonal $d(f)$ is a reduced Joyal equivalence in $\mathbf{sSet}_0$.

Our method to prove that d creates weak equivalences is essentially a modification of the proof method of [Moe89] Proposition 1.2, which was used to show the existence of the diagonal model structure on simplicial spaces. The proof of this result uses the following Theorem on Reedy model structures and functor tensor products.

Theorem 2.59 ([Hir03] 18.4.11)**.** *The functor tensor product*

$$- \otimes_\Delta - : \boldsymbol{sSet}^{\Delta^{\mathrm{op}}} \times \left((\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C\right)^{\Delta} \to (\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C$$

is a Quillen adjunction of two variables when $\boldsymbol{sSet}^{\Delta^{\mathrm{op}}}$ *and* $\left((\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C\right)^{\Delta}$ *have the Reedy model structures determined by the Quillen and column-wise structures respectively.*

Here we have used that $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ is a simplicial model category. The role of this theorem comes from the following observations

$$A_*\square\Delta^0 = \int^{[n]\in\Delta} (S^n{}_*\square\Delta^0)\ {}_*\!\otimes A_n$$

$$d^*(A) = \int^{[n]\in\Delta} (S^n{}_*\square\Delta^n)\ {}_*\!\otimes A_n$$

The second of these is just the definition of d^*. The first comes from the observation that there is an isomorphism of reduced simplicial spaces

$$(S^n{}_*\square\Delta^0)\ {}_*\!\otimes A_n \cong (S^n{}_*\!\times A_n)_*\square\Delta^0$$

and the fact that $(-)_*\square\Delta^0$ is a left adjoint, so it preserves coends.

We will define a natural transformation $\gamma : d^* \to (-)_*\Box\Delta^0$ by taking the realization of the map of cosimplicial reduced simplicial spaces

$$S^\bullet{}_*\Box\Delta^\bullet \to S^\bullet{}_*\Box\Delta^0 \tag{29}$$

All simplicial spaces are cofibrant in the Reedy model structure on $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ induced by the Kan model structure on $\mathbf{sSet}$, so to show that $\gamma_A : d^*(A) \to A_*\Box\Delta^0$ is a weak equivalence in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$, and so by the localization of Theorem 2.50 in the Segal monoid model structure it is sufficient to show that the map (29) is a Reedy weak equivalence of Reedy cofibrant cosimplicial reduced simplicial spaces.

Proposition 2.60. *The map $S^\bullet{}_*\Box\Delta^\bullet \to S^\bullet{}_*\Box\Delta^0$ is a Reedy weak equivalence between Reedy cofibrant cosimplicial reduced simplicial spaces in $\big((\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_C\big)^\Delta$.*

Proof. The maps $S^n{}_*\Box\Delta^n \to S^n{}_*\Box\Delta^0$ are column-wise weak equivalences, so the map (29) is a Reedy weak equivalence in $\big((\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C\big)^\Delta$.

We calculate the latching map for $S^\bullet{}_*\Box\Delta^0$ and show that it is an inclusion, that is a cofibration in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$. For a cosimplicial reduced simplicial space $X^\bullet$ the t^{th} latching space is

$$L^t(X^\bullet) = \int^{[n]\in\Delta} X^n \;{}_*\!\otimes \partial\Delta^t_n$$

Applied to $S^\bullet{}_*\Box\Delta^0$ we have

$$L^t(S^\bullet{}_*\Box\Delta^0) = \partial S^t{}_*\Box\Delta^0 \hookrightarrow S^t{}_*\Box\Delta^0$$

Hence $S^\bullet{}_*\Box\Delta^0$ is Reedy cofibrant.

For $S^\bullet{}_*\Box\Delta^\bullet$, following [Moe89], the t^{th} latching space has (l,m)-bisimplices

$$L^t(S^\bullet{}_*\Box\Delta^\bullet)_{l,m} = \{(\theta,\varphi) \in S^t_l \;{}_*\!\times \Delta^t_m \;\mid\; \exists j \in [t] \text{ s.t. both } \theta \text{ and } \varphi \text{ miss j}\}$$

So $S^\bullet{}_*\Box\Delta^\bullet$ is Reedy cofibrant. □

By Theorem 2.59, the coend preserves Reedy weak equivalences between Reedy cofibrant cosimplicial reduced simplicial spaces. Hence we can conclude the following.

Corollary 2.61. *There is a natural transformation*

$$\gamma : d^* \to (-)_*\Box\Delta^0$$

whose components are weak equivalences in the Segal monoid model structure on $\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}}$.

Proof. The components of the natural transformation for a reduced simplicial set A are the maps $\gamma_A : d^*(A) \to A_*\Box\Delta^0$ determined by taking the coend with A of the map of cosimplicial reduced simplicial spaces (29). Naturality of γ in A follows from the construction of the coends. □

The naturality square for γ shows that $d^*(f)$ is a weak equivalence in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ if and only if $f_*\Box\Delta^0$ is

$$\begin{tikzcd}
d^*(A) \arrow[r, "\sim"] \arrow[d, "d^*(f)"'] & A_*\Box\Delta^0 \arrow[d, "f_*\Box\Delta^0"] \\
d^*(B) \arrow[r, "\sim"] & B_*\Box\Delta^0
\end{tikzcd} \tag{30}$$

We can now show that the two adjunctions $d^* \dashv d \dashv d_*$ are Quillen adjunctions for the Segal monoid and quasi-monoid model structures.

Proposition 2.62. *$d^* \dashv d$ is a Quillen adjunction.*

Proof. The definition of d^* as the realization of the Reedy cofibrant cosimplicial reduced simplicial space $S^{\bullet}{}_{*}\square\Delta^{\bullet}$ shows that it preserves inclusions, which are the cofibrations in $\mathbf{sSet}_0$. By the square (30) and Proposition 2.56 d^* preserves acyclic cofibrations. $\square$

Proposition 2.63. *$d \dashv d_*$ is a Quillen adjunction.*

Proof. The diagonal preserves cofibrations, it only remains to show it preserves acyclic cofibrations. We will show it preserves all weak equivalences of the Segal monoid model structure. Let $f : Y \to Z$ be a weak equivalence of the Segal monoid model structure. Let X be a quasi-monoid in $\mathbf{sSet}_0$. The n-simplices of $(X^{d(Y)})_*$ are the maps

$$d(Y)\ {}_{*}\!\times \Delta^n \to X$$

Since $d(Y\ {}_{*}\!\otimes \Delta^n) = d(Y)\ {}_{*}\!\times \Delta^n$ these maps correspond to maps

$$Y\ {}_{*}\!\otimes \Delta^n \to d_*(X)$$

which are the n-simplices of $(d_*(X)^Y)_*$. Hence there are natural isomorphisms which make the diagram

$$\begin{tikzcd}
(X^{d(Z)})_* \arrow[r, equal, "\sim"] \arrow[d, "d(f)^*"'] & (d_*(X)^Z)_* \arrow[d, "f^*"] \\
(X^{d(Y)})_* \arrow[r, equal, "\sim"] & (d_*(X)^Y)_*
\end{tikzcd}$$

commute. To show d preserves weak equivalences it is sufficient to show $d_*(X)$ is a Segal monoid when X is a quasi-monoid. $d_*(X)$ is a Segal monoid if and only if it has the right lifting property for $0 < l < m$ and $0 \leq k \leq n$ against all maps

$$(\lambda^m_l{}_{*}\square\Delta^n) \cup (S^m{}_{*}\square\partial\Delta^n) \hookrightarrow S^m{}_{*}\square\Delta^n$$

$$(\partial S^m{}_{*}\square\Delta^n) \cup (S^m{}_{*}\square\Lambda^n_k) \hookrightarrow S^m{}_{*}\square\Delta^n$$

Since $d(A_*\square K) = A\ {}_{*}\!\times K$ for all $A \in \mathbf{sSet}_0$ and $K \in \mathbf{sSet}$ these lifting problems correspond by adjunction to

$$\begin{tikzcd}
(\partial S^m\ {}_{*}\!\times \Delta^n) \cup (S^m\ {}_{*}\!\times \Lambda^n_k) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
S^m\ {}_{*}\!\times \Delta^n \arrow[r] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
(\lambda^m_l\ {}_{*}\!\times \Delta^n) \cup (S^m\ {}_{*}\!\times \partial\Delta^n) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
S^m\ {}_{*}\!\times \Delta^n \arrow[r] & *
\end{tikzcd}$$

These have solutions, so $d_*(X)$ is a Segal monoid and the result holds. $\square$

Next we will show that the adjunction unit for $d_* \dashv d$ is a weak equivalence in the Segal monoid model structure. This will be important in proving that this adjunction is a Quillen equivalence, but it also allows us to replace any Segal monoid up to Segal monoid weak equivalence with a Segal monoid in the image of the functor $d_* : \mathbf{sSet}_0 \to \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$. To do this we will use the characterization of weak equivalences between Segal monoids in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ given in Proposition 2.58. We will also find another use for the Reedy acyclic cofibration of cofibrant cosimplicial reduced simplicial spaces from Proposition 2.60.

Proposition 2.64. *Let Z be a Segal monoid. The unit map $Z \to d_*d(Z)$ of the adjunction $d \dashv d_*$ is a weak equivalence in the Segal monoid model structure between Segal monoids.*

Proof. First we will show that $d_*d(Z)$ is a Segal monoid. d and d_* are a right adjoints of Quillen adjunctions by Propositions 2.62 and 2.63 so they preserve fibrant objects.

To show that this map is a weak equivalence between Segal monoids it is sufficient to show $Z_{\bullet,0} \to d_*d(Z)_{\bullet,0}$ is a reduced Joyal equivalence by Proposition 2.58. By the definition of d_* we have

$$d_*d(Z)_{\bullet,0} = \mathrm{Hom}_{\mathbf{sSet}_0}(S^{\bullet}\ {}_*\!\times \Delta^0, d(Z)) = d(Z)$$

Consider the map $Z_{\bullet,0\,*}\square\Delta^0 \to Z$. This is a row-wise weak equivalence as the maps between rows are the maps $Z_{\bullet,0} \to Z_{\bullet,n}$ that were found to be reduced Joyal equivalences in the proof of Proposition 2.58. Hence it is a weak equivalence of the Segal monoid model structure by Corollary 2.57. The diagonal of this map is a reduced Joyal equivalence, since it is a weak equivalence between cofibrant objects of the Segal monoid model structure and d is a left Quillen functor by Proposition 2.63. But $d(Z_{\bullet,0\,*}\square\Delta^0) = Z_{\bullet,0}$ so the map $Z_{\bullet,0} \to d(Z)$ is a reduced Joyal equivalence. □

We can use this approximation of Segal monoids by quasi-monoids under the functor d_* to prove that the diagonal functor creates weak equivalences for the Segal monoid model structure when $\mathbf{sSet}_0$ has the quasi-monoid model structure.

Proposition 2.65. *A map $f : Y \to Z$ between reduced simplicial spaces is a weak equivalence in the Segal monoid model structure if and only if $d(f)$ is a reduced Joyal equivalence.*

Proof. The only if part is given by Proposition 2.63, which shows that d preserves weak equivalences. To show the other direction, we use Proposition 2.64 to replace up to Segal monoid weak equivalence any Segal monoid by a Segal monoid obtained by applying d_* to a quasi-monoid. Let $f : Y \to Z$ be a map of reduced simplicial spaces such that $d(f) : d(Y) \to d(Z)$ is a reduced Joyal equivalence. Let Γ be a Segal monoid. We must show that the top map of the diagram

$$\begin{tikzcd}
(\Gamma^Z)_* \arrow[r, "f^*"] \arrow[d] & (\Gamma^Y)_* \arrow[d] \\
(d_*d(\Gamma)^Z)_* \arrow[r, "f^*"] \arrow[d, equal, "\wr"] & (d_*d(\Gamma)^Y)_* \arrow[d, equal, "\wr"] \\
(d(\Gamma)^{d(Z)})_* \arrow[r, "d(f)_*"'] & (d(\Gamma)^{d(Y)})_*
\end{tikzcd}$$

is a Kan weak equivalence of simplicial sets. The lower vertical isomorphisms are the maps described in the proof of Proposition 2.63. The bottom map $d(f)_*$ is a weak equivalence because d is a right adjoint of a Quillen adjunction by Proposition 2.62, so it preserves fibrant objects, so $d(\Gamma)$ is a quasi-monoid and by assumption $d(f)$ is a reduced Joyal equivalence. To show that the top map f^* is a weak equivalence it is sufficient to show that

$$(\Gamma^Z)_* \to (d_*d(\Gamma)^Z)_* \tag{31}$$

is a weak equivalence. The simplicial set $(\Gamma^Z)_*$ is given by

$$\mathrm{Hom}_{\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}}(Z\ {}_*\!\otimes \Delta^{\bullet}, \Gamma)$$

We claim that the cosimplicial reduced simplicial space $Z\ {}_*\!\otimes\ \Delta^\bullet$ is a cofibrant approximation of Z, the constant diagram, in the Reedy model structure on cosimplicial reduced simplicial spaces determined by the Segal monoid model structure. This follows because the maps $Z\ {}_*\!\otimes\ \Delta^n \to Z\ {}_*\!\otimes\ \Delta^0 = Z$ are weak equivalences in $(\mathbf{sSet}_0^{\Delta^{\mathrm{op}}})_C$ by the simplicial model structure, so they are weak equivalences in the Segal monoid model structure. Hence by [Hir03] Corollary 16.5.5 the map (31) is a weak equivalence since $\Gamma \to d_*d(\Gamma)$ is a weak equivalence of Segal monoids. □

Since d sends the copowering of $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ over $\mathbf{sSet}$ to the copowering of $\mathbf{sSet}_0$ over $\mathbf{sSet}$ we can show that the Segal monoid model structure is a simplicial model structure from the fact that d creates weak equivalences.

Corollary 2.66. *The Segal monoid model structure on* $\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}}$ *is a simplicial model structure.*

Proof. Since $d(Z\ {}_*\!\otimes\ K) = d(Z)\ {}_*\!\times\ K$ and d preserves colimits it is clear the functor $-\ {}_*\!\otimes\ - : \mathbf{sSet}_0^{\Delta^{\mathrm{op}}} \times \mathbf{sSet} \to \mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ is a Quillen functor. □

Proposition 2.65 reduces the problem of identifying weak equivalences in the Segal monoid model category to the problem of identifying reduced Joyal equivalences in $\mathbf{sSet}_0$. It also allows us to identify both Quillen adjunctions between $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ and $\mathbf{sSet}_0$ as Quillen equivalences.

Proposition 2.67. *Both adjunctions*

$$\begin{tikzcd} (\boldsymbol{sSet}_0^{\Delta^{\mathrm{op}}})_B \arrow[r, "d" description] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & (\boldsymbol{sSet}_0)_J \arrow[l, bend right=40, "d^*"'] \arrow[l, bend left=40, "d_*"] \end{tikzcd}$$

are Quillen equivalences.

Proof. Since d creates weak equivalences in $\mathbf{sSet}_0^{\Delta^{\mathrm{op}}}$ by Proposition 2.65 we only have to check that the maps

$$A \to dd^*(A)$$

$$dd_*(X) \to X$$

are reduced Joyal equivalences for all reduced simplicial sets A and all quasi-monoids X. The first of these has a retract given by d applied to the weak equivalence $\gamma_A : d^*(A) \to A_*\Box\Delta^0$ of Corollary 2.61, so it is a reduced Joyal equivalence.

For the second map, consider the map of reduced simplicial spaces

$$\mathrm{Hom}_{\mathbf{sSet}_0}(S^n\ {}_*\!\times\ \Delta^0, X) = (X_*\Box\Delta^0)_{n,m} \to d_*(X)_{n,m} = \mathrm{Hom}_{\mathbf{sSet}_0}(S^n\ {}_*\!\times\ \Delta^m, X) \tag{32}$$

determined by the unique maps $\Delta^m \to \Delta^0$. Row-wise, this map has the components

$$X = (X_*\Box\Delta^0)_{\bullet,m} \to d_*(X)_{\bullet,m} = \mathrm{Hom}_{\mathbf{sSet}_0}(S^\bullet\ {}_*\!\times\ \Delta^m, X) = E_0(X^{\Delta^m})$$

which are reduced Joyal equivalences. The diagonal of (32) gives a section for the map $dd_*(X) \to X$ that is a Segal monoid weak equivalence. □

3 Realizing ∞-Monoids and Groups

In this section we will show that the models of ∞-monoids we described in the previous chapter are equivalent to the model given by simplicial monoids. A simplicial monoid is a monoid in the category of simplicial sets with the cartesian monoidal product. There is a model structure for simplicial monoids transferred along the free-forgetful adjunction

$$\begin{tikzcd} \mathbf{sMon} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathbf{sSet} \arrow[l, bend left, "F_\times"] \end{tikzcd}$$

from the Kan model structure on **sSet** as described in §2 of [SS00]. We will show that there is a Quillen equivalence between **sMon** and $\mathbf{sSet}_0$ with the quasi-monoid model structure which we call the homotopy coherent nerve and realization

$$\begin{tikzcd} \mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_J \arrow[l, bend left, "\mathbb{C}"] \end{tikzcd} \tag{33}$$

This adjunction is determined by a cosimplicial simplicial monoid, which we call the homotopy coherent simplices and define in Section 3.1. It is the single object restriction of the homotopy coherent nerve defined in [Cor82] between categories enriched in simplicial sets and simplicial sets.

The first sections of this chapter, from Section 3.1 to Section 3.6 are dedicated to a new proof of the fact that the homotopy coherent nerve-realization adjunction (33) is a Quillen equivalence. This new proof is a hybrid of techniques from [Lur09] and [DS11a] and [DS11b] along with some new constructions that unify the perspectives of these references. The outline of the proof in these sections is as follows. We use the basic set-up of Lurie, where a core part of the proof reduces to showing that when X is a quasi-monoid both maps in the span of simplicial sets defined by Lurie in §2.2 of [Lur09]

$$\begin{tikzcd} (X^{S^1})_* & H(X) \arrow[l] \arrow[r] & U\mathbb{C}(X) \end{tikzcd}$$

are Kan weak equivalences. The definitions of the spaces $H(X)$ and the maps of this span are given below in Sections 3.3 and 3.4. We use Lurie's results to show that the left map is a weak equivalence but for the right map we develop our own approach using tools from Dugger and Spivak's papers [DS11a] and [DS11b]. In particular, we show that the right hand map can be factored using a filtration of the simplicial set $U\mathbb{C}(X)$ underlying the simplicial monoid $\mathbb{C}(X)$. This filtration is by a quantity called spine length, which we define in Section 3.3. This definition and the next sections where we show that each stage of the filtration is a Kan weak equivalence of simplicial sets when X is a quasi-monoid are the new content of this proof.

At the end of this chapter we consider ∞-groups among ∞-monoids. We show that the localization of the quasi-monoid model structure given in Section 2.5 is equivalent on homotopy categories to the adjunction for the reflective subcategory of simplicial groups within simplicial monoids. This is the adjunction on homotopy categories arising from the Quillen adjunction

$$\begin{tikzcd} \mathbf{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "L"] \end{tikzcd}$$

where I is the inclusion functor and L is the group completion functor. We will show that this adjunction is equivalent to the localization adjunction (20) by applying the homotopy coherent nerve-realization adjunction, which is a Quillen equivalence by the first sections of this chapter. In particular, we will show that localizing **sMon** at the realization by $\mathbb{C}$ of the map $S^1 \hookrightarrow R^1$ of reduced simplicial sets from Section 2.5 gives a model structure on the category of simplicial monoids that has fibrant objects the Kan fibrant simplicial monoids whose monoid of connected components is a group. We will show that this model structure on simplicial monoids is Quillen equivalent to the model structure for simplicial groups.

Finally, we use this equivalence of adjunctions on homotopy categories to recover a case of the result of Dwyer and Kan in [DK80] about localizing simplicial categories. In particular, we show that the derived functor of group completion for a Kan fibrant simplicial monoid M is isomorphic to the homotopy type of the loop space on $\mathbb{N}(M)$. We will do this by showing that the localization of the homotopy coherent nerve-realization adjunction described above is equivalent to the Kan loop group adjunction between **sGp** and the Kan model structure on reduced simplicial sets $(\mathbf{sSet}_0)_K$ described in Theorem 2.1.

3.1 Homotopy Coherent Simplices

In this section we will construct a cosimplicial simplicial monoid $\mathbb{C}^\bullet : \Delta \to \mathbf{sMon}$ whose components $\mathbb{C}^n$ for $n \geq 0$ we will call the homotopy coherent simplices. These cosimplicial simplicial monoids are the collapse to a single object of the cosimplicial simplicial categories that determine the homotopy coherent nerve defined in [Cor82]. We will present a new definition of these simplicial monoids, however, as the nerves of partially ordered monoids, rather than relying on previous constructions for simplicial categories. Finally we will show that the simplicial monoids that make up $\mathbb{C}^\bullet$ for all $n \geq 0$ are cofibrant, and furthermore have monoids of m-simplices that are freely generated as monoids for all $m \geq 0$. The constructions and proofs of this section are self-contained but they are based on the methods of Dugger and Spivak in [DS11a] and [DS11b]. The relation to equivalent proofs for simplicial categories will be highlighted.

As mentioned we will construct the cosimplicial simplicial monoid for this nerve-realization adjunction by first constructing a cosimplicial partially ordered monoid and then taking its nerve. A partially ordered monoid is a 2-category where there is one object, the 1-cells are the elements of the monoid and there is at most one 2-cell between 1-cells, corresponding to the partial order. The identities of a 2-category correspond to the requirement that the partial order respects the monoid multiplication.

To construct our partially ordered monoids we will start with 2-categories and collapse their 2-cells so that there is at most one 2-cell joining any two 1-cells. The 2-categories we collapse will be freely generated, a construction that we will describe in the particular case of 2-categories here, but which will be generalized to describe **computads**, or freely generated n-categories for all $n \geq 0$, in Section 4.5.

Definition 3.1. *Let $n \geq 1$. The 2-category A^n has a unique object, 1-cells freely generated by an alphabet*

$$\Sigma_n = \{x_{ij} \mid 0 \leq i < j \leq n\}$$

and 2-cells freely generated by the alphabet

$$R^n = \{x_{ik} \Rightarrow x_{ij}x_{jk} \mid 0 \leq i < j < k \leq n\}$$

with source and target as indicated.

We write 0-composition in A^n as juxtaposition because it is viewed as multiplication in the free monoid Σ_n^* generated by the set Σ_n. We write 1-composition of 2-cells in A^n with $*_1$. Composition of all kinds is read from left to right, as for words of a monoid, unlike the usual convention for composing morphisms of a category.

Such a freely generated 2-category can be constructed for example by following the construction of §2.4.3 of [GM18]. The 1-cells of A_n are words in the free monoid Σ_n^* on the alphabet Σ_n. The 2-cells of A^n are classes of the free category on generators $u\alpha v$ for u and v words in Σ_n^* and $\alpha \in R^n$ under the equivalence relation that identifies cells to establish the exchange identity. This relation identifies the words in the free category

$$(\alpha w v) *_1 (u' w \beta) \sim (u w \beta) *_1 (\alpha w v')$$

where $\alpha : u \implies u'$ and $\beta : v \implies v'$ are elements of R^n and w is a word in Σ_n^*. The 2-category A^n therefore has 2-cells given by 1-composites of the 2-cells

$$\begin{tikzcd}
* \arrow[r, "u"] & * \arrow[r, bend left, "x_{ik}"] \arrow[r, bend right, "x_{ij}x_{ik}"'] \arrow[r, phantom, "\Downarrow \alpha"] & * \arrow[r, "v"] & *
\end{tikzcd}$$

where $u, v \in \Sigma_n^*$, $\alpha \in R$, and the source of this 2-cell is $ux_{ik}v$ and its target is $ux_{ik}x_{kj}v$. A general 2-cell in A^n is of the form

$$u_1\alpha_1u_1' *_1 u_2\alpha_2u_2' *_1 \cdots *_1 u_n\alpha_nu_n'$$

where $u_i, u_i' \in \Sigma_n^*$, $\alpha_i \in R$. We say a 2-cell in A^n is **atomic** if it is of the form

$$\alpha_1 *_1 u_2\alpha_2u_2' *_1 \cdots *_1 u_n\alpha_nu_n'$$

so the source of the 2-cell is a single letter x_{ij} of Σ_n^*. The target of an atomic 2-cell is a word of the form

$$x_{il_1}x_{l_1l_2}\cdots x_{l_mj}$$

where $0 \leq i < l_1 < l_2 < \cdots < l_m < j \leq n$.

We define our partially ordered monoids with reference to 2-cells of A^n.

Definition 3.2. *Let $n \geq 0$. The partially ordered monoid C^n has its underlying monoid equal to Σ_n^* and its partial order determined by*

$$u \leq v \iff \text{there exists a 2-cell } \gamma : u \Rightarrow v \in (A^n)_2$$

Note that the source of any non-identity 2-cell of A^n is necessarily a shorter word in Σ_n^* than its target, so there are no loops of 2-cells in A^n and this is a well-defined partial order. Since $\Sigma_0 = \emptyset$ the partially ordered monoid C^0 is the trivial monoid. The partially ordered monoid C^1 is the free monoid on one generator with trivial partial order.

There is a natural 2-functor $\varphi : A^n \to C^n$ that is the identity on 1-cells and sends a 2-cell $\gamma : u \Rightarrow v$ of A^n to the unique relation $u \leq v$ it implies. Under this 2-functor an atomic 2-cell of A^n determines to an **atomic relation**

$$x_{ij} \leq x_{il_1}x_{l_1l_2}\cdots x_{l_mj}$$

in the partially ordered monoid C^n. For the purposes of the next result we will include the trivial relations $u \leq u$ for words $u \in \Sigma_n^*$ as atomic relations for C^n. This result will allow us to show that the simplicial monoids $\mathbb{C}^n$ we obtain from C^n by taking the nerve are cofibrant in the model structure on **sMon**.

Proposition 3.3. *The relation $u \leq v$ in C^n for $u, v \in \Sigma_n^*$ can be written uniquely as the product of a sequence of atomic relations $x_{l_i r_i} \leq v_i$, that is*

$$u \leq v = \prod_i (x_{l_i r_i} \leq v_i)$$

Proof. First we will show that such a decomposition exists by showing that all relations of C^n can be written as a product of atomic arrows. A generic 2-cell in A^n is an equivalence class of a composite

$$u_1 \alpha_1 u_1' *_1 u_2 \alpha_2 u_2' *_1 \cdots *_1 u_n \alpha_n u_n' \tag{34}$$

under the exchange relation. We will prove this result by induction on the length of such a word, that is the number of 2-indeterminates from R that appear in any composite equivalent to (34) under the exchange identity.

A single 2-cell $u\alpha v$ is clearly sent by φ to a product of atomic 2-cells

$$\varphi(1_u)\varphi(\alpha)\varphi(1_v) : u x_{ik} v \leq u x_{ij} x_{jk} v$$

as $\alpha : x_{ik} \Rightarrow x_{ij} x_{jk}$ is atomic. Now let $m \geq 1$ and consider a 2-cell of A^n

$$f *_1 u\alpha v = u_1 \alpha_1 u_1' *_1 u_2 \alpha_2 u_2' *_1 \cdots *_1 u_m \alpha_m u_m' *_1 u\alpha v$$

where $\alpha : x_{ik} \Rightarrow x_{ij} x_{jk}$ is a 2-indeterminate from R. By induction we can write

$$\varphi(f) = \prod_{i=1}^{l} \varphi(\gamma_i)$$

for atomic 2-cells γ_i of A^n. Let x_{ik} be the source of α. By studying the word

$$d_1^- u\alpha v = d_1^+ f$$

in Σ_n^* we can identify the copy of x_{ik} occurring in $d_1^+ f$ that is the source of α in the 2-cell $u\alpha v$. Let $1 \leq t \leq m$ be the index of the 2-cell γ_t such that the source x_{ik} of α occurs in the target of γ_t. If γ_t is a non-trivial 2-cell then let

$$d_1^+ \gamma_t = u' x_{ik} v'$$

We can write

$$\varphi(f *_1 u\alpha v) = \prod_{i=1}^{l} \varphi(\gamma_i')$$

where

$$\gamma_i' = \begin{cases} \gamma_i & i \neq t \\ \gamma_t *_1 u'\alpha v' & i = t \end{cases}$$

and $d_1^- \gamma_i' = d_1^- \gamma_i$ for all i so γ_i' are all atomic 2-cells. If γ_t is an identity 2-cell then we must take the atomic decomposition

$$\varphi(f *_1 u\alpha v) = \left(\prod_{i=1}^{t-1} \varphi(\gamma_i) \right) \varphi(1_u')\varphi(\alpha)\varphi(1_v') \left(\prod_{i=t+1}^{l} \varphi(\gamma_i) \right)$$

So we are done by induction.

Now we show that given two decompositions of a relation in C^n as products of atomic arrows, the atomic arrows involved must have the same sources and targets. Suppose

$$\prod_{i=1}^{n} \gamma_i = \prod_{j=1}^{m} \delta_j$$

Considering the sources, which are words in free monoid Σ_n^*, we must have $n = m$ and $s(\gamma_i) = s(\delta_i)$ for all i. By induction it is sufficient to consider a product $\gamma\delta = \gamma'\delta'$. We must have

$$\begin{tikzcd}
x_{ij} \arrow[r, "\gamma"] & x_{ii_1}x_{i_1i_2}\cdots x_{i_rj} & x_{ij} \arrow[r, "\gamma'"] & x_{ii_1'}x_{i_1'i_2'}\cdots x_{i_r'j} \\
x_{kl} \arrow[r, "\delta"] & x_{kk_1}x_{k_1k_2}\cdots x_{k_tl} & x_{kl} \arrow[r, "\delta"] & x_{kk_1'}x_{k_1'k_2'}\cdots x_{k_t'l}
\end{tikzcd}$$

but unique factorization in Σ_n^* and the requirement that $i_n < i_{n+1}$ and similar for k requires that $i_n = i_n'$ and $k_n = k_n'$, so the result holds. □

We can now define the simplicial monoids $\mathbb{C}^n$ by taking the nerve of these partially ordered monoids with respect to the partial order. The nerve preserves the monoid structure of the poset so the simplicial set we obtain from the nerve is a simplicial monoid.

Definition 3.4. *Let $n \geq 0$. The **homotopy coherent** n**-simplex** is the simplicial monoid $\mathbb{C}^n = NC^n$ that is the nerve of C^n with respect to the partial order.*

An **atomic m-simplex** of $\mathbb{C}^n$ is a simplex of the form

$$x_{ij} \leq \omega_1 \leq \cdots \leq \omega_m$$

for words $\omega_i \in \Sigma_n^*$. Using the characterization of relations in C^n given in Proposition 3.3 we can now show that the atomic m-simplices freely generate the monoid of m-simplices of $\mathbb{C}^n$.

Proposition 3.5. *The monoid of m-simplices of $\mathbb{C}^n$ is freely generated by the atomic m-simplices. In particular, $\mathbb{C}^n$ is a cofibrant simplicial monoid.*

Proof. For the 0-simplices this is by definition of $\mathbb{C}^n$, for the 1-simplices this follows by Proposition 3.3. Let $n \geq 1$ and suppose the result holds for n. Consider an $n+1$-simplex

$$\omega = \omega_0 \leq \omega_1 \leq \cdots \leq \omega_{n+1}$$

Its $n + 1^{\text{th}}$ face $d_{n+1}(\omega)$ can be written uniquely as a product of atomic n-simplices by induction hypothesis. Similarly $d_0^n(\omega)$ is a 1-simplex so it can be written uniquely as a product of atomic 1-simplices

$$d_{n+1}(\omega) = \omega_0 \leq \omega_1 \leq \cdots \omega_n = \prod_{s=1}^{t} (x_{i_sj_s} \leq \gamma_1^s \leq \cdots \leq \gamma_n^s)$$

$$d_0^n(\omega) = \omega_n \leq \omega_{n+1} = \prod_{r=1}^{u} (x_{k_r l_r} \leq \delta^r)$$

Define a partition of $\{1, 2, \cdots, u\}$ into t intervals $\varphi(s)$ for $1 \leq s \leq t$ by the identifications

$$\prod_{r=1}^{u} x_{k_r l_r} = \omega_n = \prod_{s=1}^{t} \gamma_n^s$$

Then

$$\prod_{s=1}^{t} \left(x_{i_s j_s} \leq \gamma_1^s \leq \cdots \leq \gamma_n^s \leq \left(\prod_{j \in \varphi(s)} \delta^j \right) \right)$$

gives a unique factorization of ω.

Finally, the degeneracy of an atomic n-simplex is obtained by repeating on of the words, so it is an atomic $n+1$-simplex and $\mathbb{C}^n$ is a cofibrant simplicial monoid. □

The simplicial monoids $\mathbb{C}^n$ for $n \geq 0$ assemble into a cosimplicial object in **sMon**. Given an order preserving map $\theta : [n] \to [m]$ in Δ we define a map $C(\theta) : C^n \to C^m$ of partially ordered monoids by $C(\theta)(x_{ij}) = x_{\theta(i)\theta(j)}$, using the convention that $x_{ii} = e$, the identity of the monoid. Since θ is order-preserving, it respects the partial order of C^n, so this gives a cosimplicial partially ordered monoid $C^\bullet$. The cosimplicial simplicial monoid $\mathbb{C}^\bullet$ is obtained by taking the nerve of $C^\bullet$. We will refer to simplices of $\mathbb{C}^n$ as **chains**.

In the next section we will describe the nerve-realization adjunction that is determined by the cosimplicial simplicial monoid $\mathbb{C}^\bullet : \Delta \to \mathbf{sMon}$.

3.2 Homotopy Coherent Nerve and Realization

The nerve-realization adjunction determined by the cosimplicial simplicial monoid of homotopy coherent simplices is between reduced simplicial sets and **sMon**

$$\begin{tikzcd}
\mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & \mathbf{sSet}_0 \arrow[l, bend left, "\mathbb{C}"]
\end{tikzcd}$$

where

$$\mathbb{C}(A) = \int^{n \in \Delta} \mathbb{C}^n \cdot A_n$$

$$\mathbb{N}(M)_n = \mathrm{Hom}_{\mathbf{sMon}}(\mathbb{C}^n, M)$$

with $\cdot$ representing the tensoring of simplicial monoids by sets given by the coproduct of $\mathbb{C}^n$ indexed by the set A_n

$$\mathbb{C}^n \cdot A_n = \coprod_{a \in A_n} \mathbb{C}^n$$

The nerve $\mathbb{N}(M)$ of a simplicial monoid M is a reduced simplicial set since $\mathbb{C}^0 = e$, the initial simplicial monoid, so there is a unique simplicial monoid map $e = \mathbb{C}^0 \to M$.

We call this the **homotopy coherent nerve** and **realization**, directly recalling the construction of Cordier [Cor82] of which it is only the restriction to reduced simplicial sets. The simplicial monoids $\mathbb{C}^n$ constructed above are exactly those obtained by collapsing all the objects of the simplicially enriched categories defining Cordier's nerve.

In this section we will show that this adjunction $\mathbb{C} \dashv \mathbb{N}$ is a Quillen adjunction when $\mathbf{sSet}_0$ has the quasi-monoid model structure. We will also give the outline of the proof the this adjunction is in fact a Quillen equivalence and we will complete this proof in the subsequent sections.

That Cordier's nerve gives a Quillen equivalence between simplicially enriched categories and simplicial sets with the Joyal model structure has been proved by at least two different methods by Lurie [Lur09] and Dugger and Spivak [DS11a] [DS11b]. The proof we give for this restricted simplicial monoids result is a hybrid of their ideas. We apply Lurie's strategy to show that there is a span of simplicial set weak equivalences relating the simplicial mapping space $(A^{S^1})_*$ of a quasi-monoid with $\mathbb{C}(A)$. We don't follow Lurie's method of constructing the straightening-unstraightening adjunction ([Lur09] §2.2), however. We will apply the ideas of [DS11a] to construct a filtration by acyclic cofibrations of $\mathbb{C}(A)$ which give the weak equivalence we want. We begin by describing the realization $\mathbb{C}(A)$ of a reduced simplicial set in more detail.

By the definition of $\mathbb{C}(A)$ as the coend

$$\mathbb{C}(A) = \int^{n \in \Delta} \mathbb{C}^n \cdot A_n$$

the monoid of r-simplices of $\mathbb{C}(A)$ is generated by equivalence classes $[a \in A_n, \gamma \in \mathbb{C}^n_r]$ of elements of $\mathbb{C}^n_r \cdot A_n$ for any value of n with the relations

$$[a \in A_n, \gamma \in \mathbb{C}^n_r] \cdot [a \in A_n, \omega \in \mathbb{C}^n_r] = [a \in A_n, \gamma\omega \in \mathbb{C}^n_r]$$

The equivalence classes are determined by the generating relations

$$(\theta^*(a) \in A_n, \gamma \in \mathbb{C}^n_r) \sim (a \in A_m, \theta(\gamma) \in \mathbb{C}^m_r)$$

for any non-decreasing map $\theta : [n] \to [m]$ of Δ. By Proposition 3.5 any chain $\gamma \in \mathbb{C}^n_r$ can be written uniquely as a product of atomic chains, that is atomic r-simplices of $\mathbb{C}^n$. Hence $\mathbb{C}(A)_r$ is generated by the classes $[a \in A_n, \gamma \in \mathbb{C}^n_r]$ where γ is an atomic chain.

To simplify the analysis these classes in $\mathbb{C}(A)_r$, and anticipating the role of necklaces later based on those used by [DS11b], we introduce the category of flagged beads over a reduced simplicial set. We will refer to the category of simplices of a reduced simplicial set A as the category of **beads** over A. A morphism of beads is a diagram

$$\begin{tikzcd}
S^n \arrow[rr, "\theta"] \arrow[dr, "a"'] & & S^m \arrow[dl, "b"] \\
& A &
\end{tikzcd}$$

where $\theta : [n] \to [m]$ is a non-decreasing map in Δ. The category of flagged beads adds extra data to each bead.

Definition 3.6. *A **flagged bead** over a simplicial set A of length r is a pair $(a : S^n \to A, \gamma \in \mathbb{C}^n_r)$ consisting of a bead $a : S^n \to A$ and an atomic chain of $\mathbb{C}^n_r$. A morphism $\theta : (a : S^n \to A, \gamma \in \mathbb{C}^n_r) \to (b : S^m \to A, \delta \in \mathbb{C}^m_r)$ is a morphisms of beads such that $\theta : [n] \to [m]$ satisfies $\delta = \theta(\gamma)$.*

This defines a category $\mathcal{B}^r_A$ of flagged beads of length r over A.

As briefly noted above, this is a modification of the definition in [DS11b] before Proposition 4.5. Dugger and Spivak were describing the realization of general simplicial sets as categories enriched in simplicial sets. Their approach use simplicial sets they call necklaces, which are sequences of standard simplicial simplices Δ^n (which they called beads) joined at their initial and final vertices (see §3 [DS11b]). For now we need only the beads of necklaces, and we don't have to be careful about the maps of beads preserving vertices, as there is exactly one vertex.

The role for the categories $\mathcal{B}^r_A$ is to categorify the relations generating the equivalence relation of the coend defining $\mathbb{C}(A)_r$. In particular pairs $(a \in A_n, \gamma \in \mathbb{C}^n_r)$ for γ atomic are exactly objects of $\mathcal{B}^r_A$ and

$$(\theta^*(a) \in A_n, \gamma \in \mathbb{C}^n_r) \sim (a \in A_m, \theta(\gamma) \in \mathbb{C}^m_r)$$

if and only if there exists a morphism $\theta : (\theta^*(a) \in A_n, \gamma \in \mathbb{C}^n_r) \to (a \in A_m, \theta(\gamma) \in \mathbb{C}^m_r)$ in $\mathcal{B}^r_A$. Hence, the classes $[a \in A_n, \gamma \in \mathbb{C}^n_r]$ generating $\mathbb{C}(A)_r$ correspond exactly to the connected components of $\mathcal{B}^r_A$.

We say an atomic chain $\gamma \in \mathbb{C}^n_r$ is **flanked** when the top vertex is $x_{01}x_{12}\cdots x_{n-1n}$. This terminology comes from the concept of flanked necklaces given by [DS11b] before Proposition 4.5. The following shows that we can restrict our attention to pairs $(a \in A_n, \gamma \in \mathbb{C}^n_r)$ where γ is flanked.

Proposition 3.7. *There is a functor $fl : \mathcal{B}^r_A \to \mathcal{B}^r_A$ that sends a flagged bead $(a \in A_n, \gamma \in \mathbb{C}^n_r)$ to a flagged bead with its atomic chain flanked. This functor acts as the identity on the set of connected components.*

Proof. Let $(a \in A_n, \gamma \in \mathbb{C}^n_r)$ be an object of $\mathcal{B}^r_A$. The top vertex of the atomic chain γ is of the form $x_{i_0i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}$ for $0 \leq i_0 < i_1 < i_2 < \cdots < i_m \leq n$. This uniquely determines an embedding $\delta_\gamma : [n'] \hookrightarrow [n]$ sending $j \in [n'] \mapsto i_j \in [n]$ such that there exists a flanked atomic r-chain $\gamma' \in \mathbb{C}^m_r$ with $\delta_\gamma(\gamma') = \gamma$. Hence we have a map in $\mathcal{B}^r_A$

$$\delta_\gamma : (\delta^*_\gamma(a) \in A_{n'}, \gamma' \in \mathbb{C}^{n'}_r) \to (a \in A_n, \gamma \in \mathbb{C}^n_r)$$

Let $fl(a \in A_n, \gamma \in \mathbb{C}^n_r) = \delta^*_\gamma(a) \in A_{n'}, \gamma' \in \mathbb{C}^{n'}_r)$.

We will define how fl acts on maps and show that the maps δ_γ determine a natural transformation $fl \implies \mathrm{Id}_{\mathcal{B}^r_A}$. Let $\theta : (\theta^*(a) \in A_n, \gamma \in \mathbb{C}^n_r) \to (a \in A_m, \theta(\gamma) \in \mathbb{C}^m_r)$ be a morphism in $\mathcal{B}^r_A$. Let $\delta_\gamma : [n'] \to [n]$ be the map determined by γ. Factor the map $\theta \circ \delta_\gamma : [n'] \to [m]$ uniquely as a surjection $\theta' : [n'] \to [m']$ followed by an injection $\delta'_\gamma : [m'] \hookrightarrow [m]$ in Δ. A surjective map preserves the property of a chain being flanked, so $\theta'(\gamma') \in \mathbb{C}^{m'}_r$ is flanked and $\delta'_\gamma(\theta'(\gamma')) = \theta(\gamma)$. Hence $\theta'(\gamma') = \theta(\gamma)'$, the unique flanked chain determining $fl(a \in A_m, \theta(\gamma) \in \mathbb{C}^m_r)$ and $\delta'_\gamma = \delta_{\theta(\gamma)} : [m'] \hookrightarrow [m]$ is the corresponding unique embedding. By unique factorization this definition respects composition and identities in $\mathcal{B}^r_A$ so fl is a functor and $\delta_{(-)} : fl \implies \mathrm{Id}_{\mathcal{B}^r_A}$ is a natural transformation.

The functor fl does nothing to flanked objects and the natural transformation means every connected component contains a flanked object, so fl acts as the identity on connected components.

□

Finally we have the following description of $\mathbb{C}(A)$.

Proposition 3.8. *The monoid $\mathbb{C}(A)_r$ of r-simplices of $\mathbb{C}(A)$ is freely generated by the classes*

$$[a \in A_n, x_{0n} \leq \cdots \leq x_{01}x_{12}\cdots x_{n-1n} \in \mathbb{C}^n_r]$$

where a is a non-degenerate simplex of A and the chain is flanked and atomic. Furthermore, the simplicial monoid $\mathbb{C}(A)$ is cofibrant.

Proof. Since $\mathbb{C}^m_r$ is a free monoid on atomic chains, the monoids $\mathbb{C}^m_r \cdot A_n$ are freely generated for all m, n by pairs $(a \in A_n, \gamma \in \mathbb{C}^m_r)$ where γ is an atomic r-chain of $\mathbb{C}^m$. The monoid of r-simplices of $\mathbb{C}(X)$ is given by the coequalizer of the diagram

$$\begin{tikzcd} \coprod\limits_{\theta:[m]\to[n]} \mathbb{C}^m_r \cdot A_n \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{n} \mathbb{C}^n_r \cdot A_n \end{tikzcd}$$

The maps of this diagram of free monoids sends generators to generators, so its coequalizer is a free monoid generated by the classes $[a \in A_n, \gamma \in \mathbb{C}^n_r]$ with γ atomic under the equivalence relation generated by

$$(\theta^*(a) \in A_m, \gamma \in \mathbb{C}^m_r) \sim (a \in A_n, \theta(\gamma) \in \mathbb{C}^n_r)$$

for all $\theta : [m] \to [n]$ in Δ.

By Proposition 3.7 every generator has a representative that is flanked. It only remains to show that each pair $(a \in A_n, \gamma \in \mathbb{C}^n_r)$ with a non-degenerate and γ flanked and atomic gives a distinct class. Suppose $[x \in X_n, \gamma \in \mathbb{C}^n_t] = [y \in X_m, \delta \in \mathbb{C}^m_t]$ where x and y are nondegenerate and γ and δ are flanked and atomic. By the discussion above, this equality of classes implies there exists a zig-zag of morphisms in $\mathcal{B}^r_A$ connecting the objects $(x \in X_n, \gamma \in \mathbb{C}^n_t)$ and $(y \in X_m, \delta \in \mathbb{C}^m_t)$. Applying the functor fl, we get a zig-zag of surjections between flanked objects connecting these two objects. Given a span

$$\begin{tikzcd} & (b \in A_l, \gamma \in \mathbb{C}^l_r) \arrow[dl, "\theta"'] \arrow[dr, "\varphi"] & \\ (a \in A_n, \theta(\gamma) \in \mathbb{C}^n_r) & & (c \in A_m, \varphi(\gamma) \in \mathbb{C}^m_r) \end{tikzcd}$$

it must be the case that $\theta^*(a) = b = \varphi^*(c)$. Since θ and φ are surjective, if a is non-degenerate then by [Hir03] Lemma 15.8.4 there exists a unique surjection ψ such that $\varphi = \psi \circ \theta$ and $\psi^*(a) = c$. Hence there exists a unique map $\psi : (a \in A_n, \theta(\gamma) \in \mathbb{C}^n_r) \to (c \in A_m, \varphi(\gamma) \in \mathbb{C}^m_r)$. A zig-zag of morphisms between flanked and atomic objects $(x \in X_n, \gamma \in \mathbb{C}^n_t)$ and $(y \in X_m, \delta \in \mathbb{C}^m_t)$ implies they are isomorphic, hence equal as $\mathcal{B}^r_A$ contains no non-identity isomorphisms.

For the final observation, we see that the face and degeneracy maps of $\mathbb{C}(A)$ act on the chain part of a class representative, and the degeneracy of a flanked atomic r-chain in $\mathbb{C}^n$ is a flanked atomic $r + 1$-chain of $\mathbb{C}^n$.

□

We can extend this observation about the structure of $\mathbb{C}(A)$ to the following.

Proposition 3.9. *$\mathbb{C}^\bullet$ is a Reedy cofibrant cosimplicial simplicial monoid. Hence, realization preserves cofibrations in $\boldsymbol{sSet}_0$.*

Proof. We must show that for all n the latching map

$$L^n(\mathbb{C}^\bullet) \to \mathbb{C}^n$$

is a cofibration of simplicial monoids. Using the expressions 3.16 of [RV14] we have

$$L^n(\mathbb{C}^\bullet) = \int^{[m]\in\Delta} \mathbb{C}^m \cdot \partial\Delta^n_m$$

Since $\mathbb{C}^0 = e$, the trivial monoid, and the coface maps $d^0, d^1 : \mathbb{C}^0 \to \mathbb{C}^1$ are equal the latching simplicial monoid is isomorphic to the realization of the 0-reduced simplicial set ∂S^n obtained from $\partial\Delta^n$ by collapsing all 0-simplices to a single 0-simplex. Hence we must show that the map $\mathbb{C}(\partial S^n) \to \mathbb{C}^n$ is a cofibration of simplicial monoids. The monoid $\mathbb{C}(\partial S^n)_r$ is generated by classes $[\delta \in \partial S^n_t, \gamma \in \mathbb{C}^t_r]$ where $\delta : [t] \hookrightarrow [n]$ is an injective map in Δ that factors through a coface map $d^i : [n-1] \hookrightarrow [n]$ for some $0 \leq i \leq n$ and γ is a flanked atomic chain. The above map sends this class to the chain $\delta(\gamma) \in \mathbb{C}^n_r$, which is atomic, but not flanked, as δ is not surjective. For any non-flanked atomic chain $\gamma \in \mathbb{C}^n_r$, the top vertex is of the form $x_{i_0i_1}x_{i_1i_2}\cdots x_{i_{m-1}i_m}$ for $0 \leq i_0 < i_1 < i_2 < \cdots < i_m \leq n$. As in the proof of Proposition 3.8, this uniquely determines an embedding $\delta : [m] \hookrightarrow [n]$ that necessarily factors through some coface map d^i since γ is not flanked. Hence the map $\mathbb{C}(\partial S^n) \to \mathbb{C}^n$ is an injection on generators, so it is a cofibration of cofibrant simplicial monoids. $\square$

To show that the homotopy coherent nerve-realization adjunction is a Quillen adjunction it only remains to show that the realization preserves acyclic cofibrations. Equivalently, we can show that the nerve preserves fibrations, or thanks to Lemma 7.14 of [JT07], fibrations between fibrant objects. Fibrations between fibrant objects in $\mathbf{sSet}_0$ with the reduced Joyal model structure are inner fibrations between quasi-monoids, so the following result is sufficient to prove that the adjunction $\mathbb{C} \dashv \mathbb{N}$ is Quillen.

Lemma 3.10. *$\mathbb{C}$ sends inner horns $\lambda^k_n \hookrightarrow S^n$ of $\boldsymbol{sSet}_0$ to acyclic cofibrations of $\boldsymbol{sMon}$.*

Proof. The monoid $\mathbb{C}(\lambda^k_n)_r$ of r-simplices is the submonoid of $\mathbb{C}(\partial S^n)_r$ generated by classes $[\delta \in \partial S^n_t, \gamma \in \mathbb{C}^t_r]$ where $\delta : [t] \hookrightarrow [n]$ is an injective map in Δ that factors through a coface map $d^i : [n-1] \hookrightarrow [n]$ for some $i \neq k$. The map $\mathbb{C}(\lambda^k_n) \hookrightarrow \mathbb{C}^n$ factors through the cofibration $\mathbb{C}(\partial S^n) \hookrightarrow \mathbb{C}^n$, so it sends this class to the chain $\delta(\gamma) \in \mathbb{C}^n_r$, which is atomic, but not flanked, as δ is not surjective. The atomic chains in the image of this inclusion are the non-flanked chains such that the top vertex either has length strictly less than $n-1$ or else has length $n-1$ and includes x_{ik} or x_{kj} for some $i < k < j$ in the top vertex.

Consider the simplicial n-cube $\square^n$ modelled as the nerve of the poset of subsets of $[n+1]$ that all contain 0 and $n+1$. There is an embedding of simplicial sets $\alpha : \square^n \hookrightarrow \mathbb{C}^{n+1}$ determined by taking the nerve of the map of posets sending

$$A = \{0 < a_1 < a_2 \cdots < a_t < n+1\} \subseteq [n+1] \mapsto x_{0a_1}x_{a_1a_2}\cdots x_{a_tn+1} \in C^{n+1}$$

Following the notation of [RV18], we claim that the pullback of this map along the inclusion $\mathbb{C}(\lambda^k_n) \hookrightarrow \mathbb{C}^n$ is the simplicial subset of $\square^n$ given by

$$\sqcap^{n,k}_t = \{A_\bullet \in \square^n_t \mid \exists\ i \neq k \text{ s.t. } i \notin A_l\ \forall\ l \text{ or } \exists\ 1 \leq i \leq n \text{ s.t. } i \in A_l\ \forall\ l\}$$

Let $\alpha(A_\bullet)$ be a t-chain in the image of $\square^n_t$ in $\mathbb{C}^n_t$. There exists $i \neq k \in [n]$ such that $i \notin A_l$ for all $0 \leq l \leq t$ if and only if there exists an order-preserving map $\theta : [t] \to [n-1]$ in Δ such that $\alpha(A_\bullet) = \theta^*(\gamma)$ for an atomic $n-1$-chain $\gamma \in \mathbb{C}^n_{n-1}$ of the form

$$\gamma = x_{0n} \leq \cdots \leq x_{01}\cdots x_{i-2\,i-1}x_{i-1\,i+1}x_{i+1\,i+2}\cdots x_{n-1\,n}$$

All these chains belong to $\mathbb{C}(\lambda^k_n)$. For the other case, there exists $i \in [n]$ such that $i \in A_l$ for all $0 \leq l \leq r$ if and only if there exists an order-preserving map $\theta : [t] \to [n-1]$ in Δ such that $\alpha(A_\bullet) = \theta^*(\gamma)$ for an atomic $n-1$-chain $\gamma \in \mathbb{C}^n_{n-1}$ of the form

$$\gamma = x_{0i}x_{in} \leq \cdots \leq x_{01}\cdots x_{n-1\,n}$$

These chains are the product of two atomic chains in $\mathbb{C}(\lambda_n^k)$. Finally, all chains in $\mathbb{C}(\lambda_n^k)$ that are of the form required to be in the image of α arise in one of these two ways.

The inclusion $\sqcap^{n,k} \hookrightarrow \square^n$ is a Kan weak equivalence because it is the inclusion of all cube faces except for the k^{th}. The following square is a pushout of simplicial monoids

$$\begin{tikzcd}
F_\times(\sqcap^{n,k}) \arrow[r] \arrow[d, hook] & \mathbb{C}(\lambda_n^k) \arrow[d, hook] \\
F_\times(\square^n) \arrow[r] & \mathbb{C}^n
\end{tikzcd}$$

where $F_\times$ is the free monoid functor on simplicial sets. So the map $\mathbb{C}(\lambda_n^k) \hookrightarrow \mathbb{C}$ is a weak equivalence of simplicial monoids.

□

Corollary 3.11. *$\mathbb{C} \dashv \mathbb{N}$ is a Quillen adjunction.*

A main goal of this chapter is to show that this adjunction is in fact a Quillen equivalence. To do this we will need the following Lemma, which identifies $U\mathbb{C}(X)$, the underlying simplicial set of the realization of X, as having the same homotopy type as the simplicial mapping space $(X^{S^1})_*$ when X is a quasi-monoid. Recall $U : \mathbf{sMon} \to \mathbf{sSet}$ denotes the forgetful functor for simplicial monoids.

Lemma 3.12. *There exists a functor $H : \boldsymbol{sSet}_0 \to \boldsymbol{sSet}$ and a span of natural transformations*

$$\begin{tikzcd}
(-^{S^1})_* & H \arrow[l, "l"'] \arrow[r, "r"] & U\mathbb{C}
\end{tikzcd}$$

such that $l_X : HX \to (X^{S^1})_*$ and $r_X : HX \to U\mathbb{C}(X)$ are Kan weak equivalences of simplicial sets when X is a quasi-monoid. Furthermore, if $\varepsilon_M : \mathbb{C}(\mathbb{N}(M)) \to M$ is the counit of the adjunction $\mathbb{C} \dashv \mathbb{N}$ then the map $U\varepsilon_M \circ r_{\mathbb{N}(M)} : H\mathbb{N}(M) \to UM$ is a Kan weak equivalence of simplicial sets when M is a fibrant simplicial monoid.

Using this Lemma, we finish this section with a proof that this adjunction is a Quillen equivalence. The sequence of results we use are essentially the same as those used in [Lur09] to prove the Quillen equivalence between the Joyal model structure on simplicial sets and the Bergner model structure on simplicial categories. In particular, our Lemma 3.12 is equivalent to the discussion of §2.2 in [Lur09]. The approach to proving this Lemma here, however, differs significantly from the approach of [Lur09]. We have not defined reduced Joyal equivalences as the maps that are sent by $\mathbb{C}$ to weak equivalences of **sMon**, which would be the equivalent approach to the definition of the Joyal model structure used by Lurie in [Lur09] Theorem 2.2.5.1. Once we have proved Lemma 3.12, however, we can show that this is the case.

Corollary 3.13. *A map $f : X \to Y$ between reduced simplicial sets is a reduced Joyal equivalence if and only if $\mathbb{C}(f)$ is a weak equivalence of simplicial monoids.*

Proof. By Corollary 2.31, we can replace f by an inner fibration between quasi-monoids, that is there exists a diagram

$$\begin{tikzcd}
X \arrow[r, hook, "\sim"] \arrow[d, "f"'] & P(R(X)) \arrow[d, "p_{R(f)}"] \\
Y \arrow[r, hook, "\sim"] & R(Y)
\end{tikzcd}$$

where the top and bottom maps are weak equivalences for the quasi-monoid model structure on $\mathbf{sSet}_0$. Hence f is a weak equivalence in the quasi-monoid model structure if and only if $p_{R(f)}$ is, which in turn is true if and only if $(p_{R(f)}{}^{S^1})_*$ is by Lemma 2.33. $\mathbf{sSet}_0$ is a category of cofibrant objects, so $\mathbb{C}$ preserves weak equivalences and $\mathbb{C}(f)$ is a weak equivalence if and only if $\mathbb{C}(p_{R(f)})$ is. By Lemma 3.12 there is a diagram

$$\begin{tikzcd}
(P(R(X))^{S^1})_* \arrow[d, "(p_{R(f)}{}^{S^1})_*"'] & H(P(R(X))) \arrow[l, "\sim", "l_{P(R(X))}"'] \arrow[r, "\sim", "r_{P(R(X))}"'] \arrow[d, "H(p_{R(X)})"] & U\mathbb{C}(P(R(X))) \arrow[d, "U\mathbb{C}(p_{R(X)})"] \\
(R(Y)^{S^1})_* & H(R(Y)) \arrow[l, "\sim", "l_{R(Y)}"'] \arrow[r, "\sim", "r_{R(Y)}"'] & U\mathbb{C}(R(Y))
\end{tikzcd}$$

Since $P(R(X))$ and $R(Y)$ are quasi-monoids all horizontal maps are Kan weak equivalences, so $\mathbb{C}(p_{R(X)})$ is a weak equivalence if and only if $(p_{R(f)}{}^{S^1})_*$ is. ☐

We are finally able to prove that the homotopy coherent nerve-realization adjunction is a Quillen equivalence using the second part of Lemma 3.12.

Proposition 3.14. $\mathbb{C} \dashv \mathbb{N}$ *is a Quillen equivalence.*

Proof. Since the left adjoint $\mathbb{C}$ creates weak equivalences by Corollary 3.13 it is sufficient to show that the counit $\varepsilon_M : \mathbb{C}(\mathbb{N}(M)) \to M$ is a weak equivalence of simplicial monoids when M is a fibrant simplicial monoid. This follows immediately from Lemma 3.12, as we have the diagram of simplicial sets

$$\begin{tikzcd}
H(\mathbb{N}(M)) \arrow[r, "\sim", "r_{\mathbb{N}(M)}"'] \arrow[dr, "\sim"'] & U\mathbb{C}(\mathbb{N}(M)) \arrow[d, "U\varepsilon_M"] \\
 & UM
\end{tikzcd}$$

where the top map is a weak equivalence because $\mathbb{N}(M)$ is a quasi-monoid when M is a fibrant simplicial monoid by Corollary 3.11. ☐

The remainder of this section is devoted to a proof of Lemma 3.12. Our approach will be the following:

1. In Subsection 3.3 we construct a filtration of $U\mathbb{C}(X)$ by simplicial subsets S_n for $n \geq 0$ such that the $n = 1$ stage of the filtration will be the space $H(X)$ and the natural transformation r will be the inclusion $H(X) = S_1 \hookrightarrow U\mathbb{C}(X)$

2. In Subsection 3.4 we define $H(X)$ using results from [Lur09] as a realization of a simplicial set dependant on X by a cosimplicial simplicial set

   $$Q_L^\bullet : \Delta \to \mathbf{sSet}$$

   We also show that there is a natural transformation

   $$(X^{S^1})_* \xleftarrow[l]{\sim} H(X)$$

   that is a Kan weak equivalence of simplicial sets when X is a quasi-monoid

3. In Subsection 3.5 we will show that the definition of $H(X)$ in Subsection 3.4 is isomorphic to the first stage S_1 of the filtration defined in Subsection 3.3. This will be done by generalizing the diagram $Q^{\bullet}_L$ and the simplicial set it realizes to give H by defining categories T^m with $T^0 = *$ and diagrams
$$Q^m : T^m \times \Delta \to \mathbf{sSet}$$
$$F^m : (T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$$
$$E^m : (T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$$
for all $m \geq 0$ such that E^m is a subdiagram of F^m for all $m \geq 0$ and the inclusions $S_m \hookrightarrow S_{m+1}$ for $n \geq 0$ can be realized as pushouts
$$\begin{tikzcd} {|E^m|_{Q^m}} \arrow[r, hook] \arrow[d] & {|F^m|_{Q^m}} \arrow[d] \\ S_m \arrow[r, hook] & S_{m+1} \end{tikzcd}$$
of the coend realization of the inclusion $E^m \hookrightarrow F^m$ of diagrams by Q^m.

4. Finally in Subsection 3.6 we show that $r : H(X) = S_1 \hookrightarrow U\mathbb{C}(X)$ is a Kan weak equivalence when X is a quasi-monoid and complete the final parts of the proof of Lemma 3.12. We will do this by analyzing the homotopy properties of the realization $| - |_{Q^m}$ and the diagrams $E^m \subseteq F^m$ defined in Subsection 3.5.

We will begin this plan by defining the filtration of $U\mathbb{C}(X)$ in the next section.

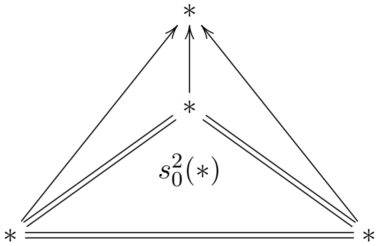
3.3 Spine Length Filtration of $\mathbb{C}(A)$

In this section we will describe a filtration of the underlying simplicial set of $\mathbb{C}(A)$ by simplicial subsets. This is part of our project of showing that $\mathbb{C} \dashv \mathbb{N}$ is a Quillen equivalence by following the approach described at the end of the previous section. In particular, once we have defined the functor H promised in Lemma 3.12 in the next section we will show that the first non-trivial stage of this filtration is equal to $H(A)$ and the inclusion
$$H(A) \hookrightarrow U\mathbb{C}(A)$$
determined by the filtration will be the natural transformation $r : H \to U\mathbb{C}$ described in Lemma 3.12.

This filtration is based on the description of words in the free monoid $\mathbb{C}(A)_r$ of r-simplices of $\mathbb{C}(A)$ and the simplicial structure of the reduced simplicial set A. In particular, it combines word length from the free monoid structure with a measure of how far a non-degenerate simplex of A is from being degenerate. The measure we will use for this property of a simplex will be called the void degree, which we will now define.

Definition 3.15. *Let A be a reduced simplicial set and let $x \in A_t$ be a simplex that is not the basepoint or a degeneracy of the basepoint. The* ***right void degree*** *is*
$$\nu(x) = \min\{m \mid d_{t-m+1} \cdots d_{t-1} d_t(x) = s_0^{t-m}(*)\}$$
We define the void degree of the basepoint and its degeneracies to be $\nu(*) = 0$.

Note that $d_1 \cdots d_t(x) = *$ for all t-simplices of A, so the set in the definition of $\nu(x)$ contains t. Hence $1 \leq \nu(x) \leq t$ for a non-basepoint t-simplex of A. The "void" we are measuring is how much of the simplex is just a degeneracy of the basepoint. Specifically, we are asking how many vertices of a t-simplex we must delete, starting from t at the right, before we encounter this degeneracy. For example, the 3-simplex below has void degree 1 as the d_3 face is $s_0^2(*)$.



The filtration of Definition 3.15 is a right filtration because we have chosen to apply face operators of the simplicial set A starting from the right. This choice is based on Lurie's choice in [Lur09]. The same constructions exist for the left case, but we won't use them, so we will suppress "right" from definitions after the first mention.

We can give an alternative description of the void degree in terms of the decalage functor Dec : **sSet** → **sSet** originally defined by Illusie in [Ill72]. The simplicial set $\mathrm{Dec}(A)$ has t-simplices A_{t+1} with face and degeneracy maps $d_i^{\mathrm{Dec}(A)} = d_i^A$ $s_i^{\mathrm{Dec}(A)} = s_i^A$ for $0 \leq i \leq t$. The remaining face map determines a map of simplicial sets $d_{\bullet+1} : \mathrm{Dec}(A) \to A$. We can repeatedly apply the Dec functor to obtain $\mathrm{Dec}^n(A)$, which has t-simplices A_{t+n} with face and degeneracy maps $d_i^{\mathrm{Dec}^n(A)} = d_i^A$ $s_i^{\mathrm{Dec}^n(A)} = s_i^A$ for $0 \leq i \leq t$. There is also a map $\mathrm{Dec}^n(A) \to A$ that sends $x \in \mathrm{Dec}^n(A)_t = A_{t+n}$ to $d_{t+1} \cdots d_{t+n}(x) \in A_t$. The fiber of this map at the basepoint of A is denoted V^n. It has t-simplices given by

$$V_t^n = \{x \in A_{t+n} \mid d_{t+1} \cdots d_{t+n}(x) = s_0^t(*)\} \tag{35}$$

Hence we can restate the void degree of a t-simplex of A as

$$\nu(x) = \min\{m \mid x \in V_{t-m}^m\} \tag{36}$$

If we define the simplicial set V^0 by $V_t^0 = \{s_0^t(*)\}$ then this extends to include the basepoint. The next proposition shows how void degree interacts with the face and degeneracy maps of A.

Proposition 3.16. *Let $x \in A_t$. Then $\nu(s_i(x)) \geq \nu(x)$ and $\nu(d_i(x)) \leq \nu(x)$.*

Proof. Let $\nu(x) = m$. We will consider two cases. First suppose $0 \leq i \leq t - m$. Then $s_i : V_{t-m}^m \to V_{t-m+1}^m$ and $d_i : V_{t-m}^m \to V_{t-m-1}^m$ are degeneracy and face maps of V^m respectively. So $d_i(x) \in V_{t-1-m}^m$ and $s_i(x) \in V_{t+1-m}^m$. Hence $\nu(d_i(x)) \leq \nu(x)$ and $\nu(s_i(x)) \leq \nu(x)$. This gives what we wanted for the face map, but for the degeneracy we must now show $\nu(s_i(x)) = \nu(x)$.

For $l < m$

$$d_{t+1-l} \cdots d_t d_{t+1}(s_i(x)) = s_i(d_{t-l} \cdots d_{t-1} d_t(x)) \neq s_0^{t-l}(*)$$

as this would contradict minimality of m for x. So $\nu(x) = \nu(s_i(x))$.

Now let $i > t - m$. We have

$$\begin{aligned} d_{t+1-m} \cdots d_t d_{t+1}(s_i(x)) &= d_{t+1-m} \cdots d_i d_{i+1} s_i d_{i+1} \cdots d_{t-1} d_t(x)) \\ &= d_{t+1-m} \cdots d_{t-1} d_t(x) \\ &\neq s_0^{t-m}(*) \end{aligned}$$

by minimality of m for x, so $\nu(s_i(x)) > \nu(x)$. For the faces

$$\begin{aligned} d_{t-m}\cdots d_{t-2}d_{t-1}(d_i(x)) &= d_{t-m}\cdots d_{i-2}d_{i-1}d_id_{i+1}\cdots d_{t-1}d_t(x)) \\ &= s_0^{t-m}(*) \end{aligned}$$

so $d_i(x) \in V_{t-m}^{m-1}$ and hence $\nu(d_i(x)) < \nu(x)$.

□

We can now define the value by which we will be filtering $\mathbb{C}(A)$. Recall from Proposition 3.8 that the monoid $\mathbb{C}(A)_t$ of t-simplices of a realization $\mathbb{C}(A)$ is freely generated by terms

$$[x \in A_n, \gamma \in \mathbb{C}_t^n]$$

where x is a non-degenerate n-simplex of A and γ is a flanked atomic chain.

Definition 3.17. *Let $e \neq \omega \in \mathbb{C}(A)_t$ be a word in the free monoid, written as*

$$\omega = [x_0 \in A_{n_0}, \gamma_0 \in \mathbb{C}_t^{n_0}][x_1 \in A_{n_1}, \gamma_1 \in \mathbb{C}_t^{n_1}]\cdots[x_l \in A_{n_l}, \gamma_l \in \mathbb{C}_t^{n_l}]$$

where all x_i are non-degenerate and all γ_i are flanked and atomic. The ***right spine length*** *(just spine length after this) of the word ω is equal to*

$$l_s(\omega) = \nu(x_0) + \sum_{i=1}^{l} n_i$$

The spine length of the identity $e \in \mathbb{C}(A)_t$ is 0.

We can bound the spine length by two more crude measures of the size of a word. Since $[x_i \in A_{n_i}, \gamma_i \in \mathbb{C}_t^{n_i}] \neq e$, $x_i \neq *$ for all i, so $n_i \geq 1$. Hence $l_s(\omega) \geq l$ and the length of the word ω, in the standard sense of a free monoid, gives a lower bound on the spine length. For a word $\omega = \prod_{i=1}^{l}[x_i \in A_{n_i}, \gamma_i \in \mathbb{C}_t^{n_1}] \in \mathbb{C}(A)_t$, the **necklace length** is

$$l_{\mathrm{Nec}}(\omega) = \sum_{i=0}^{n} n_i$$

Since $\nu(x) \leq t$ for a t-simplex, the necklace length of a word is an upper bound for our definition of spine length $l_s(\omega) \leq l_{\mathrm{Nec}}(\omega)$.

Finally, we will show that the spine length of words determines a filtration of $\mathbb{C}(A)$.

Proposition 3.18. *The sets*

$$(S_n)_t = \{\omega \in \mathbb{C}(A)_t \mid l_s(\omega) \leq n\}$$

define simplicial subsets $S_n \subseteq U\mathbb{C}(A)$ such that

$$U\mathbb{C}(A) = \bigcup_{n\geq 0} S_n$$

Proof. We need to show that if $\omega \in \mathbb{C}(A)_t$ such that $l_s(\omega) \leq n$ then $l_s(d_i(\omega)) \leq n$ and $l_s(s_i(\omega)) \leq n$ for all $0 \leq i \leq t$. Now s_i for all i and d_i for $0 < i < t$ send generating simplices to generating simplices and don't change the simplex representing the class, so they preserve spine length. It remains to consider d_0 and d_t.

Let $\omega, \omega' \in \mathbb{C}(A)_t$. If $\omega = \prod_{0 \leq i \leq n} \alpha_i$ and $\omega' = \prod_{0 \leq j \leq m} \beta_i$ for generators $\alpha_i = [x_i \in A_{n_i}, \gamma_i \in \mathbb{C}_t^{n_i}]$ and $\beta_j = [x'_j \in A_{m_j}, \gamma'_j \in \mathbb{C}_t^{m_j}]$ of $\mathbb{C}(A)_t$. Then

$$l_s(\omega) + l_s(\omega') = \nu(x_0) + \sum_{i=1}^{n} n_i + \nu(x'_0) + \sum_{j=1}^{m} m_j \leq \nu(x_0) + \sum_{i=1}^{n} n_i + m_0 + \sum_{j=1}^{m} m_j = l_s(\omega\omega')$$

by unique factorization. So it is sufficient to show for any generator $\alpha \in \mathbb{C}(A)_t$ that $l_s(d_t(\alpha)) \leq l_s(\alpha)$ and $l_s(d_0(\alpha)) \leq l_s(\alpha)$.

Consider $[x \in A_n, \gamma \in \mathbb{C}_t^n]$ with x non-degenerate and γ flanked and atomic. In particular let

$$\gamma = x_{0n} \leq \cdots \leq x_{0a_1} x_{a_1a_2} \cdots x_{a_{m-2}n} \leq x_{01}x_{12} \cdots x_{n-1n}$$

The inclusion $\{0 < a_1 < \cdots a_{m-2} < n\} \subseteq [n]$ determines a unique embedding $\delta : [m] \hookrightarrow [n]$ that sends 0 to 0 and m to n such that $d_t(\gamma) = \delta(\gamma')$ for a unique $\gamma' \in \mathbb{C}_{t-1}^m$ that is flanked and atomic. It is clear from the definition of $\mathbb{C}(A)$ that

$$d_t([x \in A_n, \gamma \in \mathbb{C}_t^n]) = [\delta^*(x) \in A_m, \gamma' \in \mathbb{C}_{t-1}^m]$$

The simplex $\delta^*(x)$ may be degenerate, so let $\sigma^*(y) = \delta^*(x)$ for some non-degenerate $y \in A_l$, where $\sigma : [m] \to [l]$ is a codegeneracy in Δ. Hence we have

$$d_t([x \in A_n, \gamma \in \mathbb{C}_t^n]) = [y \in A_l, \sigma(\gamma') \in \mathbb{C}_{t-1}^l]$$

where y is non-degenerate and $\sigma(\gamma')$ is flanked and atomic. If the generator is not the first letter in the word ω then since $l \leq n$ its contribution to the spine length is not increased by d_t. If it is the first letter, then

$$\nu(x) \geq \nu(\delta^*(x)) = \nu(\sigma^*(y)) \geq \nu(y)$$

by Proposition 3.16. Hence $l_s(d_t([x \in A_n, \gamma \in \mathbb{C}_t^n])) \leq l_s([x \in A_n, \gamma \in \mathbb{C}_t^n])$.

Now consider $d_0([x \in A_n, \gamma \in \mathbb{C}_t^n])$. This is the class of $(x \in A_n, d_0(\gamma) \in \mathbb{C}_{t-1}^n)$, but $d_0(\gamma)$ is no longer necessarily atomic. If

$$\gamma = x_{0n} \leq x_{0b_1} x_{b_1b_2} \cdots x_{b_mn} \leq \cdots \leq x_{01}x_{12} \cdots x_{n-1\,n}$$

then letting $b_0 = 0$ and $b_{m+1} = n$ we can write $d_0(\gamma) = \gamma_0\gamma_1 \cdots \gamma_r$ where each

$$\gamma_i = x_{b_ib_{i+1}} \leq \cdots \leq x_{b_ib_i+1} \cdots x_{b_{i+1}-1\,b_{i+1}} \in \mathbb{C}_{t-1}^n$$

is atomic. The elements $b_i \in [n]$ determine a partition of $[n]$ so that $[n] = [0, b_1] \vee [b_1, b_2] \vee \cdots \vee [b_m, n]$, where we are using the wedge operation of Definition 3.27. Let $\delta_i : [b_{i+1} - b_i] \hookrightarrow [n]$ be the embedding sending $i \in [b_{i+1} - b_i] \mapsto i + b_i \in [n]$ determined by this partition. Each γ_i is the image of a flanked atomic $t-1$-chain $\gamma'_i \in \mathbb{C}^{n_i}$ under the embedding $\delta_i : [b_{i+1} - b_i] \hookrightarrow [n]$. Let $n_i = b_{i+1} - b_i$, then we can write

$$d_0([x \in A_n, \gamma \in \mathbb{C}_t^n]) = [\delta_0^*(x) \in A_{n_0}, \gamma'_0 \in \mathbb{C}_t^{n_0}][\delta_1^*(x) \in A_{n_1}, \gamma'_1 \in \mathbb{C}_t^{n_1}] \cdots [\delta_r^*(x) \in A_{n_r}, \gamma'_r \in \mathbb{C}_t^{n_r}]$$

Now $\delta_0^*(x) = d_{b_1+1} \cdots d_{n-1} d_n(x)$ so by the definition of void degree $\nu(\delta_0^*(x)) \leq \nu(x) - n + b_1$.

Let surjections $\sigma_i : [n_i] \to [m_i]$ account for any degeneracy of the simplices $\delta_i^*(x)$, so $\delta_i^*(x) = \sigma_i^*(y_i)$ for $y \in A_{m_i}$ nondegenerate. Then

$$d_0([x \in A_n, \gamma \in \mathbb{C}_t^n]) = \prod_{i=0}^{r} [y_i \in A_{m_i}, \sigma_i(\gamma_i') \in \mathbb{C}_t^{m_i}]$$

is the unique factorization in $\mathbb{C}(A)_{t-1}$. It may be the case that $y_0 = *$, in which case $l_s(d_0(x)) = \nu(y_j) + \sum_{i=j+1}^{n} n_i$ where j is the first index such that $m_j \neq 0$. Since $\nu(y_j) \leq m_j$ we have

$$\begin{aligned} l_s(d_0([x \in A_n, \gamma \in \mathbb{C}_t^n]) &\leq \nu(y_0) + \sum_{i=1}^{r} m_i \\ &\leq \nu(\sigma_0^*(y_0)) + \sum_{i=1}^{r} n_i \\ &= \nu(\delta_0^*(x)) + \sum_{i=1}^{r} n_i \\ &\leq \nu(x) - n + b_1 + \sum_{i=1}^{r} n_i \\ &= \nu(x) = l_s([x \in A_n, \gamma \in \mathbb{C}_t^n]) \end{aligned}$$

since $\sum_{1 \leq i \leq n} n_i = n - b_1$. Finally, for any word ω, if n is the necklace length of ω then $l_s(\omega) \leq n$, so $\omega \in S_n$. □

As described above, this filtration will be a key tool in proving Lemma 3.12. In particular, we will take $H(X) = S_1$, the $m = 1$ stage of this filtration. In the next section we relate this simplicial set S_1 to the simplicial mapping space $(X^{S^1})_*$.

3.4 The Functor H

In this section we will define the functor $H : \mathbf{sSet}_0 \to \mathbf{sSet}$ described in Lemma 3.12 and show that there is a natural transformation

$$l : H \to (-^{S^1})_*$$

that is a weak equivalence for quasi-monoids X. This is the functor that will mediate in Lemma 3.12 between the simplicial mapping space $(X^{S^1})_*$ of a quasi-monoid X and the underlying simplicial set $U\mathbb{C}(X)$ of the homotopy coherent realization of X and allow us to prove that these spaces have the same homotopy type. The material in this section comes from §2 of [Lur09].

We recall the definition of the cosimplicial simplicial set $Q^\bullet$ from §2.2.2 of [Lur09], which we will denote by $Q_L^\bullet$ to distinguish it from later definitions. The simplicial sets Q_L^m are quotients of subsets of the simplicial cube $(\Delta^1)^n$ constructed in the following way.

Definition 3.19. *Let $n \geq 0$. The poset $P_{[n]}$ consists of non-empty subsets of the totally ordered set $[n] \in \Delta$ ordered by inclusion.*

As noted in [Lur09] taking the nerve $NP_{[n]}$ gives a simplicial set that is isomorphic to a simplicial subset of the n-cube $(\Delta^1)^n$ via the identification

$$S \subseteq [n] \leftrightarrow \vec{v}_S \in \{0,1\}^{n+1} \text{ where } \vec{v}_S(i) = 1 \text{ if and only if } i \in S$$

Definition 3.20. *Let $m \geq 0$. The simplicial set Q^m_L is obtained as the pushout*

$$\begin{tikzcd}
\coprod\limits_{0\leq i\leq n} (\Delta^1)^{\{j \mid 0\leq j<i\}} \times \{1\} \times (\Delta^1)^{j \mid i<j\leq n} \arrow[r] \arrow[d] & (\Delta^1)^{n+1} \arrow[d] \\
\coprod\limits_{0\leq i\leq n} (\Delta^1)^{\{j \mid 0\leq j<i\}} \arrow[r] & Q^m_L
\end{tikzcd}$$

where the left vertical map is the disjoint union of the natural projections of the product.

By the same analysis of the t-simplices of $Q^{m,n}$ given after Definition 3.28 a t-simplex of Q^n_L has a unique representative such that its 0^{th} subset in the chain is a singleton. We will denote this representative of a simplex $[A_\bullet] \in (Q^n_L)_t$ by $\rho_L(A)_\bullet$. This representative has $\rho(A)_0 = \{a\}$, a singleton such that $a = \min \rho(A)_i$ for all $0 \leq i \leq t$.

Lurie shows in Proposition 2.2.2.9 that this cosimplicial simplicial set induces a nerve-realization adjunction with realization

$$|K|_Q = \int^{[n]\in\Delta} K_n \times Q^n_L$$

Lurie shows that there is a natural equivalence between the realization of a simplicial set by $Q^\bullet_L$ and the original simplicial set.

Proposition 3.21 ([Lur09] Prop. 2.2.2.7)**.** *For all simplicial sets K there is a natural weak equivalence*

$$|K|_Q \xrightarrow{\sim} K$$

The connection of this realization to the spine length filtration of $\mathbb{C}(X)$ comes from applying it to the simplicial set

$$\mathrm{Hom}^R_X(*,*) = \mathrm{Hom}_{\mathbf{sSet}}(\Sigma\Delta^\bullet, X) \tag{37}$$

where $\Sigma\Delta^n$ is the suspension of the standard simplicial n-simplex Δ^n given by the pushout

$$\begin{tikzcd}
\Delta^n \arrow[r, hook, "d^{n+1}"] \arrow[d] & \Delta^{n+1} \arrow[d] \\
* \arrow[r] & \Sigma\Delta^n
\end{tikzcd}$$

The notation $\mathrm{Hom}^R_X(*,*)$ is from [Lur09], where this is the space of right morphisms from $*$ to $*$ viewed as objects of the quasi-monoid X seen as a quasi-category. The adjective right here refers to the choice of d^{n+1} as the face of Δ^{n+1} to be collapsed to form $\Sigma\Delta^n$, rather than d^0, which would yield the space of left morphisms. The definition of $\mathrm{Hom}_S(*,*)$ given in §1.2.2 of [Lur09] corresponds exactly to our definition of $(X^{S^1})_*$ for a reduced simplicial set in Section 2.1. Lurie shows that when X is a quasi-monoid these mapping spaces are Kan weakly equivalent.

Proposition 3.22 ([Lur09] Cor. 4.2.1.8)**.** *Let X be a quasi-monoid. There is a map of simplicial sets*

$$\mathrm{Hom}_{\boldsymbol{sSet}}(\Sigma\Delta^{\bullet}, X) = \mathrm{Hom}^R_X(*, *) \hookrightarrow \mathrm{Hom}_S(*, *) = (X^{S^1})_*$$

that is a weak equivalence of Kan complexes.

Hence we have natural maps in **sSet**

$$|\,\mathrm{Hom}^R_X(*, *)|_Q \xrightarrow[\text{Prop. 3.21}]{\sim} \mathrm{Hom}^R_X(*, *) \xrightarrow[\text{Prop. 3.22}]{\sim} (X^{S^1})_* \tag{38}$$

that are Kan weak equivalences when X is a quasi-monoid. This will form half of the span described in Lemma 3.12. To this end, we define the functor H as follows.

Definition 3.23. *The functor $H : \boldsymbol{sSet}_0 \to \boldsymbol{sSet}$ is defined by*

$$H(A) = |\,\mathrm{Hom}(\Sigma\Delta^{\bullet}, A)|_Q$$

for a reduced simplicial set A.

We can conclude this section, therefore, with the following result, which encompasses part of Lemma 3.12.

Lemma 3.24. *There exists a natural transformation*

$$l : H \to (-^{S^1})_*$$

such that l_X is a Kan weak equivalence when X is a quasi-monoid.

To construct the other half of Lemma 3.12 we need to relate

$$H(X) = |\,\mathrm{Hom}^R_X(*, *))|_Q$$

to the underlying simplicial set $U\mathbb{C}(X)$ of the realization of X. We will do this in the next section by identifying this space with the $m = 1$ stage of the spine length filtration of $U\mathbb{C}(X)$.

3.5 Realizing the Spine Length Filtration of $\mathbb{C}(A)$

In this section we will fix a reduced simplicial set A. We will show that the filtration

$$e = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_m \subseteq \cdots \subseteq U\mathbb{C}(A) \tag{39}$$

of the underlying simplicial set of the simplicial set $\mathbb{C}(A)$ by spine length described in Proposition 3.18 can be described in terms of colimits in **sSet**. Furthermore, we will show that there is a natural isomorphism of simplicial sets

$$H(A) \cong S_1$$

identifying the functor H defined in Definition 3.23 as the first non-trivial stage of the spine length filtration. This will yield the span of natural transformations promised by Lemma 3.12

$$(X^{S^1})_* \xleftarrow{l_X} H(X) = |\ F^0\ |_{Q^0} \cong S_1 \hookrightarrow \mathbb{C}(X)$$

We will also set the stage for the final parts of the proof of Lemma 3.12 where we will show that the inclusion $H(X) = S_1 \hookrightarrow U\mathbb{C}(X)$ is a Kan weak equivalence of simplicial sets when X is a quasi-monoid. To do this and show that $H(A) \cong S_1$ we will generalize the tools from [Lur09] used in defining $H(A)$ in Definition 3.23. In particular, we will define a sequence of categories indexed by the natural numbers

$$* = T^0 \hookrightarrow T^1 \hookrightarrow \cdots \hookrightarrow T^n \hookrightarrow \cdots$$

and diagrams

$$F^m : (T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$$

$$Q^m : T^m \times \Delta \to \mathbf{sSet}$$

as well as subdiagrams $E^m \subseteq F^m$ that together define spaces forming a pushout of simplicial sets for all $m \geq 1$

$$\begin{tikzcd}
{|E^m|_{Q^m}} \arrow[r, hook] \arrow[d] & {|F^m|_{Q^m}} \arrow[d] \\
S_m \arrow[r, hook] & S_{m+1}
\end{tikzcd} \tag{40}$$

where the bottom map is the inclusion $S_m \subseteq S_{m+1}$ of the spine length filtration (39) of $\mathbb{C}(A)$ that was proved to exist in Proposition 3.18. We will refer to this square as the **filtration colimit**. The notation $|F^m|_{Q^m}$ here represents the coend

$$|F^m|_{Q^m} = \int^{(\gamma,[n]) \in T^m \times \Delta} F^m(\gamma)_n \times Q^{(\gamma)n}$$

and similar for the diagram E^m.

These diagrams will be shown to be generalizations of the diagram $Q^{\bullet}_L : \Delta \to \mathbf{sSet}$ and the simplicial set $\mathrm{Hom}^R_A(*,*)$ for a reduced simplicial set A described in the previous section. They will be defined below in Subsections 3.5.1, 3.5.2, and 3.5.3 along with the realization coend $|-|_{Q^m}$. This will give a description of the colimits in the top half of the filtration colimit (40). The final step in Subsection 3.5.4 will be to show that the square in the diagram (40) is a pushout square and identify $H(A) \cong S_1$.

We begin by defining the categories T^m that form part of the domain of these diagrams Q^m, F^m, and E^m. Let Δ^+_{inj} be the subcategory of Δ with all objects $[n]$ for $n \geq 0$ and morphisms the injective order-preserving maps that preserve the top element. So $\gamma : [a] \hookrightarrow [r]$ is a morphism if and only if $\gamma(a) = r$. For $m \geq 0$ let $(\Delta^+_{\mathrm{inj}})^{\leq m}$ be the full subcategory of Δ^+_{inj} with the objects $[i]$ for $i \leq m$.

Definition 3.25. *The category $T^m = \mathrm{tw}((\Delta^+_{\mathrm{inj}})^{\leq m})$ is the twisted arrow category on $(\Delta^+_{\mathrm{inj}})^{\leq m}$*

Objects of T^m are injective maps $\alpha : [a] \hookrightarrow [r]$ such that $\alpha(a) = r$ and $r \leq m$. Since α preserves the top element we have a partition of $[r]$ into intervals

$$[r] = [0, \alpha(0)] \vee [\alpha(0), \alpha(1)] \vee \cdots \vee [\alpha(a-1), \alpha(a)] \tag{41}$$

Note that there is a unique morphism $[0] \hookrightarrow [r]$ in Δ^+_{inj} for all $r \geq 0$. We will denote these objects of T^m by $\varepsilon_r : [0] \hookrightarrow [r]$ for $0 \leq r \leq m$.

A morphism $(f, v) : \alpha \to \beta$ of T^m is a diagram

$$\begin{tikzcd}
{[r]} \arrow[r, hook, "f"] & {[s]} \\
{[a]} \arrow[u, hook, "\alpha"] & {[b]} \arrow[u, hook, "\beta"'] \arrow[l, hook, "v"]
\end{tikzcd}$$

The maps α and β determine partitions of $[r]$ and $[s]$ respectively into intervals. The maps f and v that make the diagram commute determine how these partitions are related, which is the data we will need when constructing diagrams F^m and Q^m shaped by T^m. We will extract from this diagram (f, v) the data that we will need for these diagrams. First we define a map $\kappa_v : [a] \to [b]$ in Δ that is a section of the injective order-preserving map v. This map has

$$\kappa_v(i) = \min \{j \in [b] \mid v(j) \geq i\}$$

for all $i \in [a]$. Note that $v(b) = a \geq i$ for all $i \in [a]$ so the set is non-empty and the minimum is defined. Restricting f to the intervals of the partition (41) of $[r]$ determined by α gives injective maps which we denote as $j_{(f,v)i}$ for $0 \leq i \leq a$

$$\begin{array}{rl}
j_{(f,v)0} : [0, \alpha(0)] \hookrightarrow [0, \beta(0)] & \\
j_{(f,v)i} : [\alpha(i-1), \alpha(i)] \hookrightarrow [0, \beta(0)] & 0 < i \leq v(0) \\
j_{(f,v)i} : [\alpha(i-1), \alpha(i)] \hookrightarrow [\beta(\kappa_v(i) - 1), \beta(\kappa_v(i))] & v(0) < i \leq a
\end{array}$$

Note that $f = \bigvee_{i=0}^{a} j_{(f,v)i}$. We will conjugate these maps by isomorphisms $\xi : [b-a] \cong [a, b]$ defined by $i \in [b-a] \mapsto i + b \in [a, b] \subseteq [n]$. These isomorphisms identify intervals with objects of Δ so that we can define for a morphism (f, v) of T^m and for all $0 \leq i < a$ the maps of Δ

$$(f, v)_i = \begin{cases} j_{(f,v)0} : [\alpha(0)] \hookrightarrow [\beta(0)] & i = 0 \\ j_{(f,v)i} \circ \xi : [\alpha(i) - \alpha(i-1)] \hookrightarrow [\beta(0)] & 0 < i \leq v(0) \\ \xi^{-1} \circ j_{(f,v)i} \circ \xi : [\alpha(i) - \alpha(i-1)] \hookrightarrow [\beta(\kappa_v(i)) - \beta(\kappa_v(i) - 1)] & v(0) < i \leq a \end{cases} \tag{42}$$

In particular,

$$\begin{array}{rl}
(f, v)_0(n) = f(n) & \\
(f, v)_i(n) = f(n + \alpha(i-1)) & 0 < i \leq v(0) \\
(f, v)_i(n) = f(n + \alpha(i-1)) - \beta(\kappa_v(i) - 1) & v(0) < i \leq a
\end{array}$$

3.5.1 Q^m and the Realization $|-|_{Q^m}$

We now define the diagrams $Q^m : T^m \times \Delta \to \mathbf{sSet}$ mentioned above. These will be viewed as cosimplicial objects in the category $\mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ so that they give realization functors

$$|-|_{Q^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$$

via the usual nerve-realization construction. We will analyze this functor $|-|_{Q^m}$ and its similarities to the realization $\mathbb{C}(A)$ at the end of this section, with a view to understanding the filtration colimit (40).

The spaces appearing in these diagrams Q^m are obtained by taking nerves of subposets of $P_{[n]}$, the poset of non-empty subsets of the totally ordered set $[n] \in \Delta$. We will now define the subposets of $P_{[n]}$ that we need for our definition of Q^m.

Definition 3.26. *For all $n \geq 1$ and $0 \leq m < n$ define the poset*

$$P_{[n]}^{[m]} = \{A \subseteq [n] \mid A \cap [m] \neq \emptyset \text{ and } n \in A\}$$

For all $l \in [n]$, define

$$W_{[n]}^{l} = \{A \subseteq [n] \mid l \in A \text{ and } n \in A\} \subseteq P_{[n]}^{[l]}$$

$$L_{[n]}^{l} = \{A \subseteq [n] \mid \min A = l \text{ and } n \in A\} \subseteq W_{[n]}^{l}$$

Note that $W_{[n]}^{0} = L_{[n]}^{0}$ necessarily for any n. This notation will hopefully clarify the analysis of posets and maps between their nerves involved in this section. We are identifying, in order of definition, the posets: of subsets of $[n]$ containing n and at least one other element less than or equal to $m < n$; of subsets of $[n]$ all containing a specific chosen $l \in [n]$ as well as n; and of subsets of $[n]$ all containing a specific chosen $l \in [n]$ as their least element as well as n.

The letter W is chosen in the second definition above because of the connection with the wedge of two subsets of $[n]$, which we now define.

Definition 3.27. *For $A = \{a_0 < a_1 < \cdots < a_m\}, B = \{b_0 < b_1 < \cdots < b_l\} \subseteq [n]$ such that $\max A = a_m = b_0 = \min B$ we define the* ***wedge*** *of A and B as*

$$A \vee B = \{a_0 < a_1 < \cdots a_{m-1} < b_0 < b_1 < \cdots < b_l\}$$

We say that $a_m = b_0$ is the **wedge point** for this wedge.

We extend this wedge operation to the simplicial set $NP_{[n]}$. For t-simplices $A_\bullet, B_\bullet \in (NP_{[n]})_t$ such that $\max A_i = \max A_j = \min B_i = \min B_j$ for all $1 \leq i, j \leq t$ we can define the wedge $A_\bullet \vee B_\bullet$ by taking the degreewise wedges

$$(A_\bullet \vee B_\bullet)_i = A_i \vee B_i$$

for all $1 \leq i \leq t$. In particular, for such simplices if $l = \max A_i$ then $A_\bullet \vee B_\bullet \in NW_{[n]}^{l}$. Furthermore, any simplex in $NW_{[n]}^{l}$ can be decomposed uniquely with wedge point at l. So $NW_{[n]}^{l}$ contains simplices that are assembled as wedges at the wedge point l.

Definition 3.28. *Let $m, n \geq 0$. $Q^{m,n}$ is the pushout*

$$\begin{tikzcd}
\coprod_{0 \leq i \leq n} NW_{n+m+1}^{i} \arrow[r] \arrow[d] & NP_{[n+m+1]}^{[n]} \arrow[d] \\
\coprod_{0 \leq i \leq n} NL_{n+m+1}^{i} \arrow[r] & Q^{m,n}
\end{tikzcd}$$

A t-simplex of $Q^{m,n}$ is a class of chains of inclusions $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_t$ of subsets of $[n+m+1]$ under the relation

$$A_\bullet \sim V_\bullet \vee A_\bullet$$

where $V_\bullet \vee A_\bullet$ has wedge point $i \leq n$. Each class $[A_\bullet]$ in $Q_t^{m,n}$ has a unique representative, which we denote $\rho(A)_\bullet \in NL^a_{n+m+1}$, such that $\rho(A)_0 \cap [n] = \{a\}$ with $a = \min \rho(A)_i$ for all i. For a class $[A_\bullet]$ the unique representative $\rho(A)_\bullet$ is constructed by taking $\rho(A)_i = \{j \in A_i \mid j \geq \max A_0 \cap [n]\}$ for $0 \leq i \leq t$. Finally we can define the diagrams $Q^m : T^m \times \Delta \to \mathbf{sSet}$. We start with the spaces $Q^{(\alpha)n}$ for $[n] \in \Delta$ and α an object of T^m.

Definition 3.29. *For $n \geq 0$ and $\alpha : [a] \hookrightarrow [r] \in T^m$ define the simplicial set $Q^{(\alpha)n}$ as*

$$Q^{(\alpha)n} = Q^{\alpha(0),n} \times \prod_{i=1}^{a} NI_i^\alpha$$

where $I_i^\alpha = \{A \subseteq [\alpha(i) - \alpha(i-1)] \mid 0 \in A$ and $\alpha(i) - \alpha(i-1) \in A\}$.

A t-simplex $\vec{A} \in Q_t^{(\alpha)n}$ consists of simplices $[A_\bullet] \in Q_t^{\alpha(0),n}$ and $A^i_\bullet \in (NI_i^\alpha)_t$ for $1 \leq i \leq a$.

We will extend this definition to a diagram $Q^m : T^m \times \Delta \to \mathbf{sSet}$ by defining maps $Q^m((f,v),\theta) : Q^{(\alpha)n} \to Q^{(\beta)l}$ of simplicial sets for morphisms $(f,v) : \alpha \to \beta$ of T^m and $\theta : [l] \to [n]$ of Δ. We will define this map by how it acts on components of the product defining $Q^{(\alpha)n}$ in Definition 3.29. Recall the maps $(f,v)_i$ for $0 \leq i < a$ defined at (42). For all $1 \leq j \leq b$ we define maps of posets

$$\begin{array}{ccc} \displaystyle\prod_{i=v(j-1)+1}^{v(j)} I_i^\alpha & \to & I_j^\beta \\ (A^i)_{v(j-1)+1 \leq i \leq v(j)} & \mapsto & \displaystyle\bigvee_{i=v(j-1)+1}^{v(j)} (f,v)_i(A^i) \end{array}$$

Taking the nerve preserves the product and defines a part of the map $Q^m((f,v),\theta)$ that we want.

To define the component mapping to $Q^{\beta(0),l}$ we will need the ordinal sum, which is the monoidal product in the category Δ. This bifunctor $- \oplus - : \Delta \times \Delta \to \Delta$ has $[n] \oplus [m] = [n+m+1]$. On morphisms $f : [m] \to [n]$, $g : [l] \to [p]$ the map $f \oplus g : [m+l+1] \to [n+p+1]$ acts on $0 \leq i \leq m$ by f and on $m+1 \leq i \leq m+l+1$ by $g(i-m)+n$.

We will use the universal property of the pushout in Definition 3.28 of $Q^{\beta(0),l}$ to construct this component. Start by defining the map of posets

$$\begin{array}{ccc} P^{[n]}_{[n+\alpha(0)+1]} \times \displaystyle\prod_{i=1}^{v(0)} I_i^\alpha & \to & P^{[l]}_{[l+\beta(0)+1]} \\ (A, (A^i)_{1 \leq i \leq v(0)}) & \mapsto & \theta \oplus (f,v)_0(A) \vee (d^0)^{m+1} \left(\displaystyle\bigvee_{i=1}^{v(0)} (f,v)_i(A^i) \right) \end{array}$$

where we have maps $\theta \oplus (f,v)_0 : [n+\alpha(0)+1] \to [l+\beta(0)+1]$ and for $1 \leq i \leq v(0)$

$$(d^0)^{m+1} \circ (f,v)_i : [\alpha(i) - \alpha(i-1)] \hookrightarrow [\beta(0)] \hookrightarrow [l+\beta(0)+1]$$

Taking the nerve of this map and the restriction to $NL^i_{[n+\alpha(0)+1]}$ for the other leg of the pushout determines a map $Q^{\alpha(0),n} \times \prod_{i=0}^{v(0)} NI_i^\alpha \to Q^{\beta(0),l}$. Together with the first map, these definitions give a map $Q^m((f,v),\theta)$ that respects composition and identities, so we have defined a functor.

Definition 3.30. *The assignment*

$$(\alpha : [a] \hookrightarrow [r], [n]) \in T^m \times \Delta \mapsto Q^{(\alpha)n}$$

defines a functor $Q^m : T^m \times \Delta \to$ ***sSet****.*

The notation Q has been chosen because we will show that Q^m is a generalization of the cosimplicial simplicial set $Q_L^\bullet$ defined in [Lur09] and used to define of the functor H in Definition 3.23. In particular, we will show in Proposition 3.57 that $Q^0 \cong Q_L^\bullet$.

As with the cosimplicial simplicial set $Q_L^\bullet$ of [Lur09] the diagrams Q^m define nerve-realization adjunctions

$$\begin{tikzcd}\mathbf{sSet}^{(T^m)^{\mathrm{op}}} \arrow[r, bend left, "|-|_{Q^m}"] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend left, "\mathrm{Sing}_{Q^m}(-)"]\end{tikzcd} \tag{43}$$

where for $D \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ and $K \in \mathbf{sSet}$

$$|D|_{Q^m} = \int^{(\alpha,[n]) \in T^m \times \Delta} D(\alpha)_n \times Q^{(\alpha)n}$$

$$\mathrm{Sing}_{Q^m}(K)(\alpha)_t = \mathbf{sSet}(Q^{(\alpha)t}, K)$$

By the standard coend definition, a t-simplex of $|D|_{Q^m}$ is an equivalence class of a pair

$$(x \in D(\alpha)_n, \vec{A} \in Q_t^{(\alpha)n})$$

under the equivalence relation generated by

$$(x \in D(\alpha)_n, Q^{(f,v)\theta}(\vec{B}) \in Q_t^{(\alpha)n}) \sim (D((f,v),\theta^*)(x) \in D(\beta)_l, \vec{B} \in Q_t^{(\beta)l})$$

for maps $(f,v) : \beta \to \alpha$ in T^m and $\theta : [l] \to [n]$ in Δ. Inspired by the corresponding notion of flanked chains for $\mathbb{C}^n$, we make the following definition.

Definition 3.31. *A simplex* $\vec{A} = ([A_\bullet], (A^i_\bullet)_{1 \le i \le a}) \in Q_t^{(\alpha)n}$ *is* ***flanked*** *when*

$$\rho(A)_0 = \{0 < n + \alpha(0) + 1\}$$

$$\rho(A)_t = [n + \alpha(0) + 1]$$

and for all $1 \le i \le a$

$$A^i_0 = \{0 < \alpha(i) - \alpha(i-1)\}$$
$$A^i_t = [\alpha(i) - \alpha(i-1)]$$

The simplices of $Q^{(\alpha)n}$ strongly resemble chains of $\mathbb{C}^{\alpha(i)-\alpha(i-1)}$, a comparison we will formalize soon, so recognizing the flanked simplices among them makes sense. Furthermore, these simplices play an important role in realizations $|F|_{Q^m}$.

Proposition 3.32. *Every class in* $(|F|_{Q^m})_t$ *has a representative* $(x \in F(\alpha)_n, \vec{A} \in Q_t^{(\alpha)n})$ *with* x *a non-degenerate simplex of* F *and* $\vec{A}$ *flanked.*

Proof. For an object $\alpha : [a] \hookrightarrow [r]$ of T^m and $n, t \geq 0$ we define the subset $\Phi_t^{(\alpha)n} \subseteq Q_t^{(\alpha)n}$ of flanked t-simplices of $Q^{(\alpha)n}$. The core of this proof is the following claim, whose proof we defer to Corollary 3.59 in the next section. We claim that for all $t \geq 0$ the map

$$\bigsqcup_{\substack{(f,v):\beta\to\alpha \\ \delta:[l]\hookrightarrow[n]}} \Phi_t^{(\beta)l} \to Q_t^{(\alpha)n} \tag{44}$$

that sends $(\vec{A} \in \Phi^{(\beta)l_t}, (f,v) : \beta \to \alpha, \delta : [l] \hookrightarrow [n])$ to $Q^m((f,v),\delta)(\vec{A})$ is a bijection.

Let $[y \in F(\alpha)_n, \vec{B} \in Q_t^{(\alpha)n}]$ be a class in $(|F|_{Q^m})_t$ with $\alpha : [a] \hookrightarrow [r]$ an object of T^m. By the previous claim there exists a unique pair of maps $((f,v) : \beta \to \alpha, \delta : [l] \hookrightarrow [n])$ and a flanked simplex $\vec{A} \in Q_t^{(\beta)l}$ such that $Q^m((f,v),\delta)(\vec{A}) = \vec{B}$. Hence

$$[y \in F(\alpha)_n, \vec{B} \in Q_t^{(\alpha)n}] = [F((f,v),\delta)(y) \in F(\beta)_l, \vec{A} \in Q_t^{(\beta)l}]$$

If $F((f,v),\delta)(y)$ is a degenerate l-simplex of $F(\beta)$ let $\sigma : [l] \to [p]$ be a map in Δ and $z \in F(\beta)_p$ a non-degenerate simplex such that $\sigma^*(z) = F((f,v),\delta)(y)$. Then

$$[F((f,v),\delta)(y) \in F(\beta)_l, \vec{A} \in Q_t^{(\beta)l}] = [z \in F(\beta)_p, Q^m(\sigma)(\vec{A}) \in Q_t^{(\beta)p}]$$

The map $Q^m(\sigma)$ only affects $[A] \in Q^{\alpha(0),l}$ in $\vec{A}$. Now σ is a surjection, so $\rho(\sigma(A))_0 = \{0 < p + \alpha(0) + 1\}$ and $\rho(\sigma(A))_t = [p + \alpha(0) + 1]$. Hence $Q^m(\sigma)(\vec{A})$ is flanked, so we are done. $\square$

We now turn our attention to defining the diagrams $F \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ whose realization by the functor $|-|_{Q^m}$ we will take in the filtration colimit (40).

3.5.2 Diagrams $F^m : (T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$

We now turn to diagrams $F^m \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$. We fix A, a reduced simplicial set, for this section. All diagrams will be constructed using A, so we will suppress mention of it where we can. These diagrams will be based on necklaces, which are sequences of simplices of varying degrees from A. The formalism of necklaces was developed in [DS11b] as a way to capture how in the realization $\mathbb{C}(A)$ a word in the monoid of r-simplices can be formed from products of terms $[x \in A_n, \gamma \in \mathbb{C}_r^n]$ for different valued of n. We will combine this feature of necklaces with the approximation of the chain structure of $\mathbb{C}^n$ given by the realization by $|-|_{Q^m}$ defined in the previous section to construct the pushout (40).

We recall the definition of necklaces used in [DS11b] §3. A **necklace** is a simplicial set of the form

$$\Delta^{n_1} \vee \Delta^{n_2} \vee \cdots \vee \Delta^{n_l}$$

where the last vertex of each simplex Δ^{n_i} is identified with the first vertex of the simplex $\Delta^{n_{i+1}}$. We call the standard simplicial simplices Δ^{n_i} the **beads** of the necklace. A necklace is in **preferred form** when $n_i > 0$ for all $1 \leq i \leq l$ unless $l = 1$. Preferred form omits all trivial beads Δ^0 from necklaces apart from the trivial necklace Δ^0.

We can view a necklace as a simplicial subset of a standard simplicial simplex with the same vertices as the necklace

$$\Delta^{n_1} \vee \Delta^{n_2} \vee \cdots \vee \Delta^{n_l} \subseteq \Delta^{\Sigma_{i=1}^{l} n_i}$$

We call the simplex with the same vertices that contains the necklace the **enveloping simplex** of the necklace. Under this identification, we can specify a necklace by listing the subset $J \subseteq [\Sigma_{i=1}^{l} n_i]$ of the vertices of the standard simplex that are **joints**, that is points where beads are joined.

The data of a necklace $L \subseteq \Delta^n$, therefore, can be specified by its subset of joints $J \subseteq [n]$. An object $\alpha : [a] \hookrightarrow [r]$ of T^m determines a necklace, which we denote $L_\alpha \subseteq \Delta^r$. This necklace has the form

$$L_\alpha = \Delta^{\alpha(0)} \vee \Delta^{\alpha(1)-\alpha(0)} \vee \cdots \vee \Delta^{\alpha(a)-\alpha(a-1)} \subseteq \Delta^r$$

Recall the join operation for simplicial sets (see for example [Rie14] §17.1). This defines a bifunctor $- \star - : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$ whose most important feature in our context is the identification $\Delta^n \star \Delta^m \cong \Delta^{n+m+1}$. This construction is derived from the ordinal sum monoidal product on Δ, so for maps $f : [n] \to [m]$, $g : [l] \to [p]$ of Δ, $f_* \star g_* = (f \oplus g)_* : \Delta^{n+l+1} \to \Delta^{m+p+1}$. For a necklace L_α we will denote the join of Δ^n to the first bead by $\Delta^n * L_\alpha$, that is

$$\Delta^n * L_\alpha = \Delta^{n+\alpha(0)+1} \vee \Delta^{\alpha(1)-\alpha(0)} \vee \cdots \vee \Delta^{\alpha(a)-\alpha(a-1)} \subseteq \Delta^{r+n+1}$$

Note that $\alpha(0) + n + 1 \geq 1$ and α is injective so $\alpha(i) - \alpha(i-1) \geq 1$, so this necklace is in preferred form. This assignment $([n], \alpha) \mapsto \Delta^n * L_\alpha$ extends to a functor $P^m : \Delta \times T^m \to \mathbf{sSet}$. Let $\theta : [n] \to [m]$ be a map in Δ and let $(f, v) : \alpha \to \beta$ be an arrow in T^m. We will show that these maps determine a map of simplicial sets $\Delta^n * L_\alpha \to \Delta^m * L_\beta$. Recall the maps $(f, v)_i$ of Δ defined at (42) above. These determine simplicial set maps

$$\begin{aligned}
&((f,v)_0)_* : \Delta^{\alpha(0)} \hookrightarrow \Delta^{\beta(0)} && \\
&((f,v)_i)_* : \Delta^{\alpha(i)-\alpha(i-1)} \hookrightarrow \Delta^{\beta(0)} && 0 < i \leq v(0) \\
&((f,v)_i)_* : \Delta^{\alpha(i)-\alpha(i-1)} \hookrightarrow \Delta^{\beta(\kappa_v(i))-\beta(\kappa_v(i)-1)} && v(0) < i \leq a
\end{aligned}$$

Hence we have maps of necklaces

$$\theta_* \star ((f,v)_0)_* : \Delta^n \star \Delta^{\alpha(0)} \to \Delta^m \star \Delta^{\beta(0)}$$

$$(d^0)^{m+1} \circ ((f,v)_i)_* : \Delta^{\alpha(i)-\alpha(i-1)} \hookrightarrow \Delta^{\beta(0)} \hookrightarrow \Delta^m \star \Delta^{\beta(0)}$$

for $0 < i \leq v(0)$ and

$$\bigvee_{i=v(j-1)+1}^{v(j)} ((f,v)_i)_* : \bigvee_{i=v(j-1)+1}^{v(j)} \Delta^{\alpha(i+1)-\alpha(i)} \hookrightarrow \Delta^{\beta(j+1)-\beta(j)}$$

for all $0 < j \leq b$. Joining these maps at endpoints gives a map $\Delta^n * L_\alpha \to \Delta^m * L_\beta$. This construction respects identities and composition so we have defined a functor.

Definition 3.33. *$P : \Delta \times T^m \to$ **sSet** is the diagram of simplicial sets with $P(\alpha)^n = \Delta^n * L_\alpha$ for $[n] \in \Delta$ and $\alpha : [a] \hookrightarrow [r]$ in T^m.*

These necklaces are the shapes we use to define diagrams $(T^m)^{\mathrm{op}} \to \mathbf{sSet}$.

Definition 3.34. *Let A be a reduced simplicial set and $m \geq 0$. Define the functor $D^m : (T^m)^{\mathrm{op}} \to$ **sSet** by*

$$D^m(\alpha)_n = \mathrm{Hom}_{\boldsymbol{sSet}}(P(\alpha)^n, A)$$

*Define a subfunctor $F^m : (T^m)^{\mathrm{op}} \to$ **sSet** by taking the pullback*

$$\begin{tikzcd}
F^m(\alpha) \arrow[r, hook] \arrow[d] & D^m(\alpha) \arrow[d] \\
* \arrow[r] & A
\end{tikzcd}$$

where the map $D^m(\alpha)_n \to A_n$ is determined by the map $\Delta^n \cong \Delta^n \star \emptyset \hookrightarrow \Delta^n * L_\alpha = P(\alpha)^n$.

From the definition of P it is clear that for $\alpha : [a] \hookrightarrow [r]$

$$D^m(\alpha) = \mathrm{Dec}^{\alpha(0)+1}(A) \times \prod_{i=1}^{a} A_{\alpha(i)-\alpha(i-1)}$$

where Dec is Illusie's decalage functor defined earlier. The simplicial $\mathrm{Dec}^{\alpha(0)+1}(A)$ set has t-simplices $\mathrm{Dec}^{\alpha(0)+1}(A)_t = A_{t+\alpha(0)+1}$ with face and degeneracy maps

$$d_i^{\mathrm{Dec}^n(A)} = d_i^A \qquad s_i^{\mathrm{Dec}^n(A)} = s_i^A$$

for $0 \leq i \leq t$. An n-simplex $\vec{x} : \Delta^n * L_\alpha \to A$ for $\alpha : [a] \hookrightarrow [r]$ therefore corresponds to the data

$$\vec{x} = (x, (x_i)_{1 \leq i \leq a})$$

where $x \in \mathrm{Dec}^{\alpha(0)+1}(A)_n$ and $x_i \in A_{\alpha(i)-\alpha(i-1)}$ for $1 \leq i \leq a$.

For maps $(f, v) : (\alpha : [a] \hookrightarrow [r]) \to (\beta : [b] \hookrightarrow [s])$ of T^m and $\theta : [n] \to [l]$ of Δ the map $D^m((f, v), \theta) : D^m(\beta)_l \to D^m(\alpha)_n$ sends an l-simplex

$$\vec{y} = (y \in \mathrm{Dec}^{\beta(0)+1}(A)_l, (y_i \in A_{\beta(i)-\beta(i-1)})_{1 \leq i \leq b}) \in D^m(\beta)_l$$

to the n-simplex of $D^m(\alpha)$ determined by

$$(\theta \oplus (f, v)_0)^*(y) \in A_{n+\alpha(0)+1} = \mathrm{Dec}^{\alpha(0)+1}(A)_n$$

$$(f, v)_i^*(d_0^{l+1}(y)) \in A_{\alpha(i)-\alpha(i-1)}$$

for $0 < i \leq v(0)$ and

$$(f, v)_i^*(y_{\kappa_v(i)}) \in A_{\alpha(i)-\alpha(i-1)}$$

for $v(0) < i \leq a$, where the maps

$$\begin{array}{rl}
(\theta \oplus (f, v)_0)^* : A_{l+\beta(0)+1} \to A_{n+\alpha(0)+1} & \\
(f, v)_i^* \circ (d_0)^{l+1} : A_{l+\beta(0)+1} \to A_{\beta(0)} \to A_{\alpha(i)-\alpha(i-1)} & 0 < i \leq v(0) \\
(f, v)_i^* : A_{\beta(\kappa_v(i))-\beta(\kappa_v(i)-1)} \to A_{\alpha(i)-\alpha(i-1)} & v(0) < i \leq a
\end{array}$$

are determined by the maps (42) of Δ arising from (f, v).

The pullback defining F^m selects simplices of $D^m(\alpha)$ where the initial segment of the first bead is a degeneracy of the basepoint, so

$$F^m(\alpha) = V^{\alpha(0)+1} \times \prod_{i=1}^{a} A_{\alpha(i)-\alpha(i-1)}$$

where the space $V^{\alpha(0)+1}$ is the fiber of the map $\mathrm{Dec}^{\alpha(0)+1}(A) \to A$ at the basepoint of A as described in (35) with t-simplices given by

$$V_t^{\alpha(0)+1} = \{x \in A_{t+\alpha(0)+1} \mid d_{t+1} \cdots d_{t+\alpha(0)+1}(x) = s_0^t(*)\}$$

For $(f, v) : \alpha \to \beta$ in T^m the maps $F^m((f, v)) : F^m(\beta) \to F^m(\alpha)$ are determined by the pullbacks

$$\begin{tikzcd}
F^m(\beta) \arrow[r, hook] \arrow[d, "F^m((f,v))"'] & D^m(\beta) \arrow[d, "D^m((f,v))"] \\
F^m(\alpha) \arrow[r, hook] & D^m(\alpha)
\end{tikzcd}$$

The face and degeneracy maps of $D^m(\alpha)$ can only act non-trivially on the $\mathrm{Dec}^{\alpha(0)+1}(A)$ component; they act as the identity on the other parts of $D^m(\alpha)$. An n-simplex of $D^m(\alpha)$ is a map $\vec{x} : \Delta^n * L_\alpha \to A$. For $0 \leq i \leq n-1$ the degeneracy maps $s^i : \Delta^n \to \Delta^{n-1}$ determine maps $s^i * 1_{L_\alpha} : \Delta^n * L_\alpha \to \Delta^{n-1} * L_\alpha$ so that $\vec{x}$ is degenerate if and only if it factors through $s^i * 1_{L_\alpha}$ for some $0 \leq i \leq n$ and similarly for the face maps.

As in the case of $Q_L^\bullet$, which we will show is isomorphic to Q^0 in Proposition 3.57, we will now relate the diagram

$$F^0 : T^0 \times \Delta^{\mathrm{op}} \cong \Delta^{\mathrm{op}} \to \mathbf{Set}$$

to a simplicial set used in defining H in Definition 3.23.

Proposition 3.35. *Let A be a reduced simplicial set. The diagram $F^0 : \Delta^{\mathrm{op}} \to$ **Set** is determined by the simplicial set*

$$F^0(\varepsilon_0) = V^1 = \mathrm{Hom}_{\boldsymbol{sSet}}(\Sigma\Delta^\bullet, A) = \mathrm{Hom}_A^R(*, *)$$

Proof. Since T^0 is the terminal category by Definition 3.25 the diagram F^0 is determined by the simplicial set $F^0(\varepsilon_0)$, where $\varepsilon_0 : [0] \to [0]$ is the unique object of T^0. By Definition 3.34 when $m = 0$ we have

$$F^0(\varepsilon_0)_t = V_t^1 = \{x \in A_{t+1} \mid d_{t+1}x = s_0(*)\} = \mathrm{Hom}_{\mathbf{sSet}}(\Sigma\Delta^t, X)$$

where $\Sigma\Delta^t$ is the suspension of the standard simplicial t-simplex obtained as the pushout

$$\begin{tikzcd}
\Delta^t \arrow[r, hook, "d^{t+1}"] \arrow[d] & \Delta^{t+1} \arrow[d] \\
* \arrow[r] & \Sigma\Delta^t
\end{tikzcd}$$

Thus we can identify $F^0 : T^0 \times \Delta^{\mathrm{op}} \to \mathbf{Set}$ with the simplicial set

$$\mathrm{Hom}_{\mathbf{sSet}}(\Sigma\Delta^\bullet, A) = \mathrm{Hom}_A^R(*, *)$$

of (37). □

3.5.3 Subdiagrams $E^m \subseteq F^m : (T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$

To complete the definition of all spaces appearing in the pushout square (40) we must define a subdiagram E^m of F^m. The subdiagram will be determined by extending the spine length filtration defined at the start of this section to the spaces of the diagram F^m and then restricting the spaces in the image of the diagram to have only simplices of spine length at most m.

Recall the spine length of a word in $\mathbb{C}(A)$ as defined in Definition 3.17. This definition was based on simplices of A occurring as representatives of letters in the word, so we can extend this definition to simplices of F^m. We will denote this new spine length by l_s^F to avoid confusion with the other spine length define in Definition 3.17. In Lemma 3.49 below we will show that these concepts are very closely linked.

Definition 3.36. *Let $\vec{x} = (x, (x_i)_{1 \le i \le a}) \in F^m(\alpha)_t$, where $x \in V_t^{\alpha(0)+1}$ and $x_i \in X_{\alpha(i)-\alpha(i-1)}$ for $1 \le i \le a$. Let $x = \sigma^*(z)$ and $x_i = \sigma_i^*(z_i)$ for $1 \le i \le a$ where $\sigma_i : [n_i] \to [\alpha(i) - \alpha(i-1)]$ and $\sigma : [n] \to [t + \alpha(0) + 1]$ are the unique degeneracy maps of Δ and $z \in X_n$ and $z_i \in X_{n_i}$ are the unique non-degenerate simplices of X determined by x and x_i. The **spine length** of $\vec{x}$ is*

$$l_s^F(\vec{x}) = \nu(z) + \sum_{i=1}^{a} n_i$$

We are using the void degree $\nu(z)$ of z as defined in Definition 3.15. By construction $x \in V_t^{\alpha(0)+1}$. So by Proposition 3.16 $\nu(z) \le \nu(\sigma^*(z)) = \nu(x) \le \alpha(0) + 1$. The last inequality can be easily seen using the alternative form of ν given in (36). Hence $\nu(\vec{x}) \le 1 + r$ for $\alpha : [a] \hookrightarrow [r]$. Therefore, for all $\vec{x}$ in the diagram F^m we have $l_s^F(\vec{x}) \le m + 1$.

We want to define a subdiagram E^m whose spaces only contain simplices with spine length at most m. The previous discussion shows that for any $\alpha : [a] \hookrightarrow [r]$ with $r < m$ when we define E^m we will have $E^m(\alpha) = F^m(\alpha)$. So we must consider simplices $\vec{x} \in F^m(\alpha)_t$ where $\alpha : [a] \hookrightarrow [m]$ and how they can have spine length strictly less than $m + 1$.

Let $\vec{x} \in F^m(\alpha)_t$ for $\alpha : [a] \hookrightarrow [m]$ and suppose that $l_s^F(\vec{x}) < m + 1$. If all $x_i \in X_{\alpha(i)-\alpha(i-1)}$ are non-degenerate as simplices of X then the deficit of the spine length must come from the first simplex $x \in V_t^{\alpha(0)+1}$. By the definition of spine length for $\vec{x}$ it must be the case that there exists a surjective map $\sigma : [t + \alpha(0) + 1] \to [t']$ and some $z \in X_{t'}$ such that $\nu(z) < \alpha(0) + 1$ and $\sigma^*(z) = x$. Hence $z \in V_{t'-\alpha(0)}^{\alpha(0)}$. By the simplicial structure of $V^{\alpha(0)}$ if $\sigma = \sigma' \oplus [\alpha(0) - 1]$ for some surjective map $\sigma' : [n + 1] \to [t' - \alpha(0)]$ then in fact $x = (\sigma')^*(z) \in V_{t+1}^{\alpha(0)}$, so $\nu(x) \le \alpha(0)$. Otherwise, there exists $t + 1 \le l \le t + \alpha(0)$ such that $\sigma = \bar{\sigma} \circ s^l$. Hence $x = s_l(\bar{\sigma}^*(z))$. In this case, however, our assumption that $\nu(z) < \alpha(0) + 1$, or equivalently $z \in V_{t'-\alpha(0)}^{\alpha(0)}$, was redundant, as the following result with $n = \alpha(0) + 1$ shows.

Lemma 3.37. *Let $x \in V_t^n$ such that $x = s_l(z)$ for $t \le l \le t + n - 1$ and $z \in X_{n+t-1}$. Then $z \in V_t^{n-1}$.*

Proof. Since $x \in V_t^n$ we have

$$\begin{aligned} s_0^t(*) &= d_{t+1}d_{t+2}\cdots d_{t+n}(x) \\ &= d_{t+1}d_{t+2}\cdots d_{t+n}(s_l(z)) \\ &= d_{t+1}\cdots d_l d_{l+1} s_l(d_{l+1}\cdots d_{t+n-1}(z)) \\ &= d_{t+1}\cdots d_{t+n-1}(z) \end{aligned}$$

So $z \in V_t^{n-1}$.

$\square$

From this analysis it is clear that for $\alpha : [a] \hookrightarrow [m]$ a simplex $\vec{x} \in F^m(\alpha)_t$ has $l_s^F(\vec{x}) < m+1$ if and only if one of the following holds:

1a. There exists $1 \leq i \leq a$ such that $x_i = s_l(z_i)$ for some $z_i \in X_{\alpha(i)-\alpha(i-1)-1}$

1b. $x = s_l(z)$ for $t+1 \leq l \leq t+\alpha(0)$ and some $z \in X_{t+\alpha(0)}$

2. $\nu(x) < \alpha(0)+1$

We will combine the first two cases by introducing some notation for maps between necklaces. In making this definition we are using Lemma 3.5 (7) of [DS11b] which implies that the image of a map $L \to \Delta^n$ from a necklace to a standard simplicial simplex is a necklace.

Definition 3.38. *Let $L \subseteq \Delta^r$ be a necklace with its enveloping simplex. For $0 \leq i \leq r-1$ define a necklace S^iL and a surjective map $S^i_L : L \to S^iL$ as the map to the image in the image factorization*

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "S^i_L"'] & \Delta^r \arrow[d, "s^i"] \\ S^iL \arrow[r, hook] & \Delta^{r-1} \end{tikzcd}$$

The map $L \hookrightarrow \Delta^r$ is an isomorphism on vertices, so $S^iL \hookrightarrow \Delta^{r-1}$ is as well and this is the inclusion of S^iL into its enveloping simplex. When the necklace $L = L_\alpha$ for some $\alpha : [a] \hookrightarrow [r]$ in T^m we will write S^i_α as shorthand for $S^i_{L_\alpha}$.

The connection to spine length becomes clear when we describe the map S^i_L and the necklace S^iL more explicitly. Suppose

$$L = \Delta^{n_0} \vee \Delta^{n_1} \vee \cdots \vee \Delta^{n_l}$$

with $r = \sum_{j=0}^{l} n_j$. Then for $0 \leq i \leq r-1$ let $0 \leq j_i \leq l$ be the index such that the vertices i and $i+1$ of Δ^r, the enveloping simplex of L, occur as vertices of $\Delta^{n_{j_i}}$, a bead of L. Let $r_i = \sum_{j=0}^{j_i-1} n_j$, then $0 \leq i - r_i \leq n_{j_i} - 1$, where we take $r_i = 0$ if $j_i = 0$. The map s^i acts as the identity on beads Δ^{n_j} for $j \neq j_i$ and acts as $s^{i-r_i} : \Delta^{n_{j_i}} \to \Delta^{n_{j_i}-1}$ on the bead $\Delta^{n_{j_i}}$. Hence

$$S^iL = \Delta^{n_1} \vee \Delta^{n_2} \vee \cdots \vee \Delta^{n_{j_i}-1} \vee \cdots \vee \Delta^{n_l}$$

and $S^i_L = 1_{\Delta^{n_1}} \vee \cdots \vee s^{i-r_i} \vee \cdots \vee 1_{\Delta^{n_l}}$.

Consider $\vec{x} \in D^m(\alpha)_n$. This corresponds to a map of simplicial sets $\vec{x} : \Delta^n * L_\alpha \to X$. Note that by the definition of L_α, $n_0 = \alpha(0)$ and $n_i = \alpha(i) - \alpha(i-1)$ for $1 \leq i \leq a$, so $r_i = \alpha(j_i - 1)$ for $j_i > 0$ and $r_i = 0$ otherwise. There exists $1 \leq i \leq a$ such that $x_i = s_l(z_i)$ for some $z_i \in X_{\alpha(i)-\alpha(i-1)-1}$ if and only if $\vec{x} : \Delta^n * L_\alpha \to X$ factors through $\Delta^n * S^k_\alpha : \Delta^t * L_\alpha \to \Delta^t * S^kL_\alpha$ for $k = l + \alpha(j_i - 1)$. This value k varies from $\alpha(0)$ to $m-1$. Similarly $x = s_l(z)$ for $t+1 \leq l \leq t+\alpha(0)$ and some $z \in X_{t+\alpha(0)}$ if and only if $\vec{x} : \Delta^t * L_\alpha \to X$ factors through $\Delta^t * S^k_\alpha : \Delta^t * L_\alpha \to \Delta^n * S^kL_\alpha$ for $k = l - t - 1$. This value of k varies between 0 and $\alpha(0) - 1$.

Using this notation, we can compile these results as follows.

Corollary 3.39. *Let $\vec{x} \in F^m(\alpha)_n$ for $\alpha : [a] \hookrightarrow [m]$. Then $l_s^F(\vec{x}) < m+1$ if and only if one of the following is true:*

1. *There exists* $0 \leq k < m$ *such that* $\vec{x} : \Delta^n * L_\alpha \to X$ *factors through* $\Delta^n * S^k_\alpha : \Delta^n * L_\alpha \to \Delta^n * S^k L_\alpha$
2. $\nu(x) < \alpha(0) + 1$

To define a subdiagram of F^m consisting of simplices with spine length at most m we must show that maps in $\Delta^{\text{op}} \times (T^m)^{\text{op}}$ do not send such a simplex to a simplex with spine length equal to $m + 1$.

Proposition 3.40. *Let* $\vec{x} \in F^m(\beta)_n$ *for* $\beta : [b] \hookrightarrow [m]$*. Let* $\theta : [l] \to [n]$ *be a map in* Δ *and* $(1_{[m]}, v) : \beta \to \alpha$ *be a morphism of* $(T^m)^{\text{op}}$ *with* $\alpha : [a] \hookrightarrow [m]$*. If* $l^F_s(\vec{x}) \leq m$ *then* $l^F_s(F^m((1_{[m]}, v), \theta^*)(\vec{x})) \leq m$

Proof. We will show that if $\vec{x}$ satisfies one of the conditions of Corollary 3.39 then so does $F^m((1_{[m]}, v), \theta^*)(\vec{x})$. We will deal with each case in turn.

Case 1: Let $\vec{x} : \Delta^n * L_\beta \to X$ factor through $\Delta^n * S^k_\alpha : \Delta^n * L_\beta \to \Delta^n * S^k L_\beta$ for some $0 \leq k < m$. Uniqueness of image factorization gives the dashed arrow making the following diagram commute.

$$\begin{tikzcd}
\Delta^m \arrow[rr, "s^i", -{Triangle[open]}] & & \Delta^{m-1} & \\
L_\alpha \arrow[u, hook] \arrow[rr, "S^i_\alpha", -{Triangle[open]}] \arrow[dr, hook, "L_{(1_{[m]},v))}"'] & & S^i L_\alpha \arrow[u, hook] \arrow[dr, hook, dashed] & \\
& L_\beta \arrow[uul, hook] \arrow[rr, "S^i_\beta"', -{Triangle[open]}] & & S^i T_\beta \arrow[uul, hook]
\end{tikzcd}$$

So if $\vec{x} : \Delta^n * L_\beta \to X$ factors through $\Delta^n * S^i_\beta$ then $F^m((1_{[m]}, v), \theta)(\vec{x}) = \vec{x} \circ \left(\theta_* * L_{(1_{[m]},v)}\right)$ factors through $\Delta^l * S^i_\alpha$.

Case 2: If $\nu(x) < \beta(0) + 1$ then $x \in V^{\beta(0)}_{n+1}$, or equivalently $d_{n+2} \cdots d_{n+\beta(0)+1}(x) = s_0^{t+1}(*)$. Note that

$$(1_{[m]}, v)_0 = d^{\beta(0)} \circ \cdots \circ d^{\alpha(0)+2} \circ d^{\alpha(0)+1} : [\alpha(0)] \hookrightarrow [\beta(0)]$$

so

$$d_{n+2} \cdots d_{n+\alpha(0)+1}((\Delta^n \star (f, v)^*_0)(x)) = d_{n+2} \cdots d_{n+\alpha(0)+1} d_{n+\alpha(0)+2} \cdots d_{n+\beta(0)+1}(x) = s_0^{n+1}(*)$$

hence $(\Delta^n \star (f, v)^*_0)(x) \in V^{\alpha(0)}_{n+1}$. Now $V^{\alpha(0)}$ is a simplicial set and $\theta^* \star \Delta^{\alpha(0)} = (\theta^* \star \Delta^0) \star \Delta^{\alpha(0)-1}$ so $\theta^* \star \Delta^{\alpha(0)}((\Delta^n \star (f, v)^*_0)(x)) \in V^{\alpha(0)}_{l+1}$. So $F^m((f, v), \theta^*)(\vec{x})$ satisfies case 2 of Corollary 3.39 and $l^F_s(F^m((f, v), \theta^*)(\vec{x})) \leq m$. □

As a corollary of this result we can make the following definition.

Definition 3.41. *Let* $E^m \subseteq F^m$ *be the subdiagram of* F^m *defined by*

$$E^m(\alpha)_n = \{\vec{x} \in F^m(\alpha)_t \mid l^F_s(\vec{x}) \leq m\}$$

3.5.4 The Filtration Colimit

We have nearly achieved our goal for this section. The realization by $|-|_{Q^m}$ of the natural transformation $E^m \hookrightarrow F^m$ determines a map between spaces

$$|E^m|_{Q^m} \to |F^m|_{Q^m}$$

that forms the top map of (40). We just have to fill in the remaining details from the square and show that it is a pushout. We will start by constructing simplicial set maps for $m \geq 0$

$$g^m : |F^m|_{Q^m} \to U\mathbb{C}(X)$$

whose image is the simplicial subset $S_{m+1} \subseteq \mathbb{C}(X)$.

We will define g^m by constructing simplicial set maps $F^m \to \mathrm{Sing}_{Q^m}(U\mathbb{C}(X))$ in $\mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ that correspond to g^m under the adjunction (43). To do this, we will need to define some simplices of $\mathrm{Sing}_{Q^m}(\mathbb{C}(X))(\alpha)$ for $\alpha : [a] \hookrightarrow [r]$ an object T^m. These simplices are maps of simplicial sets

$$Q^{(\alpha)n} \to U\mathbb{C}(X)$$

Recall Definition 3.29 of the spaces $Q^{(\alpha)n}$. Since $\mathbb{C}(X)$ is a simplicial monoid, we can define maps

$$Q^{(\alpha)n} = Q^{\alpha(0),n} \times \prod_{i=1}^{a} NI_i^{\alpha} \to U\mathbb{C}(X)$$

by defining simplicial set maps

$$Q^{\alpha(0),n} \to U\mathbb{C}(X)$$

and

$$NI_i^{\alpha} \to U\mathbb{C}(X)$$

for $1 \leq i \leq a$ and then composing their product with the unique map $\prod_{[a]} \mathbb{C}(X) \to \mathbb{C}(X)$ determined by the associative multiplication of $\mathbb{C}(X)$.

To define these maps we will need to relate the simplices of $Q^{(\alpha)n}$ to the chains of $\mathbb{C}^n$. Our main tool is the following map. Recall from Definition 3.26 that the t-simplices of $NL^l_{[n]}$ are the inclusions of subsets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_t$ of $[n]$ such that $l = \min A_i$ and $n \in A_i$ for all $0 \leq i \leq t$.

Definition 3.42. *Let $0 \leq l \leq n$, define a simplicial embedding $\gamma : NL^l_{[n]} \hookrightarrow U\mathbb{C}^n$ by taking the nerve of the poset map*

$$A = \{l < a_1 < \cdots < a_t < n\} \in L^l_{[n]} \mapsto \gamma_A = x_{la_1}x_{a_1a_2}\cdots x_{a_t\,n} \in C^n$$

This map is an embedding because we are just changing the notation; a simplex of $NL^m_{[n]}$ is uniquely determined by the same data the specifies a chain in the image of γ in $\mathbb{C}^n$. We will now define the first map $Q^{\alpha(0),n} \to \mathbb{C}(X)$.

Lemma 3.43. *Let $x \in \mathrm{Dec}^{m+1}(X)_n$ with $m \geq 0$. There exists a simplicial set map $\varphi_x : Q^{m,n} \to U\mathbb{C}(X)$ such that*

$$\varphi_x : [A_\bullet] \in Q^{m,n}_t \mapsto [x \in X_{n+m+1}, \gamma_{\rho(A)} \in \mathbb{C}^{n+m+1}_t] \in \mathbb{C}(X)_t$$

where $\{a\} = \rho(A)_0 \cap [n]$.

Proof. We will construct the map $\varphi_x : Q^{m,n} \to U\mathbb{C}(X)$ using the pushout of Definition 3.28 for $Q^{m,n}$. For $A_\bullet \in (NR^{[n]}_{[n+m]})_t$ let $a = \max A_0 \cap [n]$. So $A_\bullet \in NW^a_{[n+m+1]}$ and $A_\bullet = V_\bullet \vee \rho(A)_\bullet$ where $\rho(A)_\bullet \in NL^a_{[n+m+1]}$ with $\rho(A)_0 \cap [n] = \{a\}$. A map $\varphi'_x : NR^{[n]}_{[n+m+1]} \to \mathbb{C}(X)$ given by

$$A_\bullet \in (NR^{[n]}_{[n+m]})_t \mapsto [x \in X_{n+m+1}, \gamma_{\rho(A)} \in \mathbb{C}^{n+m+1}_t] \in \mathbb{C}(X)$$

is clearly what is needed to give the desired map by adjointness. Restricting this to a map $NL^i_{[n+m+1]} \to \mathbb{C}(X)$ defines the map making the pushout diagram in Definition 3.28 commute. It only remains to show that this map is a map of simplicial sets, that is $\varphi'_x(d_i(A_\bullet)) = d_i(\varphi'_x(A_\bullet))$ and $\varphi'_x(s_i(A_\bullet)) = s_i(\varphi'_x(A_\bullet))$ for all $0 \leq i \leq t$. This is clear for all degeneracies s_i and the faces d_i for $i \neq 0$ since these preserve the set A_0. In particular for $i \neq 0$ $\alpha(d_i(A_\bullet)) = d_i(\alpha(A))_\bullet$ and similarly for all s_i.

To complete the definition of φ'_x it only remains to show that $d_0(\varphi'_x(A_\bullet)) = \varphi'_x(d_0(A_\bullet)$. We have

$$\begin{aligned} d_0(\varphi'_x(A_\bullet)) &= [x \in X_{n+m+1}, d_0(\gamma_{\rho(A)}) \in \mathbb{C}^{n+m+1}_{t-1}] \\ \varphi'_x(d_0(A_\bullet)) &= [x \in X_{n+m+1}, \gamma_{\rho(d_0(A))} \in \mathbb{C}^{n+m+1}_{t-1}] \end{aligned}$$

Now $\gamma : NL^a_{[n+m+1]} \hookrightarrow \mathbb{C}^{n+m+1}$ is a simplicial map so

$$d_0(\gamma_{\rho(A)}) = \gamma_{d_0(\rho(A))}$$

Let $b = \max A_1 \cap [n]$, then $d_0(\rho(A)_\bullet) \in NW^b_{[n+m+1]}$. We can write

$$d_0(\rho(A)_\bullet) = d^{n+m+1} \cdots d^{b+1}(W_\bullet) \vee \rho(d_0(\rho(A)))_\bullet$$

where $\rho(d_0(\rho(A)))_\bullet \in (NL^b_{[n+m+1]})_{t-1}$ and the embedding

$$d^{n+m+1} \circ \cdots \circ d^{b+1} : [b] \hookrightarrow [n+m+1]$$

takes $W_\bullet \in (NL^a_{[b]})_{t-1}$ into $NW^b_{[n+m+1]}$. Then

$$\begin{aligned} \gamma_{\rho(d_0(A))} &= \gamma_{d^{n+m+1} \cdots d^{b+1}(W) \vee \rho(d_0(\rho(A)))} \\ &= d^{n+m+1} \cdots d^{n+1}(\gamma_W)\gamma_{\rho(d_0(\rho(A)))} \end{aligned}$$

Since $x \in \mathrm{Dec}^{m+1}(X)_n$

$$d_{n+1} \cdots d_{n+m+1}(x) = (s_0)^n(*)$$

so since $b \leq n$

$$\begin{aligned} d_0(\varphi'_x(A_\bullet)) &= [x \in X_{n+m+1}, \gamma_{\rho(d_0(A))} \in \mathbb{C}^{n+m+1}_{t-1}] \\ &= [x \in X_{n+m+1}, d^{n+m+1} \cdots d^{n+1}(\gamma_W)\gamma_{\rho(d_0(\rho(A)))} \in \mathbb{C}^{n+m+1}_{t-1}] \\ &= [d_{b+1} \cdots d_{n+m+1}(x) \in X_n, \gamma_W \in \mathbb{C}^b_{t-1}][x \in X_{n+m+1}, \gamma_{\rho(d_0(\rho(A)))} \in \mathbb{C}^{n+m+1}_{t-1}] \\ &= [(s_0)^b(*) \in X_b, \gamma_W \in \mathbb{C}^b_{t-1}][x \in X_{n+m+1}, \gamma_{\rho(d_0(\rho(A)))} \in \mathbb{C}^{n+m+1}_{t-1}] \\ &= [x \in X_{n+m+1}, \gamma_{\rho(d_0(\rho(A)))} \in \mathbb{C}^{n+m+1}_{t-1}] \end{aligned}$$

but the construction of ρ gives $\rho(d_0(A))_\bullet = \rho(d_0(\rho(A)))_\bullet$, so we are done. $\square$

The maps $NI_i^\alpha \to U\mathbb{C}(X)$ are much easier to construct. The notation I_i^α was chosen in Definition 3.29 to compactly denote that these simplicial sets involve intervals determined by $\alpha : [a] \hookrightarrow [r]$ and $i \in [a]$. Note, however, that

$$I_i^\alpha = \{A \subseteq [\alpha(i) - \alpha(i-1)] \mid 0, \gamma(i) - \gamma(i-1) \in A\} = L^0_{\alpha(i)-\alpha(i-1)}$$

so there are maps $\gamma : NI_i^\alpha \hookrightarrow U\mathbb{C}^{\alpha(i)-\alpha(i-1)}$ arising from Definition 3.42. The following definition is immediate from the simplicial structures of NI_i^α and $U\mathbb{C}^{\alpha(i)-\alpha(i-1)}$.

Definition 3.44. *Let $x \in X_{\alpha(i)-\alpha(i-1)}$. There is a map $\tau_x : NI_i^\alpha \to U\mathbb{C}(X)$ given by*

$$\tau_x(A_\bullet) = \left[x \in X_{\alpha(i)-\alpha(i-1)}, \gamma_A \in \mathbb{C}_t^{\alpha(i)-\alpha(i-1)}\right]$$

Combining these simplicial set maps we have the following.

Definition 3.45. *Let $\vec{x} = (x, (x_i)_{1\leq i\leq a}) \in F^m(\alpha)_n$ for $\alpha : [a] \hookrightarrow [r]$ an object of T^m. Define a simplicial set map*

$$\varphi_\alpha(\vec{x}) = \mu \circ \left(\varphi_x \times \prod_{i=1}^{a} \tau_{x_i}\right) : Q^{\alpha(0),n} \times \prod_{i=1}^{a} NI_i^\alpha = Q^{(\alpha)n} \to \mathbb{C}(X)$$

where $\mu : \prod_{[a]} \mathbb{C}(X) \to \mathbb{C}(X)$ is the unique map arising from the associative multiplication of $\mathbb{C}(X)$ and φ_x and τ_{x_i} are the simplicial set maps defined in Lemma 3.43 and Definition 3.44 respectively.

For $\vec{A} = ([A_\bullet], (A_\bullet^i)_{1\leq i\leq a}) \in Q^{(\alpha)n}$ where $[A_\bullet] \in Q^{\alpha(0),n}$ and $A_\bullet^i \in NI_i^\alpha$ for $1 \leq i \leq a$ we have

$$\varphi_\alpha(\vec{x})(\vec{A}) = \left[x \in X_{n+\alpha(0)+1}, \gamma_{\rho(A)} \in \mathbb{C}_t^{n+\alpha(0)+1}\right] \prod_{i=1}^{a} \left[x_i \in X_{\alpha(i)-\alpha(i-1)}, \gamma_{A^i} \in \mathbb{C}_t^{\alpha(i)-\alpha(i-1)}\right] \tag{45}$$

Proposition 3.46. *The assignment $\vec{x} \mapsto \varphi_\alpha(\vec{x})$ defines a map $\varphi : F^m \to \mathrm{Sing}_{Q^m}(\mathbb{C}(X))$ in $\boldsymbol{sSet}^{(T^m)^{\mathrm{op}}}$.*

Proof. We must show that for all maps $(f, v) : \alpha \to \beta$ in T^m and $\theta : [n] \to [l]$ in Δ

$$\varphi_\alpha(F^m((f, v), \theta)(\vec{x}))(\vec{A}) = \varphi_\beta(\vec{x}) \circ Q^m((f, v), \theta)(\vec{A}) \tag{46}$$

for all $\vec{x} \in F^m(\beta)_l$ and all $\vec{A} = (A, (A^i)_{1\leq i\leq a}) \in Q_t^{(\alpha)n}$. Now by the definition of γ we have

$$\gamma_{\theta\oplus(f,v)_0(A)\vee(d^0)^{m+1}\left(\bigvee_{i=1}^{v(0)}(f,v)_i(A^i)\right)} = (\theta \oplus (f, v)_0)_*(\gamma_{\rho(A)}) \prod_{i=1}^{v(0)} (d^0)^{l+1}(((f, v)_i)_*(\gamma_{A^i})$$

$$\gamma_{\left(\bigvee_{i=v(j-1)+1}^{v(j)}(f,v)_i(A^i)\right)} = \prod_{i=v(j-1)+1}^{v(j)} (f, v)_{i*}(\gamma_{A^i})$$

for all $1 \leq j \leq b$. By the definitions of F^m and Q^m and the equivalence relation defining classes of $\mathbb{C}(X)$, therefore, the identity (46) holds. $\square$

We define the map $g^m : |F^m|_{Q^m} \to \mathbb{C}(X)$ to be the adjoint of the map defined in Proposition 3.46. This map sends a t-simplex $[\vec{x} \in F^m(\alpha)_n, \vec{A} \in Q_t^{(\alpha)n}]$ of $|F^m|_{Q^m}$ to the word $\varphi_\alpha(\vec{x})(\vec{A}) \in \mathbb{C}(X)_t$, which was given in (45) above. Recall from Proposition 3.32 that every class in $(|F^m|_{Q^m})_t$ has a representative $(\vec{x} \in F^m(\alpha)_n, \vec{A} \in Q_t^{(\alpha)n})$ where $\vec{x}$ is non-degenerate and $\vec{A}$ is flanked. Earlier, when we defined flanked simplices of $Q^{(\alpha)n}$ in Definition 3.31, we promised to connect this to the notion of flanked chains in $\mathbb{C}^n$. We now make this connection.

Proposition 3.47. *A simplex $\vec{A} \in Q_t^{(\alpha)n}$ is flanked if and only if $\gamma_{\rho(A)} \in \mathbb{C}^{n+\alpha(0)+1}$ and $\gamma_{A^i} \in \mathbb{C}^{\alpha(i)-\alpha(i-1)}$ are flanked chains for all $1 \leq i \leq a$.*

This is obvious from the two notions of being flanked and from the construction of γ. The difference between classes in $(|F^m|_{Q^m})_t$ and classes of generators in $\mathbb{C}(X)_t$ comes down to the fact that not all degeneracies of X are accessible to the coend defining $|F^m|_{Q^m}$. To account for this we make the following definitions for simplices of these spaces.

Definition 3.48. *Let $\vec{x} = (x, (x_i)_{1 \leq i \leq a}) \in F^m(\alpha)_n$. We say $\vec{x}$ is **internally non-degenerate** when it is a non-degenerate n-simplex of $F^m(\alpha)$. We say a simplex $\vec{x} \in F^m(\alpha)_n$ is **totally non-degenerate** when it is internally non-degenerate and $x \in X_{n+\alpha(0)+1}$ and $x_i \in X_{\alpha(i)-\alpha(i-1)}$ for all $1 \leq i \leq a$ are non-degenerate simplices of X.*

The description of these simplices as totally non-degenerate is taken from the same notion for necklaces in [DS11b] §4.

Recall the spine length l_s^F defined for $\vec{x}$ in Definition 3.36 and the spine length for words of $\mathbb{C}(X)_t$ defined at Definition 3.17. We will now relate these with the map g^m.

Lemma 3.49. *Let $\vec{x} \in F^m(\alpha)_n$ and $\vec{A} \in Q_t^{(\alpha)n}$. Then $l_s^F(\vec{x}) = l_s(\varphi_\alpha(\vec{x})(\vec{A}))$ if $\vec{A}$ is flanked.*

Proof. Recall from Proposition 3.5 that $\mathbb{C}(X)_t$ is freely generated as a monoid by the classes $[x \in X_n, \gamma \in \mathbb{C}_t^n]$ such that x is a non-degenerate simplex of X and γ is flanked. The definition (45) of $\varphi_\alpha(\vec{x})(\vec{A})$ is a product of classes in $\mathbb{C}(X)_t$ whose representatives come from the data of $\vec{x}$ and $\vec{A}$. As noted above in Proposition 3.47, if $\vec{A}$ is flanked then all the chains in the class representatives of the product $\varphi_\alpha(\vec{x})(\vec{A})$ are flanked. If $\vec{x}$ is not necessarily totally non-degenerate the classes

$$\left[x \in X_{n+\alpha(0)+1}, \gamma_{\rho(A)} \in \mathbb{C}_t^{n+\alpha(0)+1}\right]$$

and

$$\left[x_i \in X_{\alpha(i)-\alpha(i-1)}, \gamma_{A^i} \in \mathbb{C}_t^{\alpha(i)-\alpha(i-1)}\right]$$

comprising $\varphi_\alpha(\vec{x})(\vec{A})$ may not be generators of $\mathbb{C}(X)_t$, as x and x_i could be degenerate as simplices of X. In general let $x = \sigma^*(z)$ and $x_i = \sigma_i^*(z_i)$ for non-degenerate simplices $z \in X_{n'}$ and $z_i \in X_{n_i}$ and surjective maps σ and σ_i for $1 \leq i \leq a$. Surjective maps acting on chains preserve the property of being flanked, so the unique factorization of $\varphi_\alpha(\vec{x})(\vec{A})$ in $\mathbb{C}(X)$ is

$$\varphi_\alpha(\vec{x})(\vec{A}) = \left[z \in X_{n'}, \sigma(\gamma_{\rho(A)}) \in \mathbb{C}_t^{n'}\right] \prod_{i=1}^{a} [z_i \in X_{n_i}, \sigma_i(\gamma_{A^i}) \in \mathbb{C}_t^{n_i}]$$

It is clear from the two definitions that $l_s^F(\vec{x}) = l_s(\varphi_\alpha(\vec{x})(\vec{A}))$ when $\vec{A}$ is flanked. □

The proof of this result leads us to the following observation.

Proposition 3.50. *Let $[\vec{x} \in F^m(\alpha)_n, \vec{A} \in Q_t^{(\alpha)n}]$ be a class in $(|F^m|_{Q^m})_t$ with $\vec{x}$ internally non-degenerate and $\vec{A}$ flanked. If $\vec{x}$ is totally non-degenerate then $\varphi_\alpha(\vec{x})(\vec{A})$ is a product of generators of the free monoid $\mathbb{C}(X)_t$ that lies in S_m.*

Proof. If $\vec{x}$ is totally non-degenerate then the representing simplices of the classes in $\mathbb{C}(X)_t$ comprising $\varphi_\alpha(\vec{x})(\vec{A})$ are non-degenerate. So by Proposition 3.5 these classes are themselves generators and the product (45) defining $\varphi_\alpha(\vec{x})(\vec{A})$ is the unique factorization of the word in the free monoid $\mathbb{C}(X)_t$. □

Proposition 3.51. *The image of g^m is the simplicial subset $S_{m+1} \subseteq U\mathbb{C}(X)$*

Proof. By Lemma 3.49 and the definition of F^m the map g^m factors through the simplicial subset $S_{m+1} \subseteq U\mathbb{C}(X)$. Now let $\omega \in (S_{m+1})_t$. Write the unique factorization

$$\omega = [x \in X_n, \gamma \in \mathbb{C}_t^n] \prod_{i=1}^{a} [x_i \in X_{n_i}, \gamma_i \in \mathbb{C}_t^{n_i}]$$

with $n, n_i \geq 1$, x and all x_i non-degenerate simplices of X, and γ and all γ_i flanked. Then $l_s(\omega) = \nu(x) + \sum_{i=1}^{a} n_i$. Let $n' = n - \nu(x) \geq 0$ since $\nu(x) \leq \deg(x) = n$. So $x \in V_{n'}^{\nu(x)}$ and since $x \neq s_0^n(*)$, $\nu(x) \geq 1$. Define $\alpha(0) = \nu(x) - 1 \geq 0$ and for all $1 \leq i \leq a$ define $\alpha(i) = n_i + \alpha(i-1)$. Now $n' + \alpha(0) + 1 = n$ and

$$\alpha(a) = \nu(x) - 1 + \sum_{i=1}^{a} n_i = l_s(\omega) - 1$$

Since $l_s(\omega) \leq m+1$, $\alpha(a) \leq m$. Define an embedding $\alpha : [a] \hookrightarrow [\alpha(a)]$ using the values $\alpha(i)$ defined above. Now we have

$$\vec{x} = (x \in V_{n'}^{\alpha(0)+1}, (x_i \in X_{\alpha(i)-\alpha(i-1)})_{1\leq i \leq a}) \in F^m(\alpha)_{n'}$$

For $\gamma_i \in \mathbb{C}_t^{n_i}$ there exists a unique $A^i_\bullet \in NL^0_{n_i}$ such that $\gamma_{A^i} = \gamma_i$, using the notation of Definition 3.42. Since $\alpha(i) - \alpha(i-1) = n_i$, $NL^0_{n_i} = NI^\alpha_i$. Similarly, for $\gamma \in \mathbb{C}_t^n$ flanked there exists a unique simplex $A_\bullet \in NL^0_n = N^0_{n'+\alpha(0)+1}$ such that $\gamma_A = \gamma$. Hence we have

$$\vec{A} = ([A_\bullet] \in Q^{\alpha(0),n'}, (A^i_\bullet \in NI^\alpha_i)_{1\leq i\leq a}) \in Q_t^{(\alpha)n'}$$

and $\varphi_\alpha(\vec{x})(\vec{A}) = \omega$.

□

Proposition 3.52. *Let $[\vec{x} \in F_n^{(\alpha)}, \vec{A} \in Q_t^{(\alpha)n}]$ and $[\vec{y} \in F_{n'}^{(\beta)}, \vec{B} \in Q_t^{(\beta)n'}]$ be classes in $(|F^m|_{Q^m})_t$ with $\alpha : [a] \hookrightarrow [m]$ and $\beta : [b] \hookrightarrow [m]$ and $\vec{x}$ and $\vec{y}$ totally non-degenerate simplices with spine length $m+1$. If $\varphi_\alpha(\vec{x})(\vec{A}) = \varphi_\beta(\vec{y})(\vec{B})$ then $[\vec{x} \in F_n^{(\alpha)}, \vec{A} \in Q_t^{(\alpha)n}] = [\vec{y} \in F_{n'}^{(\beta)}, \vec{B} \in Q_t^{(\beta)n'}]$.*

Proof. By Proposition 3.50 $\varphi_\alpha(\vec{x})(\vec{A})$ and $\varphi_\beta(\vec{y})(\vec{B})$ are the respective unique factorizations of these words into products of generators in $\mathbb{C}(X)_t$, so by (45) $a = b$ and for all $1 \leq i \leq a$

$$\begin{aligned}\left[x \in X_{n+\alpha(0)+1}, \gamma_{\rho(A)} \in \mathbb{C}_t^{n+\alpha(0)+1}\right] &= \left[y \in X_{n'+\beta(0)+1}, \gamma_{\rho(B)} \in \mathbb{C}_t^{n'+\beta(0)+1}\right] \\ \left[x_i \in X_{\alpha(i)-\alpha(i-1)}, \gamma_{A^i} \in \mathbb{C}_t^{\alpha(i)-\alpha(i-1)}\right] &= \left[y_i \in X_{\beta(i)-\beta(i-1)}, \gamma_{B^i} \in \mathbb{C}_t^{\beta(i)-\beta(i-1)}\right]\end{aligned}$$

These classes are all generators of $\mathbb{C}(X)_t$, so $x = y$, $\rho(A) = \rho(B)$, $x_i = y_i$, and $A^i = B^i$ for all i. Furthermore $\beta(0) + 1 = \nu(x) = \alpha(0) + 1$ since the spine lengths are both $m + 1$ and $\vec{x}$ and $\vec{y}$ are both totally non-degenerate. Hence $n = n'$. Now $\alpha(a) = \beta(b) = m$ so $\alpha(i) = \beta(i)$ for all i and so $\alpha = \beta$. Hence the classes in $|F^m|_{Q^m}$ must be equal, as required. $\square$ This proposition shows that, while the map g^m does collapse to the same words in S_{m+1} classes that do not have totally non-degenerate representatives, the classes with spine length $m + 1$ are preserved exactly by g^m. Hence we can finally conclude that the square (40) is a pushout.

Corollary 3.53. *The square* (40) *for* $m \geq 0$

$$\begin{tikzcd}
{|E^m|_{Q^m}} \arrow[r, hook] \arrow[d] & {|F^m|_{Q^m}} \arrow[d, "g^m"] \\
S_m \arrow[r, hook] & S_{m+1}
\end{tikzcd}$$

is a pullback and a pushout.

Proof. We defer until Corollary 3.67 in the next section the proof that the top map is an injection, so that classes in $|F^m|_{Q^m}$ that have a flanked and internally non-degenerate representative with $\vec{x} \in E^m(\alpha)_n$ cannot have another representative with spine length greater than m.

By Proposition 3.51 a class $[\vec{x} \in F_n^{(\alpha)}, \vec{A} \in Q_t^{(\alpha)n}] \in (|F^m|_{Q^m})_t$ with representative flanked and internally non-degenerate is sent to a word with spine length $m + 1$ if and only if the spine length of $\vec{x}$ is $m + 1$. Hence $\varphi_\alpha(\vec{x})(\vec{A}) \in S_m$ if and only if $[\vec{x} \in E_n^{(\alpha)}, \vec{A} \in Q_t^{(\alpha)n}] \in (|E^m|_{Q^m})_t$, so this is a pullback. By Proposition 3.51 the right vertical map is pointwise surjective, so the left vertical map is also a pointwise surjection. By Proposition 3.52 $(S_{m+1})_t$ is isomorphic to the disjoint union of $(S_m)_t$ and $(|F^m|_{Q^m})_t \setminus (|E^m|_{Q^m})_t$, so this square is a pushout. $\square$

Finally, we can use this colimit description of the spine length filtration to connect $H(X)$, the image of X under the functor defined in Definition 3.23. Consider the case $m = 0$ of the pushout square in Corollary 3.53. By Definition 3.41 $E^0(\varepsilon_0) = V^0 = *$ as only the basepoint has void degree 0. Now $S_0 = *$ as well, so the pushout square

$$\begin{tikzcd}
* \arrow[r, hook] \arrow[d, equal] & {|F^0|_{Q^0}} \arrow[d, "g^0"] \\
* \arrow[r, hook] & S_1
\end{tikzcd}$$

implies that for all reduced simplicial sets A

$$|\operatorname{Hom}_A^R(*, *)|_{Q^0} = |F^0|_{Q^0} \cong S_1$$

By Proposition 3.35 we have

$$F^0(\varepsilon_0) = \mathrm{Hom}^R_X(*,*)$$

and so since we will show in Proposition 3.57 that $Q^0 \cong Q^\bullet_L$ as cosimplicial simplicial sets we have

$$H(A) = |\,\mathrm{Hom}^R_A(*,*)|_Q \cong |F^0|_{Q^0} \cong S_1 \tag{47}$$

Hence by Lemma 3.24 for all reduced simplicial sets X we have a span

$$\begin{tikzcd} (X^{S^1})_* & H(X) = |\ F^0\ |_{Q^0} \cong S_1 \arrow[l, "l_X"'] \arrow[r, hook] & \mathbb{C}(X) \end{tikzcd}$$

where the map l_X is a Kan weak equivalence when X is a quasi-monoid. As proposed above, we will take l as the first natural transformation of Lemma 3.12 and the inclusion $S_1 \hookrightarrow U\mathbb{C}(X)$ as the second natural transformation r. In the next section we will use the characterization of the inclusions $S_m \hookrightarrow S_{m+1}$ of the spine length filtration of $U\mathbb{C}(X)$ from the filtration colimit (40) to show that $r_X : H(X) \hookrightarrow U\mathbb{C}(X)$ is a weak equivalence for all reduced simplicial sets X.

3.6 $\mathbb{C}(X)$ as a Mapping Space

Our goal for this section is to complete the proof of another part of Lemma 3.12 by proving that the simplicial set inclusion

$$H(X) \cong S_1 \subseteq U\mathbb{C}(X)$$

is a Kan weak equivalence when X is a quasi-monoid. Here we are using the identification (47) of $H(X)$ with the first non-trivial stage S_1 of the spine length filtration of $U\mathbb{C}(X)$. We will prove this map is a weak equivalence by using the whole spine length filtration $S_m \subseteq S_{m+1}$ of $U\mathbb{C}(X)$ and the filtration colimit (40) from Section 3.5. Recall that this is a pushout square

$$\begin{tikzcd} |E^m|_{Q^m} \arrow[r, hook] \arrow[d] & |F^m|_{Q^m} \arrow[d, "g^m"] \\ S_m \arrow[r, hook] & S_{m+1} \end{tikzcd}$$

where $|-|_{Q^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$ is defined as part of the adjunction (43). Our goal for this section is to show that the inclusion of diagrams

$$E^m \hookrightarrow F^m \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}} \tag{48}$$

is a weak equivalence and the functor $|-|_{Q^m}$ preserves this weak equivalence, sending it to an acyclic cofibration in **sSet** with the Kan model structure. This will then imply that the pushout inclusion $S_m \subseteq S_{m+1}$ is a Kan weak equivalence and hence $H(X) \hookrightarrow U\mathbb{C}(X)$ is a weak equivalence by the spine length filtration of Proposition 3.18 and the identification $H(X) \cong S_1$ of (47).

To make sense of this plan, we must first define a model structure on the category $\mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ of which the map (48) will be a weak equivalence and for which the functor $|-|_{Q^m}$ will preserve this weak equivalence. We will use the Reedy model structure that exists as a result of the Reedy category structure of the category T^m.

Proposition 3.54. *Let $m \geq 0$. The category T^m is a Reedy category with all non-identity morphisms increasing the degree.*

Proof. Let $\varepsilon : T^m \to \mathbb{Q}$ send $\alpha : [a] \hookrightarrow [r]$ to $\frac{r+1}{a+1}$. A morphism

$$\begin{tikzcd}
{[r]} \arrow[r, hook, "f"] & {[s]} \\
{[a]} \arrow[u, hook, "\alpha"] & {[b]} \arrow[l, hook', "v"] \arrow[u, hook, "\beta"']
\end{tikzcd}$$

of T^m has $b \leq a$ and $s \geq r$ so $\frac{r+1}{a+1} \leq \frac{s+1}{b+1}$ and this defines a functor. Since T^m is finite we can restrict the image of this functor to a finite subset of $\mathbb{Q}$, which is a finite linear order and so a well order. Hence T^m is Reedy with every non-identity morphism increasing the degree. □

Hence we can define Reedy model structures on functor categories $\mathcal{C}^{(T^m)^{\mathrm{op}}}$ and $\mathcal{C}^{(T^m)}$ when $\mathcal{C}$ is a model category. The Reedy structure of the category T^m allows us to easily identify cofibrations in the model structure on $\mathcal{C}^{(T^m)^{\mathrm{op}}}$.

Proposition 3.55. *Let $\mathcal{C}$ be a model category. A map of diagrams $\varphi : D \to D'$ in the Reedy model structure on $\mathcal{C}^{(T^m)^{\mathrm{op}}}$ is a Reedy cofibration if and only if it is a pointwise cofibration.*

Proof. All non-identity maps in $(T^m)^{\mathrm{op}}$ decrease the degree, so $\partial((T^m)^{\mathrm{op}})^\alpha = \emptyset$. Hence $L^\alpha D = \emptyset$ for all α in $(T^m)^{\mathrm{op}}$ and so the relative latching map of $\varphi : D \to D'$ is just $\varphi(\alpha) \in \mathcal{C}$. Hence φ is a Reedy cofibration if and only if it is a pointwise cofibrant map. □

We will use the cases $\mathcal{C} = \mathbf{sSet}$ with the Kan model structure and $\mathcal{C} = \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ with the diagonal model structure ([GJ99] IV Theorem 3.13), whose weak equivalences are the maps of bisimplicial sets that are sent to Kan weak equivalences by the diagonal functor and whose cofibrations are the monomorphisms ([Jar13] Theorem 1.4). These are simplicial model categories, with the tensor for bisimplicial sets given by

$$(K \otimes X)_{\bullet n} = K_\bullet \times X_{\bullet n} \tag{49}$$

for $K \in \mathbf{sSet}$ and $X \in \mathbf{sSet}^{\Delta^{\mathrm{op}}}$.

In Subsection 3.5.4 we claimed that there is a natural isomorphism

$$|-|_{Q^0} \cong |-|_Q : \mathbf{sSet} \to \mathbf{sSet}$$

where the second functor is the realization in [Lur09] Proposition 2.2.2.9. We will prove this claim here. It will come out as a special case of a more general result that all realization functors $|-|_{Q^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$ factor through the realization $|-|_Q$. We will show this by first showing that the diagram

$$Q^m : T^m \times \Delta \to \mathbf{sSet}$$

factors as a product of a functor $T^m \to \mathbf{sSet}$ with the cosimplicial simplicial set $Q_L^\bullet$ from [Lur09], here defined in Definition 3.20. The functor in $\mathbf{sSet}^{T^m}$ making up the other part of Q^m is defined by taking the nerves of posets. The following definition generalizes the posets $W_{[n]}^l$ defined in Definition 3.26 to define a functor $W^m : T^m \to \mathbf{Pos}$.

Definition 3.56. *Let $m \geq 0$. The functor $W^m : T^m \to \boldsymbol{Pos}$ sends an object $\alpha : [a] \hookrightarrow [r]$ of T^m to*

$$W^{(\alpha)} = \{A \subseteq [r] \mid \alpha(i) \in A \ \forall \ i \in [a]\} \subseteq P_{[n]}$$

and a map $(f, v) : \alpha \to \beta$ of T^m to the map of posets

$$\begin{array}{rccc} W^{(f,v)} : & W^{(\alpha)} & \to & W^{(\beta)} \\ & A & \mapsto & f(A) \end{array}$$

By the Definition 3.25 of T^m if $\alpha : [a] \hookrightarrow [r]$ and $\beta : [b] \hookrightarrow [s]$ are the objects of T^m for all $i \in [b]$

$$\beta(i) = f(\alpha(v(i))) \in f(A)$$

so $f(A) \in W^{(\beta)}$ and this is a well-defined functor.

This functor $W^m : T^m \to \mathbf{Pos}$ composes with the nerve functor $N : \mathbf{Pos} \to \mathbf{sSet}$ to give the functor

$$NW^m : T^m \to \mathbf{sSet}$$

for all $m \geq 0$. It is this functor that makes up the T^m part of the diagram $Q^m : \Delta \times T^m \to \mathbf{sSet}$. The cosimplicial part will come from the cosimplicial simplicial set from [Lur09] given in Definition 3.20 above.

Proposition 3.57. *There is a natural isomorphism of functors*

$$Q^m \cong Q_L^\bullet \times NW^m : \Delta \times T^m \to \boldsymbol{sSet} \tag{50}$$

Proof. The cosimplicial simplicial sets $Q_L^\bullet$ of Definition 3.20 have spaces Q_L^n for $n \geq 0$ given by the pushouts

$$\begin{tikzcd}
\coprod\limits_{0 \leq i \leq n} (\Delta^1)^{\{j \mid 0 \leq j < i\}} \times \{1\} \times (\Delta^1)^{\{j \mid i < j \leq n\}} \arrow[r] \arrow[d] & NP_{[n]} \arrow[d] \\
\coprod\limits_{0 \leq i \leq n} (\Delta^1)^{\{j \mid i < j \leq n\}} \arrow[r] & Q_L^n
\end{tikzcd} \tag{51}$$

where as in Definition 3.26 $P_{[n]}$ is the poset of non-empty subsets of $[n]$. The top map of (51) is isomorphic to the top map of the pushout in Definition 3.20 by the identification of $NP_{[n]}$ with a simplicial subset of the cube $(\Delta^1)^{n+1}$.

Let $\alpha : [a] \hookrightarrow [r]$ be an object of T^m. We construct a map of posets

$$P_{[n]} \times W^{(\alpha)} \to P^{[n]}_{[n+\alpha(0)+1]} \times \prod_{i=1}^{a} I_i^\alpha$$

by

$$(A, B) \mapsto (A \oplus B^0, (B^i)_{1 \leq i \leq a})$$

where we have partitioned the set $B \in W^{(\alpha)}$ by taking

$$B^0 = \{b \in B \mid b \leq \alpha(0)\}$$

and

$$B^i = \{b - \alpha(i-1) \in [\alpha(i) - \alpha(i-1)] \mid b \in B \cap [\alpha(i-1), \alpha(i)]\}$$

for all $1 \leq i \leq a$. Now $\alpha(0) \in B_0$ and $A \oplus B_0 \cap [n]A \neq \emptyset$, so $A \oplus B_0$ does belong to $P^{[n]}_{[n+\alpha(0)+1]}$. This map is clearly a bijection, so taking the nerve gives a bijection of simplicial sets. Furthermore, restricting this map to the subspaces of the cubes on the left side of the pushout diagram defining Q_L^m in Definition 3.20 identified by the top map gives bijections with the simplicial sets $NW^i_{n+m+1} \times$

$\prod_{i=1}^{a} I_i^\alpha$ and $NL_{n+m+1}^i \times \prod_{i=1}^{a} I_i^\alpha$ in the corresponding positions of the pushout definition of $Q^{(\alpha)n}$. The simplicial set isomorphism $Q^{(\alpha)n} \cong Q_L^n \times NW^{(\alpha)}$ that these maps define is clearly natural. □

By this result and Definition 3.28 there are isomorphisms

$$Q_L^n \cong Q^{0,n}$$

that send a class $[A_\bullet] \in (Q_L^n)_t$ to the corresponding class of $Q^{0,n}$ by simply adding $n+1$ to the top of every subset in $A_\bullet$, a simplex of $NP_{[n]}$. Hence by Definition 3.29 the cosimplicial simplicial sets $Q_L^\bullet$ and Q^0 are isomorphic. This proves the deferred proposition that was used in identifying $H(A) \cong S_1$ in (47).

Recall that a t-simplex $[A_\bullet] \in (Q_L^n)_t$ has a representative

$$\rho_L(A)_\bullet = (\{a\} = \rho(A)_0 \subseteq \rho(A)_1 \subseteq \cdots \subseteq \rho(A)_t) \in (NP_{[n]})_t$$

where $a = \min \rho(A)_i$ for all $0 \leq i \leq t$. The isomorphism (50) identifies

$$([\rho_L(A)_\bullet], B_\bullet) \in Q_L^n \times NW^{(\alpha)} \quad \leftrightarrow \quad ([\rho_L(A)_\bullet \oplus B_\bullet^0], (B_\bullet^i)_{1 \leq i \leq a}) \in Q^{(\alpha)n}$$

where $\rho_L(A)_\bullet \oplus B_\bullet^0$ is the unique representative of the class in $Q^{\alpha(0),n}$ with $\rho_L(A) \oplus B_\bullet^0 \cap [n] = \{a\}$, a singleton such that $a = \min(\rho_L(A)_i \oplus B_i^0)$ for all $0 \leq i \leq t$.

Recall the definition of flanked simplices of Q^m given in Definition 3.31. The following proposition describes the corresponding flanked simplices of $Q_L^n \times NW^{(\alpha)}$ using the isomorphism (50) of Proposition 3.57.

Proposition 3.58. *A simplex* $([\rho_L(A)_\bullet \oplus B_\bullet^0], (B_\bullet^i)_{1 \leq i \leq a}) \in Q_t^{(\alpha)n}$ *for* $\alpha : [a] \hookrightarrow [r]$ *is flanked if and only if its corresponding simplex* $([\rho_L(A)_\bullet], B_\bullet) \in Q_L^n \times NW^{(\alpha)}$ *has*

1. $\rho_L(A)_0 = \{0\}$
2. $\rho_L(A)_t = [n]$
3. $B_0 = \alpha([a]) \subseteq [r]$ *and*
4. $B_t = [r]$.

So for all $t \geq 0$ the set of flanked t-simplices of $Q_t^{(\alpha)n}$ corresponds to the subset $U_t^n \times Z_t^{(\alpha)} \subseteq Q_L^n \times NW_t^{(\alpha)}$ where

$$U_t^n = \{[\rho(A)_\bullet] \in (Q_L^n)_t \mid \rho_L(A)_0 = \{0\} \text{ and } \rho_L(A)_t = [n]\} \subseteq (Q_L^n)_t$$

and $Z_t^{(\alpha)}$ is the subset of $NW_t^{(\alpha)}$

$$Z_t^{(\alpha)} = \{A_\bullet \in NW^{(\alpha)} \mid A_0 = \alpha([a]) \text{ and } A_t = [r]\} \subseteq NW_t^{(\alpha)}$$

We can therefore prove the following claim that was used in the proof of Proposition 3.32.

Corollary 3.59. *Let* $\Phi_t^{(\alpha)n} \subseteq Q_t^{(\alpha)n}$ *be the subset of flanked* t*-simplices of* $Q^{(\alpha)n}$*. For all* $t \geq 0$ *the map*

$$\coprod_{\substack{(f,v):\beta\to\alpha \\ \delta:[l]\hookrightarrow[n]}} \Phi_t^{(\beta)l} \to Q_t^{(\alpha)n}$$

that sends $(\vec{A} \in \Phi^{(\beta)l_t}, (f,v) : \beta \to \alpha, \delta : [l] \hookrightarrow [n])$ *to* $Q^m((f,v),\delta)(\vec{A})$ *is a bijection.*

Proof. Using the isomorphism of Proposition 3.57 and Proposition 3.58 we can prove the equivalent statement that the map

$$\left(\bigsqcup_{\delta:[l]\hookrightarrow[n]} U_t^l \right) \times \left(\bigsqcup_{(f,v):\beta\to\alpha} Z_t^{(\alpha)} \right) \to (Q_L^n \times NW^{(\alpha)})_t$$

is a bijection for all $n, t \geq 0$ and $\alpha : [a] \hookrightarrow [r]$. This is the product of the map (56), which was shown to be an isomorphism in the proof of Proposition 3.61, and the map that sends $([\rho_L(A)_\bullet] \in U_t^l, \delta : [l] \hookrightarrow [n])$ to $([\delta(\rho(A))_\bullet] \in (Q_L^n)_t$. Consider $[\rho_L(A)_\bullet] \in (Q_L^n)_t$. The inclusions $\rho(A)_i \subseteq \rho_L(A)_t \subseteq [n]$ for all $0 \leq i < t$ determine a unique embedding $\delta : [l] \hookrightarrow [n]$ for $l = |\rho_L(A)_t|$ and a simplex $\rho_L(A)'_\bullet \in U_t^l$ such that $\delta(\rho_L(A)')_\bullet = \rho_L(A)_\bullet$. Hence this map is a bijection. $\square$

We will now proceed with the homotopical analysis of the realization functor $|-|_{Q^m}$. The homotopy behaviour of the realization $|-|_Q$ has been described in [Lur09] §2, so our goal in this section is to exploit this by using the isomorphism (50) to break the realization $|-|_{Q^m}$ into two composed realizations.

To achieve this goal, we must consider the more general case of the realization of diagrams $D \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ by $Q^m \in \mathbf{sSet}^{(T^m)}$ via the coend

$$\|D\|_{Q^m} = \int^{(\alpha,[n])\in T^m\times\Delta} Q^{(\alpha)n} \times D(\alpha)_{\bullet n}$$

This determines a functor

$$\|-\|_{Q^m} : (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$$

We use similar notation as the realization functor $|-|_{Q^m}$ from the adjunction (43) because this functor extends that realization. In particular, we recover the original realization functor for diagrams $D \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ by composing the diagram $D : (T^m)^{\mathrm{op}} \to \mathbf{sSet}$ with the inclusion

$$(-)^t : \mathbf{sSet} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}} \tag{52}$$

that sends a simplicial set K to the bisimplicial set $K^t : \Delta^{\mathrm{op}} \to \mathbf{sSet}$ with $K^t_{mn} = K_n$. If $D \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ then we denote the composite with $(-)^t$ by

$$D^t : (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$$

Hence for $m \geq 0$ and $(\alpha, [n]) \in T^m \times \Delta$

$$(Q^{(\alpha)n} \times D^t(\alpha))_{mn} = Q^{(\alpha)n}_m \times D^t(\alpha)_{mn} = Q^{(\alpha)n}_m \times D(\alpha)_n$$

so by the definition of the realization functor of the adjunction (43) we have

$$\|D^t\|_{Q^m} = |D|_{Q^m}$$

We now define a realization that will factor the realization $|-|_{Q^m} : (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$. Recall the functor $NW^m : T^m \to \mathbf{sSet}$ defined by composing the nerve functor $N : \mathbf{Pos} \to \mathbf{sSet}$ with the functor of Definition 3.6. We define a functor

$$|-|_{NW^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}} \tag{53}$$

by the coend

$$|D|_{NW^m} = \int^{\alpha \in T^m} NW^{(\alpha)} \otimes D(\alpha)$$

for $D : (T^m)^{\mathrm{op}} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ with $\otimes$ the tensoring of $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ over $\mathbf{sSet}$ and

$$\|-\|_Q : \mathbf{sSet}^{\Delta^{\mathrm{op}}} \to \mathbf{sSet} \tag{54}$$

defined as the coend

$$\|X\|_Q = \int^{[n] \in \Delta} Q_L^n \times X_{\bullet n}$$

for $X_{\bullet\bullet}$ a bisimplicial set and $\times$ the cartesian product in $\mathbf{sSet}$. This realization (54) extends the functor $|-|_Q : \mathbf{sSet} \to \mathbf{sSet}$ of [Lur09] Proposition 2.2.2.9 in the same way that we extended $|-|_{Q^m}$ to bisimplicial diagrams above. As was shown for that case, pre-composing $\|-\|_Q$ with the inclusion functor $(-)^t : \mathbf{sSet} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ recovers the realization $|-|_Q$

$$\begin{aligned} \|K^t\|_Q &= \int^{[n] \in \Delta} Q_L^n \times K^t_{\bullet n} \\ &= \int^{[n] \in \Delta} Q_L^n \times K_n \\ &= |K|_Q \end{aligned}$$

The isomorphism (50) of Proposition 3.57 allows the realization $\|-\|_{Q^m}$ to be split into these parts.

Proposition 3.60. *Let D be a diagram in $\mathbf{sSet}^{(T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$. There is a natural isomorphism of simplicial sets*

$$\|D\|_{Q^m} \cong \||D|_{NW^m}\|_Q$$

Proof. By the definition of the realization (53) we have

$$|D|_{NW^m} = \int^{\alpha \in T^m} NW^{(\alpha)} \otimes D(\alpha)$$

where $D(\alpha) \in \mathbf{sSet}^{\Delta^{\mathrm{op}}}$. This is a bisimplicial set $|D|_{NW^m} : \Delta^{\mathrm{op}} \to \mathbf{sSet}$ where

$$(|D|_{NW^m})_{mn} = \int^{\alpha \in T^m} NW^{(\alpha)}_m \times D(\alpha)_{mn}$$

for all $m, n \geq 0$ by the definition (49) of the tensoring of $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ over $\mathbf{sSet}$. By Proposition 3.57 therefore we have the isomorphism

$$\begin{aligned} \|D\|_{Q^m} &= \int^{(\alpha,[n]) \in T^m \times \Delta} Q^{(\alpha)n} \times D(\alpha)_{\bullet n} \\ &\cong \int^{(\alpha,[n]) \in T^m \times \Delta} (Q_L^n \times NW^m) \times D(\alpha)_{\bullet n} \\ &= \int^{[n] \in \Delta} Q_L^n \times \left(\int^{\alpha \in T^m} NW^m \times D(\alpha)_{\bullet n} \right) \\ &= \int^{[n] \in \Delta} Q_L^n \times (|D|_{NW^m})_{\bullet n} \\ &= \||D|_{NW^m}\|_Q \end{aligned}$$

□

Observe that if we take $m = 0$ and restrict to the case of a diagram $D^t \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ we recover the isomorphism

$$|D|_Q \cong |D|_{Q^0}$$

that we showed earlier. Hence this factorization of $\|-\|_{Q^m}$ can be seen as a generalization of this result.

For now we will restrict our attention to the functor $|-|_{NW^m}$. Recall that we are considering $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ with the diagonal model structure and $(\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ with the Reedy model structure induced by the diagonal model structure on $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$. We will show that the realization $|-|_{NW^m}$ is a left Quillen functor by showing that the diagram NW^m is a Reedy cofibrant diagram in the Reedy model structure on $\mathbf{sSet}^{(T^n)^{\mathrm{op}}}$.

Proposition 3.61. *NW^m is a Reedy cofibrant replacement of the constant diagram $*$ in $\boldsymbol{sSet}^{(T^n)^{\mathrm{op}}}$*

Proof. For all objects $\alpha : [a] \hookrightarrow [r]$ of T^m we define $Z_t^{(\alpha)} \subseteq NW_t^{(\alpha)}$ as the following set of t-simplices of $NW^{(\alpha)}$

$$Z_t^{(\alpha)} = \{A_\bullet \in NW^{(\alpha)} \mid A_0 = \alpha([a]) \text{ and } A_t = [r]\} \tag{55}$$

These are the t-simplices of $NW^{(\alpha)}$ that are not in the image of any non-identity morphism in the diagram $NW^m : T^m \to \mathbf{sSet}$. Hence the latching object for α is given by

$$L^{(\alpha)}NW_t^m = \coprod_{(f,v):\beta\to\alpha\neq 1_\alpha} Z_t^{(\beta)}$$

Now for a fixed object $\beta : [b] \hookrightarrow [s]$ of T^m we claim that the map

$$\coprod_{(f,v):\alpha\to\beta} Z_t^{(\alpha)} \to NW_t^{(\beta)} \tag{56}$$

that sends $((f,v) : \alpha \to \beta, A_\bullet \in Z_t^{(\alpha)})$ to $f(A)_\bullet \in NW_t^{(\beta)}$ is an isomorphism. This will imply that the latching map $L^{(\alpha)}NW^m \to NW^{(\alpha)}$ is an injection and so NW^m is cofibrant in the Reedy model structure.

Let $A_\bullet \in NW_t^{(\beta)}$. Then $A_t \subseteq [s]$ uniquely determines an embedding $f : [r] \hookrightarrow [s]$ where $f([r]) = A_t$. The inclusions $A_i \subseteq A_t$ determine $A'_i = \{j \in [r] \mid f(j) \in A_i\} \subseteq [r]$, so there is a unique chain $A'_\bullet$ with $A'_t = [r]$ and such that $f(A'_i) = A_i$.

The inclusion $A'_0 \subseteq A'_t$ gives unique embedding $\alpha : [a] \hookrightarrow [r]$ where $\alpha([a]) = A'_0 \subseteq [r]$. Since $\beta([b]) \subseteq A_0 = f(\alpha([a]))$ there is a unique embedding $v : [b] \hookrightarrow [a]$ such that $f \circ \alpha \circ v = \beta$. So $(f,v) : \alpha \to \beta$ is the unique morphism of T^m and $A'_\bullet$ the unique chain of $Z^{(\alpha)}$ such that $f(A'_\bullet) = A_\bullet$. Finally, $NW^{(\alpha)}$ is isomorphic to a product of cubes, so $NW^{(\alpha)} \simeq *$. □

The realization $|-|_{NW^m}$ is therefore a left Quillen functor by Theorem 18.4.11 of [Hir03].

Corollary 3.62. *The realization*

$$|-|_{NW^m} : \boldsymbol{sSet}^{(T^m)^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to \boldsymbol{sSet}^{\Delta^{\mathrm{op}}}$$

is a left Quillen functor when $\boldsymbol{sSet}^{\Delta^{\mathrm{op}}}$ has the diagonal model structure and $\boldsymbol{sSet}^{(T^m)^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ has the Reedy model structure for the Reedy category $(T^m)^{\mathrm{op}}$ and the diagonal model structure on $\boldsymbol{sSet}^{\Delta^{\mathrm{op}}}$.

We now turn our attention to a class of diagrams $D \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ for which the realization $|D|_{NW^m}$ has the same homotopy type as one of the bisimplicial sets in the image of D. The diagrams F^m and E^m will be shown to be of this type, so this allows us to easily recognize weak equivalences between them. We will call these collapsible diagrams. Recall that for all $r \geq 0$ the map $\varepsilon_r : [0] \hookrightarrow [r]$ is the unique object with domain $[0]$ and target $[r]$ in Δ^+.

Definition 3.63. *A diagram $D : (T^m)^{\mathrm{op}} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ is **collapsible** if for all objects $\alpha : [a] \hookrightarrow [r]$ and all maps $(1_{[r]}, \varepsilon_a) : \varepsilon_r \to \alpha$ in T^m of the form*

$$\begin{tikzcd}
{[r]} \arrow[r, equal] & {[r]} \\
{[a]} \arrow[u, hook, "\alpha"] & {[0]} \arrow[l, hook', "\varepsilon_a"'] \arrow[u, hook', "\varepsilon_r"']
\end{tikzcd}$$

the maps $D((1_{[r]}, \varepsilon_a)) : D(\varepsilon_r) \to D(\alpha)$ are diagonal weak equivalences of bisimplicial sets.

There is a functor $F : T^m \to (\Delta^+)^{\leq m}$ that sends $\alpha : [a] \hookrightarrow [r]$ to $[r]$ and a morphism $(f, v) : \alpha \to \beta$ to f. This functor has a section $i : (\Delta^+)^{\leq m} \hookrightarrow T^m$ that sends $[r]$ to $\varepsilon_r : [0] \hookrightarrow [r]$ and a morphism $f : [r] \to [s]$ to the morphism $(f, 1_{[0]}) : \varepsilon_r \to \varepsilon_s$

$$\begin{tikzcd}
{[r]} \arrow[r, hook, "f"] & {[s]} \\
{[0]} \arrow[u, hook, "\varepsilon_r"] \arrow[r, equal] & {[0]} \arrow[u, hook', "\varepsilon_s"']
\end{tikzcd}$$

Pre-composing a diagram $D : (T^m)^{\mathrm{op}} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ with i^{op} and F^{op} gives

$$D \circ (i^{\mathrm{op}} \circ F^{\mathrm{op}})(\alpha : [a] \hookrightarrow [r]) = D(\varepsilon_r)$$

There is a natural transformation $\eta : D \circ (i^{\mathrm{op}} \circ F^{\mathrm{op}}) \to D$ whose components are

$$\eta_\alpha = D((1_{[r]}, \varepsilon_a)) : D(\varepsilon_r) \to D(\alpha)$$

Clearly in a collapsible diagram this natural transformation is a weak equivalence. We call these diagrams collapsible because this weak equivalence allows us to reduce the homotopy colimit of the diagram to the homotopy type of $D(\varepsilon_0)$.

Proposition 3.64. *If $D : (T^m)^{\mathrm{op}} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ is a diagram then there is a map of bisimplicial sets*

$$D(\varepsilon_0) \to |D|_{NW^m} \tag{57}$$

If D is collapsible then this map is a diagonal weak equivalence. Furthermore, a natural transformation $\alpha : D \to D'$ of collapsible diagrams induces a weak equivalence $|D|_{NW^m} \to |D'|_{NW^m}$ if and only if $\alpha_{\varepsilon_0} : D(\varepsilon_0) \to D'(\varepsilon_0)$ is a weak equivalence.

Proof. By Theorem 1.4 of [Jar13] all objects of $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ are cofibrant so by Proposition 3.55 all diagrams $D \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ are cofibrant in the Reedy model structure. By the proof of Proposition 3.62 the functor $NW^m : T^m \to \mathbf{sSet}$ is a cofibrant replacement of the constant diagram

$*$ in $\mathbf{sSet}^{(T^m)}$ with the Reedy model structure determined by the Kan model structure on $\mathbf{sSet}$. Hence for a diagram $D \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$ the realization $|D|_{NW^m}$ is a model for the homotopy colimit of D.

Consider the diagram of bisimplicial sets

$$\begin{tikzcd}
\operatorname{hocolim}_{(T^m)^{\mathrm{op}}} D & \operatorname{hocolim}_{(T^m)^{\mathrm{op}}} (D \circ i^{\mathrm{op}} \circ F^{\mathrm{op}}) \arrow[l, "\eta"'] \arrow[r] & \operatorname{hocolim}_{((\Delta^+)^{\leq m})^{\mathrm{op}}} (D \circ i^{\mathrm{op}}) \\
\operatorname{hocolim}_* D(\varepsilon_0) \arrow[u] \arrow[r, equal] & \operatorname{hocolim}_* D(i(F(\varepsilon_0))) \arrow[u] \arrow[r, equal] & \operatorname{hocolim}_* D(i([0])) \arrow[u]
\end{tikzcd} \tag{58}$$

Since the realization $|D|_{NW^m}$ is a model for the homotopy colimit of the diagram D taking this model gives the map (57) as the left vertical map in (58). We will show that all maps in this diagram are weak equivalences.

The left vertical map of (58) is induced by the functor $\varepsilon_0 : * \to (T^m)$ from the terminal category $*$ that picks out the object $1_{[0]} = \varepsilon_0$ of T^m. The left map of the top row is induced by the natural transformation $\eta : D \circ i^{\mathrm{op}} \circ F^{\mathrm{op}} \to D$. The right map of the top row is induced by precomposing by the functor F^{op}. Now $D(\varepsilon_0) = D(i(F(\varepsilon_0))$ and $\varepsilon_0 : * \to T^m$ factors through the functor $i : (\Delta^+)^{\leq m} \hookrightarrow T^m$ defined above. Hence the diagram commutes. We will show first that when D is a collapsible diagram, the maps of the top row are weak equivalences.

The map on the top left is a weak equivalence because it arises from the natural transformation η, which is a weak equivalence for a collapsible diagram. To show that the right map is a weak equivalence it is sufficient to show F^{op} is homotopy final by Theorem 8.5.6 of [Rie14]. Equivalently, we can show that $F : T^m \to (\Delta^+)^{\leq m}$ is homotopy initial, that is that for every object $[n] \in (\Delta^+)^{\leq m}$ the simplicial set $N(F/[n])$ is contractible. We will show that for fixed n the category $F/[n]$ has a terminal object. Quillen's Theorem A ([Qui73] §1) then implies that $N(F/[n])$ is contractible.

Objects of $F/[n]$ are composable pairs $(\alpha : [a] \hookrightarrow [r], \chi : [r] \hookrightarrow [n])$ in $(\Delta^+)^{\leq m}$ since $F(\alpha) = [r]$. Morphisms are pairs $((f, v) : \alpha \to \beta, g : F(\beta) \to [n])$ where $((f, v), g)$ has source $(\alpha : [a] \hookrightarrow [r], \chi : [r] \hookrightarrow [n])$ and target $(\beta : [b] \hookrightarrow [s], \xi : [s] \hookrightarrow [n])$ where $\chi = \xi \circ f$. A morphism, therefore consists of a diagram

$$\begin{tikzcd}
{[r]} \arrow[r, hook, "f"] \arrow[rr, hook, bend left, "\chi"] & {[s]} \arrow[r, hook, "\xi"] & {[n]} \\
{[a]} \arrow[u, hook, "\alpha"] & {[b]} \arrow[u, hook, "\beta"'] \arrow[l, hook', "v"]
\end{tikzcd}$$

Consider the object $(\varepsilon_n : [0] \hookrightarrow [n], 1_{[n]} : [n] \to [n])$. For any object $(\alpha : [a] \hookrightarrow [r], \chi : [r] \hookrightarrow [n])$ of $F/[n]$, the following diagram determines the unique morphism from (α, χ) to $(\varepsilon_n, 1_{[n]})$

$$\begin{tikzcd}
{[r]} \arrow[r, hook, "\chi"] \arrow[rr, hook, bend left, "\chi"] & {[n]} \arrow[r, equal] & {[n]} \\
{[a]} \arrow[u, hook, "\alpha"] & {[0]} \arrow[u, hook, "\varepsilon_n"'] \arrow[l, hook', "\varepsilon_a"]
\end{tikzcd}$$

The object $[0]$ of $(\Delta^+)^{\leq m}$ is initial, so by Theorem 8.5.6 of [Rie14] the right vertical map of diagram (58) is a weak equivalence. Hence all maps in the diagram (58) are weak equivalences.

Finally, naturality of the homotopy colimit and the weak equivalence just identified shows that $|\alpha|_{NW^m} : |D|_{NW^m} \to |D'|_{NW^m}$ is a weak equivalence for collapsible diagrams D and D' if and only if α_{ε_0} is a weak equivalence.

$\square$

Returning to the realization $\|-\|_{Q^m}$, by Proposition 3.60 there is a map of simplicial sets

$$\|D(\varepsilon_0)\|_Q \to \||D|_{NW^m}\|_Q \cong \|D\|_{Q^m}$$

obtained by applying the realization $\|-\|_Q$ defined at (54) above to the map of bisimplicial sets (57). When D is collapsible this map is a weak equivalence.

Corollary 3.65. *If $D : (T^m)^{\mathrm{op}} \to \boldsymbol{sSet}^{\Delta^{\mathrm{op}}}$ is a collapsible diagram then the map*

$$\|D(\varepsilon_0)\|_Q \to \|D\|_{Q^m} \tag{59}$$

is a weak equivalence of simplicial sets. Furthermore, a natural transformation $\alpha : D \to D'$ of collapsible diagrams induces a weak equivalence $\|D\|_{Q^m} \to \|D'\|_{Q^m}$ if and only if $\alpha_{\varepsilon_0} : D(\varepsilon_0) \to D'(\varepsilon_0)$ is a diagonal weak equivalence.

Proof. The first part follows from Proposition 3.64 directly, since all bisimplicial sets are cofibrant in the diagonal model structure and the cosimplicial space $Q^\bullet_L$ is Reedy cofibrant by [GJ99] §VII Proposition 4.16 since the equalizer of

$$\begin{tikzcd} Q^0_L \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & Q^1_L \end{tikzcd}$$

is $\emptyset$.

For the last part, when D and D' are collapsible applying Proposition 3.64 gives that $\|D\|_{Q^m} \to \|D'\|_{Q^m}$ is a weak equivalence of simplicial sets if and only if $\|D(\varepsilon_0)\|_Q \to \|D'(\varepsilon_0)\|_Q$ is. By Proposition 2.2.2.7 of [Lur09], there is a natural transformation $\pi : Q^\bullet_L \to \Delta^\bullet$ that is a weak equivalence of Reedy cofibrant cosimplicial spaces. Hence for any bisimplicial set X, since X is Reedy cofibrant in the Reedy model structure on bisimplicial sets, by [Hir03] Corollary 18.4.13

$$\|X\|_Q \to |X|_\Delta \cong d(X)$$

is a weak equivalence, where $d(X)$ is the diagonal. Hence $D \to D'$ is sent to a weak equivalence by $\|-\|_{Q^m}$ if and only if it is a diagonal weak equivalence. $\square$

Since $Q^\bullet_L$ is a Reedy cofibrant cosimplicial simplicial set, by [Hir03] Theorem 18.4.11 the realization $\|-\|_Q : \mathbf{sSet}^{\Delta^{\mathrm{op}}} \to \mathbf{sSet}$ of (54) preserves injective maps, that is cofibrations of the diagonal model structure on $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$. As noted in the proof above of Corollary 3.65 $\|-\|_Q$ sends diagonal weak equivalences to Kan weak equivalences of $\mathbf{sSet}$, so $\|-\|_Q$ is left Quillen when $\mathbf{sSet}^{\Delta^{\mathrm{op}}}$ has the diagonal model structure and $\mathbf{sSet}$ has the Kan model structure.

So far we have considered the realization by Q^m of diagrams $D \in (\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$. The diagrams we will actually use, however, are the diagrams defined in Subsections 3.5.2 and 3.5.3, which belong to $\mathbf{sSet}^{(T^m)^{\mathrm{op}}}$. Composing these diagrams with the functor

$$(-)^t : \mathbf{sSet} \to \mathbf{sSet}^{\Delta^{\mathrm{op}}}$$

described at (52) recovers the realization defined as part of the adjunction (43)

$$\|(-)^t\|_{Q^m} = |-|_{Q^m}$$

Hence this adjunction is Quillen.

Proposition 3.66. *The adjunction* $|-|_{Q^m} \dashv \mathrm{Sing}_{Q^m}(-)$ *is a Quillen adjunction when* $\boldsymbol{sSet}^{(T^m)^{\mathrm{op}}}$ *has the Reedy model structure and* ***sSet*** *has the Kan model structure.*

Proof. The realizations $|-|_{NW^m}$ and $\|-\|_Q$ are left Quillen, so they preserve cofibrations and acyclic cofibrations. Hence

$$\|-\|_{Q^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \to \mathbf{sSet}$$

is left Quillen. Cofibrations and acyclic cofibrations are preserved by the inclusion $(-)^t : \mathbf{sSet} \hookrightarrow \mathbf{sSet}^{\Delta^{\mathrm{op}}}$ since the diagonal model structure has injective maps as cofibrations and $d(K^t) = K$ for all simplicial sets K. Hence the left adjoint $|-|_{Q^m} : \mathbf{sSet}^{(T^m)^{\mathrm{op}}} \to \mathbf{sSet}$ is left Quillen. □

Recall the sequence of diagrams

$$E^m \subseteq F^m \subseteq D^m : (T^m)^{\mathrm{op}} \to \mathbf{sSet}$$

for $m \geq 0$ from Definitions 3.34 and 3.41. Proposition 3.55 shows that the image of the inclusion $E^m \subseteq F^m$ under the functor $(-)^t$ is a cofibration in the model structure on $(\mathbf{sSet}^{\Delta^{\mathrm{op}}})^{(T^m)^{\mathrm{op}}}$, so we have the following corollary to Proposition 3.66, which was a claim whose proof we deferred in the proof of Corollary 3.53.

Corollary 3.67. $|E^m|_{Q^m} \hookrightarrow |F^m|_{Q^m}$ *is an injective map of simplicial sets.*

For $m \geq 1$ $E^m(\varepsilon_0) = F^m(\varepsilon_0)$, so if we can show that the diagrams E^m and F^m are collapsible, then $|E^m|_{Q^m} \hookrightarrow |F^m|_{Q^m}$ is a Kan weak equivalence by Corollary 3.65. We will show that this is the case when X is a quasi-monoid.

Proposition 3.68. *Let* X *be a quasi-monoid and let* $m \geq 1$*. The diagrams*

$$(E^m)^t \subseteq (F^m)^t \subseteq (D^m)^t$$

for X *are all collapsible.*

Proof. A diagram $F \in \mathbf{sSet}^{(T^m)^{\mathrm{op}}}$ is sent by $(-)^t$ to a collapsible diagram collapsible in the sense of Definition 3.63 if for all maps $(1_{[r]}, \varepsilon_a) : \varepsilon_r \to \alpha$ of $(T^m)^{\mathrm{op}}$ the maps $F((1_{[r]}, \varepsilon_a)) : F(\varepsilon_r) \to F(\alpha)$ are Kan weak equivalences. This follows because the diagonal of K^t is isomorphic to K. We will show that for the diagrams E^m, F^m, and D^m these maps are Kan weak equivalences.

Let $\alpha : [a] \hookrightarrow [r]$ and let $\alpha \to \varepsilon_r$ be the unique morphism in T^m for some $m \geq r$. We will show that the maps

$$\begin{gathered} D^m(\varepsilon_r) \to D^m(\alpha) \\ F^m(\varepsilon_r) \to F^m(\alpha) \\ E^m(\varepsilon_r) \to E^m(\alpha) \end{gathered}$$

are all acyclic fibrations of simplicial sets for $m \geq 1$. To show this we must solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & D^m(\varepsilon_r) \arrow[d] \\ \Delta^n \arrow[r, "\vec{x}"'] & D^m(\alpha) \end{tikzcd} \tag{60}$$

The map $\varepsilon_r : [0] \hookrightarrow [r]$ sends 0 to r so $D^m(\varepsilon_r) = \mathrm{Dec}^{r+1}(X)$. Hence for $n \geq 1$ diagram (60) corresponds to solving the lift

$$\begin{tikzcd}
\partial\Delta^n \star \Delta^r \bigcup\limits_{\partial\Delta^n * L_\alpha} \Delta^n * L_\alpha \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^{n+r+1} \arrow[r] & *
\end{tikzcd} \tag{61}$$

Since X is a reduced simplicial set, solving this is equivalent to solving the lift in $\mathbf{sSet}_0$ after reflecting the left map into $\mathbf{sSet}$. Let $\mathrm{red} : \mathbf{sSet} \to \mathbf{sSet}_0$ be the left adjoint of the inclusion $\mathbf{sSet}_0 \subseteq \mathbf{sSet}$. By Theorem 3.17 of [Joy08], for all $l \geq 0$ $\mathrm{red}(\partial\Delta^n \star \Delta^l) \hookrightarrow S^{n+l+1}$ is an inner anodyne map between reduced simplicial sets, hence a weak equivalence of the reduced Joyal model structure on $\mathbf{sSet}_0$ by Corollary 2.23. The reflected version of the left map in (61) is therefore a weak equivalence and cofibration in $\mathbf{sSet}_0$, so since X is a quasi-monoid a lift exists.

For $n = 0$, the problem (60) corresponds to finding a lift of $\Delta^0 * L_\alpha \hookrightarrow \Delta^{r+1}$ against $X \to *$. The inclusion on the left is a necklace inclusion in its enveloping simplex, so it is inner anodyne and the lift exists.

The maps $F^m(\varepsilon_r) \to F^m(\alpha)$ are pullbacks of the maps $D^m(\varepsilon_r) \to D^m(\alpha)$, so they are acyclic fibrations as well. We now turn to the maps $E^m(\varepsilon_r) \to E^m(\alpha)$. When $r < m$ these are the same maps as for F^m, so we only have to consider the case of $\alpha : [a] \hookrightarrow [m]$. Recall that Corollary 3.39 determines the conditions under which a simplex $\vec{x} = (x, (x_i)_{1 \leq i \leq a}) \in E^m(\alpha)_n$ for $\alpha : [a] \hookrightarrow [m]$ can have $l^F_s(\vec{x}) < m + 1$. We will consider each case for $\vec{x} : \Delta^n \to E^m(\alpha)$ in the following diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] & E^m(\varepsilon_m) \arrow[d] \\
\Delta^n \arrow[r, "\vec{x}"'] \arrow[ur, dashed, "\vec{y}"] & E^m(\alpha)
\end{tikzcd}$$

and show that a lift $\vec{y}$ exists in each case.

Suppose $\vec{x} \in E^m(\alpha)_n$ satisfies case 2 of Corollary 3.39, so $d_{n+2} \cdots d_{n+1+\alpha(0)}(x) = s_0^{n+1}(*)$. Note that the inclusion

$$d^{n+m+1} \circ \cdots \circ d^{n+\alpha(0)+2} : \Delta^{n+\alpha(0)+1} \star \emptyset \hookrightarrow \Delta^{n+m+1}$$

factors through the inclusion of the necklace into its enveloping simplex $\Delta^n * L_\alpha \subseteq \Delta^{n+r+1}$. Hence it factors through the left hand map of (61). A lift $\vec{y} : \Delta^n \to D^m(\varepsilon_m)$ for this diagram has $y \in X_{n+m+1}$ with $d_{n+\alpha(0)+2} \cdots d_{n+m+1}(y) = x$. So

$$\begin{aligned}
d_{n+2} \cdots d_{n+\alpha(0)+1} d_{n+\alpha(0)+2} \cdots d_{n+m+1}(y) &= d_{n+2} \cdots d_{n+\alpha(0)+1}(x) \\
&= s_0^{n+1}(*)
\end{aligned}$$

So $\vec{y}$ satisfies case 2 as well and $\vec{y} \in E^m(\varepsilon_m)$.

Suppose $\vec{x} \in E^m(\alpha)_n$ for $\alpha : [a] \hookrightarrow [m]$ satisfies case 1 of Corollary 3.39. So $\vec{x} : \Delta^n * L_\alpha \to X$ factors through $\Delta^n * S^i_\alpha$ for some $0 \leq i \leq m$. Solving the lift above when $n \geq 1$ corresponds to finding a lift for the outer square of the diagram

$$\begin{tikzcd}
\partial\Delta^n \star \Delta^r \displaystyle\bigcup_{\partial\Delta^n * L_\alpha} \Delta^n * L_\alpha \arrow[r] \arrow[d, hook] & \partial\Delta^n \star \Delta^{r-1} \displaystyle\bigcup_{\partial\Delta^n * S^i L_\alpha} \Delta^n * S^i L_\alpha \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^{n+r+1} \arrow[r, "\Delta^n \star s^i"'] & \Delta^{n+r} \arrow[r] & *
\end{tikzcd}$$

Since $S^i L_\alpha$ is a necklace a lift exists for the right square of this diagram by the same argument as above for D^m. The lift then factors through $\Delta^n \star s^i$, so it satisfies case 1 of Corollary 3.39. The $n = 0$ case corresponds to solving the lifting problem of the right square of the diagram

$$\begin{tikzcd}
\Delta^0 * L_\alpha \arrow[r] \arrow[d, hook] & \Delta^0 * S^i L_\alpha \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^{r+1} \arrow[r, "s^i"'] & \Delta^r \arrow[r] & *
\end{tikzcd}$$

which can be done again by the same arguments as above. ☐

This finally completes the proof of the main result we wanted for this section.

Corollary 3.69. *For all $m \geq 1$ and X a quasi-monoid the map $|E^m|_{Q^m} \hookrightarrow |F^m|_{Q^m}$ is a Kan acyclic cofibration of simplicial sets.*

There remains only one part of Lemma 3.12 to prove, namely that the map $S_1 \to UM$ of simplicial sets determined by the counit $\varepsilon_M : \mathbb{C}(\mathbb{N}(M)) \to M$ is a Kan weak equivalence when M is a fibrant simplicial monoid. Recall there is an isomorphism $g^0 : S_1 \to |F^0|_{Q^0}$ of simplicial sets, where $F^0(\varepsilon_0)_t = V_t^1 = \{x \in X_{t+1} \mid d_{t+1}(x) = s_0^{t-1}(*)\}$. When $X = \mathbb{N}(M)$, the homotopy coherent nerve of a simplicial monoid M we have $F^0(\varepsilon_0) = \mathrm{Hom}_{\mathbf{sMon}}(\Sigma\mathbb{C}^\bullet, M)$ where $\Sigma\mathbb{C}^n$ is defined by the pushout

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, hook, "d^{n+1}"] \arrow[d] & \mathbb{C}^{n+1} \arrow[d] \\
e \arrow[r, hook] & \Sigma\mathbb{C}^n
\end{tikzcd}$$

The inclusion $d^{n+1} : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ is a cofibration, so the simplicial monoid $\Sigma\mathbb{C}^n$ is cofibrant. The monoid $\Sigma\mathbb{C}_t^n$ of t-simplices is generated by atomic chains $\gamma \in \mathbb{C}_t^{n+1}$ such that $\gamma_0 = x_{an+1}$ for some $a \in [n]$.

Recall the simplicial set maps $\varphi_x : Q^{m,n} \to U\mathbb{C}(X)$ defined in Lemma 3.43 for any simplex $x \in \mathrm{Dec}^{m+1}(X)_n$. For $1_{[n+1]} \in \Delta_{n+1}^{n+1} = \mathrm{Dec}^1(\Delta^{n+1})_{n+1}$ we get a map $\varphi_{1_{[n+1]}} : Q^{0,n} \to U\mathbb{C}^{n+1}$ that sends $[A_\bullet] \in Q_t^{0,n}$ to $\gamma_{\rho(A)} \in \mathbb{C}^{n+1}$. The chains $\gamma_{\rho(A)}$ are all atomic and since $\rho(A)_0 = \{a < n+1\}$ they have $(\gamma_{\rho(A)})_0 = x_{an+1}$. Hence composing with the map $\mathbb{C}^{n+1} \to \Sigma\mathbb{C}^n$ we have the injective map

$$\varphi_{n+1} : Q^{0,n} \hookrightarrow U\Sigma\mathbb{C}^n$$

that sends each t-simplex of $Q^{0,n}$ to an atomic chain generating $\Sigma\mathbb{C}_t^n$. All such generators are the image of a simplex of $Q^{0,n}$, so $F_\times Q^{0,n} \cong \Sigma\mathbb{C}^n$, where $F_\times$ is the free simplicial monoid functor left adjoint to the forgetful functor U. Hence we have the following.

Proposition 3.70. *For the reduced simplicial set $\mathbb{N}(M)$ there is an isomorphism of simplicial sets*

$$F^0 = \mathrm{Hom}_{\boldsymbol{sMon}}(\Sigma\mathbb{C}^\bullet, M) \cong \mathrm{Hom}_{\boldsymbol{sSet}}(Q^{0,\bullet}, UM) = \mathrm{Sing}_{Q^0}(UM)$$

We can finally prove the remaining part of Lemma 3.12.

Proposition 3.71. *Let M be a fibrant simplicial monoid. The restriction of $\varepsilon_M : \mathbb{C}(\mathbb{N}(M)) \to M$ to the simplicial subset $S_1 \subseteq U\mathbb{C}(\mathbb{N}(M))$ is a Kan weak equivalence of simplicial sets.*

Proof. Applying the isomorphism $g^0 : S_1 \to |F^0|_{Q^0}$ and the isomorphism $F^0 \cong \mathrm{Sing}_{Q^0}(UM)$ from Proposition 3.70 we have the diagram

$$\begin{tikzcd}
\lvert \mathrm{Sing}_{Q^0}(UM)\rvert_{Q^0} \arrow[r, phantom, "\cong"] \arrow[drr] & S_1 \arrow[r, hook] \arrow[dr] & U\mathbb{C}(\mathbb{N}(M)) \arrow[d, "\varepsilon_M"] \\
& & UM
\end{tikzcd}$$

The map $|\mathrm{Sing}_{Q^0}(UM)|_{Q^0} \to UM$ that this diagram defines sends

$$[f : Q^{0,n} \to UM, [A_\bullet] \in Q_t^{0,n}] \in |\mathrm{Sing}_{Q^0}(UM)|_{Q^0} \mapsto \tilde{f}(\gamma_{\rho(A)}) = f([A_\bullet]) \in UM$$

where $\tilde{f} : FQ^{0,n} = \Sigma\mathbb{C}^n \to M$ is the adjoint of the map f. Hence this map is the counit of the adjunction $|-|_{Q^0} \dashv \mathrm{Sing}_{Q^0}$. By Proposition 3.57 this adjunction is isomorphic to the adjunction $|-|_Q \dashv \mathrm{Sing}_Q$ of Proposition 2.2.2.9 in [Lur09]. This proposition shows that this adjunction is a Quillen equivalence of **sSet** with the Kan model structure with itself, so if M is a fibrant simplicial monoid then UM is a Kan complex and $|\mathrm{Sing}_{Q^0}(UM)|_{Q^0} \to UM$ is a Kan weak equivalence. □

3.7 Localization and Group Completion

The goal of this section is to show that the adjunction

$$\begin{tikzcd}
\mathbf{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "L"]
\end{tikzcd} \tag{62}$$

showing simplicial groups as a reflective subcategory of simplicial monoids is, up to homotopy, the realization by $\mathbb{C}$ of the localization adjunction (20) for the quasi-monoid model structure. This adjunction determines a reflective subcategory of the homotopy category of **sMon** when the category **sGp** of simplicial groups has the model structure of Theorem 2 §II.3 [Qui67]. This is the model structure transferred from **sMon** via the inclusion functor I, so fibrations and weak equivalences of simplicial groups are created by the functor I and this adjunction $L \dashv I$ is Quillen.

To show that the adjunction (62) is the realization of the adjunction (20) we will use the homotopy coherent nerve-realization Quillen equivalence to transfer the localization at $S^1 \hookrightarrow R^1$ of $(\mathbf{sSet}_0)_J$ to a localization of **sMon**. This localization of **sMon** is a model structure with fibrant objects the Kan fibrant simplicial monoids M whose monoid of connected components is a group. We will show that the group completion functor induces a Quillen equivalence of this model structure with **sGp**. Finally, we will show that the model categories and Quillen equivalences constructed in this chapter and Chapter 2 recover the result of Dwyer and Kan in [DK80] that when M is a Kan fibrant simplicial monoid the total derived functor of L is a model for the homotopy type of the loop space on the reduced simplicial set $\mathbb{N}(M)$.

3.7.1 Localizing the Simplicial Monoid Model Structure

In this section we will show that it is possible to take a left Bousfield localization of **sMon**. As described at the start of Chapter 3 the model structure on **sMon** is transferred from the Kan model structure on **sSet** via the free-forgetful adjunction

$$\begin{tikzcd} \mathbf{sMon} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathbf{sSet} \arrow[l, bend left, "F_\times"] \end{tikzcd}$$

We will use this characterization and the properties of **sSet** to show that **sMon** satisfies the conditions for a left Bousfield localization at a set of maps of **sMon** to exist. The theorem we will use to guarantee the existence of a left Bousfield localization is the following.

Theorem 3.72 (J. Smith - [Bar10] Theorem 4.7)**.** *Let $\mathcal{V}$ be a combinatorial left proper model category. The left Bousfield localization of $\mathcal{V}$ at any set of morphisms exists.*

We will define the two conditions required in this theorem. A model category is **left proper** when the pushout of a weak equivalence along a cofibration is a weak equivalence, so in the following diagram

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"', "\wr"] & Y \arrow[d, "g'"] \\ Z \arrow[r, hook, "f'"'] & P \end{tikzcd}$$

if f is a cofibration and g is a weak equivalence then g', the pushout of g along f, is also a weak equivalence.

We will show that **sMon** is left proper by using the results of [BB17] which give conditions on a model category $\mathcal{V}$ that guarantee that the category of monoids $\mathbf{Mon}(\mathcal{V})$ is left proper. In our case $\mathcal{V} = \mathbf{sSet}$ so we will be checking that **sSet** satisfies the conditions of [BB17]. They define an ***h*-cofibration** as a cofibration of $\mathcal{V}$ such that all pushouts of weak equivalences along it are weak equivalences. A **trivial *h*-cofibration** is an h-cofibration that is also a weak equivalence. Clearly a left proper model category is a model category where all cofibrations are h-cofibrations. Using this concept they make the following definitions that strengthen the notion of left properness.

Definition 3.73 ([BB17] Definition 1.11)**.** *A model category $\mathcal{V}$ is **h-monoidal** if it is a monoidal model category and the tensor product $- \otimes X$ with an object sends cofibrations to h-cofibrations and trivial cofibrations to trivial h-cofibrations. A model category is **strongly h-monoidal** if it is h-monoidal and the tensor product preserves all weak equivalences.*

To state the result of [BB17] about left properness of the transferred model structure we will first need to recall the notion of a combinatorial model category. These are model categories where objects can be constructed from a set of generating objects and the homotopical structure is determined by sets of maps. We will need the concepts of ordinals and cardinals for the next few definitions. An **ordinal** is the well-ordered set of smaller ordinals, starting from $0 = \emptyset$. The **cardinality** of a set S is denoted $|S|$ and is the smallest ordinal such that there exists a bijection of sets $|S| \to S$. A **cardinal** is an ordinal λ that is its own cardinality, so $|\lambda| \cong \lambda$. A **regular cardinal** λ is an infinite cardinal that is not a union of a smaller cardinality of smaller cardinals.

This means that λ has the following property: for a map of sets $f : P \to S$ such that $|S| < \lambda$, if $P_x = \{y \in P \mid f(y) = x\}$ for all $x \in S$ and $|P_x| < \lambda$ then $|X| < \lambda$. The smallest regular cardinal is $\aleph_0 = |\mathbb{N}|$.

Definition 3.74. *Let λ be a regular cardinal. A poset is **λ-filtered** if every subset of cardinality strictly less than λ has an upper bound in the poset. A poset is **filtered** if it is $\aleph_0$-filtered.*

Definition 3.75. *An object X of a category $\mathcal{V}$ is **λ-small** if the hom functor $\mathrm{Hom}_{\mathcal{V}}(X, -) : \mathcal{V} \to$ **Set** preserves colimits of λ-filtered posets. An object is **finite** if it is $\aleph_0$-small, or equivalently $\mathrm{Hom}_{\mathcal{V}}(X, -)$ preserves colimits of filtered posets. An object is **small** if it is λ-small for some regular cardinal λ.*

Definition 3.76. *A category $\mathcal{V}$ is **locally presentable** if it is cocomplete and has a small set $\mathcal{A}$ of small objects such that every object of $\mathcal{V}$ is a colimit of objects from $\mathcal{A}$.*

There are several equivalent conditions for a category to be locally presentable, as described in Chapter 1 [AR94]. We are using the definitions as organized in [Lur09] A.1.1 but we use the original terminology of locally presentable from [AR94] rather than just presentable as Lurie calls them. We can now define a combinatorial model category.

Definition 3.77. *A model category $\mathcal{V}$ is **combinatorial** if the underlying category is locally presentable and there exist small sets of morphisms I and J such that*

1. *a morphism of $\mathcal{V}$ is an acyclic fibration if and only if it has the right lifting property against all maps in I*
2. *a morphism of $\mathcal{V}$ is a fibration if and only if it has the right lifting property against all maps in J*

We need one more definition before we can state the theorem from [BB17] that will allow us to conclude that **sMon** is left proper. The class of weak equivalences of a model category $\mathcal{V}$ is **perfect** when for any filtered category D and all diagrams $D \to \mathcal{V}^{\Delta^1}$ in the arrow category of $\mathcal{V}$ such that $D(d)$ is a weak equivalence for all d, the colimit $\operatorname{colim} D$ is a weak equivalence of $\mathcal{V}$.

The result we will use from [BB17] is the following.

Theorem 3.78 ([BB17] Theorem 3.1)**.** *Let $\mathcal{V}$ be a combinatorial model category such that the class of weak equivalences of $\mathcal{V}$ is perfect. If $\mathcal{V}$ is strongly h-monoidal then the transferred model structure on **Mon**$(\mathcal{V})$ exists and is left proper.*

Finally, we come to the result we promised about the existence of left Bousfield localizations of model structures of monoids.

Proposition 3.79. *The model structure on **sMon** is left proper and combinatorial.*

Proof. We will use [BB17] Theorem 3.1 to show that the transferred model structure on **sMon** exists and is left proper. Lemma 1.12 of [BB17] states that a model category where all objects are cofibrant is strongly h-monoidal, so we can immediately conclude that **sSet** with the Kan model structure is strongly h-monoidal. In Chapter 3 of [Hov99] Lemma 3.1.1 shows that all simplicial sets are small and Lemma 3.1.3 gives that every simplicial set is the colimit of a diagram of standard simplicial simplices Δ^n for $n \geq 0$, so **sSet** is locally presentable. The Quillen model structure on **sSet** is has generating cofibrations $\partial\Delta^n \hookrightarrow \Delta^n$ and generating acyclic cofibrations $\Lambda^k_n \hookrightarrow \Delta^n$ for $0 \leq n$ and $0 \leq k \leq n$. Hence **sSet** is combinatorial. By Remark A.3.2.3 of [Lur09] class of weak equivalences of **sSet** is perfect. Hence by Theorem 3.78 since **sSet** is strongly h-monoidal, combinatorial, and has perfect weak equivalences the transferred model structure on **sMon** exists and is left proper.

The model structure on **sMon** has generating cofibrations $F_\times(I)$ and generating acyclic cofibrations $F_\times(J)$, where I and J are the respective sets of morphisms for the combinatorial model structure on **sSet**. Since the monoidal structure of **sSet** is biclosed $X \times -$ has a right adjoint and in particular preserves all colimits. Hence, by the Corollary on page 7 of [Por08] **sMon** is locally presentable. So the model structure on **sMon** is combinatorial. □

By Theorem 3.72, therefore, we can localize **sMon** at any set of simplicial monoid maps.

3.7.2 Grouplike Simplicial Monoids

We say that a simplicial monoid is **group-like** when its monoid of connected components is a group. In this section we will show that there is a model structure on the category of simplicial monoids that is Quillen equivalent to $(\mathbf{sSet}_0)_K$ the Kan model structure on reduced simplicial sets and for which the fibrant objects are the Kan fibrant group-like simplicial monoids.

We construct this model structure by localizing the transferred model structure on **sMon** using the results of the previous section that show that left Bousfield localizations of **sMon** at sets of simplicial monoid maps exist. The localization will fill in the model category $(\mathbf{sMon})_G$ in the bottom left corner of the following diagram as well as the two adjunctions

$$\begin{tikzcd}
\mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\vdash"] & (\mathbf{sSet}_0)_J \arrow[l, bend left, "\mathbb{C}"] \arrow[d, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\vdash"] \\
(\mathbf{sMon})_G \arrow[u, bend left, "\mathrm{Id}"] \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "\mathbb{C}"] \arrow[u, bend left, "\mathrm{Id}"]
\end{tikzcd} \tag{63}$$

The top adjunction of this diagram is the homotopy coherent nerve-realization Quillen equivalence (33) and the right vertical adjunction is the adjunction from the left Bousfield localization (20) of the quasi-monoid model structure at the map $S^1 \hookrightarrow R^1$.

The model category $(\mathbf{sMon})_G$ is the left Bousfield localization of **sMon** at $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$, which is the image under $\mathbb{C}$ of the map for the localization of $(\mathbf{sSet}_0)_J$. Since $\mathbb{C} \dashv \mathbb{N}$ is a Quillen equivalence, localizing **sMon** at $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$ will give a model structure Quillen equivalent to $(\mathbf{sSet}_0)_K$. We record this construction in the next result.

Proposition 3.80. *There exists a simplicial model structure on the category of simplicial monoids such that*

- *the cofibrations are the same as those of the transferred model structure for* ***sMon***
- *the fibrant objects are Kan fibrant group-like simplicial monoids*

We denote this model structure by $(\boldsymbol{sMon})_G$. *Furthermore, the adjunction*

$$\begin{tikzcd}(\boldsymbol{sMon})_G \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\boldsymbol{sSet}_0)_K \arrow[l, bend left, "\mathbb{C}"]\end{tikzcd} \tag{64}$$

where $(\boldsymbol{sSet}_0)_K$ *has the Kan model structure of Theorem 2.1 is a Quillen equivalence.*

Proof. By Proposition 3.79 the model category **sMon** satisfies the conditions of Theorem 3.72 so we can localize at the simplicial monoid map $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$. Since $(\mathbf{sSet}_0)_K$ is the left Bousfield localization of $(\mathbf{sSet}_0)_J$ at $S^1 \hookrightarrow R^1$ and $\mathbb{C} \dashv \mathbb{N}$ gives a Quillen equivalence between **sMon** and $\mathbf{sSet}_0$ by Proposition 3.14, Theorem 3.3.20 of [Hir03] gives that $\mathbb{C} : (\mathbf{sMon})_G \leftrightarrow (\mathbf{sSet}_0)_K : \mathbb{N}$ is a Quillen equivalence.

The cofibrations of $(\mathbf{sMon})_G$ are the same as those of **sMon** by the definition of the left Bousfield localization as in [Hir03] Definition 3.3.1. The fibrant objects are the Kan fibrant simplicial monoids that are also $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(S^1)$-local by [Bar10] Theorem 2.11. These are the Kan fibrant simplicial monoids M such that the induced map of homotopy function complexes

$$\underline{\mathrm{Hom}}_{\mathbf{sMon}}(\mathbb{C}(R^1), M) \to \underline{\mathrm{Hom}}_{\mathbf{sMon}}(\mathbb{C}(S^1), M) \tag{65}$$

is a Kan weak equivalence. Since all reduced simplicial sets are cofibrant in $(\mathbf{sSet}_0)_J$ Corollary 16.1.4 of [Hir03] gives that $A \;{}_*\!\!\times \Delta^\bullet \to A$ is a cofibrant replacement of A in the Reedy model structure on $(\mathbf{sSet}_0)_J^{\Delta}$. Hence by Proposition 16.2.1 of [Hir03] $\mathbb{C}(A \;{}_*\!\!\times \Delta^\bullet) \to \mathbb{C}(A)$ is a cofibrant replacement of $\mathbb{C}(A)$ since $\mathbb{C}$ is left Quillen. Since M is a Kan fibrant simplicial monoid, therefore, we can take

$$\underline{\mathrm{Hom}}_{\mathbf{sMon}}(\mathbb{C}(A), M)_n = \mathrm{Hom}_{\mathbf{sMon}}(\mathbb{C}(A \;{}_*\!\!\times \Delta^n), M)$$

By adjointness, therefore, the map (65) is equal to

$$\mathrm{Hom}_{\mathbf{sSet}_0}(R^1 \;{}_*\!\!\times \Delta^\bullet, \mathbb{N}(M)) \to \mathrm{Hom}_{\mathbf{sSet}_0}(S^1 \;{}_*\!\!\times \Delta^\bullet, \mathbb{N}(M))$$

By adjointness of the half reduced product this is the simplicial set map $(\mathbb{N}(M)^{R^1})_* \to (\mathbb{N}(M)^{S^1})_*$. This map is a Kan weak equivalence if and only if $\mathbb{N}(M)$ is a $S^1 \hookrightarrow R^1$-local reduced simplicial set. By Proposition 2.41 this holds if and only if $\mathbb{N}(M)$ has the right lifting property against $S^1 \hookrightarrow R^1$. Hence M is $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$-local if and only if $M \to *$ has the right lifting property against $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$.

It only remains to show that simplicial monoids with the right lifting property against this map are exactly the group-like simplicial monoids. The simplicial monoid $\mathbb{C}(S^1)$ is isomorphic to the free monoid on one generator viewed as a discrete simplicial monoid. Hence a simplicial monoid map $\mathbb{C}(S^1) \to M$ corresponds to a 0-simplex $m \in M_0$. The simplicial monoid $\mathbb{C}(R^1)$ has 0-simplices $\mathbb{C}(R^1)_0$ isomorphic to the free monoid on two generators $x, y \in \mathbb{C}(R^1)_0$ and 1-simplices generated by a non-degenerate 1-simplex $\alpha : e \to xy$ and the degeneracies of x and y. Since R^1 is 2-skeletal a solution to the lifting problem

$$\begin{tikzcd}\mathbb{C}(S^1) \arrow[r, "m"] \arrow[d, hook] & M \arrow[d] \\ \mathbb{C}(R^1) \arrow[r] \arrow[ur, dashed, "\beta"] & *\end{tikzcd}$$

corresponds to specifying a 0-simplex $\bar{m} \in M_0$ and a 1-simplex $\beta \in M_1$ such that $\beta : e \to m\bar{m}$. This specifies a right inverse for the class $[m] \in \pi_0(M)$. Conversely, if $[m'] \in \pi_0(M)$ has a right inverse $[\bar{m}'] \in \pi_0(M)$, since M is Kan fibrant there exists a 1-simplex $e \to m'\bar{m}'$ in M. Hence a Kan fibrant simplicial monoid has the right lifting property against $\mathbb{C}(S^1) \hookrightarrow \mathbb{C}(R^1)$ if and only if all classes $[m] \in \pi_0(M)$ have right inverses. But this is equivalent to $\pi_0(M)$ being a group.

□

We will refer to $(\mathbf{sMon})_G$ as the **group-like model structure** for simplicial monoids. By Proposition 3.80 a fibrant object of the group-like model structure is a group-like fibrant simplicial monoid, so no confusion should arise by calling monoids and the model structure group-like. We will refer to the transferred model structure for simplicial monoids as the **standard model structure** and reserve the notation **sMon** to refer to the category of simplicial monoids with this model structure.

In the diagram (63) the horizontal adjunctions are Quillen equivalences and the vertical adjunctions are left Bousfield localizations. We want to show that the left vertical adjunction is equivalent to the group completion adjunction

$$\begin{tikzcd} \mathbf{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "L"] \end{tikzcd} \tag{66}$$

We will do this by showing that **sGp** is Quillen equivalent to the group-like model structure $(\mathbf{sMon})_G$.

We can factor the adjunction (66) between **sMon** and **sGp** through the localization adjunction for $(\mathbf{sMon})_G$ from (63). The functor $I : \mathbf{sGp} \to (\mathbf{sMon})_G$ is right Quillen because all simplicial groups are Kan fibrant group-like simplicial monoids, so they are fibrant objects of the group-like model structure. All fibrations of **sGp** are therefore sent by the inclusion functor to standard fibrations of simplicial monoids between group-like fibrant simplicial monoids, which by Proposition 3.3.16 of [Hir03] are group-like fibrations. Hence both adjunctions of the following factorization of (66) are Quillen adjunctions.

$$\begin{tikzcd} \mathbf{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & (\mathbf{sMon})_G \arrow[l, bend left, "L"] \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "\mathrm{Id}"] \end{tikzcd}$$

Proposition 3.81. *The adjunction*

$$\begin{tikzcd} \boldsymbol{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & (\boldsymbol{sMon})_G \arrow[l, bend left, "L"] \end{tikzcd} \tag{67}$$

is a Quillen equivalence.

Proof. This adjunction is Quillen by the previous discussion. A simplicial group is sent by I to a Kan fibrant group-like simplicial monoid, so by Theorem 3.2.18 of [Hir03] a map of simplicial groups $f : G \to H$ is a weak equivalence if and only if it is sent by $I : \mathbf{sGp} \to (\mathbf{sMon})_G$ to a group-like weak equivalence. Hence the right adjoint of the adjunction (67) creates weak equivalences and it is sufficient to show that the unit $\eta_X : X \to ILX$ of this adjunction is a weak equivalence for all cofibrant simplicial monoids X. Dwyer and Kan showed in Proposition 9.5 of [DK80] that for a cofibrant simplicial monoid X the unit map η_X is a weak equivalence of the standard model structure if and only if X is grouplike, so this completes the proof. □

We can compose the localized homotopy coherent nerve-realization adjunction (64) with the adjunction (67) between $(\mathbf{sMon})_G$ and $\mathbf{sGp}$ to obtain a Quillen equivalence between $\mathbf{sGp}$ and $(\mathbf{sSet}_0)_K$.

$$\begin{tikzcd}
\mathbf{sGp} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & (\mathbf{sMon})_G \arrow[l, bend left, "L"] \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "\mathbb{C}"]
\end{tikzcd}$$

This Quillen equivalence factors the composite adjunction $L\mathbb{C} \dashv \mathbb{N}I$ between $\mathbf{sGp}$ and $(\mathbf{sSet}_0)_J$ through the left Bousfield localization of the quasi-monoid model structure (20) as follows.

$$\begin{tikzcd}
\mathbf{sMon} \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "L"] \arrow[d, phantom, "\vdash"] & (\mathbf{sSet}_0)_J \arrow[l, bend left, "\mathbb{C}"] \arrow[d, bend left, "\mathrm{Id}"] \arrow[d, phantom, "\vdash"] \\
\mathbf{sGp} \arrow[u, bend left, "I"] \arrow[r, bend left, "\mathbb{N}I"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "L\mathbb{C}"] \arrow[u, bend left, "\mathrm{Id}"]
\end{tikzcd} \tag{68}$$

There is a well-known Quillen equivalence between $\mathbf{sGp}$ and $(\mathbf{sSet}_0)_K$ given by Kan's loop group functor. In the next section we will compare our Quillen equivalence to this one.

3.7.3 Derived Functors of Group Completion

We will now show that the derived functor of group completion $L : \mathbf{sMon} \to \mathbf{sGp}$ is the loop space of the homotopy coherent nerve of a Kan fibrant simplicial monoid. To do this we will show that the Quillen equivalence

$$\begin{tikzcd}
\mathbf{sGp} \arrow[r, bend left, "\mathbb{N}I"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "L\mathbb{C}"]
\end{tikzcd} \tag{69}$$

that was shown in the previous section is weakly equivalent to the Kan loop group adjunction $G \dashv \bar{W}$. The functor G defined by [Kan58] is not only the left adjoint of a Quillen equivalence between $\mathbf{sGp}$ and $(\mathbf{sSet}_0)_K$ but also gives a simplicial group model of the loop space of a reduced simplicial set. We will show that for all reduced simplicial sets A there is a natural weak equivalence

$$G(A) \xrightarrow{\sim} L\mathbb{C}(A)$$

so that when $A = \mathbb{N}(M)$ for a Kan fibrant simplicial monoid M the simplicial group $G\mathbb{N}(M)$ has the homotopy type of the total derived functor $\mathbb{L}L(M)$. To construct the weak equivalence $G(A) \to L\mathbb{C}(A)$ we will described the cosimplicial simplicial group that determines the adjunction $G \dashv \bar{W}$ via nerve-realization and show that it is Reedy cofibrant and weakly equivalent to $L\mathbb{C}^\bullet$.

We begin by describing the cosimplicial simplicial group that determines the Kan loop group adjunction. This construction comes from §2.6 of [Hin15], where a cosimplicial simplicial groupoid determining the groupoid version of the Kan loop group functor is defined. We are using the opposite convention to [Hin15] to agree with the convention chosen in the construction of $\mathbb{C}^\bullet$ in Definition 3.4 but our constructions are based on their results.

Definition 3.82. *Let $n \geq 0$. Let $G^0 = e$ be the trivial group and for $n \geq 1$ define a simplicial group G^n by the disjoint union of $n-i$-simplices*

$$G^n = \coprod_{1 \leq i \leq n} LF_\times(\Delta^{n-i})$$

for $1 \leq i \leq n$. We denote by g_i^n the $n-i$-simplex of G^n corresponding to the unique non-degenerate generating $n-i$-simplex of $LF_\times(\Delta^{n-i})$.

For $0 \leq i \leq n$ the coface maps $d^i : G^{n-1} \to G^n$ are the unique simplicial group maps that act on the generators g_j^{n-1} of G^{n-1} for $1 \leq j \leq n-1$ by

$$d^i(g_j^{n-1}) = \begin{cases} d_{n-i}(g_j^n) & 1 \leq j < i \leq n \\ g_{j+1}^n & 0 \leq i < j \leq n-1 \\ d_{n-i}(g_i^n)g_{i+1}^n & i = j \end{cases}$$

Note that when $n = 1$ since $G^0 = 2$ the coface maps d^0 and d^1 are both equal to the unique map including the trivial group in G^1

$$d^0 = d^1 = e \hookrightarrow G^1$$

For $0 \leq i \leq n-1$ the codegeneracy maps $s^i : G^n \to G^{n-1}$ are the unique simplicial group maps that act on the generators g_j^n of G^n for $1 \leq j \leq n$ by

$$s^i(g_j^n) = \begin{cases} s_{n-i-1}(g_j^{n-1}) & 1 \leq j \leq i \leq n-1 \\ e & j = i+1 \\ g_{j-1}^{n-1} & 1 \leq i+1 < j \leq n \end{cases}$$

We observe that G^n is the group completion of the nerve of a partially ordered monoid, just like $L\mathbb{C}^n$, the cosimplicial simplicial group that determines the Quillen equivalence (69). The free simplicial monoid on the simplicial n-simplex is the nerve of the free partially ordered group on $[n]$

$$F_\times(\Delta^n) = N(F([n]))$$

This holds since words in the free partially ordered monoid $F([n])$ are sequences of integers

$$i = i_1 i_2 \cdots i_m$$

for $i_l \in [n]$ and for words $i, j \in F([n])$ we have $i \leq j$ if and only if the two words are the same length m and $i_l \leq j_l$ for all $1 \leq l \leq m$. The nerve of a poset preserves disjoint unions, so

$$\begin{aligned} G^n &= \coprod_{1\leq i\leq n} LF_\times(\Delta^{n-i}) \\ &= L\left(\coprod_{1\leq i\leq n} F_\times(\Delta^{n-i})\right) \\ &= L\left(\coprod_{1\leq i\leq n} NF([n-i])\right) \\ &= LN\left(\coprod_{1\leq i\leq n} F([n-i])\right) \end{aligned}$$

We will denote

$$\Gamma^n = \coprod_{1\leq i\leq n} F([n-i])$$

so that $G^n = LN\Gamma^n$. We denote the generators of the submonoid

$$F([n-i]) \leq \Gamma^n$$

for $1 \leq i \leq n$ by $\nu_j^{i,n}$ for $0 \leq j \leq n-i$, so the generating $n-i$-simplices of G^n identified as g_j^n in Definition 3.82 correspond to the chain

$$\nu_0^{j,n} < \nu_1^{j,n} < \cdots < \nu_{n-j}^{j,n} \in (N\Gamma^n)_{n-j} \quad \leftrightarrow \quad g_j^n \in G_{n-j}^n$$

The coface and codegeneracy maps for $G^\bullet$ arise as the group completion of the nerve of the following maps of partially ordered monoids

$$d^i(\nu_r^{j,n-1}) = \begin{cases} \nu_{r+1}^{j,n} & 1 \leq j < i \leq n \text{ and } n-1-i < r \\ \nu_r^{j,n} & 1 \leq j < i \leq n \text{ and } n-1-i \geq r \\ \nu_r^{j+1,n} & 0 \leq i < j \leq n-1 \\ \nu_r^{i,n}\nu_r^{i+1,n} & i = j \end{cases}$$

$$s^i(\nu_r^{j,n}) = \begin{cases} \nu_{r-1}^{j,n-1} & 1 \leq j \leq i \leq n-1 \text{ and } n-i \leq r \\ \nu_r^{j,n-1} & 1 \leq j \leq i \leq n-1 \text{ and } n-i > r \\ e & j = i+1 \\ \nu_r^{j-1,n-1} & 1 \leq i+1 < j \leq n \end{cases}$$

We will now show that the nerve-realization adjunction for this cosimplicial simplicial group $G^\bullet$ is the Kan loop group adjunction originally defined in [Kan58] and described in detail in [GJ99] §V. The realization of a reduced simplicial set A by $G^\bullet$ is the coend

$$G(A) = \int^{[n]\in\Delta} G^n \cdot A_n$$

So the group $G(A)_r$ of r-simplices of $G(A)$ is a quotient of the free group

$$\coprod_{[n]\in\Delta}\left(\coprod_{a\in A_n} G^n_r\right)$$

by the relations

$$(\theta^*(a)\in A_n, g^n_j \in G^n_{n-j}) \sim (a\in A_m, \theta(g^n_j)\in G^m_{n-j})$$

for any non-decreasing map $\theta : [n] \to [m]$ of Δ. By Definition 3.82

$$g^n_{j+1} = d^0(g^{n-1}_j)$$

for all $1 \le j \le n-1$ so for all $l > 1$

$$g^n_l = d^0(g^{n-1}_{l-1}) = (d^0)^2(g^{n-2}_{l-2}) = \cdots = (d^0)^{l-1}(g^{n-l+1}_1)$$

Hence the group $G(A)_r$ is freely generated by classes of the form

$$[a\in A_{r+1}, g^{r+1}_1 \in G^{r+1}_r]$$

which agrees with the definition of the Kan's loop group functor G from [Kan58]. Hence this cosimplicial simplicial group $G^\bullet$ determines via the nerve-realization construction the Kan loop group adjunction

$$\begin{tikzcd}
\mathbf{sGp} \arrow[r, bend left, "\bar{W}"] \arrow[r, phantom, "\top"] & (\mathbf{sSet}_0)_K \arrow[l, bend left, "G"]
\end{tikzcd}$$

which is a Quillen equivalence when the category of reduced simplicial sets has the Kan model structure as depicted above ([GJ99] §V Proposition 6.3). Hence we can conclude the following.

Proposition 3.83. *$G^\bullet$ is a Reedy cofibrant cosimplicial simplicial group.*

Proof. As we showed in the proof of Proposition 3.9, since $G^0 = e$ and the coface maps $d^0, d^1 : G^0 \to G^1$ are equal the latching map

$$L^n(G^\bullet) \to G^n$$

is equal to $G(\partial S^n) \to G^n$, where ∂S^n is the reduced simplicial set obtained from $\partial\Delta^n$ by collapsing all 0-simplices to a single 0-simplex. Hence we must show that the map $G(\partial S^n) \to G^n$ is a cofibration of simplicial groups. But this follows by the fact that $\partial S^n \hookrightarrow S^n$ is a cofibration of the model structure $(\mathbf{sSet}_0)_K$ and the nerve-realization adjunction determined by $G^\bullet$ is the Kan loop group adjunction which is a Quillen equivalence by [GJ99] §V Proposition 6.3. □

As described in [Hin15] 2.6.1 there is a map of cosimplicial simplicial monoids $\tau : L\mathbb{C}^\bullet \to G^\bullet$. This map will be obtained as the group completion of the nerve of a map of cosimplicial partially ordered monoids. Recall from Definition 3.4 that $\mathbb{C}^n$ is the nerve of the partially ordered monoid C^n freely generated by x_{ij} for $0 \le i < j \le n$ and subject to the partial order

$$x_{ik} < x_{ij}x_{jk}$$

for $0 \leq i < j < k \leq n$. The cosimplicial maps of the cosimplicial simplicial monoid $\mathbb{C}^\bullet$ are obtained as the nerves of maps of partially ordered monoids determined by

$$d^k : x_{ij} \in C^{n-1} \mapsto x_{d^k(i)d^k(j)} \in C^n$$

$$s^k : x_{ij} \in C^n \mapsto x_{s^k(i)s^k(j)} \in C^{n-1}$$

where we use the convention that $x_{ii} = e$, the monoid identity of C^{n-1}. We will define a map of cosimplicial partially ordered monoids

$$\rho : C^\bullet \to \Gamma^\bullet$$

by

$$\rho : x_{ij} \in C^n \mapsto \nu^{i+1,n}_{n-j} \nu^{i+2,n}_{n-j} \cdots \nu^{j,n}_{n-j}$$

This map respects the partial order as for $0 \leq i < j < k \leq n$

$$\begin{array}{rcl} \rho(x_{ij}x_{jk}) & = & \nu^{i+1,n}_{n-j} \nu^{i+2,n}_{n-j} \cdots \nu^{j,n}_{n-j} \nu^{j+1,n}_{n-k} \nu^{j+2,n}_{n-k} \cdots \nu^{k,n}_{n-k} \\ & > & \nu^{i+1,n}_{n-k} \nu^{i+2,n}_{n-k} \cdots \nu^{j,n}_{n-k} \nu^{j+1,n}_{n-k} \nu^{j+2,n}_{n-k} \cdots \nu^{k,n}_{n-k} \\ & = & \rho(x_{ik}) \end{array}$$

It also respects the cosimplicial structure. For $x_{ij} \in C^{n-1}$ and $0 \leq l \leq n$ if $i < j < l$ then $d^l(i) = i$, $d^l(j) = j$, and $n - 1 - j > n - 1 - l$ so

$$\begin{array}{rcl} d^l(\rho(x_{ij})) & = & d^l(\nu^{i+1,n-1}_{n-1-j} \nu^{i+2,n-1}_{n-j} \cdots \nu^{j,n-1}_{n-1-j}) \\ & = & d^l(\nu^{i+1,n-1}_{n-1-j}) d^l(\nu^{i+2,n-1}_{n-1-j}) \cdots d^l(\nu^{j,n-1}_{n-1-j}) \\ & = & \nu^{i+1,n}_{n-j} \nu^{i+2,n}_{n-j} \cdots \nu^{j,n}_{n-j} \\ & = & \nu^{d^l(i)+1,n}_{n-d^l(j)} \nu^{d^l(i)+2,n}_{n-d^l(j)} \cdots \nu^{d^l(j),n}_{n-d^l(j)} \\ & = & \rho(x_{d^l(i)d^l(j)}) \\ & = & \rho(d^l(x_{ij})) \end{array}$$

The other identities are similar.

Applying the nerve and the group completion functors to this map ρ of cosimplicial partially ordered monoids gives the map $\tau : L\mathbb{C}^\bullet \to G^\bullet$. This map is a weak equivalence of cosimplicial simplicial groups as G and $L\mathbb{C}$ are left Quillen and so preserve the weak equivalences $S^n \to *$ for all $n \geq 0$. Hence by Propositions 3.9 and 3.83 τ is a weak equivalence of Reedy cofibrant cosimplicial simplicial groups, so by Corollary 18.4.13 of [Hir03] for all reduced simplicial sets the map of realizations induced by τ

$$L\mathbb{C}(A) \to G(A)$$

is a weak equivalence of simplicial groups.

We can now give a new proof of the fact originally shown by Dwyer and Kan in [DK80] that the loop space of the classifying space of a Kan fibrant simplicial monoid has the homotopy type of the derived functor of group completion.

Proposition 3.84. *The total derived functor of group completion of a Kan fibrant simplicial monoid M is the homotopy type of the loop space on $\mathbb{N}(M)$.*

Proof. Let M be a Kan fibrant simplicial monoid. Then by the proof of Proposition 3.14 the counit of the homotopy coherent nerve-realization adjunction $\mathbb{CN}(M) \to M$ is a weak equivalence of simplicial monoids. Since all reduced simplicial sets are cofibrant in the quasi-monoid model structure by Theorem 2.37 and $\mathbb{C}$ is a left Quillen functor for this model structure by Corollary 3.11 the simplicial monoid $\mathbb{CN}(M)$ is a cofibrant replacement of M in the standard model structure on **sMon**. Hence in the homotopy category of **sGp**

$$\mathbb{L}L(M) = L\mathbb{CN}(M) \cong G\mathbb{N}(M)$$

by the previous discussion. □

4 Strict ω-Groupoids and Homotopy Types

In this chapter we will summarize elements of the theory of strict ω-groupoids from the literature. We will use this theory in the next chapter as a partial model for homotopy types from which we will construct a new partial model of ∞-monoids. Strict ω-groupoids are higher groupoids, meaning that they have n-morphisms whose source and target are $n-1$-morphisms for $n \geq 1$ and there are groupoid structures on these sets of n-morphisms. We call these morphisms of strict ω-groupoids n-cells for all $n \geq 0$. The category ω**Gpd** of strict ω-groupoids was shown by Brown and Higgins in [BH81b] to be equivalent to the category of crossed complexes, and hence to a variety of other definitions of higher groupoids. Using this equivalence the theory of crossed complexes, which has been developed by Brown and others and is covered in the book [Bro11], can be transferred to strict ω-groupoids. In this chapter we summarize and adapt the work of Ara and Métayer in their paper [AM11] which describes the model structure on the category of strict ω-groupoids that corresponds to the model structure on crossed complexes proved in [BG89]. We also make use of the Street nerve of strict ω-groupoids to connect the homotopy theory of strict ω-groupoids to that of spaces and show that they provide a model of homotopy 2-types. The material in this chapter is a recollection of facts from the literature that we will need in the next chapter, where we will study monoids in the category of strict ω-groupoids.

Results for higher groupoids, including the model structure mentioned above, have historically been mainly based in the category of crossed complexes. The study of globular models of higher categories on the other hand has mainly focused on the more general case of strict ω-categories, as in [Ste04], [LMW10a], [AM14], and others. Many results in this chapter, therefore, have been adapted from the language of crossed modules to that of strict ω-groupoids using the equivalence of [BH81b] or specialized to the case of strict ω-groupoids from the more general case of strict ω-categories.

We begin with the definition of our basic tools for this section: the categories of strict ω-categories and groupoids. The definitions and results in this section originate with [BH81b] but we use more modern notation along the lines of [AM11].

4.1 Strict ω-Categories and Groupoids

We will recall the definitions of strict ω-categories and groupoids and some of the basic tools for these structures that we will use. Strict ω-categories are globular sets with operations, so we start by defining globular sets.

Definition 4.1. *A* ***globular set*** *A is a set graded by the monoid $\mathbb{N}$ with pairs of maps*

$$\begin{tikzcd}A_{n+1} \arrow[r, shift left, "d_n^-"] \arrow[r, shift right, "d_n^+"'] & A_n\end{tikzcd}$$

for all $n \geq 0$ that satisfy the globular relations

$$d_n^- \circ d_{n+1}^- = d_n^- \circ d_{n+1}^+ \qquad\qquad d_n^+ \circ d_{n+1}^- = d_n^+ \circ d_{n+1}^+$$

for all $n \geq 0$.

The elements of the set A_n are called n-cells. A map of globular sets $f : A \to B$ is a collection of set maps $f_n : A_n \to B_n$ between sets of n-cells that satisfy

$$f_{n-1} \circ d_n^\varepsilon = d_{n-1}^\varepsilon \circ f_n$$

for all $n > 1$ and $\varepsilon \in \{-, +\}$.

Globular sets are equivalently presheaves on the globular category $\mathbb{O}$, which is the category generated by the graph

$$\begin{tikzcd}0 \arrow[r, shift left, "s^0"] \arrow[r, shift right, "t^0"'] & 1 \arrow[r, shift left, "s^1"] \arrow[r, shift right, "t^1"'] & 2 \arrow[r, shift left, "s^2"] \arrow[r, shift right, "t^2"'] & \cdots \arrow[r, shift left, "s^{n-1}"] \arrow[r, shift right, "t^{n-1}"'] & n \arrow[r, shift left, "s^n"] \arrow[r, shift right, "t^n"'] & \cdots\end{tikzcd}$$

with the maps satisfying coglobular relations that are the duals of the globular relations given above. Together with the maps defined above globular sets form a category **Glob**, which is the presheaf category $\mathbf{Set}^{\mathbb{O}^{\mathrm{op}}}$ on $\mathbb{O}$.

For a globular set A when $0 \leq m < n$ there exist maps $d_{m,n}^\varepsilon : A_n \to A_m$ for $\varepsilon \in \{-, +\}$ given by

$$d_{m,n}^\varepsilon = d_m^\varepsilon \circ d_{m+1}^\varepsilon \circ \cdots \circ d_n^\varepsilon$$

These maps satisfy extensions of the globular relations, so for $0 \leq l < m < n$

$$d_{l,m}^- \circ d_{m,n}^- = d_{l,m}^- \circ d_{m,n}^+ \qquad\qquad d_{l,m}^+ \circ d_{m,n}^- = d_{l,m}^+ \circ d_{m,n}^+$$

The maps $d_{m,n}^-$ and $d_{n,m}^+$ determine the m-source and m-target respectively of an n-cell. We say a pair of n-cells $x, y \in A_n$ is m**-composable** when $d_{m,n}^+ x = d_{m,n}^- y$. The set of all m-composable pairs in a globular set A is given by the pullback

$$\begin{tikzcd}A_n \times_{A_i} A_n \arrow[r] \arrow[d] & A_n \arrow[d, "d_{m,n}^-"] \\ A_n \arrow[r, "d_{m,n}^+"'] & A_m\end{tikzcd}$$

Composable pairs are the inputs for m-composition operations, which are part of the definition of a strict ω-category structure on a globular set.

Definition 4.2. *A* ***strict*** ω***-category*** X *is a globular set with composition operations for all* $n \geq 1$ *and all* $0 \leq m \leq n$ *given by maps*

$$*_m : X_n \times_{X_m} X_n \to X_n$$

defined on the set of m*-composable pairs of* n*-cells and for all* $n \geq 0$ *identity operations given by maps*

$$1_n : A_{n-1} \to A_n$$

that satisfy the following identities

1. *for all* $x \in X_n$ *and* $\varepsilon \in \{-, +\}$
$$d^{\varepsilon}_n(1_{n+1}(x)) = x$$

2. *for* $0 \leq m < n$ *and* $(x, y) \in X_n \times_{X_m} X_n$ *for* $0 \leq i < n$
$$d^{\varepsilon}_{i,n}(x *_m y) = \begin{cases} (d^{\varepsilon}_{i,n}x) *_m (d^{\varepsilon}_{i,n}y) & m < i < n \\ d^{\varepsilon}_{i,n}x & \varepsilon = - \textit{ and } i = m \\ d^{\varepsilon}_{i,n}y & \varepsilon = + \textit{ and } i = m \\ d^{\varepsilon}_{i,n}x = d^{\varepsilon}_{i,n}y & i < m \end{cases}$$

3. *for* $x \in X_n$ *and* $n > 0$
$$x *_{n-1} 1_n(d^{+}_{n-1}x) = x \qquad 1_n(d^{-}_{n-1}x) *_{n-1} x = x$$

4. *for* $0 \leq m < n$ *and* $x, y, z \in X_n$ *with* $(x, y), (y, z) \in X_n \times_{X_m} X_n$
$$(x *_m y) *_m z = x *_m (y *_m z)$$

5. *for* $0 \leq l < m < n$ *and* $x, y, x', y' \in X_n$ *with* $(x, y), (x', y') \in X_n \times_{X_m} X_n$ *and* $(x, x'), (y, y') \in X_n \times_{X_l} X_n$
$$(x *_m y) *_l (x' *_m y') = (x *_l x') *_m (y *_l y')$$

6. *for* $0 \leq m < n$ *and* $(x, y) \in X_n \times_{X_m} X_n$
$$1_{n+1}(x *_n y) = 1_{n+1}(x) *_m 1_{n+1}(y)$$

Several implicit assumptions about composability are made in stating these identities, but they can easily be verified. For an ω-category X and any $n \geq 0$ we can define an n-category by simply ignoring all cells of degree greater than n. A 0-category, therefore, is simply a set X_0 and a 1-category is a small category with set of objects X_0, morphisms X_1, and composition given by $*_1$. In the case $n = 2$ we recover the definition of a 2-category with the exchange law that relates the two composition laws $*_1$ and $*_2$. The definition of a strict ω-category is the generalization to all degrees of these cases, so from now on we will identify n-categories as ω-categories for which the m-cells are all identity cells when $m > n$.

Just as there are iterated versions of the source and target maps for a globular set, in a strict ω-category there are iterated identity maps $1_{m,n} : X_m \to X_n$ given for $0 \leq m < n$ by

$$1_{m,n} = 1_m \circ 1_{m+1} \circ \cdots \circ 1_{n-1}$$

By identity 1 of Definition 4.2 1_n is a section of the maps d_n^- and d_n^+, so it is injective. The compatibility of identities with composition given by identity 6 allows us to view the n-cells X_n of a strict ω-category as having a copy of the m-cells inside it it for all $m < n$.

An ω-functor $f : X \to Y$ between strict ω-categories is a map of globular sets that respects the composition and identity operations. This means that for n-cells $(x, y) \in X_n \times_{X_m} X_n$ with $0 \leq m < n$

$$f(x *_m y) = f(x) *_m f(y)$$

and $f(1_{n+1}(x)) = 1_{n+1}(f(x))$. Together with ω-functors ω-categories form a category $\omega\mathbf{Cat}$.

There is a forgetful functor $U : \omega\mathbf{Cat} \to \mathbf{Glob}$ that simply forgets the identity and composition structures on a strict ω-category. This functor has a left adjoint $F : \mathbf{Glob} \to \omega\mathbf{Cat}$ that sends a globular set to a free strict ω-category generated by its cells. We will describe freely generated strict ω-categories and groupoids below in more detail, but for now we note that by [Mét08] the monad of the adjunction $F \dashv U$ is finitary monadic so since $\mathbf{Glob}$ is a presheaf category and so is complete and cocomplete hence so is $\omega\mathbf{Cat}$

As with ordinary categories, we can defined the notion of inverses for composition in a strict ω-category. As there are multiple ways to compose cells in a strict ω-category there are many ways for a cell to have an inverse.

Definition 4.3. *Let $0 \leq m$. A strict ω-category X is a* ***strict*** (ω, m)***-category*** *if for all $m + 1 \leq i < n$ there exist maps*

$$k^i : X_n \to X_n$$

such that

$$d_{i-1,n}^- k^i(x) = d_{i-1,n}^+ x \qquad d_{i-1,n}^+ k^i(x) = d_{i-1,n}^- x$$

so x and $k^i(x)$ are $i-1$-composable in both permutations and their composites are given by

$$k^i(x) *_{i-1} x = 1_{i,n}(d_{i-1,n}^+ x) \qquad x *_{i-1} k^n(x) = 1_{i,n}(d_{i-1,n}^- x)$$

*An $(\omega, 0)$-category will be called an ω-**groupoid**.*

We call the n-cell $k^m(x)$ the m-inverse for x. A strict (n, m)-category for $n \geq m \geq 0$ is an (ω, m)-category with only identity p-cells for $p > n$.

Strict (ω, m)-categories form full reflective subcategories of $\omega\mathbf{Cat}$ for all $m \geq 0$. We will denote these categories by $(\omega, m)\mathbf{Cat}$ in general and for $n = 0$ we will also denote the full subcategory of ω-groupoids by $\omega\mathbf{Gpd}$. The left adjoints $L^m : \omega\mathbf{Cat} \to (\omega, m)\mathbf{Cat}$ to the inclusion functors for $m \geq 0$ freely add inverses to all cells in degree m and above of a strict ω-category. Hence for all $m \geq 0$ the categories $(\omega, m)\mathbf{Cat}$, and hence in particular $\omega\mathbf{Gpd}$, are complete and cocomplete.

Now that we have stated the full definition of strict (ω, m)-categories we will relax our notation from now on to simplify expressions. We will refer to the iterated source and target maps $d_{m,n}^\varepsilon : X_n \to X_m$ for $0 \leq m < n$ simply as $d_m^\varepsilon : X_n \to X_m$ with the source of the map determined by the degree of the cell it is acting on. As well, when we wish to emphasize that an n-cell is an identity of an m-cell for $0 \leq m < n$ we will denote the iterated identities of $x \in X_m$ by

$$1_{n,m}(x) = 1_x \in X_n$$

Often, however, when composing cells we will suppress the identity notation and write $x *_l y$ for the composite $1_x *_l y$ where x is an m-cell for $0 \leq l < m < n$ and $(1_x, y) \in X_n \times_{X_l} X_n$.

Finally, we note that the adjective strict in the definition of strict (ω, m)-categories is added to distinguish these structures from other globular definitions where some of the identities of Definition 4.2 are not exact but only hold weakly in some sense. There exist many ways to weaken these axioms and obtain a weak ω-groupoid. One collection of definitions is given in [Lei02]. We will only deal with strict ω-groupoids and categories, however, so from now on we will only refer to them as (ω, m)-categories and ω-groupoids.

4.1.1 n-Globes

The underlying structure of (ω, m)-categories as presheaves on $\mathbb{O}$ allows us to apply the Yoneda lemma to describe n-cells via ω-functors. Our tools for this are the n-globes, which we define now.

Definition 4.4. *Let $n \geq 0$. The n-**globe** is the free ω-category generated by the representable globular set*

$$D^n = \mathbb{O}(-, n) : \mathbb{O}^{\mathrm{op}} \to \boldsymbol{Set}$$

This globular set has two m-cells in each set $\mathbb{O}(m, n)$ for $m < n$, one given by the iterated source maps $s^m \circ \cdots \circ s^{n-1}$ and the other by the iterated target maps $t^m \circ \cdots \circ t^{n-1}$. There is a unique n-cell given by the identity map and no higher cells. Therefore there are no composable pairs and so the globular set D^n is already an ω-category. The first three n-globes have the following forms

$$\begin{tikzcd}
D^0 : & \iota_0 & D^1 : & d_0^- \iota_1 \arrow[r, "\iota_1"'] & d_0^+ \iota_1 & D^2 : & d_0^- \iota_2 \arrow[r, bend left, "d_1^- \iota_2", ""{name=U, below}] \arrow[r, bend right, "d_1^- \iota_2"', ""{name=D, above}] & d_0^- \iota_2 \arrow[Rightarrow, from=U, to=D, "\iota_2"]
\end{tikzcd}$$

The Yoneda lemma implies that an ω-functor $D^n \to X$ of ω-categories is uniquely determined by the image of their unique n-cell $\iota_n \in D^n_n$, so there are bijections of sets

$$\mathrm{Hom}_{\omega\mathbf{Cat}}(D^n, X) = \mathrm{Hom}_{\mathbf{Glob}}(\mathbb{O}(-, n), UX) \cong X_n$$

We will denote by ∂D^n the sub-ω-category of D^n with the same cells as D^n below degree n but with no non-identity n-cells. The first three ∂D^n are

$$\begin{tikzcd}
\partial D^0 : & \emptyset & \partial D^1 : & d_0^- \iota_1 & d_0^+ \iota_1 & \partial D^2 : & d_0^- \iota_2 \arrow[r, bend left, "d_1^- \iota_2"] \arrow[r, bend right, "d_1^- \iota_2"'] & d_0^- \iota_2
\end{tikzcd}$$

Since ω-functors $D^n \to X$ are uniquely determined by n-cells of X, ω-functors $\partial D^n \to X$ for $n > 0$ are uniquely determined by pairs of parallel $n-1$-cells of X.

We denote the inclusions of ∂D^n in D^n by

$$i_n : \partial D^n \hookrightarrow D^n$$

for $n \geq 0$. The ω-functors $L^0 i_n$ will be generating cofibrations for a cofibrantly generated model structure ω-groupoids that will be described in Section 4.4.

Note that when X is an (ω, m)-category since L^m is left adjoint to the inclusion functor of $(\omega, m)\mathbf{Cat}$ into $\omega\mathbf{Cat}$ the n-cells of X correspond bijectively to ω-functors

$$L^m D^n \to X$$

and parallel pairs of $n-1$-cells of X correspond to ω-functors

$$\partial L^m D^n \to X$$

where $\partial L^m D^n$ is the sub-(ω, m)-category of $L^m D^n$ which has the same p-cells below degree n and only identity cells in degree n and above. The image under the adjunction unit

$$\eta_{D^n} : D^n \to I L^m(D^n)$$

of the unique n-cell $\iota_n \in D^n_n$ in $L^m D^n$ will be called the **principal cell** of $L^m D^n$.

Hence $L^m D^n$ play the same role for (ω, m)-categories as D^n does for general ω-categories; ω-functors $L^m D^n \to X$ are uniquely determined by the image of their principal cell. Since we will mostly be working with ω-groupoids, we define special notation for ω-groupoid versions of the n-globes for all $n \geq 0$

$$I^n = L^0 D^n$$

The constructions of the (ω, m)-categories $\partial L^m D^n$ are examples of the truncation construction for ω-categories, which we will define in the next section.

4.1.2 Truncation and Cotruncation of (ω, m)-Categories

In this section we will describe the truncation and cotruncation constructions for ω-groupoids. These give adjoint approaches to restricting an ω-groupoid to be an n-groupoid for $n \geq 0$. Our goal in defining these constructions is to make the construction in Definition 4.9 that will allow us to describe the functor restricting an ω-groupoid to its sub-ω-groupoid based at a single 0-cell. We will start with the notion of truncation and work in the more general case of an (ω, m)-groupoid for $m \geq 0$. For $n \geq 0$ the n-truncation of an (ω, m)-category discards all non-identity cells above degree n.

Definition 4.5. *For an (ω, m)-category X and $n \geq m \geq 0$ we define the n-**truncation** $\tau^n(X)$ of X as the sub-(ω, m)-category of X containing only the identity cells above degree n*

$$\tau^n(X)_i = \begin{cases} X_i & i \leq n \\ X_n & i > n \end{cases}$$

This defines a functor $\tau^n : (\omega, m)\mathbf{Cat} \to (\omega, m)\mathbf{Cat}$ that acts on ω-functors $f : X \to Y$ by restriction to the sub-(ω, m)-category $\tau^n f : \tau^n(X) \to \tau^n(Y)$.

This construction is given for *cubical* ω-groupoids in [BH81a]. These are another model of higher groupoids that start from a cubical rather than globular base. In the same paper these are shown to be equivalent to crossed complexes and hence to ω-groupoids as we defined them here. We also adapt the adjoint construction, called the coskeleton of a cubical ω-groupoid in [BH81a], to our category of ω-groupoids. We will call this the cotruncation functor.

Definition 4.6. *Let $n \geq m \geq 0$. The* ***cotruncation*** *functor $c^n : (\omega, m)\boldsymbol{Cat} \to (\omega, m)\boldsymbol{Cat}$ sends an (ω, m)-category X to the (ω, m)-category $c^n X$ with*

$$c_n X_i = \begin{cases} X_i & i \leq n \\ \{(x, y) \in X_n \times X_n \mid d^-_{n-1} x = d^-_{n-1} y \text{ and } d^+_{n-1} x = d^+_{n-1} y\} = \partial X_{n+1} & i > n \end{cases}$$

with source and target maps $d^\varepsilon_n : c^n X_{n+1} \to c^n X_n$ acting by

$$d^-_n(x, y) = x \qquad d^+_n(x, y) = y$$

and identity maps for all higher degrees. For an n-cell $x \in X_n$ the identity $n + 1$-cell is given by the pair $(x, x) \in (c^n X)_{n+1}$. The $n + 1$ inverse of an $n + 1$-cell (x, y) is given by (y, x).

This (ω, m)-category is analogous to the coskeleton of a simplicial set. It has a unique $n+1$-cell for every pair of parallel n-cells just as the $n + 1$-coskeleton of a simplicial set has unique $n + 1$-simplices with a given boundary.

Proposition 4.7. *Let $n \geq 0$. The n-truncation functor is left adjoint to the n-cotruncation functor*

$$\tau^n \dashv c^n$$

Proof. We will construct the unit and counit ω-functors and show they satisfy the triangle identities. First observe that since $c^n X$ is identical to X below degree $n + 1$, $\tau^n(c^n X) = \tau^n X$. So we define a counit ω-functor

$$\varepsilon_X : \tau^n(c^n X) \hookrightarrow X$$

by inclusion. An (ω, m)-category is an (n, m)-category if and only if this counit ω-functor is an isomorphism.

Similarly, since $\tau^n X$ is identical to X below degree $n + 1$ and the construction of $c^n X$ depends only on the n-truncation of X we define a unit

$$\eta_X : X \to c^n(\tau^n X)$$

by the identity below degree $n + 1$ and for $i > n$

$$(\varepsilon_X)_i(x) = (d^-_n x, d^+_n x) \in c^n(\tau^n X)_i$$

using the fact that $d^-_{n-1} d^-_n x = d^-_{n-1} d^+_n x$ and $d^+_{n-1} d^-_n x = d^+_{n-1} d^+_n x$ for $n + 1$-cells of X. We will say that an (ω, m)-category is n-**cotruncated** if this counit ω-functor is an isomorphism.

Finally, it is clear that

$$\tau^n(c^n(\tau^n X)) = \tau^n(X) \qquad c^n(\tau^n(c^n X)) = c^n(X)$$

as both constructions only depend on the cells below degree $n+1$ of their input and also do not change these cells in their output. Hence the triangle identities hold and this defines an adjunction $\tau^n \dashv c^n$.

$\square$

The following result shows that cotruncation and truncation are much more similar than are the skeleton and coskeleton functors for simplicial sets.

Proposition 4.8. *Let X be an (ω, m)-category. If X is an (n, m)-category then it is $n+1$-cotruncated.*

Proof. We will show that the unit ω-functor $\eta_X : X \to c^{n+1}(\tau^{n+1}(X))$ is an isomorphism. First since X is an (n, m)-category $X \cong \tau^n(X)$ so the $n+1$-cells of X are all identities. Hence by Definition 4.6 the $n+1$-cotruncation $c^{n+1}(\tau^{n+1}(X))$ has m-cells for $m > n+1$ given by pairs $(x, y) \in \partial X_{n+2}$ where $x, y \in X_{n+1}$ and

$$d_n^- x = d_n^- y \qquad d_n^+ x = d_n^+ y$$

But x and y must be identity $n+1$-cells since X is an (n, m)-category, so this implies $x = y = 1_{d_n^- x}$. Thus the ω-functor η_X is an isomorphism and X is $n+1$-cotruncated. $\square$

Using the unit ω-functor of this adjunction we make the following definition., which will help with the notation in later constructions. This notation is not standard to other presentations of this material, however, the notion of restricting a category to a subset of its objects is certainly a well-known construction.

Definition 4.9. *Let X be an (ω, m)-category and let $e \in X_0$ be a 0-cell of X. For $n \geq 0$ define $R_e^n X$ by the pullback*

$$\begin{tikzcd}
R_e^n X \arrow[r, hook, "r_{e,X}^n"] \arrow[d] & X \arrow[d, "\eta_X"] \\
I^0 \arrow[r, hook, "e"'] & c^n(\tau^n X)
\end{tikzcd}$$

*We say an (ω, m)-category is n-**reduced** when $r_{e,X}^n : R_e^n X \hookrightarrow X$ is an isomorphism.*

The (ω, m)-category $R_e^n X$ contains only those i-cells $x \in X_i$ such that $d_n^- x = 1_e = d_n^+ x$, so in particular it contains a unique i-cell 1_e for all $0 \leq i \leq n$. The full subcategory of $(\omega, m)\mathbf{Cat}$ on the n-reduced ω-groupoids will be denoted $(\omega, m)\mathbf{Cat}_{>n}$ or by $\omega\mathbf{Gpd}_{>0}$ when $m = 0$. This construction R^n determines a functor

$$R^n : D^0/(\omega, m)\mathbf{Cat} \to (\omega, m)\mathbf{Cat}_{>n}$$

from the coslice category of $(\omega, m)\mathbf{Cat}$ under D^0 to the category of n-reduced (ω, m)-categories by sending an ω-functor $f : X \to Y$ to the ω-functor $R_e^n f : R_e^n X \to R_{f(e)}^n Y$ determined by the universal property of the pullback. This functor has a left adjoint given by the inclusion of n-reduced (ω, m)-categories into $(\omega, m)\mathbf{Cat}$ with the unique choice of ω-functor $D^0 \to X$. The counit natural transformation of this adjunction is the map

$$r_{e,X}^n : R_e^n X \hookrightarrow X$$

from the pullback in Definition 4.9 above.

4.2 Homotopy Groups and Weak Equivalences of ω-Groupoids

The model structure on ω**Gpd** is based on the homotopy groups of an ω-groupoid. These were originally defined for crossed complexes in [BH81a]. Here we present the definition of homotopy groups for ω-groupoids given in [AM11], which is analogous to that taken for the homotopy theory of Kan complexes, as described in [May67] Section 1. We start by defining a homotopy relation on n-cells of an ω-groupoid and use this to define homotopy groups $\pi_n(X)$ for all $n \geq 0$. When we see the definition of the model structure on ω**Gpd** in Section 4.4 a weak equivalence will be an ω-functor that induces an isomorphism on homotopy groups.

To begin, we say that two n-cells $x, y \in X_n$ for $n > 0$ of an ω-groupoid are **parallel** when

$$d^-_{n-1}x = d^-_{n-1}y \qquad d^+_{n-1}x = d^+_{n-1}y$$

We extend this definition to 0-cells by saying that all 0-cells are parallel. The condition of being parallel for $n \geq 0$, therefore, is a necessary condition for two n-cells to be the source and target of an $n+1$-cell of X. If this is the case then we say that the parallel n-cells are n-homotopic.

Definition 4.10. *Let $n \geq 0$ and $x, y \in X_n$ be a parallel pair of n-cells of an ω-groupoid X. These cells are **n-homotopic** if there exists an $n+1$-cell $z \in X_{n+1}$ such that $d^-_n z = x$ and $d^+_n z = y$.*

Homotopy of n-cells defines an equivalence relation $\sim$ on X_n, the set of n-cells of an ω-groupoid X; the existence of identity $n+1$-cells gives symmetry, reflexivity comes from the existence of k^{n+1} inverses of $n+1$-cells and $n+1$-composition gives transitivity. Let $\bar{X}_n$ denote the set of homotopy classes of n-cells of X with a map of sets

$$\begin{array}{rcl} \pi_n : X_n & \to & \bar{X}_n \\ x & \mapsto & [x] \end{array}$$

sending an n-cell x to its corresponding class $[x]$ under the homotopy relation for n-cells.

For a pair of parallel $n-1$-cells $u, v \in X_{n-1}$ denote the set of homotopy classes of n-cells with source u and target v by

$$\pi_n(X, u, v)$$

Since n-cells can be homotopic only if they are parallel, for $n > 0$ the $n-1$-source and target maps are well-defined on homotopy classes, in particular for $\varepsilon \in \{-, +\}$

$$d^\varepsilon_{n-1}([x]) = d^\varepsilon_{n-1}x$$

Furthermore, if $0 \leq i < n$ and $d^+_i x = d^-_i y$ for $x, y \in X_n$ the $*_i$-composite of n-cells given by

$$[x] *_i [y] = [x *_i y]$$

is well-defined, as any homotopy $n+1$-cell z with $d^-_n z = x$ and $d^+_n z = x'$ determines a homotopy $z *_i 1_y : x *_i y \to x' *_i y$ and similarly for homotopies of y. Hence we can make the following definition.

Definition 4.11. *Let X be an ω-category. For $n \geq 0$ the **smart n-truncation** of X is the n-category $\bar{\tau}^n(X)$ with cells*

$$\bar{\tau}^n_i = \begin{cases} X_i & 0 \leq i < n \\ \bar{X}_n & i \geq n \end{cases}$$

This defines a functor $\bar{\tau}^n : \omega\textbf{Cat} \to \textbf{nCat}$ that is left adjoint to the inclusion $\textbf{nCat} \to \omega\textbf{Cat}$.

The unit of this adjunction is the ω-functor $\eta_X : X \to \bar{\tau}^n(X)$ which is the identity below degree n, is equal to $\pi_n : X_n \to \bar{X}_n$ in degree n and is given by

$$\pi_n \circ d_n^- : X_m \to \bar{X}_n$$

in degree $m > n$. We call this the smart truncation after the same functor in [AM14] where it is called the *tronqué intelligent*. We will see that while the ordinary truncation of Definition 4.5 preserves homotopy data from an ω-groupoid up to degree $n-1$, while the smart truncation preserves homotopy information in degree n as well.

This homotopy information is the homotopy groups of an ω-groupoid, which we will now define. These groups are defined using the structure of the n-groupoid $\bar{\tau}^n(X)$. In particular, for all $n \geq 1$ there is the structure of a groupoid on $\bar{X}_n$ with objects the $n-1$-cells and composition given by $*_{n-1}$. We record this construction in the next definition.

Definition 4.12. *We denote by $\varpi_n(X)$ the 1-groupoid*

$$\begin{tikzcd} \bar{X}_n \arrow[r, shift left, "d_{n-1}^-"] \arrow[r, shift right, "d_{n-1}^+"'] & X_{n-1} \end{tikzcd}$$

with objects the $n-1$-cells of X and morphisms the homotopy classes of n-cells of X.

For a pair of parallel $n-1$-cells $u, v \in X_{n-1}$ viewed as objects of $\varpi_n(X)$ the set of morphisms with source u and target v in the groupoid $\varpi_n(X)$ is

$$\mathrm{Hom}_{\varpi_n(X)}(u, v) = \pi_n(X, u, v)$$

While X is an ω-groupoid, $\varpi_n(X)$ is only a 1-groupoid, since all compositions and cells below degree $n-1$ are forgotten, except in the form of identity cells in X_{n-1} and $\bar{X}_n$. When $n = 1$ this construction only forgets higher cells and agrees with the smart 1-truncation

$$\bar{\tau}^1(X) = \varpi_1(X)$$

The homotopy groups of an ω-groupoid are defined using these 1-groupoids $\varpi_n(X)$ of homotopy classes.

Definition 4.13. *Let X be an ω-groupoid and $x \in X_0$ be a 0-cell of X. The* ***set of connected components*** *of X is given by*

$$\pi_0(X) = \bar{X}_0$$

For $n \geq 1$ the n^{th} ***homotopy group of*** *X* ***at*** *x is the group of automorphisms of the $n-1$-cell $1_x \in X_{n-1}$ viewed as an object of $\varpi_n(X)$*

$$\pi_n(X, x) = \pi_n(X, 1_x, 1_x) = \mathrm{Hom}_{\varpi_n(X)}(1_x, 1_x)$$

We will say an ω-groupoid X is **connected** when $\pi_0(X) = *$. In this case, for all $x, y \in X_0$ there exists a 1-cell $a \in X_1$ with $d_0^- a = x$ and $d_0^+ a = y$. Given a 0-cell $x \in X_0$, the preimage of $[x] \in \pi_0(X)$ is the subset of the 0-cells given by

$$\pi_0^{-1}([x]) = \{y \in X_0 \mid \exists\, a : y \to x \in X_1\}$$

The **connected component** of x is the sub-ω-groupoid whose n-cells all have their 0-source (and hence 0-target) belonging to $\pi_0^{-1}(X)$. We denote the connected component of a 0-cell $x \in X_0$ by X_x and observe that it is the pullback of the counit of the cotruncation-truncation adjunction $c^0 \dashv \tau^0$ along the inclusion of the set of 0-cells connected to x by a 1-cell given above.

$$\begin{tikzcd}
X_x \arrow[r, hook] \arrow[d] & X \arrow[d, "\varepsilon_X"] \\
\pi_0^{-1}([x]) \arrow[r, hook] & c^0\tau^0(X)
\end{tikzcd}$$

All n-cells belong to a connected component and it is clear that n-cells are m-composable only if they belong to the same connected component. Hence an ω-groupoid is the disjoint union of its connected components

$$X = \bigsqcup_{[x]\in\pi_0(X)} X_x$$

Hence it is often easy to restrict to a connected ω-groupoid X without loss of generality by working with each connected component of an ω-groupoid separately.

As stated in Definition 4.13, the homotopy groups $\pi_n(X, x)$ for $n \geq 1$ depend on a choice of 0-cell $x \in X_0$ as basepoint. However, all choices of basepoint within the same connected component give isomorphic groups.

Proposition 4.14. *Let X be an ω-groupoid. If there exists a 1-cell $a : x \to y$ in X then*

$$\pi_n(X, x) \cong \pi_n(X, y)$$

Proof. We define an isomorphism $\gamma_a : \pi_1(X, x) \cong \pi_1(X, y)$ determined by a. This map is given by

$$\gamma_a : [x] \in \pi_n(X, x) \mapsto [k^1 1_a *_0 x *_0 1_a] \in \pi_n(X, y)$$

This is well-defined as if $x \sim x'$ by an $n+1$-cell z then $k^1 1_a *_0 z *_0 1_a$ gives a homotopy $k^1 1_a *_0 x *_0 1_a \sim k^1 1_a *_0 x' *_0 1_a$. This is an isomorphism by the invertibility of a. □

This result allows us to define the basepoint independent homotopy groups $\pi_n(X)$ of a connected ω-groupoid, as all choices of 0-cell basepoint for $\pi_n(X, x)$ will give isomorphic groups. We can therefore drop the reference to a 0-cell when referring to homotopy groups $\pi_n(X)$ of a connected ω-groupoid.

Recall the unit ω-functor $\varpi : X \to \bar{\tau}^1(X) = \varpi_1(X)$ of the 1-truncation of X. If X is connected, the previous result shows that all automorphism groups of $\varpi_1(X)$ are isomorphic. So any choice of 0-cell $x \in X_0$ determines an isomorphic group $\pi_1(X, x)$ which is a subgroup of the groupoid $\varpi_1(X)$.

For a given 0-cell x we can choose, following the method of [Kan58] §9, a maximal tree in X_1, consisting of 1-cells $\gamma_y : x \to y$ for each $y \in X_0$. Collapsing this tree defines a functor $\varpi_1(X) \to \pi_1(X, x)$ that is a retract for the inclusion of $\pi_1(X, x)$. When X is 0-reduced there is a unique choice of such a retract and we have a unique natural ω-functor

$$u_X : X \to \pi_1(X) \tag{70}$$

When X is not 0-reduced different choices of maximal tree give homotopic retracts, so we can still obtain a retract $X \to \pi_1(X)$ by compose the ω-functor $\varpi : X \to \varpi_1(X)$ with a choice of collapse. We refer to $\pi_1(X)$ for a connected ω-groupoid X as the **fundamental group of** X.

For $n \geq 1$ we say an ω-groupoid X is **n-connected** when it is connected and $\pi_m(X) = 0$ for all $1 \leq m \leq n$. We say an ω-groupoid X is **weakly contractible** when it is connected and n-connected for all $n \geq 1$, equivalently $\pi_n(X) = 0$ for all $n \geq 0$.

An ω-functor $f : X \to Y$ preserves the property of two parallel n-cells being homotopic. If $x \sim x'$ in X_n then an $n+1$-cell z witnessing this homotopy is sent by f to an $n+1$-cell of Y witnessing the homotopy $f(x) \sim f(x')$. Hence an ω-functor determines maps

$$f^* : \pi_0(X) \to \pi_0(Y)$$

and for $n > 0$ and $x \in X_0$

$$f^* : \pi_n(X, u, v) \to \pi_n(Y, f(u), f(v))$$

Since homotopy groups of X and Y are automorphism groups of $\varpi_n(X)$ and $\varpi_n(Y)$ for $n \geq 1$ their group structures are determined by ω-groupoid structures of X and Y. Since f preserves the ω-groupoid structure of X the map f^* is a group homomorphism when $n \geq 1$.

We can now define the weak equivalences for the model structure on ω**Gpd**. This definition originally comes from [BH81a] but is presented here in the context of ω-groupoids as in [AM11].

Definition 4.15. *An ω-functor $f : X \to Y$ is a weak equivalence if induces a bijection $f^* : \pi_0(X) \to \pi_0(Y)$ on connected components and group isomorphisms*

$$f^* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

for all $n > 0$ and all 0-cells $x \in X_0$.

When X is n-connected for $n \geq 0$ we observe that the ω-functor $R^n_e(X) \hookrightarrow X$ induces an isomorphism

$$\pi_m(R^n_e(X), e) \to \pi_m(X, e)$$

for all $e \in X_0$ and $m > n$. Hence this ω-functor is a weak equivalence if and only if X is n-connected.

These weak equivalences form part of the definition of a model structure on ω**Gpd** which we will describe in Section 4.4. Finally, we will record the following useful equivalent characterization of weak equivalences of ω-groupoids from [AM11].

Proposition 4.16 ([AM11] Prop. 1.7)**.** *Let $f : X \to Y$ be an ω-functor. The following are equivalent.*

1. *f is a weak equivalence*

2. *$f^* : \varpi_1(X) \to \varpi_1(Y)$ is an equivalence of categories and for $n > 1$ and all pairs of parallel $n-1$-cells $u, v \in X_{n-1}$ the map*

   $$f^* : \pi_n(X, u, v) \to \pi_n(Y, f(u), f(v))$$

   is a bijection.

4.3 Augmented Directed Complexes and the Gray Tensor Product

In this section we will describe the Gray tensor product for ω-groupoids, which endows the category ω**Gpd** with a symmetric monoidal structure. This tensor product was introduced for ω-categories in [AAS93] as a generalization to ω-categories of the tensor product of 2-categories defined by Gray in [Gra74]. The Gray tensor product for ω-categories can be reflected to the subcategory of ω-groupoids using Day's reflection theorem, where it becomes symmetric and interacts well with the model structure that we will define in Section 4.4. To describe the Gray tensor product of ω-groupoids we will start by describing the Gray tensor product of ω-categories.

The description we give is that of [Ste04] based on the theory of augmented directed complexes as models of certain ω-categories. These augmented directed complexes are chain complexes of abelian groups with additional structure. The Gray tensor product is then determined on all ω-categories by the chain complex tensor product of these augmented directed complexes. We note that Brown and Higgins in [BH87] defined a tensor product for crossed complexes, which by the equivalence [BH81b] of ω-groupoids with crossed complexes also defines a symmetric monoidal structure on ω**Gpd**. It seems likely that this tensor product is the same as the Gray tensor product, however we do not investigate this here.

We will start by detailing Steiner's theory of augmented directed complexes from [Ste04] and their relation to ω-categories and the Gray tensor product.

4.3.1 Augmented Directed Complexes

We are recording here the parts of the theory of augmented directed complexes given in [Ste04] that we will need. These structures are chain complexes of abelian groups with extra structure that give models for some ω-categories. In particular, we describe Steiner's adjunction

$$\begin{tikzcd}\mathbf{ADC} \arrow[r, bend left, "\nu"] \arrow[r, phantom, "\top"] & \omega\mathbf{Cat} \arrow[l, bend left, "\lambda"]\end{tikzcd}$$

that shows how to construct an ω-category out of an ADC. We will also give Steiner's conditions defining a subcategory of the category **ADC** of ADCs on which this adjunction becomes an inclusion of a reflective subcategory of ω**Cat**. The ADCs in this subcategory are called the strong Steiner complexes and we will use them in the next section to define the Gray tensor product. Finally, we will give a useful result in Lemma 4.33 on colimits of strong Steiner complexes that will be used in Section 4.7 to characterize ω-groupoids arising from simplicial sets.

We start by presenting Steiner's definition from [Ste04] and some important examples.

Definition 4.17. *A non-negative chain complex K is **augmented** when there exists a map of chain complexes $\varepsilon : K \to \mathbb{Z}$ to the chain complex that consists of only the free abelian group on one generator $\mathbb{Z}$ in degree 0. An **augmented directed chain complex** or **ADC** consists of the data of an augmented chain complex (K, ε) together with submonoids $K_n^* \subseteq K_n$ of the chain groups for all $n \in \mathbb{N}$. A morphism of ADCs is an augmentation-preserving chain map $f : K \to L$ that also respects the submonoid structure, so that $f(K_n^*) \subseteq L_n^*$ for all $n \in \mathbb{N}$.*

Example 4.18. Consider the non-negative chain complex $\mathbb{Z}[\Delta^n]$ freely generated by the simplicial n-simplex Δ^n. This chain complex is augmented by the map $\varepsilon : \mathbb{Z}[\Delta^n] \to \mathbb{Z}$ sending all generators

$[i] \in \mathbb{Z}[\Delta^n]_0$ for $0 \leq i \leq n$ to $1 \in \mathbb{Z}$. If we take $\mathbb{Z}[\Delta^n]^*_m$ to be the free abelian monoid generated by the generators $[\theta] \in \mathbb{Z}[\Delta^n]_i$ for all injective maps $\theta : [m] \to [n]$ in Δ then $\mathbb{Z}[\Delta^n]$ is an ADC.

Example 4.19. The n-globe $G[n]$ is the ADC

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \end{tikzcd}$$

with $G[n]_n = \mathbb{Z}$, $G[n]_m = \mathbb{Z} \oplus \mathbb{Z}$ for $m < n$ and 0 elsewhere. The boundary maps are given by

$$\begin{array}{rcl} \partial : \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} \\ 1 & \mapsto & (1, -1) \end{array} \qquad \begin{array}{rcl} \partial : \mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} \\ (1, 0) & \mapsto & (1, -1) \\ (0, 1) & \mapsto & (1, -1) \end{array}$$

The distinguished submonoids are $G[n]^*_n = \mathbb{N} \subseteq \mathbb{Z}$ and $G[n]_m = \mathbb{N} \oplus \mathbb{N} \subseteq \mathbb{Z} \oplus \mathbb{Z}$ and the augmentation sends $(1, 0)$ and $(0, 1)$ to $1 \in \mathbb{Z}$.

The connection between ω-categories and ADCs can be motivated by the linearization of an ω-category. This is a construction that assigns to an ω-category X a chain complex λX whose n^{th} chain group is the abelian group generated by chains $[x]$ for all n-cells $x \in X_n$ with the relations

$$[x *_m y] = [x] + [y]$$

for m-composable pairs $x, y \in X_n$. We will study this construction in more detail in Section 4.6 but for now we observe that $\lambda X_0 = \mathbb{Z}[X_0]$ and the map

$$\varepsilon : \lambda X_0 \to \mathbb{Z}$$

given $\varepsilon([x]) = 1$ and the submonoids

$$\lambda X^*_n = \langle [x] \mid x \in X_n \rangle \leq \lambda X_n$$

make λX an ADC. Definition 4.17 therefore captures two properties of the linearization an ω-category X that are the result of the ω-category structure on X.

Steiner shows in [Ste04] that there is a right adjoint $\nu : \mathbf{ADC} \to \omega\mathbf{Cat}$ to the linearization functor. To define this functor we first define the abelian group μK assembled from double sequences of chains.

Definition 4.20. *Let (K, ε) be an ADC. Let μK be the abelian group of double sequences*

$$x = \begin{pmatrix} x^-_0, x^-_1, \cdots, x^-_{n-1}, x, 0, 0, \cdots \\ x^+_0, x^+_1, \cdots, x^+_{n-1}, x, 0, 0, \cdots \end{pmatrix}$$

such that

- $x^-_i, x^+_i \in K_i$ *for all* $i \in \mathbb{N}$
- *There exists* $n \in \mathbb{N}$ *such that* $x^-_n = x = x^+_n$ *and* $x^+_i = x^-_i = 0$ *for all* $i > n$
- $\partial x^+_i = \partial x^-_i = x^+_{i-1} - x^-_{i-1}$ *for all* $i \geq 1$

Define a filtration of μK by the subgroups

$$\mu K_n = \{x \in \mu K \mid x_i^\xi = 0 \text{ for all } i \geq n+1 \text{ and } \xi \in \{-,+\}\}$$

for $n \geq 0$ and $\mu K_{-1} = 0$ the trivial group. Let $0 \geq r$. We say that elements $x, y \in \mu K$ are ***congruent mod*** *$(\mu K)_r$ when they are equal in the quotient group $\mu K/\mu K_r$.*

From the definition it is clear that $x \equiv y \mod (\mu K)_r$ when $x_i^\xi = y_i^\xi$ for all $r+1 \leq i$ and all $\xi \in \{-,+\}$. This extends to the case $r = -1$ thanks to our definition of $\mu K_{-1} = 0$, so that two elements are congruent mod μK_{-1} if and only if they are equal. The sets of cells of the ω-category νK are subsets of the underlying set of elements of the abelian group μK.

Definition 4.21. *Let (K, ε) be an ADC. Define the ω-category νK with n-cells νK_n given by double sequences $x \in \mu K_n$ such that*

- $x_i^-, x_i^+ \in K_i^*$ *for all* $i \in \mathbb{N}$
- $\varepsilon(x_0^-) = \varepsilon(x_0^+) = 1$

Given $x \in \nu K_n$, define for $0 \leq i \leq n-1$ and $\xi \in \{-,+\}$

$$d_i^\xi = \begin{pmatrix} x_0^-, x_1^-, \cdots, x_{i-1}^-, x_i^\xi, 0, 0, \cdots \\ x_0^+, x_1^+, \cdots, x_{i-1}^+, x_i^\xi, 0, 0, \cdots \end{pmatrix}$$

If $x, y \in \nu K_n$ and $d_i^+ x = d_i^- y$ for some $0 \leq i < n$ then define

$$x *_i y = \begin{pmatrix} x_0^-, x_1^-, \cdots, x_{i-1}^-, x_i^-, x_{i+1}^- + y_{i+1}^-, \cdots, x_{n-1}^- + y_{n-1}^-, x+y, 0, \cdots \\ x_0^+, x_1^+, \cdots, x_{i-1}^+, x_i^+, x_{i+1}^+ + y_{i+1}^+, \cdots, x_{n-1}^+ + y_{n-1}^+, x+y, 0, \cdots \end{pmatrix}$$

Since n-cells of νK are also elements of μK_n we can still speak of congruences mod $(\mu K)_r$ for cells of νK. However, νK is not an abelian subgroup of μK.

Steiner shows that this construction defines a functor that is right adjoint to λ and that by requiring some additional conditions on an ADC this adjunction gives an adjoint equivalence between subcategories of **ADC** and ω**Cat**. The additional conditions needed on ADCs are the following.

Definition 4.22. *A* ***basis*** *of an ADC is a graded set $B_n \subseteq K_n$ such that each abelian group K_n is free with basis B_n and each K_n^* is the free monoid generated by B_n. The* ***degree*** *of a basis element $b \in B_n$ is n.*

When K is an ADC with a basis there is a partial order on the abelian group K_n defined by $x \leq y$ when $y - x \in K_n^*$. A basis is required to assure anti-symmetry of the partial order. A free abelian monoid has no invertible elements except the identity, so if $y - x$ and $x - y$ belong to K_n^* then $x = y$. This partial order gives K_n a lattice structure. The greatest lower bound $x \wedge y$ of $x, y \in K_n$ is the sum of all terms common to both words as positive terms in x and y and all terms occurring in either word as negative terms. Hence, if $x, y \in K_n^*$ then $x \wedge y = 0$ if and only if no terms are the same between the two words.

When an ADC has a basis we can require the following additional conditions to make what is called a strong Steiner complex in [AM20].

Definition 4.23. *A basis B for an ADC K is **unital** if $\varepsilon(\langle b\rangle_0^\xi) = 1$ for all $b \in B_n$ and all $n \in \mathbb{N}$.*

Definition 4.24. *A basis B for an ADC K is **strongly loop-free** if there is a partial order $<_N$ on the set of all basis elements of all degrees $\sqcup_{n\in\mathbb{N}} B_n$ such that for $a \in B_n$ and $b \in B_m$ it is the case that $a <_N b$ when either of the following hold*

- $m = n - 1$ *and* $a \leq \partial^- b$
- $n = m - 1$ *and* $\partial^+ a \geq b$

Definition 4.25. *An ADC that has a basis that is strongly loop-free and unital is a **strong Steiner complex**. The category **SSt** is the full subcategory of **ADC** on the strong Steiner complexes.*

Example 4.26. The n-globe $G[n]$ is a strong Steiner complex. Its basis elements are $(1,0),(0,1) \in \mathbb{Z}\oplus\mathbb{Z} = G[n]_m$ for $m < n$ and $1 \in \mathbb{Z} = G[n]_n$. This basis is unital and strongly loop free under the total order $<_N$ on all basis elements that has

$$G[n]_{m-1}(0,1) <_N (0,1) \in G[n]_m <_N 1 \in G[n]_n <_N (1,0) \in G[n]_m <_N (1,0) \in G[n]_{m-1}$$

for $0 < m < n$.

Example 4.27. The ADC $\mathbb{Z}[\Delta^n]$ is also a strong Steiner complex, with basis given by $[\theta] \in \mathbb{Z}[\Delta^n]_m$ for all injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ in Δ. There is a total order on injective non-decreasing maps in Δ^n that is described in [Ste04] Example 3.8. In Appendix B we will describe the total order in detail in Propositions B.3, B.4, and B.5.

One of the main results of [Ste04] is that strong Steiner complexes give a fully faithful model in **ADC** of a subcategory of ω**Cat**.

Theorem 4.28 ([Ste04] Theorem 5.11)**.** *The right adjoint ν : **ADC** $\to \omega$**Cat** to λ is fully faithful when restricted to the full subcategory of strong Steiner complexes.*

ADCs are simpler to manipulate that ω-categories, so this gives an important tool for working with some ω-categories, namely those in the image under ν of the strong Steiner complexes. We have already seen that these include the globes and ω-categories corresponding to the standard simplicial simplices. Steiner shows that the strong Steiner complexes determine a dense subcategory of ω**Cat** in the following sense.

Theorem 4.29 ([Ste04] Theorem 7.1)**.** *Every ω-category is the colimit of a small diagram of ω-categories in the image under ν of **SSt**.*

This theorem follows easily from the recognition of certain strong Steiner complexes corresponding to important shapes of ω-categories. The n-globes D^n belong to the image of ν as $\nu G[n]$ is isomorphic to D^n by the ω-functor corresponding to the unique n-cell of $\nu G[n]$

$$\begin{pmatrix} (0,1) & (0,1) & \cdots & (0,1) & 1 & 0 & \cdots \\ (1,0) & (1,0) & \cdots & (1,0) & 1 & 0 & \cdots \end{pmatrix} \in \nu G[n]_n$$

There are similar ADCs giving rise to ω-categories classifying composable pairs just as ω-functors $D^n \to X$ classify n-cells and maps between ADCs that encode the operations. Hence the structure of an arbitrary ω-category can be expressed by ω-functors whose source lies in the image of ν in $\omega\mathbf{Cat}$. The colimit in $\omega\mathbf{Cat}$ of the diagram of these ω-functors is therefore isomorphic to X.

Since Steiner complexes are a fully-faithful subcategory of $\omega\mathbf{Cat}$ we can calculate some colimits of ω-categories as colimits of ADCs. In [AM20] 3.1 colimits are defined for $\mathbf{ADC}$ by taking the colimit of the underlying chain complexes and submonoids of abelian groups of chains. Hence the functor $\mathbb{Z}[-] : \mathbf{sSet} \to \mathbf{ADC}$ preserves colimits when $\mathbb{Z}[L]$ for a simplicial set L is taken as the ADC with monoids $\mathbb{Z}[L]^*_n = \mathbb{N}[L_n] \subseteq \mathbb{Z}[L]_n$ and augmentation $\varepsilon : \mathbb{Z}[L]_0 \to \mathbb{Z}$ sending $[x] \in \mathbb{Z}[L_0]$ to 1 for all $x \in L_0$. The functor ν does not necessarily preserve all colimits, however. In Theorem 3.8 of [AM20] the authors exploit the fully-faithful embedding of strong Steiner complexes to show that ν preserves colimits of what they call strong Steiner systems. We record their description of these diagrams and their colimits here. First we must recall one more piece of terminology from [Ste04].

Definition 4.30. *Let K be an ADC with a basis. For $b \in B_n$ we define the* ***atom*** *$\langle b\rangle \in \nu K_n$ to be the double sequence*

$$\langle b\rangle = \begin{pmatrix} \langle b\rangle^-_0, \langle b\rangle^-_1, \cdots, b, 0, \cdots \\ \langle b\rangle^+_0, \langle b\rangle^+_1, \cdots, b, 0, \cdots \end{pmatrix}$$

where $\langle b\rangle^\xi_i \in K^*_i$ for $\xi \in \{-,+\}$ and $i \in \mathbb{N}$ is the word in the abelian monoid K^*_i defined inductively by

$$\langle b\rangle^\xi_i = \begin{cases} 0 & i > n \\ b & i = n \\ \partial^\xi \langle b\rangle^\xi_{i+1} & i < n \end{cases}$$

The strong Steiner systems of [AM20] are diagrams that preserve the features of strong Steiner complexes.

Definition 4.31. *A map $f : C \to D$ of ADCs is* ***rigid*** *when $\nu(f) : \nu(C) \to \nu(D)$ sends atoms of $\nu(C)$ to atoms of $\nu(D)$. A* ***strong Steiner system*** *is a diagram of strong Steiner complexes such that all maps between them are rigid and the colimit of the diagram is a strong Steiner complex with all ADC maps in the colimiting cocone rigid.*

Theorem 4.32 ([AM20] Theorem 3.8)**.** *Colimits of strong Steiner systems are preserved by ν.*

The following lemma that makes use of this result will be helpful in realizing simplicial sets as ω-groupoids in later sections. Recall that a simplicial set X is canonically the colimit of its category of simplices $(\Delta \downarrow X)$, which is the comma category for the functors $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ and the Yoneda embedding $\Delta^{\mathrm{op}} \to \mathbf{Set}$. For some simplicial sets this colimit is preserved by ν.

Lemma 4.33. *Let X be a simplicial set. If every non-degenerate simplex of X corresponds to an injective map $\Delta^n \hookrightarrow X$ and $\mathbb{Z}[X]$ with the canonical basis is a strong Steiner complex then*

$$\nu\mathbb{Z}[X] = \mathrm{colim}_{\Delta\downarrow X}\, \nu\mathbb{Z}[\Delta^n]$$

Proof. The simplicial set X is the colimit of the diagram

$$\begin{aligned} \Delta X : (\Delta \downarrow X) &\rightarrow \mathbf{sSet} \\ x : \Delta^n \to X &\mapsto \Delta^n \end{aligned}$$

where $(\Delta \downarrow X)$ is the category of simplices of X. There is a full subcategory of the category of simplices $(\Delta \downarrow X)$ whose objects are simplicial set maps $x : \Delta^n \to X$ corresponding to non-degenerate simplices $x \in X_n$. A morphism between objects of $(\Delta \downarrow X)^\Delta$

$$\begin{tikzcd} \Delta^n \arrow[rr, hook, "\theta"] \arrow[dr, "x"'] & & \Delta^m \arrow[dl, "\theta^*(x)"] \\ & X & \end{tikzcd}$$

is necessarily injective, as the source is a non-degenerate simplex and the target simplex must be non-degenerate as well. We denote this category by $(\Delta \downarrow X)^\Delta$.

An object $x : \Delta^n \to X$ of $(\Delta \downarrow X)$ corresponds to an n-simplex of X, which can be written uniquely up to reordering of degeneracies with the simplicial identities as $x = s_{m_1} s_{m_2} \cdots s_{m_j}(x_{ND})$ for a non-degenerate $n - j$-simplex $x_{ND} \in X_{n-j}$. In the category this $(\Delta \downarrow X)$ means that there is a unique map

$$\begin{tikzcd} \Delta^n \arrow[rr, "s^{m_j} \ldots s^{m_1}"] \arrow[dr, "x"'] & & \Delta^{n-j} \arrow[dl, "x_{ND}"] \\ & X & \end{tikzcd}$$

Since any map in Δ factors uniquely up to rearrangement via simplicial identities as a sequence of codegeneracy maps followed by a sequence of coface maps this determines a functor $ND : (X \downarrow \Delta) \to (X \downarrow \Delta)^\Delta$ sending objects and maps of $(X \downarrow \Delta)$ to their non-degenerate part.

The faces of a non-degenerate n-simplex $x : \Delta^n \hookrightarrow X$ must also correspond to injective simplicial set maps, so they are non-degenerate too. This property makes the functor ND a left adjoint for the inclusion of $(\Delta \downarrow X)^\Delta$ into $(\Delta \downarrow X)$. Let $y : \Delta^n \hookrightarrow X$ be an object in the image of the inclusion of $(\Delta \downarrow X)^\Delta$ and $\theta : x \to y$ be a map in $(\Delta \downarrow X)$. The corresponding map $\theta : [m] \to [n]$ in Δ factors as a sequence of codegeneracy maps followed by a sequence of coface maps as follows

$$\begin{tikzcd} \Delta^m \arrow[r, "s^{m_j} \ldots s^{m_1}"] \arrow[dr, "x"'] & \Delta^{m-j} \arrow[r, hook, "d^{l_{n-m+j}} \ldots d^{l_1}"] \arrow[d, "x_{ND}"'] & \Delta^n \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

where $x_{ND} = d_{l-1} \cdots d_{l_{n-m+j}}(y)$ is a non-degenerate $m - j$-simplex by the property of X. So $ND(x) = x_{ND}$ belongs to the image of the inclusion functor.

As a right adjoint the inclusion functor is therefore a final functor. Hence $\partial \Delta^n$ is the colimit of the restricted diagram

$$\Delta' X : (\Delta \downarrow X)^\Delta \to \mathbf{sSet}$$

Now $\mathbb{Z}[-]$ is a left adjoint, so it preserves colimits and hence $\mathbb{Z}[X]$ is the colimit in **Ch** of the diagram that sends $x : \Delta^m \to X$ in $(\Delta \downarrow X)^\Delta$ to $\nu\mathbb{Z}[\Delta^n]$. By the definition of colimits in **ADC**, the ADC $\mathbb{Z}[X]$ is the colimit of the corresponding diagram of ADCs. Every ADC in this diagram is $\mathbb{Z}[\Delta^n]$ for some n so this is a diagram of strong Steiner complexes. Furthermore, the maps between ADCs correspond to injective non-increasing maps $\theta : [m] \hookrightarrow [n]$ of Δ, so they act on atoms by

$$\langle \varphi \rangle \in \nu\mathbb{Z}[\Delta^m]_l \mapsto \langle \theta \circ \varphi \rangle \in \mathbb{Z}[\Delta^n]_l$$

Hence these maps are rigid. Since X has the property that the simplicial set maps $x : \Delta^n \to X$ corresponding to non-degenerated n-simplices are injective, the corresponding maps of the colimiting cocone under the diagram $\mathbb{Z}[-] \circ \Delta' X$ in **ADC** are rigid. Finally $\mathbb{Z}[X]$ is a strong Steiner complex by assumption, so this diagram is a strong Steiner system and hence ν preserves its colimit. □

4.3.2 Gray Tensor Products of ω-Categories and Groupoids

In this section we apply the definitions of the previous section to define the Gray tensor product of ω-categories and ω-groupoids. We will also prove two results that will be useful in manipulating tensor products in later sections. In particular, we prove a result about identity cells in Gray tensor products of ω-categories in Lemma 4.36 and how the linearization functor respects tensor products with strong Steiner complexes in Proposition 4.37.

Using results quoted in the previous section that show the embedding of **SSt** in ω**Cat** by the functor ν and that strong Steiner complexes are dense in ω**Cat**, Steiner shows in [Ste04] that the Gray tensor product extends a tensor product of ADCs. This tensor product of ADCs is determined by the standard tensor product of chain complexes. For ADCs C and D

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes C_j$$

with boundary maps acting on $c \otimes d \in C_i \otimes C_j$ by

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^i c \otimes (\partial d)$$

The augmentation of $C \otimes D$ is the map

$$e_C \otimes e_D : (c \otimes d) \in (C \otimes D)_0 = C_0 \otimes D_0 \mapsto e_C(c) e_D(d) \in \mathbb{Z}$$

and the monoids of positive elements are generated by

$$(C \otimes D)^*_n = \langle c \otimes d \mid c \in C^*_i,\, d \in D^*_j \text{ for } i + j = n \rangle \subseteq \bigoplus_{i+j=n} C_i \otimes D_j$$

This tensor product makes **ADC** a non-symmetric monoidal category, with monoidal unit the ADC $\mathbb{Z}[0]$. Furthermore, the tensor product of strong Steiner complexes is a strong Steiner complex, so this defines a monoidal structure on **SSt**, and hence on its image in ω**Cat**. The fact that the image of strong Steiner complexes by ν in ω**Cat** is dense in ω**Cat** gives the following characterization of the Gray tensor product.

Theorem 4.34 ([Ste04] 7.3 and [AM20] A.15)**.** *There is a functor*

$$- \otimes - : \omega\boldsymbol{Cat} \times \omega\boldsymbol{Cat} \to \omega\boldsymbol{Cat}$$

that extends the tensor product on strong Steiner complexes and preserves colimits in each variable. This functor is unique up to isomorphism and is given by

$$X \otimes Y = \operatorname*{colim}_{\substack{\nu(C)\to X\ C\in \boldsymbol{SSt}\\ \nu(D)\to Y\ D\in \boldsymbol{SSt}}} \nu(C \otimes D)$$

This functor makes $\omega\boldsymbol{Cat}$ a biclosed monoidal category.

The unit for this monoidal structure is the terminal ω-category D^0, which has one 0-cell and no non-identity higher cells. A discrete ω-category S is simply a set of 0-cells $S_0 = S$, so it is equal to a disjoint union of copies of D^0. Hence discrete ω-categories the Gray tensor product is simply the cartesian product

$$S \otimes X = \left(\bigsqcup_{s\in S} D^0 \right) \otimes X \cong \bigsqcup_{s\in S} (D^0 \otimes X) \cong \bigsqcup_{s\in S} X \cong S \times X$$

since $- \otimes X$ is a left adjoint, so it commutes with colimits. In fact, the Gray tensor product and the cartesian product of ω-groupoids are identical in their actions on 0-cells.

Lemma 4.35. *Let X and Y be ω-categories. The set of 0-cells of the Gray tensor product $X \otimes Y$ is $X_0 \times Y_0$.*

Proof. Recall the definition of $X \otimes Y$ in Theorem 4.34 as the colimit

$$X \otimes Y = \operatorname*{colim}_{\substack{\nu(C)\to X\ C\in \mathbf{SSt}\\ \nu(D)\to Y\ D\in \mathbf{SSt}}} \nu(C \otimes D)$$

By the discussion at Definitions 4.5 and 4.6 the functor $(-)_0 : \omega\mathbf{Gpd} \to \mathbf{Set}$ that sends an ω-groupoid to its set of 0-cells is left adjoint to the indiscrete ω-groupoid functor, which sends a set to the 1-groupoid with a unique 1-cell for every pair of elements of S. Hence $(-)_0$ preserves colimits and so

$$\begin{aligned}
(X \otimes Y)_0 &= \operatorname*{colim}_{\substack{\nu(C)\to X\ C\in \mathbf{SSt}\\ \nu(D)\to Y\ D\in \mathbf{SSt}}} \nu(C \otimes D)_0 \\
&= \operatorname*{colim}_{\substack{\nu(C)\to X\ C\in \mathbf{SSt}\\ \nu(D)\to Y\ D\in \mathbf{SSt}}} C_0 \times D_0 \\
&= \operatorname*{colim}_{\nu(C)\to X\ C\in \mathbf{SSt}} \left(C_0 \times \left(\operatorname*{colim}_{\nu(D)\to X\ D\in \mathbf{SSt}} D_0 \right) \right) \\
&= \left(\operatorname*{colim}_{\nu(C)\to X\ C\in \mathbf{SSt}} C_0 \right) \times \left(\operatorname*{colim}_{\nu(D)\to X\ D\in \mathbf{SSt}} D_0 \right) \\
&= \left(\operatorname*{colim}_{\nu(C)\to X\ C\in \mathbf{SSt}} C \right)_0 \times \left(\operatorname*{colim}_{\nu(D)\to X\ D\in \mathbf{SSt}} D \right)_0 \\
&= X_0 \times Y_0
\end{aligned}$$

since **Set** is cartesian closed so products preserve colimits. □

As a first non-trivial example of a Gray tensor product we will consider $D^1 \otimes D^1$. Since the n-globes are strong Steiner complexes we have $D^1 \otimes D^1 = \nu(G[1] \otimes G[1])$. Studying this tensor product we see that it has the following form

$$\begin{tikzcd}
0 \otimes 0 \arrow[r, "0\otimes\alpha_1"] \arrow[d, "\alpha_1\otimes 0"'] \arrow[dr, phantom, "\alpha_1\otimes\alpha_1 \Swarrow"] & 0 \otimes 1 \arrow[d, "\alpha_1\otimes 1"] \\
1 \otimes 0 \arrow[r, "1\otimes\alpha_1"'] & 1 \otimes 1
\end{tikzcd} \tag{71}$$

where α_1 is the 1-cell of $\nu G[1]$ corresponding to the generator $1 \in \mathbb{Z}$ of the abelian group of 1-chains of $G[1]$ and 0 and 1 are its source and target respectively. The 2-chain $\alpha_1 \otimes \alpha_1 \in G[1] \otimes G[1]_2$ gives a unique 2-cell of the tensor product. In the cartesian product $D^1 \times D^1$ the composites of 1-cells corresponding to the top and bottom paths of the square (71) are equal

$$(\alpha_1, 1_0) *_0 (1_1, \alpha_1) = \alpha_1 *_0 \alpha_1 = (1_0, \alpha_1) *_0 (\alpha_1, 1_1)$$

while in the tensor product they are different but connected by the unique 2-cell. For $n \geq 1$ therefore, the structure of the Gray tensor product of ω-categories differs significantly from that of the cartesian product.

For $n, m \geq 0$ and cells $x \in X_n$ and $y \in Y_m$ of ω-categories X and Y by the definition of the Gray tensor product in Theorem 4.34 there is an ω-functor

$$D^n \otimes D^m \to X \otimes Y$$

This ω-functor is the leg of the colimiting cocone under the diagram over $\mathbf{SSt}/X \times \mathbf{SSt}/Y$ for the colimit defining $X \otimes Y$ corresponding to the object $(x : D^n \to X, y : D^m \to Y)$. We will denote by $x \otimes y \in (X \otimes Y)_{n+m}$ the image of the unique $n+m$-cell $\iota_n \otimes \iota_m \in (D^n \otimes D^m)_{n+m}$ in $X \otimes Y$. When one of x or y is an identity cell the following result shows that the tensor $x \otimes y$ must be as well.

Lemma 4.36. *Let C and D be strong Steiner complexes and let X and Y be ω-categories with ω-functors $f : \nu C \to X$ and $g : \nu D \to Y$. Let $\langle c \rangle \in \nu C_p$ and $\langle d \rangle \in \nu D_q$ be atoms such that at least one of $f(\langle c \rangle)$ and $g(\langle d \rangle)$ is an identity cell of X and Y respectively, say $f(\langle c \rangle) = 1_x$ for $x \in X_m$ with $m \leq p$ and $g(\langle d \rangle) = 1_y$ for $y \in Y_l$ with $l \leq q$ where it is not the case that $m = p$ and $l = q$. Then*

$$f \otimes g(\langle c \otimes d \rangle) = 1_{d^-_{m+l} f \otimes g(\langle c \otimes d \rangle)}$$

Proof. First suppose that $f(\langle c \rangle) = 1_x$. Consider the diagram in $\omega\mathbf{Cat}$

$$\begin{tikzcd}
D^{p+q} \arrow[r, "id^{p-m}"] \arrow[d, "\iota_p\otimes\iota_q"'] & D^{m+q} \arrow[d, "\iota_{p-1}\otimes\iota_q"] & \\
D^p \otimes D^q \arrow[r, "id^{p-m}\otimes 1"] \arrow[d, "\langle c\rangle\otimes\langle d\rangle"'] & D^m \otimes D^q \arrow[d, "x\otimes\langle d\rangle"] & \\
\nu C \otimes \nu D \arrow[r, "f\otimes 1"'] & X \otimes \nu D \arrow[r, "1\otimes g"'] & X \otimes Y
\end{tikzcd}$$

The lower square commutes by the definition of f and the monoidal structure of $\omega\mathbf{Cat}$. We need to show that the top square commutes as well. For this it is sufficient to show that the square

$$\begin{tikzcd}
D^{p+q} \arrow[r, "id"] \arrow[d, "\iota_p \otimes \iota_q"'] & D^{p+q-1} \arrow[d, "\iota_{p-1} \otimes \iota_q"] \\
D^p \otimes D^q \arrow[r, "id \otimes 1"] & D^{p-1} \otimes D^q
\end{tikzcd} \tag{72}$$

commutes, as the top square is made by horizontal pasting of $p - m$ copies of this square. Once the commutativity of this square is shown the result follows since the ω-functor $D^{p+q} \to X \otimes Y$ corresponding to the image of $\langle c \rangle \otimes \langle d \rangle$ under $f \otimes g$ factors through the iterated identity ω-functor $id^{p-m} : D^{p+q} \to D^{m+q}$.

Since ω-functors $D^{p+q} \to D^{p-1} \otimes D^q$ correspond to $p + q$-cells of $D^{p-1} \otimes D^q$ to show that the top square commutes it is sufficient to show that both paths of the square send the principal cell $\iota_{p+q} \in D^{p+q}_{p+q}$ to the same $p + q$-cell of $D^{p-1} \otimes D^q$. The top path sends this cell to $1_{\iota_{p-1} \otimes \iota_q}$, so it remains to show that this is the image of ι_{p+q} under the ω-functor of the lower path as well.

The square (72) is obtained by applying ν to the diagram in **ADC**

$$\begin{tikzcd}
G[p+q] \arrow[r, "id"] \arrow[d, "\iota_p \otimes \iota_q"'] & G[p+q-1] \arrow[d, "\iota_{p-1} \otimes \iota_q"] \\
G[p] \otimes G[q] \arrow[r, "id \otimes 1"'] & G[p-1] \otimes G[q]
\end{tikzcd} \tag{73}$$

The ADC map $id : G[n] \to G[n-1]$ is defined by

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & 0 \arrow[r, "\partial"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "0"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\partial"] \arrow[d, "id_{n-1}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\partial"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "\partial"'] & 0 \arrow[r, "\partial"'] & 0 \arrow[r, "\partial"'] & \mathbb{Z} \arrow[r, "\partial"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

where $id_{n-1}(1,0) = id_{n-1}(0,1) = 1 \in \mathbb{Z}$. Hence $id \otimes 1_{G[q]} : G[p] \otimes G[q] \to G[p-1] \otimes G[q]$ acts on the $p + q$-chain $(1) \otimes (1) \in (G[p] \otimes G[q])_{p+q} = \bigoplus_{i+j=p+q} G[p]_i \otimes G[q]_j$ by

$$id \otimes 1_{G[q]} : (1) \otimes (1) \mapsto 0 \otimes (1) = 0 \in (G[p-1] \otimes G[q])_{p+q}$$

Hence when ν is applied to the diagram (73) the lower path sends $\iota_{p+q} \in D^{p+q}_{p+q}$ to an identity $p+q$-cell of $D^{p-1} \otimes D^q$. It will be the identity on the image of the target of ι_{p+q} under this ω-functor. Hence it is sufficient to show that the ω-functor

$$\begin{tikzcd}
D^{p+q-1} \arrow[r, hook, "j^+_{p+q}"] & D^{p+q} \arrow[r, "\iota_q \otimes \iota_q"] & D^p \otimes D^q \arrow[r, "id \otimes 1"] & D^{p-1} \otimes D^q
\end{tikzcd}$$

sends ι_{p+q-1} to $\iota_{p-1} \otimes \iota_q$. Equivalently, it is sufficient to show that the map of ADCs

$$\begin{tikzcd}
G[p+q-1] \arrow[r, hook, "\lambda j^+_{p+q}"] & G[p+q] \arrow[r, "\lambda(\iota_p \otimes \iota_q)"] & G[p] \otimes G[q] \arrow[r, "\lambda(id \otimes 1)"] & G[p-1] \otimes G[q]
\end{tikzcd}$$

sends $1 \in \mathbb{Z} = G[p+q-1]_{p+q-1}$ to $(1) \otimes (1) \in (G[p-1] \otimes G[q])_{p+q-1}$. This holds as

$$\begin{aligned}
\lambda(id \otimes 1)(\lambda(\iota_p \otimes \iota_q)(\lambda j^{+}_{p+q}(1))) &= \lambda(id \otimes 1)(\lambda(\iota_p \otimes \iota_q)(1,0)) \\
&= \begin{cases} \lambda(id \otimes 1)((1,0) \otimes (1) + (1) \otimes (1,0) & p \equiv 0 \pmod 2 \\ \lambda(id \otimes 1)((1,0) \otimes (1) + (1) \otimes (0,1) & p \equiv 1 \pmod 2 \end{cases} \\
&= (1) \otimes (1)
\end{aligned}$$

since

$$\partial((1) \otimes (1)) = ((1,0) - (0,1)) \otimes (1) + (-1)^p((1) \otimes ((1,0) - (0,1)))$$

This completes the proof the case when $f(\langle c \rangle)$ is an identity cell. The case when g sends $\langle d \rangle$ to an identity cell follows by the same arguments. Combining the two cases for f and g gives the result.

□

We record the following useful fact which will allow us to compare model structures of chain complexes and ω-groupoids in Section 4.6. Recall the linearization functor $\lambda : \omega\mathbf{Cat} \to \mathbf{ADC}$.

Proposition 4.37. *Let C be a strong Steiner complex and Y be an ω-category. Then*

$$\lambda(\nu(C) \otimes Y) = C \otimes \lambda(Y)$$

Proof. The linearization functor $\lambda : \omega\mathbf{Cat} \to \mathbf{ADC}$ is left adjoint to ν so it preserves colimits. Applying λ to a Gray tensor product of ω-categories therefore gives

$$\lambda(\nu(C) \otimes Y) \quad = \quad \underset{\substack{\nu(C) \to \nu(C)\ C \in \mathbf{SSt} \\ \nu(D) \to Y\ D \in \mathbf{SSt}}}{\operatorname{colim}} \lambda\nu(C \otimes D)$$

The diagram over ω-functors $\nu(C) \to \nu(C)$ has a terminal object since ν is fully faithful on strong Steiner complexes. So we have

$$\begin{aligned}
\lambda(\nu(C) \otimes Y) &= \underset{\nu(D) \to Y\ D \in \mathbf{SSt}}{\operatorname{colim}} C \otimes D \\
&= C \otimes \left(\underset{\nu(D) \to Y\ D \in \mathbf{SSt}}{\operatorname{colim}} D \right) \\
&\cong C \otimes \left(\underset{\nu(D) \to Y\ D \in \mathbf{SSt}}{\operatorname{colim}} \lambda\nu(D) \right) \\
&= C \otimes \lambda \left(\underset{\nu(D) \to Y\ D \in \mathbf{SSt}}{\operatorname{colim}} \nu(D) \right) \\
&= C \otimes \lambda Y
\end{aligned}$$

where we are again using that ν is fully faithful on strong Steiner complexes so $\lambda\nu(C \otimes D) \cong C \otimes D$ for C and D strong Steiner complexes and also the fact that $C \otimes -$ preserves colimits as a left adjoint.

□

The right adjoints of left and right tensoring that form part of the biclosed monoidal structure on $\omega\mathbf{Cat}$ are denoted $\hom_{\text{lax}}$ and $\hom_{\text{oplax}}$ respectively. These are functors

$$\hom_{\text{lax}}(-,-) : \omega\mathbf{Cat}^{\text{op}} \times \omega\mathbf{Cat} \to \omega\mathbf{Cat}$$

$$\hom_{\text{oplax}}(-,-) : \omega\mathbf{Cat}^{\text{op}} \times \omega\mathbf{Cat} \to \omega\mathbf{Cat}$$

such that there are isomorphisms

$$\operatorname{Hom}_{\omega\mathbf{Cat}}(X, \hom_{\text{oplax}}(Y,Z)) \cong \operatorname{Hom}_{\omega\mathbf{Cat}}(X \otimes Y, Z) \cong \operatorname{Hom}_{\omega\mathbf{Cat}}(Y, \hom_{\text{lax}}(X,Z))$$

natural in all variables. From these natural isomorphisms we see that the sets of n-cells of $\hom_{\text{lax}}(X,Y)$ and $\hom_{\text{oplax}}(X,Y)$ can be given by

$$\hom_{\text{lax}}(X,Y)_n = \operatorname{Hom}_{\omega\mathbf{Cat}}(D^n \otimes X, Y) \qquad \hom_{\text{oplax}}(X,Y)_n = \operatorname{Hom}_{\omega\mathbf{Cat}}(X \otimes D^n, Y)$$

By Proposition 6.12 of [AL20] these functors restrict to functors

$$\hom_{\text{lax}}(-,-) : (\omega,m)\mathbf{Cat}^{\text{op}} \times (\omega,m)\mathbf{Cat} \to (\omega,m)\mathbf{Cat}$$

$$\hom_{\text{oplax}}(-,-) : (\omega,m)\mathbf{Cat}^{\text{op}} \times (\omega,m)\mathbf{Cat} \to (\omega,m)\mathbf{Cat}$$

Hence the biclosed monoidal structure on $\omega\mathbf{Cat}$ satisfies one of the equivalent conditions of Day's reflection theorem [Day72] for the left adjoints $L^m : \omega\mathbf{Cat} \to (\omega,m)\mathbf{Cat}$ to the inclusion of (ω,m)-categories in ω-categories. Hence $(\omega,m)\mathbf{Cat}$ has a biclosed monoidal structure for all $m \geq 0$ with the (ω,m)-category Gray tensor product given by

$$X \otimes_{\omega,m} Y = L^m(X \otimes Y)$$

for all X and Y (ω,m)-categories.

In [AL20] Ara and Lucas also show that if X and Y are ω-groupoids then their Gray tensor product as ω-categories is in fact an ω-groupoid, so

$$X \otimes_{\omega,0} Y = X \otimes Y$$

Finally they show that there are natural isomorphisms $X \otimes Y \cong Y \otimes X$ when X and Y are ω-groupoids, which gives the following theorem.

Theorem 4.38 ([AL20] §6)**.** *There is a closed symmetric monoidal structure on ω**Gpd**. The tensor product of ω-groupoids is the Gray tensor product $X \otimes Y$ viewing X and Y as ω-categories. The internal hom is* $\hom_{\text{lax}}$*, which is naturally isomorphic to* $\hom_{\text{oplax}}$ *via the symmetry isomorphisms.*

The main result of [AL20] that we want to use is that this monoidal structure makes ω**Gpd** into a monoidal model category. To state this, however, we must first describe the model structure on ω**Gpd**.

4.4 The Folk Model Structure on ωGpd

As the homotopy theory of ω-groupoids we will take the homotopy category of the folk model structure. This model structure was originally described by Brown and Golasiński in [BG89] for the category of crossed complexes, which is equivalent to the category of ω-groupoids by [BH81b]. In [AM11] Ara and Métayer described the model structure on ω**Gpd** induced by the equivalence of categories with crossed complexes. They also gave a description of this model structure independent of the existence of a model structure on crossed complexes and showed that it is the same as the model structure transferred from the folk model structure on ω**Cat** given in [LMW10b]. We state their model structure in the next theorem and describe it in the rest of this section. Our main goal will be to record in Theorem 4.43 the result of [AL20] that ω**Gpd** with this model structure is a monoidal model category. This result will be critical in Chapter 5 where the model structure for monoids in ω**Gpd** will be analyzed.

Theorem 4.39 ([AM11] Theorem 3.19)**.** *There is a cofibrantly generated model structure on the category ω* ***Gpd*** *of ω-groupoids such that the weak equivalences are weak equivalences of ω-groupoids and the cofibrations are generated by the set*

$$I = \{L^0 i_n : \partial I^n \hookrightarrow I^n \mid n \geq 0\}$$

All ω-groupoids are fibrant in this model structure.

A lifting problem for an ω-functor $f : X \to Y$ against $L^0 i_n : \partial I^n \hookrightarrow I^n$ is determined by a parallel pair of $n-1$-cells $u, v \in X_{n-1}$ and an n-cell $y \in Y_n$ such that $d^-_{n-1} y = f(u)$ and $d^+_{n-1} y = f(v)$.

$$\begin{tikzcd}
\partial I^n \arrow[r, "\langle u,v\rangle"] \arrow[d, hook, "L^0 i_n"'] & X \arrow[d, "f"] \\
I^n \arrow[r, "y"'] & Y
\end{tikzcd}$$

A pushout of a generating cofibration $L^0 i_n : \partial I^n \hookrightarrow I^n$ freely attaches an n-cell to an ω-groupoid X with source and target a pair of parallel $n-1$-cells determined by a map $\partial I^n \to X$. Relative cell complexes for this model structure, therefore, are ω-functors $f : X \hookrightarrow Y$ that are injective on sets of cells and for which Y_n is freely generated outside of $f(X)_n \subseteq Y_n$.

The ω-groupoid globes defined in Section 4.1 give a supply of cofibrant objects in this model structure.

Proposition 4.40. *For all $n \geq 0$ I^n and ∂I^n are cofibrant in the folk model structure on ω* ***Gpd****.*

Proof. Since $L^0 i_n : \partial I^n \hookrightarrow I^n$ is a cofibration it is sufficient to show that ∂I^n is cofibrant for all $n \geq 0$. For $n = 0$ we have $\partial I^0 = \emptyset$ so this holds trivially. Now for $n > 0$ there are pushouts

$$\begin{tikzcd}
\partial I^{n-1} \arrow[r, hook, "L^0 i_{n-1}"] \arrow[d, hook, "L^0 i_{n-1}"'] & I^{n-1} \arrow[d, hook, "d^+_{n-1}\iota_n"] \\
I^{n-1} \arrow[r, hook, "d^-_{n-1}\iota_n"'] & \partial I^n
\end{tikzcd} \tag{74}$$

Recall that ω-functors $I^n \to X$ correspond to n-cells of X, so in this diagram we are labelling the ω-functors by the image of their principal cell. The pushout of a cofibration is a cofibration, so by induction if I^{n-1} is cofibrant then so is ∂I^n. $\square$

Since the folk model structure on ω**Gpd** is transferred from the folk model structure on ω**Cat**, the set of generating acyclic cofibrations is given by $L^0(J)$, where J is the set of generating acyclic cofibrations for ω**Cat**. The acyclic cofibrations of the folk model structure on ω**Cat** given in [LMW10b] are defined by applying the small object argument cofibration-acyclic fibration factorization of the ω-functors $j^-_n : D^{n-1} \hookrightarrow D^n$ in ω**Cat** determined by the $n-1$-cell $d^-_{n-1}\iota_n$ of D^n. There are also ω-functors $j^+_n : D^{n-1} \hookrightarrow D^n$ for $n \geq 1$ that are determined by the other $n-1$-cell $d^+_{n-1}\iota_n$ of D^n. We will show that for either choice of $\varepsilon \in \{-, +\}$ the ω-functors $L^0(j^\varepsilon_n)$ themselves give a set of generating acyclic cofibrations for the folk model structure on ω**Gpd**.

First we observe that applying L^0 to j_n^ε gives a cofibration of ω-groupoids since it factors as the bottom horizontal or right vertical cofibration $I^{n-1} \hookrightarrow \partial I^n$ of diagram (74) followed by $i_n : \partial I^n \hookrightarrow I^n$ for $\varepsilon = -$ or $+$ respectively. We define the sets of ω-functors of $\omega\mathbf{Gpd}$

$$J_0^- = \{L^0 j_n^- : I^{n-1} \hookrightarrow I^n \mid n \geq 1\}$$

$$J_0^+ = \{L^0 j_n^+ : I^{n-1} \hookrightarrow I^n \mid n \geq 1\}$$

To prove that for ω-groupoids the sets J_0^- and J_0^+ are each sets of generating acyclic cofibrations we can use Lemma 1.8 of [Bek00], the solution set condition of J. Smith. However, to apply this result we will first need to show that the ω-functors j_n^ε are all acyclic cofibrations. We have shown above that they are cofibrations, but to show that they are weak equivalences we will need a tool from the next section.

Proposition 4.41. *The ω-functors $L^0 j_n^\varepsilon : I^{n-1} \hookrightarrow I^n$ for $n \geq 1$ and $\varepsilon \in \{-, +\}$ are acyclic cofibrations.*

Proof. Deferred until Section 4.6. □

Pending this deferred proof, we can now show that the sets J_0^ε give generating acyclic cofibrations for the folk model structure on $\omega\mathbf{Gpd}$.

Proposition 4.42. *The sets J_0^ε for $\varepsilon \in \{-, +\}$ are sets of generating acyclic cofibrations of the folk model structure on ω**Gpd**.*

Proof. By Proposition 4.41 we have $J_0^\varepsilon \subseteq \overset{\sim}{\hookrightarrow} (I) \cap W$ where I is the set of generating cofibrations $L^0 i_n : \partial I_n \hookrightarrow I_n$ for $n \geq 0$ and W is the class of weak equivalences in $\omega\mathbf{Gpd}$. By Lemma 1.8 of [Bek00] it is sufficient to show that for a weak equivalence $f : X \to Y$ any diagram

$$\begin{tikzcd}
\partial I^n \arrow[r, "\langle u,v \rangle"] \arrow[d, hook, "L^0 i_n"'] & X \arrow[d, "f"] \\
I^n \arrow[r, "y"'] & Y
\end{tikzcd}$$

can be factored through a lifting problem against $L^0 j_{n+1}^\varepsilon$.

Suppose such a lifting problem is given. Then it is determined by parallel $n-1$-cells $u, v \in X_{n-1}$ and an n-cell $y \in Y_n$ such that $f(u) = d_{n-1}^- y$ and $f(v) = d_{n-1}^+ y$. Now f is a weak equivalence, so by Proposition 4.16 f induces a bijection on sets

$$\pi_n(X, u, v) \to \pi_n(Y, f(u), f(v))$$

for all $n \geq 1$. Hence there exists an n-cell $z \in X_n$ with $d_{n-1}^- z = u$, $d_{n-1}^+ z = v$, and such that $f(z)$ is homotopic to y. This means that there exists an $n+1$-cell $w \in Y_{n+1}$ such that $d_n^\varepsilon w = f(z)$ and $d_n^{-\varepsilon} w = y$. These cells define a lifting problem

$$\begin{tikzcd}
\partial I^n \arrow[r, hook, "i_n"] \arrow[d, hook, "L^0 i_n"'] & I^n \arrow[r, "z"] \arrow[d, hook, "L^0 j_{n+1}^\varepsilon"] & X \arrow[d, "f"] \\
I^n \arrow[r, hook, "j_{n+1}^{-\varepsilon}"'] & I^{n+1} \arrow[r, "w"'] & Y
\end{tikzcd}$$

so f satisfies the solution set condition and so $\overset{\sim}{\hookrightarrow}(J_0^\varepsilon) = \overset{\sim}{\hookrightarrow}(I) \cap W$ $\square$

We will now describe how the folk model structure on $\omega\mathbf{Gpd}$ gives a monoidal model category structure with the Gray tensor product of Section 4.3.

Theorem 4.43 ([AL20] Theorem 6.21)**.** *The Brown-Golasiński model structure with the Gray tensor product is a monoidal model category.*

This means that the functor

$$- \otimes - : \omega\mathbf{Gpd} \times \omega\mathbf{Gpd} \to \omega\mathbf{Gpd}$$

is a left Quillen bifunctor, so for any cofibrations $i : X \hookrightarrow Y$ and $i' : W \hookrightarrow Z$ their pushout product for the Gray tensor product

$$(X \otimes Z) \cup_{X \otimes W} (Y \otimes W) \hookrightarrow Y \otimes Z$$

is a cofibration that is acyclic if either i or i' is. Equivalently, since $\mathrm{hom}_{\mathrm{lax}}(-, X)$ is a right adjoint for $- \otimes X$, for all fibrations $f : X \to Y$ and all cofibrations $i : W \hookrightarrow Z$ the ω-functor

$$\mathrm{hom}_{\mathrm{lax}}(W, X) \to \mathrm{hom}_{\mathrm{lax}}(Z, X) \times_{\mathrm{hom}_{\mathrm{lax}}(Z,Y)} \mathrm{hom}_{\mathrm{lax}}(W, Y)$$

is a fibration which is acyclic if either f or i is.

We can immediately use the monoidal model category structure of $\omega\mathbf{Gpd}$ to determine a path space for all ω-groupoids X in the standard way using the internal hom functor $\mathrm{hom}_{\mathrm{lax}}$. Since $\partial I^1 = I^0 \sqcup I^0$ we have

$$\mathrm{hom}_{\mathrm{lax}}(\partial I^1, X) = \mathrm{hom}_{\mathrm{lax}}(I^0 \sqcup I^0, X) = \mathrm{hom}_{\mathrm{lax}}(I^0, X) \times \mathrm{hom}_{\mathrm{lax}}(I^0, X) = X \times X$$

since $\mathrm{hom}_{\mathrm{lax}}$ takes colimits to limits in the first entry. We will denote $\mathrm{hom}_{\mathrm{lax}}(i_1, X)$ by

$$\pi_X : \mathrm{hom}_{\mathrm{lax}}(I^1, X) \to X \times X$$

The ω-functor $! : I^1 \to I^0$ is the unique ω-functor to the terminal ω-groupoid $I^0 = D^0$ with a single 0-cell. Since I^0 is the unit for the Gray tensor product of ω-groupoids we have

$$I^0 \otimes I^n \cong I^n$$

for all $n \geq 0$ and so $\mathrm{hom}_{\mathrm{lax}}(I^0, X) \cong X$. Hence $\mathrm{hom}_{\mathrm{lax}}(!, X)$ is an ω-functor with source X which we will denote by $\tau_X : X \to \mathrm{hom}_{\mathrm{lax}}(I^1, X)$. The two ω-functors $L^0 j_1^\varepsilon : I^0 \hookrightarrow I^1$ for $\varepsilon \in \{-, +\}$ are both sections for $!$, so they determine retracts $\mathrm{hom}_{\mathrm{lax}}(L^0 j_1^\varepsilon, X)$ of τ_X, which we denote π_X^- and π_X^+. Applying $\mathrm{hom}_{\mathrm{lax}}$ to the composite

$$\partial I^1 \xhookrightarrow{L^0 i_0} I^1 \xrightarrow{!} I^0$$

gives the ω-functor that acts on n-cells by

$$x : I^0 \otimes I^n \to X \in \mathrm{hom}_{\mathrm{lax}}(I^0, X)_n \mapsto x \sqcup x : (I^0 \otimes I^n) \sqcup (I^0 \otimes I^n) \to X \in \mathrm{hom}_{\mathrm{lax}}(\partial I^1, X)_n$$

so this is $\Delta : X \to X \times X$. Hence we have a diagram of ω-functors

$$\begin{tikzcd}
X \arrow[r, "\tau_X"] \arrow[dr, equal, bend right=10] \arrow[dr, equal, bend right=35] & \mathrm{hom}_{\mathrm{lax}}(I^1, X) \arrow[r, "\pi_X"] \arrow[d, "\pi_X^-"'] \arrow[drr, "\pi_X^+"' near end] & X \times X \arrow[dl] \arrow[dr] & \\
& X \arrow[rr, equal, bend right=10] & & X
\end{tikzcd}$$

This determines a path space fibration for any ω-functor X.

Proposition 4.44. *Let X be an ω-groupoid. The ω-functors*

$$\begin{tikzcd}
X \arrow[r, hook, "\tau_X"] \arrow[dr, "\Delta"'] & \hom_{\mathrm{lax}}(I^1, X) \arrow[d, "\pi_X"] \\
& X \times X
\end{tikzcd}$$

make $\hom_{\mathrm{lax}}(I_1, X)$ *a path space object for X.*

Proof. We must show that the horizontal map is a weak equivalence and the vertical map is a fibration. This will follow from the monoidal model category structure on $\omega\mathbf{Gpd}$ once we have shown that I^1 is weakly contractible.

We will show that the unique ω-functor $! : I^1 \to I^0$ is an acyclic fibration. A lifting problem

$$\begin{tikzcd}
\partial I^n \arrow[r, "\langle u,v \rangle"] \arrow[d, hook, "L^0 i_n"'] & I^1 \arrow[d, "!"] \\
I^n \arrow[r, "1_{\iota_0}"'] & I^0
\end{tikzcd}$$

is determined by a pair of parallel $n-1$-cells $u, v \in I^1_{n-1}$ and an n-cell of I^0, which must be 1_{ι_0} if $n \geq 1$. Hence a solution is simply an n-cell of I^1 whose source and target are u and v respectively. If $n > 2$ then u and v are identity cells, since I^1 has no non-identity n-cells for $n > 1$. So $u = v$ as parallel identity cells are equal and so $1_u \in I^1_n$ is a solution to the lifting problem. If $n = 2$ then $u = v$ again, as 1-cells of I^1 are freely generated by ι_1 and its inverse, so there are no non-equal parallel 1-cells. Hence 1_u is again a solution. Finally if $n = 0$ then it is possible that $u \neq v$, but in this case either ι_1 or $k^1\iota_1$ give a solution. Hence ! has the right lifting property against all $L^0 i_n$ and is an acyclic fibration.

The ω-functors $L^0 j_1^- : I^0 \hookrightarrow I^1$ and $L^0 j_1^+$ are sections of !, so they are acyclic cofibrations. Hence by the monoidal model structure on $\omega\mathbf{Gpd}$ the ω-functors for $\varepsilon \in \{-, +\}$

$$\pi_X^\varepsilon = \hom_{\mathrm{lax}}(L^0 j_1^\varepsilon, X) : \hom_{\mathrm{lax}}(I^1, X) \to X$$

is an acyclic fibration, since X is fibrant. The ω-functor $\tau_X = \hom_{\mathrm{lax}}(!, X)$ is a retract of π_X^ε, so it is a weak equivalence. Finally the ω-functor

$$\pi_X = \hom_{\mathrm{lax}}(L^0 i_1, X) : \hom_{\mathrm{lax}}(I^1, X) \to X \times X$$

is a fibration since $L^0 i_1 : \partial I^1 \hookrightarrow I^1$ is a cofibration and X is fibrant.

□

In the next section we will describe free ω-groupoids, which will provide a supply of cofibrant objects in the folk model structure on $\omega\mathbf{Gpd}$.

4.5 Free ω-Categories

In this section we will describe ω-categories that are free on sets of generating n-cells for all $n \geq 0$. These structures are called computads by Street, who introduced them for 2-categories in [Str76]. This construction was later generalized to n-categories by Street and Burroni, who calls them polygraphs in [Bur93]. Computads are defined in these sources for ω-categories, however the definition can be easily adapted to (ω, n)-categories for $n \geq 0$ and in particular to ω-groupoids. We will give the basic definitions of these constructions in this section and use them in Lemma 4.46 to give a cofibrant replacement construction for ω-groupoids in the folk model structure on $\omega\mathbf{Gpd}$.

Computads are ω-categories that are constructed by freely attaching n-cells in every dimension. To make this definition, therefore, we must describe how to attach n-cells to an $n-1$-category. This description is from [Mak05] §4 and 5. Let $n \geq 0$. An $n+1$**-extension frame** of an ω-category X is a tuple (X, U, s, t) consisting of X, a set U and set maps

$$s, t : U \to X_n$$

such that $s(u)$ and $t(u)$ are parallel n-cells of X, meaning

$$d^-_{n-1} \circ s = d^-_{n-1} \circ t \qquad d^+_{n-1} \circ t = d^+_{n-1} \circ s$$

An $n+1$-extension frame, therefore, consists of an ω-category an a labelling of pairs of parallel n-cells by a set of indeterminates.

An $n+1$**-extension** of an $n+1$-extension frame (X, U, s, t) is a pair $(f : X \to Y, \gamma : U \to Y_{n+1})$ consisting of an ω-functor $f : X \to Y$ and a set map $\gamma : U \to Y_{n+1}$ such that for all $u \in U$

$$d^-_n\gamma(u) = f(s(u)) \qquad d^+_n\gamma(u) = f(t(u))$$

An extension realizes the set U of indeterminates of an $n+1$-extension frame as $n+1$-cells of Y in a way consistent with the ω-category structure of X. A map $g : (f, \gamma) \to (f', \gamma')$ of $n+1$-extensions of an ω-category X consists of an ω-functor $g : Y \to Y'$ such that the diagrams

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Y \arrow[rr, "g"'] & & Y' \end{tikzcd} \qquad \begin{tikzcd} & U \arrow[dl, "\gamma"'] \arrow[dr, "\gamma'"] & \\ Y_{n+1} \arrow[rr, "g_{n+1}"'] & & Y'_{n+1} \end{tikzcd}$$

of ω-categories and sets respectively, commute. Thus there is a category of $n+1$-extensions of an ω-category X. A **free** $n+1$**-extension** is an initial object in this category. Such free extensions exist by [Mak05] §4 and we will denote the ω-category component of a free extension by $X[U]$. By [Mak05] 4.2 and 4.4 X is contained as a sub-ω-category of $X[U]$ and $U \subseteq X[U]_{n+1}$ is a subset of the $n+1$-cells of $X[U]$ that is disjoint from X.

The free $n+1$-extension has the following universal property: for any ω-functor $f : X \to Y$ and set map $\gamma : U \to Y_{n+1}$ such that $f(s(u)) = d^-_n\gamma(u)$ and $f(s(u)) = d^+_n\gamma(u)$ there exists a unique ω-functor $f' : X[U] \to Y$ such that $f'|_X = f$ and $f'(u) = \gamma(u) \in Y_{n+1}$ for all $u \in U \subseteq X[U]_{n+1}$. This characterizes free $n+1$-extensions up to isomorphism. In particular, a free $n+1$-extension is a pushout in $\omega\mathbf{Cat}$

$$\begin{tikzcd} \coprod_U \partial D^{n+1} \arrow[r] \arrow[d, hook, "\coprod_U i_{n+1}"'] & X \arrow[d, hook] \\ \coprod_U D^{n+1} \arrow[r] & X[U] \end{tikzcd}$$

If X is an (ω, m)-category then we take as the free $(n+1, m)$-extension of X by a set of indeterminates U the (ω, m)-category $L^m X[U]$ obtained by by applying the left adjoint functor L^m of the inclusion of $(\omega, m)\mathbf{Cat}$ to the free $n+1$-extension $X[U]$. If $m = 0$ and we are working with ω-groupoids we will denote the free $n+1$-groupoid extension by

$$X(U) = L^0 X[U]$$

Since L^m is a left adjoint it preserves pushouts, so a free $(n+1, m)$-extension of an (ω, m)-category is a pushout in $(\omega, m)\mathbf{Cat}$

$$\begin{tikzcd}
\coprod_U \partial L^m D^{n+1} \arrow[r] \arrow[d, hook, "\coprod_U L^m i_{n+1}"'] & X \arrow[d, hook] \\
\coprod_U L^m D^{n+1} \arrow[r] & L^m X[U]
\end{tikzcd}$$

This (ω, m)-category has the following universal property: for any ω-functor $f : X \to Y$ with Y an (ω, m)-category and any set map $\gamma : U \to Y_{n+1}$ such that $f(s(u)) = d_n^- \gamma(u)$ and $f(t(u)) = d_n^+ \gamma(u)$ there exists a unique ω-functor $f' : L^m X[U] \to Y$ such that $f'|_X = f$ and $f'(u) = \gamma(u)$ for all $u \in U \subseteq L^m X[U]_{n+1}$. This is immediate from the fact that L^m is left adjoint to the inclusion functor.

A computad in (ω, m)-categories for $m \geq 0$ is an iterated free extension of a set of 0-cells by sets of indeterminates.

Definition 4.45. *For $n \geq 0$ an **n-computad** in (ω, m)**Cat** is defined by induction on n*

- *a **0-computad** is a set viewed as a discrete ω-category*
- *for $n > 0$ an **n-computad** X **in** (ω, m)**Cat** is an (n, m)-category such that $\tau^{n-1}X$ is an $(n-1, m)$-computad and there is a pushout diagram in (ω, m)**Cat***

$$\begin{tikzcd}
\coprod_{U_n} L^m \partial D^n \arrow[r] \arrow[d, hook, "\coprod_{U_m} L^m i_n"'] & \tau^{n-1}X \arrow[d, hook] \\
\coprod_{U_n} L^m D^n \arrow[r] & X
\end{tikzcd}$$

*so that X is isomorphic a free (n, m)-extension $L^m \tau^{n-1} X[U_n]$ of an $(n-1, m)$-computad $\tau^{n-1}X$ in (ω, m)**Cat** by a set of indeterminates $U_n \subseteq X_n$*

*A **computad** in (ω, m)**Cat** is an (ω, m)-category X such that $\tau^n(X)$ is an n-computad for all $n \geq 0$.*

A computad, therefore, is specified by an (ω, m)-category X as well as sets $U_n \subseteq X_n$ for all $n \geq 0$ of indeterminates such that the diagrams in the definition are all pushouts. We can denote this schematically by a diagram

$$\begin{tikzcd}
 & U_{n+1} \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & U_n \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & & U_1 \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \\
\cdots \arrow[r, shift left] \arrow[r, shift right] & L^m\tau^n(X)[U_{n+1}] \arrow[r, shift left, "d_n^+"] \arrow[r, shift right, "d_n^-"'] & L^m\tau^{n-1}(X)[U_n] \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[r, shift left] \arrow[r, shift right] & L^m\tau^0(X)[U_1] \arrow[r, shift left, "d_0^+"] \arrow[r, shift right, "d_0^-"'] & \tau^0(X)
\end{tikzcd}$$

where the set U_n of n-indeterminates is attached to the $n-1$-computad $L^m\tau^{n-1}(X)$.

As shown in [Mak05] §5, an ω-category computad has a unique choice of sets U_n of indeterminates. These are the **indecomposable cells**, which are non-identity n-cells $x \in X_n$ such that for $y, z \in X_n$ if $x = y *_m z$ for $m < n$ then one of y or z is an identity and the other is equal to x. The example of free groups, which are 1-computads in $\omega\mathbf{Gpd}$, shows that there is no hope for such a unique choice of indeterminates for ω-groupoid computads.

It is clear from the definition that ω-groupoid computads are cofibrant objects in $\omega\mathbf{Gpd}$ with the folk model structure of Theorem 4.39. In [Mét08] the author shows that computads are exactly the cofibrant ω-categories in the folk model structure on $\omega\mathbf{Cat}$. Such a characterization for the cofibrant ω-groupoids in the folk model structure on $\omega\mathbf{Gpd}$ is almost certainly possible, using higher groupoid versions of the fact that a subgroup of a free group is free. However, we do not attempt this here. We are satisfied with the fact that ω-groupoid computads form a class of cofibrant ω-groupoids in the folk model structure, as for any ω-groupoid X we can construct a computad that is a cofibrant replacement of X.

Lemma 4.46. *For any ω-groupoid X there exists a cofibrant replacement of X by a computad X^* such that the map $X^* \to X$ is an acyclic fibration and induces a bijection on 0-cells.*

Proof. Let X be an ω-groupoid. We will use an explicit description of the small object argument for the cofibrantly generated model category $\omega\mathbf{Gpd}$, which is an adaptation for ω-groupoids of the version for ω-categories described in §4.2 of [Mét03]. We will construct n-groupoids $(X^*)^n$ with maps $p_n : (X^*)^n \to X$ such that the lifting problems

$$\begin{tikzcd}
\partial I^m \arrow[r] \arrow[d, hook, "L^0 i_m"'] & (X^*)^n \arrow[d, "p_n"] \\
I^m \arrow[r] & X
\end{tikzcd}$$

have solutions for all $m \leq n$ and the diagram

$$\begin{tikzcd}
(X^*)^0 \arrow[r, hook] \arrow[d, "p_0"'] & (X^*)^1 \arrow[r, hook] \arrow[dl, "p_1"'] & (X^*)^2 \arrow[r, hook] \arrow[dll, "p_2"'] & \cdots \arrow[r, hook] & (X^*)^n \arrow[r, hook] \arrow[dllll, "p_n"] & \cdots \\
X & & & & &
\end{tikzcd}$$

commutes. Each $(X^*)^n$ will be an n-groupoid since all m-cells for $m > n$ are identities and $(X^*)^{n+1}$ is obtained from $(X^*)^n$ by freely attaching a set of $n+1$-cells. We then take $X^* \to X$ as the filtered colimit of these computads with the universal map to X corresponding to this cocone. The cofibrant ω-groupoid X^* is therefore a computad as $\tau^n(X^*) = (X^*)^n$ for all $n \geq 0$.

For $n = 0$, take $(X^*)^0 = X_0$, the set of objects of X. The lifting problems

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & X_0 \arrow[d] \\
I^0 \arrow[r] & X
\end{tikzcd}$$

clearly all have solutions. Now let $n \geq 0$ and suppose $p_n : (X^*)^n \to X$ is defined for $(X^*)^n$ an n-computad with p_n a bijection on sets of 0-cells. We will label all lifting problems

$$\begin{tikzcd}
\partial I^{n+1} \arrow[r, "\langle x,y \rangle"] \arrow[d, hook, "L^0 i_{n+1}"'] & (X^*)^n \arrow[d, "p_n"] \\
I^{n+1} \arrow[r, "z"'] & X
\end{tikzcd}$$

by the data that specifies them: an $n+1$-cell $z \in X_{n+1}$ and n-cells $x, y \in (X^*)^n_n$ such that $p_n(x) = d^-_n z$ and $p_n(y) = d^+_n z$. We will say that a lifting problem is trivial when $y = x$ and $z = 1_{p(x)}$. In this case, there is an obvious lift given by $1_x \in (X^*)^n_{n+1}$.

For each tuple (z, x, y) specifying a non-trivial lifting problem we attach an $n+1$-cell to $(X^*)^n$ with source and target x and y to obtain $(X^*)^{n+1}$. By hypothesis $(X^*)^n$ is an n-computad, so $(X^*)^{n+1}$ is an $n+1$-computad. We define $p : X^* \to X$ as the unique ω-functor from the colimit X^* of these ω-functors determined by the ω-functors p_n for $n \geq 0$. We also observe that the n-cells of X^* are the same as those of $(X^*)^n$

$$(X^*)^n_n = X^*_n$$

The map $p : X^* \to X$ has the right lifting property against all generating cofibrations $L^0 i_n : \partial I^n \hookrightarrow I^n$ since a lifting problem

$$\begin{tikzcd}
\partial I^n \arrow[r, dashed] \arrow[rrr, bend left] \arrow[d, hook, "L^0 i_n"'] & (X^*)^{n-1} \arrow[r, hook] \arrow[drr, "p^{n-1}"'] & (X^*)^n \arrow[r, hook] \arrow[dr, "p^n"] & X^* \arrow[d, "p"] \\
I^n \arrow[rrr] & & & X
\end{tikzcd}$$

factors through $(X^*)^{n-1} \hookrightarrow (X^*)^n$ as shown and a solution to any non-trivial lifting problem against p^{n-1} exists in $(X^*)^n$ by construction. □

This cofibrant replacement by a computad will allow us to deal only with cofibrant objects that are computads. When taking this replacement we will also preserve some properties of the original ω-groupoid. Recall that an ω-groupoid is n-reduced if it has a unique 0-cell and no non-identity m-cells for $1 \leq m \leq n$.

Lemma 4.47. *If X is an n-reduced ω-groupoid then X^* is also n-reduced.*

Proof. This is clear since if X is n-reduced with unique object $e \in X_0$ then there are no non-trivial lifting problems up to degree $n+1$, hence $(X^*)_m = X_m = \{1_e\}$ for all $m \leq n$ by the construction of Lemma 4.46. Since $(X^*)^m_m = X^*_m$ we have the result. □

4.6 ω-Groupoids and Chain Complexes

In Section 4.3 we described Steiner's theory of ADCs and the connection they give between ω-categories and chain complexes. As motivation we considered the linearization functor $\lambda : \omega\mathbf{Cat} \to \mathbf{ADC}$, which constructs an ADC from an ω-category. Recall that an ADC is a chain complex of abelian groups with additional structure consisting of distinguished submonoids of the chain groups and an augmentation map $e : C \to \mathbb{Z}[0]$. When the ω-category is an ω-groupoid, the ADC λX is only barely more than a chain complex. In particular, its distinguished submonoids for $n > 0$ are equal to the chain groups, as for $x \in X_n$ and $0 \leq m < n$

$$[x] + [k^m x] = [x *_m k^m x] = [1_{d_m^- x}] = 0$$

so $-[x] = [k^m x] \in \lambda X_n^*$. Forgetting the augmentation and submonoid of λX_0 we have a functor $\lambda : \omega\mathbf{Gpd} \to \mathbf{Ch}$. In this section we will show that this functor is part of a Quillen adjunction for $\omega\mathbf{Gpd}$ with the folk model structure and $\mathbf{Ch}$, the category of non-negative chain complexes of abelian groups with the standard model structure. This has been shown in [Gue21] in the case of the corresponding functor $\lambda : \omega\mathbf{Cat} \to \mathbf{Ch}$, but we will adapt this result to our case of ω-groupoids. We will also use this adjunction to give the proof of Proposition 4.41.

We begin with a version of the Dold-Kan correspondence for ω-groupoids. The Dold-Kan correspondence [Dol58] establishes an equivalence of categories between $\mathbf{Ch}$ and $\mathbf{sAb}$ the category of simplicial abelian groups. The corresponding result for ω-groupoids says that there is an equivalence of categories between $\omega\mathbf{Gpd}(\mathbf{Ab})$, the category of ω-groupoids in abelian groups, and $\mathbf{Ch}$. This is given in [Bro11] Theorem 14.8.1 for crossed complexes, which are equivalent to ω-groupoids. The explicit construction of the functors of this equivalence which we repeat here is given in §4 of [Ara13].

Let X be an ω-groupoid in abelian groups. We define a chain complex CX whose 0^{th} chain group is X_0 and whose n^{th} chain group is the kernel of $d_{n-1}^- : X_n \to X_{n-1}$. The identity maps $1_{n-1} : X_{n-1} \to X_n$ are sections for d_{n-1}^ε so

$$X_n \cong CX_n \oplus X_{n-1}$$

and we can take CX_n as the quotient of X_n by the abelian subgroup generated by the identity cells. Since ω-functors preserve identities an ω-functor determines a chain map and $C : \omega\mathbf{Gpd}(\mathbf{Ab}) \to \mathbf{Ch}$ defines a functor.

If X is an ω-groupoid in abelian groups then the structure homomorphisms of the ω-groupoid are abelian group homomorphisms. In particular, for $(x, x'), (y, y') \in X_n \times_{X_i} X_n$ with $d_i^+ x = d_i^- x'$ and $d_i^+ y = d_i^- y'$

$$\begin{aligned}(x + y) *_i (x' + y') &= m_i((x + y, x' + y')) \\ &= m_i((x, x') + (y, y')) \\ &= m_i(x, x') + m_i(y, y') \\ &= (x *_i x') + (y *_i y')\end{aligned}$$

Hence

$$x + y = (x *_i 1_{d_i^+ x}) + (1_{d_i^- y} *_i y) = (x + 1_{d_i^- y}) *_i (1_{d_i^+ x} + y)$$

so if $d_i^- y = d_i^+ x$ then

$$(x - 1_{d_i^+ x}) + y = x *_i y \tag{75}$$

Hence the abelian group CX_n is generated by classes of n-cells $x, y \in X_n$ subject to the relations

$$[x *_i y] = [x] + [y]$$

Hence if X is a computad as an ω-groupoid with set of generating n-cells S_n then CX_n is the free abelian group generated by S_n.

Now for a chain complex K we define an ω-groupoid in abelian groups AK with group of n-cells given by

$$AK_n = K_n \oplus K_{n-1} \oplus \cdots \oplus K_0$$

so $AK_n = K_n \oplus AK_{n-1}$ and source, target, and identity maps are given by

$$\begin{aligned} d^-_{n-1} : (x_n, x_{n-1}, \cdots, x_0) \in K_n \oplus AK_{n-1} &\mapsto (x_{n-1}, \cdots, x_0) \in AK_{n-1} \\ d^+_{n-1} : (x_n, x_{n-1}, \cdots, x_0) \in K_n \oplus AK_{n-1} &\mapsto (\partial x_n + x_{n-1}, \cdots, x_0) \in AK_{n-1} \\ 1_{n-1} : (x_{n-1}, \cdots, x_0) \in AK_{n-1} &\mapsto (0, x_{n-1}, \cdots, x_0) \in K_n \oplus AK_{n-1} \end{aligned}$$

Composition is defined for $0 \leq m < n$

$$(x_n, \cdots, x_0) *_m (y_n, \cdots, y_0) = (y_n + x_n, \cdots y_{m+1} + x_{m+1}, y_m, \cdots, y_0)$$

when

$$d^+_m(x_n, \cdots, x_0) = (\partial x_{m+1} + x_m, x_{m-1}, \cdots, x_0) = (y_m, \cdots, y_0) = d^-_m(y_n, \cdots, y_0)$$

The k^m inverse of $(x_n, \cdots, x_0)$ is given by

$$(-x_n, \cdots, -x_{m+1}, \partial x_{m+1} + x_m, x_{m-1}, \cdots, x_0)$$

so this defines an ω-groupoid in abelian groups. In [Ara13], Ara shows that these functors define an equivalence of categories and that this equivalence respects weak equivalences.

Proposition 4.48 ([Ara13] Prop. 4.7 and 4.8)**.** *The functors C and A define an equivalence of categories $\omega\,\boldsymbol{Gpd}(\boldsymbol{Ab}) \cong \boldsymbol{Ch}$ between ω-groupoids in abelian groups and non-negative chain complexes in abelian groups. Under this equivalence ω-functors in $\omega\,\boldsymbol{Gpd}(\boldsymbol{Ab})$ that are weak equivalences of ω-groupoids correspond exactly to quasi-isomorphisms of chain complexes.*

A key observation made in [Ara13] is that above degree 2 the exchange law makes an ω-groupoid already essentially abelian. The precise statement is this.

Proposition 4.49 ([Ara13] Prop. 4.5)**.** *Let X be an ω-groupoid with a single 0-cell and no non-identity 1-cells. Then X is an ω-groupoid in abelian groups.*

For $n > 1$ $*_1$-composition and $*_0$-composition are two binary operations on n-cells $x, y \in X_n$ which are compatible in the following way by the exchange law

$$(x *_1 y) *_0 (x' *_1 y') = (x *_0 x') *_1 (y *_0 y')$$

The Eckmann-Hilton argument shows that these operations are the same and define an abelian group operation on all sets of n-cells for $n > 1$

$$x + y = x *_0 y = x *_1 y$$

Since the group operation on cells is determined by the ω-groupoid composition all structure maps of the ω-groupoid respect it and so are abelian group homomorphisms. The structure of an ω-groupoid in abelian groups is completed by giving $X_0 = \{*\}$ and $X_1 = \{1_*\}$ the trivial group structure, which is preserved by the identity maps as $1_* \in X_n$ for $n > 1$ is the group identity.

Recall the linearization functor λ for ω-categories that we described in Section 4.3. By the discussion at the start of this section, restricting to ω-groupoids and forgetting the extra structure on the chain complex of the resulting ADC, which is anyway relatively meagre for ω-groupoids, this defines a functor $\lambda : \omega\mathbf{Gpd} \to \mathbf{Ch}$. We will now show that this functor is the left adjoint of a Quillen adjunction between ω-groupoids and chain complexes. This is an adaptation to the case of ω-groupoids of Proposition 4.8 of [Gue21].

Proposition 4.50. *There is Quillen adjunction*

$$\begin{tikzcd}
\boldsymbol{Ch} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \omega\boldsymbol{Gpd} \arrow[l, bend left, "\lambda"]
\end{tikzcd} \tag{76}$$

*between ω **Gpd** with the folk model structure and the category of non-negative chain complexes of abelian groups with the standard model structure.*

Proof. We will construct this adjunction by composing the adjunction

$$\begin{tikzcd}
\omega\mathbf{Gpd}(\mathbf{Ab}) \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \omega\mathbf{Gpd} \arrow[l, bend left, "F"]
\end{tikzcd} \tag{77}$$

between ω-groupoids and ω-groupoids in the category of abelian groups arising from the free-forgetful adjunction between sets and abelian groups with the equivalence of categories

$$\omega\mathbf{Gpd}(\mathbf{Ab}) \cong \mathbf{Ch}$$

of Proposition 4.48. This certainly defines an adjunction, but we must show that the left adjoint is the functor λ we have described and that this adjunction is Quillen for the two model structures.

The left adjoint F of the adjunction (77) sends an ω-groupoid X to $F(X)$, the ω-groupoid whose abelian group of n-cells is generated by the set X_n of n-cells of X such that for $x, y \in X_n$ with $d_i^+ x = d_i^- y$

$$[x] *_i [y] = [g *_i y]$$

and subject to the relations needed to make the structure homomorphisms of the ω-groupoid into abelian group homomorphisms. The functor $C : \omega\mathbf{Gpd}(\mathbf{Ab}) \to \mathbf{Ch}$ for the equivalence of categories of Proposition 4.48 sends the ω-groupoid $F(X)$ in $\mathbf{Ab}$ to the chain complex with chain groups $CF(X)_0 = F(X)_0$ and $CF(X)_n$ the quotient of the abelian group $F(X)_n$ by the subgroup generated by identities for all $n > 0$. So by (75) $CF(X)$ is the chain complex whose group of n-chains is generated by classes $[x]$ for $x \in X_n$ subject to the relations

$$[x *_i y] = [x] + [x]$$

Hence $CF(X) = \lambda(X)$.

This shows that λ is a left adjoint, to show it is part of a Quillen adjunction we must show that it preserves cofibrations and acyclic cofibrations. The ω-groupoids I^n are sent to the chain complexes

$$\begin{tikzcd}\lambda(I^n) = \cdots 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}\end{tikzcd}$$

underlying the ADC $G[n]$ of Definition 4.19. The generating cofibrations $L^0 i_n : \partial I^n \hookrightarrow I^n$ are sent to the maps of chain complexes

$$\begin{tikzcd}
\lambda(\partial I^n) \arrow[d, hook, "\lambda(i_n)"'] & & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, hook] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, equal] \\
\lambda(I^n) & = & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z}\oplus\mathbb{Z}
\end{tikzcd}$$

which are cofibrations of the chain complex model structure since they are injective with quotient having $\mathbb{Z}$ in degree n and 0 elsewhere. So λ preserves cofibrations.

It only remains to prove that λ preserves acyclic cofibrations. However, we cannot use the characterization of generating acyclic cofibrations from Proposition 4.42 because we will use this Quillen adjunction to prove Proposition 4.41 that all $j_n^\varepsilon : I^{n-1} \hookrightarrow I^n$ are acyclic cofibrations. We will instead show that λ sends acyclic cofibrations to quasi-isomorphisms using the monoidal model structure on $\omega\mathbf{Gpd}$.

Let i be an acyclic cofibration in the folk model structure. Then since all ω-groupoids are fibrant in the folk model structure, there is a solution to the lifting problem

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, hook, "i"', "\wr"] & X \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed, "r"'] & I^0
\end{tikzcd}$$

which determines a retract $r : Y \to X$ for i. By the monoidal model structure for $\omega\mathbf{Gpd}$ the lifting problem

$$\begin{tikzcd}
X\otimes I^1 \cup_{X\otimes\partial I^1} Y\otimes\partial I^1 \arrow[r] \arrow[d, hook, "\wr"'] & Y \arrow[d] \\
Y\otimes I^1 \arrow[r] \arrow[ur, dashed, "h"'] & I^0
\end{tikzcd}$$

has a solution h that determines an ω-functor h making the following diagrams commute

$$\begin{tikzcd}
Y \arrow[d, "j_0^+\otimes Y"'] \arrow[dr, "i\circ r"] & \\
I^1\otimes Y \arrow[r, "h"] & Y \\
Y \arrow[u, "j_0^-\otimes Y"] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I^1\otimes X \arrow[r, "t\otimes X"] \arrow[d, hook, "I^1\otimes i"'] & X \arrow[d, hook, "i"] \\
I^1\otimes Y \arrow[r, "h"'] & Y
\end{tikzcd}$$

Pre-composing with the ω-functor of ω-categories $D^1 \to I^1$ that corresponds to the principal 1-cell of I^1 gives the same diagrams with I^1 replaced by D^1, since $I^0 = D^0$, the terminal ω-category/groupoid. Since $D^1 = \nu(G[1])$ for $G[1]$, the strong Steiner complex defined in Definition

4.19, by Proposition 4.37 applying λ to these diagrams gives diagrams of chain complexes

$$\begin{tikzcd}
\lambda(Y) \arrow[d, "\lambda(j_0^+)\otimes\lambda(Y)"'] \arrow[dr, "\lambda(i\circ r)"] & \\
G[1]\otimes\lambda(Y) \arrow[r, "\lambda(h)"] & \lambda(Y) \\
\lambda(Y) \arrow[u, "\lambda(j_0^-)\otimes\lambda(Y)"] \arrow[ur, equal] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G[1]\otimes\lambda(Y) \arrow[r, "\lambda(t)\otimes\lambda(X)"] \arrow[d, hook, "G[1]\otimes\lambda(i)"'] & \lambda(X) \arrow[d, hook, "\lambda(i)"] \\
G[1]\otimes\lambda(Y) \arrow[r, "\lambda(h)"'] & \lambda(Y)
\end{tikzcd}$$

where all tensor products are the tensor product of chain complexes. We observe that $G[1]$ is the chain complex corresponding to the simplicial abelian group $\mathbb{Z}[\Delta^1]$ under the Dold-Kan correspondence, so these diagrams show that $\lambda(i)$ is a chain homotopy equivalence and so is a quasi-isomorphism.

□

This result was the main goal of this section. We will finish off by proving a simple consequence of the results of Ara from [Ara13] that lets us complete the proof of Proposition 4.41. Recall that by Propositions 4.48 and 4.49, when an ω-groupoid has a unique 0-cell and no non-identity 1-cells it is equivalent to a chain complex. The next proposition shows that up to homotopy 1-connected ω-groupoids also behave like chain complexes.

Proposition 4.51. *An ω-functor between 1-connected cofibrant ω-groupoids is a weak equivalence if and only if $\lambda(f)$ is a quasi-isomorphism of chain complexes.*

Proof. One direction is clear, since by Proposition 4.50 a weak equivalence between cofibrant ω-groupoids is sent to a quasi-isomorphism. For the other direction, we will start by reducing to the case when the source and target of the ω-functor are 1-reduced computads. Let X be a simply connected cofibrant ω-groupoid. Pick an object $e \in X_0$ and consider $R^1_e X$. The inclusion

$$\varepsilon_{e,X} : R^1_e X \hookrightarrow X$$

is clearly a weak equivalence, since X is simply connected. The computad replacement $(R^1_e X)^*$ of $R^1_e X$ of Lemma 4.46 is 1-reduced by Lemma 4.47, so $(R^1_e X)^*$ is a cofibrant 1-reduced computad. Furthermore, the ω-functor

$$\begin{tikzcd}
(R^1_e X)^* \arrow[r, "p"] & R^1_e X \arrow[r, hook] & X
\end{tikzcd}$$

is a weak equivalence between cofibrant ω-groupoids, so it is sent to a quasi-isomorphism by λ. Finally, for any ω-functor $f : X \to Y$ such that X and Y are simply connected cofibrant ω-groupoids a solution to the lifting problem in the top square below exists because $(R^1_e X)^*$ is cofibrant and $(R^1_{f(e)} Y)^* \to R^1_{f(e)} Y$ is an acyclic fibration.

$$\begin{tikzcd}
(R^1_e X)^* \arrow[r, dashed] \arrow[d] & (R^1_{f(e)} Y)^* \arrow[d] \\
R^1_e X \arrow[r, "R^1_e f"] \arrow[d, hook] & R^1_{f(e)} Y \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

All vertical ω-functors in this diagram are weak equivalences, so showing that $(R^1_e X)^* \to (R^1_{f(e)} Y)^*$ is a weak equivalence is equivalent to showing that f is. Thus we can reduce to the case of ω-functors between 1-reduced computads.

To complete the proof we must show that an ω-functor between 1-reduced computads is a weak equivalence if and only if it is sent to a quasi-isomorphism by λ. Recall from Proposition 4.49 that a 1-reduced ω-groupoid already has the structure of an ω-groupoid in abelian groups with

$$x + y = x *_0 y = x *_0 y$$

for $x, y \in X_n$ with $n > 0$. The identity for the group operation on n-cells is the identity n-cell 1_* for the unique 0-cell $*$ and the group structure on 0-cells and 1-cells is the trivial group. If we apply λ to an ω-groupoid with a single object and no non-identity 1-cells like this, the chain complex we obtain has groups of n-chains for $n \geq 2$ generated by classes $[x]$ of n-cells of X subject to the relations

$$[x *_n y] = [x] + [y]$$

Since the abelian group structure on X is defined using the composition we have

$$[x + y] = [x *_1 y] = [x] + [y]$$

and the chain group λX_n for $n > 2$ is given by CX_n, the n^{th} chain group of the chain complex corresponding under the equivalence of categories between $\omega\mathbf{Gpd}(\mathbf{Ab})$ and $\mathbf{Ch}$ of Proposition 4.49. For 1-cells we also have $\lambda X_1 = CX_1 = 0$ as there are only identity 1-cells. In the case of 0-cells is there a subtle distinction between these two constructions, as $\lambda X_0 = \mathbb{Z}$, the free abelian group with the single 0-cell of X as generator, while $CX_0 = 0$ since X as an ω-groupoid in abelian groups has the trivial group structure on its group of 0-cells. The map

$$i : CX \hookrightarrow \lambda X$$

that is the identity above degree 0 and in degree 0 is the inclusion of the trivial group in $\mathbb{Z}$ is a chain map as $CX_1 = \lambda X_1 = 0$ and so the diagram

$$\begin{tikzcd}
0 \arrow[r, equal, "i_1"] \arrow[d, "\partial"'] & 0 \arrow[d, "\partial"] \\
0 \arrow[r, hook, "i_0"'] & \mathbb{Z}
\end{tikzcd}$$

of abelian groups commutes. This map is a quasi-isomorphism and defines a natural transformation $i : C \to \lambda$ between functors C and λ restricted to the subcategory of 1-reduced ω-groupoids. By Proposition 4.49, since an ω-functor $f : X \to Y$ between 1-reduced ω-groupoids is a weak equivalence if and only if Cf is a quasi-isomorphism. Hence the diagram

$$\begin{tikzcd}
CX \arrow[r, hook, "\sim"] \arrow[d, "Cf"'] & \lambda X \arrow[d, "\lambda(f)"] \\
CY \arrow[r, hook, "\sim"'] & \lambda Y
\end{tikzcd}$$

where horizontal maps are quasi-isomorphisms shows that $\lambda(f)$ is a quasi-isomorphism if and only if f is a weak equivalence.

$\square$

With this tool for identifying weak equivalences we can prove that the ω-functors j_n^ε for $n \geq 1$ and $\varepsilon \in \{-,+\}$ are acyclic cofibrations as was claimed in Proposition 4.41.

Proof of Prop. 4.41. Let $n \geq 1$ and $\varepsilon \in \{-,+\}$. We will show that $j_n^\varepsilon : I^{n-1} \hookrightarrow I^n$ is an acyclic cofibration by first showing it is a cofibration between cofibrant, 1-connected ω-groupoids and then showing that $\lambda(j_n^\varepsilon)$ is a quasi-isomorphism. By Proposition 4.51 this makes j_n^ε a weak equivalence.

We will start with showing that this ω-functor is a cofibration. The ω-functors j_n^ε for $n \geq 1$ factor as

$$\begin{tikzcd}
\partial I^{n-1} \arrow[r, hook, "L^0 i_{n-1}"] \arrow[d, hook, "L^0 i_{n-1}"'] & I^{n-1} \arrow[d, hook, "d_{n-1}^{-\varepsilon}\iota_n"] & \\
I^{n-1} \arrow[r, hook, "d_{n-1}^{\varepsilon}\iota n"'] \arrow[rr, hook, bend right, "L^0 j_n^\varepsilon"'] & \partial I^n \arrow[r, hook, "L^0 i_n"] & I^n
\end{tikzcd}$$

where the square is a pushout, so j_n^ε is a cofibration. Furthermore, I^n is cofibrant for all $n \geq 1$ as

$$\begin{tikzcd}
\emptyset \arrow[r, hook, "L^0 i_0"] & I^0 \arrow[r, hook, "L^0 i_1"] & \cdots \arrow[r, hook, "L^0 i_n"] & I^n
\end{tikzcd}$$

are cofibrations.

Now we will show that I^n is 1-connected for all $n \geq 0$, so we can apply Proposition 4.51. All I^n for $n \geq 0$ are connected and I^0 is clearly weakly contractible. For $n = 1$ there is no non-trivial loop at any 0-cell so $\pi_1(I^1) = 0$. This already shows that $j_1^\varepsilon : I^0 \hookrightarrow I^1$ is an acyclic cofibration. If $n > 1$ then I_1^n has unique non-trivial loops at the 0-cell $d_0^- \iota_n$ given by

$$\begin{tikzcd}
d_0^- \iota_n \arrow[r, "d_1^- \iota_n"] & d_0^+ \iota_n \arrow[r, "k^1 d_1^+ \iota_n"] & d_0^- \iota_n
\end{tikzcd}$$

and its inverse. But the 2-cell $d_2^- \iota_n *_1 k^1 d_1^+ \iota_n$ gives a homotopy so that this loop is equivalent to the identity loop in $\varpi_x(X)$. Hence $\pi_1(I^n) = 0$ for all $n > 1$ since I^n is connected. Now let $n > 1$ and apply λ to the ω-functors j_n^ε. This gives

$$\begin{tikzcd}
\lambda(I^{n-1}) \arrow[dd, hook, "\lambda(j_n^\varepsilon)"'] & & \cdots \arrow[r] & 0 \arrow[r] \arrow[dd, equal] & 0 \arrow[r] \arrow[dd, hook] & \mathbb{Z} \arrow[r] \arrow[dd, hook, "i_\varepsilon"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[dd, equal] & \cdots \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[dd, equal] \\
& = & & & & & & & \\
\lambda(I^n) & & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}
\end{tikzcd}$$

where $i_\varepsilon : \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ sends $1 \in \mathbb{Z}$ to $(1,0)$ when $\varepsilon = +$ and to $(0,1)$ when $\varepsilon = -$. Both chain complexes have trivial homology except in degree 0 where $H_0(\lambda I^n) = H_0(\lambda I^{n-1}) = \mathbb{Z}$. So this is a quasi-isomorphism and hence by Proposition 4.51 j_n^ε is an acyclic cofibration. $\square$

4.7 The Street Nerve-Realization Adjunction

In [Str87] Street defined a cosimplicial object in ω**Cat** consisting of ω-categories called oriented simplices. In this section we will show that the nerve-realization adjunction between ω**Gpd** and

sSet determined by this cosimplicial object is a Quillen adjunction between ω**Gpd** with the folk mode structure and **sSet** with the Kan model structure. I have not been able to find an explicit statement for this result in literature for strict ω-groupoids or crossed complexes, however it is essentially contained in the results for crossed modules of Brown and Higgins in [BH91]. In particular, their Proposition 2.6 shows that the nerve creates weak equivalences and their Remark 2.5 shows that it preserves fibrant objects. In this section, for completeness, we will give self-contained proofs of these results from [BH91] in the language of ω-groupoids.

Street's definition of the oriented simplices was given in terms of parity complexes, which are structures defined to allow defining and manipulating cells of freely generated ω-categories. However, in [FM19] the authors show that not all ω-categories determined by parity complexes are computads. In [Ste04] the oriented simplices are constructed as ω-categories arising from strong Steiner complexes, which the author shows are computads. We will use this alternative definition using ADCs to present oriented simplices here.

Recall the ADCs $\mathbb{Z}[\Delta^n]$ defined in Example 4.18. As we show in Appendix B Proposition B.5 these are strong Steiner complexes with strongly loop-free basis consisting of the chains $[\theta] \in \mathbb{Z}[\Delta^n]_m$ corresponding to injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ in Δ. With the images of coface and codegeneracy maps of the standard simplicial simplices in **Ch** these ADCs form a cosimplicial object in **ADC**, which in turn determines a cosimplicial object in ω**Cat** by applying the functor ν. These are Street's oriented simplices. Since the ω-categories $\nu\mathbb{Z}[\Delta^n]$ are strong Steiner complexes, the following result of Steiner shows that they are computads in ω**Cat**.

Theorem 4.52 ([Ste04] Theorem 6.1)**.** *If K is a strong Steiner complex then νK is freely generated as a computad by its atoms.*

The ω-categories $\nu\mathbb{Z}[\Delta^n]$ therefore are freely generated as computads by the atoms $\langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m$ for $\theta : [m] \hookrightarrow [n]$ injective non-decreasing maps in Δ. In particular, there is a unique n-atom $\langle 1_{[n]}\rangle \in \nu\mathbb{Z}[\Delta^n]_n$ corresponding to the identity map on $[n]$ in Δ. The coface and codegeneracy ω-functors act on atoms by composing the map in Δ that determines the atom with the corresponding coface or codegeneracy map of Δ. Hence $d^i : \nu\mathbb{Z}[\Delta^n] \to \nu\mathbb{Z}[\Delta^{n+1}]$ for $0 \leq i \leq n+1$ always sends atoms to atoms as the coface maps $d^i : [n] \hookrightarrow [n+1]$ are injective and non-decreasing while $s^i : \nu\mathbb{Z}[\Delta^n] \to \nu\mathbb{Z}[\Delta^{n-1}]$ for $0 \leq i \leq n-1$ sends some atoms to identity cells, for example

$$s^i(\langle 1_{[n]}\rangle) = \langle s^i\rangle = 1_{\langle 1_{[n-1]}\rangle}$$

The structure of the oriented simplices arising from the results of [Ste04] is described in detail in Appendix B.

This cosimplicial ω-category gives rise to a nerve-realization adjunction between ω**Cat** and **sSet** in the usual way. The Street nerve of an ω-category is the simplicial set with set of n-simplices the set of ω-functors

$$\nu\mathbb{Z}[\Delta^n] \to X$$

The Street nerve for ω-categories has been studied in detail as a way to understand the homotopy theory of ω-categories, for example in [AM14] and [Gag18] as a potential source of a Thomason model structure on ω**Cat** and in [Ver08], which identifies exactly the simplicial sets that are the Street nerves of ω-categories.

Applying the functor $L^0 : \omega\mathbf{Cat} \to \omega\mathbf{Gpd}$ to the oriented simplices gives a cosimplicial object in ω**Gpd** which we will denote by

$$\mathcal{O}^n = L^0\nu\mathbb{Z}[\Delta^n]$$

When X is an ω-groupoid, since L^0 is left adjoint to the inclusion of $\omega\mathbf{Gpd}$ in $\omega\mathbf{Cat}$ we have as n-simplices of the Street nerve

$$N_\omega(X)_n = \{\alpha : \mathcal{O}^n \to X\}$$

This defines a functor $N_\omega : \omega\mathbf{Gpd} \to \mathbf{sSet}$, which we will call the Street nerve. Since we will primarily be concerned with ω-groupoids we will make it clear when any Street nerve is that of an ω-category rather than an ω-groupoid. The main goal of this section is to study the nerve-realization adjunction determined by the cosimplicial ω-groupoid $\mathcal{O}^n$

$$\begin{tikzcd} \omega\mathbf{Gpd} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & \mathbf{sSet} \arrow[l, bend left, "C_\omega"] \end{tikzcd} \tag{78}$$

which we will call the **Street nerve-realization adjunction**. The left adjoint of the Street nerve is the realization of a simplicial set defined in the usual way as a coend in $\omega\mathbf{Gpd}$. That is, for a simplicial set L

$$C_\omega(L) = \int^{n\in\Delta} L_n \cdot \mathcal{O}^n = \mathrm{coeq}\left(\bigsqcup_{\theta:[n]\to[m]} L_m \cdot \mathcal{O}^n \rightrightarrows \bigsqcup_{n} L_n \cdot \mathcal{O}^n \right) \tag{79}$$

Since L^0 is a left adjoint, the realization $C_\omega(L)$ is equal to L^0 applied to the corresponding ω-category realization determined by the oriented simplices $\nu\mathbb{Z}[\Delta^n]$. To distinguish this ω-category construction from the ω-groupoid construction we denote the realization of a simplicial set L as an ω-category by oriented simplices as $F_\omega(L)$. We can get a lot of information about the ω-groupoid $C_\omega(L)$ therefore by studying the corresponding ω-category realization $F_\omega(L)$.

This left adjoint realization functor F_ω for ω-categories seems to be less well-studied than the corresponding nerve construction. In [Ver08] Observation 245 there is a sketch of the proof that the realization of a simplicial set by this left adjoint is a computad, but this is based on parity complexes, so suffers from the same uncertainty mentioned at the start of this section arising from the results of [FM19]. We present another proof that the realization of a simplicial set is a computad that is not based on the formalism of parity complexes.

We begin by describing the m-cells of the ω-category $F_\omega(L)$ for a simplicial set L that will be the m-indeterminates. By the definition of $F_\omega(L)$ as a coend the ω-category $F_\omega(L)$ contains m-cells

$$[x \in L_n, \langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m]$$

which are the image of $(x \in L_n, \langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m) \in (L_n \cdot \nu\mathbb{Z}[\Delta^n])_m$. These cells satisfy the following identities as a result of the coequalizer

$$\begin{aligned} [d_i x \in L_n, \langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m] &= [x \in L_{n+1}, \langle d^i \circ \theta\rangle \in \nu\mathbb{Z}[\Delta^{n+1}]_m] \\ [s_j x \in L_n, \langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m] &= [x \in L_{n-1}, \langle s^j \circ \theta\rangle \in \nu\mathbb{Z}[\Delta^{n-1}]_m] \end{aligned}$$

where $0 \le i \le n+1$ and $0 \le j \le n-1$. Hence any such cell has a representative

$$[x \in L_m, \langle 1_{[m]}\rangle \in \nu\mathbb{Z}[\Delta^m]_m]$$

which is not an identity m-cell if x is non-degenerate. If $x = \sigma^*(y)$ for a surjective map $\sigma : [m] \to [p]$ then this cell is the m-cell identity on the p-cell $[x \in L_p, \langle 1_{[p]} \rangle \in \nu\mathbb{Z}[\Delta^p]_p]$ since

$$s^i(\langle 1_{[m]} \rangle) = \langle s^i \rangle = 1_{\langle 1_{[m-1]} \rangle} \in \nu\mathbb{Z}[\Delta^{m-1}]_m$$

A simplicial set L is the filtered colimit of its n-skeleta for $0 \leq n$, where the n-skeleton of L is obtained from the $n-1$-skeleton by the pushout

$$\begin{tikzcd}
\coprod_{x \in ND(L)_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}^{n-1} L \arrow[d, hook] \\
\coprod_{x \in ND(L)_n} \Delta^n \arrow[r] & \mathrm{sk}^n L
\end{tikzcd} \tag{80}$$

with $ND(X)_n$ the set of non-degenerate n-simplices of L. Since F_ω preserves colimits we can describe $F_\omega(L)$ by determining the image of the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ under the functor F_ω.

Proposition 4.53. *For all $n \geq 0$ the F_ω realization of the inclusion $\delta_n : \partial\Delta^n \hookrightarrow \Delta^n$ is the ω-functor*

$$\nu\mathbb{Z}[\partial\Delta^n] \hookrightarrow \nu\mathbb{Z}[\Delta^n]$$

arising from the functor ν of Definition 4.21 applied to the ADC map $\mathbb{Z}[\delta_n]$.

Proof. The simplicial set $\partial\Delta^n$ has the property that its non-degenerate simplices correspond to injective simplicial set maps $\Delta^m \hookrightarrow \partial\Delta^n$. Furthermore, $\mathbb{Z}[\partial\Delta^n]$ is a strong Steiner complex via the restriction of the basis and its total order from Example 4.27 for $\mathbb{Z}[\Delta^n]$. Hence by Lemma 4.33

$$\nu\mathbb{Z}[\partial\Delta^n] = \mathrm{colim}_{(\Delta \downarrow \partial\Delta^n)} \nu\mathbb{Z}[\Delta^m]$$

Since $F_\omega(\Delta^m) = \nu\mathbb{Z}[\Delta^m]$ therefore $\nu\mathbb{Z}[\partial\Delta^n]$ is the colimit of the diagram

$$F_\omega \circ \Delta\partial\Delta^n : (\Delta \downarrow \partial\Delta^n) \to \omega\mathbf{Cat}$$

where $\Delta\partial\Delta^n : (\Delta \downarrow \partial\Delta^n) \to \mathbf{sSet}$ is the canonical diagram that gives $\partial\Delta^n$ as a colimit over its category of simplices. Since F_ω is a left adjoint it preserves colimits and so $F_\omega(\partial\Delta^n) = \nu\mathbb{Z}[\partial\Delta^n]$. Furthermore, $\nu\mathbb{Z}[\delta_n] : \nu\mathbb{Z}[\partial\Delta^n] \hookrightarrow \nu\mathbb{Z}[\Delta^n]$ is the unique ω-functor corresponding to the cocone under the diagram $F_\omega \circ \Delta'\partial\Delta^n$ for $\nu\mathbb{Z}[\Delta^n]$, so $F_\omega(\delta_n) = \nu\mathbb{Z}[\delta_n]$. □

As strong Steiner complexes, $\mathbb{Z}[\partial\Delta^n]$ and $\mathbb{Z}[\Delta^n]$ both give ω-categories under the functor ν that are computads with m-indeterminates given by the atoms $\langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m$ for injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ in Δ. Hence there is a pushout of ω-categories

$$\begin{tikzcd}
\partial D^n \arrow[r, "{\langle d^-_{n-1}\langle 1_{[n]}\rangle, d^+_{n-1}\langle 1_{[n]}\rangle\rangle}"] \arrow[d, hook, "i_n"'] & \nu\mathbb{Z}[\partial\Delta^n] \arrow[d, hook] \\
D^n \arrow[r, "\langle 1_{[n]}\rangle"'] & \nu\mathbb{Z}[\Delta^n]
\end{tikzcd} \tag{81}$$

attaching the unique n-atom $\langle 1_{[n]} \rangle$ of $\nu\mathbb{Z}[\Delta^n]$, which is the only atom missing from $\nu\mathbb{Z}[\partial\Delta^n]$. Combined with the previous discussion of skeleta of simplicial sets we have the following description of $F_\omega(L)$ and hence $C_\omega(L)$, since L_0 preserves computads.

Proposition 4.54. *Let L be a simplicial set. The ω-category $F_\omega(L)$ is a computad with m-indeterminates the m-cells*

$$[x \in L_m, \langle 1_{[m]} \rangle \in \nu\mathbb{Z}[L]_m]$$

for $x \in L_m$ a non-degenerate m-simplex.

The ω-groupoid $C_\omega(L)$ is a computad with a set of m-indeterminates given by the m-cells

$$[x \in L_m, \langle 1_{[m]} \rangle \in \mathcal{O}^n_m]$$

for $x \in L_m$ a non-degenerate m-simplex.

Proof. Since F_ω preserves colimits there is a pushout

$$\begin{tikzcd}
\coprod\limits_{x \in ND(L)_n} \partial D^n \arrow[r, "{\langle d^-_{n-1}\langle 1_{[n]}\rangle, d^+_{n-1}\langle 1_{[n]}\rangle\rangle}"] \arrow[d, hook, "\coprod i_n"'] & \coprod\limits_{x \in ND(L)_n} \nu\mathbb{Z}[\partial\Delta^n] \arrow[r] \arrow[d, hook] & F_\omega(\mathrm{sk}^{n-1} L) \arrow[d, hook] \\
\coprod\limits_{x \in ND(L)_n} D^n \arrow[r, "\langle 1_{[n]} \rangle"'] & \coprod\limits_{x \in ND(L)_n} \nu\mathbb{Z}[\Delta^n] \arrow[r] & F_\omega(\mathrm{sk}^n L)
\end{tikzcd}$$

obtained by combining the pushouts (81) and F_ω applied to (80). This attaches the n-cells $[x \in L_m, \langle 1_{[m]} \rangle \in \nu\mathbb{Z}[L]_m]$ to the $n-1$-skeleton and since all n-cells of $F_\omega(L)$ are determined by the n-skeleton of L in the colimit $F_\omega(L)$ is a computad with n-indeterminates given by these cells.

Since L^0 preserves colimits the description of $C_\omega(L) = L^0 F_\omega(L)$ follows from the fact $F_\omega(L)$ is a computad. □

For our proof that the Street nerve-realization adjunction is Quillen we will make use of the Quillen adjunction $\lambda \dashv U$ that was proved in the previous section in Proposition 4.50.

Proposition 4.55. *The Street nerve-realization adjunction* (78) *is a Quillen adjunction between ω**Gpd** with the folk model structure and* **sSet** *with the standard Quillen model structure.*

Proof. We will show that C_ω preserves cofibrations and acyclic cofibrations by showing that it sends generating sets of each of these classes to the corresponding class of ω-functors. The simplicial set cofibrations

$$\partial\Delta^n \hookrightarrow \Delta^n$$

for $n \geq 0$ are sent by C_ω to the ω-functors

$$\partial\mathcal{O}^n \to \mathcal{O}^n$$

where $\mathcal{O}^n$ is the standard Street oriented simplex and $\partial\mathcal{O}^n = L^0\nu\mathbb{Z}[\partial\Delta^n]$ by Proposition 4.53. Applying L^0 to the pushout (81) shows that $\partial\mathcal{O}^n \to \mathcal{O}^n$ is a cofibration of ω**Gpd** because L^0 preserves pushouts and $L^0 i_n : \partial I^n \hookrightarrow I^n$ is a generating cofibration for the model structure on ω**Gpd**.

Now consider the generating acyclic cofibrations $\Lambda^k_n \hookrightarrow \Delta^n$ of **sSet** for $n \geq 1$ and $0 \leq k \leq n$. The ω-functors $C_\omega(\Lambda^k_n) \hookrightarrow C_\omega(\Delta^n)$ are sent by λ to the chain maps

$$\mathbb{Z}[\Lambda^k_n] \hookrightarrow \mathbb{Z}[\Delta^n]$$

which are known to be quasi-isomorphisms. Hence by Proposition 4.51 it is sufficient to show that $C_\omega(\Lambda^k_n)$ and $C_\omega(\Delta^n) = \mathcal{O}^n$ are 1-connected.

It is clear that $\mathcal{O}^n$ is 0-connected, since for all $i \in [n]$ there exists a 1-cell $\langle 0, i\rangle : 0 \to i$. We will show that $\pi_1(\mathcal{O}^n, 0)$ is trivial. Take a 1-cell $\alpha \in \mathcal{O}^n_1$ that is a loop at 0. It must be of the form

$$0 \xrightarrow{\langle 0,j_1\rangle} j_1 \xrightarrow{\alpha_1} j_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{m-1}} j_m \xrightarrow{k^1\langle 0,j_m\rangle} 0 \tag{82}$$

for $m \geq 0$ with $j_i \neq j_{i+1}$ for all $1 \leq i < m$ and where each 1-cell $\alpha_i : j_i \to j_{i+1}$ is either $\langle j_i, j_{i+1}\rangle$ or $k^1\langle j_{i+1}, j_i\rangle$ depending on whether $j_i < j_{i+1}$ or $j_i > j_{i+1}$. We will prove that these are all homotopic to $1_0 : 0 \to 0$ by induction on m, the number of 0-cells other than 0 that the loop passes through.

If $m = 0$ this is clearly the identity 1-cell $1_0 : 0 \to 0$. If $m = 1$ then $j_1 = j_m$ and so this loop of two 1-cells is the composite of a 1-cell and its inverse, which is again the identity 1-cell at 0. Now let $m \geq 2$ and suppose that a loop at 0 that is a composite of m 1-cells is homotopic to $1_{[0]}$. Consider α which is a loop at 0 and a composite of $m+1$ 1-cells passing through j_i for $1 \leq i \leq m$ as in (82). As $m \geq 2$ we at least have $j_1, j_2 \in [n]$ with $j_1 \neq j_2$. Now either $j_1 < j_2$ or $j_1 > j_2$. In the first case we have $\alpha_1 = \langle j_1, j_2\rangle$. Consider the 2-cell $\langle 0, j_1, j_2\rangle : \langle 0, j_2\rangle \implies \langle 0, j_1\rangle *_0 \langle j_1, j_2\rangle$. Whiskering this 2-cell on the right by α_i for $i \geq 2$ and $k^1\langle 0, j_m\rangle$ gives a 2-cell

$$\begin{tikzcd}
& j_1 \arrow[dr, "\alpha_1"] \arrow[d, phantom, "\Uparrow"] \\
0 \arrow[ur, "\langle 0,j_1\rangle"] \arrow[rr, "\langle 0,j_2\rangle"'] & {} & j_2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{m-1}"] & j_m \arrow[r, "k^1\langle 0,j_m\rangle"] & 0
\end{tikzcd}$$

The top path is α, so α belongs to the same homotopy class as another loop at 0 which is a composite of m 1-cells. By the induction hypothesis therefore it is homotopic to 1_0. If $j_1 > j_2$ then $\alpha_1 = k^1\langle j_2, j_1\rangle$. Consider the 2-cell

$$\langle 0, j_2, j_1\rangle *_0 k^1\langle j_2, j_1\rangle : \langle 0, j_2\rangle *_0 k^1\langle j_2, j_1\rangle \implies \langle 0, j_1\rangle$$

Again whiskering by α_i for $i \geq 2$ and $k^1\langle 0, j_m\rangle$ gives a 2-cell

$$\begin{tikzcd}
& j_1 \arrow[dr, "\alpha_1"] \arrow[d, phantom, "\Uparrow"] \\
0 \arrow[ur, "\langle 0,j_1\rangle"] \arrow[rr, "\langle 0,j_2\rangle"'] & {} & j_2 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{m-1}"] & j_m \arrow[r, "k^1\langle 0,j_m\rangle"] & 0
\end{tikzcd}$$

The top path is α, so α belongs to the same homotopy class as another loop at 0 which is a composite m 1-cells. By the induction hypothesis therefore it is homotopic to 1_0.

For $C_\omega(\Lambda^k_n) = L^0\nu\mathbb{Z}[\partial\Delta^n]$ if $n > 2$ then the map $C_\omega(\partial\Delta^n) \hookrightarrow C_\omega(\Delta^n)$ is the identity on 0, 1, and 2-cells, so it induces an isomorphism on fundamental groups. If $n = 1$ then $\Lambda^k_1 = \Delta^0$ so it is clear that $C_\omega(\Lambda^k_1)$ is 1-connected. It only remains to show that $C_\omega(\Lambda^k_2)$ is 1-connected for all $0 \leq k \leq 2$. The simplicial set Λ^k_2 is 1-skeletal, so $C_\omega(\Lambda^k_2)$ has no non-identity 2-cells and its only 1-cells are two out of $\langle 0, 1\rangle$, $\langle 1, 2\rangle$, and $\langle 0, 2\rangle$. So $C_\omega(\Lambda^k_2)$ is 1-connected as there are no non-identity loops at any object and all 0-cells have a 1-cell connecting them to $k \in [2]$. □

This proof shows that the fact that $\lambda \dashv U$ is Quillen goes a long way to proving that the Street nerve-realization adjunction is Quillen. The only further details to check are found in the restriction to the 2-groupoid truncation. The Street nerve for 2-groupoids was shown to be Quillen by Moerdijk and Svensson in [MS93], so this result can be seen as a small generalization of that result by the results of [Gue21] that we adapted in Proposition 4.50.

Combining the Street nerve-realization adjunction with the linearization adjunction of Proposition 4.50 gives an adjunction between **sSet** and **Ch**. In the next result we see that this is simply the free-forgetful adjunction, aided by the Dold-Kan equivalence.

Proposition 4.56. *The following diagram of Quillen adjunctions commutes*

$$\begin{tikzcd}
\boldsymbol{sSet} \arrow[r, bend right, "C_\omega"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\mathbb{Z}[-]"'] \arrow[d, phantom, "\dashv"] & \omega\boldsymbol{Gpd} \arrow[l, bend right, "N_\omega"'] \arrow[d, bend right, "\lambda"'] \arrow[d, phantom, "\dashv"] \\
\boldsymbol{sAb} \arrow[u, bend right, "U"'] \arrow[r, bend right, "N"'] \arrow[r, phantom, "\top"] & \boldsymbol{Ch} \arrow[l, bend right, "\Gamma"'] \arrow[u, bend right, "U"']
\end{tikzcd}$$

where $N \dashv \Gamma$ *is the Dold-Kan equivalence.*

Proof. We will show that $\lambda C_\omega(L) = N\mathbb{Z}[L]$ for all simplicial sets L, the right adjoints will then be equal by uniqueness of adjoints. First we observe that $\lambda\mathcal{O}^n$ is the chain complex freely generated by m-chains $[\theta] \in \lambda\mathcal{O}^n_m$ corresponding to injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ in Δ. These are the non-degenerate simplices of Δ^n so

$$\lambda\mathcal{O}^n = N\mathbb{Z}[\Delta^n]$$

the normalized chain complex of the free simplicial abelian group on Δ^n. Now recall the definition of $C_\omega(L)$ as the coend (79). Therefore $\lambda C_\omega(L)$ is given by

$$\begin{aligned}
\lambda C_\omega(L) &= \int^{n\in\Delta} L_n \cdot \lambda\mathcal{O}^n \\
&= \int^{n\in\Delta} L_n \cdot N\mathbb{Z}[\Delta^n] \\
&= N\mathbb{Z}\left[\int^{n\in\Delta} L_n \cdot \Delta^n\right] \\
&= N\mathbb{Z}[L]
\end{aligned}$$

since λ and $\mathbb{Z}$ are left adjoints so they commute with colimits and N is part of an equivalence of categories so it does as well. □

We now have seen two notions of homotopy groups for an ω-groupoid X: the algebraic definition given in Definition 4.13 using the structure of the ω-groupoid and those of the simplicial set $N_\omega(X)$, which is a Kan complex since all ω-groupoids are fibrant in the folk model structure and N_ω is a right Quillen functor. As mentioned above, Proposition 2.6 of [BH91] for crossed complexes shows that these two notions give isomorphic groups. In the next result we give a proof this result in the context of ω-groupoids.

Proposition 4.57. *Let* X *be an* ω*-groupoid. The simplicial set* $N_\omega(X)$ *is a Kan complex and for any* 0*-simplex* $\alpha_x : \mathcal{O}^0 \to X$ *of* $N_\omega(X)$ *the groups* $\pi_n(N_\omega(X), \alpha_x)$ *are isomorphic to* $\pi_n(X, 1_x)$*.*

Proof. All ω-groupoids are fibrant in the folk model structure on $\omega\mathbf{Gpd}$ by Theorem 4.39, so since N_ω is a right Quillen functor $N_\omega(X)$ is a fibrant simplicial set.

The homotopy group $\pi_n(N_\omega(X), \alpha_x)$ is defined in Definition 3.6 of [May67] as the quotient of the set of n-simplices $\alpha \in N_\omega(X)_n$ such that $d_i\alpha = s_0^{n-1}(\alpha_x)$ for all $0 \leq i \leq n$ by the restriction to this subset of the equivalence relation that identifies $\alpha, \beta : \mathcal{O}^n \to X$ if

1. $d_i\alpha = d_i\beta$ for all $0 \leq i \leq n$ and
2. there exists $\gamma : \mathcal{O}^{n+1} \to X$ such that $d_0\gamma = \alpha$, $d_1\gamma = \beta$ and
$$d_i\gamma = s_{n-1}d_i\alpha = s_{n-1}d_i\beta$$
for all $2 \leq i \leq n+1$

The n-simplex $s_0^{n-1}\alpha_x : \mathcal{O}^n \to X$ is the composite

$$\mathcal{O}^n \xrightarrow{(s^0)^{n-1}} \mathcal{O}^0 \xrightarrow{\alpha_x} X$$

which sends all atoms $\langle\theta\rangle \in \mathcal{O}^n_m$ to the identity m-cell $1_x \in X_m$ on the 0-cell $x \in X_0$. An n-cell $\alpha : \mathcal{O}^n \to X$ with $d_i\alpha = s_0^{n-1}\alpha_x$ for all $0 \leq i \leq n$, therefore, is an ω-functor that sends every m-atom $\langle\theta\rangle \in \mathcal{O}^n_m$ for $m < n$ to the identity m-cell $1_x \in X_m$ on $x \in X_m$. Such an n-simplex therefore is uniquely determined by where it sends the unique n-atom $\langle 1_{[n]}\rangle$. By Lemma A.2 since $\alpha(\langle d^i\rangle) = 1_x$ for all $0 \leq i \leq n$

$$\begin{aligned} \alpha(d^-_{n-1}\langle 1_{[n]}\rangle) &= 1_x \\ \alpha(d^+_{n-1}\langle 1_{[n]}\rangle) &= 1_x \end{aligned}$$

This establishes a bijection between n-simplices $\alpha : \mathcal{O}^n \to X$ such that $d_i\alpha = s_0^{n-1}(\alpha_x)$ and n-cells $z \in X_n$ such that $d^-_{n-1}z = d^+_{n-1}z = 1_x$.

Let $\gamma : \mathcal{O}^{n+1} \to X$ be an $n+1$-simplex of $N_\omega(X)$ such that $d_i\gamma = s_0^n(\alpha_x)$ for all $2 \leq i \leq n+1$. This corresponds to an ω-functor $\mathcal{O}^{n+1} \to X$ that sends all m-atoms $\langle\theta\rangle \in \mathcal{O}^{n+1}_m$ corresponding to injective non-decreasing maps $\theta : [m] \hookrightarrow [n+1]$ except d^0, d^1, and $1_{[n+1]}$ to 1_x. By the previous discussion, there is a bijection between n-cells of X with source and target equal to 1_x and possible values of $\gamma(\langle d^0\rangle)$ and $\gamma(\langle d^1\rangle)$. For $\gamma(\langle 1_{[n+1]}\rangle)$ we have by Proposition B.10

$$\gamma(d^-_{n-1}\langle 1_{[n+1]}\rangle) = \gamma(\omega_n^{(1)}) *_{n-1} \gamma(\omega_n^{(3)}) *_{n-1} \cdots *_{n-1} \gamma(\omega_n^{(2\lfloor {}^n/{}_2\rfloor+1)})$$

$$\gamma(d^+_{n-1}\langle 1_{[n+1]}\rangle) = \gamma(\omega_n^{(2\lfloor {}^{n+1}/{}_2\rfloor)}) *_{n-1} \cdots *_{n-1} \gamma(\omega_n^{(2)}) *_{n-1} \gamma(\omega_n^{(0)})$$

Again by Lemma A.2 since $\gamma(\langle d^i\rangle) = 1_x$ for all $2 \leq i \leq n+1$ we have

$$\gamma(d^-_{n-1}\langle 1_{[n+1]}\rangle) = \gamma(\omega_n^{(1)})$$

$$\gamma(d^+_{n-1}\langle 1_{[n+1]}\rangle) = \gamma(\omega_n^{(0)})$$

Using the notation of Definition B.7, for all $\vec{j} \in J^{(n)t}_{n+1,m}$ we have $\gamma(\langle d^{\vec{j}}\rangle) = 1_x$ so by Corollary B.12

$$\gamma(\omega_n^{(1)}) = \gamma(\langle d^1\rangle)$$

and similarly

$$\gamma(\omega_n^{(0)}) = \gamma(\langle d^0\rangle)$$

Hence the cell $\gamma(\langle 1_{[n+1]}\rangle)$ is an $n+1$-cell of X with source and target given by $\gamma(\langle d^0\rangle)$ and $\gamma(\langle d^1\rangle)$ respectively. Thus a simplicial homotopy between n-simplices $\alpha, \beta \in N_\omega(X)_n$ whose faces are all α_x corresponds exactly to an $n+1$-cell of X that witnesses a homotopy between n-cells $\alpha(\langle 1_{[n]}\rangle)$ and $\beta(\langle 1_{[n]}\rangle)$ of X.

Definition 4.13 of the homotopy groups of X now shows that there is a well-defined isomorphism

$$\alpha : \pi_n(X, 1_x) \to \pi_n(N_\omega(X), \alpha_x)$$

that sends the class of an n-cell $z \in X_n$ with $d^-_{n-1}z = d^+_{n-1}z = 1_x$ to the class of the n-simplex $\alpha(z) : \mathcal{O}^n \to X$ determined by the ω-functor

$$\alpha(z)(\langle\theta\rangle) = \begin{cases} z & \theta = 1_{[n]} \\ 1_x & \theta \neq 1_{[n]} \end{cases}$$

We must show that this is a group homomorphism when $n \geq 1$. Let $[z], [w] \in \pi_n(X, 1_x)$ and consider $[z][w] = [z *_{n-1} w]$ in the group $\pi_n(X, 1_x)$. We will show that $[\alpha(z *_{n-1} w)] = [\alpha(z)][\alpha(w)]$ in the group structure for the homotopy group $\pi_n(N_\omega(X), \alpha_x)$. This means we must show that $\alpha(z *_{n-1} w)$ is a filler for the horn $\Lambda^n_{n+1} \to N_\omega(X)$ determined by

$$\begin{array}{rcll} d_0 & = & \alpha(w) & \\ d_1 & = & ? & \\ d_2 & = & \alpha(z) & \\ d_i & = & \alpha_x & 3 \leq i \leq n+1 \end{array}$$

A filler for this horn would be an $n+1$-simplex $\gamma : \mathcal{O}^{n+1} \to X$ that sends $\langle d^i\rangle$ to 1_x for all $3 \leq i \leq n+1$ and sends

$$\begin{array}{rcl} \gamma(\langle d_0\rangle) & = & z \\ \gamma(\langle d_2\rangle) & = & w \end{array}$$

The $n+1$-cell $\gamma(\langle 1_{[n+1]}\rangle) \in X_{n+1}$ must have its source and target consistent with the images of these atoms. Since $\gamma(\langle\theta\rangle) = 1_x$ for all injective non-decreasing maps θ of Δ except for d^0, d^1, d^2, and $1_{[n+1]}$ by Lemma A.2 and Corollary B.12

$$\begin{array}{rcl} \gamma(d^+_n\langle 1_{[n+1]}\rangle) & = & \gamma(\omega^{(2)}_n) *_{n-1} \gamma(\omega^{(0)}_n) \\ & = & \gamma(\langle d^{(2)}\rangle) *_{n-1} \gamma(\langle d^{(0)}\rangle) \\ & = & z *_{n-1} w \end{array}$$

$$\begin{array}{rcl} \gamma(d^-_n\langle 1_{[n+1]}\rangle) & = & \gamma(\omega^{(1)}_n) \\ & = & \gamma(\langle d^{(1)}\rangle) \end{array}$$

Hence taking $\gamma(\langle d_1\rangle) = z *_{n-1} w$ and $\gamma(\langle 1_{[n+1]}\rangle) = 1_{z *_{n-1} w}$ defines an ω-functor $\gamma : \mathcal{O}^{n+1} \to X$ that makes $\alpha(z *_{n-1} w)$ a filler for the horn and hence $[\alpha(z)][\alpha(w)] = [\alpha(z *_{n-1} w)]$. □

As an immediate corollary we recover the result of [BH91] Proposition 2.6 that N_ω creates weak equivalences of ω-groupoids in the following sense.

Corollary 4.58. *An ω-functor $f : X \to Y$ in ω**Gpd** is a weak equivalence if and only if $N_\omega(f)$ is a weak equivalence of simplicial sets.*

4.8 Homotopy 2-Types and ω-Groupoids

It is well-known (see for example [Sim98]) that the adjunction $C_\omega \dashv N_\omega$ does not induce a Quillen equivalence between $\omega\mathbf{Gpd}$ with the folk model structure and the Kan model structure on simplicial sets. In fact, Simpson shows that when $n \geq 3$ there is no reasonable realization functor from the category of n-groupoids to the category of topological spaces that realizes all n-types. However, as mentioned before, in [MS93] it is shown that 2-groupoids do model all 2-types. In this section we will show that the adjunction $C_\omega \dashv N_\omega$ is a Quillen equivalence when $\omega\mathbf{Gpd}$ and $\mathbf{sSet}$ are localized to ignore all homotopy data above degree 2. This is the best we can hope for out of the folk model structure on $\omega\mathbf{Gpd}$ as the next result shows. This result further extends the connections described in [Ara13] between 1-reduced ω-groupoids and chain complexes discussed in Section 4.6.

Proposition 4.59. *Let K be a simply-connected simplicial set. The unit map $\eta_K : K \to N_\omega C_\omega(K)$ of the Street nerve-realization adjunction is weakly equivalent to the Hurewicz map $K \to U\mathbb{Z}[K]$ arising as the unit of the free-forgetful adjunction $U \dashv \mathbb{Z}[-]$ between* ***sSet*** *and* ***sAb****.*

Proof. Since K is simply connected for any 0-simplex $x \in K_0$ the inclusion of the simplicial subset K_x with unique 0-simplex x and only degenerate 1-simplices is a weak equivalence. Since $C_\omega \dashv N_\omega$ is a Quillen adjunction by Proposition 4.55 and all simplicial sets are cofibrant and all ω-groupoids are fibrant in the folk model structure, the endofunctor $N_\omega C_\omega : \mathbf{sSet} \to \mathbf{sSet}$ preserves weak equivalences. Hence we may assume that K is 1-reduced in the sense of having a unique 0-simplex and no non-degenerate 1-simplices.

If K is 1-reduced then so is the ω-groupoid $C_\omega(K)$ as the 0-cells of $C_\omega(K)$ are the 0-simplices of K and degenerate simplices correspond to identity cells of $C_\omega(K)$. Hence by Proposition 4.49 $C_\omega(K)$ has the structure of an ω-groupoid in abelian groups. There is an isomorphism

$$\begin{array}{rcll}
N_\omega(UC_\omega(K)) & = & \mathrm{Hom}_{\omega\mathbf{Gpd}}(\mathcal{O}^\bullet, UC_\omega(K)) & \\
& \cong & \mathrm{Hom}_{\omega\mathbf{Gpd(Ab)}}(F(\mathcal{O}^\bullet), C_\omega(K)) & \text{by the adjunction } F \dashv U \ (77) \\
& \cong & \mathrm{Hom}_{\mathbf{Ch}}(CF(\mathcal{O}^\bullet), C(C_\omega(K)) & \text{by Prop. 4.48} \\
& \cong & \mathrm{Hom}_{\mathbf{Ch}}(\lambda(\mathcal{O}^\bullet), C(C_\omega(K)) & \text{by Prop. 4.50} \\
& \cong & \mathrm{Hom}_{\mathbf{Ch}}(\mathbb{Z}[\Delta^\bullet], C(C_\omega(K)) & \\
& = & U\Gamma(C(C_\omega(K))) & \\
& = & U\Gamma(N\mathbb{Z}[K]) &
\end{array}$$

since $C_\omega(K)$ is a computad and an ω-groupoid in abelian groups with generating sets of n-cells the non-degenerate n-simplices of K. Since $N \dashv \Gamma$ is an adjoint equivalence the map $K \to U\Gamma(N\mathbb{Z}[K])$ is isomorphic to the unit of the free-forgetful adjunction $U \dashv \mathbb{Z}[-]$. □

This shows that there is no hope of the unit map for the adjunction $C_\omega \dashv N_\omega$ being a weak equivalence for all simplicial sets K. A concrete counterexample is the sphere S^2 modelled as the 1-reduced simplicial set with a single 2-simplex and no non-degenerate higher simplices. The chain complex $\mathbb{Z}[S^2]$ has non-trivial homology in degree 2 only, while the higher homotopy groups of S^2 are certainly not all 0. By [use] this unit map is a weak equivalence only for an extremely restricted class of Eilenberg-Mac Lane spaces.

The Quillen equivalence between 2-groupoids and homotopy 2-types from [MS93] can be realized using the Quillen adjunction $C_\omega \dashv N_\omega$ by localizing the model structures on $\mathbf{sSet}$ and $\omega\mathbf{Gpd}$. The results of [MS93] show that the unit of this adjunction induces isomorphisms of connected components and the first two homotopy groups.

Proposition 4.60 ([MS93] Corollary 2.6)**.** *The unit map $\eta_K : K \to N_\omega C_\omega(K)$ of the Street nerve-realization adjunction induces isomorphisms on π_i for $0 \leq i \leq 2$.*

Proof. The functor W defined in [MS93] is the composite of $C_\omega : \mathbf{sSet} \to \omega\mathbf{Gpd}$ with the smart 2-truncation functor of Definition 4.11. The smart truncation functor preserves homotopy groups in dimensions $n \leq 2$, so the claimed result follows immediately from Corollary 2.6 of [MS93]. □

Instead of truncating to 2-groupoids to recover the Quillen equivalence of 2-types with 2-groupoids we take the approach of localizing the model structure on $\omega\mathbf{Gpd}$ at ω-functors that ignore all homotopy information above degree 2. This is done by the applying the methods of [Lac11] §6 for **sSet** to $\omega\mathbf{Gpd}$. As in [Lac11] §6 let $g_{n+2} : \partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 1$ be the generating cofibration for the Kan model structure on **sSet**. The model structure on **sSet** is left proper and cellular, so by [Hir03] Theorem 4.1.1 there exists a localization of **sSet** at the single map g_n, which we will denote by $P_n\mathbf{sSet}$ following [Lac11] §6. A simplicial set map f is a weak equivalence of $P_n\mathbf{sSet}$ if and only if it induces isomorphisms on π_i for all $0 \leq i \leq n$. We will now define the localization of $\omega\mathbf{Gpd}$ that determines a model structure Quillen equivalent to $P_2\mathbf{sSet}$ via the Street nerve-realization adjunction.

Proposition 4.61. *The left Bousfield localization of $\omega\mathbf{Gpd}$ with the folk model structure at the map $C_\omega(g_2) : \partial\mathcal{O}^4 \hookrightarrow \mathcal{O}^4$ exists. It as a model structure $P_2\omega\mathbf{Gpd}$ with*

- *the same cofibrations as the folk model structure on $\omega\mathbf{Gpd}$*
- *fibrant objects the ω-groupoids X such that $\pi_n X = 0$ for $n > 2$*
- *weak equivalences the ω-functors that induce isomorphisms on π_n for $0 \leq n \leq 2$*

Proof. First, Theorems 4.2 and 4.4 of [Sau03] show that the model structure on $\omega\mathbf{Gpd}$ is proper and cellular via the equivalence [BH81b] of the category of ω-groupoids with the category of crossed complexes. Hence Theorem 4.1.1 of [Hir03] shows that the left Bousfield localization of $\omega\mathbf{Gpd}$ at any set of maps exist. Let $P_2\omega\mathbf{Gpd}$ denote the left Bousfield localization of the folk model structure at the map $C_\omega(g_2) : \partial\mathcal{O}^4 \hookrightarrow \mathcal{O}^4$. This model structure has the same cofibrations as the folk model structure on $\omega\mathbf{Gpd}$ by the definition of left Bousfield localization. By Lemma 3.3.11 and Definition 3.3.8 of [Hir03] we can identify the fibrant objects of $P_2\omega\mathbf{Gpd}$ by taking a cosimplicial resolution of $C_\omega(g_2)$ in $\omega\mathbf{Gpd}$ that is also a Reedy cofibration of $\omega\mathbf{Gpd}^\Delta$.

By Corollary 16.2.2 of [Hir03] since C_ω is a left Quillen functor by Proposition 4.55 it is sufficient to take a cosimplicial resolution of g_2 that is a Reedy cofibration in $\mathbf{sSet}^\Delta$ and apply C_ω to it. Since $g_2 : \partial\Delta^4 \hookrightarrow \Delta^4$ is a cofibration in **sSet** the map of cosimplicial simplicial sets

$$g_2 \times \Delta^\bullet : \partial\Delta^4 \times \Delta^\bullet \hookrightarrow \Delta^4 \times \Delta^\bullet$$

is a cosimplicial resolution of g_2 that is also a Reedy cofibration in $\mathbf{sSet}^\Delta$ by Corollaries 15.9.10 and 15.9.11 of [Hir03]. By Lemma 3.3.11, therefore, an ω-groupoid X is fibrant in the model structure $P_2\omega\mathbf{Gpd}$ if and only if $N_\omega(X)$ is fibrant in the model structure $P_2\mathbf{sSet}$. Hence by Proposition 1.5.1 of [Hir03] and Proposition 4.57 X is a P_2-fibrant ω-groupoid if and only if $\pi_n(X) = 0$ for all $n \geq 3$.

Recall the truncation functor τ^n defined in Definition 4.5. For an ω-groupoid X the 3-truncation has only identity n-cells for $n > 3$, so $\pi_n(\tau^3(X)) = 0$ for all $n > 2$ and so $\tau^3(X)$ is P_2-fibrant. We will show that for all ω-groupoids X the inclusion $\tau^3(X) \hookrightarrow X$ is a P_2-weak equivalence. Recall the cofibrant replacement for the folk model structure on $\omega\mathbf{Gpd}$ given in Lemma 4.46. This construction builds X^*, the cofibrant replacement of an ω-groupoid X, by successively attaching n-cells to cofibrant $n-1$-groupoids. Hence

$$\tau^3(X^*) = (X^*)^3 = (\tau^3(X))^*$$

using the notation of Lemma 4.46 for the construction of X^* as the filtered colimit of inclusions $(X^*)^{n-1} \hookrightarrow (X^*)^n$. Furthermore the functor τ^3 preserves weak equivalences so if $X^* \xrightarrow{\sim} X$ is the cofibrant replacement of X in the folk model structure on $\omega\mathbf{Gpd}$ then the diagram

$$\begin{tikzcd}
\tau^3(X^*) \arrow[r, hook] \arrow[d, "\wr"'] & X^* \arrow[d, "\wr"] \\
\tau^3(X) \arrow[r, hook] & X
\end{tikzcd}$$

commutes and shows that $\tau^3(X^*)$ is the computad replacement of $\tau^3(X)$. Hence we may assume X is a computad and hence $\tau^3(X) \hookrightarrow X$ is a cofibration. By Proposition 3.5.3 of [Hir03] we must show that for every P_2-fibrant ω-groupoid Y the map of homotopy classes of maps

$$\pi_{\omega\mathbf{Gpd}}(X, Y) \to \pi_{\omega\mathbf{Gpd}}(\tau^3(X), Y) \tag{83}$$

induced by the inclusion $\tau^3(X) \hookrightarrow X$ is an isomorphism.

If Y is P_2-fibrant then the natural map $Y \to c^3(Y)$ to the cotruncation of Y is a weak equivalence, since all parallel n-cells of $c^3(Y)$ are homotopic if $n > 2$ by the Definition 4.6 of the cotruncation functor. Hence we may assume that Y is 3-cotruncated, so the natural map $Y \cong c^3(Y)$ is an isomorphism. Hence the map (83) is surjective as all lifting problems

$$\begin{tikzcd}
\tau^3(X) \arrow[r] \arrow[d, hook] & c^3(Y) \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

have solutions by the adjointness $\tau^3 \dashv c^3$. It only remains to show that this map is injective. For this we will use the fact that $\hom_{\mathrm{lax}}(I^1, Y)$ is a path space for any ω-groupoid Y by Proposition 4.44. By the symmetric closed monoidal structure on $\omega\mathbf{Gpd}$ and the fact that L_0 is left adjoint to the inclusion of $\omega\mathbf{Gpd}$ in $\omega\mathbf{Cat}$ the n-cells of $\hom_{\mathrm{lax}}(I^1, Y)$ correspond to ω-functors $D^1 \otimes D^n \to Y$. By Proposition B.1.15 of [AM20] if Y is a 2-category then all n-cells of $\hom_{\mathrm{lax}}(I^1, Y)$ are identity cells for $n > 2$. Hence $\hom_{\mathrm{lax}}(I^1, \bar{\tau}^2(Y))$ is a 2-category. Since Y is P_2-fibrant the natural map $Y \to \bar{\tau}^2(Y)$ is a weak equivalence, so it is sufficient to show that

$$\pi_{\omega\mathbf{Gpd}}(X, \bar{\tau}^2(Y)) \to \pi_{\omega\mathbf{Gpd}}(\tau^3(X), \bar{\tau}^2(Y))$$

is injective. Since $\hom_{\mathrm{lax}}(I^1, \bar{\tau}^2(Y))$ is a path space it is sufficient to show that the lifting problems

$$\begin{tikzcd}
\tau^3(X) \arrow[r] \arrow[d, hook] & \hom_{\mathrm{lax}}(I^1, \bar{\tau}^2(Y)) \arrow[d] \\
X \arrow[r] \arrow[ru, dashed] & \bar{\tau}^2(Y) \times \bar{\tau}^2(Y)
\end{tikzcd}$$

all have solutions, which is the case because the inclusion $\tau^3(X) \hookrightarrow X$ is bijection below degree 3 and $\bar{\tau}^2(Y) \times \bar{\tau}^2(Y)$ and $\hom_{\mathrm{lax}}(I^1, \bar{\tau}^2(Y))$ are 2-categories. Hence we have showed that $\tau^3(X) \hookrightarrow X$ is a P_2-weak equivalence.

Let $f : X \to Y$ be an ω-functor. Consider the diagram

$$\begin{tikzcd}
\tau^3(X) \arrow[r, hook, "\sim"] \arrow[d, "\tau^3(f)"'] & X \arrow[d, "f"] \\
\tau^3(Y) \arrow[r, hook, "\sim"] & Y
\end{tikzcd}$$

where the horizontal maps are P_2-weak equivalences. It is clear that f is a P_2-weak equivalence if and only if $\tau^3(f)$ is, but since $\tau^3(X)$ and $\tau^3(Y)$ are P_2-fibrant $\tau^3(f)$ is a P_2-weak equivalence if and only if it is a folk weak equivalence. Hence f is a P_2-weak equivalence if and only if it induces isomorphisms on π_n for $0 \leq n \leq 2$. □

Before showing that $P_2\omega$**Gpd** and P_2**sSet** are Quillen equivalent via the Street nerve-realization adjunction we will describe a fibrant replacement functor for $P_2\omega$**Gpd**. By Proposition 4.61 the ω-functors $i_n : \partial I^n \hookrightarrow I^n$ are acyclic cofibrations in the model structure $P_2\omega$**Gpd** when $n \geq 4$ since they are the identity in degree 3 and below. Therefore any P_2-fibrant ω-groupoid has the right lifting property against $L^0 i_n$ for all $n \geq 4$. We claim that applying the small object argument to the set of ω-functors $L^0 i_n$ for $n \geq 4$ gives a functorial P_2-fibrant replacement of an ω-groupoid X. Let $\hat{X}$ be the ω-groupoid obtained from this construction. The ω-functor $X \to \hat{X}$ is the transfinite composition of ω-functors

$$\begin{tikzcd}
X \arrow[r, hook, "\sim"] & \hat{X}^1 \arrow[r, hook, "\sim"] & \dots \arrow[r, hook, "\sim"] & \hat{X}^n \arrow[r, hook, "\sim"] & \dots \arrow[r, hook] & \hat{X}
\end{tikzcd}$$

obtained as the pushouts of diagrams indexed by lifting problems of $L^0 i_n$ against $\hat{X}^i \to *$ for $n \geq 4$ and $i \geq 0$

$$\begin{tikzcd}
\partial I^n \arrow[r, "\langle u,v \rangle"] \arrow[d, hook, "L^0 i_n"'] & \hat{X}^i \arrow[d] \\
I^n \arrow[r] & *
\end{tikzcd}$$

which correspond to parallel pairs of $n-1$-cells $u, v \in \hat{X}^i_{n-1}$. In particular, $\hat{X}^0 = X$ and for $i \geq 0$ and all pairs of parallel $n-1$-cells $\langle u, v \rangle$ in $\hat{X}^i$ with $n \geq 4$, $\hat{X}^{i+1}$ is the pushout

$$\begin{tikzcd}
\coprod\limits_{\langle u,v \rangle} \partial I^n \arrow[r] \arrow[d, hook, "\wr"] & \hat{X}^i \arrow[d, hook, "\wr"] \\
\coprod\limits_{\langle u,v \rangle} I^n \arrow[r] & \hat{X}^{i+1}
\end{tikzcd}$$

Since $P_2\omega$**Gpd** is a model category and $L^0 i_n$ are acyclic cofibrations for all $n \geq 4$ the pushout of the left vertical map is an acyclic cofibration. Hence the transfinite composite $X \hookrightarrow \hat{X}$ is an acyclic cofibration of $P_2\omega$**Gpd**. By the small object argument $\hat{X}$ has the right lifting property against all $L^0 i_n$ for $n \geq 4$, so all parallel pairs of $n-1$-cells are homotopic when $n \geq 4$ in $\hat{X}$. Thus $\pi_n(\hat{X}, x) = 0$ for all $x \in X_0$ and all $n \geq 3$ and so $\hat{X}$ is P_2-fibrant by Proposition 4.61. We note that this replacement does not change the n-cells of X for $0 \leq n \leq 3$, so $X \hookrightarrow \hat{X}$ is the identity in degree 3 and below.

Now we proceed with showing the promised Quillen equivalence between $P_2\omega\mathbf{Gpd}$ and $P_2\mathbf{sSet}$.

Proposition 4.62. *The adjunction $C_\omega \dashv N_\omega$ is a Quillen equivalence between $P_2\omega\mathbf{Gpd}$ and $P_2\mathbf{sSet}$.*

Proof. By Theorem 3.3.20 of [Hir03] since $g_2 : \partial\Delta^4 \hookrightarrow \Delta^4$ is a cofibration between cofibrant objects of **sSet** the functors $C_\omega \dashv N_\omega$ determine a Quillen adjunction

$$\begin{tikzcd}
P_2\omega\mathbf{Gpd} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & P_2\mathbf{sSet} \arrow[l, bend left, "C_\omega"]
\end{tikzcd}$$

The unit map $\eta_K : K \to N_\omega C_\omega(K)$ is a weak equivalence of $P_2\mathbf{sSet}$ by Proposition 4.60, so to show this adjunction is a Quillen equivalence it only remains to show that N_ω creates weak equivalences. This is clear by Proposition 4.61 and Proposition 4.57 since an ω-functor f is a P_2-weak equivalence if and only if it induces isomorphisms on π_n for $0 \leq n \leq 2$. □

It is clear from these results that the smart 2-truncation adjunction of Definition 4.11

$$\begin{tikzcd}
\mathbf{2Gpd} \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & P_2\omega\mathbf{Gpd} \arrow[l, bend left, "\overline{\tau}^2"]
\end{tikzcd}$$

where **2Gpd** is the category of 2-groupoids with the model structure of Theorem 1.2 of [MS93] is a Quillen equivalence. Composing this adjunction with that of Proposition 4.62 gives the Quillen equivalence of [MS93] Corollary 2.6.

5 Monoids in ω-Groupoids

In this chapter we will use a model based in ω-groupoids for the homotopy theory of 2-types of monoids and their group completions to calculate the first derived functor of group completion of a monoid M. By Proposition 3.84, therefore, we will have calculated $\pi_2(BM)$, the second homotopy group of the classifying space of M. In the previous chapter we showed that strict ω-groupoids only model all homotopy 2-types of spaces, so this seems to be the limit of this approach to calculating homotopy groups of BM.

The connection of strict ω-groupoids with the homotopy theory of monoids comes from the Street nerve, which we saw in Section 4.7 determines a Quillen adjunction with the category of simplicial sets. We will show that this nerve is lax monoidal for the Gray tensor product, so that the Street nerve extends to a Quillen adjunction between monoids for the Gray tensor product on strict ω-groupoids and simplicial monoids. For 2-types we show that this monoid Street nerve adjunction is also a Quillen equivalence, using the localized Quillen equivalence proved in Section 4.8. From this equivalence we can calculate the first derived functor of group completion using monoids in strict ω-groupoids. Our formula for $\pi_2(BM)$ is based on this calculation and the theory of rewriting systems for monoids, which obtains homotopy invariants from the data of a presentation of a monoid M by generators and relations. This formula is reminiscent of the Hopf formula for the second group homology group, so we call it a Hopf formula for $\pi_2(BM)$.

5.1 Gray Monoids and Gray Groups

In this section we will define the categories of Gray monoids, which are monoids in the category ω**Gpd** with the Gray tensor product, and of Gray groups, which are Gray monoids with an extra invertibility condition. We will show that these categories have model structures determined by the folk model structure on ω**Gpd** and that there is a left Quillen group completion functor that is left adjoint to the inclusion of Gray groups into Gray monoids. In the next section we will use the Street nerve to show that these Gray monoids and groups with these model structures form models of truncated homotopy types of simplicial monoids and their group completions.

Recall that Theorem 4.43 (coming from §6 of [AL20]) shows that the folk model structure on ω**Gpd** is a monoidal model structure for the Gray tensor product. Proposition 6.23 and remark 6.24 of [AL20] show that ω**Gpd** with the folk model structure satisfies the monoid axiom (Definition 3.3 of [SS00]) which requires that transfinite compositions of tensor products of an ω-groupoid with an acyclic cofibration are weak equivalences. This allows the results of [SS00] to be applied to transfer the folk model structure of ω**Gpd** along the free-forgetful adjunction to the category of monoids for the Gray tensor product on ω**Gpd**

$$\begin{tikzcd}
\mathbf{Mon}_{\otimes}(\omega\mathbf{Gpd}) \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \omega\mathbf{Gpd} \arrow[l, bend right, "F_{\otimes}"']
\end{tikzcd}$$

We will call such monoids **Gray monoids**, extending the name for monoids in the category of 2-groupoids with the Gray tensor product. The category of these monoids with the model structure transferred along this adjunction will be denoted by **Gray**.

Definition 5.1. *The category **Gray** of Gray monoids is a cofibrantly generated model category with*

- *weak equivalences the monoid ω-functors $f : M \to N$ that are sent to weak equivalences in ω**Gpd** by the forgetful functor U : **Gray** $\to \omega$**Gpd***
- *generating cofibrations the monoid ω-functors*

  $$F_{\otimes}(L^0 i_n) : F_{\otimes}(\partial I^n) \hookrightarrow F_{\otimes}(I^n)$$

  for $n \geq 0$
- *generating acyclic cofibrations the monoid ω-functors*

  $$F_{\otimes}(j_n^{\varepsilon}) : F_{\otimes}(I^{n-1}) \hookrightarrow F_{\otimes}(I^n)$$

  for $n \geq 1$ and for either choice of $\varepsilon \in \{-, +\}$

A Gray monoid M is an ω-groupoid with a multiplication ω-functor $m : M \otimes M \to M$ and a unit ω-functor $e : I^0 \to M$ that satisfy the usual monoid identities. By Lemma 4.35, 0-cells of $M \otimes M$ are pairs $m \otimes m'$ of 0-cells $m, m' \in M_0$. Hence on 0-cells, a Gray monoid has the structure of an ordinary monoid in **Set**. We can extend this ordinary multiplication of 0-cells of a Gray monoid to an action of 0-cells on n-cells for $n \geq 0$ from the left and right by restricting the multiplication ω-functor m to the 0-cells of M in one entry

$$\begin{tikzcd} M_0 \times M = M_0 \otimes M \arrow[r, hook] & M \otimes M \arrow[r, "m"] & M \end{tikzcd}$$

We will denote the left and right actions of a 0-cell $x \in M_0$ on an n-cell $a \in M_n$ by

$$x \cdot a = m(1_x \otimes a) \qquad\qquad a \cdot x = m(a \otimes 1_x)$$

We observe that this action respects the structure maps of the ω-groupoid M, so for $0 \leq i < n$ and $\varepsilon \in \{-, +\}$

$$d_i^\varepsilon(x \cdot a) = x \cdot a \qquad\qquad x \cdot (a *_i b) = (x \cdot a) *_i (x \cdot y) \qquad\qquad 1_{x \cdot a} = x \cdot 1_a$$

If $n = 0$ then the action coincides with the multiplication of the monoid M_0 of 0-cells.

When the monoid of 0-cells M_0 of a Gray monoid M is a group we will say that M is a Gray group.

Definition 5.2. *A Gray monoid is a* ***Gray group*** *if its ordinary monoid of* 0*-cells is a group.*

We will denote by **GrayGp** the full subcategory of **Gray** on Gray groups. The inclusion of **GrayGp** in **Gray** has a left adjoint L : **Gray** $\to$ **GrayGp** which sends a Gray monoid M to the pushout in **Gray**

$$\begin{tikzcd} M_0 \arrow[r] \arrow[d, hook] & LM_0 \arrow[d, hook] \\ M \arrow[r] & LM \end{tikzcd} \tag{84}$$

where LM_0 is the group completion of the **Set** monoid of 0-cells of M.

We can construct a path space for Gray monoids using the path space construction of Proposition 4.44 by putting a Gray monoid structure on $\hom_{\text{lax}}(I^1, M)$ when M is a Gray monoid. To do this, we follow the example of the path space for differential graded algebras, which are monoids for the chain complex tensor product, as discussed in §5 of [SS00].

Proposition 5.3. *Let M be a Gray monoid. The ω-groupoid* $\hom_{\text{lax}}(I^1, M)$ *is a Gray monoid and a path space for the model structure on* ***Gray****.*

Proof. Let M be a Gray monoid. We will show that the factorization of Definition 4.44 for M is a diagram of Gray monoids and ω-functors that preserve Gray monoid structures. For simplicity we will denote the path space for an ω-groupoid M by

$$M^I = \hom_{\text{lax}}(I^1, M)$$

We begin by describing the Gray monoid structure on the ω-groupoid M^I. Let $m : M \otimes M \to M$ be the multiplication for the Gray monoid M. We define the multiplication $\mu : M^I \otimes M^I \to M^I$

on M^I by the adjoint of the map $\bar{\mu} : I^1 \otimes (M^I \otimes M^I) \to M$ given by the following composite

$$\begin{tikzcd}
I^1 \otimes (M^I \otimes M^I) \arrow[r, "s\otimes(M^I\otimes M^I)"] \arrow[rrdd, "\bar{\mu}"'] & (I^1 \otimes I^1) \otimes (M^I \otimes M^I) \arrow[r, "\gamma"] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^1) \arrow[d, "\varepsilon^{\otimes}_M \otimes \varepsilon^{\otimes}_M"] \\
& & M \otimes M \arrow[d, "m"] \\
& & M
\end{tikzcd}$$

This ω-functor $\bar{\mu} : I^1 \otimes (M^I \otimes M^I) \to M$ involves the following ω-functors

- the counit $\varepsilon^{\otimes}_M : I^1 \otimes M^I \to M$ for the adjunction $I^1 \otimes - \dashv \hom_{\text{lax}}(I^1, -)$ of the monoidal structure on ω**Gpd**
- $s : I^1 \to I^1 \otimes I^1$ is the ω-functor corresponding to the 1 source of the unique 2-cell of $D^1 \otimes D^1$

$$\begin{tikzcd}
D^1 \arrow[r, "0\otimes\alpha_1 *_0 \alpha_1 \otimes 1"] \arrow[d, "\eta"'] & D^1 \otimes D^1 \arrow[d, "\eta\otimes\eta"] \\
I^1 \arrow[r, "s"'] & I^1 \otimes I^1
\end{tikzcd}$$

  where η is the unit of the adjunction $L \dashv I$ with I the inclusion of ω**Gpd** in ω**Cat**
- $\gamma : (I^1 \otimes I^1) \otimes (M^I \otimes M^I) \to (I^1 \otimes M^I) \otimes (I^1 \otimes M^1)$ is composed of the following components of the natural transformations that define the symmetric monoidal structure on ω**Gpd**

$$(I^1 \otimes I^1) \otimes (M^I \otimes M^I) \overset{\alpha}{\to} I^1 \otimes (I^1 \otimes (M^I \otimes M^I))$$

$$I^1 \otimes (I^1 \otimes (M^I \otimes M^I)) \overset{1\otimes\alpha}{\to} I^1 \otimes (I^1 \otimes M^I) \otimes M^I$$

$$I^1 \otimes (I^1 \otimes M^I) \otimes M^I \overset{1\otimes c\otimes 1}{\to} I^1 \otimes (M^I \otimes I^1) \otimes M^I$$

$$I^1 \otimes (M^I \otimes I^1) \otimes M^I \overset{1\otimes\alpha^{-1}}{\to} I^1 \otimes (M^I \otimes (I^1 \otimes M^I))$$

$$I^1 \otimes (M^I \otimes (I^1 \otimes M^I)) \overset{\alpha^{-1}}{\to} (I^1 \otimes M^I) \otimes (I^1 \otimes M^I)$$

  where c is the symmetry natural isomorphism and α is the associativity natural isomorphism for the Gray tensor product on ω**Gpd**.

Identity and associativity hold for M^I by the corresponding properties for M. For associativity, in the following diagrams the top paths are the adjoints of $\mu \circ (\mu \otimes 1) \circ \alpha$ and $\mu \circ (1 \otimes \mu)$ respectively under the adjunction $I^1 \otimes - \dashv \hom_{\text{lax}}(I^1, -)$

$$\begin{tikzcd}
I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[r, "1\otimes\alpha"] \arrow[d, "s\otimes 1"'] & I^1 \otimes ((M^I \otimes M^I) \otimes M^I) \arrow[r, "1\otimes(\mu\otimes 1)"] \arrow[d, "s\otimes 1"'] & I^1 \otimes (M^I \otimes M^I) \arrow[d, "1\otimes\mu"] \arrow[ddl, bend right, "s\otimes 1"] \\
I^1 \otimes I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[r, "1\otimes\alpha"] & I^1 \otimes I^1 \otimes ((M^I \otimes M^I) \otimes M^I) \arrow[dl, "\gamma"] \arrow[d, "1\otimes(\mu\otimes 1)"] & I^1 \otimes M^I \arrow[ddddd, "\varepsilon_M^{\otimes}"] \\
(I^1 \otimes (M^I \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "(s\otimes 1)\otimes 1"'] \arrow[dr, "(1\otimes\mu)\otimes 1"] & I^1 \otimes I^1 \otimes (M^I \otimes M^I) \arrow[d, "\gamma"] & \\
((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "\gamma\otimes 1"'] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^I) \arrow[dd, "\varepsilon_M^{\otimes}\otimes 1"] & \\
((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "(\varepsilon_M^{\otimes}\otimes\varepsilon_M^{\otimes})\otimes 1"'] & & \\
(M \otimes M) \otimes (I^1 \otimes M^I) \arrow[r, "m\otimes 1"'] \arrow[d, "1\otimes\varepsilon_M^{\otimes}"'] & M \otimes (I^1 \otimes M^I) \arrow[d, "1\otimes\varepsilon_M^{\otimes}"] & \\
(M \otimes M) \otimes M \arrow[r, "m\otimes 1"'] \arrow[d, "\alpha"'] & M \otimes M \arrow[r, "m"'] & M \\
M \otimes (M \otimes M) \arrow[r, "1\otimes m"'] & M \otimes M \arrow[ur, "m"'] &
\end{tikzcd} \tag{85}$$

$$\begin{tikzcd}
I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[rr, "1\otimes(1\otimes\mu)"] \arrow[d, "s\otimes 1"'] & & I^1 \otimes (M^I \otimes M^I) \arrow[d, "1\otimes\mu"] \arrow[ddl, "s\otimes 1"] \\
I^1 \otimes I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[dr, "1\otimes(1\otimes\mu)"] \arrow[d, "\gamma"'] & & I^1 \otimes M^I \arrow[ddddd, "\varepsilon_M^{\otimes}"] \\
(I^1 \otimes M^I) \otimes (I^1 \otimes (M^I \otimes M^I)) \arrow[dr, "1\otimes(1\otimes\mu)"] \arrow[d, "1\otimes(s\otimes 1)"'] & I^1 \otimes I^1 \otimes (M^I \otimes M^I) \arrow[d, "\gamma"] & \\
(I^1 \otimes M) \otimes ((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \arrow[d, "1\otimes\gamma"'] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^I) \arrow[dd, "1\otimes\varepsilon_M^{\otimes}"] & \\
(I^1 \otimes M^I) \otimes ((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \arrow[d, "1\otimes(\varepsilon_M^{\otimes}\otimes\varepsilon_M^{\otimes})"'] & & \\
(I^1 \otimes M^I) \otimes (M \otimes M) \arrow[r, "1\otimes m"'] \arrow[d, "\varepsilon_M^{\otimes}\otimes 1"'] & (I^1 \otimes M^I) \otimes M \arrow[d, "\varepsilon_M^{\otimes}\otimes 1"] & \\
M \otimes (M \otimes M) \arrow[r, "1\otimes m"'] & M \otimes M \arrow[r, "m"'] & M
\end{tikzcd} \tag{86}$$

Now we must show that the two bottom paths of these diagrams are equal. For (85) this composite, omitting $m \circ (1 \otimes m)$ which occurs at the end of both paths, is given by the left-most path of the following diagram.

$$\begin{tikzcd}
I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[d, swap, "s\otimes 1"] & \\
(I^1 \otimes I^1) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[r, "(s\otimes 1)\otimes 1"] \arrow[d, "1\otimes\alpha"] & ((I^1 \otimes I^1) \otimes I^1) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[d, "1\otimes\alpha"] \\
(I^1 \otimes I^1) \otimes ((M^I \otimes M^I) \otimes M^I) \arrow[r, "(s\otimes 1)\otimes 1"] \arrow[d, "\gamma"] & ((I^1 \otimes I^1) \otimes I^1) \otimes ((M^I \otimes M^I) \otimes M^I) \arrow[d, "\gamma"] \\
(I^1 \otimes (M^I \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[r, "(s\otimes 1)\otimes 1"] & ((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "\gamma\otimes 1"] \\
& ((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "\alpha"] \arrow[ddl, bend right, swap, "(\varepsilon^{\otimes}_M \otimes \varepsilon^{\otimes}_M)\otimes\varepsilon^{\otimes}_M"] \\
& (I^1 \otimes M^I) \otimes ((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \arrow[d, "\varepsilon^{\otimes}_M \otimes (\varepsilon^{\otimes}_M \otimes \varepsilon^{\otimes}_M)"] \\
(M \otimes M) \otimes M \arrow[r, swap, "\alpha"] & M \otimes (M \otimes M)
\end{tikzcd}$$

The bottom path of (86), again omitting $m \circ (1 \otimes m)$, is the left-most path of the following diagram

$$\begin{tikzcd}
I^1 \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[d, swap, "s\otimes 1"] & \\
(I^1 \otimes I^1) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[r, "(1\otimes s)\otimes 1"] \arrow[d, swap, "\gamma"] & (I^1 \otimes (I^1 \otimes I^1)) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[ddl, bend left, "\gamma"] \\
(I^1 \otimes M^I) \otimes (I^1 \otimes (M^I \otimes M^I)) \arrow[d, swap, "1\otimes(s\otimes 1)"] & \\
(I^1 \otimes M) \otimes ((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \arrow[d, swap, "1\otimes\gamma"] & \\
(I^1 \otimes M^I) \otimes ((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \arrow[d, swap, "1\otimes(\varepsilon^{\otimes}_M \otimes \varepsilon^{\otimes}_M)"] & \\
(I^1 \otimes M^I) \otimes (M \otimes M) \arrow[d, swap, "\varepsilon^{\otimes}_M \otimes 1"] & \\
M \otimes (M \otimes M) &
\end{tikzcd}$$

Finally coherence for symmetric monoidal categories gives that the following diagram commutes and this completes the associativity identity.

$$\begin{tikzcd}
((I^1 \otimes I^1) \otimes I^1) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[d, "1\otimes\alpha"] \arrow[dr, bend left, "c\otimes 1"] & \\
((I^1 \otimes I^1) \otimes I^1) \otimes ((M^I \otimes M^I) \otimes M^I) \arrow[d, "\gamma"] & (I^1 \otimes (I^1 \otimes I^1)) \otimes (M^I \otimes (M^I \otimes M^I)) \arrow[dd, "\gamma"] \\
((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "\gamma\otimes 1"] & \\
((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) \otimes (I^1 \otimes M^I) \arrow[d, "\alpha"] & (I^1 \otimes M^I) \otimes ((I^1 \otimes I^1) \otimes (M^I \otimes M^I)) \arrow[dl, bend left, "1\otimes\gamma"] \\
(I^1 \otimes M^I) \otimes ((I^1 \otimes M^I) \otimes (I^1 \otimes M^I)) &
\end{tikzcd}$$

The identity ω-functor $e^I : I^0 \to M^I$ is determined by the 1-cell $1_e \in M_1$ which is the identity on the 0-cell $e : I^0 \to M$ determined by identity ω-functor for the Gray monoid M. The adjoint ω-functor $\bar{e}^I : I^1 \otimes I^0 \to M$ determined by the adjunction $I^1 \otimes - \dashv \hom_{\mathrm{lax}}(I^1, -)$ is the composite

$$I^0 \otimes I^1 \xrightarrow{!\otimes 1} I^0 \otimes I_0 \xrightarrow{e} M$$

where the first ω-functor is the unique map to the terminal ω-groupoid I^0. The left identity property for this unit ω-functor follows from the fact that e is a unit for M as a Gray monoid. In the following diagram the top right path is the adjoint of $\mu \circ (e^I \otimes 1)$ under the adjunction $I^1 \otimes - \dashv \hom_{\mathrm{lax}}(I^1, -)$.

$$\begin{tikzcd}
I^1 \otimes (I^0 \otimes M^I) \arrow[r, "1\otimes(e^I\otimes 1)"] \arrow[d, "s\otimes 1"'] & I^1 \otimes (M^I \otimes M^I) \arrow[r, "1\otimes\mu"] \arrow[d, "s\otimes 1"] & I^1 \otimes M^I \arrow[ddd, "\varepsilon^{\otimes}_M"] \\
(I^1 \otimes I^1) \otimes (I^0 \otimes M^I) \arrow[r, "1\otimes(e^I\otimes 1)"'] \arrow[d, "\gamma"'] & (I^1 \otimes I^1) \otimes (M^I \otimes M^I) \arrow[d, "\gamma"] & \\
(I^1 \otimes I^0) \otimes (I^1 \otimes M^I) \arrow[r, "(1\otimes e^I)\otimes 1"'] \arrow[d, "!\otimes 1"'] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^I) \arrow[d, "\varepsilon^{\otimes}_M\otimes\varepsilon^{\otimes}_M"] & \\
I^0 \otimes (I^1 \otimes M^I) \arrow[r, "e\otimes 1"'] \arrow[d, "1\otimes\varepsilon^{\otimes}_M"'] & M \otimes M \arrow[r, "m"'] & M \\
I^0 \otimes M \arrow[urr, bend right, "\lambda_M"'] & &
\end{tikzcd}$$

The bottom left path of this diagram is the bottom left path of the next commutative diagram and the top right path is the adjoint of λ_{M^I}, so this proves the left identity condition.

$$\begin{tikzcd}
 & I^1 \otimes (I^0 \otimes M^I) \arrow[dl, "s\otimes 1"'] \arrow[d, "\lambda^{-1}_{I^1}\otimes 1"] \arrow[dddr, bend left, "1\otimes(\lambda_{M^I})"] & \\
(I^1 \otimes I^1) \otimes (I^0 \otimes M^I) \arrow[r, "(!\otimes 1)\otimes 1"] \arrow[d, "\gamma"'] & (I^0 \otimes I^1) \otimes (I^0 \otimes M^I) \arrow[d, "\gamma"] & \\
(I^1 \otimes I^0) \otimes (I^1 \otimes M^I) \arrow[r, "(!\otimes 1)\otimes 1"'] & (I^0 \otimes I^0) \otimes (I^1 \otimes M^I) \arrow[d, "\lambda_{I^0}\otimes 1"'] & \\
 & I^0 \otimes (I^1 \otimes M^I) \arrow[r, "\lambda_{M^I}"'] \arrow[d, "1\otimes\varepsilon^{\otimes}_M"'] & I^1 \otimes M^I \arrow[d, "\varepsilon^{\otimes}_M"] \\
 & I^0 \otimes M \arrow[r, "\lambda_M"'] & M
\end{tikzcd}$$

The right identity condition follows from the same arguments.

Hence $\hom_{\mathrm{lax}}(I^1, M)$ is a Gray monoid. Now we must show that the ω-functors τ_M and π^{ε}_M for $\varepsilon \in \{-,+\}$ of Definition 4.44 are Gray monoid ω-functors. We start with $\tau_M = \hom_{\mathrm{lax}}(!, M)$ by showing that the following diagram commutes.

$$\begin{tikzcd}
M \otimes M \arrow[r, "\tau_M\otimes\tau_M"] \arrow[d, "m"'] & M^I \otimes M^I \arrow[d, "\mu"] \\
M \arrow[r, "\tau_M"'] & M^I
\end{tikzcd} \tag{87}$$

Applying the adjoint gives a diagram

$$\begin{tikzcd}
I^1 \otimes (M \otimes M) \arrow[r, "1\otimes(\tau_M\otimes\tau_M)"] \arrow[d, "s\otimes 1"'] & I^1 \otimes (M^I \otimes M^I) \arrow[r, "1\otimes\mu"] \arrow[d, "s\otimes 1"] & I^1 \otimes M^I \arrow[ddd, "\varepsilon^{\otimes}_M"] \\
(I^1 \otimes I^1) \otimes (M \otimes M) \arrow[r, "1\otimes(\tau_M\otimes\tau_M)"] \arrow[d, "\gamma"'] & (I^1 \otimes I^1) \otimes (M^I \otimes M^I) \arrow[d, "\gamma"] & \\
(I^1 \otimes M) \otimes (I^1 \otimes M) \arrow[r, "(1\otimes\tau_M)\otimes(1\otimes\tau_M)"] \arrow[d, "(!\otimes M)\otimes(!\otimes M)"'] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^I) \arrow[d, "\varepsilon^{\otimes}_M\otimes\varepsilon^{\otimes}_M"] & \\
(I^0 \otimes M) \otimes (I^0 \otimes M) \arrow[r, "\lambda_M\otimes\lambda_M"'] & M \otimes M \arrow[r, "m"'] & M
\end{tikzcd}$$

where λ is the natural isomorphism for the unit condition on I^0. The bottom left square of this diagram commutes because $\tau_M : M \to M^I$ is the internal composition ω-functor for the internal hom of $\omega\mathbf{Gpd}$ when we observe that

$$M \cong \hom_{\mathrm{lax}}(I^1, I^0) \otimes \hom_{\mathrm{lax}}(I^0, M)$$

Now consider the following diagram, where the lower left path is the same as the lower left path of the above square.

$$\begin{tikzcd}
I^1 \otimes (M \otimes M) \arrow[d, "s\otimes 1"'] \arrow[rrd, bend left=10, "!\otimes 1"] & & & \\
(I^1 \otimes I^1) \otimes (M \otimes M) \arrow[r, "(!\otimes !)\otimes 1"'] \arrow[d, "\gamma"'] & (I^0 \otimes I^0) \otimes (M \otimes M) \arrow[r, "\lambda_{I^0}"'] \arrow[d, "\gamma"] & I^0 \otimes (M \otimes M) \arrow[d, "\lambda_{M\otimes M}"] & \\
(I^1 \otimes M) \otimes (I^1 \otimes M) \arrow[r, "(!\otimes 1)\otimes(!\otimes 1)"'] & (I^0 \otimes M) \otimes (I^0 \otimes M) \arrow[r, "\lambda_M \otimes \lambda_M"'] & M \otimes M \arrow[r, "m"'] & M
\end{tikzcd}$$

Hence the adjoint of the top path in (5.1) $m \circ (! \otimes (M \otimes M))$ which is the adjoint of $\tau_M \circ m$ as well by the following diagram.

$$\begin{tikzcd}
I^1 \otimes (M \otimes M) \arrow[r, "1\otimes m"] \arrow[d, "!\otimes 1"'] & I^1 \otimes M \arrow[r, "1\otimes \tau_M"] \arrow[d, "!\otimes 1"'] & I^1 \otimes M^I \arrow[d, "\varepsilon^{\otimes}_M"] \\
I^0 \otimes (M \otimes M) \arrow[r, "1\otimes m"'] \arrow[rd, "\lambda_{M\otimes M}"'] & I^0 \otimes M \arrow[r, "\lambda_M"'] & M \\
& M \otimes M \arrow[ru, "m"'] &
\end{tikzcd}$$

This ω-functor also respects the identity. The diagram

$$\begin{tikzcd}
I^0 \arrow[r, "e"] \arrow[rd, "e^I"'] & M \arrow[d, "\tau_M"] \\
& M^I
\end{tikzcd}$$

commutes since τ_M sends a 0-cell of M to the 0-cell of M^I that corresponds to the identity 1-cell on it.

Finally we must show that the ω-functors π^{ε}_M of Definition 4.44 are also Gray monoid ω-functors. First we will show that the following diagram commutes.

$$\begin{tikzcd}
M^I \otimes M^I \arrow[r, "\pi^{\varepsilon}_M \otimes \pi^{\varepsilon}_M"] \arrow[d, "\mu"'] & M \otimes M \arrow[d, "m"] \\
M^I \arrow[r, "\pi^{\varepsilon}_M"'] & M
\end{tikzcd} \tag{88}$$

where $\pi^{\varepsilon}_M = \hom_{\mathrm{lax}}(j^{\varepsilon}_1, M) : M^I \to M$ is the ω-functor of Definition 4.44. The following diagram shows that this does commute, where we are using the notation $\hom_{\mathrm{lax}}(I^1, X) = X^I$ generally.

$$\begin{tikzcd}
M^I \otimes M^I \arrow[r, "\pi^\varepsilon_M \otimes \pi^\varepsilon_M"] \arrow[ddd, "\eta_{M^I \otimes M^I}"'] & M \otimes M \arrow[d, "\eta_{M\otimes M}"] \arrow[rddd, bend left, "\lambda^{-1}_{M\otimes M}"] & \\
 & (I^1 \otimes (M \otimes M))^I \arrow[d, "\mathrm{hom}_{\mathrm{lax}}(j^\varepsilon_1{,}1)"] & \\
 & I^1 \otimes (M \otimes M) \arrow[rd, "!\otimes 1"] & \\
(I^1 \otimes (M^I \otimes M^I))^I \arrow[ruu, bend left, "\mathrm{hom}_{\mathrm{lax}}(I^1{,}I^1\otimes(\pi^\varepsilon_M\otimes\pi^\varepsilon_M))"] \arrow[r, "\pi^\varepsilon_{I^1\otimes(M\otimes M)}"'] \arrow[d, "\mathrm{hom}_{\mathrm{lax}}(I^1{,}s\otimes 1)"'] & I^1 \otimes (M^I \otimes M^I) \arrow[r, "!\otimes(\pi^\varepsilon_M\otimes\pi^\varepsilon_M)"'] \arrow[d, "s\otimes 1"] & I^0 \otimes (M \otimes M) \arrow[d, "\lambda^{-1}_{I^0}\otimes 1"] \\
((I^1 \otimes I^1) \otimes (M^I \otimes M^I))^I \arrow[r, "\pi^\varepsilon_{(I^1\otimes I^1)\otimes(M\otimes M)}"'] \arrow[d, "\mathrm{hom}_{\mathrm{lax}}(I^1{,}\gamma)"'] & (I^1 \otimes I^1) \otimes (M^I \otimes M^I) \arrow[r, "!\otimes(\pi^\varepsilon_M\otimes\pi^\varepsilon_M)"'] \arrow[d, "\gamma"] & (I^0 \otimes I^0) \otimes (M \otimes M) \arrow[d, "\gamma"] \\
((I^1 \otimes M^I) \otimes (I^1 \otimes M^I))^I \arrow[r, "\pi^\varepsilon_{(I^1\otimes M)\otimes(I^1\otimes M)}"'] \arrow[d, "\mathrm{hom}_{\mathrm{lax}}(I^1{,}\varepsilon^\otimes_M\otimes\varepsilon^\otimes_M)"'] & (I^1 \otimes M^I) \otimes (I^1 \otimes M^I) \arrow[r, "(!\otimes\pi^\varepsilon_M)\otimes(!\otimes\pi^\varepsilon_M)"'] \arrow[d, "\varepsilon^\otimes_M\otimes\varepsilon^\otimes_M"] & (I^0 \otimes M) \otimes (I^0 \otimes M) \arrow[ld, bend left, "(\lambda_M\otimes\lambda_M)"] \\
(M \otimes M)^I \arrow[r, "\pi^\varepsilon_{M\otimes M}"'] \arrow[d, "\mathrm{hom}_{\mathrm{lax}}(I^1{,}m)"'] & M \otimes M \arrow[d, "m"] & \\
M^I \arrow[r, "\pi^\varepsilon_M"'] & M &
\end{tikzcd}$$

Finally, π^ε_M also respects the identity because τ_M does and π^ε_M is a retract of τ_M.

□

This path space construction allows us to define a model structure on the subcategory of Gray groups in **Gray** that is transferred via the inclusion adjunction. This is inspired by the approach of [Lac11] to proving a model structure for groupoids enriched in **2Gpd** with the Gray tensor product.

Proposition 5.4. *The category **GrayGp** has a model structure transferred from **Gray** by the adjunction*

$$\begin{tikzcd}
\textbf{\textit{GrayGp}} \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & \textbf{\textit{Gray}} \arrow[l, bend right, "L"']
\end{tikzcd}$$

*With this model structure on **GrayGp** this adjunction is a Quillen adjunction.*

Proof. The model structure on **Gray** is transferred from ω**Gpd** by the free-forgetful monoid adjunction, so the transferred model structure for **GrayGp** will have fibrations and weak equivalences the Gray monoid ω-functors that are fibrations and weak equivalences respectively as ω-functors in the folk model structure in ω**Gpd**. We will temporarily refer to such Gray monoid ω-functors between Gray groups as **transferred fibrations** and **transferred weak equivalences** until we have shown that they are in fact fibrations and weak equivalences of a model structure. To show this we must show that Gray monoid ω-functors between Gray groups that have the left lifting property against all transferred fibrations of **GrayGp** are transferred weak equivalences. For this we will use the path space construction of Proposition 5.3. We begin by showing that the path space of Gray group is a Gray group.

Let M be a Gray group. The 0-cells of $\hom_{\text{lax}}(I^1, M)$ are the ω-functors $a : I^1 \to M$, which in turn correspond to 1-cells $a : x \to y$ of M. Multiplication of 0-cells is given by

$$(a : x \to y) \cdot (b : w \to z) = (a \cdot w) *_1 (y \cdot b)$$

Given a 1-cell $a : x \to y$ its inverse in $(M^I)_0$ is the 1-cell

$$a^{-1} = y^{-1} \cdot k^1 a \cdot x^{-1} : x^{-1} \to y^{-1}$$

since

$$\begin{aligned}
a^{-1} \cdot a &= (a^{-1} \cdot x) *_1 (y^{-1} \cdot a) \\
&= ((y^{-1} \cdot k^1 a \cdot x^{-1}) \cdot x) *_1 (y^{-1} \cdot a) \\
&= (y^{-1} \cdot k^1 a) *_1 (y^{-1} \cdot a) \\
&= 1_e \\
a \cdot a^{-1} &= (a \cdot x^{-1}) *_1 (y \cdot a^{-1}) \\
&= (a \cdot x^{-1}) *_1 (y \cdot (y^{-1} \cdot k^1 a \cdot x^{-1})) \\
&= (a \cdot x^{-1}) *_1 (a \cdot x^{-1}) \\
&= 1_e
\end{aligned}$$

Hence if M is a Gray group then so is M^I.

Now we can adapt the argument of [Qui73] §II that was also used by [SS00] in the proof of Lemma 2.3. This argument uses the path space fibration of **GrayGp** to show that any Gray monoid ω-functor between Gray groups with the left lifting property against all transferred fibrations of **GrayGp** is a transferred weak equivalence. Let $f : M \to N$ be a Gray monoid ω-functor between Gray groups that has the left lifting property against transferred fibrations. All Gray groups are transferred fibrant so the lift of the following diagram gives a retract for f

$$\begin{tikzcd}
M \arrow[r, equal] \arrow[d, "f"'] & M \arrow[d] \\
N \arrow[r] \arrow[ur, dashed, "r"'] & I^0
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
N \arrow[r, "f"] \arrow[d, "f"'] & N \arrow[r, "\tau_N"] & N^I \arrow[d, "\pi_M"] \\
M \arrow[rr, "\langle 1_M, f \circ r \rangle"'] \arrow[urr, dashed, "H"'] & & M \times M
\end{tikzcd}$$

The outer square commutes by the fact that $r \circ f = 1_M$ and $\pi_M \circ \tau_M = \Delta : M \to M \times M$ by Proposition 4.44. This is a diagram of Gray groups and π_M is a transferred fibration so there exists a lift H by the condition on f. The ω-functor H makes f an isomorphism in the homotopy category of ω**Gpd**, so it is a transferred weak equivalence. □

In the next section, we consider how the monoidal structure of ω**Gpd** interacts with the Street nerve functor and establish a Quillen adjunction between the categories of monoids in ω**Gpd** and in **sSet**.

5.2 Street Nerves of Gray Monoids and Groups

To extend the Quillen adjunction $C_\omega \dashv N_\omega$ to **Gray** we must understand how the functor N_ω interacts with the Gray tensor product. We will show that N_ω is lax monoidal for the Gray tensor product, so it determines a functor $N_\omega : \mathbf{Gray} \to \mathbf{sMon}$ to the category of simplicial monoids. We will also show that the Street nerve of a Gray monoid is a simplicial group if and only if it is a Gray group, and the group completion functor we defined in Proposition 5.4 commutes with the nerve to give the group completion functor for simplicial monoids defined in Section 3.7. We will show that the Street nerve has these properties by constructing an ω-groupoid analogue of the Alexander-Whitney map of chain complexes. We start by defining some necessary maps of the simplex category Δ.

Definition 5.5. *Let $p, q \geq 0$. Define non-decreasing injective maps $\top\!\!\top^{p,q}$ and $\perp\!\!\perp^{p,q}$ of Δ by*

$$\begin{array}{rccc} \top\!\!\top^{p,q} : & [p] & \hookrightarrow & [p+q] \\ & i & \mapsto & i \end{array} \qquad\qquad \begin{array}{rccc} \perp\!\!\perp^{p,q} : & [q] & \hookrightarrow & [p+q] \\ & i & \mapsto & i+p \end{array}$$

The maps $\top\!\!\top^{p,q}$ and $\perp\!\!\perp^{p,q}$ are the inclusions of the first $p+1$ elements or the last $q+1$ elements of $[p+q]$ respectively. Hence we have

$$\begin{aligned} \top\!\!\top^{p,q} &= d^{p+q} \circ d^{p+q-1} \circ \cdots \circ d^{p+1} \\ \perp\!\!\perp^{p,q} &= d^{p-1} \circ d^{p-2} \circ \cdots \circ d^{0} \end{aligned}$$

We can also express these maps by simply listing the elements of the image of the injective non-decreasing map in $[p+q]$, so

$$\top\!\!\top^{p,q} = 012 \cdots p \qquad\qquad \perp\!\!\perp^{p,q} = p\; p+1 \cdots p+q$$

Definition 5.6 ([May67] 29.7, [Mac63] VIII 8.5)**.** *The Alexander-Whitney maps on chain complexes are maps*

$$AW^{\boldsymbol{Ch}} : \mathbb{Z}[\Delta^n \times \Delta^n] \to \mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]$$

where the chain complexes appearing are the normalized chain complexes of simplicial sets. These maps are given by

$$\begin{array}{rcl} \mathbb{Z}[\Delta^n \times \Delta^n] & \to & \mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n] \\ [[\theta : [m] \to [n], \varphi : [m] \to [n]] & \mapsto & \displaystyle\sum_{p+q=m} [\theta \circ \top\!\!\top^{p,q} : [p] \to [n]] \otimes [\varphi \circ \perp\!\!\perp^{p,q} : [q] \to [n]] \end{array}$$

As is shown in [SS03] §2.3, the Alexander-Whitney maps are co-lax monoidal for the tensor product of chain complexes and the tensor product of simplicial abelian groups. This holds as

$$\mathbb{Z}[\Delta^n \times \Delta^n] = N(F_{\mathrm{ab}}(\Delta^n) \otimes_{\mathbf{sAb}} F_{\mathrm{ab}}(\Delta^n))$$

where $N : \mathbf{sAb} \to \mathbf{Ch}$ is the normalization functor, $F_{\mathrm{ab}} : \mathbf{sSet} \to \mathbf{sAb}$ is the free abelian group functor, and $\otimes_{\mathbf{sAb}}$ is the degree-wise tensor product for simplicial abelian groups. This co-lax monoidal structure determines a natural transformation

$$\Gamma(C) \otimes_{\mathbf{sAb}} \Gamma(D) \to \Gamma(C \otimes D) \tag{89}$$

for chain complexes C and D, with $\Gamma : \mathbf{Ch} \to \mathbf{sAb}$ the right adjoint to the normalization functor, which is determined by

$$\Gamma(C)_n = \mathrm{Hom}_{\mathbf{Ch}}(\mathbb{Z}[\Delta^n], C)$$

The tensor product $\Gamma(C) \otimes_{\mathbf{sAb}} \Gamma(C)$ is the simplicial abelian group with group of n-simplices generated by pairs

$$(c : \mathbb{Z}[\Delta^n] \to C) \otimes (d : \mathbb{Z}[\Delta^n] \to D)$$

The natural transformation (89) sends such a pair to the chain map

$$\begin{tikzcd}\mathbb{Z}[\Delta^n] \arrow[r, "\mathbb{Z}[\Delta]"] & \mathbb{Z}[\Delta^n \times \Delta^n] \arrow[r, "AW"] & \mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n] \arrow[r, "c\otimes d"] & C \otimes D\end{tikzcd}$$

We note the very clear similarity between the functor Γ of the Dold-Kan equivalence and the Street nerve that arises from the definition $\mathcal{O}^n = L_0\nu\mathbb{Z}[\Delta^n]$. These functors are both nerves determined by the cosimplicial chain complex $\mathbb{Z}[\Delta^n]$

$$N_\omega(X) = \mathrm{Hom}_{\omega\mathbf{Gpd}}(\mathcal{O}^\bullet, X) \qquad\qquad \Gamma(C) = \mathrm{Hom}_{\mathbf{Ch}}(\mathbb{Z}[\Delta^\bullet], X)$$

To obtain a similar natural transformation for a lax monoidal structure on the Street nerve with respect to the Gray tensor product we will need an ω-groupoid version of the Alexander-Whitney map.

We have a tool for obtaining ω-categories and functors from chain complexes: Steiner's functor ν ([Ste04]) as defined in Definition 4.21. Both $\mathbb{Z}[\Delta^n \times \Delta^n]$ and $\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]$ are strong Steiner complexes and the Alexander-Whitney map on chain complexes determines an ADC map between them. By the definition above it sends elements of the submonoids $\mathbb{N}[\Delta^n \times \Delta^n]_m$ to elements of the submonoids

$$(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])^*_m = \{(\theta : [p] \to [n]) \otimes (\varphi : [q] \to [n]) \mid p + q = m\} \leq \bigoplus_{p+q=m} \mathbb{Z}[\Delta^n_p \times \Delta^n_q]$$

where $(\mathbb{Z}[\Delta^n]\otimes\mathbb{Z}[\Delta^n])_m = \bigoplus_{p+q=m} \mathbb{Z}[\Delta^n_p\times\Delta^n_q]$. Furthermore, the Alexander-Whitney map respects augmentations as $\top\!\!\top^{0,0} = 1_{[0]} = \perp\!\!\!\perp^{0,0}$ so it is the identity on 0-chains.

Now $\Delta^n \times \Delta^n$ is the nerve of a poset, so the same analysis that determines the total order on elements of the basis of $\mathbb{Z}[\Delta^n]$ as a strong Steiner complex determines a partial order on basis elements of $\mathbb{Z}[\Delta^n \times \Delta^n]$. Thus $\mathbb{Z}[\Delta^n \times \Delta^n]$ is a strong Steiner complex with the canonical basis. Furthermore non-degenerate m-simplices of $\Delta^n \times \Delta^n$ are determined by their 0-simplices, so they correspond to injective maps $\Delta^m \to \Delta^n \times \Delta^n$. So by Lemma 4.33 we have

$$\nu\mathbb{Z}[\Delta^n \times \Delta^n] = \mathrm{colim}_{(\Delta\downarrow\Delta^n\times\Delta^n)}\, \nu\mathbb{Z}[\Delta^m] = F_\omega(\Delta^n \times \Delta^n)$$

By the definition of the tensor product in Theorem 4.34

$$\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]) = \nu(\mathbb{Z}[\Delta^n]) \otimes \nu(\mathbb{Z}[\Delta^n])$$

Hence applying ν to $AW^{\mathbf{Ch}}$ determines an ω-functor

$$\nu(AW^{\mathbf{Ch}}) : F_\omega(\Delta^n \times \Delta^n) \to \nu(\mathbb{Z}[\Delta^n]) \otimes \nu(\mathbb{Z}[\Delta^n])$$

Applying L^0, the left adjoint to the inclusion of $\omega\mathbf{Gpd}$ in $\omega\mathbf{Cat}$ gives the Alexander-Whitney ω-functor.

Definition 5.7. *The Alexander-Whitney ω-functor is*

$$AW = L_0\nu(AW^{\boldsymbol{Ch}}) : C_\omega(\Delta^n \times \Delta^n) \to \mathcal{O}^n \otimes \mathcal{O}^n$$

Note that when X is an ω-groupoid by [AL20] Proposition 6.12 the ω-categories

$$\hom_{\text{lax}}(\nu(\mathbb{Z}[\Delta^n]), X) \qquad \hom_{\text{oplax}}(\nu(\mathbb{Z}[\Delta^n]), X)$$

are ω-groupoids, so by the closed monoidal structure on ω**Cat**

$$L^0(\nu(\mathbb{Z}[\Delta^n]) \otimes \nu(\mathbb{Z}[\Delta^n])) = \mathcal{O}^n \otimes \mathcal{O}^n$$

When $n = 0$ this is the identity ω-functor as $C_\omega(\Delta^0 \times \Delta^0) = \mathcal{O}^0 \otimes \mathcal{O}^0$. The first non-trivial case is when $n = 1$. The Alexander-Whitney map on chain complexes sends $[1_{[1]}, 1_{[1]}] \in \mathbb{Z}[\Delta^1 \times \Delta^1]_1$ to the sum

$$\begin{aligned}\sum_{p+q=1} [\Top^{p,q} : [p] \to [1]] \otimes [\Bot^{p,q} : [q] \to [1]] \quad &= \quad [\Top^{1,1} : [1] \to [1]] \otimes [\Bot^{1,0} : [0] \to [1]] \\ &\quad\;\; + [\Top^{1,0} : [0] \to [1]] \otimes [\Bot^{1,1} : [1] \to [1]] \\ &= \quad [01] \otimes [1] + [0] \otimes [01]\end{aligned}$$

since $\Top^{1,1} = 1_{[1]} = \Bot^{1,1}$, $\Top^{0,1} = d^1$, and $\Bot^{1,0} = d^0$. The ω-functor $\nu(AW^{\mathbf{Ch}})$ therefore sends $\langle 1_{[1]}, 1_{[1]} \rangle \in \nu\mathbb{Z}[\Delta^1 \times \Delta^1]_1$ to

$$\left(\begin{array}{cc} 0 \otimes 0 & 0 \otimes 01 + 01 \otimes 1 \\ 1 \otimes 1 & 0 \otimes 01 + 01 \otimes 1 \end{array} \right) = \langle 0 \otimes 01 \rangle *_0 \langle 01 \otimes 1 \rangle$$

We observe that $\mathcal{O}^1 = I^1$ and so composing the diagonal ω-functor $C_\omega(\Delta) : \mathcal{O}^1 \to C_\omega(\Delta^1 \times \Delta^1)$ gives the ω-functor $s : I^1 \to I^1 \otimes I^1$ we used in defining the Gray monoid structure on M^I in the proof of Proposition 5.3.

This ω-functor AW was defined in [Ver08] Theorem 255, where Verity shows that the ω-functor

$$F_\omega(\Delta^n \times \Delta^n) \to \nu\mathbb{Z}[\Delta^n] \otimes \nu\mathbb{Z}[\Delta^n]$$

is the universal map for the collapse of certain cells of $F_\omega(\Delta^n \times \Delta^n)$ to identities.

We now show that these ω-functors make N_ω lax monoidal for the Gray tensor product.

Proposition 5.8. *The Street nerve functor $N_\omega : \omega$ **Gpd** $\to$ **sSet** is lax monoidal for the Gray tensor product and the cartesian product on simplicial sets.*

Proof. The Gray tensor unit of ω**Gpd** is I^0, the terminal ω-groupoid so $N_\omega(I^0) = \Delta^0$, the terminal simplicial set, which is the unit for the cartesian product. Now we must define a natural transformation

$$\rho_{X,Y} : N_\omega(X) \times N_\omega(Y) \to N_\omega(X \otimes Y)$$

We will do this using the ω-functors

$$AW : C_\omega(\Delta^n \times \Delta^n) \to \mathcal{O}^n \otimes \mathcal{O}^n$$

for all $n \geq 0$. An n-simplex of $N_\omega(X) \times N_\omega(Y)$ is a pair

$$(x : \mathcal{O}^n \to X, y : \mathcal{O}^n \to Y)$$

of ω-functors. The simplicial set map $\rho_{X,Y}$ sends such a pair to the ω-functor

$$\mathcal{O}^n \xrightarrow{C_\omega(\Delta)} C_\omega(\Delta^n \times \Delta^n) \xrightarrow{AW} \mathcal{O}^n \otimes \mathcal{O}^n \xrightarrow{x \otimes y} X \otimes Y$$

which is an n-simplex of $N_\omega(X \otimes Y)$. This is a simplicial set map as for $\theta : [m] \to [n]$ a non-decreasing map in Δ the diagram

$$\begin{tikzcd}
\mathcal{O}^n \arrow[r, "C_\omega(\Delta)"] \arrow[d, "\theta^*"'] & C_\omega(\Delta^n \times \Delta^n) \arrow[r, "AW"] \arrow[d, "C_\omega(\theta^* \times \theta^*)"'] & \mathcal{O}^n \otimes \mathcal{O}^n \arrow[r, "\theta^*(x) \otimes \theta^*(y)"] \arrow[d, "\theta^* \otimes \theta^*"'] & X \otimes Y \\
\mathcal{O}^m \arrow[r, "C_\omega(\Delta)"'] & C_\omega(\Delta^m \times \Delta^m) \arrow[r, "AW"'] & \mathcal{O}^m \otimes \mathcal{O}^m \arrow[ur, "x \otimes y"'] &
\end{tikzcd}$$

commutes. $\square$

If M is a Gray monoid, therefore, then $N_\omega(M)$ has the structure of a simplicial monoid with multiplication given by the simplicial set map

$$N_\omega(M) \times N_\omega(M) \xrightarrow{\rho_{M,M}} N_\omega(M \otimes M) \xrightarrow{N_\omega(m)} N_\omega(M) \tag{90}$$

The Street nerve therefore determines a functor $N_\omega : \mathbf{Gray} \to \mathbf{sMon}$. By the general discussion for lax monoidal right adjoints of §3.3 of [SS03] there exists a left adjoint $C_\omega^\otimes : \mathbf{sMon} \to \mathbf{Gray}$ to N_ω since $\omega\mathbf{Gpd}$ has colimits and a free-forgetful adjunction with $\mathbf{Gray}$. This is the functor $C_\omega^\otimes$ that acts on a simplicial monoid by taking the coequalizer in $\mathbf{Gray}$

$$\begin{tikzcd}
F_\otimes C_\omega U F_\times U(X) \arrow[r, shift left, "\varepsilon^\otimes_{F_\otimes C_\omega U(X)} \circ F_\otimes \varphi"] \arrow[r, shift right, "F_\otimes C_\omega U(\varepsilon^\times_X)"'] & F_\otimes C_\omega U(X) \arrow[r] & C_\omega^\otimes(X)
\end{tikzcd}$$

where $F_\otimes : \omega\mathbf{Gpd} \to \mathbf{Gray}$ and $F_\times : \mathbf{sSet} \to \mathbf{sMon}$ are the free monoid functors left adjoint to the respective forgetful functors, $\varepsilon^\otimes : UF_\otimes \to 1$ and $\varepsilon^\times : UF_\times \to 1$ are the counits of the free-forgetful adjunctions, and $\varphi : C_\omega U F_\times U(X) \to UF_\otimes C_\omega U(X)$ is the ω-functor

$$C_\omega(UF_\times UX) = \bigsqcup_{n \geq 0} C_\omega U(X^n) \xrightarrow{\sqcup \tilde{\varphi}} \bigsqcup_{n \geq 0} C_\omega U(X^{\otimes n}) = UF_\otimes C_\omega U(X)$$

determined by the ω-functors $\tilde{\varphi} : C_\omega(UX \times UX) \to C_\omega UX \otimes C_\omega UX$ that are adjoint to

$$UX \times UX \xrightarrow{\varepsilon_{UX} \times \varepsilon_{UX}} N_\omega C_\omega U(X) \times N_\omega C_\omega U(X) \xrightarrow{AW} N_\omega(C_\omega U(X) \otimes C_\omega U(X))$$

The adjunction

$$\begin{tikzcd}
\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & \mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"]
\end{tikzcd}$$

is a Quillen adjunction for the transferred model structures since fibrations and weak equivalences of **Gray** and **sMon** are created by the forgetful functors to $\omega\mathbf{Gpd}$ and **sSet** and N_ω preserves fibrations and weak equivalences of $\omega\mathbf{Gpd}$.

By the general results for adjunctions with lax-monoidal right adjoints of [SS03] §3 the adjunction $C_\omega^\otimes \dashv N_\omega$ makes the following square of adjunctions commute

$$\begin{tikzcd}
\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] \arrow[d, bend right, "U"'] \arrow[d, phantom, "\vdash"] & \mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[d, bend right, "U"'] \arrow[d, phantom, "\vdash"] \\
\omega\mathbf{Gpd} \arrow[u, bend right, "F_\otimes"'] \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & \mathbf{sSet} \arrow[l, bend left, "C_\omega"] \arrow[u, bend right, "F_\times"']
\end{tikzcd} \tag{91}$$

In particular, for a simplicial set K, $F_\otimes C_\omega(K) \cong C_\omega^\otimes F_\times(K)$. Since $UN_\omega C_\omega^\otimes(X) \cong N_\omega UC_\omega^\otimes(X)$ we can apply the forgetful functor to the simplicial monoid map η_X from the unit of the adjunction $C_\omega^\otimes \dashv N_\omega$ to obtain a simplicial set map

$$U\eta_X : UX \to UN_\omega C_\omega^\otimes(X) \cong N_\omega UC_\omega^\otimes(X)$$

The adjoint of this simplicial set map for the adjunction $C_\omega \dashv N_\omega$ is an ω-functor

$$\chi_X : C_\omega(UX) \to UC_\omega^\otimes(X)$$

We will show later in Section 5.3 that for a class of almost cofibrant simplicial monoids this ω-functor is a weak equivalence. For now we observe that on 0-cells this ω-functor is always a bijection.

Lemma 5.9. *If X is a simplicial monoid then the monoid $C_\omega^\otimes(X)_0$ is equal to X_0.*

Proof. The truncation construction of Definition 4.5 determines a functor $\tau^0 : \mathbf{Gray} \to \mathbf{Mon}$ that has a right adjoint sending a monoid M to the discrete Gray monoid on M. Hence τ^0 preserves colimits and so $C_\omega^\otimes(X)_0$ is the coequalizer in the category of monoids

$$\begin{tikzcd}
F_\times UF_\times U(X_0) \arrow[r, shift left, "\varepsilon^\times_{F_\times U(X_0)}"] \arrow[r, shift right, "F_\times U(\varepsilon^\times_{X_0})"'] & F_\times U(X_0) \arrow[r] & X_0
\end{tikzcd}$$

since for a Gray monoid X, $F_\otimes(X)_0$ is the free monoid on X_0 and $C_\omega(X)_0 = X_0$. □

If M is a Gray monoid we will denote the multiplication of n-simplices of the simplicial monoid $N_\omega(M)$ given by (90) above by $\alpha \cdot \beta$ for $\alpha, \beta : \mathcal{O}^n \to M \in N_\omega(M)_n$. Recall we used similar notation for the action of the monoid of 0-cells on n-cells in a Gray monoid M at the start of Section 5.1. This is justified by the fact that 0-simplices $x : \mathcal{O}^0 \to M$ of $N_\omega(M)$ are the same as 0-cells $x \in M_0$ of M and for $\alpha : \mathcal{O}^n \to M$ an n-simplex of $N_\omega(M)$ the n-simplex $s_0^n(x) \cdot \alpha : \mathcal{O}^n \to M$ is determined by the action of $x \in M_0$ on α in the following way

$$(s_0^n(x) \cdot \alpha)(\langle\theta\rangle) = x \cdot \alpha(\langle\theta\rangle) \in M_n$$

for all $\langle\theta\rangle \in \mathcal{O}^n_m$. A similar identity holds for multiplication on the right. In particular, the monoid structure on the set of 0-simplices is exactly the monoid structure on the set of 0-cells of the Gray monoid.

When M is a Gray group we can show that all simplices of $N_\omega(M)$ have multiplicative inverses, so that $N_\omega(M)$ is a simplicial group. The key step for proving this will be reducing to a special case as described in Appendix C in Lemma C.1.

Proposition 5.10. *Let M be a Gray monoid. The nerve $N_\omega(M)$ is a simplicial group if and only if M is a Gray group.*

Proof. One direction is clear, since for $n = 0$ the Alexander-Whitney ω-functor is an identity so the monoid structure on 0-simplices of $N_\omega(M)$ is the same as the monoid structure on the set of 0-cells M_0. Hence if $N_\omega(M)$ is a simplicial group then M_0 is a group and so M is a Gray group.

Let M be a Gray group. We will show by induction on $n \geq 0$ that the monoids $N_\omega(M)_n$ of n-simplices are groups. For $n = 0$ this holds by the identification of M_0 with $N_\omega(M)_0$ described above. Let $n = 1$ and $\alpha : \mathcal{O}^1 \to M$ be a 1-simplex of $N_\omega(M)$. The Alexander-Whitney ω-functor sends $C_\omega(\Delta)(\langle 1_{[1]}\rangle) = \langle(1_{[1]}, 1_{[1]})\rangle \in C_\omega(\Delta^1 \times \Delta^1)$ to

$$\langle 0 \otimes 1_{[1]}\rangle *_0 \langle 1_{[1]} \otimes 1\rangle \in (\mathcal{O}^1 \otimes \mathcal{O}^1)_1$$

Hence for any $\beta : \mathcal{O}^1 \to M$ the product $\alpha \cdot \beta$ in the monoid of 1-simplices $N_\omega(M)_1$ is the 1-simplex

$$\mathcal{O}^1 \xrightarrow{C_\omega(\Delta)} C_\omega(\Delta^1 \times \Delta^1) \xrightarrow{AW} \mathcal{O}^1 \otimes \mathcal{O}^1 \xrightarrow{\alpha\otimes\beta} M \otimes M \xrightarrow{m} M$$

This corresponds to the 1-simplex $(s_0(d_0^-\beta) \cdot \alpha) *_0 (\beta \cdot s_0(d_0^+\alpha)) \in M_1$. Hence the inverse of $\alpha \in N_\omega(M)_1$ is the 1-simplex $\alpha^{-1} : \mathcal{O}^1 \to M$ determined by the image of $\langle 1_{[1]}\rangle$ which is the 1-cell

$$d_0^-\alpha^{-1} \cdot k^1\alpha \cdot d_0^+\alpha^{-1} : d_0^-\alpha^{-1} \to d_0^+\alpha^{-1}$$

As noted above this is the ω-functor $s : I^1 \to I^1 \otimes I^1$ used in defining the multiplication structure on M^I in Proposition 5.3.

Now let $n > 1$ and suppose that $N_\omega(M)_m$ is a group for all $m < n$. Let $\alpha : \mathcal{O}^n \to M$ be an n-simplex of $N_\omega(M)$. We will start by showing that without loss of generality we may assume that for all $1 \leq i \leq n$ $d_i\alpha = s_0^{n-1}(e)$, the multiplicative identity for the monoid of $n-1$-simplices. By the induction hypothesis there exist multiplicative inverses in $N_\omega(M)_{n-1}$ for all the faces of α. Define

$$\alpha_{(n)} = s_{n-1}(d_n\alpha^{-1}) \cdot \alpha$$

and for all $0 < i < n$

$$\alpha_{(i)} = s_{i-1}(d_i\alpha^{-1}_{(i+1)}) \cdot \alpha_{(i+1)}$$

Hence

$$\begin{aligned} d_n\alpha_{(n)} &= d_ns_{n-1}(d_n\alpha^{-1}) \cdot d_n\alpha \\ &= d_n\alpha^{-1} \cdot d_n\alpha \\ &= s_0^{n-1}(e) \end{aligned}$$

Now let $i > 0$ and assume that $d_j\alpha_{(i+1)} = s_0^{n-1}(e)$ for all $i+1 \leq j \leq n$. Then for $i < j \leq n$

$$\begin{aligned} d_j\alpha_{(i)} &= d_js_{i-1}(d_i\alpha^{-1}_{(i+1)}) \cdot d_j\alpha_{(i+1)} \\ &= s_{i-1}(d_{j-1}d_i\alpha^{-1}_{(i-1)}) \cdot s_0^{n-1}(e) \\ &= s_{i-1}(d_id_j\alpha^{-1}_{(i-1)}) \\ &= s_0^{n-1}(e) \end{aligned}$$

and

$$\begin{aligned} d_i\alpha_{(i)} &= d_i s_{i-1}(d_i\alpha^{-1}_{(i-1)}) \cdot d_i\alpha_{(i-1)} \\ &= d_i\alpha^{-1}_{(i-1)}) \cdot d_i\alpha_{(i-1)} \\ &= s_0^{n-1}(e) \end{aligned}$$

Hence $\alpha_{(1)}$ has $d_i\alpha_{(1)} = s_0^{n-1}(e)$ for all $0 < i \leq n$. Since the degenerate n-simplices used to obtain $\alpha_{(1)}$ from α have inverses by the induction hypothesis, to show α has a multiplicative inverse it is sufficient to show that $\alpha_{(1)}$ has an inverse. We will now show that when $d_i\alpha = s_0^{n-1}(e)$ for all $1 \leq i \leq n$ we are in the special case of Lemma C.1 and in this case α has a multiplicative inverse.

Let $\theta : [m] \hookrightarrow [n]$ be an injective non-decreasing map of Δ with $1 \leq m < n-1$ or such that $m = n-1$ and $\theta(0) = 0$. Then there exists $0 < i \leq n$ such that $\theta = d^i \circ \theta'$ with $\theta' : [m] \hookrightarrow [n-1]$ an injective non-decreasing map. Hence

$$\begin{aligned} \alpha(\langle\theta\rangle) &= \theta^*\alpha(\langle 1_{[m]}\rangle) \\ &= (\theta')^*(d_i\alpha(\langle 1_{[m]}\rangle)) \\ &= (\theta')^*(s_0^{n-1}\alpha(\langle 1_{[m]}\rangle)) \\ &= (s_0^m e(\langle 1_{[m]}\rangle \\ &= 1_e \end{aligned}$$

Since $n > 1$ this means that $\alpha(\langle i\rangle) = e$ for all $i \in [n]$. This means that α is an n-simplex of the special case given in Lemma C.1. We will now construct the inverse of α. The simplex α determines $\alpha(\langle 1_{[n]}\rangle) \in M_n$ with

$$d^-_{n-1}\alpha(\langle 1_{[n]}\rangle) = 1_e \qquad d^+_{n-1}\alpha(\langle 1_{[n]}\rangle) = \alpha(\langle d^0\rangle)$$

by Lemma A.2 since $\alpha(\langle\theta\rangle) = 1_e$ for all $\theta : [m] \hookrightarrow [n]$ injective non-decreasing maps with $m < n-1$ and $\alpha(\langle d^i\rangle) = 1_e$ for all $0 < i \leq n$ where

$$d^-_{n-1}\langle 1_{[n]}\rangle \equiv \sum_{i=0}^{\lfloor {}^{n-1}/_{2}\rfloor} \langle d^{2i+1}\rangle \mod (\mu\mathbb{Z}[\Delta^n])_{n-2}$$

in $\mathcal{O}^n$ by Proposition B.2. Define an n-simplex $\alpha' : \mathcal{O}^n \to M$ of $N_\omega(M)$ that acts on m-atoms of $\mathcal{O}^n$ by

$$\alpha'(\langle\theta\rangle) = \begin{cases} e & m = 0 \\ 1_e & 1 \leq m < n-1 \text{ or } \theta = d^i \text{ for } 0 < i \leq n \\ k^1\alpha(\langle d^0\rangle) & \theta = d^0 \\ k^1\alpha(\langle 1_{[n]}\rangle) & \theta = 1_{[n]} \end{cases}$$

This is a well-defined ω-functor $\mathcal{O}^n \to M$ by Lemma A.2 and Proposition B.2. This determines a multiplicative inverse for α by Lemma C.1 since α' is an n-simplex satisfying the conditions of this lemma, so $\alpha' \cdot \alpha(\langle\theta\rangle) = 1_e$ for $1 \leq m$ and $m < n-1$ or $m = n-1$ and $\theta(0) = 0$ and

$$\begin{aligned} \alpha \cdot \alpha'(\langle\theta\rangle) &= \alpha'(\langle\theta\rangle) *_0 \alpha(\langle\theta\rangle) \\ &= k^1\alpha(\langle\theta\rangle) *_0 \alpha(\langle\theta\rangle) \\ &= 1_e \end{aligned}$$

when $\theta = d^0$ or $1_{[n]}$. Hence $\alpha \cdot \alpha' = s_0^n(e)$ and similarly for $\alpha \cdot \alpha'$. Hence α' is a multiplicative inverse for α. □

The Street nerve and Gray group completion therefore commute with the group completion functor for simplicial monoids as in the following diagram.

$$\begin{tikzcd}
\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] & \mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{GrayGp} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] \arrow[u, bend right, "i"'] & \mathbf{sGp} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[u, bend right, "i"']
\end{tikzcd} \tag{92}$$

In the next section we will describe localizations of these model categories that make the horizontal adjunctions Quillen equivalences.

5.3 Homotopy 2-Types and Gray Monoids

In this section we extend the Quillen equivalence of Proposition 4.62 to homotopy types of simplicial and Gray monoids and groups. In particular, our main goal will be to show that the Quillen equivalence for 2-types of Proposition 4.62 induces Quillen equivalences between similarly localized model structures on **Gray** and **sMon** and on **GrayGp** and **sGp**. We will again use localization of model categories to restrict to 2-types within larger model structures. We start by adapting the arguments of Proposition 4.62 to show a Quillen equivalence between localized categories of Gray monoids and simplicial monoids.

Proposition 5.11. *There exist model structures P_2**Gray** and P_2**sMon** on the categories of Gray monoids and simplicial monoids respectively that are transferred via the free-forgetful monoid adjunctions with $P_2\omega$**Gpd** and P_2**sSet** respectively. The adjunction*

$$\begin{tikzcd}
P_2\boldsymbol{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & P_2\boldsymbol{sMon} \arrow[l, bend left, "C_\omega^\otimes"]
\end{tikzcd}$$

is a Quillen equivalence.

Proof. We will use the results for left Bousfield localizations of monoidal model categories from [Whi18]. In particular, we observe that both ω**Gpd** and **sSet** are tractable cofibrantly generated model categories in the terminology of this reference since the domains of their generating cofibrations are cofibrant. These model categories are also monoidal model categories. Theorem 4.5 of [Whi18] therefore implies that the model categories P_2**sSet** and $P_2\omega$**Gpd** are monoidal model categories that satisfy the monoid axiom and so allow the transfer of their model structures along the free-forgetful adjunction to their categories of monoids if and only if for the maps g_2 and $C_\omega(g_2)$

$$g_2 \times \partial\Delta^n : \partial\Delta^4 \times \partial\Delta^n \hookrightarrow \Delta^4 \times \partial\Delta^n \qquad C_\omega(g_2) \otimes \partial I^n : C_\omega(\partial\Delta^4) \otimes \partial I^n \hookrightarrow \mathcal{O}^4 \otimes \partial I^n$$

$$g_2 \times \Delta^n : \partial\Delta^4 \times \Delta^n \hookrightarrow \Delta^4 \times \Delta^n \qquad C_\omega(g_2) \otimes I^n : C_\omega(\partial\Delta^4) \otimes I^n \hookrightarrow \mathcal{O}^4 \otimes I^n$$

are P_2-weak equivalences of their respective model categories for all $n \geq 0$. This holds because in all cases these maps are identities in degrees less than 4 and weak equivalences for the P_2 model structures are those maps or ω-functors that induce isomorphisms on π_n for $0 \leq n \leq 2$. Hence by [SS03] Theorem 3.12 and Proposition 3.16 the adjunction $C_\omega^\otimes \dashv N_\omega$ is a Quillen equivalence since $C_\omega(\Delta^0) = I^0$ and the cosimplicial space $\Delta^\bullet$ is a cosimplicial resolution of Δ^0. □

A Gray monoid M is P_2-fibrant if and only if it has trivial homotopy groups above degree 2 or equivalently if and only if it has the right lifting property against $F_\otimes(L^0 i_n) : F_\otimes(\partial I^n) \hookrightarrow F_\otimes(I^n)$ for all $n \geq 4$, where $F_\otimes : \omega\mathbf{Gpd} \to \mathbf{Gray}$ is the free Gray monoid functor left adjoint to the forgetful functor. Applying the small object argument to the Gray monoid ω-functors $F_\otimes(L^0 i_n)$ for $n \geq 4$ gives a P_2-fibrant replacement $\hat{M}$ for a Gray monoid M. The Gray monoid ω-functor $M \overset{\sim}{\hookrightarrow} \hat{M}$ is the identity on n-cells for $0 \leq n \leq 3$, so in particular if M is a Gray group then so is $\hat{M}$. Furthermore, the path space construction for Gray monoids of Proposition 5.3 gives a path space for P_2-fibrant Gray monoids, so that we can transfer the model structure of $P_2\mathbf{Gray}$ to a model structure $P_2\mathbf{GrayGp}$ on the category of Gray groups via the inclusion functor $I : \mathbf{GrayGp} \to \mathbf{Gray}$.

Proposition 5.12. *The model structure $P_2\boldsymbol{Gray}$ restricts to a model structure $P_2\boldsymbol{GrayGp}$ on the category of Gray groups such that the adjunction*

$$\begin{tikzcd}
P_2\boldsymbol{GrayGp} \arrow[r, bend right, "I"'] \arrow[r, phantom, "\perp"] & P_2\boldsymbol{Gray} \arrow[l, bend right, "L"']
\end{tikzcd}$$

is a Quillen adjunction.

Proof. We will again adapt the argument of [Qui73] §II as was done in the proof of Proposition 5.4. As described above we have a P_2-fibrant replacement for Gray monoids that preserves the property of being a Gray group. The path space construction of Proposition 5.3 factors the diagonal Gray monoid ω-functor $\Delta : M \to M \times M$ as a folk weak equivalence τ_M followed by a folk fibration π_M

$$\begin{tikzcd}
M \arrow[r, "\sim", "\tau_M"'] \arrow[dr, "\Delta"'] & \hom_{\mathrm{lax}}(I^1, M) \arrow[d, "\pi_M"] \\
 & M \times M
\end{tikzcd}$$

If M is P_2-fibrant then so is $\hom_{\mathrm{lax}}(I^1, M)$ since τ_M is a folk weak equivalence and Gray monoids are P_2-fibrant if and only if their homotopy groups are trivial above degree 2. Hence if M is P_2-fibrant then τ_M is a P_2-weak equivalence and π_M is a P_2-fibration. As was shown the proof of Proposition 5.4, if M is a Gray group then so is the path space $\hom_{\mathrm{lax}}(I^1, M)$. Hence there is a P_2-fibrant replacement functor and a P_2-fibration and P_2-weak equivalence path space construction for Gray groups, so by the same argument as in the proof of Proposition 5.4 the transferred model structure $P_2\mathbf{GrayGp}$ exists and $L \dashv I$ is a Quillen adjunction. □

The same argument for simplicial groups establishes a P_2-model structure on the category of simplicial groups that is transferred from the model structure of $P_2\mathbf{sMon}$ via the adjunction $I \dashv L$.

Hence we have a homotopy 2-type localized version of the square of Quillen adjunctions (92)

$$\begin{tikzcd}
P_2\mathbf{Gray} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] & P_2\mathbf{sMon} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
P_2\mathbf{GrayGp} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] \arrow[u, bend right, "i"'] & P_2\mathbf{sGp} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[u, bend right, "i"']
\end{tikzcd} \tag{93}$$

In this square the top adjunction is a Quillen equivalence by Proposition 4.62. In the rest of this section our goal will be to show that the bottom adjunction is also a Quillen equivalence.

In [SS03] the authors prove that a general Quillen equivalence between monoidal model categories where the right adjoint is lax monoidal can be extended to a Quillen equivalence between categories of monoids under certain conditions. We have already used a lot of their technology in studying the specific case of ω-groupoids and Gray monoids, including the ω-functor $\chi_X : C_\omega U(X) \to UC_\omega^\otimes(X)$ defined for a simplicial monoid X. Proposition 5.1 of [SS03] shows that this ω-functor is a weak equivalence when X is constructed as a transfinite composition of pushouts attaching cells via generating cofibrations for **sMon** starting from the initial simplicial monoid. In fact, in the specific case we are dealing with of the adjunction $C_\omega \dashv N_\omega$ we will show that we can extend this result slightly to more simplicial monoids X. The simplicial monoids we will allow are the following.

Definition 5.13. *Let N be a discrete monoid viewed as a simplicial monoid. An **N-cell simplicial monoid** is a simplicial monoid that is obtained from N as a possibly infinite transfinite composition of pushouts along simplicial monoid maps of the generating cofibrations of **sMon***

$$F_\times(\partial\Delta^n) \hookrightarrow F_\times(\Delta^n)$$

for all $n \geq 1$.

This definition is a small modification of the concept of a **cell monoid** from Definition 3.2 of [SS03]. Note that if N is a free monoid then an N-cell monoid is a cell monoid in the terminology of [SS03].

Here we will be taking advantage of the fact that by Lemma 5.9 $C_\omega^\otimes$ simply turns 0-simplices of simplicial monoids into 0-cells of Gray monoids. In particular, we will not need the monoid of 0-simplices of a simplicial monoid X to be cofibrant to get the same result as [SS03] for the ω-functor $\chi_X : C_\omega U(X) \to UC_\omega^\otimes(X)$.

Lemma 5.14. *Let X be an N-cell simplicial monoid for a discrete monoid N. The ω-functor $\chi_X : C_\omega(X) \to UC_\omega^\otimes(X)$ is a weak equivalence.*

Proof. We will adapt the argument from the proof of Proposition 5.1 of [SS03]. Since X is an N-cell simplicial monoid it is the filtered colimit of a sequence of simplicial monoid cofibrations $X^n \hookrightarrow X^{n+1}$ for $n \geq 0$ where $X^0 = N$, the discrete monoid, and the cofibrations are obtained as pushouts of the generating cofibrations of **sMon** for $n \geq 1$

$$\begin{tikzcd}
\coprod_{t \in T_n} F_\times(\partial\Delta^n) \arrow[r, hook] \arrow[d] & \coprod_{t \in T_n} F_\times(\Delta^n) \arrow[d] \\
X^n \arrow[r, hook] & X^{n+1}
\end{tikzcd}$$

for sets T_n of generators of the free monoids of non-degenerate n-simplices of X. Applying the left adjoint $C^{\otimes}_{\omega}$ preserves colimits, so $C^{\otimes}_{\omega}(X)$ is the filtered colimit in **Gray** of cofibrations $C^{\otimes}_{\omega}(X^n) \hookrightarrow C^{\otimes}_{\omega}(X^{n+1})$ in **Gray** that are obtained as pushouts

$$\begin{tikzcd}
\coprod_{t \in T_n} F_\otimes C_\omega(\partial\Delta^n) \arrow[r, hook] \arrow[d] & \coprod_{t \in T_n} F_\otimes C_\omega(\Delta^n) \arrow[d] \\
C^{\otimes}_{\omega}(X^n) \arrow[r, hook] & C^{\otimes}_{\omega}(X^{n+1})
\end{tikzcd}$$

where we are using the observation from the square of adjunctions (91) that $F_\otimes C_\omega(K) \cong C^{\otimes}_{\omega} F_\times(K)$ for all simplicial sets K.

For all $n \geq 0$ the simplicial monoids X^n are N-cell monoids and it is sufficient to show that the ω-functor

$$\chi_{X^n} : C_\omega U(X^n) \to UC^{\otimes}_{\omega}(X^n)$$

is a weak equivalence for all $n \geq 0$. We will proceed by induction on $n \geq 0$ to show that χ_{X^n} is a weak equivalence and $UC^{\otimes}_{\omega}(X^n)$ is a cofibrant ω-groupoid. By Lemma 5.9 χ_N is an isomorphism, hence it is a weak equivalence. Furthermore all discrete ω-groupoids are cofibrant, so $UC^{\otimes}_{\omega}(X^0) = UN$ is a cofibrant ω-groupoid.

Now let $n \geq 0$ and suppose that χ_{X^n} is a weak equivalence and $UC^{\otimes}_{\omega}(X^n)$ is a cofibrant ω-groupoid. As in the proof of Proposition 5.1 of [SS03] we will adapt the descriptions of the pushouts

$$\begin{tikzcd}
F_\times(\partial\Delta^n) \arrow[r, hook] \arrow[d] & F_\times(\Delta^n) \arrow[d] \\
UX^n \arrow[r, hook] & P
\end{tikzcd}
\qquad
\begin{tikzcd}
F_\otimes C_\omega(\partial\Delta^n) \arrow[r, hook] \arrow[d] & F_\otimes(\mathcal{O}^n) \arrow[d] \\
C^{\otimes}_{\omega}(X^n) \arrow[r, hook] & R
\end{tikzcd}$$

of a free cofibration in **sMon** and **Gray** respectively that is given in the proof of Lemma 6.2 of [SS00]. In this construction the simplicial monoid P is obtained as the filtered colimit in **sSet** of a sequence of simplicial sets and simplicial set cofibrations

$$\begin{tikzcd}
X^n = P_0 \arrow[r, hook] & P_1 \arrow[r, hook] & \cdots \arrow[r, hook] & P_m \arrow[r, hook] & \cdots
\end{tikzcd}$$

The simplicial set cofibrations $P_m \hookrightarrow P_{m+1}$ for $m \geq 1$ are obtained as pushouts of simplicial set cofibrations

$$\begin{tikzcd}
Q^{\times}_m(\partial\Delta^n, \Delta^n, X^n) \arrow[r, hook] \arrow[d] & (X^n \times \Delta^n)^{\times m} \times X^n \arrow[d] \\
P_{m-1} \arrow[r, hook] & P_m
\end{tikzcd}$$

where the top map is the inclusion of a simplicial subset of $(X^n \times \Delta^n)^{\times m} \times X^n$ with l-simplices consisting of length m-sequences of l-simplices $((x^i \in X^n_l, \theta^i \in \Delta^n_l)^m_{i=1}, x \in X^n_l)$ such that $\theta^i \in \partial\Delta^n_l$ for at least one value $1 \leq i \leq m$. There is a similar description of the Gray monoid R as the filtered colimit in $\omega\mathbf{Gpd}$ of folk cofibrations $R_{m-1} \hookrightarrow R_m$ for $m \geq 1$

$$\begin{tikzcd}
UC^{\otimes}_{\omega}(X^n) = R_0 \arrow[r, hook] & R_1 \arrow[r, hook] & \cdots \arrow[r, hook] & R_m \arrow[r, hook] & \cdots
\end{tikzcd}$$

obtained as pushouts in $\omega\mathbf{Gpd}$

$$\begin{tikzcd}
Q^{\otimes}_m(C_\omega(\partial\Delta^n), \mathcal{O}^n, C^{\otimes}_{\omega}(X^n)) \arrow[r, hook] \arrow[d] & (C^{\otimes}_{\omega}(X^n) \otimes \mathcal{O}^n)^{\otimes m} \otimes C^{\otimes}_{\omega}(X^n) \arrow[d] \\
R_{m-1} \arrow[r, hook] & R_m
\end{tikzcd}$$

Applying the functor C_ω to the simplicial set pushouts and filtered colimits defining P preserves these colimits of cofibrations between cofibrant objects, so it is sufficient to show that for all $m \geq 1$ the ω-functors $C_\omega(P_m) \to R_m$ obtained from the universal property of the pushouts that form the front and back faces of the following cube

$$\begin{tikzcd}
C_\omega(Q^{\times}_m(\partial\Delta^n, \Delta^n, X^n)) \arrow[rr, hook] \arrow[dr] \arrow[dd] & & C_\omega((X^n \times \Delta^n)^{\times m} \times X^n) \arrow[dr] \arrow[dd] & \\
& Q^{\otimes}_m(C_\omega(\partial\Delta^n), \mathcal{O}^n, C^{\otimes}_{\omega}(X^n)) \arrow[rr, hook] \arrow[dd] & & (C^{\otimes}_{\omega}(X^n) \otimes \mathcal{O}^n)^{\otimes m} \otimes C^{\otimes}_{\omega}(X^n) \arrow[dd] \\
C_\omega(P_{m-1}) \arrow[rr, hook] \arrow[dr] & & C_\omega(P_m) \arrow[dr] & \\
& R_{m-1} \arrow[rr, hook] & & R_m
\end{tikzcd}$$

All simplicial sets are cofibrant and C_ω is a left Quillen functor by Proposition 4.55, so the ω-groupoids at the corners of the back face are cofibrant. The ω-groupoids R_m are cofibrant for all $m \geq 0$ since there is a cofibration $UC^{\otimes}_{\omega}(X^n) = R_0 \hookrightarrow R_m$ and $UC^{\otimes}_{\omega}(X^n)$ is cofibrant by the induction hypothesis for the induction on n. Since $\omega\mathbf{Gpd}$ is a monoidal model category the tensor product of cofibrant ω-groupoids is cofibrant, so $(C^{\otimes}_{\omega}(X^n) \otimes \mathcal{O}^n)^{\otimes m} \otimes C^{\otimes}_{\omega}(X^n)$ is cofibrant for all m. Finally $Q^{\otimes}_m(C_\omega(\partial\Delta^n), \mathcal{O}^n, C^{\otimes}_{\omega}(X^n))$ is a colimit of a punctured cube diagram where all ω-groupoids are cofibrant and all ω-functors are cofibrations, so it is a cofibrant ω-groupoid.

Hence by Theorem 5.2.5 of [Hov99] to show that the ω-functor $C_\omega(P_m) \to R_m$ is a weak equivalence it is sufficient to show that the three corner ω-functors between the front and back faces are weak equivalences. We will proceed by induction on $m \geq 1$. For $m = 1$ since $P_0 = UX^n$ and $R_0 = UC^{\otimes}_{\omega}(X^n)$ the bottom left corner ω-functor is $\chi_{X^n} : C_\omega U(X^n) \to UC^{\otimes}_{\omega}(X)$, which is a weak equivalence by the induction hypothesis for the induction on n. The two remaining corner comparison ω-functors are weak equivalences by the arguments given in the proof of Proposition 5.1 (1) of [SS03] since $\Delta^\bullet$ is a cosimplicial resolution of Δ^0 so by Proposition 3.16 $C_\omega \dashv N_\omega$ is a weak monoidal Quillen pair. Hence the ω-functor $C_\omega(P_m) \to R_m$ is a weak equivalence for all $m \geq 0$ and so $C_\omega(UX^{n+1}) \to UC^{\otimes}_{\omega}(X^{n+1})$ is a weak equivalence and since $UC^{\otimes}_{\omega}(X^n)$ is cofibrant by induction hypothesis and the ω-functors $R_{m-1} \hookrightarrow R_m$ are cofibrations for all $m \geq 1$ the ω-groupoid $UC_\omega^\otimes(X^n)$ is cofibrant. □

Recall from Section 3.7.2 that a simplicial monoid X is grouplike when the monoid $\pi_0(X)$ is a group. By Proposition 9.5 of [DK80] if X is a cofibrant grouplike simplicial monoid then the unit map $\eta_X : X \to ILX$ for the adjunction $L \dashv I$ is a weak equivalence of simplicial monoids. We will next prove that $C_\omega^\otimes$ sends this group completion map to the unit map for the Gray group completion of the Gray monoid $C_\omega^\otimes(X)$ and that this Gray monoid ω-functor is a folk weak equivalence of Gray monoids. Before doing so we begin with a comparison of the group completion functors for Gray monoids and simplicial monoids.

The group completion of a simplicial monoid freely attaches inverses for all cells in all simplicial degrees. The Gray group completion, however, only attaches multiplicative inverses for the 0-cells of a Gray monoid via the pushhout (84). While we have already observed that for a simplicial monoid X there is an isomorphism $C_\omega^\otimes(LX) \cong LC_\omega^\otimes(X)$ of Gray groups arising from the commutative square of adjunctions (92) we will now show that we can get $LC_\omega^\otimes(X)$ for less from the simplicial monoid X. In particular, if we only freely attach inverses for the 0-simplices of X via a pushout

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, hook] & LX_0 \arrow[d] \\
X \arrow[r] & L_0X
\end{tikzcd} \tag{94}$$

to obtain a simplicial monoid L_0X then $C_\omega^\otimes(L_0X) \cong LC_\omega^\otimes(X)$. This follows from Lemma 5.9 and the fact that $C_\omega^\otimes$ is a left adjoint and so preserves pushouts, so when it is applied to (94) we obtain the pushout (84).

We can now proceed to the proof that $C_\omega^\otimes$ preserves the group completion weak equivalence for grouplike cofibrant simplicial monoids.

Proposition 5.15. *Let X be a cofibrant grouplike simplicial monoid. The Gray monoid ω-functor*

$$C_\omega^\otimes(\eta_X) : C_\omega^\otimes(X) \to C_\omega^\otimes(LX)$$

is a folk weak equivalence.

Proof. By the previous discussion $C_\omega^\otimes$ sends the map $X \to L_0X$ for a simplicial monoid X to the universal Gray monoid ω-functor $C_\omega^\otimes(X) \to LC_\omega^\otimes(X)$. Hence it is sufficient to show that when X is a grouplike cofibrant simplicial monoid $C_\omega^\otimes$ preserves the weak equivalence $X \to L_0X$ shown in the previous lemma. By Lemma 5.14 since X is a cofibrant simplicial monoid the vertical ω-functors in the diagram

$$\begin{tikzcd}
C_\omega U(X) \arrow[r] \arrow[d, "\wr"'] & C_\omega U(L_0X) \arrow[d, "\wr"] \\
UC_\omega^\otimes(X) \arrow[r] & UC_\omega^\otimes(L_0X)
\end{tikzcd}$$

are weak equivalences. By Proposition 10.4 of [DK80] since X is a grouplike cofibrant simplicial monoid the map $X \to L_0X$, which is the localization of X at the discrete cofibrant simplicial submonoid of 0-simplices X_0, is a weak equivalence. Since C_ω preserves all weak equivalences of simplicial sets the top horizontal ω-functor of this diagram is a weak equivalence and the result follows. □

Finally we can prove the Quillen equivalence for Gray groups that we are seeking.

Proposition 5.16. *The Quillen adjunction*

$$\begin{tikzcd}
P_2\boldsymbol{GrayGp} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & P_2\boldsymbol{sGp} \arrow[l, bend left, "C_\omega^\otimes"]
\end{tikzcd}$$

is a Quillen equivalence.

Proof. The model structures on P_2**GrayGp** and P_2**sGp** are transferred from P_2**Gray** and P_2**sMon** respectively via the inclusion functors. Hence Gray monoid ω-functors in P_2**GrayGp** are P_2-weak equivalences or P_2-fibrations if and only if they are P_2-weak equivalences or P_2-fibrations respectively of P_2**Gray**. Since $N_\omega : P_2\mathbf{Gray} \to P_2\mathbf{sMon}$ is right Quillen therefore $N_\omega : P_2\mathbf{GrayGp} \to P_2\mathbf{sGp}$ is also right Quillen. Furthermore $N_\omega : P_2\mathbf{GrayGp} \to P_2\mathbf{sGp}$ creates P_2-weak equivalences because $N_\omega : P_2\mathbf{Gray} \to P_2\mathbf{sMon}$ does, so to show $C_\omega^\otimes \dashv N_\omega$ is a Quillen equivalence it only remains to show that for a cofibrant simplicial group G the unit map $G \to N_\omega C_\omega^\otimes(G)$ is a P_2-weak equivalence. We will show this by replacing G with a weakly equivalent simplicial monoid and using the fact that the adjunction unit for simplicial monoids is a P_2-weak equivalence by Proposition 5.11.

Consider the image IG of a cofibrant simplicial group G under the inclusion functor $I : P_2\mathbf{sGp} \to P_2\mathbf{sMon}$ into simplicial monoids. Take a cofibrant replacement $X_G \overset{\sim}{\to} IG$ of IG in **sMon**, which will be a cofibrant replacement of IG in P_2**sMon** as well since the P_2-localized model category has the same cofibrations as **sMon**. There is a diagram

$$\begin{tikzcd}
X_G \arrow[r, "\eta_{X_G}", "\sim"'] \arrow[d, "\wr"'] & N_\omega C_\omega^\otimes(X_G) \arrow[d] \\
G \arrow[r, "\eta_G"] & N_\omega C_\omega^\otimes(G)
\end{tikzcd}$$

in P_2**sMon** where the top horizontal map is a P_2-weak equivalence by Proposition 5.11. To show that η_G is a P_2-weak equivalence it is sufficient to show that $C_\omega^\otimes(X_G) \to C_\omega^\otimes(G)$ is a weak equivalence, since N_ω preserves all weak equivalences. The rest of the proof will be dedicated to showing this.

The simplicial monoid map $X_G \overset{\sim}{\to} IG$ factors as

$$\begin{tikzcd}
& LX_G \arrow[dr, "\wr"] & \\
X_G \arrow[ur, "\eta_{X_G}", "\wr"'] \arrow[rr, "\sim"'] & & IG
\end{tikzcd}$$

where the map η_{X_G} is a weak equivalence by [DK80] Proposition 9.5 so $LX_G \to IG$ is a weak equivalence between cofibrant simplicial groups since L is a left Quillen functor. Since $C_\omega^\otimes : \mathbf{sGp} \to \mathbf{GrayGp}$ is a left Quillen functor it preserves weak equivalences between cofibrant simplicial groups, hence applying $C_\omega^\otimes$ to this factorization gives

$$\begin{tikzcd}
& C_\omega^\otimes L(X_G) \arrow[dr, "\wr"] & \\
C_\omega^\otimes(X_G) \arrow[ur, "C_\omega^\otimes(\eta_{X_G})", "\wr"'] \arrow[rr, "\sim"'] & & C_\omega^\otimes(G)
\end{tikzcd}$$

where we are using the commutativity of the square of adjunctions (92) that implies $C_\omega^\otimes L(X_G) \cong LC_\omega^\otimes(X_G)$ and suppressing the inclusion functors of Gray groups into Gray monoids and simplicial groups into simplicial monoids. As mentioned $C_\omega^\otimes$ preserves the weak equivalence $LX_G \xrightarrow{\sim} G$. Since $\pi_0(X_G) = \pi_0(G)$ is a group, X_G is a grouplike cofibrant simplicial monoid so by Proposition 5.15 $C_\omega^\otimes(\eta_{X_G})$ is a folk weak equivalence and so $C_\omega^\otimes(X_G) \to C_\omega^\otimes(G)$ is a folk weak equivalence. $\square$

We remark that composing the adjunction $N_\omega \dashv C_\omega^\otimes$ for Gray groups and simplicial groups with the adjunction $L\mathbb{C} \dashv \mathbb{N}$ for simplicial groups and 0-reduced simplicial sets establishes a Quillen equivalence between $P_2\mathbf{GrayGp}$ and $P_3\mathbf{sSet}_0$, the model category of 3-types of reduced simplicial sets.

$$\begin{tikzcd}
P_2\mathbf{GrayGp} \arrow[r, bend left, "N_\omega"] \arrow[r, phantom, "\top"] & P_2\mathbf{sGp} \arrow[l, bend left, "C_\omega^\otimes"] \arrow[r, bend left, "\mathbb{N}"] \arrow[r, phantom, "\top"] & P_3\mathbf{sSet}_0 \arrow[l, bend left, "L\mathbb{C}"]
\end{tikzcd}$$

The Quillen equivalence $L\mathbb{C} \dashv \mathbb{N}$ determines a Quillen equivalence of 2-types of simplicial groups with 3-types of reduced simplicial sets by adapting the results of [Bie08] for the Quillen equivalence $G \dashv \bar{W}$. Studying the right adjoint of this adjunction we notice it resembles the functor N_3 defined in Lemma 2.9 of [Ber99]. The left adjoint of N_3 defined by Berger puts a Gray monoid structure on the realization $C_\omega\mathbb{C}(X)$ rather than using the construction $C_\omega^\otimes$ of [SS03] for a left adjoint for the lax monoidal functor N_ω. The Gray monoid structure given in [Ber99] appears to be based on an ω-groupoid version of the Eilenberg-Zilber or shuffle map, which is a section of the Alexander-Whitney map of chain complexes. The construction of a section for AW is complicated by the fact that the shuffle map on chain complexes does not give an ADC map like $AW^{\mathbf{Ch}}$ does. This chain complex map sends some positive chains to sums including negative chains, so it does not respect the ADC structure of submonoids $\mathbb{N}[\Delta^n \times \Delta^n]_m \leq \mathbb{Z}[\Delta^n \times \Delta^n]_m$ of $\mathbb{Z}[\Delta^n \times \Delta^n]$. It does seem possible, however, to construct a section for the ω-groupoid maps AW. For $n = 1$ there is an obvious ω-functor $\mathcal{O}^1 \otimes \mathcal{O}^1 \to C_\omega(\Delta^1 \times \Delta^1)$ arising from the comparison of the 2-groupoids of each ω-groupoid.

$$\begin{tikzcd}
0\otimes 0 \arrow[r, "\langle 0\otimes 01\rangle"] \arrow[d, "\langle 01\otimes 0\rangle"'] \arrow[dr, phantom, "\overset{\langle 01\otimes 01\rangle}{\Swarrow}"] & 0\otimes 1 \arrow[d, "\langle 01\otimes 1\rangle"] & & (0,0) \arrow[r, "{(00,01)}"] \arrow[d, "{(01,00)}"'] \arrow[dr, "{(01,01)}" description] \arrow[dr, phantom, shift left=3ex, "\overset{(001,011)}{\Nearrow}"] \arrow[dr, phantom, shift right=3ex, "\underset{(011,001)}{\Swarrow}"] & (0,1) \arrow[d, "{(01,11)}"] \\
1\otimes 0 \arrow[r, "\langle 1\otimes 01\rangle"'] & 1\otimes 1 \arrow[r, phantom, "\longmapsto"] & {} & (1,0) \arrow[r, "{(11,01)}"'] & (1,1)
\end{tikzcd}$$

Applying λ recovers the shuffle map, since it sends the 2-chain $01 \otimes 01$ of $\mathcal{O}^1 \otimes \mathcal{O}^1$ to the sum $(011, 001) - (001, 011) \in \mathbb{Z}[\Delta^1 \times \Delta^1]_2$.

Extending this definition to higher values of n is difficult, however, due to the lack of a construction for ω-groupoids corresponding to the functor ν sending ADCs to ω-categories. In [Luc17] §4.2 the author describes (ω, p)-ADCs, which are ADCs where for $n > p$ the distinguished submonoids of the chain groups are equal to the entire abelian chain group. Applying ν to an (ω, p)-ADC gives an (ω, p)-category, however these will retain some of the abelian character of the chain complex, as there is not enough order information for the cells to recover a unique decomposition of all cells up to the axioms of an (ω, p)-category. In particular any pair of loops necessarily commute, so for example in $\nu L\mathbb{Z}[\Delta^3]$, where $L\mathbb{Z}[\Delta^3]$ is the $(\omega, 0)$-ADC arising from the ADC $\mathbb{Z}[\Delta^3]$ the loops

$$\alpha_{013} = \langle 01\rangle *_0 \langle 13\rangle *_0 k^1\langle 03\rangle \qquad\qquad \alpha_{023} = \langle 02\rangle *_0 \langle 23\rangle *_0 k^1\langle 03\rangle$$

are equal no matter the order they are composed in since

$$\begin{aligned}
\alpha_{023} *_0 \alpha_{013} &= \begin{pmatrix} 0 & \emptyset \\ 0 & 02+23-03 \end{pmatrix} *_0 \begin{pmatrix} 0 & \emptyset \\ 0 & 01+13-03 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \emptyset \\ 0 & 01+13-03+02+23-03 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \emptyset \\ 0 & 01+13-03 \end{pmatrix} *_0 \begin{pmatrix} 0 & \emptyset \\ 0 & 02+23-03 \end{pmatrix} \\
&= \alpha_{013} *_0 \alpha_{023}
\end{aligned}$$

Therefore a version of the functor ν to ω-groupoids that distinguishes between these composites will require some extra data. There are Eilenberg-Zilber and Alexander-Whitney maps defined for crossed complexes in [Ton03], which will give rise to ω-functors by the equivalence of the categories of crossed complexes and ω-groupoids. Whether these ω-functors agree with the ω-functors AW defined in Definition 5.7 has not been determined however.

5.4 Coherent Presentations of Monoids and $\pi_2(BM)$

In this final section we will take advantage of the Quillen equivalences from (93) to calculate the first derived functor of group completion of a monoid M. These Quillen equivalences allow us to calculate this derived functor as $\mathbb{L}L_1^{\mathbf{Gray}}(M)$, the first derived functor of the group completion functor $L : \mathbf{Gray} \to \mathbf{GrayGp}$ for Gray monoids. In this section we will show that this derived functor is obtained from two known constructions arising from a presentation of the monoid M by generators and relations: the group of identities among relations and a homotopy basis for the presentation. To do this we will show that the first derived functor of group completion for Gray monoids can also be calculated as the first derived functor of

$$L : \mathbf{Mon}(\omega\mathbf{Gpd}) \to \mathbf{Gp}(\omega\mathbf{Gpd})$$

the group completion functor for monoids in $\omega\mathbf{Gpd}$ for the cartesian product of ω-groupoids. This allows us to enter the world of polygraphic resolutions, which have been developed in the theory of rewriting systems for monoids. These have been used to compute the homology groups of the classifying space BM of a monoid and in this section we show that they can be used to calculate the derived functor of group completion of M, or equivalently by the results of Section 3.7, the homotopy group $\pi_2(BM)$.

We will begin by describing the construction of polygraphic resolutions of monoids from the data of a presentation of a monoid by generators and relations.

5.4.1 Presentations of Monoids and Polygraphic Resolutions

In this section we will show how data from the presentation of a monoid by generators and relations can be used to construct a resolution of the monoid as an object of $(\omega, 1)\mathbf{Cat}$ with the folk model structure. This construction is called a polygraphic resolution in [GM12], where they are defined as a generalization to (ω, m)-categories of the polygraphic resolutions for $\omega\mathbf{Cat}$ defined in [Mét03] and of the construction of Squier in [SOK94]. We will use this resolution in subsequent sections to calculate $\pi_2(BM)$. In fact, by the definition of homotopy groups of ω-groupoids we will only need the 3-truncation of this resolution to calculate the desired derived functor. The data required for this truncation of the polygraphic resolution is called a coherent presentation of M and we will describe it in this section. For this discussion we follow the approach and notation of the survey [Laf07] with reference to the original work of Squier in [SOK94].

A **presentation of a monoid** M is a pair (Σ, R, s, t) consisting of a set Σ of **generators** (or alphabet), a set R of **relations** and maps $s, t : R \to \Sigma^*$ from the set R to the underlying set of the free monoid Σ^* on the generators Σ. Hence each $r \in R$ determines pairs $s(r), t(r) \in \Sigma^*$ of words from the free monoid on the set of generators Σ. This set R generates a **congruence relation** defined as the reflexive and transitive closure $\sim_R^*$ of the symmetric relation

$$uxv \sim_R uyv$$

for $u, v \in \Sigma^*$ and $r \in R$ such that $s(r) = x$ and $t(r) = y$ or $s(r) = y$ and $t(r) = x$. This relation is called the **reduction relation** induced by R in [SOK94]. The monoid presented by this presentation is the quotient of Σ^* by this relation. This quotient is a monoid by the compatibility of the congruence relation with multiplication in Σ^* and the natural map $\pi_R : \Sigma^* \to \Sigma^*/\sim_R^*$ is a map of monoids. Furthermore, for any monoid map $f : \Sigma^* \to M$ such that $f(x) = f(y)$ for all $(x, y) \in R$ there exists a unique monoid map $\bar{f} : \Sigma/\sim_R^* \to M$ such that $f = \bar{f} \circ \pi_R$. The presentation (Σ, R) presents the monoid M if $M \cong \Sigma^*/\sim_R^*$.

The connection to ω-categories comes from the computads defined earlier in Definition 4.45. Recall that a 0-computad is a set viewed as 0-cells of an ω-category and an n-computad for $n \geq 1$ consists of an $n-1$-computad as well as a set of n-indeterminates that are freely attached via pushouts along attaching maps that specify their source and target. The data of a presentation of a monoid M, therefore, determines a $(2,1)$-computad

$$\begin{tikzcd}
R \arrow[d] \arrow[dr, shift left, "t"] \arrow[dr, shift right, "s"'] & \Sigma \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \\
\Sigma^*(R) \arrow[r, shift left, "d_1^+"] \arrow[r, shift right, "d_1^-"'] & \Sigma^* \arrow[r, shift left, "d_0^+"] \arrow[r, shift right, "d_0^-"'] & *
\end{tikzcd} \tag{95}$$

A 1-computad in $\omega\mathbf{Cat}$ with a single 0-cell is a free monoid, so the set Σ of 1-indeterminates produces the free monoid Σ^* viewed as a 1-computad. The maps $s, t : R \to \Sigma^*$ from the presentation determine the attachment of 2-cells in the diagram above. We will denote this $(2,1)$-computad by $(\Sigma, R)^*$ and we will usually suppress the map $s, t : R \to \Sigma^*$ when referring to a presentation (Σ, R) of a monoid in the future.

The map π_R can be extended to an ω-functor of $(2,1)$-categories

$$\begin{tikzcd}
\Sigma^*(R) \arrow[r, shift left, "d_1^+"] \arrow[r, shift right, "d_1^-"'] \arrow[d, "\pi_R \circ d_1^-"'] & \Sigma^* \arrow[r, shift left, "d_0^+"] \arrow[r, shift right, "d_0^-"'] \arrow[d, "\pi_R"'] & * \arrow[d, equal] \\
M \arrow[r, equal] & M \arrow[r, shift left, "d_0^+"] \arrow[r, shift right, "d_0^-"'] & *
\end{tikzcd} \tag{96}$$

It follows from the universal property of a computad that $\pi_R \circ d_1^- = \pi_R \circ d_1^+$ since $\pi_R(x) = \pi_R(y)$ for all $(x, y) \in R$. This structure is the same as the free 2-category generated by the data described in the graph of Definition 3.1 in [SOK94].

Let $\pi : (\Sigma, R)^* \to M$ be the map (96). Then π has the right lifting property against $\partial D^n \hookrightarrow D^n$ for all $0 \leq n \leq 2$. The cases $n = 0$ and 1 are clear. For $n = 2$ a lifting problem against π corresponds to two 1-cells $x, y \in \Sigma^*$ that are sent by π_R to the same element of $M \cong \Sigma^* / \sim_R^*$. By the definition of the quotient by the congruence relation this means there is a sequence

$$x = x_0 \sim_R x_1 \sim_R x_2 \sim_R \cdots \sim_R x_n = y$$

of $x_i \in \Sigma^*$, which is equivalent to the existence of a 2-cell in $\Sigma^*(R)$ joining x and y since M is a $(2, 1)$-category. Hence a solution to the lifting problem exists.

What we have done, therefore, is begin the process of applying the small object argument to the ω-category M for the set of ω-functors, as we did in Lemma 4.46 for ω-groupoids. The set R of relations is a generating set for all pairs of 1-cells, that is words in Σ^*, that must be sent to the same element of M. The congruence relation constructs all parallel pairs of 1-cells sent to the same element of M to give the right lifting property for $(\Sigma, R)^* \to M$ against $\partial D^2 \hookrightarrow D^2$. Without using the terminology of computads (or their other name of polygraphs) this was the process Squier described in [SOK94] and was continued in the context of the folk model structure on $(\omega, 1)$**Cat** in [GM12].

As described in §2.2.2 of [GM12], the next stages of the small object argument for a monoid presentation require freely attaching n-cells to the $(n, 1)$-computad already defined, starting with $\Sigma^*(R)$ when $n = 2$. To construct the next stage of a cofibrant replacement of M enough 3-cells must be freely attached to $\Sigma^*(R)$ so that all parallel pairs of 2-cells of $(\Sigma, R)^*$ are joined by a 3-cell. A **homotopy basis** P for $(\Sigma, R)^*$ is a set together with maps of sets $s, t : P \to \Sigma^*(R)$ that identify parallel pairs of 2-cells of $(\Sigma, R)^*$ such that the $(3, 1)$-computad $(\Sigma, R, P)^*$

$$\begin{tikzcd}
P \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & R \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \Sigma \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \\
\Sigma^*(R)(P) \arrow[r, shift left, "d_2^+"] \arrow[r, shift right, "d_2^-"'] & \Sigma^*(R) \arrow[r, shift left, "d_1^+"] \arrow[r, shift right, "d_1^-"'] & \Sigma^* \arrow[r, shift left, "d_0^+"] \arrow[r, shift right, "d_0^-"'] & *
\end{tikzcd}$$

has a natural ω-functor $(\Sigma, R, P)^* \to M$ with the right lifting property against $\partial D^n \hookrightarrow D^n$ for $0 \leq n \leq 3$.

Given homotopy bases for all $n \geq 3$ the $(\omega, 1)$-category obtained by attaching all of these cells will be a cofibrant replacement of M in $((\omega, 1)\mathbf{Cat})_{>0}$ by the small object argument. This produces the notion of a polygraphic resolution of a monoid, which we record here from [GM12] §2.3.3

Definition 5.17. *Let M be a monoid. A* ***polygraphic resolution*** *of M is an $(\omega, 1)$-computad X_M together with an acyclic fibration $\pi_M : X_M \xrightarrow{\sim} M$ of the folk model structure on $(\omega, 1)$**Cat** such that $(X_M)_0 = *$.*

The specific case of a presentation (Σ, R) and a homotopy basis P that produces a $(3, 1)$-computad replacement of M will be our main interest in this section, so we record its definition as a coherent presentation of a monoid M from [GMM13] Definitions 4 and 5.

Definition 5.18. *A* ***coherent presentation*** *of a monoid M is a triple (Σ, R, P) such that (Σ, R) is a presentation of the monoid M and P is a homotopy basis of the $(2,1)$-computad $(\Sigma, R)^*$.*

A polygraphic resolution of a monoid M determines a coherent presentation from the 3-truncation $\tau^3(X_M)$. By the discussion after the definition of computads in (ω, m)**Cat** in Definition 4.45, there is a unique choice of the set Σ arising from a polygraphic resolution $\tau^3(X_M)$, however there may be many different possible choices of sets of indeterminates generating the 1 and 2-cells of X_M, which are sets R of relations and homotopy bases P for the coherent presentation. We are mainly concerned with the homotopy properties of these presentations, which are not changed by varying the set of indeterminates of a $(3,1)$-computad, so we will refer to a 3-truncation $\tau^3(X_M)$ of a cofibrant replacement of the discrete monoid M as a coherent presentation without specifying sets R and P of relations and a homotopy basis.

These definitions were first developed by Squier in [SOK94] to study the case when these sets R and P can be chosen to be finite. Such monoids are said to have **finite 3-derivation type** and higher finiteness conditions are imposed in [GM12] by requiring the existence of n-truncated polygraphic resolutions of M that have finite homotopy bases in dimensions up to n for $n \geq 3$.

Polygraphic resolutions have also been used to connect data about the presentation of a monoid to its homotopical and homological properties. In particular, in [LM09] the authors show that the polygraphic homology, defined as the derived functors of abelianization $\lambda : (\omega, 1)\mathbf{Cat} \to \mathbf{Ch}$, for monoids viewed as $(\omega, 1)$-categories is the same as the standard definition of monoid homology as the integral homology of the classifying space BM. Our goal in this section is to use polygraphic resolutions to calculate a derived functor of group completion for Gray monoids in ω**Gpd**. To do this we will need to change our perspective of M and its polygraphic resolutions, from objects of $(\omega, 1)$**Cat** to monoids in the category of ω-groupoids.

5.4.2 Strict Monoids and Groups in ωGpd

In this section we will shift our perspective of monoids and their polygraphic resolutions from objects of $(\omega, 1)$**Cat** to monoid objects of ω**Gpd** for the cartesian product. This will allow us to connect these constructions to the model structures for Gray monoids and Gray groups defined in Section 5.1. We will show that the first derived functor of group completion for Gray monoids $\mathbb{L}L_1^{\mathbf{Gray}}$ can be calculated from the functor

$$L : (\omega, 1)\mathbf{Cat} \to \omega\mathbf{Gpd}$$

that adds inverses for 1-cells of an $(\omega, 1)$-category. To do this we will identify the full subcategory $((\omega, 1)\mathbf{Cat})_{>0}$ of 0-reduced $(\omega, 1)$-categories with the category $\mathbf{Mon}(\omega\mathbf{Gpd})$ of monoids in ω**Gpd** with the cartesian product, which we will call **strict monoids**. We will show that this category has a model structure that is the transferred model structure for monoids with respect to the cartesian product. These strict monoids are a special case of Gray monoids and this model structure is a restriction of the model structure on **Gray** from Definition 5.1. The main result of this section shows that the first derived functor of group completion for Gray monoids can be calculated using strict monoids, in particular we will show that $\mathbb{L}L_1^{\mathbf{Gray}}(M)$ can be calculated using the resolution of M by a coherent presentation defined in the previous section.

We can illustrate the equivalence between $((\omega, 1)\mathbf{Cat})_{>0}$ and $\mathbf{Mon}(\omega\mathbf{Gpd})$ with the following example. Recall the computad X_M of a polygraphic resolution for a monoid M. For all $n \geq 1$ the

1-category structure

$$\begin{tikzcd}
(X_M)_n \arrow[r, shift left, "d_0^-"] \arrow[r, shift right, "d_0^+"'] & (X_M)_0 = *
\end{tikzcd}$$

of X_M makes the set $(X_M)_n$ of n-cells of X_M into a monoid. Hence by omitting $(X_M)_0 = *$ and shifting down the degrees of the remaining sets of cells X_M can be viewed as a monoid object for the cartesian product in ω**Gpd**. Clearly the monoid M can also be seen as a monoid object for the cartesian product in ω**Gpd** and the ω-functor $\pi_M : X_M \to M$ is an ω-functor of cartesian product monoids in ω**Gpd**.

This example identifies an isomorphism of categories between monoids in ω**Gpd** with the cartesian product and the full subcategory of $(\omega, 1)$**Cat** of 0-reduced $(\omega, 1)$-categories. When the monoid in ω**Gpd** is a group then the corresponding ω-category with a single object is an ω-groupoid.

Lemma 5.19. *There are isomorphisms S of categories making the diagram*

$$\begin{tikzcd}
((\omega,1)\,\boldsymbol{Cat})_{>0} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "S"', "\cong"] & (\omega\,\boldsymbol{Gpd})_{>0} \arrow[l, bend left, "I"] \arrow[d, "S"', "\cong"] \\
\boldsymbol{Mon}(\omega\,\boldsymbol{Gpd}) \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \boldsymbol{Gp}(\omega\,\boldsymbol{Gpd}) \arrow[l, bend left, "I"]
\end{tikzcd}$$

commute where the functors I are inclusions and the functors L are the left adjoints to inclusion that add inverses.

Proof. These are instances of the isomorphism of Observation 85 of [Ver08] which identifies ω-categories as categories enriched in ω-categories. In this case the category has a single object, so it is a monoid, and it is an $(\omega, 1)$-category so it is enriched in ω-groupoids. When the ω-category with a single object is an ω-groupoid its monoid of 1-cells is a group. $\square$

We previously studied monoids in ω**Gpd** with the Gray tensor product, rather than the cartesian product, so to make use of the theory of polygraphic resolutions we must understand the relationship between these two kinds of monoids in ω**Gpd**.

Recall from Section 5.1 that we called monoids in ω**Gpd** with the Gray tensor product Gray monoids. We will call monoids for the cartesian product **strict monoids**. As this name suggests, strict monoids can be seen as a restricted class of Gray monoids. To justify this, we observe that since the terminal object I^0 of ω**Gpd** is the Gray tensor unit there are projections

$$\begin{tikzcd}
 & X \otimes Y \arrow[dl, "X\otimes !"'] \arrow[dr, "!\otimes Y"] & \\
X \cong X \otimes D^0 & & D^0 \otimes Y \cong Y
\end{tikzcd}$$

Hence the identity functor on ω**Gpd** is lax monoidal for the Gray and cartesian products, with a natural transformation

$$\varphi_{X,Y} = \langle X\otimes !, ! \otimes Y\rangle : X \otimes Y \to X \times Y$$

so by the general arguments of [SS03] §3.3 since $\omega\mathbf{Gpd}$ is cartesian closed by Theorem 5.1 of [How79] and the equivalence of the categories of crossed complexes and ω-groupoids there is an adjunction

$$\begin{tikzcd}
\mathbf{Mon}(\omega\mathbf{Gpd}) \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \mathbf{Gray} \arrow[l, bend left, "L^{mon}"]
\end{tikzcd} \tag{97}$$

The functor R sends a strict monoid M with multiplication ω-functor $m : M \times M \to M$ and unit $e : I^0 \to M$ to the Gray monoid RM with underlying ω-groupoid M, unit $e : I^0 \to M$ and multiplication given by the ω-functor

$$M \otimes M \xrightarrow{\varphi_{M,M}} M \times M \xrightarrow{m} M$$

If a monoid M in $\mathbf{Mon}(\omega\mathbf{Gpd})$ is a group then since R preserves the monoid of 0-cells of M it is sent by R to a Gray group. By the same argument as in Lemma 5.9 L^{mon} preserves the monoid of 0-cells so it sends a Gray group to a group in $\omega\mathbf{Gpd}$. Hence the adjunction (97) extends to a square of adjunctions

$$\begin{tikzcd}
\mathbf{Mon}(\omega\mathbf{Gpd}) \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "L"] \arrow[d, phantom, "\vdash"] & \mathbf{Gray} \arrow[l, bend left, "L^{mon}"] \arrow[d, bend left, "L"] \arrow[d, phantom, "\vdash"] \\
\mathbf{Gp}(\omega\mathbf{Gpd}) \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] \arrow[u, bend left, "I"] & \mathbf{GrayGp} \arrow[l, bend left, "L^{mon}"] \arrow[u, bend left, "I"]
\end{tikzcd} \tag{98}$$

The polygraphic resolutions described above in Section 5.4.1 live in $\mathbf{Mon}(\omega\mathbf{Gpd})$, so we would like to extend the homotopical structures of Gray monoids to strict monoids by putting a model structure on $\mathbf{Mon}(\omega\mathbf{Gpd})$ and $\mathbf{Gp}(\omega\mathbf{Gpd})$ and making the adjunctions in (98) a Quillen adjunction.

Recall that in Section 5.1 we used the fact that $\omega\mathbf{Gpd}$ is a monoidal model category by [AL20] §6 to show that there was a model structure on $\mathbf{Gray}$ that was transferred from $\omega\mathbf{Gpd}$ via the free-forgetful adjunction $F_\otimes \dashv U$ with $\omega\mathbf{Gpd}$. We cannot use the same approach for strict monoids because $\omega\mathbf{Gpd}$ is not a monoidal model category with the cartesian product. By Example 7.2 of [Lac02] the pushout corner ω-functor

$$(I^1 \times \partial I^1) \cup_{\partial I^1 \times \partial I^1} (\partial I^1 \times I^1) \to I^1 \times I^1$$

is not a cofibration of $\omega\mathbf{Gpd}$ so $- \times - : \omega\mathbf{Gpd} \times \omega\mathbf{Gpd} \to \omega\mathbf{Gpd}$ cannot be a left Quillen bifunctor.

We will show that there is a transferred model structure on $\mathbf{Mon}(\omega\mathbf{Gpd})$ from the free-forgetful adjunction with $\omega\mathbf{Gpd}$ and a transferred model structure on $\mathbf{Gp}(\omega\mathbf{Gpd})$ via the adjunction $I \dashv L$ or equivalently the composite adjunction $LL^{mon} \dashv RI$. We will do this by making use of the isomorphisms of Lemma 5.19. First we record the following lemma that describes the ω-functors in $((\omega,1)\mathbf{Cat})_{>0}$ and $(\omega\mathbf{Gpd})_{>0}$ that correspond under S to the proposed weak equivalences of $\mathbf{Mon}(\omega\mathbf{Gpd})$ and $\mathbf{Gp}(\omega\mathbf{Gpd})$.

Lemma 5.20. *Let $f : X \to Y$ be an ω-functor of $((\omega,1)\boldsymbol{Cat})_{>0}$ (respectively of $(\omega\boldsymbol{Gpd})_{>0}$. The following are equivalent:*

1. $If : IX \to IY$ is a weak equivalence of $(\omega, 1)\boldsymbol{Cat}$ (respectively $\omega\boldsymbol{Gpd}$)

2. for all $n \geq 0$ and all parallel n-cells $x, x' \in X_n$ if there exists an $n+1$-cell $v : f(x) \to f(x')$ of Y then there exists an $n+1$-cell $u : x \to x'$ of X and an $n+2$-cell $w : f(u) \to v$ of Y

3. $USf : USX \to USX$ is a weak equivalence of $\omega\boldsymbol{Gpd}$

where $I : ((\omega, 1)\boldsymbol{Cat})_{>0} \to (\omega, 1)\boldsymbol{Cat}$ is the inclusion functor, S is the isomorphism of Lemma 5.19 and $U : \boldsymbol{Mon}(\omega\boldsymbol{Gpd}) \to \omega\boldsymbol{Gpd}$ is the forgetful functor.

Proof. By Remark 3.20 of [AM11] the ω-functor $If : IX \to IY$ of $(\omega, 1)$**Cat** is a weak equivalence if and only it is an ω-weak equivalence of ω**Cat**. The ω-weak equivalences of ω**Cat** are defined in Definition 8 of §4.3 of [LMW10b]. To describe these we must first identify the reversible cells of an $(\omega, 1)$-category in the sense of Definition 6 of §4.1 of [LMW10b].

Let X be an $(\omega, 1)$-category. All n-cells of X are reversible in the sense of Definition 6 for $n \geq 2$ since they are invertible as n-cells. In particular, for $x \in X_n$ with $n \geq 2$ there exists $k^n x \in X_n$ such that $k^n x *_{n-1} x = 1_{d^+_n-1x}$ and $x *_{n-1} k^n x = 1_{d^-_n-1x}$, so x is reversible since by Proposition 6 of §4.2 of [LMW10b] all identity $n+1$-cells are reversible. Hence for $n \geq 1$ two parallel n-cells of X are ω-equivalent by Definition 6 of §4.2 of [LMW10b] if and only if there exists a $n+1$-cell of X joining them.

Let $f : X \to Y$ be an ω-functor of $((\omega, 1)\mathbf{Cat})_{>0}$, so X and Y are $(\omega, 1)$-categories each with a unique object. Then by Definition 8 of §4.3 of [LMW10b] If is a weak equivalence if and only if for all $n \geq 0$ and all parallel n-cells $x, x' \in X_n$ if there exists an $n+1$-cell $v : f(x) \to f(x')$ of Y then there exists an $n+1$-cell $u : x \to x'$ of X and an $n+2$-cell $w : f(u) \to v$ of Y. Under the isomorphism S of Lemma 5.19 n-cells of X for $n \geq 1$ correspond to $n-1$-cells of USX. Therefore the conditions for If to be a weak equivalence of $(\omega, 1)$**Cat** are the same as for USf to satisfy the second characterization of Proposition 4.16 of weak equivalences of ω**Gpd**.

When $f : X \to Y$ is an ω-functor of $(\omega\mathbf{Gpd})_{>0}$ the equivalence of the first two conditions is Proposition 4.16. Hence f is a weak equivalence of ω**Gpd** if and only if it is a weak equivalence of $(\omega, 1)$**Cat**, which completes the proof. □

Proposition 5.21. *There is are model structures on the categories $\boldsymbol{Mon}(\omega\boldsymbol{Gpd})$ of strict monoids in $\omega\boldsymbol{Gpd}$ and subcategory of groups $\boldsymbol{Gp}(\omega\boldsymbol{Gpd})$ in $\omega\boldsymbol{Gpd}$ that are transferred via the adjunctions with $\omega\boldsymbol{Gpd}$*

$$\begin{tikzcd}
\boldsymbol{Gp}(\omega\boldsymbol{Gpd}) \arrow[r, bend left, "I"] \arrow[r, phantom, "\top"] & \boldsymbol{Mon}(\omega\boldsymbol{Gpd}) \arrow[l, bend left, "L"] \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \omega\boldsymbol{Gpd} \arrow[l, bend left, "F_\times"]
\end{tikzcd}$$

Proof. First ω**Gpd** is cartesian closed by Theorem 5.1 of [How79] and the equivalence of the categories of crossed complexes and ω-groupoids. Hence for any ω-groupoid X the functor $X \times -$ preserves colimits and so by the corollary of §2.6 of [Por08] the forgetful functor $U : \mathbf{Mon}(\omega\mathbf{Gpd}) \to \omega\mathbf{Gpd}$ has a left adjoint which we will denote by $F_\times$.

We will show that $\mathbf{Mon}(\omega\mathbf{Gpd})$ has the transferred model structure by showing that it is a cofibrantly generated model category with set of generating cofibrations

$$F_\times(I) = \{F_\times(L^0 i_n) : F_\times(\partial I^n) \hookrightarrow F_\times(I^n) \mid n \geq 0\} \tag{99}$$

set of generating acyclic cofibrations

$$F_\times(J_0^-) = \{F_\times(L^0 j_n^-) : F_\times(I^{n-1}) \hookrightarrow F_\times(I^n) \mid n \geq 1\} \tag{100}$$

and weak equivalences the strict monoid ω-functors sent by $U : \mathbf{Mon}(\omega\mathbf{Gpd}) \to \omega\mathbf{Gpd}$ to weak equivalences of $\omega\mathbf{Gpd}$. We will do this by showing the corresponding cofibrantly generated model structure on $((\omega,1)\mathbf{Cat})_{>0}$ determined by the isomorphism of categories of Lemma 5.19. We must therefore show that $((\omega,1)\mathbf{Cat})_{>0}$ is a cofibrantly generated model category with generating cofibrations the set $S^{-1}F_\times(I)$, generating acyclic cofibrations the set $F_\times(J_0^-)$ and by Lemma 5.20 weak equivalences the ω-functors that are sent by I to weak equivalences of $(\omega,1)\mathbf{Cat}$.

The category $((\omega,1)\mathbf{Cat})_{>0}$ is the full subcategory of $(\omega,1)\mathbf{Cat}$ on the $(\omega,1)$-categories with a single object. The left adjoint R_0 of the inclusion functor $I : ((\omega,1)\mathbf{Cat})_{>0} \to (\omega,1)\mathbf{Cat}$ sends a $(\omega,1)$-category X to the pushout in $(\omega,1)\mathbf{Cat}$

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, hook] & * \arrow[d] \\
X \arrow[r, "\eta_X"'] & IR_0X
\end{tikzcd}$$

where the left vertical map is the inclusion of the set of 0-cells of X viewed as a discrete $(\omega,1)$-category and the bottom horizontal map is the unit of the adjunction $R_0 \dashv I$. We can recognize the ω-functors (99) and (100) under the isomorphism S as the images of ω-functors of $(\omega,1)\mathbf{Cat}$ under R_0. Let $n \geq 0$. For X an $(\omega,1)$-category with a single 0-cell there are isomorphisms

$$\begin{array}{rcl}
X_{n+1} & \cong & \mathrm{Hom}_{(\omega,1)\mathbf{Cat}}(L^1D^{n+1}, IX) \\
 & \cong & \mathrm{Hom}_{((\omega,1)\mathbf{Cat})_{>0}}(R_0L^1D^{n+1}, X)
\end{array}$$

$$\begin{array}{rcl}
X_{n+1} & = & SX_n \\
 & \cong & \mathrm{Hom}_{\mathbf{Mon}(\omega\mathbf{Gpd})}(F_\times(I^n), SX) \\
 & \cong & \mathrm{Hom}_{((\omega,1)\mathbf{Cat})_{>0}}(S^{-1}F_\times(I^n), X)
\end{array}$$

Hence for all $n \geq 0$ by the Yoneda lemma $S^{-1}F_\times(I^n) \cong R_0L^1D^{n+1}$ in $((\omega,1)\mathbf{Cat})_{>0}$ and so for $n \geq 0$

$$S^{-1}F_\times(L^0 i_n) : S^{-1}F_\times(\partial I^n) \hookrightarrow S^{-1}F_\times(I^n) \cong R_0(L^1 i_{n+1}) : R_0(\partial L^1D^{n+1} \hookrightarrow L^1D^{n+1})$$

and for $n \geq 1$

$$S^{-1}F_\times(L^0 j_n^\varepsilon) : S^{-1}F_\times(I^{n-1}) \hookrightarrow S^{-1}F_\times(I^n) \cong R_0(L^1 j_{n+1}^\varepsilon) : R_0(L^1D^n \hookrightarrow L^1D^{n+1})$$

Therefore it is sufficient to show that $((\omega,1)\mathbf{Cat})_{>0}$ is a cofibrantly generated model category with generating cofibrations the set of ω-functors

$$I_{>0} = \{R_0(L^1 i_n) \mid n \geq 1\}$$

and generating acyclic cofibrations the set of ω-functors

$$J_{>0} = \{R_0(L^1 j_n^-) \mid n \geq 2\}$$

Let $\mathcal{W}_{>0}$ be the class of ω-functors of $((\omega,1)\mathbf{Cat})_{>0}$ satisfying the equivalent conditions of Lemma 5.20. We show this data determines a cofibrantly generated model structure on $((\omega,1)\mathbf{Cat})_{>0}$ using J. Smith's theorem as stated in Theorem 1.7 of [Bek00]. By Proposition 5 of §3.2 of [LMW10b] $\omega\mathbf{Cat}$ is locally finitely presentable, hence so are its reflective subcategories $(\omega,1)\mathbf{Cat}$ and $((\omega,1)\mathbf{Cat})_{>0}$. If $f : X \to Y$ has the right lifting property against all ω-functors in $I_{>0}$ then for all pairs of parallel n-cells of X with $n \geq 0$ if there exists $y \in Y_{n+1}$ such that $d_n^- y = f(x)$ and $d_n^+ y = f(x')$ then there is a lifting problem in $(\omega,1)\mathbf{Cat}$

$$\begin{tikzcd}
\partial L^1 D^n \arrow[r, "\langle x,x' \rangle"] \arrow[d, hook, "L^1 i_n"'] & IX \arrow[d, "If"] \\
L^1 D^n \arrow[r, "y"'] & IY
\end{tikzcd}$$

which has a solution by adjointness of $R_0 \dashv I$ and $I_{>0}$-injectivity of f. Hence f satisfies the second condition of Lemma 5.20 so it belongs to $\mathcal{W}_{>0}$. The class $\overset{\sim}{\hookrightarrow}(I_{>0}) \cap \mathcal{W}_{>0}$ is closed under transfinite composition and pushout because $\overset{\sim}{\hookrightarrow}(I_{>0})$ is and the inclusion functor I preserves both and sends ω-functors in $\overset{\sim}{\hookrightarrow}(I_{>0}) \cap \mathcal{W}_{>0}$ to acyclic cofibrations of $(\omega,1)\mathbf{Cat}$, which are weak equivalences. Finally by the second characterization of weak equivalences in Lemma 5.20 $\mathcal{W}_{>0}$ satisfies the solution set condition at $I_{>0}$.

The model structure on $\mathbf{Gp}(\omega\mathbf{Gpd})$ is transferred from $\mathbf{Mon}(\omega\mathbf{Gpd})$ via the adjunction $L \dashv I$ if and only if it is transferred from $\omega\mathbf{Gpd}$ via the composite adjunction $LF_\times \dashv UI$. This transferred model structure exists by the same arguments as for $\mathbf{Mon}(\omega\mathbf{Gpd})$.

□

As an immediate corollary the adjunctions appearing in the square (98) are all Quillen.

Corollary 5.22. *The adjunctions*

$$\begin{tikzcd}
\boldsymbol{Mon}(\omega\,\boldsymbol{Gpd}) \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "L"] \arrow[d, phantom, "\vdash"] & \boldsymbol{Gray} \arrow[l, bend left, "L^{mon}"] \arrow[d, bend left, "L"] \arrow[d, phantom, "\vdash"] \\
\boldsymbol{Gp}(\omega\,\boldsymbol{Gpd}) \arrow[u, bend left, "I"] \arrow[r, bend left, "R"] \arrow[r, phantom, "\top"] & \boldsymbol{GrayGp} \arrow[l, bend left, "L^{mon}"] \arrow[u, bend left, "I"]
\end{tikzcd}$$

are Quillen adjunctions when the categories of monoids and groups have the transferred model structures.

Proof. The right adjoints R of the top and bottom adjunctions acts as the identity on ω-functors so by the definition of the transferred model structures on $\mathbf{Mon}(\omega\mathbf{Gpd})$ and $\mathbf{Gray}$ it preserves fibrations and weak equivalences. The left vertical adjunction is the adjunction transferring the model structure for $\mathbf{Gp}(\omega\mathbf{Gpd})$ by Proposition 5.21 so it is Quillen. The right vertical adjunction is Quillen by Proposition 5.4. □

The next result shows that if we wish to calculate π_1 of a Gray monoid or group then we may as well make it strict first with the functor L^{mon}.

Proposition 5.23. *The units of the adjunctions $L^{mon} \dashv R$ between **Gray** and **Mon**$(\omega$**Gpd**$)$ and between **GrayGp** and **Gp**$(\omega$**Gpd**$)$ induce isomorphisms on π_n for $n = 0, 1$.*

Proof. Let M be a Gray monoid with multiplication ω-functor $\mu_M : M \otimes M \to M$. Recall the smart n-truncation defined in Definition 4.11. Taking $n = 1$ we consider the 1-groupoid $\bar{\tau}^1(M)$. We claim that this is a strict monoid in ω**Gpd**. Define a multiplication

$$m_M : \bar{\tau}^1(M) \times \bar{\tau}^1(M) \to \bar{\tau}^1(M)$$

that is the same as μ_M on 0-cells. The 1-cells of $\bar{\tau}^1(M)$ are homotopy classes of 1-cells of M, so let $[x], [y] \in \bar{\tau}^1(M)_1$ be two such classes with representatives $x, y \in M_1$. The ω-functor m_M sends $([x], [y]) \in (\bar{\tau}^1(M) \times \bar{\tau}^1(M))_1$ to the 1-cell

$$I^1 \xrightarrow{s} I^1 \otimes I^1 \xrightarrow{x \otimes y} M \otimes M \xrightarrow{\mu_M} M \longrightarrow \bar{\tau}^1(M)$$

where $s : I^1 \to I^1 \otimes I^1$ is the ω-functor given in Proposition 5.3 or equivalently the composite $AW \circ C_\omega(\Delta)$ of the Alexander-Whitney ω-functor of Definition 5.7 with C_ω applied to the diagonal $\Delta^1 \to \Delta^1 \times \Delta^1$. Hence

$$m_M([x], [y]) = [(d_0^- x \cdot y) *_0 (x \cdot d_0^+ y)]$$

This is a well-defined composite as for any 2-cell $z : x \implies x'$ of M there is a 2-cell

$$(d_0^- x \cdot y) *_0 (z \cdot d_0^+ y) : (d_0^- x \cdot y) *_0 (x \cdot d_0^+ y) \implies (d_0^- x \cdot y) *_0 (x' \cdot d_0^+ y)$$

and similarly for a homotopy of y.

With this strict monoid structure on $\bar{\tau}^1(M)$ the ω-functor $\gamma : M \to RI\bar{\tau}^1(M)$ determined by the unit of the adjunction $\bar{\tau}^1 \dashv I$ for the inclusion functor $I :$ **Gpd** $\to \omega$**Gpd** of 1-groupoids into ω-groupoids is a Gray monoid ω-functor. The diagram

$$\begin{tikzcd}
M \otimes M \arrow[r, "\gamma \otimes \gamma"] \arrow[d, "\mu_M"'] & \bar{\tau}^1(M) \otimes \bar{\tau}^1(M) \arrow[d] \\
M \arrow[dr, "\gamma"'] & \bar{\tau}^1(M) \times \bar{\tau}^1(M) \arrow[d, "m_M"] \\
& \bar{\tau}^1(M)
\end{tikzcd}$$

commutes by the definition of m_M. Furthermore, γ has the universal property that for all Gray monoid ω-functors $f : M \to N$ if N is a 1-groupoid then there exists a unique Gray monoid ω-functor $\bar{f} : RI\bar{\tau}^1(M) \to N$ such that $f = \bar{f} \circ \gamma$. This holds as a 1-groupoid Gray monoid is necessarily the image under R of a strict monoid.

Now consider the unit $\eta_M : M \to RL^{mon}(M)$ of the adjunction (98). Compose this with the Gray monoid ω-functor

$$RL^{mon}(M) \to RI\bar{\tau}^1(L^{mon}(M))$$

and by the universal property of γ there exists a unique ω-functor $RI\bar{\tau}^1(L^{mon}(M)) \to RI\bar{\tau}^1(M)$ making the outer path of the diagram

$$\begin{tikzcd}
 & RL^{mon}(M) \arrow[r, "R\gamma"] \arrow[dr, dashed] & RI\bar{\tau}^1(L^{mon}(M)) \\
M \arrow[ur, "\eta_M"] \arrow[rr, "\gamma"'] & & RI\bar{\tau}^1(M) \arrow[u, dashed]
\end{tikzcd}$$

commute. By the universal property of η_M there exists a unique Gray monoid ω-functor

$$RL^{mon}(M) \to RI\bar{\tau}^1(M)$$

making the right triangle of the diagram commute. Finally the universal property of $R(\gamma)$ shows that $\bar{\tau}^1(M) \cong \bar{\tau}^1(L^{mon}(M))$. But since $\bar{\tau}^1(M) = \varpi_1(M)$ Definition 4.13 implies that π_n of M and $L^{mon}(M)$ are isomorphic for $n = 0, 1$. The same arguments give the result for the adjunction $L^{mon} \dashv R$ between **GrayGp** and **Gp**(ω**Gpd**).

$\square$

It seems likely that this statement can be promoted to a Quillen equivalence between the model categories P_1**Gray** of 1-types of Gray monoids and the corresponding localized model structure of strict monoids. Composing this with the previous Quillen equivalences with reduced simplicial sets would recover the 2-group case of the known Quillen equivalence (Proposition 4.62) between homotopy 2-types and 2-groupoids. However, we do not show the necessary conditions to construct the localization for 1-types on **Mon**(ω**Gpd**).

Finally we will use these isomorphisms to complete the goal of this section. We will show that the first derived functor of group completion of a discrete monoid M viewed as a Gray monoid is the same as the corresponding derived functor of group completion for M viewed as a strict monoid.

Proposition 5.24. *Let M be a discrete monoid. The first derived functors of group completion for M in **Gray** and in **Mon**(ω**Gpd**) are equal*

$$\mathbb{L}L_1^{\textbf{Gray}}(M) = \mathbb{L}L_1^{\textbf{Mon}(\omega\textbf{Gpd})}(M)$$

Proof. The first derived functor $\mathbb{L}L_1^{\textbf{Gray}}(M)$ of M is calculated by taking a cofibrant replacement of M as a Gray monoid and applying the functor $L :$ **Gray** $\to$ **GrayGp** followed by the fundamental group functor π_1. We will construct a cofibrant replacement of M in **Gray** using the small object argument for the generating cofibrations $F_{\otimes}(i_n)$ of **Gray**. This determines a Gray monoid ω-functor $f : X_M \to M$, which is an acyclic fibration by the usual small object argument.

We will now consider the image under L^{mon} of this Gray monoid ω-functor $f : X_M \to M$. Since M is a discrete monoid $L^{mon}(M) = M$. Since L^{mon} is left Quillen by Corollary 5.22 $L^{mon}X_M$ is a cofibrant strict monoid. We claim that $L^{mon}(X_M) \to M$ induces isomorphisms on π_n for $n = 0, 1$. By Proposition 5.23 the horizontal Gray monoid ω-functor below induces an isomorphism on π_n for $n = 0, 1$, so since X_M is a resolution of the discrete monoid M the diagonal Gray monoid ω-functor in the diagram

$$\begin{tikzcd}
X_M \arrow[r] \arrow[d, "\wr"'] & RL^{mon}(X_M) \arrow[dl] \\
M &
\end{tikzcd}$$

induces an isomorphism on π_n for $n = 0, 1$, as claimed.

The 2-truncation $\tau^2(L^{mon}(X_M)) \to M$ in $\mathbf{Mon}(\omega\mathbf{Gpd})$ of the image of the resolution under L^{mon} can be extended by the small object argument to a cofibrant replacement of M in $\mathbf{Mon}(\mathbf{Gpd})$, which we denote by $T_M \overset{\sim}{\to} M$. Therefore the first derived functor of $L : \mathbf{Mon}(\omega\mathbf{Gpd}) \to \mathbf{Gp}(\omega\mathbf{Gpd})$ acting on M gives

$$\mathbb{L}L_1^{\mathbf{Mon}(\omega\mathbf{Gpd})}(M) = \pi_1(LT_M) = \pi_1(L\tau^2(L^{mon}X_M)) = \pi_1(LL^{mon}(X_M))$$

since π_1 only depends on the 2-truncation of an ω-groupoid. Now

$$\begin{aligned} \pi_1(LX_M) &\cong \pi_1(L^{mon}L(X_M)) && \text{Prop. 5.23} \\ &= \pi_1(LL^{mon}X_M)) && \text{Cor. 5.22} \end{aligned}$$

so since X_M is a cofibrant replacement of M in **Gray** the first derived functors of the group completion functors L for **Gray** and $\mathbf{Mon}(\omega\mathbf{Gpd})$ are isomorphic. □

5.4.3 A Hopf Formula for $\pi_2(BM)$

In this section we give a formula for $\pi_2(BM)$ based on the data of a presentation of the monoid M. We will use a coherent presentation of a monoid from Definition 5.18 as a truncated polygraphic resolution and the results of the previous section identifying $((\omega, 1)\mathbf{Cat})_{>0}$ with $\mathbf{Mon}(\omega\mathbf{Gpd})$ and the derived functors

$$\mathbb{L}L_1^{\mathbf{Gray}}(M) = \mathbb{L}L_1^{\mathbf{Mon}(\omega\mathbf{Gpd})}(M)$$

to calculate the first derived functor of group completion of M, which by Section 3.7 is isomorphic to $\pi_2(BM)$. In performing this calculation we recognize that the groups involved have already been studied, so we can benefit from methods of calculation already devised for them. We will use some of these methods to do some example calculations at the end of this section.

The group $\pi_2(BM)$ will be shown to be a quotient of the group of **identities among relations**. This is an abelian group that is defined in [BH82] §1 from the data of a presentation of a group as a quotient of a free group by a normal subgroup. For a group G presented by a free group F and a normal subgroup $N \trianglelefteq F$ this group is defined as the kernel of the map of a crossed module $H \to F$ freely generated by F in degree 1 and generators R of N in degree 2. We will modify this definition by applying it to monoid presentations using the group completion functor and replacing crossed modules with ω-groupoids via the equivalence of categories.

Definition 5.25. *Let M be a monoid with presentation (Σ, R, s, t). The group of* ***identities among relations*** *$N(R)$ is the automorphism group of 2-cells*

$$N(R) = \mathrm{Hom}_{L(\Sigma,R)^*}(1_*, 1_*)$$

of the 2-groupoid $L(\Sigma, R)^*$ obtained by applying L to the $(2,1)$-computad $(\Sigma, R)^*$ given by (95).

By the discussion following Proposition 4.49 the group of identities among relations is abelian, with operation $*_1 = *_0$ inherited from the ω-groupoid structure of $L(\Sigma, R)^*$.

In [BH82] §5 the group of identities among relations is identified as π_2 of a space $X(2)$ constructed as a geometric realization of a presentation of a group by generators and relations. This space is constructed by taking a wedge of circles $X(1)$ indexed by the set of generators of the free group F and attaching 2-cells along boundaries determined by the relations generating the normal subgroup $N \trianglelefteq F$. In [Lod00] Loday shows that the group of identities among relations is isomorphic to the group $\pi_2(X(2))$. Loday also shows that there is an isomorphism with the group of Igusa pictures, which has as generators finite oriented planar graphs determined by generators and relations. These constructions give more tools for calculating the group of identities among relations, which we will now show has the group $\pi_2(BM)$ as a quotient.

Theorem 5.26. *Let M be a discrete monoid with coherent presentation (Σ, R, P). The second homotopy group $\pi_2(BM)$ of the classifying space of M is the quotient*

$$\pi_2(BM) = N(R)/N(P)$$

where $N(R)$ is the group of identities among relations for the presentation (Σ, R) and $N(P)$ is the subgroup of identities among relations determined by the set P.

Proof. Let M be a monoid with a coherent presentation (Σ, R, P). As described in Definition 5.18, the coherent presentation of M determines $(3,1)$-computad replacement of M in $\mathbf{Mon}(\omega\mathbf{Gpd})$ that can be extended to a full cofibrant replacement $X \xrightarrow{\sim} M$ in $\mathbf{Mon}(\omega\mathbf{Gpd})$. This means that

$$\tau^3(X) = (\Sigma, R, P)^*$$

Applying L to this cofibrant strict monoid gives a cofibrant strict group, which is equivalently an ω-groupoid with a single 0-cell under the isomorphism of Lemma 5.19. The automorphism group of 1-cells at the group identity $e \in LX$ is therefore isomorphic to the group of identities among relations $N(R)$ by Definition 5.25.

The fundamental group of LX is the quotient of the automorphism group $N(R)$ by the homotopy relation. The 2-cells of L are generated by the homotopy basis P, which has maps $s, t : P \to \Sigma^*(R)$ sending $p \in P$ to pairs $s(p), t(p) \in (\Sigma, R)^*_2$ of parallel 2-cells of $(\Sigma, R)^*$. These pairs give rise to identities among relations

$$(k^2 s(p) *_1 t(p)) *_0 k^1 d_1^+(t(p)) : 1_* \implies 1_*$$

so $\mathbb{L}L_1^{\mathbf{Mon}(\omega\mathbf{Gpd})}(M) = \pi_1(LX)$ is the quotient of the group of identities among relations by the normal subgroup $N(P) \trianglelefteq N(R)$ generated by these relations from the coherent presentation of M.

By Proposition 5.24 $\mathbb{L}L_1^{\mathbf{Mon}(\omega\mathbf{Gpd})}(M)$ is equal to the first derived functor of M for the group completion functor of Gray monoids. Now consider $X' \xrightarrow{\sim} M$ a cofibrant replacement for the discrete monoid M viewed as a discrete simplicial monoid in $\mathbf{sMon}$. This is also a cofibrant replacement of M in $P_2\mathbf{sMon}$. Applying $C_\omega^\otimes$ gives a Gray monoid ω-functor $C_\omega^\otimes(X) \to M$ since $C_\omega^\otimes(M) = M$ by Lemma 5.9. Consider the diagram of simplicial monoids

$$\begin{tikzcd}
M \arrow[d, equal] & X' \arrow[l, "\sim"'] \arrow[r, hook] \arrow[d, "\wr"] & LX' \arrow[d, "\wr"] \\
N_\omega C_\omega^\otimes M & N_\omega C_\omega^\otimes(X') \arrow[l] \arrow[r, hook] & N_\omega L C_\omega^\otimes(X')
\end{tikzcd}$$

where the indicated weak equivalences are P_2-weak equivalences by by Propositions 5.11 and5.16. Since N_ω creates weak equivalences for $P_2\mathbf{Gray}$ by the proof of Proposition 4.62 and the lower left horizontal map is a P_2-weak equivalence the Gray monoid ω-functor $C_\omega^\otimes(X') \to M$ is a P_2-weak equivalence, so it is a cofibrant replacement of M in $P_2\mathbf{Gray}$ and can be used to calculate the derived functor of L

$$\begin{aligned} \mathbb{L}L_1^{\mathbf{Gray}}(M) &= \pi_1 L C_\omega^{\otimes}(X') & \\ &= \pi_1 N_\omega L C_\omega^{\otimes}(X') & \text{Prop. 4.57} \\ &= \pi_1 L X' & \text{Prop. 5.16} \\ &= \mathbb{L}L_1^{\mathbf{sMon}}(M) & \end{aligned}$$

The last group is the first derived functor of group completion for simplicial monoids acting on M. Finally, by Proposition 3.84 there is an isomorphism

$$\mathbb{L}L_1^{\mathbf{sMon}}(M) \cong \pi_2(BM)$$

□

Recall that the Hopf formula ([Hop41]) gives the second homology $H_2(BG)$ of a group G with presentation by a free group F and a normal subgroup of relations $R \trianglelefteq F$ as the quotient

$$H_2(BG) = (R \cap [F,F])/[R,F]$$

of the subgroup of the commutator group $[F,F]$ of commutators among the relations R by the commutator group of R in F. This subgroup $[R,F]$ seems to represent the commutators always present in R that arise because R is a normal subgroup and so closed under conjugation. In particular, if $x \in F$ and $uv^{-1} \in R$ then

$$uv^{-1}(xuv^{-1}x^{-1}) = [uv^{-1}, x] \in [R,F]$$

The formula we give for $\pi_2(BM)$ seems analogous to this. We take the quotient of the group of identities among relations by the subgroup of identities among relations that were already present in P before we allowed inverses for 0-cells of the cofibrant strict monoid X replacement of M in $\mathbf{Mon}(\omega\mathbf{Gpd})$ determined by the coherent presentation (Σ, R, P). Hence $\pi_2(BM)$ seems to measure the identities among relations that are created for the monoid M by group completion.

Theorem 5.26 has been expressed in terms of two known constructions: the group of identities among relations and the set P from a coherent presentation of the monoid M. We can therefore apply methods of calculation already devised for these constructions. In particular, there is a description of a general procedure in [BS99] for presenting by generators and relations the group $N(R)$ of identities among relations. This procedure involves choice, so it does not give an algorithm for calculating this group in all cases. We will adapt this method to the case when a monoid M has trivial group completion and use it to give some example calculations using Theorem 5.26.

Let (Σ, R) be a monoid presentation for a monoid M such that $LM = e$ the trivial group. For example, any monoid M such that all elements are idempotent satisfies this condition. When L is applied to the $(2,1)$-computad $(\Sigma, R)^*$, therefore, for all $x \in \Sigma$ there exist 2-cells $\gamma_x \in L(\Sigma, R)^*_2$ such that $\gamma_x : x \implies 1_*$. We will denote by $\gamma_{x^{-1}}$ the 2-cell

$$k^1\gamma_x : k^1 x \implies 1_*$$

where we are abusing notation for inverses in $L(\Sigma, R)^*$ viewed as a $(\omega, 1)$-category with a single object or as a strict monoid in $\omega\mathbf{Gpd}$. We will further abuse notation by denoting

$$\gamma_{xy} = \gamma_x *_0 \gamma_y : x *_0 y \implies 1_*$$

so that $\gamma_u : u \implies 1_*$ is defined for all words $u \in L\Sigma^*$, the free group on the set Σ.

Recall the linearization functor $\lambda : \omega\mathbf{Cat} \to \mathbf{Ch}$ from Section 4.3. The right adjoint ν of this functor determines a unit ω-functor $X \to \nu\lambda(X)$ for all ω-categories X. Since $\nu\lambda(X)$ is a sub-ω-category of $\mu\lambda(X)$ there is a natural map for all $n \geq 1$ to the quotient by the subgroup $\mu\lambda(X)_{n-1}$ of cells in degree $n-1$ and below

$$X \to \mu\lambda(X)/\mu\lambda(K)_{n-1}$$

The n-cells component of this map is a set map from the set X_n of n-cells of X to the abelian group generated by classes $[x]$ of n-cells and subject to the relations $[x *_i y] = [x] + [y]$. When X is a computad with set S_n of n-indeterminates this is a map

$$\lambda : \tau^{n-1}(X)[S_n] \to \mathbb{Z}[S_n]$$

to the free abelian group generated by the set S_n, which we will denote by λ as above. This map plays a role in [BS99] determining a presentation of $N(R)$, the group of identities among relations in terms of generators and relations of its own. We will reproduce their description of $N(R)$ in the case that LM is the trivial group in the next proposition.

Proposition 5.27. *Let M be a monoid such that the group completion LM is the trivial group. Let (Σ, R) be a presentation of the monoid M There is a surjective abelian group homomorphism*

$$\rho : \mathbb{Z}[R] \to N(R)$$

that sends $r \in R$ to the 2-cell

$$\rho_r = k^2(\gamma_{s(r)}) *_1 r *_1 \gamma_{t(r)} : 1_* \implies 1_* \in L(\Sigma, R)^*_2$$

where $s(r), t(r) \in (\Sigma, R)^*_1$ are the elements of Σ^* joined by the generating 2-cell labelled by the relation $r \in R$. The kernel of this map is generated by

$$\lambda(\gamma_x) \in \mathbb{Z}[R]$$

for all $x \in \Sigma$.

Proof. This is simply a translation of Theorems 1.1 and 1.2 of [BS99] in the case of a presentation of the trivial group $G = e$ from the context of crossed modules to that of ω-groupoids. The choice of 2-cells γ_x for all elements of the alphabet $x \in \Sigma$ described above corresponds to the choice of a map h_1 of [BS99] paragraph 2.10. The abelian group homomorphism ρ corresponds to δ_3 in [BS99] paragraph 2.11, which is equal to $\tilde{\delta}_3$ defined in [BS99] paragraph 2.12 when the group being presented is trivial. The map λ defined above corresponds to h_2 defined in [BS99] paragraph 2.12. Hence ρ is surjective by Theorem 1.1 of [BS99] and

$$\rho(\lambda(\alpha)) = k^2(\gamma_{d_1^- \alpha}) *_1 \alpha *_1 \gamma_{d_1^+ \alpha}$$

for all $\alpha \in L(\Sigma, R)^*_2$. We will now show that the kernel of ρ is generated by the terms $\lambda(\gamma_x)$ for all $x \in \Sigma$. This is an extension of the results of [BS99] that is possible because we have restricted to the case of a presentation of the trivial group.

Let $w \in \mathrm{Ker}(\rho)$. Then $w = \sum_{r\in R} a_r r \in \mathbb{Z}[R]$ with $a_r \in \mathbb{Z}$ such that only finitely many entries are not 0. If $\rho(w) = 0$ then

$$0 = \lambda\rho(w) = \lambda\rho\left(\sum_{r\in R} a_r r\right) = \sum_{r\in R} a_r \lambda\rho_r$$

so

$$w = \sum_{r\in R} a_r(r - \lambda\rho_r)$$

and hence $r - \lambda\rho_r$ generates the kernel of ρ. Now

$$\begin{aligned}\lambda(\rho_r) &= \lambda(k^2(\gamma_{s(r)}) *_1 r *_1 \gamma_{t(r)}) \\ &= -\lambda(\gamma_{s(r)}) + r + \lambda(\gamma_{t(r)})\end{aligned}$$

so

$$r - \lambda\rho_r = \lambda(\gamma_{t(r)}) - \lambda(\gamma_{s(r)})$$

By the definition of γ_u for all $u \in L\Sigma^*$ $\lambda(\gamma_u)$ is a sum of terms $\pm\lambda(\gamma_x)$ for $x \in \Sigma$. Now

$$\begin{aligned}\rho(\lambda(\gamma_x)) &= k^2(\gamma_{d_1^-\gamma_x}) *_1 \gamma_x *_1 \gamma_{d_1^+\gamma_x} \\ &= k^2(\gamma_x) *_1 \gamma_x *_1 \gamma_{1_*} \\ &= 1_*\end{aligned}$$

so all terms $\lambda(\gamma_x)$ belong to the kernel of ρ for all $x \in \Sigma$. Hence these terms generate the kernel of ρ. □

There are also results describing the construction of a homotopy basis for a presentation (Σ, R) of a monoid M. There has been much effort put into the case when the presentation of M has certain additional nice properties for rewriting computations in the quotient $\Sigma^*/\sim_R^* \cong M$. These are given in terms of the 2-computad $[\Sigma, R]^*$, where the 2-cells are not invertible, unlike in the $(2, 1)$-computad $(\Sigma, R)^*$ defined in (95).

For a presentation (Σ, R) a **rewriting step** is a 2-cell of the 2-computad $[\Sigma, R]^*$ of the form

$$\begin{tikzcd} * \arrow[r, "u"] & * \arrow[r, bend left, "s(r)", ""{name=U, below}] \arrow[r, bend right, "t(r)"', ""{name=D, above}] & * \arrow[r, "v"] & * \arrow[Rightarrow, from=U, to=D, "r"] \end{tikzcd}$$

for $u, v \in \Sigma^*$ and $r \in R$. We say a presentation is **Noetherian** when there does not exist an infinite sequence of composable rewriting steps in the 2-computad $[\Sigma, R]^*$. A **branching** is a pair of 2-cells of the 2-computad $[\Sigma, R]^*$ that have the same source. There are three kinds of branchings possible:

1. **aspherical** branchings are those where the two 2-cells are the same

$$\begin{tikzcd} & d_1^+\gamma \\ d_1^-\gamma \arrow[ur, Rightarrow, bend left, "\gamma"] \arrow[dr, Rightarrow, bend right, "\gamma"'] & \\ & d_1^+\gamma \end{tikzcd}$$

2. **Peiffer** branchings

$$\begin{tikzcd}
 & d_1^+\gamma *_0 d_1^-\delta \\
d_1^-\gamma *_0 d_1^-\delta \arrow[ur, Rightarrow, "\gamma *_0 d_1^-\delta"] \arrow[dr, Rightarrow, "d_1^-\gamma *_0 \delta"'] & \\
 & d_1^-\gamma *_0 d_1^+\delta
\end{tikzcd}$$

3. and finally **overlapping** branchings are all other branchings

A branching is **critical** when it is overlapping and minimal with respect to the partial order on branchings determined by

$$(\gamma, \delta) \leq (u *_0 \gamma *_0 v, u *_0 \delta *_0 v)$$

for $u, v \in \Sigma^*$. A branching is **local** when the two 2-cells are both rewriting steps of $[\Sigma, R]^*$. A branching (γ, δ) is **confluent** when there exist 2-cells $\sigma, \rho \in [\Sigma, R]_2^*$ with common target completing the diagram

$$\begin{tikzcd}
 & d_1^+\gamma \arrow[dr, Rightarrow, "\sigma"] & \\
d_1^-\gamma \arrow[ur, Rightarrow, "\gamma"] \arrow[dr, Rightarrow, "\delta"'] & & d_1^+\sigma \\
 & d_1^+\delta \arrow[ur, Rightarrow, "\rho"'] &
\end{tikzcd} \tag{101}$$

A presentation is **convergent** when it is Noetherian and all branchings are confluent. If a presentation is Noetherian then to show it is convergent it is sufficient by Newman's lemma to check that all local branchings are confluent. A **family of generating confluences** of a convergent presentation (Σ, R) is a set P together with set maps $s, t : P \to (\Sigma, R)_2^*$ such that for all critical branchings (γ, δ) of $[\Sigma, R]^*$ there exists $p \in P$ and a confluence $\sigma, \rho \in [\Sigma, R]_2^*$ for the branching (γ, δ) as in (101) such that

$$s(p) = \gamma *_1 \sigma \qquad t(p) = \delta *_1 \rho$$

In other words, P consists of indeterminates for 3-cells that fill in parallel 2-cells arising from a confluence of each critical branching. Squier proved in [SOK94] that such a set of generating confluences determines a homotopy basis for $(\Sigma, R)^*$.

Theorem 5.28 ([SOK94] Theorem 5.2)**.** *Let (Σ, R) be a convergent presentation of a monoid M. A family of generating confluences (P, s, t) is a homotopy basis for the $(2,1)$-computad $(\Sigma, R)^*$.*

Hence (Σ, R, P) is a coherent presentation when (Σ, R) is a convergent presentation and P is a family of generating confluences. Together with the tools for calculating the group of identities among relations discussed above, this allows for some calculations of $\pi_2(BM)$. We present two example calculations from cases where π_2 is already known. Of course, since in both these examples $\pi_1(BM)$ is trivial the Hurewicz theorem implies that $\pi_2(BM) = H_2(BM)$ and the homology group can be calculated from the abelianization of a coherent presentation of M by [LM09] Corollary 4 of §3.4. However, in these cases we show that $\pi_2(BM)$ is also obtained by the formula of Theorem 5.26.

Example 5.29. Consider the monoid I generated by a single element $\Sigma_I = \{a\}$ and with the unique relation

$$r : aa \to a$$

Hence I has a unique non-identity element that is idempotent and so $LI = e$. Proposition 8 of [Rab05] shows that $BI \simeq *$, since the element a determines a natural transformation between the identity monoid map on I viewed as a functor between categories with a single object and the constant functor at 1_*. Hence we should find that $\pi_2(BI) = 0$.

We choose

$$\gamma_a = r *_0 k^1 a : a \implies 1_* \in L(\{a\}, \{r\})_2^*$$

as the 2-cell of $L(\{a\}, \{r\})^*$ connecting $a \in \{a\}^*$ to the identity 1_* on the unique 0-cell $*$. As described in Proposition 5.27 the group of identities among relations is isomorphic to the quotient of $\mathbb{Z}[r]$ by $\lambda(\gamma_a) = r$ so it is the trivial group. Hence $\pi_2(BI)$ must be trivial as expected.

Example 5.30. Consider the monoid Π described in [Fie02] which is generated by the set $\Sigma_\Pi = \{x_{ij} \mid i, j \in \{1, 2\}\}$ and has the relations

$$r_{ij,kl} : x_{ij}x_{kl} \to x_{il}$$

for all $i, j, k, l \in \{1, 2\}$. This monoid has trivial group completion as $x_{ij}x_{ij} = x_{ij}$ for $i, j \in \{1, 2\}$ so all elements are idempotent. In [Fie02] it is shown that the classifying space $B\Pi$ has the homotopy type of the 2-sphere S^2. Hence in particular we expect that $\pi_2(B\Pi) = \mathbb{Z}$.

We define 2-cells

$$\gamma_{ij} = r_{ij,ij} *_0 k^1 x_{ij} : x_{ij} \implies 1_*$$

for all $i, j \in \{1, 2\}$. Hence by Proposition 5.27 since $L\Pi$ is the trivial group, the group of identities among relations $N(R)$ is isomorphic to the quotient of the free abelian group generated by $r_{ij,kl}$ for $i, j, k, l \in \{1, 2\}$ by the subgroup generated by

$$\lambda(\gamma_{ij}) = r_{ij,ij}$$

So $N(R)$ is the free abelian group generated by $r_{ij,kl}$ such that $i \neq k$ or $j \neq l$.

The critical branchings for this presentation are the branchings

$$\begin{tikzcd}
& x_{il}x_{mn} \arrow[dr, Rightarrow, "r_{il,mn}"] & \\
x_{ij}x_{kl}x_{mn} \arrow[ur, Rightarrow, "r_{ij,kl}*_0x_{mn}"] \arrow[dr, Rightarrow, "x_{ij}*_0r_{kl,mn}"'] & & x_{in} \\
& x_{ij}x_{kn} \arrow[ur, Rightarrow, "r_{ij,kn}"'] &
\end{tikzcd}$$

which are confluent with the 2-cells as indicated in the diagram. By Theorem 5.28 therefore the subgroup $N(P) \trianglelefteq N(R)$ is generated by

$$r_{ij,kn} + r_{kl,mn} - r_{ij,kl} - r_{il,mn} \in N(R)$$

for all $i, j, k, l, m, n \in \{1, 2\}$. There is a large amount of redundancy amongst these generators, however. If $i = k$ then the generator is

$$r_{ij,in} + r_{il,mn} - r_{ij,il} - r_{il,mn} = r_{ij,in} - r_{ij,il}$$

which has no dependence on m so there are two copies of this generator from the generating confluences. Similarly, if $l = n$ then the generator is

$$r_{ij,kl} + r_{kl,ml} - r_{ij,kl} - r_{il,ml} = r_{kl,ml} - r_{il,ml}$$

which has no dependence on j so again there are two copies of this generator. Now we define

$$N(P)_0 = \langle r_{ij,kn} + r_{kl,mn} - r_{ij,kl} - r_{il,mn} \in N(R) \mid i = k \text{ or } l = n\rangle \trianglelefteq N(P) \trianglelefteq N(R)$$

and since $r_{ij,ij} = 0$

$$N(P)_0 = \langle r_{ij,kl} \mid i = k \text{ or } j = l\rangle$$

The remaining generators of $N(P)$ from the generating confluences not present in $N(P)_0$ have $i \neq k$ and $l \neq n$, so there are 16 of these.

Now we define

$$\begin{aligned} N(P)_1 &= N(P)_0 \cup \langle r_{ij,kn} + r_{kl,mn} - r_{ij,kl} - r_{il,mn} \in N(R) \mid i = m \text{ and } j = n\rangle \\ &= \langle r_{ij,kl} \mid i = k \text{ or } j = l\rangle \cup \langle r_{ij,kl} - j_{kl,ij} \mid i, j, k, l \in \{1, 2\}\rangle \end{aligned}$$

There are now only 12 generators of $N(P)$ not accounted for in $N(P)_1$: those with $i \neq k$, $l \neq n$, and $i \neq m$ or $j \neq n$. These are listed below, along with their consequences when added to the subgroup $N(P)_1$ of $N(R)$, which contains $r_{ij,kl}$ for $i = k$ and $j = l$ and $r_{ij,kl} - r_{kl,ij}$ for all $i, j, k, l \in \{1, 2\}$. These consequences are labelled by symbols $(*)$, $(\dagger)$, and $(\ddagger)$ to indicate repetitions.

$$\begin{array}{rcrc}
r_{11,22} + r_{21,12} - r_{11,21} - r_{11,12} & \Longrightarrow & r_{11,22} + r_{21,12} & (*) \\
r_{11,22} + r_{21,22} - r_{11,21} - r_{11,22} & \Longrightarrow & r_{11,22} - r_{11,22} & 0 \\
r_{11,21} + r_{22,21} - r_{11,22} - r_{12,21} & \Longrightarrow & -r_{11,22} - r_{12,21} & -(\dagger) \\
r_{12,22} + r_{21,22} - r_{12,21} - r_{11,22} & \Longrightarrow & -r_{12,21} - r_{11,22} & -(\dagger) \\
r_{12,21} + r_{22,11} - r_{12,22} - r_{12,11} & \Longrightarrow & r_{12,21} + r_{22,11} & (\ddagger) \\
r_{12,21} + r_{22,21} - r_{12,22} - r_{12,21} & \Longrightarrow & r_{12,21} - r_{12,21} & 0 \\
r_{21,12} + r_{11,12} - r_{21,11} - r_{21,12} & \Longrightarrow & r_{21,12} - r_{21,12} & 0 \\
r_{21,12} + r_{11,22} - r_{21,11} - r_{21,22} & \Longrightarrow & r_{21,12} + r_{11,22} & (*) \\
r_{21,11} + r_{12,11} - r_{21,12} - r_{22,11} & \Longrightarrow & -r_{21,12} - r_{22,11} & -(*) \\
r_{22,12} + r_{11,12} - r_{22,11} - r_{21,12} & \Longrightarrow & -r_{22,11} - r_{21,12} & -(*) \\
r_{22,11} + r_{12,11} - r_{22,12} - r_{22,11} & \Longrightarrow & r_{22,11} - r_{22,11} & 0 \\
r_{22,11} + r_{12,21} - r_{22,12} - r_{22,21} & \Longrightarrow & r_{22,11} + r_{12,21} & (\ddagger)
\end{array}$$

Observe that since $r_{22,11} - r_{11,22} \in N(P)_1$ and the relations $(\dagger)$ and $(\ddagger)$ satisfy

$$(\ddagger) - (\dagger) = (r_{22,11} + r_{12,21}) - (r_{11,22} + r_{12,21}) = r_{22,11} - r_{11,22}$$

it is redundant to add $(\dagger)$ and $(\ddagger)$ to $N(P)_1$. Hence

$$N(P) = \langle r_{ij,kl} \mid i = k \text{ or } j = l\rangle \cup \langle r_{11,22} - r_{22,11}, r_{11,22} + r_{21,12}, r_{11,22} + r_{12,21}\rangle$$

Hence it is clear that $N(R)/N(P)$ is the free group on a single generator as expected.

Appendices

A ω-Functors and Identities

In this appendix we will study ω-functors of the form

$$f : \nu K \to X \tag{102}$$

for K a strong Steiner complex (Definition 4.25). Recall from Theorem 4.52, which comes from [Ste04], that when K is a strong Steiner complex the ω-category νK is a computad that is freely generated by its atoms $\langle k \rangle \in \nu K_n$ (Definition 4.30). Hence ω-functors f like (102) are uniquely determined by the images of the atoms $f(\langle k \rangle) \in X_n$. In particular we will prove that the images of cells of νK under f depend only on the cells modulo $(\mu K)_n$.

To prove the results we want in this appendix we need the notion from [Ste04] of decomposition index for cells in the strict ω-category arising from an ADC.

Definition A.1. *Let K be an ADC with a basis. Let $x \in \nu K_m$ be an m-cell. The* ***decomposition index*** *of x is the smallest integer $-1 \leq r \leq m$ such that $x \equiv 0 \mod (\mu K)_{r+1}$ or x is congruent to an atom mod $(\mu K)_{r+1}$.*

The result we want to prove is the following.

Lemma A.2. *Let K be a strong Steiner complex and $n \geq 1$. Let $f : \nu K \to X$ be an ω-functor and $x \in X_0$ be a 0-cell of X. If for all all m-atoms $\langle a \rangle \in \nu K_m$ for $1 \leq m \leq n$*

$$f(\langle a \rangle) = 1_x$$

then $f(k) = 1_x$ for all m-cells $k \in \nu K_m$ with $1 \leq m \leq n$. Furthermore, if $k \in \nu K_r$ is an r-cell with

$$k \equiv \langle a_1 \rangle + \cdots + \langle a_t \rangle \mod (\mu K)_n$$

for atoms $\langle a_i \rangle$ of νK then

$$f(k) = f(\langle a_1 \rangle) *_0 \cdots *_0 f(\langle a_t \rangle)$$

Proof. We will prove this by induction on $n \geq 1$. Let $n = 1$ and suppose $f : \nu K \to X$ is an ω-functor that sends all 1-atoms of νK to the 1-cell $1_x \in X_1$ for a chosen $x \in X_0$. For a 1-cell $k \in K_1$ there exists a decomposition $k = \langle a_1 \rangle *_0 \cdots *_0 \langle a_t \rangle$ for some 1-atoms $\langle a_i \rangle$ of νK since νK is a computad generated by its atoms by Theorem 4.52. Hence all 1-cells are sent to $1_x \in X_1$ by f so the image of f is contained in $R^1_x(X)$, which has the structure of an ω-groupoid in abelian groups by Proposition 4.49. In particular, for all $*_1$-composable r-cells $k, k' \in \nu K_r$

$$f(k *_1 k') = f(k) *_1 f(k') = f(k') *_1 f(k) = f(k) *_0 f(k') = f(k') *_0 f(k)$$

Consider an r-cell $k \in \nu K_r$ such that

$$k \equiv \langle a_1 \rangle + \cdots + \langle a_t \rangle \mod (\mu K)_1 \tag{103}$$

for atoms $\langle a_i \rangle$ of νK. Since K is a strong Steiner complex there exists an order of these atoms such that $\langle a_i \rangle_1^+ \wedge \langle a_j \rangle_1^- = 0$ for $i > j$. By the previous discussion we may assume that the sum was originally given in this order, as in the image of f in $R_x^1(X)$ we are free to rearrange terms of any $*_1$-composite of cells. Proposition 5.1 of [Ste04] therefore gives that

$$k = k_1 *_1 k_2 *_1 \cdots *_1 k_t$$

with $k_1 \equiv \langle a_i \rangle \mod (\mu K)_1$. Now since $k_i \equiv \langle a_i \rangle \mod (\mu K)_1$ there exist 1-atoms a_j^i for $1 \leq j \leq l$ such that

$$k_i \equiv \sum_{j=1}^{l} \langle a_j^i \rangle + \langle a_i \rangle \mod (\mu K)_0$$

Since K is a strong Steiner complex there exists an order on these atoms such that

$$k_i = \alpha *_0 \langle a_i \rangle *_0 \alpha'$$

with $\alpha, \alpha' \in \nu K_1$ composites of the 1-atoms $\langle a_j^i \rangle$, where we are using Proposition 5.5 of [Ste04], which says that cells that are congruent to an atom modulo $(\mu K)_0$ are equal to that atom. The ω-functor sends all 1-cells to 1_x so

$$f(k_i) = f(\langle a_i \rangle)$$

and so

$$f(k) = f(\langle a_1 \rangle) *_1 f(\langle a_2 \rangle) *_1 \cdots *_1 f(\langle a_t \rangle) = f(\langle a_1 \rangle) *_0 f(\langle a_2 \rangle) *_0 \cdots *_0 f(\langle a_t \rangle)$$

since the image of f lies in $R_x^1(X)$.

Now let $n \geq 1$ and suppose that the result holds for all ω-functors from νK to X sending m-atoms of νK to identities of a chosen 0-cell of X if $m \leq n$. Let $f : \nu K \to X$ be an ω-functor such that $f(\langle a \rangle) = 1_x$ for some $x \in X_0$ and all m-atoms $\langle a \rangle \in \nu K_m$ for $1 \leq m \leq n+1$. Let k be an r-cell of νK with

$$k \equiv \langle a_1 \rangle + \cdots + \langle a_t \rangle \mod (\mu K)_{n+1}$$

for atoms a_i of K. Since K is a strong Steiner complex we may rearrange the order of terms in the sum so that $\langle a_i \rangle_{n+1}^+ \wedge \langle a_j \rangle_{n+1}^- = 0$ for $i > j$. We will denote the new order of these terms in the sum by

$$k \equiv \langle \bar{a}_1 \rangle + \cdots + \langle \bar{a}_t \rangle \mod (\mu K)_{n+1}$$

Proposition 5.1 of [Ste04] gives a decomposition in νK

$$k = k_1 *_{n+1} \cdots *_{n+1} k_t$$

with $k_i \equiv \langle \bar{a}_i \rangle \mod (\nu K)_{n+1}$. Hence by Proposition 5.2 (ii) of [Ste04] for $1 \leq i \leq t$ there is a congruence

$$k_i \equiv \langle \bar{a}_1^i \rangle + \cdots + \langle \bar{a}_u^i \rangle \mod (\mu K)_n \tag{104}$$

for atoms $\bar{a}_j^i$ of K with $1 \leq j \leq u$ where $\bar{a}_l^i = \bar{a}_i$ for some $1 \leq l \leq u$ and all other $\bar{a}_j^i$ for $j \neq l$ are $n+1$-atoms of νK. Since K is a strong Steiner complex we can choose the order of this sum to satisfy the condition $\langle \bar{a}_j^i \rangle_n^+ \wedge \langle \bar{a}_{j'}^i \rangle_n^- = 0$ for $j > j'$

The ω-functor f sends all m-atoms for $1 \leq m \leq n$ to 1_x, so we can apply the induction hypothesis to the congruence (104) for k_i to obtain

$$f(k_i) = f(\langle a^i_1 \rangle) *_0 \cdots *_0 f(\langle a^i_u \rangle)$$

but since $\bar{a}^i_j$ are $n+1$-atoms for $j \neq l$, $f(\langle \bar{a}^i_j \rangle) = 1_x$ so

$$f(k_i) = f(\langle \bar{a}^i_l \rangle) = f(\langle \bar{a}_i \rangle)$$

Hence

$$f(k) = f(k_1) *_{n+1} \cdots *_{n+1} f(k_t) = f(\langle \bar{a}_1 \rangle) *_{n+1} \cdots *_{n+1} f(\langle \bar{a}_t \rangle)$$

Any $n+1$-cell of νK is congruent modulo $(\mu K)_n$ to a sum of $n+1$-atoms by Proposition 5.2 (i) of [Ste04]. Since f sends all m-atoms of νK for $1 \leq m \leq n$ to 1_x the induction hypothesis gives that f sends any $n+1$-cell of νK to the $*_0$-composite of the $n+1$-atoms in the congruence. These $n+1$-atoms are all sent to 1_x by f so f sends all $n+1$-cells of νK to 1_x. The induction hypothesis gives that f sends all m-cells of νK to 1_x for $1 \leq m \leq n$ so this completes the proof of the first part of the result.

Now $d^\varepsilon_{n+1}\langle \bar{a}_i \rangle$ is an $n+1$-cell of νK so by the previous discussion

$$f(d^\varepsilon_{n+1}\langle \bar{a}_i \rangle) = 1_x$$

Hence for $1 \leq i < t$

$$\begin{aligned} 1_{d^-_{n+1} f(\langle \bar{a}_i \rangle)} &= 1_x = 1_{d^+_0 f(\langle \bar{a}_i \rangle)} \\ 1_{d^+_{n+1} f(\langle \bar{a}_i \rangle)} &= 1_x = 1_{d^-_0 f(\langle \bar{a}_i \rangle)} \end{aligned}$$

and so

$$\begin{array}{rcl} f(\langle \bar{a}_i \rangle) *_{n+1} f(\langle \bar{a}_{i+1} \rangle) & = & \left(f(\langle \bar{a}_i \rangle) *_0 1_{d^+_0 f(\langle \bar{a}_i \rangle)} \right) *_{n+1} \left(1_{d^-_0 f(\langle \bar{a}_{i+1} \rangle)} *_{n+1} f(\langle \bar{a}_{i+1} \rangle) \right) \\ & = & \left(f(\langle \bar{a}_i \rangle) *_0 1_{d^-_{n+1} f(\langle \bar{a}_i \rangle)} \right) *_{n+1} \left(1_{d^+_{n+1} f(\langle \bar{a}_{i+1} \rangle)} *_0 f(\langle \bar{a}_{i+1} \rangle) \right) \\ & = & \left(f(\langle \bar{a}_i \rangle) *_{n+1} 1_{d^+_{n+1} f(\langle \bar{a}_i \rangle)} \right) *_0 \left(1_{d^-_{n+1} f(\langle a_{i+1} \rangle)} *_{n+1} f(\langle \bar{a}_{i+1} \rangle) \right) \\ & = & f(\langle \bar{a}_i \rangle) *_0 f(\langle \bar{a}_{i+1} \rangle) \end{array}$$

Hence

$$f(k) = f(\langle \bar{a}_1 \rangle) *_{n+1} \cdots *_{n+1} f(\langle \bar{a}_t \rangle) = f(\langle \bar{a}_1 \rangle) *_0 \cdots *_0 f(\langle \bar{a}_t \rangle) = f(\langle a_1 \rangle) *_0 \cdots *_0 f(\langle a_t \rangle)$$

since the image of f lies in $R^1_x(X)$ where $*_0$ is a commutative binary operation on cells. □

B The Structure of Oriented Simplices

Recall Street's oriented simplices, which are the ω-categories $\nu\mathbb{Z}[\Delta^n]$ for $n \geq 0$. In this appendix we will use the characterization from [Ste04] of ω-categories in the image of the functor ν from **ADC** to describe the cells of $\nu\mathbb{Z}[\Delta^n]$. We will show that the chain complex $\mathbb{Z}[\Delta^n]$ is a strong Steiner complex and so by Theorem 4.52 $\nu\mathbb{Z}[\Delta^n]$ is freely generated as a computad by its atoms. We will use this fact and the results of [Ste04] to describe the source and target of the atoms $\langle 1_{[n]} \rangle$ corresponding to the unique non-degenerate n-simplex $1_{[n]} \in \Delta^n_n$.

The atoms of $\nu\mathbb{Z}[\Delta^n]$ are of the form $\langle\theta\rangle \in \nu\mathbb{Z}[\Delta^n]_m$ for $\theta : [m] \to [n]$ an injective map in Δ. Injective maps of Δ can be uniquely written as composites of coface maps δ^i with increasing order on the omitted integers i. The next definition gives notation for the sets of increasing chains in $[n]$ that will appear in our descriptions of the atoms of $\nu\mathbb{Z}[\Delta^n]$.

Definition B.1. *Define for $n \geq 0$ and $1 \leq m \leq n$ the set of sequences*

$$J_{n,m} = \{\vec{j} = j_1 < j_2 < \cdots j_m \mid j_l \in [n] \text{ for all } 1 \leq l \leq m\}$$

Define the subsets of $J_{n,m}$

$$J^+_{n,m} = \{\vec{j} \in J_{n,m} \mid j_l - l \equiv 1 \pmod 2 \text{ for all } 1 \leq l \leq m\}$$

$$J^-_{n,m} = \{\vec{j} \in J_{n,m} \mid j_l - l \equiv 0 \pmod 2 \text{ for all } 1 \leq l \leq m\}$$

and for all $0 \leq t \leq m$ and all $0 \leq p \leq n$

Proposition B.2. *Let $\theta : [m] \to [n]$ be an injective map in Δ corresponding to a non-degenerate m-simplex of Δ^n. Then for all $0 \leq i < m$*

$$\langle\theta\rangle^\xi_i = \sum_{\vec{j} \in J^\xi_{n,m-i}} [d_{j_1} d_{j_2} \cdots d_{j_{m-i}} \theta]$$

Proof. We will prove this inductively by applying Definition 4.30. For the base case, $i = m - 1$ we have

$$\langle\theta\rangle^-_{m-1} = \partial^+[\theta] = \sum_{i=0}^{\lfloor m-1/2 \rfloor} [d_{2i+1}\theta] \qquad \langle\theta\rangle^+_{m-1} = \partial^-[\theta] = \sum_{i=0}^{\lfloor m/2 \rfloor} [d_{2i}\theta]$$

This proves the base case. Now let $1 \leq i \leq m - 1$ and suppose

$$\langle\theta\rangle^\xi_i = \sum_{\vec{j} \in J^\xi_{n,m-i}} [d_{j_1} d_{j_2} \cdots d_{j_{m-i}} \theta]$$

Then

$$\partial\langle\theta\rangle^\xi_i = \sum_{t=0}^{i} \sum_{\vec{j} \in J^\xi_{n,m-i}} (-1)^t [d_t d_{j_1} d_{j_2} \cdots d_{j_{m-i}} \theta] \tag{105}$$

We will make use of the fact that the coface maps generate the injective maps in Δ with the cosimplicial identities. Hence, a non-degenerate i-simplex of Δ^n can be written uniquely as $d_{l_1} \cdots d_{l_{n-i}} \iota$ for $l_1 < l_2 < \cdots < l_{n-i}$ and ι the unique non-degenerate n-simplex of Δ^n. Returning to our specific case of the sum (105), suppose $t > j_1$ and define

$$p = \max\{1 \leq l \leq m - i \mid j_l \leq t + l - 1\}$$

Then $j_p \leq t + l - 1$ and if $p < m - i$ then $p + t < j_{p+1}$ Furthermore

$$d_t d_{j_1} d_{j_2} \cdots d_{j_{m-i}} = \begin{cases} d_{j_1} d_{j_2} \cdots d_{j_p} d_{t+p} d_{j_{p+1}} \cdots d_{j_{m-i}} & p < m - i \\ d_{j_1} d_{j_2} \cdots d_{j_{m-i}} d_{p+m-i} & p = m - i \end{cases}$$

This allows us to put all the terms from the sum not already in their unique form into that form. We will show that these terms with $t > j_1$ all cancel in the sum (105).

Suppose $t \equiv 0 \pmod 2$. Then $t + p \equiv p \pmod 2$ and so

$$(j_1 < j_2 < \cdots < j_{p-1} < t + p < j_{p+1} < \cdots < j_{m-i}) \in J_i^-$$

Now

$$d_t d_{j_1} d_{j_2} \cdots d_{j_{m-i}} = \begin{cases} d_{j_p - p+1} d_{j_1} d_{j_2} \cdots d_{j_{p-1}} d_{t+p} d_{j_{p+1}} \cdots d_{j_{m-i}} & p < m - i \\ d_{j_{m-i} - m+i+1} d_{j_1} d_{j_2} \cdots d_{j_{m-i-1}} d_{t+m-i} & p = m - i \end{cases}$$

Hence all terms $[d_{2k} d_{j_1} d_{j_2} \cdots d_{j_{m-i}} \theta]$ for $\vec{j} \in J^-_{n,m-i}$ and $2k \geq j_1$ in the sum (105) defining $\partial \langle \theta \rangle_i^-$ cancel.

Now suppose $t \equiv 1 \pmod 2$. Then $t + p \equiv p + 1 \pmod 2$ and so

$$(j_1 < j_2 < \cdots < j_p < t + p < j_{p+2} < \cdots < j_{m-i}) \in J^-_{n,m-i}$$

In this case it is not possible that $p = m - i$ since $t \geq 1$. Now

$$d_t d_{j_1} d_{j_2} \cdots d_{j_{m-i}} = d_{j_{p+1} - p-1} d_{j_1} d_{j_2} \cdots d_{j_p} d_{t+p} d_{j_{p+2}} \cdots d_{j_{m-i}}$$

and since $j_{p+1} \equiv p+1 \pmod 2$, $j_{p+1} - p + 1 \equiv 0 \pmod 2$. Hence all terms $-[d_{2k+1} d_{j_1} d_{j_2} \cdots d_{j_{m-i}} \theta]$ for $\vec{j} \in J^-_{n,m-i}$ and $2k + 1 \geq j_1$ in the sum (105) defining $\partial \langle \theta \rangle_i^-$ cancel.

The only remaining terms in the sum are those for which $t < j_1$ and there can be no cancellation between these as they are already in the unique form specified above. Hence

$$\langle \theta \rangle^-_{i-1} = \partial^- \langle \theta \rangle^\xi_m = \sum_{\vec{j} \in J^-_{n,m-i+1}} [d_{j_1} d_{j_2} \cdots d_{j_{m-i+1}} \theta]$$

The same argument gives the result for $\langle \theta \rangle^+_{i-1}$. □

The basis described above for the ADC $\mathbb{Z}[\Delta^n]$ satisfies the conditions of Definition 4.25, making $\mathbb{Z}[\Delta^n]$ a strong Steiner complex. For any injective map $\theta : [m] \to [n]$ by Proposition B.2

$$\langle \theta \rangle^-_0 = [\theta(0)] \qquad \langle \theta \rangle^+_0 = [\theta(m)]$$

so this basis is unital. The basis of $\mathbb{Z}[\Delta^n]$ described above is strongly loop-free as there exists a total order on the set of all injective maps in Δ with codomain $[n]$. We define this order recursively as in Example 3.8 of [Ste04].

Proposition B.3. *Let $n \geq 0$. There exists a total order on injective maps with codomain $[n]$ in Δ such that $\theta : [m] \to [n] <_N \varphi : [p] \to [n]$ if and only if one of the following conditions hold*

1. $\theta(0) < \varphi(0)$ *in* $[n]$
2. $\theta(0) = \varphi(0)$ *and* $m = 0$ *and* $p > 0$
3. $\theta(0) = \varphi(0)$ *and* $m, p > 0$ *and* $\theta \circ d^0 >_N \varphi \circ d^0$

where $d^0 : [m-1] \to [m]$ and $d^0 : [p-1] \to [p]$ are the injective maps in Δ that omit 0.

Proof. We will show this relation is anti-symmetric by induction on $\min\{m,p\}$. Suppose one of m or p is 0 and $\theta \leq_N \varphi$ and $\theta \geq_N \varphi$. If $\theta <_N \varphi$ because $\theta(0) < \varphi(0)$ then it is not possible that $\theta \geq_N \varphi$ as $\theta \neq \varphi$ and none of the conditions for the strict relation allow for $\theta >_N \varphi$. Hence $\theta(0) = \varphi(0)$. If $\theta <_N \varphi$ then it must be the case that $p > 0$. But in this case it is clearly not possible for $\theta \geq_n \varphi$. Hence if $\theta \leq_N \varphi$ and $\theta \geq_N \varphi$ then $\theta = \varphi$. Now let $l \geq 0$ and suppose that $\theta \leq_N \varphi$ and $\theta \geq_N \varphi$ implies $\theta = \varphi$ when $\min\{m,p\} \leq l$.

Let $\theta : [m] \to [n]$ and $\varphi : [p] \to [n]$ be injective order-preserving maps and suppose that $\min\{m,p\} = l+1$ and $\theta \leq_n \varphi$ and $\theta \geq_n \varphi$. If $\theta(0) < \varphi(0)$ then it is not possible that $\theta \geq_N \varphi$ as none of the conditions allow this possibility. Hence $\theta(0) = \varphi(0)$. Since $m, p > l \geq 0$, it must be the case that $\theta \circ d^0 \leq_N \varphi \circ d^0$ and $\theta \circ d^0 \geq_N \varphi \circ d^0$. But by the induction hypothesis this implies $\theta \circ d^0 = \varphi \circ d^0$, so $\theta = \varphi$ since $\theta(0) = \varphi(0)$ as well.

We will show transitivity by induction on $\min\{m,p,q\}$ for $\theta : [m] \to [n]$, $\varphi : [p] \to [n]$, and $\psi : [q] \to [n]$ injective maps in Δ such that $\theta <_N \varphi <_N \psi$. Suppose one of m, p, or q is equal to 0. Suppose $q = 0$. Then it must be the case that $\varphi(0) < \psi(0)$, so $\theta(0) \leq \varphi(0) < \psi(0)$ and $\theta <_N \psi$. Now suppose $q > 0$ and $p = 0$. Then it must be the case that $\theta(0) < \varphi(0)$, so $\theta(0) < \varphi(0) \leq \psi(0)$ and $\theta <_N \psi$. Finally, suppose $m = 0$ and $p, q > 0$. Then $\theta <_N \psi$ by the second condition of the base case. Now let $l \geq 0$ and suppose that for any $\theta : [m] \to [n]$, $\varphi : [p] \to [n]$, and $\psi : [q] \to [n]$ injective maps in Δ such that $\theta <_N \varphi <_N \psi$ if $\min\{m,p,q\} \leq l$ then $\theta <_N \psi$.

Let $\theta : [m] \to [n]$, $\varphi : [p] \to [n]$, and $\psi : [q] \to [n]$ be injective maps in Δ such that $\theta <_N \varphi <_N \psi$ and $\min\{m,p,q\} = l+1$. Suppose $\theta(0) < \varphi(0)$ or $\varphi(0) < \psi(0)$ in $[n]$. Then $\theta(0) < \psi(0)$ and so $\theta <_N \psi$. Otherwise, it must be the case that $\theta \circ d^0 >_N \varphi \circ d^0 >_N \psi \circ d^0$. Hence by the induction hypothesis, $\theta \circ d^0 >_N \psi \circ d^0$ so $\theta <_N \psi$.

Finally, we will show this is a total order. Again this will be by induction on $\min\{m,p\}$. Suppose $\theta \neq \varphi$ and one of m or p is equal to 0. If $\theta(0) \neq \varphi(0)$ then either $\theta <_N \varphi$ or $\theta >_N \varphi$. If $\theta(0) = \varphi(0)$ then it cannot be the case that $m = p = 0$, as $\theta \neq \varphi$, so one is strictly larger and again $\theta <_N \varphi$ or $\theta >_N \varphi$. Now let $l \geq 0$ and suppose that for all $\theta : [m] \to [n]$ and $\varphi : [p] \to [n]$ such that $\theta \neq \varphi$ with $\min\{m,p\} \leq l$, either $\theta <_N \varphi$ or $\theta >_N \varphi$.

Let $\theta \neq \varphi$ with $\min\{m,p\} = l+1$. If $\theta(0) \neq \varphi(0)$ then either $\theta <_N \varphi$ or $\theta >_N \varphi$. If $\theta(0) = \varphi(0)$ then it must be the case that $\theta \circ d^0 \neq \varphi \circ d^0$ and so by the induction hypothesis $\theta \circ d^0 <_N \varphi \circ d^0$ or $\theta \circ d^0 >_N \varphi \circ d^0$. Hence $\theta(0) \neq \varphi(0)$. $\square$

The injective maps $\theta : [m] \to [n]$ are the m-simplices of Δ^n, so this determines a total order on the basis elements of the ADC $\mathbb{Z}[\Delta^n]$. We will use the following alternative characterization of this total order to prove that the basis for $\mathbb{Z}[\Delta^n]$ is strongly loop-free.

Proposition B.4. *Let $1_{[n]} \in \Delta^n_n$ be the unique non-degenerate n-simplex. Let*

$$d_{i_1} \cdots d_{i_m} 1_{[n]} \in \Delta^n_{n-m} \qquad d_{j_1} \cdots d_{j_p} 1_{[n]} \in \Delta^n_{n-p}$$

for $\vec{i} = i_1 < i_2 < \cdots < i_m$ and $\vec{j} = j_1 < j_2 < \cdots < j_p$ be distinct non-degenerate simplices of Δ^n. If $1 \leq t \leq \min\{m,p\}$ is the first position where $\vec{i}$ and $\vec{j}$ differ then

$$d_{i_1} \cdots d_{i_m} 1_{[n]} <_N d_{j_1} \cdots d_{j_p} 1_{[n]} \iff \begin{array}{ccc} i_t > j_t & \text{and} & j_t - t \equiv 1 \pmod 2 \\ & \text{or} & \\ i_t < j_t & \text{and} & i_t - t \equiv 0 \pmod 2 \end{array}$$

If $i_l = j_l$ for all $1 \leq l \leq m$ and $m < p$ then

$$d_{i_1} \cdots d_{i_m} 1_{[n]} <_N d_{j_1} \cdots d_{j_p} 1_{[n]} \iff j_{m+1} - (m+1) \equiv 1 \pmod 2$$

Proof. This is just a change in perspective from injective maps in Δ to simplicial operators. The simplex $d_{i_1} \cdots d_{i_m} 1_{[n]}$ corresponds to an injective map

$$d^{i_m} \circ d^{i_{m-1}} \circ \cdots \circ d^{j_1} : [n-m] \hookrightarrow [n]$$

that omits $i_1 < i_2 < \cdots i_m$. Similarly, $d_{j_1} \cdots d_{j_p} 1_{[n]}$ corresponds to an injective map

$$d^{j_p} \circ d^{j_{p-1}} \circ \cdots \circ d^{j_1} : [n-m] \hookrightarrow [n]$$

that omits $j_1 < j_2 < \cdots j_p$.

Suppose $t \leq \min\{m, p\}$ is the first position where $\vec{i}$ and $\vec{j}$ differ and $i_t > j_t$. Then $\theta_{\vec{i}}(l) = \theta_{\vec{j}}(l)$ for all $0 \leq l \leq j_t - t$ and $\theta_{\vec{i}}(j_t - t + 1) \neq \theta_{\vec{j}}(j_t - t + 1)$. Hence to compare these maps we must apply d_0 $j_t - t + 1$ times. Since $j_1 < j_2 < \cdots j_p$, for all $1 \leq l \leq t$

$$j_l \leq j_t - t + l$$

Hence $j_l \leq j_t - t + l$ for all $l \leq t$ and so since $i_l = j_l$ for all $1 \leq l \leq t$

$$\begin{aligned} d_0^{b-t+1} d_{i_1} \cdots d_{i_m} &= d_0^{b-t+2} d_{i_2} \cdots d_{i_{t+1}} \cdots d_{i_m} \\ &= d_0^{j_t} d_{i_t} \cdots d_{i_m} \\ &= d_0 d_1 \cdots d_{j_t - 1} d_{i_t} d_{i_{t+1}} \cdots d_{i_m} \end{aligned}$$

$$\begin{aligned} d_0^{b-t+1} d_{j_1} \cdots d_{j_p} &= d_0^{j_t} d_{j_t} d_{j_{t+1}} \cdots d_{j_p} \\ &= d_0 d_1 \cdots d_{j_t - 1} d_{j_t} d_{j_{t+1}} \cdots d_{j_p} \end{aligned}$$

The injective map corresponding to

$$0 < 1 < \cdots < j_t - 1 < j_t < j_{t+1} < \cdots j_p$$

sends 0 to a value strictly greater than j_t while the injective map corresponding to

$$0 < 1 < \cdots < j_t - 1 < i_t < i_{t+1} < \cdots i_m$$

sends 0 to j_t since $i_t > j_t$. Hence

$$d_0^{b-t+1} d_{i_1} \cdots d_{i_m} 1_{[n]} <_N d_0^{b-t+1} d_{j_1} \cdots d_{j_p} 1_{[n]}$$

Therefore, by Proposition B.4

$$d_{i_1} \cdots d_{i_m} 1_{[n]} <_N d_{j_1} \cdots d_{j_p} 1_{[n]} \iff j_t \equiv t - 1 \pmod 2$$

Now suppose $i_l = j_l$ for all $1 \leq l \leq m$ and $m < p$. Hence the sequence $\vec{i}$ is just a truncation of $\vec{j}$. Now $j_{m+1} > i_m$ so $j_{m+1} - m + l - 1 \geq i_l$ for all $1 \leq l \leq m$. Consider

$$\begin{aligned} d_0^{j_{m+1}-m} d_{i_1} d_{i_2} \cdots d_{i_m} &= d_0^{j_{m+1}} \\ &= d_0 d_1 d_2 \cdots d_{j_{m+1}-1} \end{aligned}$$

$$\begin{aligned} d_0^{j_{m+1}-m} d_{j_1} d_{j_2} \cdots d_{j_p} &= d_0^{j_{m+1}-m} d_{i_1} d_{i_2} \cdots d_{i_m} d_{j_{m+1}} \cdots d_{j_p} \\ &= d_0^{j_{m+1}} d_{j_{m+1}} \cdots d_{j_p} \\ &= d_0 d_1 d_2 \cdots d_{j_{m+1}-1} d_{j_{m+1}} \cdots d_{j_p} \end{aligned}$$

Hence the injective map corresponding to

$$0 < 1 < 2 < \cdots < j_{m+1} - 1$$

sends 0 to j_{m+1} and the injective map corresponding to

$$0 < 1 < 2 < \cdots < j_{m+1} - 1 < j_{m+1} < j_{m+2} < \cdots < j_p$$

sends 0 to a value strictly greater than j_{m+1}. We have applied d_0 to our original functions $j_{m+1} - m$ times, so by Proposition B.4

$$d_0^{j_{m+1}-m} d_{i_1} d_{i_2} \cdots d_{i_m} 1_{[n]} <_N d_0^{j_{m+1}-m} d_{j_1} d_{j_2} \cdots d_{j_p} 1_{[n]}$$

if and only if $j_{m+1} - m$ is even. Or equivalently,

$$d_{i_1} \cdots d_{i_m} 1_{[n]} <_N d_{j_1} \cdots d_{j_p} 1_{[n]} \iff j_{m+1} - (m+1) \equiv 1 \pmod 2$$

□

Proposition B.5. *With the total order $<_N$ from Proposition B.3 $\mathbb{Z}[\Delta^n]$ has a strongly loop-free basis.*

Proof. We will prove the first of the conditions in Definition 4.24, the proof of the second condition uses the same argument.

Suppose $\delta \in \Delta^n_{t-1}$ and $\varphi \in \Delta^n_t$ are simplices such that $[\delta] \leq \partial^-[\varphi]$ in $\mathbb{Z}[\Delta^n]$. Then $\delta = d_j \varphi$ for some $j \equiv 1 \pmod 2$. Write for $\vec{i} = i_1 < j_2 < \cdots < i_m \leq t$

$$\varphi = d_{i_1} \cdots d_{i_m} 1_{[n]}$$

where $1_{[n]}$ is the unique non-degenerate n-simplex of Δ^n. We consider two cases.

Suppose there exists $1 \leq l \leq m$ such that $j + l - 1 < i_l$ and for all $1 \leq r \leq l - 1$, $j + r - 1 \geq i_r$. Then

$$\delta = d_j \varphi = d_j d_{i_1} \cdots d_{i_m} = d_{i_1} \cdots d_{i_{l-1}} d_{j+l-1} d_{i_l} \cdots d_{i_m}$$

Since j is odd $j + l - 1 \equiv l \pmod 2$. Hence $\delta <_N \varphi$ by Proposition B.4 since $i_l > j + l - 1$.

Now suppose $j + r - 1 \geq i_r$ for all $1 \leq r \leq m$. Then

$$\delta = d_j \varphi = d_j d_{i_1} \cdots d_{i_m} = d_{i_1} \cdots d_{i_m} d_{j+m}$$

Since j is odd, $j + m \equiv m + 1 \pmod 2$ and so by Proposition B.4 $\delta <_N \varphi$. □

The key result from [Ste04] for ADCs with unital strongly loop-free bases that we will apply to $\mathbb{Z}[\Delta^n]$ is the following.

Proposition B.6. *Let K be an ADC with a strongly loop-free basis. Let $x \in \nu K_n$ be an n-cell that is congruent modulo $(\mu K)_r$ for some $0 \leq r < n$ to a sum of atoms*

$$x \equiv \langle b_1 \rangle + \langle b_2 \rangle + \cdots + \langle b_k \rangle \mod (\mu K)_r$$

where $b_1 <_N b_2 <_N \cdots <_N b_k$ in the partial order on basis elements. Then there is a decomposition in νK

$$x = x_1 *_r x_2 *_r \cdots *_r x_k$$

with $x_i \equiv \langle b_i \rangle \mod (\mu K)_r$.

We will use this proposition to verify that cells we construct in the strict ω-category $\nu\mathbb{Z}[\Delta^n]$ describe the source an target of the n-cell $\langle 1_{[n]} \rangle$ of $\nu\mathbb{Z}[\Delta^n]$. Before this we will give a brief description of what can generally be said about the cells x_i in the decomposition given by this result.

The cells x_i for $1 \leq i \leq k$ will be n_i-cells, where n_i is the degree of the basis element b_i. It must be the case that $n_i > r$, as $\langle b \rangle \equiv 0 \mod (\mu K)_r$ if b is a basis element of degree less than or equal to r. Now $d^\xi_{r-1}(x) = d^\xi_{r-1}(x_i)$, so $(x_i)^\xi_j = x^\xi_j$ for all $0 \leq j < r$ and all $\xi \in \{-,+\}$. As well, for all $r < j \leq n_i$ and all $\xi \in \{-,+\}$ we have $(x_i)^\xi_j = \langle b_i \rangle^\xi_j$ since $x_i \equiv \langle b_i \rangle \mod (\mu K_r)$. So the double sequences for the cells x_i look like

$$x_i = \begin{pmatrix} x^-_0, \cdots, x^-_{r-1}, (x_i)^-_r, \langle b_i \rangle^-_{r+1}, \langle b_i \rangle^-_{r+2}, \cdots, \langle b_i \rangle^-_{n_i-1}, b, 0, \cdots \\ x^+_0, \cdots, x^+_{r-1}, (x_i)^+_r, \langle b_i \rangle^+_{r+1}, \langle b_i \rangle^+_{r+2}, \cdots, \langle b_i \rangle^+_{n_i-1}, b, 0, \cdots \end{pmatrix}$$

with only the entries on the r^{th} level not already determined. Furthermore, $d^-_r(x) = d^-_r(x_1)$ and $d^+_r(x) = d^+_r(x_k)$ so $(x_1)^-_r = x^-_r$ and $(x_k)^+_r = x^+_r$.

We will now proceed to define the cells in $\nu\mathbb{Z}[\Delta^n]$ that will form the source and target of $\langle 1_{[n]} \rangle$. First we must define certain sequences of values from $[n]$ that allow us to compactly describe certain simplicial operators and hence certain cells of $\nu\mathbb{Z}[\Delta^n]$.

Definition B.7. *Define for $n \geq 0$ and $1 \leq m \leq n$ the set of sequences*

$$J_{n,m} = \{\vec{j} = j_1 < j_2 < \cdots j_m \mid j_l \in [n] \text{ for all } 1 \leq l \leq m\}$$

Define the subsets of $J_{n,m}$

$$J^+_{n,m} = \{\vec{j} \in J_{n,m} \mid j_l - l \equiv 1 \pmod 2 \text{ for all } 1 \leq l \leq m\}$$

$$J^-_{n,m} = \{\vec{j} \in J_{n,m} \mid j_l - l \equiv 0 \pmod 2 \text{ for all } 1 \leq l \leq m\}$$

and for all $0 \leq t \leq m$ and all $0 \leq p \leq n$

$$J^{(p)t}_{n,m} = \left\{ \vec{j} \in J_{n,m} \mid j_t < p < j_{t+1} \text{ and } j_l - l \equiv \begin{cases} t+p & 1 \leq l \leq t \\ t+p+1 & t+1 \leq l \leq m \end{cases} \right\}$$

Note that for $\vec{j} \in J^{(p)t}_{n,m}$

$$j_t - t \equiv t + p \equiv j_{t+1} - (t+1) - 1 \equiv j_{t+1} - t \pmod 2$$

so $j_t \equiv j_{t+1} \equiv p \pmod 2$.

For $\vec{j} \in J_{n,m}$ we denote by

$$d_{\vec{j}} = d_{j_1} d_{j_2} \cdots d_{j_m}$$

the corresponding simplicial operator for an n-simplex.

The following Definition provides an explicit description of cells $\omega^{(p)}_{n-1}$ for $0 \leq p \leq n$ that will make up the source and target of the n-cell $\langle 1_{[n]} \rangle$ of $\nu\mathbb{Z}[\Delta^n]$.

Definition B.8. *Let $n \geq 3$. For all $0 \leq p \leq n$ and all $2 \leq m \leq n-1$ define the $n-m$-cells $\gamma^{(p)}_{n-m}$ and $\delta^{(p)}_{n-m}$ of $\nu\mathbb{Z}[\Delta^n]$ such that*

$$\gamma^{(p)}_{n-m} \equiv \sum_{\substack{t \in [m] \ s.t. \\ t \equiv p \pmod 2}} \ \sum_{\vec{j} \in J^{(p)t}_{n,m}} \langle d_{\vec{j}} 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-m-1}$$

$$\delta^{(p)}_{n-m} \equiv \sum_{\substack{t \in [m] \ s.t. \\ t \equiv p+1 \pmod 2}} \ \sum_{\vec{j} \in J^{(p)t}_{n,m}} \langle d_{\vec{j}} 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-m-1}$$

$$(\gamma^{(p)}_{n-m})^{\xi}_{j} = \begin{cases} \displaystyle\sum_{t=0}^{m+1} \sum_{\vec{j} \in J^{(p)t}_{n,m+1}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle^{-}_{n-m-1} & \xi = + \text{ and } j = n-m-1 \\ \langle 1_{[n]} \rangle^{-}_{n-m-1} & \xi = - \text{ and } j = n-m-1 \\ \langle 1_{[n]} \rangle^{\xi}_{j} & \xi \in \{-,+\} \text{ and } 0 \leq j \leq n-m-2 \end{cases}$$

$$(\delta^{(p)}_{n-m})^{\xi}_{j} = \begin{cases} \langle 1_{[n]} \rangle^{+}_{n-m-1} & \xi = + \text{ and } j = n-m-1 \\ \displaystyle\sum_{t=0}^{m+1} \sum_{\vec{j} \in J^{(p)t}_{n,m+1}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle^{+}_{n-m-1} & \xi = - \text{ and } j = n-m-1 \\ \langle 1_{[n]} \rangle^{\xi}_{j} & \xi \in \{-,+\} \text{ and } 0 \leq j \leq n-m-2 \end{cases}$$

For $0 \leq p \leq n$ define

$$\omega^{(p)}_{1} = \langle d_p \rangle$$

and for $2 \leq m \leq n-1$ define

$$\omega^{(p)}_{n-m+1} = \gamma^{(p)}_{n-m} *_{n-m-1} \omega^{(p)}_{n-m} *_{n-m-1} \delta^{(p)}_{n-m}$$

with

$$\omega^{(p)}_{n-m+1} \equiv \langle d_p 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-m}$$

Proof. We must show that the cells in this definition are well-defined. For $\gamma^{(p)}_{n-m}$ and $\delta^{(p)}_{n-m}$ for $0 \leq p \leq n$ and $2 \leq m \leq n-1$. We must show that the definition of the entries in the double sequence are valid for the conditions defining $\nu\mathbb{Z}[\Delta^n]$ in Definition 4.21 and Definition 4.20. For the conditions on $\mu\mathbb{Z}[\Delta^n]$, we only have to show that

$$\partial(\gamma^{(p)}_{n-m})^{+}_{n-m} = \partial(\gamma^{(p)}_{n-m})^{-}_{n-m} = (\gamma^{(p)}_{n-m})^{+}_{n-m-1} - (\gamma^{(p)}_{n-m})^{-}_{n-m-1}$$

as the other cases for lower entries in the double sequence follow from the fact that $\langle 1_{[n]}\rangle$ is a valid n-cell of $\nu\mathbb{Z}[\Delta^n]$.

For a fixed value of p, the cells $\gamma^{(p)}_{n-m}$ and $\delta^{(p)}_{n-m}$ are distinguished by the choice of parity for $t+p$, which must remain constant as $t \in [m]$ varies. Let $a \in \{0,1\}$ and define $\alpha \in \{-,+\}$ to be the sign of $(-1)^{a+1}$. With this definition, for all $n \geq 3$ and $2 \leq m \leq n-1$, $\vec{j} \in J^{\alpha}_{n,m}$ if and only if $j_k - k \equiv a \pmod 2$ for all $1 \leq k \leq m$. Then we want to show that

$$\partial\left(\sum_{\substack{t\in[m] \text{ s.t.}\\ t+p\equiv a \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,m}} \langle d_{\vec{j}} 1_{[n]}\rangle\right) = (-1)^a \left(\sum_{t=0}^{m+1} \sum_{\vec{j}\in J^{(p)t}_{n,m+1}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle^{\alpha}_{n-m-1} - \langle 1_{[n]}\rangle^{\alpha}_{n-m-1}\right)$$

We will start by analyzing the quantity on the left, which we will call π_a

$$\pi_a = \partial\left(\sum_{\substack{t\in[m] \text{ s.t.}\\ t+p\equiv a \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,m}} \langle d_{\vec{j}} 1_{[n]}\rangle\right)$$

where

$$\pi_0 = \partial(\gamma^{(p)}_{n-m})$$

$$\pi_1 = \partial(\delta^{(p)}_{n-m})$$

Let $a \in \{0,1\}$ be fixed. We want to identify which terms do not cancel in the sum π_a. We will do this by analyzing cases.

Suppose $j \equiv a+1 \pmod 2$. We consider several sub-cases.

(O1) Suppose $j \equiv a+1 \pmod 2$ and $l < t$. Consider

$$\begin{aligned} d_j d_{j_1} \cdots d_{j_m} &= d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+1}} \cdots d_{j_m} \\ &= d_{j_{l+1}-(l+1)} d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+2}} \cdots d_{j_m} \end{aligned}$$

Define $\vec{j'} \in J_{n,m}$ by

$$j'_k = \begin{cases} j_k & 1 \leq k \leq l \\ j+l & k = l+1 \\ j_k & l+2 \leq k \leq m \end{cases}$$

This sequence satisfies

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t+p \pmod 2 & 1 \leq k \leq l \\ j+l-(l+1) \equiv j-1 \equiv a \equiv t+p \pmod 2 & k = l+1 \\ j_k - k \equiv t+p \pmod 2 & l+2 \leq k \leq t \\ j_k - k \equiv t+p+1 \pmod 2 & t+1 \leq k \leq m+1 \end{cases}$$

so $\vec{j'} \in J^{(p)t}_{n,m}$, but $j_{l+1} - (l+1) \equiv t+p \equiv j+1 \pmod 2$ since $l+1 \leq t$, so the term $(-1)^j [d_j d_{\vec{j}}]$ cancels in the sum π_a.

(O2) Suppose$j \equiv a+1 \pmod 2$ and $l = t$. Then the sequence

$$d_j d_{j_1} \cdots d_{j_m} = d_{j_1} d_{j_2} \cdots d_{j_t} d_{j+t} d_{j_{t+1}} \cdots d_{j_m}$$

cannot arise from $d_r d_{\vec{j'}}$ for any $r \equiv a \pmod 2$ and any $\vec{j'} \in J_{n,m}^{(p)t'}$. For all $1 \le k \le t$

$$j_k - (k-1) \equiv t + p + 1 \equiv a + 1 \pmod 2$$

and for all $t+1 \le k \le m$

$$j_k - k \equiv t + p + 1 \equiv a + 1 \pmod 2$$

so all such terms appearing in the sum π_a have the sign $(-1)^{a+1}$. Hence none of the terms with $j \equiv a+1 \pmod 2$ and $l = t$ cancel.

(O3) Suppose $j \equiv a+1$ and $l > t$.

$$\begin{array}{rcl} d_j d_{j_1} \cdots d_{j_m} & = & d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+1}} \cdots d_{j_m} \\ & = & d_{j_l-(l-1)} d_{j_1} d_{j_2} \cdots d_{j_{l-1}} d_{j+l} d_{j_{l+2}} \cdots d_{j_m} \end{array}$$

Define

$$j'_k = \begin{cases} j_k & 1 \le k \le l-1 \\ j+l & k = l \\ j_k & l+1 \le k \le m \end{cases}$$

then

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t + p \pmod 2 & 1 \le k \le t \\ j_k - k \equiv t + p + 1 \pmod 2 & t+1 \le k \le l-1 \\ j + l - l \equiv j \equiv a + 1 \equiv t + p + 1 \pmod 2 & k = l \\ j_k - k \equiv t + p + 1 \pmod 2 & l+1 \le k \le m \end{cases}$$

Hence $\vec{j'} \in J_{n,m}^{(p)t}$ and since $l > t$, $j_l - (l-1) \equiv (t+p+1) + 1 \equiv t + p \pmod 2$ the term $(-1)^j [d_j d_{j_1} \cdots d_{j_m}]$ cancels in the sum π_a.

Now suppose $j \equiv a \pmod 2$, we will again consider several sub-cases.

(E1) Suppose $j \equiv a \pmod 2$, $l \ge 1$, and $j + l < p$. Then $j_l < j + l < p$ so $l \le t$.

$$\begin{array}{rcl} d_j d_{j_1} \cdots d_{j_m} & = & d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+1}} \cdots d_{j_m} \\ & = & d_{j_l-l-1} d_{j_1} d_{j_2} \cdots d_{j_{l-1}} d_{j+l} d_{j_{l+1}} \cdots d_{j_m} \end{array}$$

If we define a sequence $\vec{j'} \in J_{n,m}$

$$j'_k = \begin{cases} j_k & 1 \le k \le l-1 \\ j+l & k = l \\ j_k & l+1 \le k \le m \end{cases}$$

then $\vec{j'} \in J_{n,m}^{(p)t}$ since

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t+p \pmod 2 & 1 \le k \le l-1 \\ j+l-l \equiv j \equiv a \equiv t+p \pmod 2 & k = l \\ j_k - k \equiv t+p \pmod 2 & l+1 \le k \le t \\ j_k - k \equiv t+p+1 \pmod 2 & t+1 \le k \le m+1 \end{cases}$$

Since $l \le t$, $j_l - l - 1 \equiv t+p+1 \equiv a+1 \equiv j+1 \pmod 2$, so this term cancels.

(E2) Suppose $j \equiv a \pmod 2$, $l = 0$, and $j < p$. Then $j < j_1$ and $j < j_1 < \cdots < j_m$ is a sequence in $J_{n,m+1}$. Suppose that the term $(-1)^a[d_j d_{j_1} \cdots d_{j_m}]$ in π_a cancels. Then it must cancel with a term of the form

$$(-1)^{a+1}[d_{j_r - r} d_j d_{j_1} \cdots d_{j_{r-1}} d_{j_{r+1}} \cdots d_{j_m}]$$

for some $1 \le r \le m$. Such a term can only occur with the opposite sign if $j_r - r \equiv t+p+1 \equiv a+1 \pmod 2$, so it must be the case that $r \ge t+1$. But then the sequence $\vec{j'} \in J_{n,m}$ with

$$j'_k = \begin{cases} j & k = 1 \\ j_{k-1} & 2 \le k \le r-1 \\ j_k & r+1 \le k \le m \end{cases}$$

has $j'_1 - 1 \equiv j - 1 \equiv a+1 \equiv t+p+1 \pmod 2$. Hence the only value of t' for which it is possible that $\vec{j'} \in J_{n,m}^{(p)t'}$ is $t' = 0$. But $j'_1 = j < p$, so this cannot occur. Hence none of these terms cancel.

(E3) Suppose $j \equiv a \pmod 2$ and $j + l = p$. So since $j_l < j + l = p < j_{l+1}$ hence $l = t$ and $j = p - t$. This term does not cancel with any term $(-1)^r[d_r d_{\vec{j'}}]$ in the sum π_a, since $\vec{j'}$ will always contain an operator d_p, which is not possible for $\vec{j'} \in J_{n,m}^{(p)t'}$ for any t'.

(E4) Suppose $j \equiv a \pmod 2$, $j + l > p$, and $l < m$. So since $p < j + l < j_{l+1}$ hence $t < l + 1$ or equivalently $t \le l$. Since $l < m$ consider

$$\begin{aligned} d_j d_{j_1} \cdots d_{j_m} &= d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+1}} \cdots d_{j_m} \\ &= d_{j_{l+1} - l - 1} d_{j_1} d_{j_2} \cdots d_{j_l} d_{j+l} d_{j_{l+2}} \cdots d_{j_m} \end{aligned}$$

Then if we define $\vec{j'} \in J_{n,m}$

$$j'_k = \begin{cases} j_k & 1 \le k \le l \\ j+t & k = l+1 \\ j_k & t+2 \le k \le m \end{cases}$$

Since $t \le l$ this sequence satisfies

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t+p \pmod 2 & 1 \le k \le t \\ j_k - k \equiv t+p+1 \pmod 2 & t+1 \le k \le l \\ j+l-(l+1) \equiv j-1 \equiv a+1 \equiv t+p+1 \pmod 2 & k = l+1 \\ j_k - k \equiv t+p+1 \pmod 2 & l+2 \le k \le m \end{cases}$$

Hence $\vec{j'} \in J_{n,m}^{(p)t}$. Since $l+1 > t$, $j_{l+1} - (l+1) \equiv t+p+1 \equiv a+1 \pmod 2$. Hence this term cancels in the sum π_a.

(E5) Suppose $j \equiv a \pmod 2$, $l = m$, and $j + m > p$. Then

$$[d_j d_{j_1} \cdots d_{j_m} 1_{[n]}] = [d_{j_1} \cdots d_{j_m} d_{j+m} 1_{[n]}]$$

If this term cancels in the sum π_a it will do so with a term of the form

$$(-1)^{a+1}[d_{j_r - r - 1} d_{j_1} \cdots d_{j_{r-1}} d_{j_{r+1}} \cdots d_{j_m} d_{j+m} 1_{[n]}]$$

for some $1 \leq r \leq m$. For this term to have the sign $(-1)^{a+1}$ in the sum, it must be the case that $j_r - r - 1 \equiv a + 1 \equiv t + p + 1 \pmod 2$, which occurs if and only if $r \leq t$, as then $j_r - r \equiv t + p \pmod 2$. Define the sequence $\vec{j'} \in J_{n,m}$ by

$$j'_k = \begin{cases} j_k & 1 \leq k \leq r - 1 \\ j_{k+1} & r \leq k \leq m - 1 \\ j + m & k = m \end{cases}$$

Then

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t + p \pmod 2 & 1 \leq k \leq r - 1 \\ j_{k+1} - k \equiv t + p + 1 \pmod 2 & r \leq k \leq t \\ j_{k+1} - k \equiv (t + p + 1) + 1 \equiv t + p \pmod 2 & t + 1 \leq k \leq m - 1 \\ j + m - m \equiv j \equiv a \equiv t + p \pmod 2 & k = m \end{cases}$$

But $j + m > p$ and $j'_m - m = j + m - m = j \equiv a \equiv t + p \pmod 2$, so $\vec{j'} \notin J^{(p)t'}_{n,m}$ for any value of t'. Hence these terms do not cancel.

The terms that are not cancelled in the sum π_a with sign $(-1)^a$ are the following. From the case (E2) we get the terms $(-1)^a[d_j d_{\vec{j}}]$ with $j < p$, $j < j_1$, $j \equiv a \pmod 2$, $\vec{j} \in J^{(p)t}_{n,m}$. If we define

$$\vec{j'} = j < j_1 < \cdots < j_m$$

then

$$j'_k - k \equiv \begin{cases} j - 1 \equiv t + p + 1 \pmod 2 & k = 1 \\ j_{k-1} - k \equiv t + p + 1 \pmod 2 & 2 \leq k \leq t + 1 \\ j_{k-1} - k \equiv (t + p + 1) + 1 \equiv t + p \pmod 2 & t + 2 \leq k \leq m + 1 \end{cases}$$

so since $j'_{t+1} = j_t < p < j_{t+1} = j'_{t+2}$, $\vec{j'}$ belongs to $J^{(p)t+1}_{n,m+1}$. Furthermore, it is clear that any $\vec{j'} \in J^{(p)t+1}_{n,m+1}$ corresponds to such a term from (E2).

From case (E3) there are the terms $(-1)^a[d_j d_{\vec{j}}]$ with $\vec{j} \in J^{(p)t}_{n,m}$ and $j + t = p$. Now

$$\begin{aligned} d_j d_{j_1} \cdots d_{j_m} &= d_{j_1} \cdots d_{j_t} d_p d_{j_{t+1}} \cdots d_{j_m} \\ &= d_{j_1} \cdots d_{j_t} d_{j_{t+1} - 1} \cdots d_{j_m - 1} d_p \end{aligned}$$

where $\vec{j'} = j_1 < \cdots < j_t < j_{t+1} - 1 < \cdots < j_m - 1 \in J_{n,m}$ since $j_t < p < j_{t+1}$ so $j_t < j_{t+1} - 1$. Now

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t + p \equiv a \pmod 2 & 1 \leq k \leq t \\ j_k - 1 - k \equiv (t + p + 1) - 1 \equiv t + p \equiv a \pmod 2 & t + 1 \leq k \leq m \end{cases}$$

Hence $\vec{j'} \in J^{\alpha}_{n,m}$ where α is the sign of $(-1)^{a+1}$ and these terms belong to $(-1)^a\langle d_p 1_{[n]}\rangle^{\alpha}_{n-m-1}$. In particular, these are the terms $(-1)^a[d_{\vec{j}}d_p]$ of $(-1)^a\langle d_p 1_{[n]}\rangle^{\alpha}_{n-m-1}$ where t, which is the largest value such that $j_t < p$, satisfies $t + p \equiv a \pmod 2$.

From case (E5) we have the terms

$$(-1)^a[d_j d_{\vec{j}} 1_{[n]}] = (-1)^a[d_{\vec{j}} d_{j+m} 1_{[n]}]$$

with $j \equiv a \pmod 2$, $\vec{j} \in J^{(p)t}_{n-1,m}$ and $j + m > p$. This gives a sequence $\vec{j'} = j_1 < j_2 < \cdots < j_m < j + m \in J_{n,m+1}$ where

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t + p \pmod 2 & 1 \le k \le t \\ j_k - k \equiv t + p + 1 \pmod 2 & t + 1 \le k \le m \\ j + m - (m+1) \equiv j - 1 \equiv a + 1 \equiv t + p + 1 \pmod 2 & k = m + 1 \end{cases}$$

so $\vec{j'} \in J^{(p)t}_{n,m+1}$ with $t < m + 1$ and $t + p \equiv a \pmod 2$. Furthermore, any $\vec{j'} \in J^{(p)t+1}_{n,m+1}$ with $t + p \equiv a \pmod 2$ corresponds to such a term from (E5). Let $\vec{j'} \in J^{(p)t}_{n,m+1}$ with $t < m + 1$ and $t + p \equiv a \pmod 2$. Then

$$\vec{j} = j'_1 < j'_2 < \cdots < j'_m$$

satisfies

$$j_k - k \equiv \begin{cases} j'_k - k \equiv t + p \pmod 2 & 1 \le k \le t \\ j'_k - k \equiv t + p + 1 \pmod 2 & t + 1 \le k \le m \end{cases}$$

and since $j_t = j'_t < p < j'_{t+1} = j_{t+1}$ hence $\vec{j} \in J^{(p)t}_{n-1,m}$. Also since $t < m + 1$, $j'_{m+1} - (m + 1) \equiv t + p + 1 \pmod 2$ so $j = j'_{m+1} - m \equiv t + p \pmod 2$. So $d_{\vec{j'}} = d_j d_{\vec{j}}$ with $\vec{j} \in J^{(p)t}_{n-1,m}$ and $j \equiv t + p$ and $j < j_1$ since $j = j'_{m+1} - m \le j'_1$ and $j'_1 - 1 \equiv t + p \pmod 2$ so $j'_1 \equiv t + p + 1 \not\equiv j \pmod 2$.

Thus we have shown that the terms with sign $(-1)^a$ that do not cancel in the sum π_a are

$$\sum_{\substack{t \in [m] \text{ s.t.} \\ t+p \equiv a \pmod 2}} \left(\sum_{\vec{j} \in J^{(p)t}_{n,m+1}} [d_{\vec{j}} 1_{[n]}] + \sum_{\vec{j} \in J^{(p)t+1}_{n,m+1}} [d_{\vec{j}} 1_{[n]}] \right) + \sum_{\substack{\vec{j} \in J^{-}_{n-1,m} \text{ s.t.} \\ t+p \equiv a \pmod 2}} [d_{\vec{j}} d_p 1_{[n]}] \tag{106}$$

The terms that are not cancelled in the sum π_a with sign $(-1)^{a+1}$ only arise from case (O2). These are the terms $(-1)^{a+1}[d_j d_{\vec{j}}]$ with $l = t$, $j \equiv a+1 \pmod 2$, and $\vec{j} \in J^{(p)t}_{n,m}$ for $t+p \equiv a \pmod 2$. Now

$$d_j d_{j_1} \cdots d_{j_m} = d_{j_1} d_{j_2} \cdots d_{j_t} d_{j+t} d_{j_{t+1}} \cdots d_{j_m}$$

so if we define

$$\vec{j'} = j_1 < j_2 < \cdots < j_t < j + t < j_{t+1} < \cdots < j_m$$

then

$$j'_k - k \equiv \begin{cases} j_k - k \equiv t + p \pmod 2 & 1 \le k \le t \\ j + t - (t+1) \equiv j - 1 \equiv a \equiv t + p \pmod 2 & k = t + 1 \\ j_{k-1} - k \equiv (t + p + 1) + 1 \equiv t + p \pmod 2 & t + 2 \le k \le m + 1 \end{cases}$$

so $\vec{j'}$ belongs to $J^{\alpha}_{n,m+1}$ where α is the sign of $(-1)^{a+1}$. Furthermore, $j'_k \neq p$ for any $1 \leq k \leq m+1$. The only position where this could be possible is $j'_{t+1} = j + t$, but if $j + t = p$ then $j \equiv t + p \equiv a \pmod 2$, which is a contradiction. Finally, it is also the case that $\vec{j'} \notin J^{(p)t'}_{n,m+1}$ for any value t'. Since $j'_k - k \equiv t + p$ for all $1 \leq k \leq m+1$ the only possible values of t' would be 0 or $m+1$. If $t' = 0$ then $j'_k > p$ for all $k \in [m+1]$, so $t = 0$ and

$$j - 1 = j'_1 - 1 \equiv t' + p + 1 \equiv t + p + 1 \equiv a + 1 \pmod 2$$

which is not possible, as $j \equiv a + 1 \pmod 2$. If $t' = m + 1$ then

$$j - 1 = j + m - (m+1) = j'_{m+1} - (m+1) \equiv t' + p \equiv m + 1 + p \equiv a + 1 \pmod 2$$

which is not possible, as $j \equiv a + 1 \pmod 2$.

Furthermore, any $\vec{j'} \in J^{\alpha}_{n,m+1}$ such that $\vec{j'} \notin J^{(p)t'}_{n,m+1}$ for any value t' and $j'_k \neq p$ for any $1 \leq k \leq m+1$ corresponds to such a term from (O2). Let $\vec{j'} \in J^{\alpha}_{n,m+1}$ be such a sequence. Let t be the largest value such that $j_t < p$. Then $p < j_{t+1}$ since $j_k \neq p$ for all $1 \leq k \leq m+1$.

Now consider the following cases.

1. Suppose $p + t \equiv a + 1 \pmod 2$. Claim that this implies $t \geq 1$. If $t = 0$ then $p \equiv a + 1 \pmod 2$, so $j'_k - k \equiv a \equiv p + 1 \equiv t + p + 1 \pmod 2$ for all $1 \leq k \leq m+1$. This implies $\vec{j'} \in J^{(p)0}_{n,m+1}$, which is a contradiction with the conditions on $\vec{j'}$. Hence $t \geq 1$, so we can consider

   $$d_{j'_1} \cdots d_{j'_{m+1}} = d_{j'_t - t - 1} d_{j'_1} \cdots d_{j'_{t-1}} d_{j'_{t+1}} \cdots d_{j'_{m+1}}$$

   The sequence $\vec{j} = j'_1 < \cdots j'_{t-1} < j'_{t+1} < \cdots < j'_{m+1} \in J_{n,m}$ satisfies

   $$j_k - k \equiv \begin{cases} j'_k - k \equiv a \equiv p + (t-1) \pmod 2 & 1 \leq k \leq t-1 \\ j'_{k+1} - k \equiv a + 1 \equiv p + (t-1) + 1 \pmod 2 & t \leq k \leq m \end{cases}$$

   hence $\vec{j} \in J^{(p)t-1}_{n,m}$, since $j'_{t-1} < j'_t < p < j'_{t+1}$. As well, $j'_t - t - 1 \equiv a + 1 \pmod 2$, so $(-1)^{a+1}[d_{\vec{j'}} 1_{[n]}] = (-1)^{a+1}[d_{j'_t - t - 1} d_{\vec{j}} 1_{[n]}]$, which is a term in π_a from case (O2).

2. Suppose $p + t \equiv a \pmod 2$. Claim that this implies that $t \leq m$. Suppose for contradiction that $p + t \equiv a \pmod 2$ and $t = m + 1$. Then $m + 1 + p \equiv a \pmod 2$ so $j'_k - k \equiv a \equiv m + 1 + p \equiv t + p \pmod 2$ for all $1 \leq k \leq m+1$. But then this implies that $\vec{j'} \in J^{(p)m+1}_{n,m+1}$, which contradicts the conditions for $\vec{j'}$. Since $t \leq m$ consider

   $$d_{j'_1} \cdots d_{j'_{m+1}} = d_{j'_{t+1} - t} d_{j'_1} \cdots d_{j'_t} d_{j'_{t+2}} \cdots d_{j'_{m+1}}$$

   The sequence $\vec{j} = j'_1 < \cdots j'_t < j'_{t+2} < \cdots < j'_{m+1} \in J_{n,m}$ satisfies

   $$j_k - k \equiv \begin{cases} j'_k - k \equiv a \equiv p + t \pmod 2 & 1 \leq k \leq t \\ j'_{k+1} - k \equiv a + 1 \equiv p + t + 1 \pmod 2 & t + 1 \leq k \leq m \end{cases}$$

   hence $\vec{j} \in J^{(p)t}_{n,m}$ since $j'_t < p < j'_{t+1} < j'_{t+2}$. As well, $j'_t - t + 1 \equiv a + 1 \pmod 2$, so $(-1)^{a+1}[d_{\vec{j'}} \iota] = (-1)^{a+1}[d_{j'_t - t + 1} d_{\vec{j}} 1_{[n]}]$, which is a term in π_a from case (O2).

This completes the analysis of π_a. Recall that $\alpha \in \{-,+\}$ is the sign of $(-1)^{a+1}$. Then we must show that

$$\pi_a = (-1)^a \left(\sum_{t=0}^{m+1} \sum_{\vec{j} \in J_{n,m+1}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-m-1}^{\alpha} - \langle 1_{[n]} \rangle_{n-m-1}^{\alpha} \right)$$

The terms with sign $(-1)^{a+1}$ in the sum on the right are those $[d_{\vec{j}} 1_{[n]}]$ for $\vec{j} \in J_{n,m+1}^{\alpha}$ such that $\vec{j} \notin J_{n,m+1}^{(p)t}$ for any $t \in [m+1]$ and $j_k \neq p$ for any $1 \leq$. These are exactly the same as the terms with sign $(-1)^{a+1}$ in π_a.

The terms with sign $(-1)^a$ in the sum on the right are those $[d_{\vec{j}} 1_{[n]}]$ for $\vec{j} \in J_{n,m+1}^{(p)t}$ such that $\vec{j} \notin J_{n,m+1}^{\alpha}$ and those $[d_{\vec{j}} d_p 1_{[n]}]$ for $\vec{j} \in J_{n-1,m}^{\alpha}$ such that the sequence

$$\vec{j'} = j_1 \cdots j_t < p < j_{t+1} + 1 < \cdots j_m + 1$$

where t is the largest value in $[m]$ such that $j_t < p$, does not belong to $J_{n,m+1}^{\alpha}$. We will show that these are exactly the terms we identified in (106).

First, we claim that for $\vec{j} \in J_{n-1,m}^{\alpha}$, $[d_{\vec{j}} d_p 1_{[n]}] = [d_{\vec{j'}} 1_{[n]}]$ for some $\vec{j'} \in J_{n,m+1}^{\alpha}$ if and only if $t + p \equiv a + 1 \pmod 2$. Consider a sequence $\vec{j} \in J_{n-1,m}^{\alpha}$. Let t be the largest value such that $j_t < p$. Then $d_{\vec{j}} d_p = d_{\vec{j'}}$ where $\vec{j'} \in J_{n,m+1}$ is defined by

$$j'_k = \begin{cases} j_k & 1 \leq k \leq t \\ p & k = t+1 \\ j_{k-1} + 1 & t+2 \leq k \leq m+1 \end{cases}$$

Hence

$$j'_k - k \equiv \begin{cases} a \pmod 2 & 1 \leq k \leq t \\ p - t \equiv t + p \pmod 2 & k = t+1 \\ j_{k-1} + 1 - k \equiv a \pmod 2 & t+2 \leq k \leq m+1 \end{cases}$$

so it is clear that $\vec{j'} \in J_{n,m+1}^{\alpha}$ if and only if $t + p \equiv a \pmod 2$.

Finally, if $\vec{j} \in J_{n,m+1}^{(p)t}$ then $\vec{j} \notin J_{n,m+1}^{\alpha}$ if and only if $p + t \equiv a \pmod 2$ and $t \leq m$ or $p + t - 1 \equiv a \pmod 2$ and $t \geq 1$. For $\vec{j} \in J_{n,m+1}^{(p)t}$

$$j_k - k \equiv \begin{cases} t + p \pmod 2 & 1 \leq k \leq t \\ t + p + 1 \pmod 2 & t+1 \leq k \leq m+1 \end{cases}$$

Clearly $\vec{j} \in J_{n,m+1}^{\alpha}$ if and only if $t = 0$ and $t+p+1 \equiv a \pmod 2$ or $t = m+1$ and $t+p \equiv a \pmod 2$. Hence we have proved the identity (106) and so $\gamma_{n-m}^{(p)}$ and $\delta_{n-m}^{(p)}$ are well-defined elements of $\mu\mathbb{Z}[\Delta^n]_{n-m}$.

Now we will show that $\gamma_{n-m}^{(p)}$ and $\delta_{n-m}^{(p)}$ are $n-m$-cells of $\nu\mathbb{Z}[\Delta^n]$ using Definition 4.21. Clearly $(\gamma_{n-m}^{(p)})_j^+$, $(\gamma_{n-m}^{(p)})_j^- \in \mathbb{Z}[\Delta^n]^*$ for all $0 \leq j \leq n-m$ and similarly for $\delta_{n-m}^{(p)}$. If $n - m \geq 2$ then $\varepsilon((\gamma_{n-m}^{(p)})_0^-) = \varepsilon(\langle 1_{[n]} \rangle_0^-) = 1$ and $\varepsilon((\gamma_{n-m}^{(p)})_0^+) = \varepsilon(\langle 1_{[n]} \rangle_0^+) = 1$ since $\langle 1_{[n]} \rangle$ is an n-cell of $\nu\mathbb{Z}[\Delta^n]$.

If $n-m=1$ then $\varepsilon((\gamma_{n-m}^{(p)})_0^+) = \varepsilon(\langle 1_{[n]}\rangle_0^-) = 1$ and $\varepsilon((\delta_{n-m}^{(p)})_0^-) = \varepsilon(\langle 1_{[n]}\rangle_0^+) = 1$. For the other cases, recall the definition of $J_{n,n}^{(p)t}$.

$$J_{n,n}^{(p)t} = \left\{ \vec{j} \in J_{n,n} \mid j_t < p < j_{t+1} \text{ and } j_l - l \equiv \begin{cases} t+p & 1 \leq l \leq t \\ t+p+1 & t+1 \leq l \leq n \end{cases} \right\}$$

The only possible choice of t for which $J_{n,n}^{(p)t}$ is not immediately seen to be empty is $t = p$ with the unique possible sequence in $J_{n,n}^{(p)t}$ being

$$0 < 1 < 2 < \cdots < p-1 < p+1 < \cdots < n$$

This sequence has $t = p$ and $t + p \equiv 0 \pmod 2$. But $p - 1 - (p) \equiv 1 \not\equiv 0 \equiv t + p \pmod 2$ and $p+1-(p+1) \equiv 0 \not\equiv 1 \equiv t+p+1 \pmod 2$ so this is not a valid sequence for $J_{n,n}^{(p)p}$. Hence $J_{n,n}^{(p)t} = \emptyset$ for all $0 \leq t \leq n+1$ and

$$\varepsilon\left(\sum_{t=0}^{m+1} \sum_{\vec{j} \in J_{n,n}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle_0^\alpha \right) = \varepsilon(\langle d_p 1_{[n]}\rangle_0^\alpha) = 1$$

So $\gamma_{n-m}^{(p)}$ and $\delta_{n-m}^{(p)}$ are $n-m$-cells of $\nu\mathbb{Z}[\Delta^n]$.

Now we will show inductively on $2 \leq n-m \leq n-1$ that the $n-1$-cells $\omega_{n-m}^{(p)}$ are well-defined and

$$(\omega_{n-m}^{(p)})_{n-m-1}^- = (\gamma_{n-m}^{(p)})_{n-m-1}^+$$
$$(\omega_{n-m}^{(p)})_{n-m-1}^+ = (\delta_{n-m}^{(p)})_{n-m-1}^-$$
$$\omega_{n-m}^{(p)} \equiv \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-m-1}$$

For the base case, $n - m = 1$, $\omega_1^{(p)} = \langle d_p\rangle$ is well-defined and clearly

$$\omega_1^{(p)} \equiv \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_0$$

Furthermore,

$$(\omega_1^{(p)})_0^- = \langle d_p\rangle_0^- = (\gamma_1^{(p)})_0^+$$
$$(\omega_1^{(p)})_0^+ = \langle d_p\rangle_0^+ = (\delta_1^{(p)})_0^-$$

This completes the base case.

Now let $0 \leq n-m < n-1$ and suppose $\omega_{n-m}^{(p)}$ is well defined and

$$(\omega_{n-m}^{(p)})_{n-m-1}^- = (\gamma_{n-m}^{(p)})_{n-m-1}^+$$

$$(\omega_{n-m}^{(p)})_{n-m-1}^+ = (\delta_{n-m}^{(p)})_{n-m-1}^-$$

$$\omega_{n-m}^{(p)} \equiv \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-m-1}$$

For $0 \leq p \leq n$ define

$$\omega_{n-m+1}^{(p)} = \gamma_{n-m}^{(p)} *_{n-m-1} \omega_{n-m}^{(p)} *_{n-m-1} \delta_{n-m}^{(p)}$$

Hence

$$\omega_{n-m+1}^{(p)} \equiv \omega_{n-m}^{(p)} \equiv \langle d_{\vec{j}} 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-m}$$

since $\gamma_{n-m}^{(p)}$ and $\delta_{n-m}^{(p)}$ are $n-m$-cells and

$$\omega_{n-m}^{(p)} \equiv \langle d_{\vec{j}} 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-m-1}$$

by the induction hypothesis. Finally, we have

$$\begin{aligned}
(\omega_{n-m+1}^{(p)})_{n-m}^{-} &= (\gamma_{n-m}^{(p)} *_{n-m-1} \omega_{n-m}^{(p)} *_{n-m-1} \delta_{n-m}^{(p)})_{n-m}^{-} \\
&= \sum_{\substack{t\in[m] \text{ s.t.} \\ t\equiv p \pmod 2}} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-m}^{-} + \sum_{\substack{t\in[m] \text{ s.t.} \\ t\equiv p+1 \pmod 2}} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] \\
&= \sum_{t\in[m]} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-m}^{-} \\
&= (\gamma_{n-m+1}^{(p)})_{n-m}^{+}
\end{aligned}$$

$$\begin{aligned}
(\omega_{n-m+1}^{(p)})_{n-m}^{+} &= (\gamma_{n-m}^{(p)} *_{n-m-1} \omega_{n-m}^{(p)} *_{n-m-1} \delta_{n-m}^{(p)})_{n-m}^{+} \\
&= \sum_{\substack{t\in[m] \text{ s.t.} \\ t\equiv p \pmod 2}} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-m}^{+} + \sum_{\substack{t\in[m] \text{ s.t.} \\ t\equiv p+1 \pmod 2}} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] \\
&= \sum_{t\in[m]} \sum_{\vec{j}\in J_{n,m}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-m}^{+} \\
&= (\delta_{n-m+1}^{(p)})_{n-m}^{-}
\end{aligned}$$

This completes the induction. □

We will now describe the $n-1$-cells $\omega_{n-1}^{(p)}$ completely as cells of $\nu\mathbb{Z}[\Delta^n]$.

Proposition B.9. *For all $n \geq 3$ and all $0 \leq p \leq n$ the $n-1$-cells $\omega_{n-1}^{(p)}$ of $\nu\mathbb{Z}[\Delta^n]$ have*

$$(\omega_{n-1}^{(p)})_{n-1}^{+} = (\omega_{n-1}^{(p)})_{n-1}^{-} = [d_p 1_{[n]}]$$

$$(\omega_{n-1}^{(p)})_{n-2}^{-} = \sum_{t\in[2]} \sum_{\vec{j}\in J_{n,2}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-2}^{-}$$

$$(\omega_{n-1}^{(p)})_{n-2}^{+} = \sum_{t\in[2]} \sum_{\vec{j}\in J_{n,2}^{(p)t}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]} \rangle_{n-2}^{+}$$

and for all $0 \leq j \leq n-3$

$$(\omega_{n-1}^{(p)})_{j}^{-} = (\gamma_{n-2}^{(p)})_{j}^{-} = \langle 1_{[n]} \rangle_{j}^{-}$$

$$(\omega_{n-1}^{(p)})_{j}^{+} = (\delta_{n-2}^{(p)})_{j}^{+} = \langle 1_{[n]} \rangle_{j}^{+}$$

Proof. By Definition B.8 we have

$$\omega^{(p)}_{n-1} = \gamma^{(p)}_{n-2} *_{n-3} \omega^{(p)}_{n-2} *_{n-3} \delta^{(p)}_{n-2}$$

where $\omega^{(p)}_{n-1} \equiv \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-2}$ so

$$(\omega^{(p)}_{n-1})^{+}_{n-1} = (\omega^{(p)}_{n-1})^{-}_{n-1} = [d_p 1_{[n]}]$$

and

$$\begin{aligned}
(\omega^{(p)}_{n-1})^{-}_{n-2} &= (\gamma^{(p)}_{n-2} *_{n-3} \omega^{(p)}_{n-2} *_{n-3} \delta^{(p)}_{n-2})^{-}_{n-2} \\
&= \sum_{\substack{t\in[2] \text{ s.t.} \\ t\equiv p \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle^{-}_{n-2} + \sum_{\substack{t\in[2] \text{ s.t.} \\ t\equiv p+1 \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] \\
&= \sum_{t\in[2]} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle^{-}_{n-2}
\end{aligned}$$

$$\begin{aligned}
(\omega^{(p)}_{n-1})^{+}_{n-2} &= (\gamma^{(p)}_{n-2} *_{n-3} \omega^{(p)}_{n-2} *_{n-3} \delta^{(p)}_{n-2})^{+}_{n-2} \\
&= \sum_{\substack{t\in[2] \text{ s.t.} \\ t\equiv p \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle^{+}_{n-2} + \sum_{\substack{t\in[2] \text{ s.t.} \\ t\equiv p+1 \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] \\
&= \sum_{t\in[2]} \sum_{\vec{j}\in J^{(p)t}_{n,2}} [d_{\vec{j}} 1_{[n]}] + \langle d_p 1_{[n]}\rangle^{+}_{n-2}
\end{aligned}$$

and for all $0 \le j \le n-3$

$$(\omega^{(p)}_{n-1})^{-}_{j} = (\gamma^{(p)}_{n-2})^{-}_{j} = \langle 1_{[n]}\rangle^{-}_{j}$$

$$(\omega^{(p)}_{n-1})^{+}_{j} = (\delta^{(p)}_{n-2})^{+}_{j} = \langle 1_{[n]}\rangle^{+}_{j}$$

□

We are now finally ready to describe the source and target of $\langle 1_{[n]}\rangle$.

Proposition B.10. *The n-cell $\langle 1_{[n]}\rangle \in \nu\mathbb{Z}[\Delta^n]$ has source and target given by*

$$d^{-}_{n-1}\langle 1_{[n]}\rangle = \omega^{(1)}_{n-1} *_{n-2} \omega^{(3)}_{n-1} *_{n-2} \cdots *_{n-2} \omega^{(2\lfloor {}^{n-1}/_2 \rfloor+1)}_{n-1}$$

$$d^{+}_{n-1}\langle 1_{[n]}\rangle = \omega^{(2\lfloor {}^{n}/_2 \rfloor)}_{n-1} *_{n-2} \cdots *_{n-2} \omega^{(2)}_{n-1} *_{n-2} \omega^{(0)}_{n-1}$$

where $\omega^{(p)}_{n-1}$ are the $n-1$-cells of $\nu\mathbb{Z}[\Delta^n]$ defined in Definition B.8 for all $0 \le p \le n$.

Proof. This is an application of Proposition B.6 to the ADC $\mathbb{Z}[\Delta^n]$, which has a unital, very strongly loop-free basis. By Proposition B.4, $d_i 1_{[n]} <_N d_{i+2} 1_{[n]}$ if and only if i is odd. Hence in the identities

$$d^{-}_{n-1} 1_{[n]} \equiv \partial^{-}[1_{[n]}] = [d_1 1_{[n]}] + [d_3 1_{[n]}] + \cdots + [d_{2k+1} 1_{[n]}] \mod (\mu\Delta^n)_{n-2}$$

$$d^{+}_{n-1} 1_{[n]} \equiv \partial^{+}[1_{[n]}] = [d_{2l} 1_{[n]}] + \cdots + [d_2 1_{[n]}] + [d_0 1_{[n]}] \mod (\mu\Delta^n)_{n-2}$$

the sums on the right are written in increasing order for the total order $<_N$ on the basis elements. Applying Proposition B.6 gives that

$$d^-_{n-1}\langle 1_{[n]}\rangle = \bar{\omega}^{(1)}_{n-1} *_{n-2} \bar{\omega}^{(3)}_{n-1} *_{n-2} \cdots *_{n-2} \bar{\omega}^{(2\lfloor {}^{n-1}/_{2}\rfloor+1)}_{n-1}$$

$$d^+_{n-1}\langle 1_{[n]}\rangle = \bar{\omega}^{(2\lfloor {}^{n}/_{2}\rfloor)}_{n-1} *_{n-2} \cdots *_{n-2} \bar{\omega}^{(2)}_{n-1} *_{n-2} \bar{\omega}^{(0)}_{n-2}$$

with

$$\bar{\omega}^{(p)}_{n-1} \equiv \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-2}$$

and

$$\begin{aligned} d^-_j\, \bar{\omega}^{(p)}_{n-1} &= d^-_j\, d^\beta_{n-1}\langle 1_{[n]}\rangle = \langle 1_{[n]}\rangle^-_j \\ d^+_j\, \bar{\omega}^{(p)}_{n-1} &= d^+_j\, d^\beta_{n-1}\langle 1_{[n]}\rangle = \langle 1_{[n]}\rangle^-_j \end{aligned}$$

for all $0 \leq j \leq n-3$ with $\beta = -$ if p is odd and $\beta = +$ if p is even. Comparing these $n-1$-cells $\bar{\omega}^{(p)}$ with the $n-1$-cells $\omega^{(p)}$ as described in Proposition B.9 we see that for all $j \neq n-2$

$$(\bar{\omega}^{(p)}_{n-1})^+_j = (\omega^{(p)}_{n-1})^+_j$$

$$(\bar{\omega}^{(p)}_{n-1})^-_j = (\omega^{(p)}_{n-1})^-_j$$

To show $\bar{\omega}^{(p)}_{n-1} = \omega^{(p)}_{n-1}$ it only remains to show that for all $0 \leq p \leq n$

$$\bar{\omega}^{(p)}_{n-1} \equiv \sum_{t=0}^{2} \sum_{\vec{j} \in J^{(p)t}_{n,2}} \langle d_{j_1} d_{j_2} 1_{[n]}\rangle + \langle d_p 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-3}$$

Observe that for all $0 \leq p \leq n$

$$\begin{aligned} \sum_{\vec{j} \in J^{(p)0}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] &= \sum_{\vec{j} \in J^{(p+2)0}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] + \sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j \equiv p+1 \pmod 2}} [d_{p+2} d_j 1_{[n]}] \\ \sum_{\vec{j} \in J^{(p)2}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] &= \sum_{\vec{j} \in J^{(p+2)2}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] - \sum_{\substack{0\leq j<p \text{ s.t.} \\ j \equiv p+1 \pmod 2}} [d_j d_p 1_{[n]}] \\ \sum_{\vec{j} \in J^{(p)1}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] &= \sum_{\vec{j} \in J^{(p+2)1}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}] - \sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j \equiv p \pmod 2}} [d_p d_j 1_{[n]}] + \sum_{\substack{0\leq j<p \text{ s.t.} \\ j \equiv p \pmod 2}} [d_j d_{p+2} 1_{[n]}] \end{aligned}$$

Hence if we write for $0 \leq p \leq n$

$$\alpha_p = \sum_{t=0}^{2} \sum_{\vec{j} \in J^{(p)t}_{n,2}} [d_{j_1} d_{j_2} 1_{[n]}]$$

then for all $0 \leq p < n$

$$\alpha_p = \alpha_{p+2} + \sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j\equiv p+1 \pmod 2}} [d_{p+2}d_j 1_{[n]}] - \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p+1 \pmod 2}} [d_j d_p 1_{[n]}]$$
$$- \sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j\equiv p \pmod 2}} [d_p d_j 1_{[n]}] + \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p \pmod 2}} [d_j d_{p+2} 1_{[n]}]$$

Now

$$\sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j\equiv p+1 \pmod 2}} [d_{p+2}d_j 1_{[n]}] + \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p \pmod 2}} [d_j d_{p+2} 1_{[n]}] = \sum_{\substack{p+1<j-1\leq n-1 \text{ s.t.} \\ j-1\equiv p \pmod 2}} [d_{j-1}d_{p+2} 1_{[n]}]$$
$$+ \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p \pmod 2}} [d_j d_{p+2} 1_{[n]}]$$

$$= \langle d_{p+2}\rangle_{n-2}^{\xi} - [d_p d_{p+2} 1_{[n]}]$$

where $\xi = +$ if p is even and $\xi = -$ if p is odd. Hence

$$\begin{aligned}\alpha_p &= \alpha_{p+2} + \langle d_{p+2}1_{[n]}\rangle_{n-2}^{\xi} - [d_p d_{p+2} 1_{[n]}] \\ &\quad - \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p+1 \pmod 2}} [d_j d_p 1_{[n]}] - \sum_{\substack{p+2<j\leq n \text{ s.t.} \\ j\equiv p \pmod 2}} [d_p d_j 1_{[n]}] \\ &= \alpha_{p+2} + \langle d_{p+2}1_{[n]}\rangle_{n-2}^{\xi} - [d_{p+1} d_p 1_{[n]}] \\ &\quad - \sum_{\substack{0\leq j<p \text{ s.t.} \\ j\equiv p+1 \pmod 2}} [d_j d_p 1_{[n]}] - \sum_{\substack{p+1<j-1\leq n-1 \text{ s.t.} \\ j-1\equiv p+1 \pmod 2}} [d_{j-1} d_p 1_{[n]}] \\ &= \alpha_{p+2} + \langle d_{p+2}1_{[n]}\rangle_{n-2}^{\xi} - \langle d_p 1_{[n]}\rangle_{n-2}^{\zeta}\end{aligned}$$

where $\zeta = +$ if p is odd and $\zeta = -$ if p is even.

Now $\bar{\omega}_{n-1}^{(p)} \equiv \langle d_p\rangle \mod (\mu\Delta^n)_{n-2}$, so

$$\bar{\omega}_{n-1}^{(p)} \equiv \sum_{t=0}^{2} \sum_{\vec{j}\in J_{n,2}^{(p)t}} \langle d_{j_1}d_{j_2}1_{[n]}\rangle + \langle d_{2i+1}1_{[n]}\rangle \mod (\mu\Delta^n)_{n-3}$$

if and only if $(\bar{\omega}_{n-1}^{(p)})_{n-2}^{-} = \alpha_p + \langle d_p 1_{[n]}\rangle_{n-2}^{-}$ or if and only if $(\bar{\omega}_{n-1}^{(p)})_{n-2}^{+} = \alpha_p + \langle d_p 1_{[n]}\rangle_{n-2}^{+}$. Since the compositions of the $\bar{\omega}_{n-1}^{(p)}$ cells go in opposite order for the superscript p if p is even or odd we will do these cases separately.

We start with the odd case. We will proceed by induction on $0 \leq i \leq \lfloor {}^{n-1}/_2 \rfloor$ to show that

$$(\bar{\omega}_{n-2}^{(2i+1)})_{n-2}^{-} = \alpha_{2i+1} + \langle d_p 1_{[n]}\rangle_{n-2}^{-}$$

Now $\bar{\omega}^{(1)}_{n-1} \equiv \langle d_1 1_{[n]} \rangle \mod (\mu\Delta^n)_{n-2}$ and since $d^-_{n-2}\bar{\omega}^{(1)}_{n-1} = d^-_{n-2}d^-_{n-1}\langle 1_{[n]}\rangle = d^-_{n-2}\langle 1_{[n]}\rangle$ we have

$$\begin{aligned}
(\bar{\omega}^{(1)}_{n-1})^-_{n-2} - \langle d_1 1_{[n]}\rangle^-_{n,2} &= \sum_{\vec{j}\in J^-_{n-2}} [d_{j_1}d_{j_2}1_{[n]}] - \sum_{j=0}^{\lfloor {}^{n-2}/_2 \rfloor} [d_{2j+1}d_1 1_{[n]}] \\
&= \sum_{\vec{j}\in J^-_{n,2}} [d_{j_1}d_{j_2}1_{[n]}] - \sum_{j=0}^{\lfloor {}^{n-2}/_2 \rfloor} [d_1 d_{2j+2} 1_{[n]}] \\
&= \sum_{\vec{j}\in J^{(1)0}_{n,2}} [d_{j_1}d_{j_2}1_{[n]}]
\end{aligned}$$

where

$$J^{(1)0}_{n,2} = \{\vec{j} = (j_1 < j_2) \mid 1 < j_1 \text{ and } j_l \equiv l \pmod 2 \text{ for all } 1 \le l \le 2\}$$

Since there are no values of $\vec{j}$ with $j_1 < j_2 < 1$ or $j_1 < 1$ and $j_1 \equiv 1 \pmod 2$, $J^{(1)1}_{n,2} = J^{(1)2}_{n,2} = \emptyset$ and so $\alpha_1 = (\bar{\omega}^{(1)}_{n-1})^-_{n-2} - \langle d_1\rangle^-_{n-2}$.

Now suppose $0 \le i < \lfloor {}^{n-1}/_2 \rfloor$ and $(\bar{\omega}^{(2i+1)}_{n-1})^-_{n-2} = \langle d_{2i+1}1_{[n]}\rangle^-_{n-2} + \alpha_{2i+1}$. Since $\bar{\omega}^{(2i+3)}_{n-1} \equiv \langle d_{2i+3}1_{[n]}\rangle \mod (\mu\Delta^n)_{n-2}$ we must calculate $(\bar{\omega}^{(2i+3)}_{n-1})^-_{n-2} - \langle d_{2i+3}1_{[n]}\rangle^-_{n-2}$. Now $d^-_{n-2}\bar{\omega}^{(2i+3)}_{n-1} = d^+_{n-2}\bar{\omega}^{(2i+1)}_{n-1}$ so

$$(\bar{\omega}^{(2i+3)}_{n-1})^-_{n-2} - \langle d_{2i+3}\rangle^-_{n-2} \quad = \quad \langle d_{2i+1}1_{[n]}\rangle^+_{n-2} + \alpha_{2i+1} - \langle d_{2i+3}1_{[n]}\rangle^-_{n-2}$$

But $\alpha_{2i+1} = \alpha_{2i+3} - \langle d_{2i+1}1_{[n]}\rangle^+_{n-2} + \langle d_{2i+3}\rangle^-_{n-2}$ since $p = 2i+1$ is odd. Hence

$$(\bar{\omega}^{(2i+3)}_{n-1})^-_{n-2} - \langle d_{2i+3}1_{[n]}\rangle^-_{n-2} = \alpha_{2i+3}$$

This completes the induction.

Now we proceed with the even case. We will use induction on $0 \le i \le \lfloor {}^n/_2 \rfloor$ to show that

$$(\bar{\omega}^{(2i)}_{n-2})^+_{n-2} = \alpha_{2i} + \langle d_p 1_{[n]}\rangle^+_{n-2}$$

Now $\bar{\omega}^{(0)}_{n-1} \equiv \langle d_0 1_{[n]}\rangle \mod (\mu\Delta^n)_{n-2}$ and since $d^+_{n-2}\bar{\omega}^{(0)}_{n-1} = d^+_{n-2}d^+_{n-1}\langle 1_{[n]}\rangle = d^+_{n-2}\langle 1_{[n]}\rangle$ we have

$$\begin{aligned}
(\bar{\omega}^{(0)}_{n-1})^+_{n-2} - \langle d_0 1_{[n]}\rangle^+_{n,2} &= \sum_{\vec{j}\in J^+_{n,2}} [d_{j_1}d_{j_2}1_{[n]}] - \sum_{j=0}^{\lfloor {}^{n-1}/_2 \rfloor} [d_{2j}d_0 1_{[n]}] \\
&= \sum_{\vec{j}\in J^+_{n,2}} [d_{j_1}d_{j_2}1_{[n]}] - \sum_{j=0}^{\lfloor {}^{n-1}/_2 \rfloor} [d_0 d_{2j+1} 1_{[n]}] \\
&= \sum_{\vec{j}\in J^{(0)0}_{n,2}} [d_{j_1}d_{j_2}1_{[n]}]
\end{aligned}$$

where

$$J^{(0)0}_{n,2} = \{\vec{j} = (j_1 < j_2) \mid 0 < j_1 \text{ and } j_l \equiv l+1 \pmod 2 \text{ for all } 1 \le l \le 2\}$$

Since there are no values of $\vec{j}$ with $j_1 < j_2 < 0$ or $j_1 < 0$, $J^{(0)1}_{n,2} = J^{(0)2}_{n,2} = \emptyset$ and so $\alpha_0 = (\bar{\omega}^{(0)}_{n-1})^+_{n-2} - \langle d_0 1_{[n]}\rangle^+_{n-2}$.

Now suppose $0 \leq i < \lfloor {}^{n-2}/_2 \rfloor$ and $(\bar{\omega}^{(2i)}_{n-1})^+_{n-2} = \langle d_{2i} 1_{[n]}\rangle^+_{n-2} + \alpha_{2i}$. Since $\bar{\omega}^{(2i+2)}_{n-1} \equiv \langle d_{2i+2} 1_{[n]}\rangle$ mod $(\mu\Delta^n)_{n-2}$ we must calculate $(\bar{\omega}^{(2i+2)}_{n-1})^+_{n-2} - \langle d_{2i+2} 1_{[n]}\rangle^+_{n-2}$. Now $d^+_{n-2}\bar{\omega}^{(2i+2)}_{n-1} = d^-_{n-2}\bar{\omega}^{(2i)}_{n-1}$ so

$$(\bar{\omega}^{(2i+2)}_{n-1})^+_{n-2} - \langle d_{2i+2} 1_{[n]}\rangle^+_{n-2} \quad = \quad \langle d_{2i} 1_{[n]}\rangle^-_{n-2} + \alpha_{2i} - \langle d_{2i+2} 1_{[n]}\rangle^+_{n-2}$$

But $\alpha_{2i} = \alpha_{2i+2} + \langle d_{2i+2} 1_{[n]}\rangle^+_{n-2} - \langle d_{2i} 1_{[n]}\rangle^-_{n-2}$ since $p = 2i$ is even. Hence

$$(\bar{\omega}^{(2i+2)}_{n-1})^+_{n-2} - \langle d_{2i+2} 1_{[n]}\rangle^+_{n-2} = \alpha_{2i+2}$$

This completes the induction. □

Finally we can use the results of Appendix A to show that when an n-simplex $x : \mathcal{O}^n \to X$ sends atoms $\langle d_{\vec{j}} 1_{[n]}\rangle \in \nu\mathbb{Z}[\Delta^n] \subseteq \mathcal{O}^n$ to identities of X then the source and target of $x(\langle 1_{[n]}\rangle)$ are greatly simplified.

Proposition B.11. *Let $n \geq 2$, $0 \leq p \leq n$, and $2 \leq a \leq b \leq n-1$. Suppose $x : \mathcal{O}^n \to G$ is a map of strict ω-categories such that $x(\langle d_{\vec{j}} 1_{[n]}\rangle)$ is an identity in G for all $\vec{j} \in J^{(p)t}_{n,m}$ with $0 \leq t \leq m$ and $a \leq m \leq b$. Then*

$$x(\omega^{(p)}_{n-a+1}) = x(\omega^{(p)}_{n-b})$$

Proof. Recall the $n-1$-cells $\omega^{(p)}_{n-m}$ of $\mathcal{O}^n$ defined for $1 \leq m \leq n-1$ in Definition B.8. We will show that $x(\omega^{(p)}_{n-c}) = x(\omega^{(p)}_{n-b})$ for all $a - 1 \leq c \leq b$ by downward induction on c. The base case $c = b$ is clear.

Now let $a < c \leq b$ and suppose that $x(\omega^{(p)}_{n-c}) = x(\omega^{(p)}_{n-b})$. Then

$$\begin{aligned} x(\omega^{(p)}_{n-c+1}) &= x(\gamma^{(p)}_{n-c}) *_{n-c-1} x(\omega^{(p)}_{n-c}) *_{n-c-1} x(\delta^{(p)}_{n-c}) \\ &= x(\gamma^{(p)}_{n-c}) *_{n-c-1} x(\omega^{(p)}_{n-b}) *_{n-c-1} x(\delta^{(p)}_{n-c}) \end{aligned}$$

By assumption, $x(\langle d_{\vec{j}} 1_{[n]}\rangle) = 1_{x(d^+_{n-m-1}\langle d_{\vec{j}} 1_{[n]}\rangle)}$ for all $\vec{j} \in J^{(p)t}_{n,m}$ with $0 \leq t \leq m$ and $a \leq m \leq b$. Since

$$\gamma^{(p)}_{n-c} \equiv \sum_{\substack{t\in[c] \text{ s.t.} \\ t\equiv p \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,c}} \langle d_{\vec{j}} 1_{[n]}\rangle \quad \text{mod } (\mu\Delta^n)_{n-c-1}$$

$$\delta^{(p)}_{n-c} \equiv \sum_{\substack{t\in[c] \text{ s.t.} \\ t\equiv p+1 \pmod 2}} \sum_{\vec{j}\in J^{(p)t}_{n,c}} \langle d_{\vec{j}} 1_{[n]}\rangle \quad \text{mod } (\mu\Delta^n)_{n-c-1}$$

Lemma A.2 therefore implies that $x(\gamma^{(p)}_{n-c}) = 1_{x(d^+_{n-c-1}\gamma^{(p)}_{n-c})}$ and $x(\delta^{(p)}_{n-c}) = 1_{x(d^-_{n-c-1}\delta^{(p)}_{n-c})}$. Hence

$$\begin{aligned} x(\omega^{(p)}_{n-c+1}) &= 1_{x(d^+_{n-c-1}\gamma^{(p)}_{n-c})} *_{n-c-1} x(\omega^{(p)}_{n-c}) *_{n-c-1} 1_{x(d^-_{n-c-1}\delta^{(p)}_{n-m})} \\ &= x(\omega^{(p)}_{n-b}) \end{aligned}$$

This completes the induction. □

There is an immediate corollary for the extremal case of this proposition, that is when $a = 2$ and $b = n-1$.

Corollary B.12. *Let $n \geq 2$ and $0 \leq p \leq n$. Suppose $x : \mathcal{O}^n \to G$ is a map of strict ω-categories such that $x(\langle d_{\vec{j}} 1_{[n]} \rangle)$ is an identity in G for all $\vec{j} \in J^{(p)t}_{n,m}$ with $0 \leq t \leq m$ and $2 \leq m \leq n-1$. Then*

$$x(\omega^{(p)}_{n-1}) = x(\omega^{(p)}_1) = x(\langle d_p 1_{[n]} \rangle)$$

C Multiplication in Nerves of Gray Monoids

In this appendix we will study multiplication of simplices in the simplicial monoid $N_\omega(M)$ determined by a Gray monoid M. An n-simplex of $N_\omega(M)$ is an ω-functor $\alpha : \mathcal{O}^n \to M$ and the simplicial set $N_\omega(M)$ is a simplicial monoid with multiplication determined by the lax monoidal structure of N_ω as shown in Proposition 5.8.

In general, for $n > 0$ the product ω-functor $\alpha \cdot \beta : \mathcal{O}^n \to M$ for n-simplices $\alpha, \beta \in N_\omega(M)_n$ sends atoms of $\mathcal{O}^n$ to composites of several cells of M. In this appendix we consider a special case, where α and β have a simple form in which the corresponding ω-functors $\mathcal{O}^n \to M$ are determined by n-cells of M. In this special case we can give the following simple description of the product ω-functor $\alpha \cdot \beta : \mathcal{O}^n \to M$. This special case forms the core of the proof of Proposition 5.10, which shows that $N_\omega(M)$ is a simplicial group if and only if M is a Gray group.

Lemma C.1. *Let $n \geq 2$ and let M be a Gray monoid. Let $\alpha, \beta : \mathcal{O}^n \to M$ be n-simplices of $N_\omega(M)$ such that $\alpha(\langle i \rangle) = \beta(\langle i \rangle) = e \in M_0$ for all $i \in [n]$ and*

$$\alpha(\langle \theta \rangle) = \beta(\langle \theta \rangle) = 1_e$$

for all $\theta : [m] \hookrightarrow [n]$ injective non-decreasing maps of Δ with $m \geq 1$ and $m < n-1$ or $m = n-1$ and $\theta(0) = 0$. Then

$$\alpha \cdot \beta(\langle \theta \rangle) = \begin{cases} e & m = 0 \\ 1_e & 1 \leq m \text{ and } m < n-1 \text{ or } m = n-1 \text{ and } \theta(0) = 0 \\ \beta(\langle \theta \rangle) *_0 \alpha(\langle \theta \rangle) & \theta = d^0 \text{ or } 1_{[n]} \end{cases}$$

Proof. Consider the following commutative diagram for all $\theta : [m] \hookrightarrow [n]$ injective non-decreasing maps of Δ

$$\begin{tikzcd}
\mathcal{O}^m \arrow[r, "C_\omega(\Delta)"] \arrow[d, hook, "C_\omega(\theta)"'] & C_\omega(\Delta^m \times \Delta^m) \arrow[r, "AW"] \arrow[d, hook, "C_\omega(\theta \times \theta)"'] & \mathcal{O}^m \otimes \mathcal{O}^m \arrow[r, "\theta^*\alpha \otimes \theta^*\beta"] \arrow[d, hook, "C_\omega(\theta) \otimes C_\omega(\theta)"'] & M \otimes M \\
\mathcal{O}^n \arrow[r, "C_\omega(\Delta)"'] & C_\omega(\Delta^n \times \Delta^n) \arrow[r, "AW"'] & \mathcal{O}^n \otimes \mathcal{O}^n \arrow[ur, "\alpha \otimes \beta"'] &
\end{tikzcd} \tag{107}$$

For all such θ

$$\alpha(\langle \theta \rangle) = \theta^* \alpha(\langle 1_{[m]} \rangle)$$

so

$$\alpha \cdot \beta(\langle \theta \rangle) = m(\theta^* \alpha \otimes \theta^* \beta(AW(C_\omega(\Delta)(1_{[m]}))))$$

If $m = 0$ then the top path of the diagram (107) factors through the 0-cells $M_0 \times M_0$ of $M \otimes M$

$$\begin{tikzcd}
\mathcal{O}^0 \arrow[r, "C_\omega(\Delta)", "\cong"'] & C_\omega(\Delta^0 \otimes \Delta^0) \arrow[r, "AW", "\cong"'] & \mathcal{O}^0 \otimes \mathcal{O}^0 \arrow[r, "\theta^*\alpha\otimes\theta^*\beta"] & M_0 \times M_0 \arrow[r, hook] \arrow[dr, "m|_0"'] & M \otimes M \arrow[d, "m"] \\
 & & & & M
\end{tikzcd}$$

so $\alpha \cdot \beta(\langle i \rangle) = e \cdot e = e$ since $e \in M_0$ is the multiplicative identity of the Gray monoid M.

To prove the result for the remaining cases with $m \geq 1$ we claim that it is sufficient to show that for $n \geq 1$ and n-simplices α and β of $N_\omega(M)$ satisfying the conditions of the lemma

$$\alpha \otimes \beta(AW(C_\omega(\Delta)(\langle 1_{[n]} \rangle))) = (e \otimes \beta(\langle 1_{[n]} \rangle)) *_0 (\alpha(\langle 1_{[n]} \rangle) \otimes e) \tag{108}$$

Suppose this identity holds, then we will show that $\alpha \cdot \beta$ acts on atoms of $\mathcal{O}^n$ as described in the lemma. First it is clear that the identity (108) implies for $n \geq 1$

$$\begin{array}{rcl}
\alpha \cdot \beta(\langle 1_{[n]} \rangle) & = & m(\alpha \otimes \beta(AW(C_\omega(\Delta)(\langle 1_{[n]} \rangle)))) \\
 & = & m((e \otimes \beta(\langle 1_{[n]} \rangle)) *_0 (\alpha(\langle 1_{[n]} \rangle) \otimes e)) \\
 & = & m(e \otimes \beta(\langle 1_{[n]} \rangle)) *_0 m(\alpha(\langle 1_{[n]} \rangle) \otimes e) \\
 & = & \beta(\langle 1_{[n]} \rangle) *_0 \alpha(\langle 1_{[n]} \rangle)
\end{array}$$

Now we observe that if $\alpha : \mathcal{O}^n \to M$ satisfies the conditions of the lemma then for all injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ of Δ with $1 \leq m < n$ so does

$$\theta^*\alpha = \alpha \circ C_\omega(\theta) : \mathcal{O}^m \hookrightarrow \mathcal{O}^m \to M$$

For all $\varphi : [m'] \hookrightarrow [m]$ injective non-decreasing maps of Δ

$$\theta^*\alpha(\langle \varphi \rangle) = \alpha(\langle \theta \circ \varphi \rangle)$$

Hence if $m' < m - 1$ then $m' < n - 1$ and so $\alpha(\langle \theta \circ \varphi \rangle) = 1_e$. If $m' = n - 1$ and $\theta(\varphi(0)) = 0$ then $\varphi(0) = 0$ since θ and φ are injective and non-decreasing. So $\theta = d^i$ for some $i > 0$ since $m < n$ and $\alpha(\langle \theta \circ \varphi \rangle) = 1_e$. Thus $\theta^*\alpha$ satisfies the conditions of the lemma when α does. Hence if α and β satisfy the conditions of the lemma we have for injective non-decreasing maps $\theta : [m] \hookrightarrow [n]$ of Δ with $m \geq 1$

$$\begin{array}{rcl}
m(\alpha \otimes \beta(AW(C_\omega(\Delta)(\langle \theta \rangle)))) & = & m(\theta^*\alpha \otimes \theta^*\beta(AW(C_\omega(\Delta)(\langle 1_{[m]} \rangle)))) \\
 & = & \theta^*\beta(\langle 1_{[m]} \rangle) *_0 \theta^*\alpha(\langle 1_{[m]} \rangle) \\
 & = & \beta(\langle \theta \rangle) *_0 \alpha(\langle \theta \rangle) \\
 & = & \begin{cases} 1_e *_0 1_e = 1_e & m < n - 1 \text{ or } m = n - 1 \text{ and } \theta(0) = 0 \\ \beta(\langle \theta \rangle) *_0 \alpha(\langle \theta \rangle) & \theta = d^0 \text{ or } 1_{[n]} \end{cases}
\end{array}$$

by the claim (108) above. To prove the result, therefore, it only remains to show this identity, which we now do.

Let $n \geq 1$. Since $\mathbb{Z}[\Delta^n]$ is a strong Steiner complex so is $\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]$. The order on the chains of the tensor product is determined by the order on the chains of $\mathbb{Z}[\Delta^n]$ from Proposition B.3. In particular, by Example 3.10 of [Ste04]

$$(\theta : [m] \hookrightarrow [n]) \otimes (\varphi : [l] \hookrightarrow [n]) <_N (\theta' : [m'] \hookrightarrow [n]) \otimes (\varphi' : [l'] \hookrightarrow [n])$$

if $\theta <_N \theta'$ in $\mathbb{Z}[\Delta^n]$ or if $\theta = \theta'$, $m = m'$ is even, and $\varphi <_N \varphi'$, or if $\theta = \theta'$, $m = m'$ is odd, and $\varphi >_N \varphi'$. We can apply this total order to the terms of the sum

$$AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))^\varepsilon_n = \sum_{p+q=n} [\top\!\!\!\top^{p,q}] \otimes [\perp\!\!\!\perp^{p,q}] = \sum_{p+q=n} [012\cdots p] \otimes [p\; p+1 \cdots n]$$

that occurs as the n-entry in the n-cell $AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))$. None of these terms have the same first entry in the tensor, so we can determine their order by looking at the first entries of the tensors. Hence the order when n is even is

$$\begin{aligned}&[\top\!\!\!\top^{0,n}] \otimes [\perp\!\!\!\perp^{0,n}] <_N [\top\!\!\!\top^{2,n-2}] \otimes [\perp\!\!\!\perp^{2,n-2}] <_N \cdots <_N [\top\!\!\!\top^{n,0}] \otimes [\perp\!\!\!\perp^{n,0}] <_N \\ &\qquad <_N [\top\!\!\!\top^{n-1,1}] \otimes [\perp\!\!\!\perp^{n-1,1}] <_N \cdots <_N [\top\!\!\!\top^{3,n-3}] \otimes [\perp\!\!\!\perp^{3,n-3}] <_N [\top\!\!\!\top^{1,n-1}] \otimes [\perp\!\!\!\perp^{1,n-1}]\end{aligned}$$

or in the equivalent notation describing the images of the injective non-decreasing maps $\top\!\!\!\top^{p,q}$ and $\perp\!\!\!\perp^{p,q}$ in $[n]$

$$\begin{aligned}&[0] \otimes [01\cdots n] <_N [012] \otimes [2\cdots n] <_N \cdots <_N [012\cdots n] \otimes [n] <_N \\ &\qquad <_N [012\cdots n-1] \otimes [n-1\; n] <_N \cdots <_N [0123] \otimes [34\cdots n] <_n [01] \otimes [12\cdots n]\end{aligned}$$

and when n is odd

$$\begin{aligned}&[\top\!\!\!\top^{0,n}] \otimes [\perp\!\!\!\perp^{0,n}] <_N [\top\!\!\!\top^{2,n-2}] \otimes [\perp\!\!\!\perp^{2,n-2}] <_N \cdots <_N [\top\!\!\!\top^{n-1,1}] \otimes [\perp\!\!\!\perp^{n-1,1}] <_N \\ &\qquad <_N [\top\!\!\!\top^{n,0}] \otimes [\perp\!\!\!\perp^{n,0}] <_N \cdots <_N [\top\!\!\!\top^{3,n-3}] \otimes [\perp\!\!\!\perp^{3,n-3}] <_N [\top\!\!\!\top^{1,n-1}] \otimes [\perp\!\!\!\perp^{1,n-1}]\end{aligned}$$

$$\begin{aligned}&[0] \otimes [01\cdots n] <_N [012] \otimes [2\cdots n] <_N \cdots <_N [012\cdots n-1] \otimes [n-1\; n] <_N \\ &\qquad <_N [012\cdots n] \otimes [n] <_N \cdots <_N [0123] \otimes [34\cdots n] <_n [01] \otimes [12\cdots n]\end{aligned}$$

This order satisfies the condition of Proposition 5.1 of [Ste04] that allows us to convert sums into composites of cells in $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$. In particular, there are decompositions when n is even in $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$

$$AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle)) = x^{0,n} *_{n-1} x^{2,n-1} *_{n-1} \cdots *_{n-1} x^{n,0} *_{n-1} x^{n-1,1} *_{n-1} x^{n-3,3} *_{n-1} \cdots *_{n-1} x^{1,n-1}$$

with

$$x^{p,q} \equiv \langle \top\!\!\!\top^{p,q} \otimes \perp\!\!\!\perp^{p,q}\rangle \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-1}$$

and when n is odd there are decompositions

$$AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle)) = y^{0,n} *_{n-1} y^{2,n-1} *_{n-1} \cdots *_{n-1} y^{n-1,1} *_{n-1} y^{n,0} *_{n-1} y^{n-3,3} *_{n-1} \cdots *_{n-1} y^{1,n-1}$$

with

$$y^{p,q} \equiv \langle \top\!\!\!\top^{p,q} \otimes \perp\!\!\!\perp^{p,q}\rangle \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-1}$$

Now $\alpha \otimes \beta(\langle 012\cdots p \otimes p\; p+1\cdots n\rangle)$ is an identity by Lemma 4.36 if $\alpha(\langle 012\cdots p\rangle)$ or $\beta(\langle p\; p+1\cdots n\rangle)$ are identities. This occurs when $0 < p < n-1$ or $0 < q < n-1$, so the only atoms $\langle 012\cdots p \otimes p\; p+1\cdots n\rangle$ not sent to identities by $\alpha \otimes \beta$ occur when $p = 0$ or $p = n$. Hence by Lemma A.2 $\alpha \otimes \beta(x^{p,q})$ is an identity n-cell unless $p = 0$ or $p = n$ and so

$$\alpha \otimes \beta(AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))) = \alpha \otimes \beta(x^{0,n}) *_{n-1} \alpha \otimes \beta(x^{n,0})$$

when n is even and

$$\alpha \otimes \beta(AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))) = \alpha \otimes \beta(y^{0,n}) *_{n-1} \alpha \otimes \beta(y^{n,0})$$

when n is odd. We will proceed in the even case, the odd case being identical up to replacing x with y.

By the decomposition of $AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))$ in the case n is even above

$$\begin{aligned}
(x^{0,n})^-_{n-1} &= AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))^-_{n-1} \\
&= \sum_{i=0}^{\lfloor {}^{n-1}/_{2} \rfloor} \sum_{p+q=n-1} [d^{2i+1} \circ \top\!\!\top^{p,q}] \otimes [d^{2i+1} \circ \perp\!\!\!\perp^{p,q}] \\
&= \sum_{p+q=n-1} \left(\sum_{i=\lceil {}^{p}/_{2} \rceil}^{\lfloor {}^{n-1}/_{2} \rfloor} [012 \cdots p] \otimes [p\; p+1 \cdots 2i \overset{\vee}{+} 1 \cdots n] \right. \\
&\qquad \left. + \sum_{i=0}^{\lfloor {}^{p}/_{2} \rfloor - 1} [012 \cdots 2i \overset{\vee}{+} 1 \cdots p] \otimes [p\; p+1 \cdots n] \right)
\end{aligned}$$

where the notation $2i \overset{\vee}{+} 1$ indicates that this value is missing from the target of the injective non-decreasing map corresponding to the sequence of integers in $[n]$. Note that there must always be overlap at p between the two sides of the tensor; the largest value in the target of the injective non-decreasing map corresponding to the left entry of the tensor must be equal to the least value in the target of the right side of the tensor. In other words, the missing number can't occur at the join.

Now by the definition of the chain complex tensor product

$$\begin{aligned}
\partial[\top\!\!\top^{p,q} \otimes \perp\!\!\!\perp^{p,q}] = &\sum_{i=0}^{p} (-1)^i [012 \cdots \overset{\vee}{i} \cdots p \otimes p\; p+1 \cdots n] \\
&+ \ (-1)^p \sum_{j=0}^{q} (-1)^j [012 \cdots p \otimes p\; p+1 \cdots \overset{\vee}{j} \cdots n]
\end{aligned}$$

For these tensors it is possible to have the missing number occur at the join, so that the sequences on either side of the tensor do not overlap. When $p = 0$ therefore

$$\begin{aligned}
\langle \top\!\!\top^{0,n} \otimes \perp\!\!\!\perp^{0,n} \rangle^-_{n-1} &= \sum_{j=0}^{\lfloor {}^{n-1}/_{2} \rfloor} [\top\!\!\top^{0,n} \otimes d^{2j+1} \circ \perp\!\!\!\perp^{0,n-1}] \\
&= \sum_{j=0}^{\lfloor {}^{n-1}/_{2} \rfloor} [0 \otimes d^{2j+1}] \\
&= \sum_{j=0}^{\lfloor {}^{n-1}/_{2} \rfloor} [0 \otimes 012 \cdots 2j \overset{\vee}{+} 1 \cdots n]
\end{aligned}$$

Hence

$$
\begin{aligned}
x^{0,n} \equiv{}& \langle 0 \otimes 012\cdots n\rangle + \sum_{p+q=n-1} \left(\sum_{i=\lceil p/2 \rceil}^{\lfloor n-1/2 \rfloor} \langle 012\cdots p \otimes p\, p+1 \cdots \overset{\vee}{2i+1} \cdots n\rangle \right.\\
&\qquad\qquad \left. + \sum_{i=0}^{\lfloor p/2 \rfloor -1} \langle 012\cdots \overset{\vee}{2i+1} \cdots p \otimes p\, p+1 \cdots n\rangle \right)\\
&\qquad - \sum_{j=0}^{\lfloor n-1/2 \rfloor} \langle 0 \otimes 012\cdots \overset{\vee}{2j+1} \cdots n\rangle \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-2}\\
\equiv{}& \langle 0 \otimes 012\cdots n\rangle + \sum_{\substack{p+q=n-1\\ \text{s.t. } p\neq 0}} \left(\sum_{i=\lceil p/2 \rceil}^{\lfloor n-1/2 \rfloor} \langle 012\cdots p \otimes p\, p+1 \cdots \overset{\vee}{2i+1} \cdots n\rangle \right. \qquad (109)\\
&\left. + \sum_{i=0}^{\lfloor p/2 \rfloor -1} \langle 012\cdots \overset{\vee}{2i+1} \cdots p \otimes p\, p+1 \cdots n\rangle \right) \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-2}
\end{aligned}
$$

Let $p > 0$ and $0 \leq i < \lfloor p/2 \rfloor$ and consider the images under $\alpha \otimes \beta$ of the atoms occurring in the second sum above

$$\alpha \otimes \beta(\langle 012\cdots \overset{\vee}{2i+1} \cdots p \otimes p\, p+1 \cdots n\rangle) \in M_{n-1}$$

Since $p > 0$ and $p \leq n-1$ therefore

$$\alpha(\langle 012\cdots \overset{\vee}{2i+1} \cdots p\rangle) = 1_e \in M_{p-1}$$

by hypothesis on α since $p-1 < n-1$. It is not possible to have $p = 1$ as this would require $0 \leq i < \lfloor 1/2 \rfloor = 0$. Thus $p > 1$ and so by hypothesis on β

$$\beta(p\, p+1 \cdots n) = 1_e \in M_q$$

since $p \leq n-1$. So by Lemma 4.36 for $p > 0$ and $0 \leq i < \lfloor n/2 \rfloor$

$$\alpha \otimes \beta(\langle 012\cdots \overset{\vee}{2i+1} \cdots p \otimes p\, p+1 \cdots n\rangle) = 1_{e\otimes e} \in \mathcal{O}^n_{n-1}$$

Now let $p < 0$ and $\lceil p/2 \rceil \leq i \leq \lfloor n-1/2 \rfloor$ and consider the images under $\alpha \otimes \beta$ of the atoms occurring in the first sum above

$$\alpha \otimes \beta(\langle 012\cdots p \otimes p\, p+1 \cdots \overset{\vee}{2i+1} \cdots n\rangle) \in M_{n-1}$$

Since $p > 0$ therefore

$$\alpha(\langle 012\cdots p\rangle) = 1_e \in M_p$$

by hypothesis on α since $p \leq n-1$ and the only $n-1$-atom of $\mathcal{O}^n$ not sent to 1_e by α is $d^0 = 12\cdots n$. As well, since $p > 0$, $q = n-1-p < n-1$ and so by hypothesis on β

$$\beta(p\, p+1 \cdots \overset{\vee}{2i+1} \cdots n) = \begin{cases} 1_e \in M_q & p < n-1 \\ e \in M_0 & p = n-1 \end{cases}$$

since if $p = n - 1$ then $q = 0$ and $\beta(\langle n-1 \rangle) = e$ by hypothesis on β. Hence Proposition 4.36 gives that for $p < 0$ and $\lceil {}^{p}/_{2} \rceil \leq i \leq \lfloor {}^{n-1}/_{2} \rfloor$

$$\alpha \otimes \beta(\langle 012 \cdots p \otimes p\; p+1 \cdots 2i \overset{\vee}{+} 1 \cdots n \rangle) = 1_{e \otimes e} \in \mathcal{O}^{n}_{n-1}$$

Now by Lemma 4.36 since α and β both send all m-cells of $\mathcal{O}^n$ for $m \leq n-2$ to the identity m-cell 1_e of M it follows that

$$\alpha \otimes \beta(\langle \theta \otimes \varphi \rangle) = 1_{e \otimes e}$$

for all $\theta : [m] \hookrightarrow [n]$ and $\varphi : [l] \hookrightarrow [n]$ injective non-decreasing maps of Δ with $m < n-1$ and $l < n-1$. Hence in particular $\alpha \otimes \beta$ sends all m-atoms of $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$ for $1 \leq m < n-1$ to $1_{e \otimes e}$, so by Lemma A.2 $\alpha \otimes \beta : \nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]) \to M \otimes M$ sends all m-cells of $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$ for $1 \leq m < n-1$ to $1_{e \otimes e}$.

Furthermore, Lemma A.2 allows us to translate the congruence expression (109) for $x^{0,n}$ modulo $(\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-2}$ directly into a composite of cells of $M \otimes M$ when $\alpha \otimes \beta$ is applied by replacing sums with $*_0$. By the previous discussion $\alpha \otimes \beta$ sends all the $n-1$ atoms in this congruence expression (109) to $1_{e \otimes e}$. Hence

$$\alpha \otimes \beta(x^{0,n}) = \alpha \otimes \beta(\langle 0 \otimes 012 \cdots n \rangle) = e \otimes \beta(\langle 1_{[n]} \rangle)$$

We will perform a very similar analysis of the image of $x^{n,0}$ under $\alpha \otimes \beta$, but in this case we can afford to be a little less precise and still obtain the result we want. Recall that

$$x^{n,0} \equiv \langle 012 \cdots n \otimes n \rangle \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-1}$$

If $n = 1$ then by Proposition 5.6 of [Ste04] $x^{0,1} = \langle 01 \otimes 1 \rangle$ and the identity (108) follows immediately. Otherwise by Proposition 5.2 (ii) of [Ste04] there exist $n-1$-atoms $\langle a_j \rangle$ of $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$ for $1 \leq j \leq r$ such that

$$x^{n,0} \equiv \langle 012 \cdots n \otimes n \rangle + \sum_{j=1}^{r} \langle a_j \rangle \mod (\mu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n]))_{n-2}$$

By Lemma A.2 therefore

$$\alpha \otimes \beta(x^{n,0}) = \alpha \otimes \beta(\langle 012 \cdots n \otimes n \rangle) *_0 \alpha \otimes \beta(\langle a_1 \rangle) *_0 \cdots *_0 \alpha \otimes \beta(\langle a_r \rangle)$$

Now

$$\alpha \otimes \beta(\langle 012 \cdots n \otimes n \rangle) = \alpha(\langle 012 \cdots n \rangle) \otimes \beta(\langle n \rangle) = \alpha(\langle 012 \cdots n \rangle) \otimes e$$

and so

$$\begin{aligned} d^{+}_{n-1}\alpha \otimes \beta(x^{0,n}) &= d^{-}_{n-1}\alpha \otimes \beta(x^{n,0}) \\ &= \big(d^{-}_{n-1}\alpha(\langle 012 \cdots n \rangle) \otimes e\big) *_0 d^{-}_{n-1}\left(\alpha \otimes \beta(\langle a_1 \rangle) *_0 \cdots *_0 \alpha \otimes \beta(\langle a_r \rangle)\right) \\ &= 1_{e \otimes e} *_0 d^{-}_{n-1}\left(\alpha \otimes \beta(\langle a_1 \rangle) *_0 \cdots *_0 \alpha \otimes \beta(\langle a_r \rangle)\right) \\ &= d^{-}_{n-1}\left(\alpha \otimes \beta(\langle a_1 \rangle) *_0 \cdots *_0 \alpha \otimes \beta(\langle a_r \rangle)\right) \\ &= \alpha \otimes \beta(\langle a_1 \rangle) *_0 \cdots *_0 \alpha \otimes \beta(\langle a_r \rangle) \end{aligned}$$

The last part of this identity holds because the atoms $\langle a_j \rangle$ are $n-1$-atoms of $\nu(\mathbb{Z}[\Delta^n] \otimes \mathbb{Z}[\Delta^n])$. The identity

$$d^{-}_{n-1}\alpha(\langle 012 \cdots n) \otimes e = 1_{e \otimes e}$$

holds because

$$d^-_{n-1}\langle 012\cdots n\rangle \equiv \sum_{i=0}^{\lfloor {}^{n-}/_2 \rfloor} \langle 01\cdots 2i \overset{\vee}{+} 1\cdots n\rangle \quad \mathrm{mod}\ (\mu(\mathbb{Z}[\Delta^n]\otimes\mathbb{Z}[\Delta^n]))_{n-2}$$

so by Lemma A.2

$$\alpha(d^-_{n-1}\langle 012\cdots n\rangle) = \alpha(\langle d^1\rangle) *_0 \cdots *_0 \langle d^{2\lfloor {}^{n-1}/_2 \rfloor+1}\rangle) = 1_e$$

since the only $n-1$-atom of $\nu\mathbb{Z}[\Delta^n]$ not sent to 1_e by α is $\langle d^0\rangle$. Hence

$$\begin{array}{rcl}
\alpha\otimes\beta(AW(C_\omega(\Delta)(\langle 1_{[n]}\rangle))) & = & \alpha\otimes\beta(x^{0,n}) *_{n-1} \alpha\otimes\beta(x^{n,0}) \\
 & = & \alpha\otimes\beta(x^{0,n}) *_{n-1} \left(1_{d^+_{n-1}(\alpha\otimes\beta(x^{0,n}))} *_0 \alpha(\langle 012\cdots n\rangle)\otimes e\right) \\
 & = & \left(\alpha\otimes\beta(x^{0,n}) *_0 1_e\right) *_{n-1} \left(1_{d^+_{n-1}(\alpha\otimes\beta(x^{0,n}))} *_0 \alpha(\langle 012\cdots n\rangle)\otimes e\right) \\
 & = & \alpha\otimes\beta(x^{0,n}) *_0 \alpha(\langle 012\cdots n\rangle)\otimes e \\
 & = & e\otimes\beta(\langle 012\cdots n\rangle) *_0 \alpha(\langle 012\cdots n\rangle)\otimes e
\end{array}$$

when $n > 1$. This completes the proof of the identity (108). □

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