UNIVERSITY OF CAMBRIDGE

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SUPERPROBABILITY ON GRAPHS

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Abstract

The classical random walk isomorphism theorems relate the local times of a continuous-time random walk to the square of a Gaussian free field. The Gaussian free field is a spin system (or sigma model) that takes values in Euclidean space; in this work, we generalise the classical isomorphism theorems to spin systems taking values in hyperbolic and spherical geometries. The corresponding random walks are no longer Markovian: they are the vertex-reinforced and vertex-diminished jump processes. We also investigate supersymmetric versions of these formulas, which give exact random walk representations.

The proofs are based on exploiting the continuous symmetries of the corresponding spin systems. The classical isomorphism theorems use the translation symmetry of Euclidean space, while in hyperbolic and spherical geometries the relevant symmetries are Lorentz boosts and rotations, respectively. These very short proofs are new even in the Euclidean case.

To illustrate the utility of these new isomorphism theorems, we present several applications. These include simple proofs of exponential decay for spin system correlations, exact formulas for the resolvents of the joint processes of random walks together with their local times, and a new derivation of the Sabot–Tarrès magic formula for the limiting local time of the vertex-reinforced jump process.

The second ingredient is a new Mermin–Wagner theorem for hyperbolic sigma models. This result is of intrinsic interest for the sigma models, and together with the aforementioned isomorphism theorems, implies our main theorem on the VRJP, namely, that it is recurrent in two dimensions for any translation invariant finite-range initial jump rates.

We also use supersymmetric hyperbolic sigma models to study the arboreal gas. This is a model of unrooted random forests on a graph, where the probability of a forest F with $|F|$ edges is multiplicatively weighted by a parameter $\beta^{|F|} > 0$. In simple terms, it can be defined to be Bernoulli bond percolation with parameter $p = \frac{\beta}{1+\beta}$ conditioned to be acyclic, or as the $q \to 0$ limit with $p = \beta q$ of the random cluster model.

It is known that on the complete graph K_N with $\beta = \alpha/N$ there is a phase transition similar to that of the Erdős–Rényi random graph: a giant tree percolates for $\alpha > 1$ and all trees have bounded size for $\alpha < 1$. In contrast to this, by exploiting an exact relationship with the hyperbolic sigma model, we show that the forest constraint is significant in two dimensions: trees do not percolate on $\mathbb{Z}^2$ for any finite $\beta > 0$. This result is again a consequence of our hyperbolic Mermin–Wagner theorem, and is used in conjunction with a version of the principle of dimensional reduction. To further illustrate our methods, we also give a spin-theoretic proof of the phase transition on the complete graph.

Preface

This thesis consists five chapters, which I have further grouped into two parts. The second part, entitled superprobability, consists of three papers. These are as follows:

- The geometry of random walk isomorphism theorems, *Ann. Inst. Henri Poincaré Probab. Stat.*, 57(1): 408-454, 2021, coauthored with Roland Bauerschmidt and Tyler Helmuth.
- Random spanning forests and hyperbolic symmetry, *Commun. Math. Phys.* 381 1223–1261, 2021, coauthored with Roland Bauerschmidt, Nicholas Crawford, and Tyler Helmuth.
- Dynkin isomorphism and Mermin–Wagner theorems for hyperbolic sigma models and recurrence of the two-dimensional vertex-reinforced jump process, *Ann. Probab.*, 47(5):3375–3396, 2019, coauthored with Roland Bauerschmidt and Tyler Helmuth.

These three can be read in any order, but I would recommend reading "The geometry of random walk isomorphism theorems" before the other two. The first part, entitled superanalysis, contains the necessary background material on supermathematics that is required for the second part. It is an expanded version of the appendix to [8], and as such, it can be used either as an introduction or an appendix to the second part, being referred back to as needed.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared above and specified in the text. It is not substantially the same as any that I have submitted, or is being concurrently submitted, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

Acknowledgements

Before we dive into mathematical matters, I thought I might take a brief moment to reflect upon my experience at Cambridge over the last five years, and to give my heart felt thanks to all those who have helped me along the way. And what a experience it has been! I have been so fortunate to have met so many wonderful people who have inspired me on both personal and professional levels.

Without doubt, I owe my greatest thanks to my supervisor, Roland Bauerschmidt. Roland has been a continuous source of guidance, support, and inspiration, and I am truly grateful for the time that he has devoted towards helping me develop as a mathematician. I could not have asked for a better supervisor.

Secondly, I would like to extend a very big thank you to both Tyler Helmuth and Nick Crawford for their enthusiastic collaboration which has ultimately lead to this thesis.

These last few years have reinforced my belief that the real joy in mathematics comes from sharing it with others, and on that note, there are two communities to which I would like to express my thanks.

The first, of course, is the Cambridge Centre for Analysis: Matthias, Erlend, Kweku, Maxime, Mo Dick, Nicolai, Fritz, Eardi, Lisa, Megan, (and Sam!), thank you all so much for such stimulating

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The second, is the 'superprobability' community (if I may call it that!): I would like to thank David Brydges, Gordon Slade, Tom Spencer, Persi Diaconis, Margarita Disertori, and particularly Christophe Sabot and Pierre Tarrès for sharing their thoughts and insights at various conferences and visits over the years. I would also like to thank Codina Cotar and Sergio Bacallado for taking the time to carefully examine this thesis, and for their helpful comments and suggestions which have surely improved the final product.

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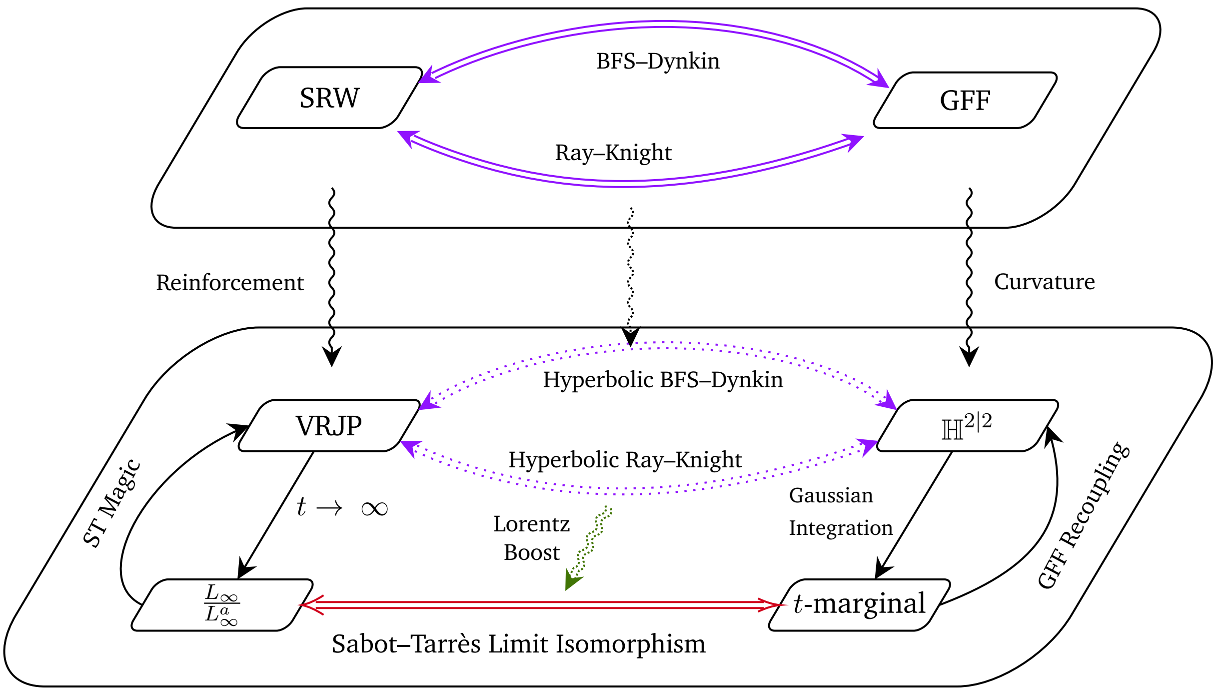
A huge thank you to my siblings, Maddie and Simon, and to my friends, James, Michael, Andy, Jim, Kal, Will, and Camilo, for all the joy you have shared with me over the years. A very special thank you to Miruna for all of her love, support, and encouragement, and for helping me carry on when things were tough.

Finally, I would like to thank Mr Ruthven for teaching me how to integrate by parts: despite my initial protests[1], it has indeed proven quite useful.



To Mum and Dad, of course.

[1]"Why can't I just integrate it like normal?"

Contents

Part I

SUPERANALYSIS

Chapter 1

Superalgebra and Supergeometry

1.1 Supervector Spaces

Supervector Spaces

A **supervector space** is a vector space which carries the additional structure of a $\mathbb{Z}_2$**-grading**, otherwise known as a **supergrading**. This means that it admits a decomposition into a direct sum of two vector spaces which are labelled by distinct elements of $\mathbb{Z}_2$,

$$V = V_0 \oplus V_1 = V_{\text{even}} \oplus V_{\text{odd}} \tag{1.1.1}$$

These subspaces are respectively called **even** and **odd**, or in physics parlance, **bosonic** and **fermionic**. Supervector spaces are equipped with natural inclusion $\iota_j : V_j \to V$ and projection maps $\pi_j : V \to V_j$ for each of their even and odd subspaces. By projecting, we can uniquely represent each element $u \in V$ as sum of its even and odd components:

$$u = u_0 + u_1 \tag{1.1.2}$$

where $u_0 \in V_0$ and $u_1 \in V_1$. The **parity** of a homogeneous element is defined as

$$\alpha(u) = \begin{cases} 0 \in \mathbb{Z}_2, & u \in V_0 \\ 1 \in \mathbb{Z}_2, & u \in V_1 \end{cases} \tag{1.1.3}$$

or simply $\alpha(u_i) = i$ when we have a parity subscript.

We will also use the incredibly useful **superbar** notation to indicate parity. The convention is that objects to the left/right of the bar should be considered even/odd. We will use the superbar in a variety of ways, for instance, rather than writing $V = V_0 \oplus V_1$, we will often write

$$V = V_0 | V_1 \tag{1.1.4}$$

or even

$$V = \frac{V_0}{V_1} \tag{1.1.5}$$

to indicate parity on the supervector space; in the second case we have used a vertical superbar, where even/odd objects appear above/below the bar. The superbar is particularly useful as it helps avoid proliferation of subscripts (which can be an issue with modules over superalgebras, as both the basis elements and coefficients are graded), and also in cases where explicit subscripts add to confusion instead of clarity; this is prone to occur whenever we wish to a consider an object that is normally considered even to be odd.

As a first example, we now define the standard **Cartesian supervector space** $\mathbb{R}^{p|q}$, where

$$\mathbb{R}^{p|q} := \mathbb{R}^p \,\big|\, \mathbb{R}^q. \tag{1.1.6}$$

An element of $v \in \mathbb{R}^{p|q}$, represented as a column vector, is given as

$$v = \left[\begin{array}{c} a_1 \\ \vdots \\ a_p \\ \hline b_1 \\ \vdots \\ b_q \end{array}\right] \tag{1.1.7}$$

where are $a_i, b_i \in \mathbb{R}$ ordinary real coefficients.

Tensor Products, Direct Sums, and the Kozsul Sign Rule

For V and W supervector spaces, their **direct sum** is simply the ordinary direct sum with grading

$$V \oplus W = V_0 \oplus W_0 \,\Big|\, V_1 \oplus W_1. \tag{1.1.8}$$

Their **tensor product**, is likewise defined as usual, but now carries the grading given by

$$\begin{aligned} V \otimes W &= (V \otimes W)_0 \,\Big|\, (V \otimes W)_1 \\ &= V_0 \otimes W_0 \oplus V_1 \otimes W_1 \,\Big|\, V_0 \otimes W_1 \oplus V_1 \otimes W_0. \end{aligned} \tag{1.1.9}$$

i.e., so that for a simple tensor we have $\alpha(v \otimes w) = \alpha(v) + \alpha(w)$, with the general case following by linearity. The tensor product is associative in the usual way,

$$\begin{gathered} (U \otimes V) \otimes W \cong U \otimes (V \otimes W) \\ (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \end{gathered} \tag{1.1.10}$$

but the braiding map which gives the natural isomorphism $V \otimes W \cong W \otimes V$, which sends $v \otimes w \mapsto w \otimes v$ in the ordinary case, is replaced by the **superbraiding**

$$\begin{gathered} \tau : V \otimes W \to W \otimes V \\ v \otimes w \mapsto (-1)^{\alpha(v)\alpha(w)} w \otimes v. \end{gathered} \tag{1.1.11}$$

This is known as the **Kozsul sign rule**, and it is this that distinguishes supervector spaces, superalgebras, etc. from those which are merely $\mathbb{Z}_2$-graded. It is for this reason that *super* is the preferred terminology.

Remark 1.1.1. Many constructions in linear superalgebra are defined on homogeneous elements and extended to the entire space by linearity; we will often omit this final step, as we have above, and simply define expressions in terms of homogeneous elements, with the understanding that one can linearly extend as required. Furthermore, as a general rule, if the parity of an object appears in an expression, we assume that that object is homogeneous.

Linear and Multilinear Maps

Let $V = V_0|V_1$ and $W = W_0|W_1$ be two supervector spaces, and let $T : V \to W$ be a linear map between them. By composing with the inclusion and projection operations,

$$V_i \xrightarrow{\iota_i} V \xrightarrow{T} W \xrightarrow{\pi_j} W_j, \tag{1.1.12}$$

we obtain linear maps defined on the subspaces $T_{ij} : V_j \to W_i$, with parity defined by

$$\alpha(T_{ij}) = \alpha(V_j) + \alpha(W_i) = i + j. \tag{1.1.13}$$

Taking direct sums over maps with the same parity allows us to decompose T into its even and odd parts,

$$\begin{aligned} T &= T_{00} \oplus T_{11} \,\Big|\, T_{01} \oplus T_{10} \\ &= T_0 \,\Big|\, T_1. \end{aligned} \tag{1.1.14}$$

Diagrammatically, we see that T_0 and T_1 are naturally represented by ribbons; even maps are flat whilst odd maps have a half twist:

$$T = \begin{array}{c} V_0 \\ \hline V_1 \end{array} \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{c} W_0 \\ \hline W_1 \end{array} \Bigg| \begin{array}{c} V_0 \\ \hline V_1 \end{array} \begin{array}{c} \searrow\!\!\!\nearrow \\ \nearrow\!\!\!\searrow \end{array} \begin{array}{c} W_0 \\ \hline W_1 \end{array} \tag{1.1.15}$$

This ribbon decomposition a induces a supervector space structure on the **space of linear maps** as

$$L(V, W) = L_{\text{flat}}(V, W) \,\Big|\, L_{\text{twist}}(V, W). \tag{1.1.16}$$

Remark 1.1.2. By using both the vertical and horizontal superbars, we obtain a useful representation of linear maps as block operators:

$$T = \left[\begin{array}{c|c} T_{00} & T_{01} \\ \hline T_{10} & T_{11} \end{array}\right] \tag{1.1.17}$$

The effect of the bars is $\mathbb{Z}_2$-additive, giving T a chequerboard structure of even and odd.

Homomorphisms. Supervector space **homomorphisms** are required to preserve the $\mathbb{Z}_2$-grading in addition to the linear structure, and hence correspond to only the even linear maps:

$$\operatorname{Hom}(V, W) := L_{\text{flat}}(V, W). \tag{1.1.18}$$

The importance of the odd maps will become clear when we examine supersymmetry.

Multilinear Maps. Multilinear maps are also graded. If $T : V^{(1)} \times \cdots \times V^{(n)} \to W$ is a multilinear map with subspace maps

$$T^i_{j_1 \ldots j_n} : V_{j_1} \times \cdots \times V_{j_n} \to W_i, \tag{1.1.19}$$

we define

$$\alpha(T^j_{i_1 \ldots i_n}) := \alpha(W_j) + \sum_k \alpha(V_{i_k}) \tag{1.1.20}$$

and set $T = T_0 + T_1$ with

$$\begin{aligned} T_0 &:= \bigoplus_{\alpha(T_I)=0} T_I, \\ T_1 &:= \bigoplus_{\alpha(T_I)=1} T_I. \end{aligned} \tag{1.1.21}$$

Multilinear maps out of products of supervector spaces correspond to linear maps out of their associated tensor product spaces, just like the usual case.

Example 1.1.3. The tensor product, considered as a bilinear map

$$\begin{aligned} \otimes : V \times W &\to V \otimes W \\ (u, v) &\mapsto u \otimes v \end{aligned} \tag{1.1.22}$$

is even, and satisfies the expected universal property: for any bilinear map $T : V \times W \to U$, there exists a unique linear map of the same parity $\widetilde{T} : V \otimes W \to U$ such that $T = \widetilde{T} \circ \otimes$.

Parity Reversal

The parity reversal of a supervector space $V = V_0|V_1$ is defined as the supervector space with the same subspace decomposition as V, but with opposite labelling:

$$\Pi V = V_1|V_0 \tag{1.1.23}$$

The parity reversal operator Π, thought of as a map $\Pi : V \to \Pi V$, satisfies the universal property that for any even/odd map $T : V \to W$, there exists a unique map of the opposite parity $\widetilde{T} : \Pi V \to W$ such that $T = \widetilde{T} \circ \Pi$. We also have a universal property in the opposite direction, namely, that for any even/odd map $S : W \to V$, we have a unique map of the opposite parity $\widetilde{S} : W \to \Pi V$ such that $\widetilde{S} = \Pi \circ S$.

The parity reversal operator thus allows us to represent odd maps in terms of even maps into (or out of) a parity reversed space, i.e.,

$$L_{\text{twist}}(V, W) \simeq \mathrm{Hom}(\Pi V, W) \simeq \mathrm{Hom}(V, \Pi W). \tag{1.1.24}$$

This is easily seen by untwisting the ribbon:

$$\begin{tikzcd} \underline{V_0} \arrow[dr] & \underline{W_0} & \simeq & \underline{V_1} \arrow[r] & \underline{W_0} & \simeq & \underline{V_0} \arrow[r] & \underline{W_1} \\ V_1 \arrow[ur] & W_1 & & V_0 \arrow[r] & W_1 & & V_1 \arrow[r] & W_0 \end{tikzcd} \tag{1.1.25}$$

Dimension and Bases

The **dimension** of a supervector space $V = V_0|V_1$ is defined as the pair of integers

$$\dim(V) = p|q \tag{1.1.26}$$

where $\dim(V_0) = p$ and $\dim(V_1) = q$ are the usual dimensions of the even and odd subspaces. The natural bases of supervector spaces are likewise graded, decomposing as $B = B_0|B_1$ where B_0 is a basis for V_0 and B_1 is a basis for V_1. Choosing a basis $e_1, \ldots, e_p|\varepsilon_1, \ldots \varepsilon_q$ for a finite dimensional supervector space gives an isomorphism with $\mathbb{R}^{p|q}$.

Dual Space

The **dual space** $V^* = V_0^*|V_1^*$ of a supervector space V is defined as the space of even and odd linear maps from $V \to \mathbb{R}^{1|0}$ with the ribbon grading:

$$V^* = L_{\text{flat}}(V, \mathbb{R}) \,\Big|\, L_{\text{twist}}(V, \mathbb{R}). \tag{1.1.27}$$

Equivalently, this is

$$V^* = \mathrm{Hom}(V_0, \mathbb{R}) \,\Big|\, \mathrm{Hom}(V_1, \Pi\mathbb{R}) \tag{1.1.28}$$

Choosing a basis $e_1, \ldots, e_n \,|\, \varepsilon_1, \ldots, \varepsilon_m$ on V then defines a dual basis $x^1, \ldots, x^n \,|\, \xi^1, \ldots, \xi^m$ on V^* where

$$x^i(e_j) = \xi^i(\varepsilon_j) = \delta^i_j. \tag{1.1.29}$$

Example 1.1.4. Let V and W be supervector spaces. Then

$$L(V, W) \simeq W \otimes V^*. \tag{1.1.30}$$

1.2 Superalgebras

A **superalgebra** is a supervector space $A = A_0 \oplus A_1$ equipped with an even bilinear multiplication operation

$$\begin{aligned} m : A \times A &\to A \\ (u, v) &\mapsto m(u, v) \end{aligned} \tag{1.2.1}$$

This means that for homogeneous elements $u, v \in A$ we have

$$\alpha(m(u, v)) = \alpha(u) + \alpha(v). \tag{1.2.2}$$

We will often leave the multiplication map implicit, and simply write uv. Just as for regular algebras, superalgebras may be unital or associative. However, as a consequence of the Koszul braiding, the natural notion of commutativity is replaced by **supercommutativity**: in a supercommutative algebra A, we have for all homogeneous elements u, v

$$uv = (-1)^{\alpha(u)\alpha(v)} vu. \tag{1.2.3}$$

This relation implies that elements in A_0 commute with all others, whereas elements in A_1 anti-commute with each other. In particular, each odd element anti-commutes with itself, and therefore squares to zero.

Supercommutative superalgebras form one of the two important classes of superalgebras, the other being Lie superalgebras. For now, we defer the discussion of Lie superalgebras, and will focus on supercommutative superalgebras. These arise as algebras of 'functions' on superspaces, and will play the role of observables of our spin systems.

In the following, we only consider supercommutative superalgebras that are associative and unital.

Homomorphisms and Superideals. All of the other concepts and constructions surrounding ordinary algebras carry over to the super case too. For superalgebra homomorphisms and ideals, the only differences are a result of the grading compatibility requirements: homomorphisms between superalgebras correspond to even linear maps

$$T : A \to B \tag{1.2.4}$$

that commute with multiplication

$$T(uv) = T(u)T(v). \tag{1.2.5}$$

If A and B have units, then we further require $T(1_A) = 1_B$.

The natural ideals of a superalgebra, **superideals**, are defined as ordinary ideals with the additional requirement that they are $\mathbb{Z}_2$-graded subspaces, splitting as[1]:

$$I = I_0 \oplus I_1, \qquad I_i = I \cap A_i. \tag{1.2.6}$$

Superideals come in the three standard flavours of left, right, and two-sided, all of which coincide on supercommutative superalgebras. Two-sided superideals correspond to the kernels of superalgebra homomorphisms, and can be used to construct quotient superalgebras following the usual procedure.

The (left) ideal generated by a collection of elements

$$(u_1, \dots, u_k)_L := \{a_1 u_1 + \cdots + a_k u_k \,|\, a_i \in A\} \tag{1.2.7}$$

generates a (left) superideal if and only if each u_i is of definite parity; as such, we often write $(u_1, \dots, u_n \,|\, \psi_1, \dots, \psi_m)_L$. The same is true for right and two-sided ideals; these are written as $(u_1, \dots, u_k)_R$ and $(u_1, \dots, u_k)$, and likewise have their usual definitions. We will never need to consider a non-super ideal in a superalgebra, so we just write ideal.

[1] One may want to check that this is non-trivial!

Tensor Products. The tensor product of two superalgebras $A \otimes B$ has a natural superalgebra structure with the induced multiplication map $m_{A\otimes B} : A \otimes B \times A \otimes B \to A \otimes B$,

$$m_{A\otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) = (-1)^{|b_1||a_2|} m_A(a_1, a_2) \otimes m_B(b_1, b_2) \tag{1.2.8}$$

where m_A and m_B are the multiplication maps of A and B. The minus factor again results from the braiding.

When A and B are both supercommutative, we have natural inclusion maps $i_A : A \to A \otimes B$, $i_B : B \to A \otimes B$, which send

$$\begin{aligned} a &\mapsto a \otimes 1 \\ b &\mapsto 1 \otimes b \end{aligned} \tag{1.2.9}$$

In the supercommutative case, it convenient to use the braiding $a \otimes b \mapsto (-1)^{\alpha(a)\alpha(b)} b \otimes a$ to 'incorporate' the tensor product into the algebra product, allowing the elements of a and b to freely supercommute. With this convention, we see that the rather complicated looking expression above is actually very simple.

$$a_1 b_1 a_2 b_2 = (-1)^{|b_1||a_2|} a_1 a_2 b_1 b_2. \tag{1.2.10}$$

The inclusion map is now totally trivial: $i_A(a) = a$, $i_B(b) = b$, with the images considered as elements of $A \otimes B$. Following the usual arguments, we find homomorphisms out of $A \otimes B$ are exactly pairs of homomorphisms (T_A, T_B) out of A and B.

Grassmann Algebras

The archetypal supercommutative superalgebras are the **Grassmann algebras**. These are thought of as polynomial rings in odd variables, which anti-commute as a result of the sign rule. Formally, we construct the real Grassmann algebra in N odd variables $\xi_1, \ldots, \xi_N$, by taking a quotient of the free associative algebra $\mathbb{R}\langle \xi_1, \ldots, \xi_N \rangle$ by the ideal generated by the $\frac{N(N+1)}{2}$ anti-commutation relations

$$\xi_i \xi_j = -\xi_j \xi_i, \qquad 1 \le i \le j \le N. \tag{1.2.11}$$

We denote the Grassmann algebra in N variables as Ω^N, or sometimes in the polynomial ring style $\mathbb{R}[\xi_1, \ldots, \xi_N]$.

Like their commutative cousins, the Grassmann algebras are spanned by monomials, but the anti-commutativity imposes a major difference: Grassmann algebras are finite dimensional, as any monomial with a square term ξ_i^2 is automatically zero. A general element $P \in \Omega^N$ is then a linear combination of square free monomials[2] $\xi_I = \xi_{i_1} \xi_{i_2} \ldots \xi_{i_k}$

$$P = \sum_{I \subseteq \{1,\ldots,N\}} p_I \xi_I, \qquad p_I \in \mathbb{R}. \tag{1.2.12}$$

This also shows that $\dim(\Omega^N) = 2^N$.

As a superalgebra, Ω^N splits as $\Omega^N_{\text{even}} \oplus \Omega^N_{\text{odd}}$ with the even/odd parts being spanned by monomials with an even/odd number of terms, i.e., $\alpha(\xi_I) = |I| \mod 2$. It is easy to see that the product respects the grading, and furthermore, is supercommutative: for monomials ξ_{I_1}, ξ_{I_2}, one can check (say, by pulling variables through one at a time) that

$$\xi_{I_1} \xi_{I_2} = (-1)^{\alpha(\xi_I)\alpha(\xi_I)} \xi_{I_2} \xi_{I_1}, \tag{1.2.13}$$

which implies supercommutativity for the entire algebra by linearity.

[2] It is sometimes said that a fixed ordering must be chosen on the Grassmann variables to obtain a valid Grassmann polynomial due to the sign ambiguity, but this is only necessary if we require a unique representation. Ordinary polynomials are really no different in this regard, just as $5x_1x_2 + x_3$ and $5x_2x_1 + x_3$ are two different representations of the same polynomial, $5\xi_1\xi_2 + \xi_3$ and $-5\xi_2\xi_1 + \xi_3$ are two different representations of the same Grassmann polynomial.

Grassmann extensions

The importance of the Grassmann algebras stems from their ability to turn any commutative algebra into a supercommutative one through the tensor product: if A is any commutative algebra, then $A \otimes \Omega^N = A \otimes \Omega_0^N \oplus A \otimes \Omega_1^N$ is a supercommutative one[3]. A general element of $A \otimes \Omega^N$ can be written as

$$F = \sum_{I \subseteq \{1,\dots,N\}} f_I \xi_I, \qquad f_I \in A. \tag{1.2.14}$$

As we will need to make frequent use of this construction, we call $A \otimes \Omega^N$ a **Grassmann extension** of A.

Example 1.2.1. The algebra of smooth differential forms on $\mathbb{R}^N$ can be represented as the Grassmann extension $C^\infty(\mathbb{R}^N) \otimes \Lambda(\mathbb{R}^N)$. Here, $\Lambda(\mathbb{R}^N) \simeq \mathbb{R}[dx_1, \dots, dx_N]$ is the Grassmann algebra generated by the coordinate differentials $dx_1, \dots, dx_N$ together with the wedge product

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \tag{1.2.15}$$

and a general differential form $F \in C^\infty(\mathbb{R}^N) \otimes \Lambda(\mathbb{R}^N)$ is written as

$$F = \sum_{I \subseteq \{1,\dots,N\}} f_I(x_1, \dots, x_N) dx_{i_1} \wedge \dots \wedge dx_{i_p}, \qquad f_I \in C^\infty(\mathbb{R}^N). \tag{1.2.16}$$

The coefficient functions commute with the differentials $f_i dx_j = dx_j f_i$ and each other $f_i f_j = f_j f_i$.

1.3 Superfunction algebras

A running theme across much of mathematics is the idea that algebra is dual to geometry. For instance, maps between smooth manifolds can equally be seen as homomorphisms between their algebras of functions, but in the opposite direction. This particular result is known as "Milnor's exercise", or **smooth Gelfand duality**:

Theorem 1.3.1. *For any two smooth manifolds M and N, there is a natural bijection between smooth functions*

$$\varphi : M \to N \tag{1.3.1}$$

and algebra homomorphisms in the opposite direction

$$\varphi^* : C^\infty(N) \to C^\infty(M) \tag{1.3.2}$$

where $C^\infty(-) := C^\infty(- \to \mathbb{R})$.

In the super world, this theme can be taken as definition: smooth supergeometry is *defined* as the dual to smooth superalgebra. The basic object of study on the algebraic side is the **smooth superfunction algebra** $C^\infty(\mathbb{R}^{n|m})$.

[3]It is interesting to interpret this idea as $\Omega^N(A) := A \otimes \Omega^N$

Definition 1.3.2 (Smooth Superfunction Algebra)**.** The smooth superfunctions algebra $C^\infty(\mathbb{R}^{n|m})$ is defined as

$$C^\infty(\mathbb{R}^{n|m}) := C^\infty(\mathbb{R}^n) \otimes \Omega^m, \tag{1.3.3}$$

It's even and odd subspaces are then

$$\begin{aligned} C^\infty_{\text{even}}(\mathbb{R}^{n|m}) &= C^\infty(\mathbb{R}^n) \otimes \Omega^m_0 \\ C^\infty_{\text{odd}}(\mathbb{R}^{n|m}) &= C^\infty(\mathbb{R}^n) \otimes \Omega^m_1 \end{aligned} \tag{1.3.4}$$

Its elements $F \in C^\infty(\mathbb{R}^{n|m})$ are called **superfunctions**, and are represented in terms of **Cartesian supercoordinates** $x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m$ as

$$F(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m) = \sum_{I \subseteq [\![m]\!]} f_I(x_1, \ldots, x_n)\xi_I \tag{1.3.5}$$

where the $f_I \in C^\infty(\mathbb{R}^n)$ are smooth functions and $\xi_I \in \Omega^m$ are Grassmann monomials. Even superfunctions are written with lowercase latin letters; odd superfunctions are written with lowercase greek letters.

As the notation suggests, we will think of $C^\infty(\mathbb{R}^{n|m})$ as the algebra of smooth functions on $\mathbb{R}^{n|m}$, but for technical reasons, can no longer think of it as the a set of points of a supervector space. Instead, we view it as a **formal superspace**, defined through duality. Before we look at the spaces themselves, we first explain how to compose superfunctions.

Grassmann Analytic Continuation

As superfunctions are not functions with a domain and codomain in a traditional sense, their composition must be defined in a different manner with a procedure known as **Grassmann analytic continuation**. The idea is as follows: consider a function $g \in C^\infty(\mathbb{R})$, and an even superfunction $f \in C^\infty(\mathbb{R}^{1|2})$, written in Cartesian coordinates as

$$\begin{aligned} g &= g(y) \\ f &= f(x \,|\, \xi_1, \xi_2) = f_b(x) + f_1(x)\xi_1\xi_2 \end{aligned} \tag{1.3.6}$$

Setting $y = f(x \,|\, \xi_1, \xi_2)$, we obtain an apparently ill-formed composite function $h = g \circ f$,

$$h(x \,|\, \xi_1, \xi_2) = g(f_b(x) + f_1(x)\xi_1\xi_2), \tag{1.3.7}$$

as we are now evaluating a smooth function on an expression involving Grassmann variables. However, if we exploit the fact that $g(y)$ can equivalently be represented using Taylor's theorem as

$$g(y) = g(y_b) + g'(y_b)(y - y_b) + r(y_b, y)(y - y_b)^2 \tag{1.3.8}$$

and only now substitute $y = f_b(x) + f_1(x)\xi_1\xi_2$ with $y_b = f_b(x)$, $y - y_b = f_1(x)\xi_1\xi_2$,

$$\begin{aligned} g(f_b + f_1\xi_1\xi_2) &= g(f_b) + g'(f_b)f_1\xi_1\xi_2 + r(f_b, f)(f_1\xi_1\xi_2)^2 \\ &= g(f_b) + g'(f_b)f_1\xi_1\xi_2, \end{aligned} \tag{1.3.9}$$

we see that the remainder term r is annihilated by $(\xi_1\xi_2)^2 = 0$, and we are left with a well defined expression containing composition only with respect to ordinary smooth functions.

Grassmann analytic continuation is performed relative to the *body* and *soul* of a superfunction, as above. For F a smooth superfunction in $C^\infty(\mathbb{R}^{n|m})$, with

$$F(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m) = \sum_{I \subseteq [\![m]\!]} f_I(x_1, \ldots, x_n)\xi_{i_p}, \tag{1.3.10}$$

the **body** of F is defined as the ordinary smooth function obtained by formally setting all Grassmann variables to zero

$$F_b = F(x_1, \ldots, x_n \,|\, 0, \ldots, 0) = f_\emptyset(x_1, \ldots, x_n). \tag{1.3.11}$$

The remaining part is referred to as the **soul**:

$$F_s = F - F_b = \sum_{I \subseteq [\![m]\!], I \neq \emptyset} f_I(x_1, \ldots, x_n)\xi_I, \tag{1.3.12}$$

Souls are always nilpotent, and for $F \in C^\infty(\mathbb{R}^{n|m})$, the nilpotency is of order less than m, i.e., $F_s^m = 0$. More generally, collections of souls are mutually nilpotent: for $F_s = (F_s^1 \ldots F_s^k)$ any collection of k souls, and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ a multi-index with $|\alpha| = a_1 + \cdots + \alpha_k \geq m$, we have

$$F_s^\alpha = (F_s^1)^{\alpha_1} \ldots (F_s^k)^{\alpha_k} = 0 \tag{1.3.13}$$

Using the body and soul decomposition $F = F_b + F_s$, we generalise the simple example above to the composition of $g \in C^\infty(\mathbb{R}^k)$ with a collection of k even superfunctions $f^1, \ldots, f^k \in C^\infty_{\text{even}}(\mathbb{R}^{n|m})$. Let $f_b^1, \ldots, f_b^k$ be their real valued bodies, and let

$$g^{(\alpha)}(x) := \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_k^{\alpha_k}} g(x), \qquad x^\alpha := x_1^{\alpha_1} \cdots x_k^{\alpha_k}. \tag{1.3.14}$$

Then, using Taylor's theorem, $g(x)$ can be represented as

$$g(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} g^{(\alpha)}(y)(x-y)^\alpha + \sum_{|\alpha| = m} r_\alpha(x,y)(x-y)^\alpha \tag{1.3.15}$$

where r_α are smooth remainder functions. Substituting $x \mapsto f = (f^1, \ldots, f^k)$ and expanding around around the body $f_b = (f_b^1, \ldots, f_b^k)$ therefore gives

$$g(f) = \sum_{|\alpha| < \infty} \frac{1}{\alpha!} g^{(\alpha)}(f_b)(f - f_b)^\alpha = \sum_{|\alpha| < \infty} \frac{1}{\alpha!} g^{(\alpha)}(f_b) f_s^\alpha \tag{1.3.16}$$

where the expansion is finite due to the nilpotence of the soul.

The procedure is simpler for odd Grassmann analytic continuation, as we can substitute directly: if $\eta_1, \ldots, \eta_k \in C^\infty_{\text{odd}}(\mathbb{R}^{n|m})$, $\eta_i = \eta_i(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m)$ are any odd superfunctions, and $G \in C^\infty(\mathbb{R}^{0|k})$ with

$$G(\psi_1, \ldots, \psi_1) = \sum_{I \subseteq [\![k]\!]} g_I \psi_I \tag{1.3.17}$$

then

$$G(\eta_1, \ldots, \eta_k) = \sum_{I \subseteq [\![k]\!]} g_I \eta_I \tag{1.3.18}$$

where

$$\eta_I = \eta_{i_1}(x \,|\, \xi)\eta_{i_2}(x \,|\, \xi) \ldots \eta_{i_p}(x \,|\, \xi) \tag{1.3.19}$$

if $\xi_I = \xi_{i_1}\xi_{i_2} \ldots \xi_{i_p}$. Combining the even and odd cases gives Grassmann analytic continuation in the general case:

Definition 1.3.3. Let $f = (f_1, \ldots, f_n)$, $f_i \in C^\infty_{\text{even}}(\mathbb{R}^{p|q})$, and $\eta = (\eta_1, \ldots, \eta_m)$ $\eta_i \in C^\infty_{\text{odd}}(\mathbb{R}^{p|q})$, be a collection of even/odd superfunctions, and let $G \in C^\infty(\mathbb{R}^{n|m})$,

$$G(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m) = \sum_{I \subseteq [\![m]\!]} g_I(x)\xi_I. \tag{1.3.20}$$

Then the composition $G(f \,|\, \eta) \in C^\infty(\mathbb{R}^{p|q})$ is defined as

$$G(f \,|\, \eta) := \sum_{I \subseteq [\![m]\!]} \left(\sum_{|\alpha| < \infty} \frac{1}{\alpha!} g_I^{(\alpha)}(f_b)(f - f_b)^\alpha \right) \eta_I. \tag{1.3.21}$$

Functions and algebra homomorphisms

Every smooth function $f \in C^\infty(\mathbb{R}^n)$

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ (x_1, \ldots, x_n) &\mapsto f(x_1, \ldots, x_n) \end{aligned} \tag{1.3.22}$$

defines an algebra homomorphism in the opposite direction

$$f^* : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^n) \tag{1.3.23}$$

by sending $g \in C^\infty(\mathbb{R})$ to its composition with f, i.e.,

$$f^*[g] = g(f(x_1, \ldots, x_n)). \tag{1.3.24}$$

Inverting this idea allows us to think of superfunctions as functions between formal superspaces. Every even superfunction $f \in C^\infty_{\text{even}}(\mathbb{R}^{n|m})$ defines a map

$$f^* : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^{n|m}) \tag{1.3.25}$$

which is again defined using composition, but now interpreted in the Grassmann analytic sense:

$$f^*[g] = g(f(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m)) \tag{1.3.26}$$

Furthermore, this map is a superalgebra homomorphism. Clearly, f^* preserves degree, and is unital and linear

$$f^*[1] = 1(f) = 1, \qquad f^*[c_1 g + c_2 h] = c_1 f^*[g] + c_2 f^*[h], \tag{1.3.27}$$

so one need only check f^* commutes with multiplication. This follows from the $C^\infty(\mathbb{R}^{n|2})$ case: letting $f = y(\boldsymbol{x}) + q(\boldsymbol{x})\xi_1\xi_2$, we have

$$\begin{aligned} f^*[gh] &= [gh](y) + [gh]'(y) f_s \\ &= g(y)h(y) + (g'(y)h(y) + g(y)h'(y)) f_s \\ &= \big(g(y) + g'(y) f_s\big)\big(h(y) + h'(y) f_s\big) \\ &= f^*[g] f^*[h] \end{aligned} \tag{1.3.28}$$

where we have used that $f_s^2 = 0$ on the third line. Likewise, for every odd superfunction $\eta \in C^\infty_{\text{odd}}(\mathbb{R}^{n|m})$ we have a superalgebra homomorphism

$$\eta^* : C^\infty(\mathbb{R}^{0|1}) \to C^\infty(\mathbb{R}^{n|m}) \tag{1.3.29}$$

defined by odd Grassmann analytic continuation. Compared to the even case, this is rather trivial as a general element $\psi \in C^\infty_{\text{odd}}(\mathbb{R}^{0|1})$ is of the form $\psi(\xi) = a\xi$, which gives

$$\eta^*[\psi] = \psi(\eta) = a\eta(x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m). \tag{1.3.30}$$

As above, η^* degree preserving, unital and linear, and trivially commutes with multiplication

$$\eta^*[\psi_1]\eta^*[\psi_2] = a_1 a_2 \eta^2 = 0 = \eta^*[0] = \eta^*[\psi_1\psi_2], \tag{1.3.31}$$

so it does indeed define a superalgebra homomorphism.

Superfunctions thus define superalgebra homomorphisms in exactly the same way as ordinary functions, and we should therefore think of them as 'maps' between **formal superspaces**:

$$\begin{aligned} f &: \mathbb{R}^{n|m} \to \mathbb{R}^{1|0} \\ \eta &: \mathbb{R}^{n|m} \to \mathbb{R}^{0|1} \end{aligned} \tag{1.3.32}$$

acting as

$$\begin{aligned} (x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) &\mapsto f(x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) \\ (x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) &\mapsto \eta(x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) \end{aligned} \tag{1.3.33}$$

in the even/odd cases. More generally, just as how a map

$$f : \mathbb{R}^n \to \mathbb{R}^m \tag{1.3.34}$$

is represented in coordinates as an m-tuple of smooth functions $f = (f_1, \dots, f_m)$, with $f_i \in C^\infty(\mathbb{R}^n)$,

$$(x_1, \dots, x_n) \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)), \tag{1.3.35}$$

a 'map' between two superspaces

$$F : \mathbb{R}^{n|m} \mapsto \mathbb{R}^{p|q} \tag{1.3.36}$$

is represented in coordinates by a $p|q$-tuple of smooth superfunctions $F = (f_1, \dots, f_p | \eta_1, \dots, \eta_q)$, with $f_i \in C^\infty_{\text{even}}(\mathbb{R}^{n|m})$, $\eta_i \in C^\infty_{\text{odd}}(\mathbb{R}^{n|m})$, so that

$$(x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) \mapsto \big(f_1(x|\xi), \dots, f_p(x|\xi) \,\big|\, \eta_1(x|\xi), \dots, \eta_q(x|\xi)\big). \tag{1.3.37}$$

On the algebraic side, every such map defines a superalgebra homomorphism in the reverse direction

$$F^* : C^\infty(\mathbb{R}^{p|q}) \to C^\infty(\mathbb{R}^{n|m}), \tag{1.3.38}$$

again defined using Grassmann analytic continuation:

$$\begin{aligned} F^*[g] &= g\big(f_1(x|\xi), \dots, f_p(x|\xi) \big|\, \eta_1(x|\xi), \dots, \eta_q(x|\xi)\big), \\ F^*[\psi] &= \psi\big(f_1(x|\xi), \dots, f_p(x|\xi) \big|\, \eta_1(x|\xi), \dots, \eta_q(x|\xi)\big). \end{aligned} \tag{1.3.39}$$

Under their interpretation, superfunctions are composed using Grassmann analytic continuation; this is of course compatible with their reversed composition as superalgebra homomorphisms, as this is also defined using Grassmann analytic continuation, i.e., $(G \circ F)^* = F^* \circ G^*$.

Remark 1.3.4. In fact, every superalgebra homomorphism $\phi : C^\infty(\mathbb{R}^{p|q}) \to C^\infty(\mathbb{R}^{n|m})$ is represented by a superfunction $F : \mathbb{R}^{n|m} \to \mathbb{R}^{p|q}$, so that $\phi = F^*$ for some F.

Coordinate Transforms

Every automorphism of $C^\infty(\mathbb{R}^{n|m})$

$$F^* : C^\infty(\mathbb{R}^{n|m}) \to C^\infty(\mathbb{R}^{n|m}) \tag{1.3.40}$$

defines a **coordinate transformation**. Geometrically, this corresponds to an invertible map

$$\begin{gathered} F : \mathbb{R}^{n|m} \to \mathbb{R}^{n|m} \\ (x'_1, \dots, x'_n \,|\, \xi'_1, \dots, \xi'_m) \mapsto (f_1, \dots, f_n \,|\, \psi_1, \cdots \psi_m) \end{gathered} \tag{1.3.41}$$

which represents the old coordinate functions (x, ξ) in terms of the new

$$x_i = f_i(x', \xi') \quad \xi_i = \psi_i(x', \xi'), \tag{1.3.42}$$

or, more concisely as $x|\xi = F(x'|\xi')$. The corresponding homomorphism acts on superfunctions as $G(x, \xi) \mapsto F^*[G] = G(F(x', \xi'))$. As the sense of a coordinate transforms follows the algebraic rather than the geometric direction, we write $x \,|\, \xi \mapsto x' \,|\, \xi'$, following the ordinary convention.

Example 1.3.5. Changes of coordinates do not need to preserve the bodies of superfunctions. For instance, the change of coordinates $(x \,|\xi, \eta) \mapsto (x' \,|\, \xi', \eta')$ given by

$$\begin{aligned} x &= x' + \xi'\eta' \\ \eta &= \eta' \\ \xi &= \xi' \end{aligned} \tag{1.3.43}$$

is coordinate transform on $\mathbb{R}^{1|2}$. The inverse transform is simply

$$\begin{aligned} x' &= x - \xi\eta \\ \eta' &= \eta \\ \xi' &= \xi \end{aligned} \tag{1.3.44}$$

Both of these maps define superalgebra homomorphisms on $C^\infty(\mathbb{R}^{1|2})$,

$$\begin{aligned} G(x \,|\, \xi, \eta) &\mapsto G(x' + \xi'\eta' \,|\, \xi', \eta') \\ F(x' \,|\xi', \eta') &\mapsto F(x - \xi\eta \,|\, \xi, \eta) \end{aligned} \tag{1.3.45}$$

and are clearly inverses of each one another in this sense too, and hence are automorphisms.

Supermanifolds

More generally, supermanifolds are defined using the following principle:

Theorem 1.3.6. *Every smooth manifold M can be smoothly embedded into a Cartesian space $\mathbb{R}^n$ of large enough dimension:*

$$i : M \to \mathbb{R}^n \tag{1.3.46}$$

Dually, this defines a surjective algebra homomorphism

$$i^* : C^\infty(\mathbb{R}^n) \to C^\infty(M). \tag{1.3.47}$$

Furthermore, this embedding can be realised as the joint zero locus of a finite collection of functions $(f_1, \dots, f_k) \in C^\infty(\mathbb{R}^n)$, so that

$$M \simeq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f_1(\boldsymbol{x}) = \cdots = f_k(\boldsymbol{x}) = 0\}. \tag{1.3.48}$$

Algebraically, this states that

$$C^\infty(M) \simeq C^\infty(\mathbb{R}^n)/(f_1, \dots, f_k) \tag{1.3.49}$$

i.e., the ideal defined by $\ker(i^*)$ is generated by $f_1, \dots, f_k$.

We take the super-analogue of this statement as a definition:

Definition 1.3.7. A superalgebra $C^\infty(M)$ is the algebra of functions on a formal superspace M if it is isomorphic to a quotient superalgebra of the form

$$C^\infty(M) \simeq C^\infty(\mathbb{R}^{n|m})/(f_1, \ldots, f_p \,|\, \rho_1, \ldots, \rho_q), \tag{1.3.50}$$

where $(f_1, \ldots, f_p \,|\, \rho_1, \ldots, \rho_q)$ is a finitely generated ideal.

Functions between superspaces are defined as follows:

Definition 1.3.8. A smooth function between two formal superspaces M and N

$$f : M \to N \tag{1.3.51}$$

is defined as a superalgebra homomorphism

$$f^* : C^\infty(N) \to C^\infty(M) \tag{1.3.52}$$

where $C^\infty(N)$ and $C^\infty(M)$ are superalgebras of the form (1.3.50).

Remark 1.3.9. The superspaces defined according to the above definition can be quite singular, and are closer to schemes than manifolds. For instance, $C^\infty(\mathbb{R})/(x^2)$ is not the algebra of functions on any manifold, although it does arise as the algebra of global sections of a scheme. Later we will give conditions on what sorts of ideals $(f_1, \ldots, f_p \,|\, \rho_1, \ldots, \rho_q)$ give rise to supermanifolds, which are superspaces that are 'locally isomorphic' to $\mathbb{R}^{p|q}$. But from the algebraic perspective, having the extra flexibility can be quite useful, so we retain it.

Example 1.3.10. Let x, y, z be Cartesian coordinates on $\mathbb{R}^3$. Then the algebra of functions on the sphere $\mathbb{S}^2$ is isomorphic to the quotient algebra

$$C^\infty(\mathbb{S}^2) \simeq C^\infty(\mathbb{R}^3)/(x^2 + y^2 + z^2 - 1), \tag{1.3.53}$$

as $\mathbb{S}^2$ can be represented as the zero locus of $x^2 + y^2 + z^2 - 1$. In analogous way, the algebra of functions on the **supersphere** $\mathbb{S}^{2|2}$ is isomorphic to

$$C^\infty(\mathbb{S}^{2|2}) \simeq C^\infty(\mathbb{R}^{3|2})/(x^2 + y^2 + z^2 - 2\xi\eta - 1), \tag{1.3.54}$$

where $x^2 + y^2 + z^2 - 2\xi\eta - 1$ is the zero locus representing the supersphere, written here in Cartesian supercoordinates $x, y, z \,|\, \xi, \eta$ on $\mathbb{R}^{3|2}$. The superfunction $x^2 + y^2 + z^2 - 2\xi\eta$ is known as the super-Euclidean quadratic form; we discuss this in further detail in section 1.6.

The algebra of functions on the **hyperbolic superspace** $\mathbb{H}^{2|2}$ is isomorphic to the quotient algebra

$$C^\infty(\mathbb{H}^{2|2}) \simeq C^\infty(\mathbb{R}^{3|2})/(x^2 + y^2 - z^2 - 2\xi\eta + 1, 1^\varepsilon_{z>0} - 1) \tag{1.3.55}$$

where $x^2 + y^2 - z^2 - 2\xi\eta$ is now the Minkowski quadratic form on $\mathbb{R}^{3|2}$, and $1_{z>0}$ is a smoothed indicator function which picks out the upper of the two branches of the hyperboloid. As we can explicitly solve $z = \sqrt{1 + x^2 + y^2 - 2\xi\eta}$, we can write a general function on $\mathbb{H}^{2|2}$ as

$$F(x, y, z \,|\, \xi, \eta) \mapsto F(x, y, \sqrt{1 + x^2 + y^2 - 2\xi\eta} \,|\, \xi, \eta) = \tilde{F}(x, y \,|\, \xi, \eta), \tag{1.3.56}$$

so we have an isomorphism $C^\infty(\mathbb{H}^{2|2}) \simeq C^\infty(\mathbb{R}^{2|2})$.

1.4 Further remarks on superspaces

Points in superspace

It worth making a few remarks about why superspaces cannot be regarded as spaces in the ordinary sense. The fundamental difference between superspaces and ordinary ones is that maps between superspaces are not uniquely determined by their values on the 'points' of the space.

The set of points underlying an ordinary manifold is equivalently realised as the set of maps from an external point into the space. Taking, say, $\mathbb{R}^n$, a point $p \in \mathbb{R}^n$ can be thought of as a map

$$\begin{aligned} p : \bullet &\to \mathbb{R}^n \\ \bullet &\mapsto (c_1, \ldots, c_n) \end{aligned} \tag{1.4.1}$$

which sends a point to its coordinates $(c_1, \ldots, c_n)$ in the space. To be clear, $(c_1, \ldots, c_n)$ is an n-tuple of real numbers, each of which can be thought of as a constant function $c_i \in C^\infty(\bullet) \simeq \mathbb{R}$. Dually, points in a space correspond to algebra homomorphisms in the other direction

$$\begin{aligned} p^* : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f(x_1, \ldots, x_n) &\mapsto f(c_1, \ldots, c_n), \end{aligned} \tag{1.4.2}$$

corresponding to the evaluation of a function on the point itself, and clearly, if we know the value of f for all $(c_1, \ldots, c_n) \in \mathbb{R}^n$, then we know f.

Pulling this idea over to the super case, we encounter a problem: a point again corresponds to $\mathbb{R}^{0|0}$ (as there are no variables), and its set of smooth functions $\mathbb{C}^\infty(\mathbb{R}^0)$ is again isomorphic to $\mathbb{R}$, and so the set of points $p \in \mathbb{R}^{n|m}$ is defined as the set of superalgebra homomorphisms

$$p^* : C^\infty(\mathbb{R}^{n|m}) \to \mathbb{R} \tag{1.4.3}$$

as these are dual to formal maps $p : \bullet \to \mathbb{R}^{n|m}$. However, because superalgebra homomorphisms are required to preserve parity[4], all odd superfunctions must map to zero as $\mathbb{R}$ has no odd component. Hence, a general homomorphism is of the form

$$p^*(F) = F(c_1, \ldots, c_n \,|\, 0, \ldots, 0). \tag{1.4.4}$$

By knowing the value of a superfunction $F \in C^\infty(\mathbb{R}^{n|m})$ on all points $p \in \mathbb{R}^{n|m}$, we are able to reconstruct the body $F_b \in C^\infty(\mathbb{R}^n)$ of F, but are unable to say anything about its soul. In this sense, souls are ephemeral as they cannot be seen directly: superfunctions, when evaluated on points, are indistinguishable from their bodies.

Visualising Superspaces

When dealing with geometric objects, it is often helpful to have a picture in mind. As the set of ordinary points in a superspace is naturally equivalent to the set of points of its body, we can imagine $\mathbb{R}^{n|m}$ looking like ordinary $\mathbb{R}^n$, but with a certain 'aura' indicating the presence of the soul. This is sometimes described as a 'Grassmann cloud' or 'fuzz', but I find it better to think of it as a sort of metallic shimmer on the space; in my view, this better reflects the rigidity of the Grassmann directions.

[4]Even removing the degree preservation requirement (like we did to obtain odd linear maps) does not resolve the issue: the image of $p^*[\xi]$ must always equal zero as if $p^*[\xi] = c$, then

$$c^2 = p^*[\xi]^2 = p^*[\xi^2] = p^*[0] = 0,$$

which is impossible for $c \neq 0$.

Body and Soul

Previously, we have discussed 'body and soul' in the context of Grassmann analytic continuation, but it is useful to understand how the concept fits into the larger picture. Our original formulation for $C^\infty(\mathbb{R}^{n|m})$, although correct, paints a somewhat misleading picture as the decomposition of a superfunction into its body and soul does not truly exist in an absolute sense, but only in a relative one.

Definition 1.4.1. The **soul** of a superalgebra $A_{\text{even}} \oplus A_{\text{odd}}$ is the ideal generated by the entire odd subspace of A:

$$A_{\text{soul}} = (A_{\text{odd}}) \tag{1.4.5}$$

The **body** of A is then defined as the quotient algebra

$$A_{\text{body}} = A/A_{\text{soul}}. \tag{1.4.6}$$

We denote the associated superalgebra homomorphism by

$$i^* : A \to A_{\text{body}}. \tag{1.4.7}$$

Remark 1.4.2. The body of a superalgebra is an ordinary algebra, and if A is supercommutative, then A_{body} is commutative. The notation $i^* : A \to |A|$ is again here to suggest that we think of this algebraic quotient as something that is induced by a geometric inclusion $i : |M| \to M$ in the opposite direction.

Here are some examples of superalgebra bodies:

- The body of the Grassmann algebra Ω^N is $|\Omega^N| \simeq \mathbb{R}$. This is also true for any quotient algebra of of a Grassmann algebra $|\Omega^N/I| \simeq \mathbb{R}$.
- If $A = B \otimes \Omega^N$ is the Grassmann extension of a commutative algebra B, then $|A| \simeq B$.
- In particular, the body of the smooth superfunction algebra

$$\left|C^\infty(\mathbb{R}^{n|m})\right| = C^\infty(\mathbb{R}^{n|m})/(\xi_1, \ldots, \xi_n) = C^\infty(\mathbb{R}^n) \tag{1.4.8}$$

  is the ordinary algebra of smooth functions on $\mathbb{R}^n$. It is useful to think of this in a geometric fashion as

$$\left|C^\infty(\mathbb{R}^{n|m})\right| = C^\infty(|\mathbb{R}^{n|m}|) = C^\infty(\mathbb{R}^n), \tag{1.4.9}$$

  so that the body of $\mathbb{R}^{n|m}$ is $\mathbb{R}^n$, and that $C^\infty(-)$ preserves this relation.

This last example motivates the following definition:

Definition 1.4.3. The **body** of a superspace M is the formal space $|M|$ associated to the ordinary algebra of functions $C^\infty(|M|) := C^\infty_{\text{body}}(M)$. The geometric inclusion is denoted,

$$i : |M| \to M \tag{1.4.10}$$

and is defined as dual to the algebraic quotient map $i^* : C^\infty(M) \to C^\infty_{\text{body}}(M)$.

When a superalgebra $A = |A| \otimes \Omega^N$ is defined as a Grassmann extension of a commutative algebra $|A|$, we have a natural embedding of the body back into the superalgebra

$$p^* : |A| \to |A| \otimes \Omega^N, \tag{1.4.11}$$

where $p^* = i_{|A|}$ is the natural inclusion associated with the tensor product (again, now thought of as the opposite of a geometric projection $p : M \to |M|$). By composing with the quotient map $i^* : |A| \otimes \Omega^N \to |A|$, we obtain the **body projection**

$$P_{\text{body}} = p^* \circ i^* : B \otimes \Omega^N \to B \to B \otimes \Omega^N \tag{1.4.12}$$

and, taking its complement, the **soul projection**:

$$P_{\text{soul}} = I - P_{\text{body}}. \tag{1.4.13}$$

The inclusion of the body back into a superfunction algebra designates a subspace of superfunctions as ordinary real valued functions, much in the same way as how the inclusion of the reals into any algebra $i_{\mathbb{R}} : \mathbb{R} \to A$ indicates which elements are real numbers, that is $\mathrm{im}(i_{\mathbb{R}}) \simeq \mathbb{R}$. For the smooth superfunction algebra $C^\infty(\mathbb{R}^{n|m}) = C^\infty(\mathbb{R}^n) \otimes \Omega^m$, the body and soul projections give the previously described decomposition of a superfunction

$$F(x_1, \dots, x_n \,|\, \xi_1, \dots, \xi_m) = \sum_{I \subseteq [\![m]\!]} f_I(x_1, \dots, x_n)\xi_I \tag{1.4.14}$$

into its body and soul $F = F_{\text{b}} + F_{\text{s}}$, where again

$$\begin{aligned} F_b &= P_{\text{body}}(F) = F(x_1, \dots, x_n \,|\, 0, \dots, 0) = f_\emptyset(x_1, \dots, x_n) \\ F_{\text{s}} &= P_{\text{soul}}(F) = F - F_b = \sum_{I \subseteq [\![m]\!], I \neq \emptyset} f_I(x_1, \dots, x_n)\xi_I. \end{aligned} \tag{1.4.15}$$

This gives a decomposition of $C^\infty(\mathbb{R}^{n|m})$ as

$$C^\infty(\mathbb{R}^{n|m}) = C^\infty_{\text{body}}(\mathbb{R}^{n|m}) \oplus C^\infty_{\text{soul}}(\mathbb{R}^{n|m}). \tag{1.4.16}$$

However, for a general superalgebra, there is *no natural embedding*

$$p^* : |A| \to A, \tag{1.4.17}$$

and hence *no natural splitting* into body and soul $A = A_{\text{body}} \oplus A_{\text{soul}}$. In particular, for a superfunction $F \in C^\infty(M)$ on a superspace M, there is no absolute sense in which we have a decomposition

$$F = F_b + F_s \tag{1.4.18}$$

into a 'real valued' body and nilpotent soul. It is only after choosing a particular embedding do we obtain a splitting. This is true even for **linear superspaces** V, as although we have

$$C^\infty(V) \simeq C^\infty(\mathbb{R}^{n|m}), \tag{1.4.19}$$

we still have to choose a *specific* isomorphism in order to identify them.

Remark 1.4.4. The body of a superspace sits inside it, but the soul is free to 'tilt' around it.

1.5 Supermodules and Supermatrices

Vector spaces generalise to modules by replacing the ground field of scalars by an arbitrary ring or algebra. A **left module** over an algebra A is a vector space M equipped with a bilinear map from $A \times M \to M$, $(a, m) \mapsto am$ called left multiplication, such that $(ab)m = a(bm)$ and $em = m$. A **right module** over A is similarly defined, but multiplication taking place on the right $M \times A \to M$, $(m, a) \mapsto ma$. If M is both a left and right A-module such that for all $m \in M$, $a_1, a_2 \in A$, $a_1(ma_2) = (a_1 m)a_2$, then M is called a **bimodule**. Any left module over a commutative algebra can be considered as a bimodule by defining the right multiplication as $ma := am$ (and vice-versa for right modules).

When A is a superalgebra, the natural class of modules are **supermodules**, which are (left or right) modules equipped with a $\mathbb{Z}_2$-grading

$$M = M_0 \oplus M_1 \tag{1.5.1}$$

that is compatible with the superalgebra structure

$$\begin{aligned} A_i M_j &\subseteq M_{i+j} \quad \text{(left supermodule)} \\ M_j A_i &\subseteq M_{i+j} \quad \text{(right supermodule)}. \end{aligned} \tag{1.5.2}$$

Supermodules over supercommutative algebras can be considered as **superbimodules** by setting

$$A_i M_j = (-1)^{ij} M_j A_i. \tag{1.5.3}$$

As we are only interested in this case, we will just say supermodule.

Remark 1.5.1. By considering $\mathbb{R}$ as a purely even superalgebra, the definition of a supervector space is identical to an $\mathbb{R}$-superbimodule. Note that unlike supervector spaces (but like superalgebras), supermodules do not in general split into a direct sum of ordinary modules because multiplication by odd algebra elements exchanges parity.

Free Supermodules

A supermodule over a superalgebra $A = A_0 \oplus A_1$ is called **free** if it can be written in the form

$$M = M_0 \oplus M_1 = (A_0^p \times A_1^q)_0 \oplus (A_1^p \times A_0^q)_1 = A^{p|q}. \tag{1.5.4}$$

We define the **rank** of such a supermodule as the superpair of natural numbers $p|q$. Free supermodules admit a $\mathbb{Z}_2$-graded basis

$$e_1, \ldots, e_p \in M_0 \qquad \varepsilon_1, \ldots, \varepsilon_q \in M_1 \tag{1.5.5}$$

with the e_i labelled even and the ε_i labelled odd. Using this basis, every element $v \in M$ is uniquely expressible as a linear combination $v = v_0 + v_1$,

$$\begin{aligned} v_0 &= \sum_{i=1}^{p} a_i e_i + \sum_{i=1}^{q} \alpha_i \varepsilon_i \\ v_1 &= \sum_{i=1}^{p} \alpha_i e_i + \sum_{i=1}^{q} a_i \varepsilon_i \end{aligned} \tag{1.5.6}$$

with $v_i \in M_i$, $a_i \in A_0$, $\alpha_i \in A_1$. When M is a superbimodule, the basis elements satisfy the supercommutativity relations

$$\begin{aligned} a e_i &= e_i a \quad & \alpha e_i &= e_i \alpha \\ a \varepsilon_i &= \varepsilon_i a \quad & \alpha \varepsilon_i &= -\varepsilon_i \alpha \end{aligned} \tag{1.5.7}$$

Supermatrix Algebras

Linear maps between free supermodules over the same algebra are most easily described by supermatrices. A **supermatrix** is a 2×2 block matrix

$$R = \begin{array}{c} \\ p\{ \\ q\{ \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{r} & \overbrace{\qquad}^{s} \end{array} \\ \left[\begin{array}{c|c} R_{00} & R_{01} \\ \hline R_{10} & R_{11} \end{array} \right] \end{array} \tag{1.5.8}$$

with entries in a supercommutative superalgebra. The dimension of a supermatrix is denoted $p|q \times r|s$ where the dimension of each block is indicated above, and we denote the collection of all such matrices over a superalgebra A by $\mathrm{Mat}_{p|q,r|s}(A)$. Under the usual matrix addition and scalar multiplication $\mathrm{Mat}_{p|q,r|s}(A)$ is a supervector space with grading $R = R_0 + R_1$,

$$R_0 = \left[\begin{array}{c|c} \text{even} & \text{odd} \\ \hline \text{odd} & \text{even} \end{array}\right], \qquad R_1 = \left[\begin{array}{c|c} \text{odd} & \text{even} \\ \hline \text{even} & \text{odd} \end{array}\right] \tag{1.5.9}$$

where even/odd indicates the blocks of homogeneous elements. Furthermore, as we can multiply matrices by elements of the algebra on the left

$$(F, R) \mapsto FR = \left[\begin{array}{c|c} FR_{00} & FR_{01} \\ \hline (-1)^{\alpha(F)} FR_{10} & (-1)^{\alpha(F)} FR_{11} \end{array}\right] \tag{1.5.10}$$

and on the right

$$(R, F) \mapsto RF = \left[\begin{array}{c|c} R_{00}F & (-1)^{\alpha(F)} R_{01}F \\ \hline R_{10}F & (-1)^{\alpha(F)} R_{11}F \end{array}\right] \tag{1.5.11}$$

in a fashion compatible with the grading and supercommutativity

$$\begin{aligned} \alpha(FR) = \alpha(RF) = \alpha(F) + \alpha(R), \\ FR = (-1)^{\alpha(F)\alpha(R)} RF, \end{aligned} \tag{1.5.12}$$

$\mathrm{Mat}_{p|q,r|s}(A)$ carries the structure of a superbimodule over A. Notice that the signs are different from ordinary matrix-scalar multiplication because the have implicitly used the graded basis with the coefficients 'in the middle':

$$\begin{aligned} R = &\sum_{i,j} e_i (R_{00})^i_j e^j + \sum_{i,j} e_i (R_{01})^i_j \varepsilon^j \\ &\sum_{i,j} \varepsilon_i (R_{10})^i_j e^j + \sum_{i,j} \varepsilon_i (R_{11})^i_j \varepsilon^j \end{aligned} \tag{1.5.13}$$

Here, the $e_i, e^j, \varepsilon_i, \varepsilon^j$ are even/odd basis vectors/covectors and which contract as $e^i(e_j) = e_j(e^i) = \delta^i_j$ and $\varepsilon^i(\varepsilon_j) = -\varepsilon_j(\varepsilon^i) = \delta^i_j$, and these supercommute with elements of A.

Supermatrix Operations

Here we introduce the three fundamental matrix operations: the supertranspose, supertrace, and the superdeterminant. The **supertranspose** of a supermatrix is defined as

$$R^{ST} = \left[\begin{array}{c|c} R^T_{00} & (-1)^{\alpha(R)} R^T_{10} \\ \hline -(-1)^{\alpha(R)} R^T_{01} & R^T_{11} \end{array}\right], \tag{1.5.14}$$

where R^T_{ij} is the usual matrix transpose, and satisfies[5]

$$(QR)^{ST} = (-1)^{\alpha(Q)\alpha(R)} R^{ST} Q^{ST}. \tag{1.5.15}$$

The **supertrace** is defined for square supermatrices, and is given by

$$\mathrm{str}(R) = \mathrm{tr}(R_{00}) - (-1)^{\alpha(R)} \mathrm{tr}(R_{11}). \tag{1.5.16}$$

[5]Interestingly, the supertranspose is not an involution and requires 4 repeated applications to return to the initial state (it behaves somewhat like the Fourier transform, or a spinor, in this regard).

Like the usual trace, it is linear

$$\mathrm{str}(aQ+bR)=a\,\mathrm{str}(Q)+b\,\mathrm{str}(R), \tag{1.5.17}$$

and satisfies the graded cyclic condition

$$\mathrm{str}(QR)=(-1)^{\alpha(Q)\alpha(R)}\,\mathrm{str}(RQ). \tag{1.5.18}$$

Finally, the **superdeterminant**, which is only defined for even invertible supermatrices, is given by the Schur complement[6] style expression

$$\mathrm{sdet} R := \det(R_{00}-R_{01}R_{11}^{-1}R_{10})\det(R_{11})^{-1}. \tag{1.5.19}$$

This satisfies

$$\mathrm{sdet}(QR)=\mathrm{sdet}(Q)\mathrm{sdet}(R) \tag{1.5.20}$$

1.6 Superspins

Let us now refer to the collection of Cartesian coordinates as a **superspin**, which we denote with the shorthand $v=(x_1,\ldots,x_n\,|\,\xi_1,\ldots,\xi_m)$. For the reasons discussed above, superspins do not correspond to the topological points of $\mathbb{R}^{n|m}$, however, it is extremely useful to pretend that they are, as it allows us to think of $\mathbb{R}^{n|m}$ as a supervector space. For instance, the formal map

$$\begin{aligned} + : \mathbb{R}^{n|m}\times\mathbb{R}^{n|m} &\to \mathbb{R}^{n|m} \\ (u,v) &\mapsto u+v \end{aligned} \tag{1.6.1}$$

where $u=(x\,|\,\xi)$, $v=(y\,|\,\eta)$ and $u+v=(x+y\,|\,\xi+\eta)$ is a well defined map of superspaces, as it corresponds to the superalgebra homomorphism

$$\begin{aligned} +^* : C^\infty(\mathbb{R}^{n|m}) \to C^\infty(\mathbb{R}^{n|m}\times\mathbb{R}^{n|m}) \simeq C^\infty(\mathbb{R}^{2n|2m}) \\ +^*[F]=F(x_1+y_1,\ldots,x_n+y_n\,|\,\xi_1+\eta_1,\ldots,\xi_m+\eta_m), \end{aligned} \tag{1.6.2}$$

or, more compactly, $+^*[F]=F(u+v)$. Using superspins, the superspace $\mathbb{R}^{n|m}$ has a 'supervector space structure', as we furthermore have a scalar multiplication map,

$$\begin{aligned} \mathbb{R}\times\mathbb{R}^{n|m} &\to \mathbb{R}^{n|m} \\ (\lambda,u) &\mapsto \lambda u=(\lambda x_1,\ldots,\lambda x_n\,|\,\lambda\xi_1,\ldots,\lambda\xi_m) \end{aligned} \tag{1.6.3}$$

which is compatible with the addition operation in the usual way; here λ should be interpreted as the coordinate function on $\mathbb{R}$, which now plays the role of a scalar. Each of the vector space axioms now corresponds to a particular commutative diagram: for instance, associativity of addition $(u+v)+w=u+(v+w)$ corresponds to

$$\begin{tikzcd} \mathbb{R}^{n|m}\times\mathbb{R}^{n|m}\times\mathbb{R}^{n|m} \arrow[r, "(+,\mathrm{id})"] \arrow[d, "(\mathrm{id},+)"] & \mathbb{R}^{n|m}\times\mathbb{R}^{n|m} \arrow[d, "+"] \\ \mathbb{R}^{n|m}\times\mathbb{R}^{n|m} \arrow[r, "+"] & \mathbb{R}^{n|m} \end{tikzcd} \tag{1.6.4}$$

Writing out all such diagrams is a tedious distraction, so we will not do it. But the essence is that superspins can be manipulated in exactly the same way as supervectors.

[6] For comparison, the Schur complement formula for the determinant of an ordinary matrix R is

$$\det R=\det(R_{00}-R_{01}R_{11}^{-1}R_{10})\det(R_{11})$$

Bold Notation

Let $\Lambda = \{1, 2, \ldots, N\}$ denote a finite set. Given a copy of $\mathbb{R}^{n|m}$, let us write

$$(\mathbb{R}^{n|m})^{\Lambda} := \prod_{i\in\Lambda} \mathbb{R}^{n|m} \simeq \mathbb{R}^{Nn|Nm} \tag{1.6.5}$$

to denote a Λ-indexed product. Given coordinates $(x \,|\, \xi)$ on $\mathbb{R}^{n|m}$, we denote the associated coordinates on $(\mathbb{R}^{n|m})^{\Lambda}$ with an additional index $(x_a \,|\, \xi_a)_{a\in\Lambda}$. Thinking of coordinates as superspins, $(\mathbb{R}^{n|m})^{\Lambda}$ is the configuration space for a collection of $|\Lambda|$ superspins $(u_a)_{a\in\Lambda} := (x_a \,|\, \xi_a)_{a\in\Lambda}$. To avoid an explosion of sub/superscripts, we use bolded terms to denote collections of objects indexed by Λ. Thus, we write

$$\boldsymbol{u} := (u_a)_{a\in\Lambda} \tag{1.6.6}$$

to denote the collection of all superspins, $\boldsymbol{x}^i := (x^i_a)_{a\in\Lambda}$ to denote the collection of all i-th coordinates functions etc. A superfunction on $(\mathbb{R}^{n|m})^{\Lambda}$ is then written in compact form as

$$F(\boldsymbol{u}) := F(x^1_1, \ldots, x^n_1, \ldots, x^1_N, \ldots, x^n_N \,|\, \xi^1_1, \ldots, \xi^m_1, \ldots, \xi^1_N, \ldots, \xi^m_N). \tag{1.6.7}$$

We sometimes use primes for the same effect when Λ is small, say writing $F(u, u')$ on $\mathbb{R}^{n|m} \times \mathbb{R}^{n|m}$ as we have above.

Norms and Forms

An **even bilinear form** on $\mathbb{R}^{n|m}$ is superfunction

$$B : \mathbb{R}^{n|m} \times \mathbb{R}^{n|m} \to \mathbb{R} \tag{1.6.8}$$

which is bilinear with respect to superspin addition

$$\begin{aligned} B(u+v, w) &= B(u, w) + B(v, u) \\ B(u, v+w) &= B(u, v) + B(u, w) \end{aligned} \tag{1.6.9}$$

and scalar multiplication

$$B(\lambda u, v) = \lambda B(u, v) = B(u, \lambda v). \tag{1.6.10}$$

The bilinearity implies that every bilinear form is represented by a quadratic superpolynomial of the form

$$B(u, u') = \sum_{i,j=1}^{n} A_{ij} x_i x'_j + \sum_{i,j=1}^{m} D_{ij} \xi_i \xi'_j. \tag{1.6.11}$$

The real coefficients A_{ij}, D_{ij} are conveniently represented by a real valued, even supermatrix,

$$B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] \tag{1.6.12}$$

which we can identify with the bilinear form itself. Here, A is an $n \times n$ matrix, and D is an $m \times m$ matrix[7].

[7] The off diagonal entries are zero because we are considering even bilinear forms. Allowing the off diagonal entries to be non-zero, and taking $A = D = 0$ gives an odd bilinear form, and would correspond to a superfunction into $\mathbb{R}^{0|1}$.

Orthosymplectic Forms

We are interested in the special case of **orthosymplectic forms**. These are even bilinear forms that are symmetric in the superspins,

$$B(u,v) = B(v,u) \tag{1.6.13}$$

and are non-degenerate

$$\det(A) \neq 0, \quad \det D \neq 0. \tag{1.6.14}$$

The anti-commutativity interacts with the symmetry condition in an interesting way. Expanding out the symmetry condition $B(u,u') - B(u',u) = 0$,

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij} x_i x'_j + \sum_{i,j=1}^{m} D_{ij} \xi_i \xi'_j - \sum_{i,j=1}^{n} A_{ij} x'_i x_j - \sum_{i,j=1}^{m} D_{ij} \xi'_i \xi_j \\
&= \sum_{i,j=1}^{n} A_{ij} x_i x'_j + \sum_{i,j=1}^{m} D_{ij} \xi_i \xi'_j - \sum_{i,j=1}^{n} A_{ij} x_j x'_i + \sum_{i,j=1}^{m} D_{ij} \xi_j \xi'_i \\
&= \sum_{i,j=1}^{n} (A_{ij} - A_{ji}) x_i x'_j + \sum_{i,j=1}^{m} (D_{ij} + D_{ji}) \xi_i \xi'_j = 0,
\end{aligned} \tag{1.6.15}$$

we see that A is required to be a symmetric matrix (as would be usual for a symmetric bilinear form), but D is required to be skew-symmetric due to the anti-commutativity of the Grassmann variables. Together with the non-degeneracy condition, this implies that A is represented by a (possibly Lorentzian) inner product, whilst D is represented by a symplectic form. The non-degeneracy of the symplectic form implies that the fermionic dimension of our superspace must be even, so only the Cartesian superspaces $\mathbb{R}^{n|2m}$ admit an orthosymplectic structure.

By performing a linear change of basis, every orthosymplectic form can be represented as

$$B = \left[\begin{array}{cc|cc} -I_p & 0 & 0 & 0 \\ 0 & I_{n-p} & 0 & 0 \\ \hline 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{array}\right] \tag{1.6.16}$$

where p is the number of 'time-like' dimensions defined by A. We will be interested in two cases: $p = 0$, which corresponds to the **super-Euclidean inner product**, and $p = 1$, corresponding to the **super-Minkowski inner product**. We will mostly denote these in a dot product style as

$$\begin{aligned}
u \cdot u' &= \sum_{i=1}^{n} x_i x'_i - \sum_{i=1}^{m} \xi_i \eta'_i - \eta_i \xi'_i \qquad \text{(Euclidean)} \\
u \cdot u' &= -zz' + \sum_{i=1}^{n} x_i x'_i - \sum_{i=1}^{m} \xi_i \eta'_i - \eta_i \xi'_i \qquad \text{(Minkowski)}
\end{aligned} \tag{1.6.17}$$

but will sometimes write $\langle u, u' \rangle$ or (u, u') to denote these, just as we would an ordinary inner product. When equipped with the appropriate inner product, we call $\mathbb{R}^{n|2m}$ a **Euclidean superspace**, and call $\mathbb{R}^{n,1|2m}$ a **Minkowski superspace**; we have used coordinates $(x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m)$ on $\mathbb{R}^{n|2m}$, and $(z, x_1, \ldots, x_n \,|\, \xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_m)$ on $\mathbb{R}^{n,1|2m}$.

In both cases, we have written the fermionic coordinates in terms of 'symplectic conjugates', so that ξ_i is paired with η_i for $i = 1, \ldots, m$; on Minkowski superspace, the time-like coordinate z will usually appear as the first entry of the superspin as above, but will sometimes appear at the end if it is convenient. In any case, we reserve the variable z explicitly for this purpose. Further, we opt for the 'mostly plus' metric signature of gravitational physics rather than the 'mostly minus' signature of particle physics.

Quadratic Forms and Energy Functionals

Every orthosymplectic form B gives rise to a quadratic form on $\mathbb{R}^{n|2m}$, represented as a superfunction

$$q_B : \mathbb{R}^{n|2m} \to \mathbb{R} \tag{1.6.18}$$

with $q_B(u) = B(u,u)$. For the Euclidean and Minkowski products, the corresponding quadratic forms are denoted

$$\begin{aligned} ||u||^2_{\text{Euc}} &= \sum_i x_i^2 - \sum_{i=1}^m 2\xi_i\eta_i, \\ ||u||^2_{\text{Min}} &= -z^2 + \sum_i x_i^2 - \sum_{i=1}^m 2\xi_i\eta_i. \end{aligned} \tag{1.6.19}$$

or just by $||u||^2$ when the context is clear. Here, we are tempted to take a square root in order to define a 'metric superfunction' $d(u,u') = \sqrt{||u-u'||^2}$. Sadly, this is not quite possible due to the lack of smoothness at the origin; on $\mathbb{R}^{1|2}$, we would for instance have

$$d(u,u') = \sqrt{(x-x')^2 - 2(\xi-\xi')(\eta-\eta')} = |x-x'| - \frac{(\xi-\xi')(\eta-\eta')}{|x-x'|}, \tag{1.6.20}$$

which is not a well defined map of superspaces. On the other hand, as long as we do not take the square root, there is no issue in defining

$$\begin{aligned} ||u-u'||^2_{\text{Euc}} &= \sum_i (x_i - x_i')^2 - \sum_{i=1}^m 2(\xi_i - \xi_i')(\eta_i - \eta_i') \\ ||u-u'||^2_{\text{Min}} &= -(z-z')^2 + \sum_i (x_i - x_i')^2 - \sum_{i=1}^m 2(\xi_i - \xi_i')(\eta_i - \eta_i'). \end{aligned} \tag{1.6.21}$$

The utility of these 'quadratic metrics' is that they allow us to define natural energy functionals on collections of superspins, that is, on spin systems. In all models we examine, the pair interaction between spins u_a, u_b is given by

$$H(u_a, u_b) = \frac{\beta_{ab}}{2} ||u_a - u_b||^2 \tag{1.6.22}$$

for some constant β_{ab}. The interpretation here is that the two superspins are thought of as connected by a spring with stiffness β_{ab}, with the Hamiltonian functional above defining Hooke's' law (or a sort of Minkowskian analogue).

Chapter 2

Supercalculus and Supersymmetries

2.1 Derivatives and Derivations

On a smooth manifold M, functions are differentiated by vector fields through the Lie derivative. This associates to every vector field $X \in \mathrm{Vect}(M)$ a **derivation** on $C^\infty(M)$, which we recall is a linear map

$$T_X \colon C^\infty(M) \to C^\infty(M) \tag{2.1.1}$$

obeying the Leibniz rule

$$T_X(fg) = T_X(f)g + fT_X(g). \tag{2.1.2}$$

Concretely, if M is n-dimensional and X is represented in local coordinates as

$$X = \sum_{j=1}^{n} g_j(x_1, \ldots, x_n) \frac{\partial}{\partial x_j}, \tag{2.1.3}$$

then its associated derivation acts on functions as

$$T_X(f) = \sum_{j=1}^{n} g_j(x_1, \ldots, x_n) \frac{\partial f}{\partial x_j}. \tag{2.1.4}$$

In fact, every derivation on $C^\infty(M)$ arises from a vector field giving a natural isomorphism $\mathrm{Vect}(M) \simeq \mathrm{Der}(C^\infty(M))$. Thus, we can replace geometric objects (vector fields) with algebraic objects (derivations). The perspective is useful for superspaces, as their definition is fundamentally algebraic rather than geometric.

Superderivations

Let $A = A_0 \oplus A_1$ be a supercommutative superalgebra. Recall that a linear map $T \colon A \to A$ is written in block form as

$$T = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix}, \tag{2.1.5}$$

and is even if $T_{01} = T_{10} = 0$, and odd if $T_{00} = T_{11} = 0$. A homogeneous linear map is one that is even or odd, and the parity map is again denoted

$$\alpha(T) = \begin{cases} 0 \in \mathbb{Z}_2, & T \text{ is even} \\ 1 \in \mathbb{Z}_2, & T \text{ is odd} \end{cases}, \tag{2.1.6}$$

and for homogeneous $F \in A$ we have $\alpha(TF) = \alpha(T) + \alpha(F)$. A **homogeneous superderivation** is then defined as a homogeneous linear map $T \colon A \to A$ that obeys the **super-Leibniz rule**

$$T(FG) = (TF)G + (-1)^{\alpha(T)\alpha(F)} F(TG). \tag{2.1.7}$$

Thus even and odd superderivations are derivations and antiderivations, respectively. A general superderivation is a sum of an even and an odd superderivation. We denote the collection of derivations of a superalgebra A by $\mathrm{Der}(A)$.

Euclidean Superspace $\mathbb{R}^{n|m}$

Consider the superalgebra $C^\infty(\mathbb{R}^{0|m})$. For each Grassmann variable $\xi_1, \dots, \xi_m$, define the **left Grassmann derivatives** $\frac{\partial}{\partial \xi_i} \colon C^\infty(\mathbb{R}^{0|m}) \to C^\infty(\mathbb{R}^{0|m})$ as the unique linear maps determined by

$$\frac{\partial}{\partial \xi_i}(\xi_i \xi_I) = \xi_I \quad \text{if } \xi_i \xi_I \neq 0, \qquad \frac{\partial}{\partial \xi_i} 1 = 0 \tag{2.1.8}$$

for ξ_I a Grassmann monomial. One can check that these define odd superderivations on $C^\infty(\mathbb{R}^{0|m})$: if F is a homogeneous Grassmann polynomial, then the above conditions imply

$$\partial_{\xi_i}(FG) = (\partial_{\xi_i} F)G + (-1)^{\alpha(F)} F(\partial_{\xi_i} G). \tag{2.1.9}$$

The super-Leibniz rule implies that Grassmann derivatives behave exactly as the usual commutative coordinate partial derivatives $\frac{\partial}{\partial x}$, except they anti-commute with each other and the Grassmann variables:

$$\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2} F = -\frac{\partial}{\partial \xi_2}\frac{\partial}{\partial \xi_1} F, \qquad \frac{\partial}{\partial \xi_1}(\xi_2 F) = -\xi_2 \frac{\partial}{\partial \xi_1} F. \tag{2.1.10}$$

The left Grassmann derivatives form a basis for $\mathrm{Der}(C^\infty(\mathbb{R}^{0|m}))$ as a rank $0|m$ $C^\infty(\mathbb{R}^{0|m})$ supermodule, with a general superderivation T having a unique representation as

$$T = \sum_{\alpha=1}^{m} F_\alpha \frac{\partial}{\partial \xi_\alpha} \tag{2.1.11}$$

with $F_\alpha \in C^\infty(\mathbb{R}^{0|m})$. If all F_α are even/odd, then T is odd/even.

More generally, every superderivation $T \in \mathrm{Der}(C^\infty(\mathbb{R}^{n|m}))$ can be realised in Cartesian coordinates $(x^1, \dots, x^n \mid \xi^1, \dots, \xi^m)$ as

$$T = \sum_{\alpha=1}^{n} F_\alpha \frac{\partial}{\partial x^\alpha} + \sum_{\alpha=1}^{m} G_\alpha \frac{\partial}{\partial \xi^\alpha} \tag{2.1.12}$$

where $F_\alpha, G_\alpha \in C^\infty(\mathbb{R}^{n|m})$ are superfunctions, $\frac{\partial}{\partial x_\alpha}$ are the usual coordinate partial derivatives on $\mathbb{R}^n$, and $\frac{\partial}{\partial \xi_i}$ are the Grassmann derivatives defined above. If T is an even/odd superderivation, then F_α are even/odd superfunctions and G_α are odd/even superfunctions. We also see that $\mathrm{Der}(C^\infty(\mathbb{R}^{n|m}))$ is a free rank $n|m$ $C^\infty(\mathbb{R}^{n|m})$-supermodule.

Chain rule

There is an analogue of the chain rule in the super setting which describes the action of derivations on composite superfunctions.

Theorem 2.1.1. *For $\boldsymbol{f} = (f_1, \dots, f_n)$, $f_i \in C^\infty_{\mathit{even}}(\mathbb{R}^{p|q})$, and $\boldsymbol{\eta} = (\eta_1, \dots, \eta_m)$, $\eta_i \in C^\infty_{\mathit{odd}}(\mathbb{R}^{p|q})$, a collection of even/odd superfunctions on $\mathbb{R}^{p|q}$, and $T \in \mathrm{Der}(C^\infty(\mathbb{R}^{p|q}))$ a superderivation, the action of T on the composite function $g(\boldsymbol{f} \mid \boldsymbol{\eta})$ for $g \in C^\infty(\mathbb{R}^{n|m})$ is described by the chain rule*

$$T(g(\boldsymbol{f} \mid \boldsymbol{\eta})) = \sum_{j=1}^{n} T(f_j) \frac{\partial}{\partial x_j} g(\boldsymbol{f} \mid \boldsymbol{\eta}) + \sum_{j=1}^{m} T(\eta_j) \frac{\partial}{\partial \xi_j} g(\boldsymbol{f} \mid \boldsymbol{\eta}) \tag{2.1.13}$$

where $\frac{\partial g}{\partial x_j}$ and $\frac{\partial g}{\partial \xi_j}$ are the derivatives of $g(x_1, \dots, x_n \mid \xi_1, \dots, \xi_m)$ with respect to its j-th even/odd component.

Remark 2.1.2. Care must be taken with the order of the factors in (2.1.13) due to the potential presence of odd quantities.

Proof. See [79, p.59]. □

2.2 Berezin Integration

In the same way that derivations are an algebraic abstraction of differentiation, spaces of linear functionals on algebras give a model of integration. Indeed, using a functional analytic approach, measure theory can be developed this way, with the Riesz–Markov theorem and its ilk providing the bridge between spaces of measures and dual spaces of functions. It is in this style that we define integration over $\mathbb{R}^{n|m}$: the space of measures (or rather, distributions) on superspace $\mathbb{R}^{n|m}$ is defined as the continuous dual of $C^\infty(\mathbb{R}^{n|m})$. We denote these functionals using an integral-style notation

$$\int_{\mathbb{R}^{n|m}} \cdot \, d\mu : C^\infty(\mathbb{R}^{n|m}) \to \mathbb{R} \tag{2.2.1}$$

and refer to them as **Berezin integrals** or **Berezin measures**.

Superfunction Spaces

Actually, the space of linear functionals on $C^\infty(\mathbb{R}^{n|m})$ is not entirely suitable because the associated distributions have compact support. In particular, the super-analogue of the Lebesgue measure is not contained in $C^\infty(\mathbb{R}^{n|m})^*$. By starting with a smaller space of test functions, we gain access to a larger dual; following the ordinary case, we focus on the space of **compactly supported smooth superfunctions**, defined as the Grassmann extension

$$C_c^\infty(\mathbb{R}^{n|m}) := C_c^\infty(\mathbb{R}^n) \otimes \Omega^m. \tag{2.2.2}$$

Lying between $C^\infty(\mathbb{R}^{n|m})$ and $C_c^\infty(\mathbb{R}^{n|m})$ are a multitude of intermediate spaces that one could consider. We list a few here, all of which are defined as Grassmann extensions of ordinary function algebras:

- Schwartz superfunctions: $\mathcal{S}(\mathbb{R}^{n|m}) := \mathcal{S}(\mathbb{R}^n) \otimes \Omega^m$
- Superfunctions vanishing at infinity: $C_0^\infty(\mathbb{R}^{n|m}) := C_0^\infty(\mathbb{R}^n) \otimes \Omega^m$
- Bounded superfunctions: $C_b^\infty(\mathbb{R}^{n|m}) := C_b^\infty(\mathbb{R}^n) \otimes \Omega^m$
- Slowly growing superfunctions: $C_{\text{slow}}^\infty(\mathbb{R}^{n|m}) := C_{\text{slow}}^\infty(\mathbb{R}^n) \otimes \Omega^m$, where

$$C_{\text{slow}}^\infty(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \text{There exists } \beta \in \mathbb{N} \text{ s.t. } \sup_{x \in \mathbb{R}^n} \frac{|\partial_x^\alpha f|}{(1 + ||x||^\beta)} < \infty, \text{ for all } \alpha \in \mathbb{N}^n \right\} \tag{2.2.3}$$

These spaces fit together in the following chain of inclusions:

$$C_c^\infty(\mathbb{R}^{n|m}) \subseteq \mathcal{S}(\mathbb{R}^{n|m}) \subseteq C_0^\infty(\mathbb{R}^{n|m}) \subseteq C_b^\infty(\mathbb{R}^{n|m}) \subseteq C_{\text{slow}}^\infty(\mathbb{R}^{n|m}) \subseteq C^\infty(\mathbb{R}^{n|m}) \tag{2.2.4}$$

Their dual spaces are nested in the reverse order. We present the general framework using $C_c^\infty(\mathbb{R}^{n|m})^*$, i.e., the most general class of distributions. Later, we will work with some of the other classes, particular Schwartz superfunctions (i.e., those with rapid decay), and also those that have slow growth. The framework we present here transfers to this setting essentially unchanged, as long as one is careful with the usual convergence issues.

Integration on $\mathbb{R}^{0|m}$

Let us first examine the fermionic superspaces, $\mathbb{R}^{0|m}$. As their underlying topological space is a single point, all superfunctions are compactly supported $C_c^\infty(\mathbb{R}^{0|m}) = C^\infty(\mathbb{R}^{0|m})$. Berezin integrals on fermionic spaces are completely determined by their values on Grassmann monomials as these form a basis for $C^\infty(\mathbb{R}^{0|m})$:

$$\begin{aligned}\int_{\mathbb{R}^{0|m}} F\,d\mu &= \int_{\mathbb{R}^{0|m}} \sum_{I\subseteq[\![m]\!]} f_I \xi_I\,d\mu \\ &= \sum_{I\subseteq[\![m]\!]} f_I \int_{\mathbb{R}^{0|m}} \xi_I\,d\mu.\end{aligned} \tag{2.2.5}$$

Choosing the Berezin measure defined by

$$\int_{\mathbb{R}^{0|m}} \xi_I\,d\xi = \begin{cases} 1 & \xi_I = \xi_1\xi_2\ldots\xi_m \\ 0 & \text{otherwise}\end{cases}, \tag{2.2.6}$$

we obtain Berezin's remarkable definition of integration as iterated differentiation:

$$\int_{\mathbb{R}^{0|m}} F\,d\xi := \frac{\partial}{\partial \xi_m}\cdots\frac{\partial}{\partial \xi_2}\frac{\partial}{\partial \xi_1}F \tag{2.2.7}$$

We call this the **Berezin–Lebesgue measure**, as it is uniquely singled out by its infinitesimal symmetry under 'fermionic translations': for all $\frac{\partial}{\partial \xi_i}$ and $F \in C^\infty(\mathbb{R}^{0|m})$, we have

$$\int_{\mathbb{R}^{0|m}} \frac{\partial}{\partial \xi_i} F\,d\xi = 0. \tag{2.2.8}$$

Every Berezin measure can be written as a density with respect to the Berezin–Lebesgue measure by precomposing on the right with a function $\rho \in C^\infty(\mathbb{R}^{0|m})$:

$$\int_{\mathbb{R}^{0|m}} F(\xi)\,d\mu(\xi) = \int_{\mathbb{R}^{0|m}} F(\xi)\,\rho(\xi) d\xi := \frac{\partial}{\partial \xi_m}\cdots\frac{\partial}{\partial \xi_2}\frac{\partial}{\partial \xi_1}(F(\xi)\rho(\xi)) \tag{2.2.9}$$

This function ρ is called the **density** of the Berezin measure.

The parity of the Berezin–Lebesgue measure on $\mathbb{R}^{0|m}$ is defined as

$$\alpha(d\xi) = \begin{cases} 0 \in \mathbb{Z}_2, & m \text{ is even} \\ 1 \in \mathbb{Z}_2, & m \text{ is odd}\end{cases}, \tag{2.2.10}$$

and the parity of a general Berezin measure $d\mu = \rho\,d\xi$ with ρ homogeneous is

$$\alpha(d\mu) = \alpha(\rho) + \alpha(d\xi) \tag{2.2.11}$$

Integration on $\mathbb{R}^{n|m}$

Continuous linear functionals on $C_c^\infty(\mathbb{R}^{n|m})$ correspond to **Berezin distributions** on $\mathbb{R}^{n|m}$, and are of the form

$$T(x,\xi) = \sum_I T_I(x)\xi_I d\xi \tag{2.2.12}$$

where the $T_I \in C_c^\infty(\mathbb{R}^n)^*$ are distributions on $\mathbb{R}^{n|0} = \mathbb{R}^n$ and $d\xi$ is the Berezin–Lebesgue measure on $\mathbb{R}^{0|m}$. If each distribution T_I is an actual measure, then we obtain a **Berezin measure** on $\mathbb{R}^{n|m}$, which we write

$$d\mu(x,\xi) = \sum_I d\nu_I(x)\xi_I d\xi. \tag{2.2.13}$$

The integral of such a measure is defined as

$$\int_{\mathbb{R}^{n|m}} F d\mu(x,\xi) := \sum_I \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_m} \cdots \frac{\partial}{\partial \xi_2} \frac{\partial}{\partial \xi_1} (F\xi_I)\, d\nu_I(x). \tag{2.2.14}$$

Taking $d\nu_\varnothing(x) = dx$ the Lebesgue measure on $\mathbb{R}^n$ and $d\nu_I = 0$ otherwise gives the **Berezin–Lebesgue measure** $dx\, d\xi$ on $\mathbb{R}^{n|m}$. This is the essentially unique super-translation invariant measure on $\mathbb{R}^{n|m}$, i.e.,

$$\int_{\mathbb{R}^{n|m}} \frac{\partial}{\partial x_i} F\, dx\, d\xi = \int_{\mathbb{R}^{n|m}} \frac{\partial}{\partial \xi_i} F\, dx\, d\xi = 0 \tag{2.2.15}$$

for all $F \in C_c^\infty(\mathbb{R}^{n|m})$.

Finally, we have the **super Fubini theorem**: for $d\mu$ and $d\nu$ homogenous Berezin measures on $\mathbb{R}^{n|m}$ and $\mathbb{R}^{p|q}$, we have for all $F \in C_c^\infty(\mathbb{R}^{n+p|m+q})$

$$\int_{\mathbb{R}^{p|q}} \int_{\mathbb{R}^{n|m}} F\, d\mu\, d\nu = (-1)^{\alpha(d\mu)\alpha(d\nu)} \int_{\mathbb{R}^{n|m}} \int_{\mathbb{R}^{p|q}} F\, d\nu\, d\mu. \tag{2.2.16}$$

Changes of variables

Under a change of coordinates $\boldsymbol{x}|\boldsymbol{\xi} \mapsto \boldsymbol{y}|\boldsymbol{\eta}$, i.e., so that $x_i = x_i(\boldsymbol{y},\boldsymbol{\eta}) \in C^\infty_{\text{even}}(\mathbb{R}^{n|m})$ and $\xi_i = \xi_i(\boldsymbol{y},\boldsymbol{\eta}) \in C^\infty_{\text{odd}}(\mathbb{R}^{n|m})$, a Berezin measure transforms according a superanalogue of the Jacobian known as the **Berezinian**:

$$d\boldsymbol{x}\, d\boldsymbol{\xi} \mapsto \mathrm{Ber}(\boldsymbol{y}|\boldsymbol{\eta}) d\boldsymbol{y}\, d\boldsymbol{\eta}. \tag{2.2.17}$$

The Berezinian, which is an even superfunction, is expressed as a superdeterminant of the **Berezinian supermatrix** of partial derivatives[1]

$$\mathrm{Ber}(\boldsymbol{y}|\boldsymbol{\eta}) = \mathrm{sdet}(M), \quad M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.2.18}$$

where

$$\begin{aligned} A_{ij} &= \frac{\partial x_j}{\partial y_i}, & B_{ij} &= \frac{\partial \xi_j}{\partial y_i}, \\ C_{ij} &= \frac{\partial x_j}{\partial \eta_i}, & D_{ij} &= \frac{\partial \xi_j}{\partial \eta_i}. \end{aligned} \tag{2.2.19}$$

This gives the change of variables formula for the Berezin–Lebesgue integral as

$$\int_{\mathbb{R}^{n|m}} F(\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{x}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^{n|m}} F(\boldsymbol{x}(\boldsymbol{y},\boldsymbol{\eta}), \boldsymbol{\xi}(\boldsymbol{y},\boldsymbol{\eta}))\ \mathrm{Ber}(\boldsymbol{y}|\boldsymbol{\eta}) d\boldsymbol{y}\, d\boldsymbol{\eta}, \tag{2.2.20}$$

and more generally,

$$\int_{\mathbb{R}^{n|m}} F(\boldsymbol{x},\boldsymbol{\xi})\, \rho(\boldsymbol{x}|\boldsymbol{\xi}) d\boldsymbol{x}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^{n|m}} F(\boldsymbol{x},\boldsymbol{\xi})\, \rho(\boldsymbol{x}|\boldsymbol{\xi})\ \mathrm{Ber}(\boldsymbol{y}|\boldsymbol{\eta}) d\boldsymbol{y}\, d\boldsymbol{\eta} \tag{2.2.21}$$

for Berezin integration against $\rho(\boldsymbol{x}|\boldsymbol{\xi}) d\boldsymbol{x}\, d\boldsymbol{\xi}$ (the dependence of $\boldsymbol{x}$ and $\boldsymbol{\xi}$ on $\boldsymbol{y}$ and $\boldsymbol{\eta}$ is not shown for clarity). See [13] for a proof of this.

[1]Note that this gives the transpose of the usual way of writing the Jacobian; we have written it this way to avoid having to take derivatives on the right.

2.3 Supersymmetries and Lie Superalgebras

Under the supercommutator map

$$[T_1, T_2] = T_1 \circ T_2 - (-1)^{\alpha(T_1)\alpha(T_2)} T_2 \circ T_1, \tag{2.3.1}$$

superderivations form a **Lie superalgebra**. The general definition is as follows:

Definition 2.3.1. A **Lie superalgebra** is a supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with an even bilinear operation

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{2.3.2}$$

called the **Lie superbracket**, which is skew-supercommutative

$$[u, v] = -(-1)^{\alpha(u)\alpha(v)}[v, u] \tag{2.3.3}$$

and satisfies the super Jacobi identity:

$$(-1)^{\alpha(u)\alpha(w)}[u, [v, w]] + (-1)^{\alpha(w)\alpha(v)}[w, [u, v]] + (-1)^{\alpha(v)\alpha(u)}[v, [w, u]] = 0 \tag{2.3.4}$$

The primary Lie superalgebra of interest is $\mathrm{Der}(C^\infty(\mathbb{R}^{n|m}))$, which describes the complete set of infinitesimal symmetries of Cartesian superspace, i.e., infinitesimal superdiffeomorphisms. Rather than examining all such symmetries, it is often useful to consider various sub-algebras of $\mathrm{Der}(C^\infty(\mathbb{R}^{n|m}))$ which preserve some additional structure on the superspace. We begin with what we call **integral symmetries**, before discussing the **orthosymplectic superalgebras**.

Integral Symmetries and Ward Identities

Infinitesimal integral symmetries are defined as follows:

Definition 2.3.2. Let $\int_{\mathbb{R}^{n|m}} \cdot \, d\mu$ be a Berezin integral on $\mathbb{R}^{n|m}$. A superderivation $T \in \mathrm{Der}(C^\infty(\mathbb{R}^{n|m}))$ is an infinitesimal symmetry of $\int_{\mathbb{R}^{n|m}} \cdot \, d\mu$ if for all $F \in C_c^\infty(\mathbb{R}^{n|m})$

$$\int_{\mathbb{R}^{n|m}} T[F] \, d\mu = 0. \tag{2.3.5}$$

Infinitesimal symmetries lead to integration by parts formulas, otherwise known as **Ward identities**: suppose T is a symmetry of $\int_{\mathbb{R}^{n|m}} \cdot \, d\mu$, and that $F, G \in C_c^\infty(\mathbb{R}^{n|m})$ have compact support. Then

$$\int_{\mathbb{R}^{n|m}} T[FG] \, d\mu = 0, \tag{2.3.6}$$

since FG is compactly supported. Since T acts as a superderivation, we obtain the Ward identity

$$\int_{\mathbb{R}^{n|m}} T[F]G \, d\mu = -(-1)^{\alpha(T)\alpha(F)} \int_{\mathbb{R}^{n|m}} FT[G] \, d\mu. \tag{2.3.7}$$

If either T or F is even, this is the usual integration by parts formula.

Remark 2.3.3. It is difficult to overstate the importance the above identity. Without doubt, it is the existence of Ward identities which gives the spin system approach its power.

Example 2.3.4. As previously mentioned, infinitesimal super-translations are symmetries of the Berezin–Lebesgue measure: for all $F \in C_c^\infty(\mathbb{R}^{n|m})$, we have

$$\int_{\mathbb{R}^{n|m}} \frac{\partial}{\partial x_i} F \, dx \, d\xi = \int_{\mathbb{R}^{n|m}} \frac{\partial}{\partial \xi_i} F \, dx \, d\xi = 0. \tag{2.3.8}$$

One can check, this property uniquely characterises the Berezin–Lebesgue measure (up to an overall scaling factor). The Berezin–Lebesgue measure has many more symmetries: if

$$T = \sum_i F_i \frac{\partial}{\partial x_i} + \sum_i G_i \frac{\partial}{\partial \xi_i} \tag{2.3.9}$$

is any superderivation satisfying the zero divergence condition

$$\operatorname{div}(T) = \sum_i \frac{\partial F_i}{\partial x_i} + \sum_i (-1)^{\alpha(G_i)} \frac{\partial G_i}{\partial \xi_i} = 0, \tag{2.3.10}$$

then T is a symmetry of $\int_{\mathbb{R}^{n|m}} \cdot \, dx \, d\xi$.

Anomalous Ward Identities

Let $\int_{\mathbb{R}^{n|m}} \cdot \, d\mu$ and $\int_{\mathbb{R}^{n|m}} \cdot \, d\nu$ be two Berezin integrals which are related as

$$d\nu = \rho \, d\mu, \qquad d\mu = \rho^{-1} d\nu \tag{2.3.11}$$

for some $\rho \in C^\infty_{\text{even}}(\mathbb{R}^{n|m})$ an even invertible superfunction[2] , and let T be a symmetry of $d\mu$. Then, as the product of ρ with any compactly supported superfunction F is also compactly supported, $F\rho \in C^\infty_c(\mathbb{R}^{n|m})$, we have

$$\int_{\mathbb{R}^{n|m}} T[F\rho] d\mu = 0. \tag{2.3.13}$$

Expanding out the associated Ward identity for $d\mu$,

$$\int_{\mathbb{R}^{n|m}} T[F] \rho d\mu = - \int_{\mathbb{R}^{n|m}} (-1)^{\alpha(T)\alpha(F)} F T[\rho] d\mu \tag{2.3.14}$$

and absorbing ρ back into the measure gives an **anomalous symmetry** for $d\nu$,

$$\int_{\mathbb{R}^{n|m}} T[F] \, d\nu = -(-1)^{\alpha(T)\alpha(F)} \int_{\mathbb{R}^{n|m}} F \frac{T[\rho]}{\rho} \, d\nu, \tag{2.3.15}$$

which in turn, gives rise to an **anomalous Ward identity**:

$$\int_{\mathbb{R}^{n|m}} T[F] G \, d\nu = (-1)^{|T||F|+1} \int_{\mathbb{R}^{n|m}} F T[G] \, d\nu + (-1)^{|T|(|F|+|G|)+1} \int_{\mathbb{R}^{n|m}} F G \frac{T[\rho]}{\rho} \, d\nu \tag{2.3.16}$$

Here it is convenient to rewrite the relative density as $\rho = \pm e^{\log \rho}$ with the choice of sign indicated by the sign of the body. For clarity, let us suppose that ρ_b is positive, so we can write[3] $\rho = e^{\log \rho}$. Defining $H = -\log \rho$, so that $\rho = e^{-H}$, we see that

$$\frac{T[\rho]}{\rho} = -T[H] \tag{2.3.18}$$

and so the anomalous Ward identity for $d\nu = e^{-H} d\mu$ can now be written as

$$\int_{\mathbb{R}^{n|m}} T[F] G \, d\nu = (-1)^{|T||F|+1} \int_{\mathbb{R}^{n|m}} F T[G] \, d\nu - (-1)^{|T|(|F|+|G|)+1} \int_{\mathbb{R}^{n|m}} F G T[H] \, d\nu. \tag{2.3.19}$$

As the $e^{-H} d\mu$ notation suggests, the Berezin measures presented in this form arise as **Berezin–Gibbs measures** in the super-analogue of statistical mechanics.

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[2]Recall that the multiplicative inverse of a superfunction $F = F_{\text{even}} + F_{\text{odd}}$, should it exist, is of the form

$$\frac{1}{F} = \frac{1}{F_{\text{even}}} - \frac{1}{F_{\text{even}}^2} F_{\text{odd}}, \tag{2.3.12}$$

with $\frac{1}{F_{\text{even}}}$ and $\frac{1}{F_{\text{even}}^2}$ understood in the Grassmann analytic sense. This inverse will indeed exist if and only if the body of F is nowhere 0, i.e., so that $\frac{1}{F_b}$ is a smooth function.

[3]Explicitly, the logarithm of $\rho = \rho_b + \rho_s$ is defined using Grassmann analytic continuation as

$$\log(\rho) = \log(\rho_b) + \frac{\rho_s}{\rho_b} - \frac{\rho_s^2}{2\rho_b^2} + \frac{\rho_s^3}{3\rho_b^3} - \dots \tag{2.3.17}$$

Orthosymplectic Symmetries

The infinitesimal symmetries of the ordinary dot product on $\mathbb{R}^n$ are described by the (special) orthogonal Lie algebra $\mathfrak{so}(n)$. Usually, this is defined as the $\frac{n(n-1)}{2}$ dimensional vector space of $n \times n$ skew-symmetric matrices, equipped with the matrix commutator $[A, B] = AB - BA$. For us, however, a more useful characterisation is given by the *derivation representation*, which expresses $\mathfrak{so}(n)$ as a Lie subalgebra of $\mathrm{Der}(C^\infty(\mathbb{R}^n))$. Here it is convenient to use the **diagonal embedding**

$$\tilde{\Delta} : \mathrm{Der}(C^\infty(\mathbb{R}^n)) \to \mathrm{Der}(C^\infty(\mathbb{R}^n \times \mathbb{R}^n)) \tag{2.3.20}$$

of $\mathrm{Der}(C^\infty(\mathbb{R}^n))$ into $C^\infty(\mathrm{Der}(C^\infty(\mathbb{R}^n \times \mathbb{R}^n))$, which sends a derivation $T = \sum_{i=1}^n F_i(x^i)\frac{\partial}{\partial x^i}$ to $\tilde{\Delta}(T) = T_a + T_b = \sum_{i=1}^n F_i(x_a^i)\frac{\partial}{\partial x_a^i} + \sum_{i=1}^n F_i(x_b^i)\frac{\partial}{\partial x_b^i}$ where $(x_a^1, \dots, x_a^n, x_b^1, \dots, x_b^n)$ are coordinates on $\mathbb{R}^n \times \mathbb{R}^n$. Then, a derivation T is contained in the $\mathfrak{so}(n)$ subalgebra of $\mathrm{Der}(C^\infty(\mathbb{R}^n))$ if the image of T under the diagonal map annihilates the dot product

$$\tilde{\Delta}(T)(u_a \cdot u_b) = 0, \tag{2.3.21}$$

where $u_a \cdot u_b = x_a^1 x_b^1 + \cdots + x_a^n x_b^n \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. One may then check that every such derivations are closed under the derivation supercommutator, and they are all of the form

$$T_A = \sum_{i,j} A_{ij} x^i \frac{\partial}{\partial x^j}, \tag{2.3.22}$$

for some skew-symmetric matrix A. This does indeed define a representation of $\mathfrak{so}(n)$ as the map $A \mapsto T_A$ preserves the Lie bracket $[T_A, T_B] = T_{[A,B]}$, and furthermore, this representation is faithful. Similarly, the derivations $T \in C^\infty(\mathbb{R}^{n,1})$ which annihilate the Minkowski inner product $u_a \cdot u_b = -z_a z_b + x_a^1 x_b^1 + \cdots + x_a^n x_b^n$, form a representation of the Lorentzian Lie algebra $\mathfrak{so}(n, 1)$. Such derivations are of the same form as (2.3.22), but A is now skew-symmetric with respect to the Minkowski inner product, i.e.,

$$A^T J_{n,1} + J_{n,1} A = 0, \quad J_{n,1} = \mathrm{diag}(-1, 1, \dots, 1). \tag{2.3.23}$$

Orthosymplectic Lie Superalgebras. Extending this idea to the super setting leads us to the **orthosymplectic Lie superalgebras**, denoted $\mathfrak{osp}(n, p|2m)$ in the general case. This can be defined as a matrix Lie superalgebra, but here we opt for an equivalent characterisation using its super-derivation representation, as this is ultimately what we are interested in.

The cases $p = 0$ and $p = 1$ describe the symmetries of the super-Euclidean and super-Minkowski inner products, which we recall are given by

$$\begin{aligned} u_a \cdot u_b &= \sum_{i=1}^n x_a^i x_b^i - \sum_{i=1}^m \xi_a^i \eta_b^i - \eta_a^i \xi_b^i \qquad &\text{(Euclidean)} \\ u_a \cdot u_b &= -z_a z_b + \sum_{i=1}^n x_a^i x_b^i - \sum_{i=1}^m \xi_a^i \eta_b^i - \eta_a^i \xi_b^i \qquad &\text{(Minkowski).} \end{aligned} \tag{2.3.24}$$

In each case, let us denote the coordinates of a superspin by $(u^1, \dots, u^{n+2m}) = (x^1, \dots x^n \mid \xi^1, \dots, \xi^m, \eta^1 \dots, \eta^m)$ and $(u^0, u^1, \dots, u^{n+2m}) = (z, x^1, \dots x^n \mid \xi^1, \dots, \xi^m, \eta^1 \dots, \eta^m)$.

As in the ordinary case, we define the diagonal map

$$\tilde{\Delta} : \mathrm{Der}(C^\infty(\mathbb{R}^{n+p|2m})) \to \mathrm{Der}(C^\infty(\mathbb{R}^{n+p|2m} \times \mathbb{R}^{n+p|2m})) \tag{2.3.25}$$

and define a symmetry of an inner product $u_a \cdot u_b \in C^\infty(\mathbb{R}^{n+p|2m} \times \mathbb{R}^{n+p|2m})$ as a super-derivation $T \in \mathrm{Der}(C^\infty(\mathbb{R}^{n+p|2m}))$ such that

$$\tilde{\Delta}(T)(u_a \cdot u_b) = 0. \tag{2.3.26}$$

Such superderivations are then of the form

$$T = \sum_{i,j} R_{ij} u^i \frac{\partial}{\partial u^j} \tag{2.3.27}$$

where R is a real $n+p|2m \times n+p|2m$ supermatrix satisfying

$$R^{ST} J + JR = 0, \tag{2.3.28}$$

where R^{ST}, the supertranspose of R, and J are given by

$$R^{ST} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{ST} = \begin{bmatrix} A^T & C^T \\ -B^T & D^T \end{bmatrix}, \qquad J = \left[\begin{array}{c|cc} J_{n,p} & 0 & 0 \\ \hline 0 & 0 & -I_m \\ 0 & I_m & 0 \end{array}\right], \tag{2.3.29}$$

and $J_{n,p} = \mathrm{diag}(-I_p, I_n)$.

It is convenient to represent the coefficient matrix R in 3×3 block form, in which case one can check that it is of the form

$$R = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline B_2^T & D_{11} & D_{12} \\ -B_1^T & D_{21} & -D_{11}^T \end{array}\right], \tag{2.3.30}$$

where $A^T J_{n,p} + J_{n,p} A = 0$, $D_{12} = D_{12}^T$, and $D_{21} = D_{21}^T$. The coefficient matrices in the even/odd cases are then

$$R_{\text{even}} = \left[\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D_{11} & D_{12} \\ 0 & D_{21} & -D_{11}^T \end{array}\right], \quad R_{\text{odd}} = \left[\begin{array}{c|cc} 0 & B_1 & B_2 \\ \hline B_2^T & 0 & 0 \\ -B_1^T & 0 & 0 \end{array}\right], \tag{2.3.31}$$

and the constraints imply that the dimension of the underlying supervector space is $\frac{1}{2}(n+p)(n+p-1) + m(2m+1)\,|\,2(n+p)m$. The even symmetries take on a familiar form: they are the direct sum of an orthogonal symmetry $A \in \mathfrak{so}(n,p)$ and a symplectic symmetry $D \in \mathfrak{sp}(2m)$. The odd symmetries, otherwise known as **supersymmetries** have no classical analogue.

2.4 Supersymmetric Localisation

Given a collection of superspins $\boldsymbol{u} = (u_a)_{a\in\Lambda} \in \mathbb{R}^{n|2m}$ indexed by a set Λ, and $A \in \mathbb{R}^{\Lambda\times\Lambda}$ a real valued matrix indexed by Λ, we define the product $A\boldsymbol{u}$ as the superspin with entries

$$[A\boldsymbol{u}]_a = \sum_{b\in\Lambda} A_{ab} u_b. \tag{2.4.1}$$

For example, if $\boldsymbol{u} = (u_1, u_2)$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A\boldsymbol{u} = (u_1 + 2u_2, 3u_1 + 4u_2)$. Let us also define the Λ-indexed super-Euclidean and super-Minkowski inner products as the sum

$$(\boldsymbol{u}, \boldsymbol{u}) = \sum_{a\in\Lambda} u_a \cdot u_a. \tag{2.4.2}$$

Given a matrix $A \in \mathbb{R}^{\Lambda\times\Lambda}$, we then have

$$(\boldsymbol{u}, A\boldsymbol{u}) = \sum_{a\in\Lambda} u_a \cdot [A\boldsymbol{u}]_a = \sum_{a,b\in\Lambda} A_{ab} u_a \cdot u_b. \tag{2.4.3}$$

This notation is useful for defining high dimensional Gaussian measures which are invariant under the action of a lower dimensional Lie group (or rather, algebra). For instance, let $A \in \mathbb{R}^{N\times N}$ be symmetric and positive definite. Then we can associate to A the $\mathfrak{so}(n)$-invariant[4] Gaussian measure on $\mathbb{R}^{nN}$

$$e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = e^{-\frac{1}{2}\sum_{a,b} A_{ab}(x_a^1 x_b^1+\cdots+x_a^n x_b^n)} \prod_{i=1}^N \frac{dx_i^1 \, \ldots \, dx_i^n}{\sqrt{2\pi}^n}, \tag{2.4.4}$$

where we have used the superspin notation $u_a = (x_a^1, \ldots, x_a^n)$ on the left hand side. As is well known, the integral of this measure is

$$\int_{\mathbb{R}^{nN}} e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = \frac{1}{\sqrt{\det(A)}^n} \tag{2.4.5}$$

In physics parlance, this is a *bosonic* Gaussian integral. We can also associate to A the $\mathfrak{sp}(2m)$-invariant *fermionic* Gaussian Berezin measure on $\mathbb{R}^{0|2mN}$

$$e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = e^{\frac{1}{2}\sum_{a,b} A_{ab}(\xi_a^1\eta_b^1-\eta_a^1\xi_b^1+\cdots+\xi_a^m\eta_b^m-\eta_a^m\xi_b^m)} \prod_{i=1}^N d\xi_i^1 d\eta_i^1 \, \ldots \, d\xi_i^m d\eta_i^m, \tag{2.4.6}$$

where we have used superspins $u_a = (\xi_a^1, \ldots, \xi_a^m, \eta_a^1, \ldots, \eta_a^m)$. A computation shows that the Berezin integral of the fermionic Gaussian is given by a positive rather than negative power of the determinant:

$$\int_{(\mathbb{R}^{0|2m})^N} e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = \det(A)^m. \tag{2.4.7}$$

Combining these two cases, we obtain the **super-Gaussian** measure on $(\mathbb{R}^{n|2m})^N$:

$$e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = e^{-\frac{1}{2}\sum_{a,b} A_{ab}(\sum_{i=1}^n x_a^i x_b^i - \sum_{i=1}^m \xi_a^i\eta_b^i-\eta_a^i\xi_b^i)} \prod_{i=1}^N \frac{dx_i^1 \, \ldots \, dx_i^n d\xi_i^1 d\eta_i^1 \, \ldots \, d\xi_i^m d\eta_i^m}{\sqrt{2\pi}^n}. \tag{2.4.8}$$

This is invariant under $\mathfrak{osp}(n|2m)$ under its diagonal representation, and has Berezin integral

$$\int_{(\mathbb{R}^{n|2m})^N} e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = \det(A)^{\frac{2m-n}{2}}. \tag{2.4.9}$$

When the bosonic and fermionic dimensions are equal, that is, when $n = 2m$, we see that the value of this integral is independent of A!

$$\int_{(\mathbb{R}^{2m|2m})^N} e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})}d\boldsymbol{u} = 1. \tag{2.4.10}$$

This most surprising result is a simple example of *supersymmetric localisation*. The rest of this section describes this phenomenon; as we shall see, this is the result of a Ward identity coming from an odd symmetry of the underlying Berezin-Lebesgue measure. For ease of exposition, we focus on the simplest case, with $n = 2m = 2$.

The odd symmetry $Q \in \mathfrak{osp}(2|2)$,

$$Q = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} - x\frac{\partial}{\partial \eta} + y\frac{\partial}{\partial \xi}, \tag{2.4.11}$$

[4] By this we mean invariant under the diagonal action $T \mapsto \sum_{a=1}^N T_a$ for $T \in \mathfrak{so}(n)$.

is the **supersymmetry generator** associated to this localisation result. Unlike an ordinary symmetry, it has the property that $Q^2 \in \mathfrak{osp}(2|2)$ is also a derivation[5]. Let us denote the image of Q under the diagonal map by the same letter,

$$Q \equiv \sum_{i=1}^{N} Q_i\,, \qquad Q_i \equiv \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i}. \tag{2.4.12}$$

so that Q is now considered as an element of $\mathrm{Der}(C^\infty((\mathbb{R}^{2|2})^N))$. In terms of superspin coordinates, Q acts as

$$Qx_i = \xi_i, \quad Qy_i = \eta_i, \quad Q\xi_i = -y_i, \quad Q\eta_i = x_i. \tag{2.4.13}$$

A superfunction $F \in C^\infty((\mathbb{R}^{2|2})^N)$ is defined to be **supersymmetric** if $QF = 0$.

Note that if $G \in C^\infty(\mathbb{R}^{p|q})$ is any superfunction and $(f_1, \dots, f_p|\psi_1, \dots, \psi_q)$ are any collection of supersymmetric superfunctions, then the composite is supersymmetric by the chain rule:

$$Q(G(f|\psi)) = \sum_i Q(f_i) \frac{\partial G}{\partial x_i} + \sum_j Q(\psi_j) \frac{\partial G}{\partial \xi_j} = 0. \tag{2.4.14}$$

As a corollary to this, we find that supersymmetric superfunctions form a subalgebra of $C^\infty((\mathbb{R}^{2|2})^N)$, which we see by taking $G(x,y) = ax + by$, $G(x,y) = xy$ etc.

Example 2.4.1. By definition, Q annihilates the super-Euclidean inner product, and so

$$u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j \tag{2.4.15}$$

is supersymmetric.

Much of the magic of supersymmetry is due to the fundamental *localisation theorem*:

Theorem 2.4.2. *Suppose $F \in C^\infty((\mathbb{R}^{2|2})^N)$ is supersymmetric and integrable against the Berezin-Lebesgue measure $d\boldsymbol{u}$ on $(\mathbb{R}^{2|2})^N$. Then*

$$\int_{(\mathbb{R}^{2|2})^N} F d\boldsymbol{u} = F_b(0) \tag{2.4.16}$$

where the right-hand side is the body of F evaluated at 0.

For a proof of this, see the Appendix of Chapter 3.

Example 2.4.3. Let $f \in C_c^\infty(\mathbb{R}^{N\times N})$, and let $(u_i \cdot u_j)_{i,j=1}^N$, be a collection of super-Euclidean inner products $u_i \cdot u_j \in C^\infty((\mathbb{R}^{2|2})^N)$. Then, by (2.4.14) and (2.4.15), the composite superfunction $f(\boldsymbol{u}\boldsymbol{u}^T) \in C_c^\infty((\mathbb{R}^{2|2})^N)$ is supersymmetric, and so by Theorem 2.4.2 we have

$$\int_{(\mathbb{R}^{2|2})^N} f(\boldsymbol{u}\boldsymbol{u}^T) d\boldsymbol{u} = f(\boldsymbol{0}) \tag{2.4.17}$$

[5] Technically, any odd symmetry has this property as the Lie super-bracket acts as an anti-commutator on odd elements $[Q,Q] = 2Q^2$. However, the super-bracket can be trivial $Q^2 = 0$. Consider, say, $Q = \xi\frac{\partial}{\partial x} - x\frac{\partial}{\partial \eta}$.

Part II

SUPERPROBABILITY

Chapter 3

The geometry of random walk isomorphism theorems

3.1 Introduction

Random walk isomorphism theorems refer to a class of distributional identities that relate the local times of Markov processes to the squares of Gaussian fields. These theorems, which connect two different types of probabilistic objects, have their origins in the work of the physicist K. Symanzik [101]. Isomorphism theorems have been useful in the investigation of a variety of phenomena, and they can be used in two directions: to study field theoretic questions in terms of random walks, and to study random walks in terms of field theory. An incomplete list of topics investigated via isomorphism theorems includes: local times of Markov processes [69] and their large deviations [17, 22]; cover times and thick points of the simple random walk [1, 34, 57]; four-dimensional self-avoiding walk [6, 15]; ϕ^4 field theory [18, 19, 49]; and random walk loop soups [63, 102].

The purpose of this article is to expand the scope of isomorphism theorems beyond Gaussian fields. Namely, we describe, and make use of, isomorphism theorems that relate *non-Markovian* stochastic processes to *non-Gaussian* spin systems. Our proofs also provide a new perspective on isomorphism theorems: they are consequences of the symmetries of the underlying spin systems.

In Section 3.1 below we give an introduction to isomorphism theorems and the processes this article is concerned with. Before doing this, we briefly summarise the new results contained in this article:

- New and efficient proofs of the Brydges–Fröhlich–Spencer–Dynkin (BFS–Dynkin), Eisenbaum, and second generalised Ray–Knight isomorphism theorems for the simple random walk (SRW). These results are all derived in a few pages from a more general Ward identity for the Gaussian free field.

- New and efficient proofs of supersymmetric versions of the isomorphism theorems for the SRW. In particular, we prove a previously unknown supersymmetric version of the generalised second Ray–Knight isomorphism. For the reader's convenience we also present an introduction to supersymmetry directed towards probabilists in an appendix.

- New isomorphism theorems connecting the vertex-reinforced jump process (VRJP) with hyperbolic sigma models, and supersymmetric versions of these theorems. The analogue of the BFS–Dynkin isomorphism previously appeared in [9], and here we also establish analogues of the Eisenbaum and Ray–Knight isomorphism theorems. Our proofs are geometric and do not rely on any particular set of coordinates. In particular, we do not use horospherical coordinates.

- New isomorphism theorems for the vertex-diminished jump process (VDJP). The VDJP is connected to a spin model taking values in the hemisphere. It previously appeared in the context of the Ray–Knight isomorphism theorem for SRW in [90].

We also give several applications of these isomorphism theorems. In Section 3.6 we show that the Sabot–Tarrès limit formula for the local time of the VRJP [89] is a direct consequence of our supersymmetric Ray–Knight theorem for the $\mathbb{H}^{2|2}$ model. In Section 3.7 we show how isomorphism theorems yield fixed-time formulas and representations of the resolvents for the joint processes of the random walks together with their local times. Lastly, we prove some results concerning exponential decay of correlation functions for the associated spin models in Section 3.8.

Isomorphism theorems for hyperbolic and spherical geometries

Let X_t be a continuous-time stochastic process on a finite state space Λ with associated local times $\boldsymbol{L}_t = (L_t^i)_{i\in\Lambda}$. The processes considered in this paper are all of the form

$$\mathbb{P}[X_{t+dt} = j \mid (X_s)_{s\leqslant t}, X_t = i] = \beta_{ij}(1+\varepsilon L_t^j)\,dt, \qquad \varepsilon \in \{-1,0,1\}, \tag{3.1.1}$$

where $\beta_{ij} \geqslant 0$ and $\beta_{ij} = \beta_{ji}$ for all $i, j \in \Lambda$.

The random walk models defined by (3.1.1) are defined more precisely below. The models have all appeared previously, though they have received varying amounts of attention. When $\varepsilon = 0$ the model is the *continuous-time simple random walk*; for $\varepsilon = 1$ it is the *vertex-reinforced jump process* (VRJP) first studied in [30, 31]; for $\varepsilon = -1$ it is the *vertex-diminished jump process* (VDJP) which appeared in [90]. As the names suggest, the VRJP is a random walk that is encouraged to revisit vertices it has visited in the past, while the VDJP is discouraged from doing so.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space, $\mathbb{H}^n$ denote n-dimensional hyperbolic space, and let $\mathbb{S}^n_+$ denote the upper hemisphere of the n-dimensional sphere. Below we will introduce spin systems that take values in these spaces, and then link these to the aforementioned random walks. The spin systems are the $\mathbb{R}^n$-valued *Gaussian free field* (GFF), corresponding to the SRW; the $\mathbb{H}^n$-valued *hyperbolic spin model*, corresponding to the VRJP; and the $\mathbb{S}^n_+$-valued *hemispherical spin model*, corresponding to the VDJP.

To give a flavour of the relationships that we will establish, recall Dynkin's formulation of an isomorphism linking the SRW and the $\mathbb{R}$-valued GFF [44]. Let $G = (\Lambda, E)$ be a finite graph with Laplacian Δ, $h > 0$, and let $\langle\cdot\rangle$ denote the expectation of a GFF $(u_i)_{i\in\Lambda}$ with covariance $(-\Delta+h)^{-1}$. This is often called the *massive GFF* with *mass* $m = \sqrt{h}$. Let $\mathbb{E}_i$ denote the expectation of a continuous-time SRW X_t with associated local time field $\boldsymbol{L}_t = (L_t^i)_{i\in\Lambda}$, started from $i \in \Lambda$, with X_t independent of the GFF. Then for all bounded $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\Big\langle u_i u_j g(\frac{1}{2}\boldsymbol{u}^2)\Big\rangle = \Big\langle \int_0^\infty \mathbb{E}_i(g(\frac{1}{2}\boldsymbol{u}^2 + \boldsymbol{L}_t)1_{X_t=j})\, e^{-ht}\,dt\Big\rangle, \qquad \boldsymbol{u}^2 \equiv (u_i^2)_{i\in\Lambda}. \tag{3.1.2}$$

The left-hand side is a generalization of the spin-spin correlation between the spins u_i and u_j of the GFF. In particular, taking $g = 1$ in (3.1.2) reveals the well-known fact that the second moments of the massive GFF are given by the Green's function of a SRW killed at rate h.

In Theorems 3.3.3 and 3.4.4 we establish analogues of (3.1.2) for the hyperbolic and hemispherical spin models; the hyperbolic case first appeared in [9]. Our methods also allow us to establish other isomorphism theorems. In particular, we give new proofs of the Eisenbaum isomorphism theorem [46] and of the generalised second Ray–Knight theorem [47] for the GFF, and we establish analogues of these results for hyperbolic and hemispherical spin models. Our proofs apply to n-component spin systems for general $n \in \mathbb{N} = \{1, 2, \dots\}$ in all cases, and even for the GFF some of these results are new when $n > 1$. To ease our exposition we will refer to the generalised second Ray–Knight theorem as the Ray–Knight isomorphism in what follows.

Supersymmetric isomorphism theorems

There is another type of isomorphism that relates the simple random walk to a spin system, in which the GFF is replaced by the *supersymmetric Gaussian free field (SUSY GFF)*. These isomorphisms originated in work of McKane [72] and Parisi and Sourlas [83]. Supersymmetry has played a role in several interesting probabilistic problems [20, 21, 28, 40], and several of the applications we mentioned in the opening paragraph of this article involve the SUSY GFF [6, 15, 17, 22, 63].

The most important aspect of the SUSY isomorphism for the SRW is immediately apparent from the statement of the result, and hence we defer a careful definition of the SUSY GFF to Section 3.5. Let $\langle\cdot\rangle$ now denote the expectation with respect to the SUSY GFF. The SUSY isomorphism theorem is that for all smooth and bounded $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\left\langle u_i^1 u_j^1 g(\frac{1}{2}|\boldsymbol{u}|^2) \right\rangle = \int_0^\infty \mathbb{E}_i(g(\boldsymbol{L}_t)1_{X_t=j})\, e^{-ht}\, dt, \qquad |\boldsymbol{u}|^2 \equiv (|u_i|^2)_{i\in\Lambda}. \tag{3.1.3}$$

The key point of (3.1.3) is that the right-hand side only involves the simple random walk, while the left-hand side involves only the components $(u_i)_{i\in\Lambda}$ of the SUSY GFF. Thus questions about the local time of random walk can be rephrased purely in terms of the SUSY GFF.

The viewpoint that isomorphism theorems arise as a consequence of continuous symmetries applies equally well to supersymmetric spin systems. Beyond proving (3.1.3), Section 3.5 also establishes results analogous to (3.1.3) for the supersymmetric $\mathbb{H}^{2|2}$ and $\mathbb{S}_+^{2|2}$ models, and moreover we prove a SUSY variant of the Ray–Knight isomorphism. This is new even for the simple random walk. We emphasise that these theorems give direct access to the local times of the non-Markovian VRJP and VDJP in terms of the spin models. The analogue of (3.1.3) for $\mathbb{H}^{2|2}$ first appeared in [9].

Proof ideas

Our proofs of isomorphism theorems all follow a common strategy. The spin systems we consider possess continuous symmetries, and as a result satisfy integration by parts formulas that are called *Ward identities* in the physics literature. Isomorphism theorems are a direct consequence of these Ward identities.

A key step is to consider a random walk X_t to be a marginal of the joint process $(X_t, \boldsymbol{L}_t)$ of the walk and its local times together. Our Ward identities can be rephrased in terms of the infinitesimal generator of this joint process, and *all* of our isomorphism theorems follow quite quickly by choosing appropriate specializations of the Ward identities. In particular, this gives a unified set of proofs of the BFS–Dynkin, Eisenbaum, and Ray–Knight isomorphism theorems for the SRW.

Structure of this article

Section 3.2 gives our new proofs of the classical isomorphism theorems that link random walks to Gaussian fields. We present our arguments in detail in this familiar context as very similar ideas are used in Sections 3.3 and 3.4, which derive isomorphism theorems for the VRJP and VDJP. We derive supersymmetric isomorphisms for the SRW, the VRJP, and the VDJP in Section 3.5, and Sections 3.6 through 3.8 concern applications of our new isomorphisms.

To keep this article self-contained, Appendix 3.A contains an introduction to the parts of supersymmetry needed to understand our supersymmetric isomorphisms and their applications. In Appendix 3.B we discuss some further aspects of symmetries and supersymmetries that are not needed for our results, but that help place the results of this article in context.

Related literature and future directions

Related literature. For monograph-length treatments of isomorphism theorems and related topics, e.g., loop soups, see [69, 102]. Many proofs of various isomorphism theorems have been given; here we mention only the recent [60, 90]. The major innovation in the present work is that we do not rely on Gaussian calculations. This is important both for obtaining results for $\mathbb{H}^n$ and $\mathbb{S}^n_+$, and for obtaining supersymmetric variants.

Future directions. This article describes isomorphism theorems that link spin systems on $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n_+$ (and the supersymmetric versions when $n = 2$) to random walks. This provides a partial answer to a question of Kozma [61], who asked if there are other spin models (beyond the $\mathbb{H}^{2|2}$ model) with associated random walks. The development of a more systematic connection between spin models and random walks would be very interesting. In particular, it is natural to wonder if there are geometric spaces beyond $\mathbb{R}^n$, $\mathbb{H}^n$, and $\mathbb{S}^n_+$ that have associated isomorphism theorems.

Another interesting future direction would be to clarify the relation between our new isomorphism theorems and loop soups. In the setting of the SRW this connection is well-developed [69, 102] — do these connections extend to the VRJP and VDJP? Similar questions can be asked about random interlacements; for recent progress in this direction see [76].

Notation and conventions

Λ will be a finite set and $\beta = (\beta_{ij})_{i,j\in\Lambda}$ will be a set of edge weights, i.e., $\beta_{ij} = \beta_{ji} \geqslant 0$. The edge weights induce a graph with vertices Λ and edge set $\{\{i,j\} \mid \beta_{ij} > 0\}$, and we will assume that this graph is connected. We also let $\boldsymbol{h} = (h_i)_{i\in\Lambda}$ denote a set of non-negative vertex weights; here we are setting a convention that bold symbols denote objects indexed by Λ. Both β and $\boldsymbol{h}$ will play the role of parameters in our models. For typographical reasons we will sometimes write h in place of $\boldsymbol{h}$ when there is no risk of confusion.

Suppose V is a set equipped with a binary operation $(x, y) \mapsto x \cdot y$. We write V^Λ for the set of maps from Λ to V, denote elements of this set by $\boldsymbol{u} = (u_i)_{i\in\Lambda}$, and let $|\boldsymbol{u}|^2 = (u_i \cdot u_i)_{i\in\Lambda}$. If elements of V are vectors, e.g., $u_i = (u_i^1, \dots, u_i^n) \in \mathbb{R}^n$, then we write $\boldsymbol{u}^\alpha = (u_i^\alpha)_{i\in\Lambda}$ for the collection of α^{th} components.

For a function $f\colon \mathbb{R} \to \mathbb{R}$ we often impose that f is smooth and has *rapid decay*. A sufficient condition is that f and its derivatives decay faster than any polynomial: for every p and k, there are constants $C_{p,k}$ such that the kth derivative satisfies $|f^{(k)}(u)| \leqslant C_{p,k}|u|^{-p}$. If $f\colon \mathbb{R}^n \to \mathbb{R}$, $(u_1, \dots, u_n) \mapsto f(u_1, \dots, u_n)$, then we say f has *rapid decay in* u_1 if $f(\cdot, u_2, \dots, u_n)$ has rapid decay with constants uniform in $u_2, \dots, u_n$. Rapid decay in u_j is defined analogously, and we say such an f has *rapid decay* if it has rapid decay in some coordinate. For a non-smooth function f, we say that f has rapid decay if the the above holds with $k = 0$.

Similarly, we often impose that $f\colon \mathbb{R}^n \to \mathbb{R}^m$ has *moderate growth*. A sufficient condition is that f has at most polynomial growth, i.e., there exists q and C_k such that $|\nabla^k f(u)| \leqslant C_k|u|^q$ for all k.

Given a function $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$, $(i, \boldsymbol{\ell}) \mapsto f(i, \boldsymbol{\ell})$ we say f is *smooth*, *rapidly decaying*, etc. if it has this property with respect to its second coordinate $\boldsymbol{\ell}$. Throughout we will assume functions are Borel measurable without making this explicit.

3.2 Isomorphism theorems for flat geometry

In this section we introduce the simple random walk, the corresponding Gaussian free field, and several well-known isomorphism theorems relating these objects. The method of proof will be used repeatedly in the remainder of the paper when we consider other spin systems. An important aspect of the proofs is that they do not rely on explicit Gaussian computations; this is essential for the generalization of these theorems to non-Gaussian spin systems. Our proofs also show that these results are true for GFFs with any number of components.

Simple random walk and Gaussian free field

Simple random walk. The continuous-time *simple random walk* (SRW) on Λ with symmetric edge weights $\beta \equiv (\beta_{ij})_{i,j\in\Lambda}$, i.e., $\beta_{ij} = \beta_{ji} \geqslant 0$, is the Markov jump process $(X_t)_{t\geqslant 0}$ with transition rates

$$\mathbb{P}[X_{t+dt} = i \mid X_t = j] = \beta_{ij}\, dt. \tag{3.2.1}$$

We write $\mathbb{P}_i$ and $\mathbb{E}_i$ for the law and expectation of X_t when it is started from the vertex i. Formally, X_t is a continuous-time Markov process with generator Δ_β, where the Laplacian Δ_β is the matrix indexed by Λ that acts on $f\colon \Lambda \to \mathbb{R}$ by

$$(\Delta_\beta f)(i) \equiv \sum_{j\in\Lambda} \beta_{ij}(f(j) - f(i)). \tag{3.2.2}$$

In what follows it will be useful to view X_t as a marginal of the Markov process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ consisting of X_t and its *local times* $\boldsymbol{L}_t \equiv (L_t^i)_{i\in\Lambda}$, which are defined by

$$L_t^i \equiv L_0^i + \int_0^t 1_{X_s=i}\, ds, \qquad i \in \Lambda, \tag{3.2.3}$$

where the vector $\boldsymbol{L}_0$ is a collection of free parameters called the *initial local time*. A short computation shows that the generator of $(X_t, \boldsymbol{L}_t)$ acts on smooth functions $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}f)(i,\boldsymbol{\ell}) = (\Delta_\beta f)(i,\ell) + \frac{\partial f(i,\boldsymbol{\ell})}{\partial \ell_i}, \quad \text{i.e.,} \quad \mathcal{L}\boldsymbol{f} = \Delta_\beta \boldsymbol{f} + \partial \boldsymbol{f}, \tag{3.2.4}$$

where Δ_β only acts on the first argument and the last equation uses the vector notation

$$\boldsymbol{f} \equiv (f(i,\boldsymbol{\ell}))_{i\in\Lambda}, \quad \partial\boldsymbol{f} \equiv (\frac{\partial f(i,\boldsymbol{\ell})}{\partial \ell_i})_{i\in\Lambda}. \tag{3.2.5}$$

We write $\mathbb{P}_{i,\boldsymbol{\ell}}$ for the law of $(X, \boldsymbol{L})$ started at $(i,\boldsymbol{\ell}) \in \Lambda\times\mathbb{R}^\Lambda$, and $\mathbb{E}_{i,\boldsymbol{\ell}}$ for its expectation. Note that $\mathbb{E}_{i,\boldsymbol{\ell}} f(X_t, \boldsymbol{L}_t) = \mathbb{E}_{i,\boldsymbol{0}} f(X_t, \boldsymbol{\ell} + \boldsymbol{L}_t)$, and in particular that $f_t(i,\boldsymbol{\ell}) \equiv \mathbb{E}_{i,\boldsymbol{\ell}} f(X_t, \boldsymbol{L}_t)$ is a smooth function with rapid decay in $\boldsymbol{\ell}$ if f is smooth with rapid decay.

Gaussian free field. The (n-component) *Gaussian free field* (*GFF* or $\mathbb{R}^n$ *model*) is a spin system taking values in $\mathbb{R}^n$. Its configurations are elements $\boldsymbol{u} \in (\mathbb{R}^n)^\Lambda$; by an abuse of notation we will write $\mathbb{R}^{n\Lambda}$ in place of $(\mathbb{R}^n)^\Lambda$. Let $\boldsymbol{h} = (h_i)_{i\in\Lambda}$, and assume $h_i \geqslant 0$. To define the probability of a configuration, let

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \qquad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + \frac{1}{2}(\boldsymbol{h}, |\boldsymbol{u}|^2), \tag{3.2.6}$$

where $(\boldsymbol{f},\boldsymbol{g}) \equiv \sum_{i\in\Lambda} f_i g_i$, $|\boldsymbol{u}|^2 \equiv (u_i\cdot u_i)_{i\in\Lambda}$, and $\cdot$ is the Euclidean inner product. In (3.2.6) the Laplacian acts diagonally on the n components of $\boldsymbol{u}$, i.e., $\Delta_\beta\boldsymbol{u} = (\Delta_\beta \boldsymbol{u}^\alpha)_{\alpha=1}^n$, and hence (3.2.6) can be rewritten using

$$(\boldsymbol{u}, -\Delta_\beta\boldsymbol{u}) = \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2, \qquad (\boldsymbol{h}, |\boldsymbol{u}|^2) = \sum_{i\in\Lambda} h_i u_i\cdot u_i, \tag{3.2.7}$$

where $(u_i-u_j)^2$ is shorthand for $(u_i-u_j)\cdot(u_i-u_j)$. Note that another common notation is $h_i = m_i^2 \geqslant 0$, and m_i is called the *mass* at the vertex i. Define the unnormalised expectation $[\cdot]_{\beta,h}$ on functions $F\colon \mathbb{R}^{n\Lambda}\to\mathbb{R}$ by

$$[F]_{\beta,h} \equiv \int_{\mathbb{R}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} d\boldsymbol{u}, \tag{3.2.8}$$

where the integral is with respect to Lebesgue measure $d\boldsymbol{u}$ on $\mathbb{R}^{n\Lambda}$. We set $[\cdot]_\beta \equiv [\cdot]_{\beta,0}$.

The *Gaussian free field* is the probability measure on $\mathbb{R}^{n\Lambda}$ defined by the normalised expectation

$$\langle F\rangle_{\beta,h} \equiv \frac{1}{Z_{\beta,h}}[F]_{\beta,h} = \frac{[Fe^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}]_\beta}{[e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}]_\beta}, \qquad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{3.2.9}$$

Note that for the expectation in (3.2.9) to be well-defined we must have $Z_{\beta,h} < \infty$; this is the case if and only if $h_i > 0$ for some i. The divergence if $\boldsymbol{h} = \boldsymbol{0}$ is due to the invariance of $H_\beta(\boldsymbol{u})$ under the simultaneous translation $u_i \mapsto u_i + s$ for any $s \in \mathbb{R}^n$.

Fundamental integration by parts identity

For any differentiable $f\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ we write

$$T_j f \equiv \frac{\partial f}{\partial u_j^1}, \qquad \boldsymbol{T} f \equiv (T_i f)_{i\in\Lambda}. \tag{3.2.10}$$

Thus T_j is the infinitesimal generator of translations of the j^{th} coordinate in the direction $e^1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. The following lemma is a consequence of the translation invariance of Lebesgue measure, and we will derive all of our isomorphism theorems from this identity. In later sections of this paper we will derive analogous results by replacing the translation symmetry by different symmetries.

Lemma 3.2.1. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be smooth with rapid decay, and let $\rho\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ be smooth with moderate growth. Then:*

$$-\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})u_j^1 \mathcal{L} f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u}) f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta. \tag{3.2.11}$$

In particular, the following integrated version holds for all $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})u_j^1 f(j, \frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t, \boldsymbol{L}_t))\,dt\Big]_\beta. \tag{3.2.12}$$

Remark 3.2.2. Using (3.2.5) and with $(\boldsymbol{T}, \boldsymbol{f}) \equiv \sum_{i\in\Lambda} T_i f_i$, (3.2.11) can be restated compactly as

$$-\Big[(\rho(\boldsymbol{u})\boldsymbol{u}^1, (\mathcal{L}\boldsymbol{f})(\frac{1}{2}|\boldsymbol{u}|^2))\Big]_\beta = \Big[(\boldsymbol{T}\rho(\boldsymbol{u}), \boldsymbol{f}(\frac{1}{2}|\boldsymbol{u}|^2))\Big]_\beta. \tag{3.2.13}$$

Proof. We first prove (3.2.11) by integration by parts. If $f_1, f_2\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ are differentiable and have rapid decay, then integration by parts implies

$$[(T_j f_1) f_2]_\beta = \big[f_1 (T_j^\star f_2)\big]_\beta, \tag{3.2.14}$$

where, for $f\colon \mathbb{R}^{n\Lambda} \to \mathbb{R}$ differentiable,

$$T_j^\star f(\boldsymbol{u}) \equiv -T_j f(\boldsymbol{u}) + (T_j H_\beta(\boldsymbol{u})) f(\boldsymbol{u}). \tag{3.2.15}$$

We now compute the right-hand side of (3.2.15). To simplify notation, let $x_i \equiv u_i^1$ and $\boldsymbol{x} \equiv (x_i)_{i\in\Lambda}$. By (3.2.6), (3.2.2), and using that T_j is the derivative in the x-component,

$$T_j H_\beta(\boldsymbol{u}) = \frac{1}{2} T_j \sum_{i\in\Lambda} u_i \cdot (-\Delta u)_i = (-\Delta_\beta \boldsymbol{x})_j, \tag{3.2.16}$$

so that for a function of the form $f(\frac{1}{2}|\boldsymbol{u}|^2)$,

$$-T_j^\star f(\frac{1}{2}|\boldsymbol{u}|^2) = (\Delta_\beta \boldsymbol{x})_j f(\frac{1}{2}|\boldsymbol{u}|^2) + x_j \frac{\partial f(\frac{1}{2}|\boldsymbol{u}|^2)}{\partial \ell_j}, \tag{3.2.17}$$

where the last term denotes a partial derivative with respect to the jth coordinate of the function f. By applying (3.2.17) to each of the functions $f(j, \frac{1}{2}|\boldsymbol{u}|^2)$ and using $(\boldsymbol{f}_1, \Delta_\beta \boldsymbol{f}_2) = (\Delta_\beta \boldsymbol{f}_1, \boldsymbol{f}_2)$,

$$-\sum_{j\in\Lambda} T_j^\star f(j, \frac{1}{2}|\boldsymbol{u}|^2) = \sum_{j\in\Lambda} x_j \left[\Delta_\beta f(j, \frac{1}{2}|\boldsymbol{u}|^2) + \frac{\partial f(j, \frac{1}{2}|\boldsymbol{u}|^2)}{\partial \ell_j}\right] = \sum_{j\in\Lambda} x_j (\mathcal{L} f)(j, \frac{1}{2}|\boldsymbol{u}|^2). \tag{3.2.18}$$

To verify (3.2.11), multiply (3.2.18) by ρ and use the result to rewrite the left-hand side of (3.2.11). The desired equation then follows by applying (3.2.14):

$$-\sum_{j\in\Lambda}\left[\rho x_j \mathcal{L} f(j, \frac{1}{2}|\boldsymbol{u}|^2)\right]_\beta = \sum_{j\in\Lambda}\left[\rho T_j^\star f(j, \frac{1}{2}|\boldsymbol{u}|^2)\right]_\beta = \sum_{j\in\Lambda}\left[(T_j\rho) f(j, \frac{1}{2}|\boldsymbol{u}|^2)\right]_\beta.$$

We now prove (3.2.12); it suffices to consider f smooth with rapid decay. Indeed, if f_ε is the convolution of f with a smooth mollifier in the second argument, one has $f_\varepsilon \to f$ pointwise and the f_ε are bounded uniformly in ε by a function with rapid decay, so by dominated convergence the result for f follows from the result for the f_ε. Let $f_t(i, \boldsymbol{\ell}) \equiv \mathbb{E}_{i,\boldsymbol{\ell}}(f(X_t, \boldsymbol{L}_t))$, and note that f_t is a smooth function with rapid decay since f has this property (see below (3.2.5)). Apply (3.2.11) to f_t and rewrite the left-hand side using Kolmogorov's backward equation, i.e., $\mathcal{L} f_t = \partial_t f_t$. The result is

$$-\frac{\partial}{\partial t}\sum_{j\in\Lambda}\left[\rho(\boldsymbol{u}) u_j^1 \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t, \boldsymbol{L}_t))\right]_\beta = \sum_{j\in\Lambda}\left[(T_j\rho)(\boldsymbol{u})\, \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_t, \boldsymbol{L}_t)\right]_\beta. \tag{3.2.19}$$

To conclude, integrate (3.2.19) over $(0, \infty)$. The result follows since the boundary term at infinity on the left-hand side vanishes. To see this last claim, recall that the graph induced by β is finite and connected, so $L_t^i \to \infty$ in probability for all vertices $i \in \Lambda$. When f has sufficient decay this implies

$$\lim_{T\to\infty} \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_T, \boldsymbol{L}_T) = 0 \tag{3.2.20}$$

for all $\boldsymbol{u}$. If f has sufficient decay and ρ has moderate growth then (3.2.20) implies

$$\lim_{T\to\infty} [\rho(\boldsymbol{u})\mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2} f(X_T, \boldsymbol{L}_T)]_\beta = 0 \tag{3.2.21}$$

by dominated convergence, as desired. This completes the proof of (3.2.12). □

Our proofs of the classical isomorphism theorems will apply Lemma 3.2.1 with the following choices of ρ and f; further details will be given in the proofs.

- BFS–Dynkin isomorphism: $\rho(\boldsymbol{u}) = u_a$ and $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell}) 1_{j=b}$ with $a, b \in \Lambda$;
- Ray–Knight isomorphism: $T_a\rho(\boldsymbol{u}) \to \delta(u_a) - \delta(u_a - s)$ and $f(j, \boldsymbol{\ell}) \to g(\boldsymbol{\ell})\delta(\ell_a - \frac{s^2}{2}) 1_{j=a}$;
- Eisenbaum isomorphism: $\rho(\boldsymbol{u}) = \exp(s(\boldsymbol{h}, \boldsymbol{u}) - \frac{s^2}{2}(\boldsymbol{h}, \boldsymbol{1}))$ and $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell}) e^{-(h,\boldsymbol{\ell})} 1_{j=a}$.

BFS–Dynkin isomorphism theorem

We now prove the BFS–Dynkin isomorphism theorem.

Theorem 3.2.3. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ have rapid decay, and let $a, b \in \Lambda$. Then:*

$$\left[u_a^1 u_b^1\, g(\frac{1}{2}|\boldsymbol{u}|^2)\right]_\beta = \left[\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t))1_{X_t=b}\,dt\right]_\beta. \tag{3.2.22}$$

Proof. Apply Lemma 3.2.1 with $\rho(\boldsymbol{u}) = u_a^1$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and use $T_j\rho(\boldsymbol{u}) = 1_{j=a}$. □

If $\boldsymbol{h} \neq \boldsymbol{0}$, after replacing $g(\boldsymbol{\ell})$ by $g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{\ell})}$ in (3.2.22) the unnormalised expectation can be normalised using (3.2.9). Since $\mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_t)) = \mathbb{E}_a(g(\boldsymbol{L}_t + \boldsymbol{\ell}))$ for the simple random walk, we immediately obtain Dynkin's formulation of this theorem as stated, e.g., in [102, Theorem 2.8].

Corollary 3.2.4. *Let $\langle\cdot\rangle_\beta$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be bounded, $a, b \in \Lambda$, and suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Then*

$$\left\langle u_a^1 u_b^1\, g(\frac{1}{2}|\boldsymbol{u}|^2)\right\rangle_{\beta,h} = \left\langle\int_0^\infty \mathbb{E}_a\left(g(\boldsymbol{L}_t + \frac{1}{2}|\boldsymbol{u}|^2)e^{-(\boldsymbol{h},\boldsymbol{L}_t)}1_{X_t=b}\right)dt\right\rangle_{\beta,h}. \tag{3.2.23}$$

We have rebranded this the BFS–Dynkin isomorphism because a version of Corollary 3.2.4 first appeared in the work of Brydges, Fröhlich, and Spencer [16, Theorem 2.2].

Ray–Knight isomorphism

The Ray–Knight isomorphism (i.e., the generalised second Ray–Knight theorem) is also a quick consequence of Lemma 3.2.1. Several other proofs of this identity exist for the 1-component GFF, see [47, 90] and references therein. For an explanation of the name, see [102, Remark 2.19].

We introduce the following notation for translations to emphasise the analogy between the classical Ray–Knight isomorphism and its hyperbolic and spherical versions. Let θ_s be the translation of all coordinates by $s \in \mathbb{R}$ in the direction $e^1 = (1, 0, \dots, 0) \in \mathbb{R}^n$, i.e., $\theta_s f(\boldsymbol{u}) \equiv f(\boldsymbol{u} + s\boldsymbol{e}^1)$ for $\boldsymbol{e}^1 = (e^1, \dots, e^1) \in \mathbb{R}^{n\Lambda}$. In particular, $\theta_s\boldsymbol{u} = \boldsymbol{u} + s\boldsymbol{e}^1$. Note that θ_s is the group action associated to the diagonal translation symmetry, which has infinitesimal generator $\sum_{j\in\Lambda} T_j$.

We will write

$$[\delta_{u_0}(u_a)F]_\beta \tag{3.2.24}$$

for the expectation of the spin model in which the spin at vertex a is fixed to $u_0 = (0, \dots, 0) \in \mathbb{R}^n$.

Theorem 3.2.5. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$\left[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\right]_\beta = \left[\mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2} g(\boldsymbol{L}_{\tau(\frac{s^2}{2})})\delta_{u_0}(u_a)\right]_\beta \tag{3.2.25}$$

where $\tau(\gamma) \equiv \inf\{t \mid L_a^t \geq \gamma\}$ and $u_0 = (0, \dots, 0) \in \mathbb{R}^n$.

Proof of Theorem 3.2.5. Since the identity is trivial if $s = 0$, assume $s \neq 0$. The proof is by applying Lemma 3.2.1 with $\rho_\varepsilon(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the functions $\rho_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}$ and $\eta_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0} - \delta_{\theta_s u_0}$ and $\delta_{\frac{1}{2}s^2}$ subject to $\rho_\varepsilon(v)\eta_\varepsilon(\frac{1}{2}|v|^2) = 0$ for all $v \in \mathbb{R}^n$. Explicitly, with $\delta^{(k)}_{\tilde{u},\varepsilon}(x)$ denoting a smooth approximation to a delta function at $\tilde{u} \in \mathbb{R}^k$ with support in the ball $|x - \tilde{u}| < \varepsilon/2$, we may take

$$\rho_\varepsilon(u_a) = \int_0^{s-\varepsilon} \delta^{(n)}_{u_0,\varepsilon}(\theta_{-r}u_a)\,dr, \quad \eta_\varepsilon(\ell) = \delta^{(1)}_{0,\varepsilon}\left(\ell - \frac{1}{2}s^2 - \frac{\varepsilon}{2}\right). \tag{3.2.26}$$

By Lemma 3.2.1, since $\rho_\varepsilon(u_a)\eta_\varepsilon(\frac{1}{2}|u_a|^2) = 0$,

$$\left[T_a\rho_\varepsilon(u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta = \left[\rho_\varepsilon(u_a)u_a^1 g(\frac{1}{2}|\boldsymbol{u}|^2)\eta_\varepsilon(\frac{1}{2}|u_a|^2)\right]_\beta = 0. \tag{3.2.27}$$

Let $dL^a = 1_{X_t=a}\,dt$. By the continuity[1] of $s \mapsto \mathbb{E}_{a,\boldsymbol{\ell}} g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)})$ and the definition of η_ε,

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{E}_{a,\boldsymbol{\ell}} \int_0^\infty g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a) 1_{X_t=a}\,dt &= \lim_{\varepsilon\to 0} \mathbb{E}_{a,\boldsymbol{\ell}} \int_0^\infty g(\boldsymbol{L}_{\tau(L^a)})\eta_\varepsilon(L^a)\,dL^a \\ &= \lim_{\varepsilon\to 0} \int_0^\infty \mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_{\tau(\gamma)}))\eta_\varepsilon(\gamma)\,d\gamma = \mathbb{E}_{a,\boldsymbol{\ell}} g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)}), \quad (3.2.28)\end{aligned}$$

uniformly in $\boldsymbol{\ell}$ with $\ell_a \leqslant \frac{1}{2}s^2$. Since $T_a\rho_\varepsilon(u_a) = \delta^{(n)}_{u_0,\varepsilon}(u_a) - \delta^{(n)}_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)$, taking the limit $\varepsilon \to 0$ in (3.2.27) yields, by (3.2.28),

$$\Big[\mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta = \Big[\mathbb{E}_{a,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta \quad (3.2.29)$$

where we have used the invariance of $[\cdot]_\beta$ under θ_s, i.e., $[F]_\beta = [\theta_s F]_\beta$. To conclude, observe

$$\Big[\mathbb{E}_{a,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}))\delta_{u_0}(u_a)\Big]_\beta = \Big[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\Big]_\beta \quad (3.2.30)$$

since $\tau(\frac{1}{2}s^2) = 0$ if $L_0^a = \frac{s^2}{2}$. □

Eisenbaum isomorphism theorem

The Eisenbaum isomorphism theorem involves a continuous-time random walk with killing. Thus let X_t be a killed random walk with killing rates $\boldsymbol{h}$, and let $\boldsymbol{L}_t$ be its local times. To be precise, the generator of the joint process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ is given by

$$(\mathcal{L}^h f)(i,\boldsymbol{\ell}) \equiv \mathcal{L}f(i,\boldsymbol{\ell}) - h_i f(i,\boldsymbol{\ell}), \quad \text{i.e.,} \quad \mathcal{L}^h = \mathcal{L} - \boldsymbol{h}. \quad (3.2.31)$$

for $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ smooth. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding (deficient) expectation, i.e., integration with respect to the density of the killed random walk, which may have measure less than 1. Note that the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t,\boldsymbol{L}_t)\big) = \mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t,\boldsymbol{L}_t)e^{-\sum_{j\in\Lambda} h_j(L_t^j - \ell_j)}\big), \quad (3.2.32)$$

and we can hence write

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t,\boldsymbol{L}_t)e^{-\sum_{j\in\Lambda} h_j L_t^j}) = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t,\boldsymbol{L}_t))e^{-\sum_{j\in\Lambda} h_j \ell_j}. \quad (3.2.33)$$

Probabilistically, the deficient law can be realised as a Markov process with state space $(\Lambda \cup \{\dagger\}) \times \mathbb{R}^{\Lambda\cup\{\dagger\}}$, where $\dagger \notin \Lambda$ is an absorbing ‘cemetery’ state. The walk jumps from i to $\dagger$ with rate h_i. The generator acts on functions that are identically zero at $\dagger$, and we identify such functions with functions on $\Lambda \times \mathbb{R}^\Lambda$. We denote the time of the one and only jump to $\dagger$ by ζ.

The following theorem is a version of Eisenbaum’s isomorphism [46].

Theorem 3.2.6. *Suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Let $\langle\cdot\rangle_{\beta,h}$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ have moderate growth, let $a \in \Lambda$, and let $s \in \mathbb{R}$. Then*

$$\Big\langle (\theta_s u_a^1) g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2) \Big\rangle_{\beta,h} = s\sum_{i\in\Lambda} h_i \Big\langle \int_0^\infty \mathbb{E}^h_{i,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)1_{X_t=a})\,dt \Big\rangle_{\beta,h}. \quad (3.2.34)$$

[1] To see continuity, since g is compactly supported, it suffices to show that for a sufficiently large T, $s \mapsto \mathbb{E}_{a,\boldsymbol{\ell}} g(\boldsymbol{L}_{\tau(\frac{1}{2}s^2)\wedge T})$ is continuous. Since g is Lipschitz, it suffices to show $\mathbb{E}_{a,\boldsymbol{\ell}}|\boldsymbol{L}_{\tau(\frac{1}{2}s^2-\delta)\wedge T} - \boldsymbol{L}_{\tau(\frac{1}{2}s^2+\delta)\wedge T}|_1 \to 0$ as $\delta \to 0$, $|\cdot|_1$ the 1-norm. Let J_δ be the event that a jump occurs in the interval $[\frac{1}{2}s^2 - \delta, \frac{1}{2}s^2 + \delta]$. Then

$$\mathbb{E}_{a,\boldsymbol{\ell}}|\boldsymbol{L}_{\tau(\frac{1}{2}s^2-\delta)\wedge T} - \boldsymbol{L}_{\tau(\frac{1}{2}s^2+\delta)\wedge T}|_1 \leqslant \delta + T\mathbb{P}_{a,\ell}(J_\delta) = O_T(\delta).$$

Proof. We apply Lemma 3.2.1 with

$$\rho(\boldsymbol{u}) \equiv e^{s(\boldsymbol{h},\boldsymbol{u})-\frac{s^2}{2}(\boldsymbol{h},\boldsymbol{1})} = e^{\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}(e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)}), \tag{3.2.35}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{\ell})}1_{j=a}. \tag{3.2.36}$$

While ρ does not have moderate growth in the sense of our conventions, the very rapid (Gaussian) decay of f is sufficient for the lemma to hold. We then use that $(T_j\rho)(\boldsymbol{u}) = sh_j\rho(\boldsymbol{u})$ to obtain

$$\begin{aligned} s\sum_{j\in\Lambda} h_j\Big[\rho(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{L}_t)})\,dt\Big]_\beta &= \sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})u_j^1 g(\frac{1}{2}|\boldsymbol{u}|^2)1_{j=a}e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta \\ &= \Big[u_a^1 g(\frac{1}{2}|\boldsymbol{u}|^2)e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)}\Big]_\beta, \end{aligned} \tag{3.2.37}$$

by inserting the definition (3.2.35). Using (3.2.33) to substitute

$$\rho(\boldsymbol{u})\mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)e^{(-\boldsymbol{h},\boldsymbol{L}_t)}) = \mathbb{E}^h_{j,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t))e^{-\frac{1}{2}(\boldsymbol{h},|\theta_{-s}\boldsymbol{u}|^2)} \tag{3.2.38}$$

and by the translation invariance of $[\cdot]_\beta$, i.e., $[\theta_s F]_\beta = [F]_\beta$, we can rewrite (3.2.37) as

$$s\sum_{j\in\Lambda} h_j\Big[\Big(\int_0^\infty \mathbb{E}^h_{j,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)1_{X_t=a})\,dt\Big)e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta = \Big[(\theta_s u_a^1)g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)e^{-\frac{1}{2}(\boldsymbol{h},|\boldsymbol{u}|^2)}\Big]_\beta. \tag{3.2.39}$$

This can be re-written in terms of $[\cdot]_{\beta,h}$ as

$$s\sum_{j\in\Lambda} h_j\Big[\int_0^\infty \mathbb{E}^h_{j,\frac{1}{2}|\theta_s\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)1_{X_t=a})\,dt\Big]_{\beta,h} = \Big[(\theta_s u_a^1)g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\Big]_{\beta,h}, \tag{3.2.40}$$

and normalising gives (3.2.34). □

We will now derive the usual formulation of the Eisenbaum isomorphism as a corollary. For notational simplicity, suppose $n = 1$, and let $u_i = u_i^1$. Writing the translations explicitly, Theorem 3.2.6 yields, for $\boldsymbol{s} = (s, s, \ldots, s) \in \mathbb{R}^\Lambda$, $s \neq 0$,

$$\begin{aligned} \Big\langle \frac{u_a+s}{s} g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2)\Big\rangle_{\beta,h} &= \sum_{i\in\Lambda} h_i\Big\langle \mathbb{E}^h_{i,\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2}\int_0^\infty g(\boldsymbol{L}_t)1_{X_t=a}\,dt\Big\rangle_{\beta,h} \\ &= \sum_{i\in\Lambda} h_i\Big\langle \mathbb{E}_i\int_0^\infty g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2+\boldsymbol{L}_t)1_{X_t=a}e^{-\sum_{j\in\Lambda}h_jL_t^j}\,dt\Big\rangle_{\beta,h} \\ &= \sum_{i\in\Lambda} h_i\Big\langle \mathbb{E}_a\int_0^\infty g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2+\boldsymbol{L}_t)1_{X_t=i}e^{-\sum_{j\in\Lambda}h_jL_t^j}\,dt\Big\rangle_{\beta,h} \end{aligned} \tag{3.2.41}$$

where in the last line we have used the reversibility of the killed random walk, i.e., the probability of a path $P_{i\to a}$ is the same as its reversal $P_{a\to i}$. Bringing the sum inside the Gaussian expectation, we recognise the conditional density that X jumps from i to † at time t, proving the following corollary. Recall ζ is the time of the jump to the cemetery state.

Corollary 3.2.7. *Suppose $\boldsymbol{h} \neq \boldsymbol{0}$. Let $\langle\cdot\rangle_{\beta,h}$ be the expectation of the $\mathbb{R}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed SRW. Suppose $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ has moderate growth, $a \in \Lambda$, and $\boldsymbol{s} = (s, s, \ldots, s) \in \mathbb{R}^\Lambda$ with $s \neq 0$. Then*

$$\Big\langle \frac{u_a+s}{s} g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2)\Big\rangle_{\beta,h} = \Big\langle \mathbb{E}^h_a\big(g(\frac{1}{2}|\boldsymbol{u}+\boldsymbol{s}|^2+\boldsymbol{L}_\zeta)\big)\Big\rangle_{\beta,h}. \tag{3.2.42}$$

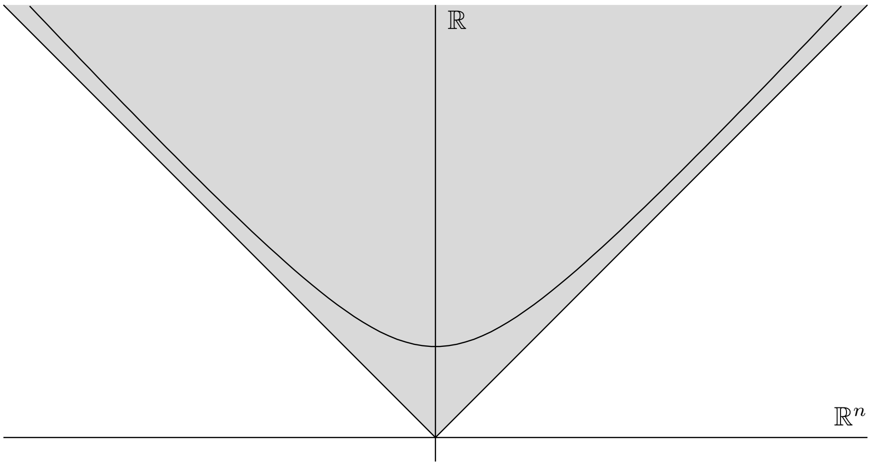

Figure 3.1. Minkowski space $\mathbb{R}^{n,1}$. The shaded area is the causal future and the hyperboloid is $\mathbb{H}^n$.

3.3 Isomorphism theorems for hyperbolic geometry

In this section we describe spin models with hyperbolic symmetry, the associated vertex-reinforced jump processes, and isomorphism theorems that link these objects. The proofs follow closely those of Section 3.2, but with the translation symmetry of $\mathbb{R}^n$ replaced by the boost symmetry of $\mathbb{H}^n$.

The vertex-reinforced jump process

The *vertex-reinforced jump process* (VRJP) X_t with *initial local time* $\boldsymbol{L}_0 \in (0,\infty)^\Lambda$ and *initial vertex* $v \in \Lambda$ is the process X_t with $X_0 = v$ and jump rates

$$\mathbb{P}_{v,\boldsymbol{L}_0}[X_{t+dt} = j \mid (X_s)_{s\leqslant t}, X_t = i] = \beta_{ij} L_t^j \, dt, \tag{3.3.1}$$

where the local times $\boldsymbol{L}_t$ of X_t are defined as in (3.2.3). Note that (3.1.1) with $\varepsilon = 1$ is the special case of (3.3.1) in which $\boldsymbol{L}_0 = \mathbf{1}$. The construction of a VRJP with given initial local times is straightforward, see [31, Section 2]. Our assumption that the graph induced by the edge weights β is connected implies that $L_t^j \to \infty$ as $t \to \infty$ in probability for all j and all sets of initial local times, see [31, Lemma 1].

As in Section 3.2, it will be helpful to view X_t as the marginal of the process $(X_t, \boldsymbol{L}_t)$ that includes the local times $\boldsymbol{L}_t$. For convenience we will also call this joint process a VRJP. Unlike X_t, the joint process $(X_t, \boldsymbol{L}_t)$ is a Markov process. The generator $\mathcal{L}$ of the joint process acts on smooth functions $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}g)(i,\boldsymbol{\ell}) = \sum_{j\in\Lambda} \beta_{ij}\ell_j (g(j,\boldsymbol{\ell}) - g(i,\boldsymbol{\ell})) + \frac{\partial g(i,\boldsymbol{\ell})}{\partial \ell_i}. \tag{3.3.2}$$

We note that $g_t(i,\boldsymbol{\ell}) = \mathbb{E}_{i,\boldsymbol{\ell}}\, g(X_t, \boldsymbol{L}_t)$ is smooth in ℓ for any $t > 0$ if g is smooth. This can be seen, for example, from the explicit construction of the VRJP in [31, Section 2].

Hyperbolic symmetry

The VRJP will be seen to be closely related with hyperbolic symmetry, i.e., the Lorentz group $O(n,1)$. In this subsection we discuss the relevant aspects of this group and its action on Minkowski and hyperbolic space.

Minkowski space. *Minkowski space* $\mathbb{R}^{n,1}$ is the vector space $\mathbb{R}^{n+1}$ equipped with the indefinite *Minkowski inner product*

$$u_1 \cdot u_2 \equiv -u_1^0 u_2^0 + \sum_{\alpha=1}^{n} u_1^\alpha u_2^\alpha, \tag{3.3.3}$$

where each $u_i = (u_i^0, u_i^1, \dots, u_i^n) \in \mathbb{R}^{n,1}$. The points $u \in \mathbb{R}^{n,1}$ with $u \cdot u < 0$ are called *time-like*. The set of time-like vectors with $u^0 > 0$ is called the *causal future*; schematically this is the shaded area in Figure 3.1. In what follows, for $u \in \mathbb{R}^{n,1}$ it will be notationally convenient to write $z = u^0$ and $x = u^1$.

The group preserving the quadratic form $u \cdot u$ given by (3.3.3) is the *Lorentz group* $O(n,1)$. The *restricted Lorentz group* $SO^+(n,1)$ is the subgroup of $T \in O(n,1)$ with $\det T = 1$ and $T_{00} > 0$. $SO^+(n,1)$ preserves the causal future, see Figure 3.1. The elements of $SO^+(n,1)$ can be written as compositions of rotations and boosts. We briefly review the aspects of these transformations needed for what follows. Rotations act on the coordinates $u^1, \dots, u^n$ exactly as in Euclidean space, while a boost θ_s by $s \in \mathbb{R}$ in the xz-plane acts by

$$\theta_s z = x \sinh s + z \cosh s, \quad \theta_s x = x \cosh s + z \sinh s, \quad \theta_s u^\alpha = u^\alpha, \quad (\alpha = 2, \dots, n), \tag{3.3.4}$$

and similarly for boosts in other planes. From (3.3.4) it follows that the infinitesimal generator T of boosts in the xz-plane is the linear differential operator satisfying

$$Tz = x, \quad Tx = z, \quad Tu^\alpha = 0, \quad (\alpha = 2, \dots, n), \tag{3.3.5}$$

i.e.,

$$T \equiv z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}. \tag{3.3.6}$$

Hyperbolic space. When given the metric induced by the Minkowski inner product, the set

$$\mathbb{H}^n \equiv \{u \in \mathbb{R}^{n,1} \mid u \cdot u = -1, z > 0\} \tag{3.3.7}$$

is a model for n-dimensional hyperbolic space. Note that (3.3.7) implies $z \geqslant 1$. For $u, v \in \mathbb{H}^n$, $-u \cdot v = \cosh(d(u,v))$, where $d(u,v)$ is the geodesic distance from u to v. In particular, $-u \cdot v \geqslant 1$. For details on why this is indeed hyperbolic space see, e.g. [24].

$\mathbb{H}^n$ is the orbit under $SO^+(n,1)$ of the point $u_0 = (1, 0, \dots, 0)$, and the stabiliser of u_0 is the subgroup $SO(n)$. Thus $\mathbb{H}^n$ can be identified with $SO^+(n,1)/SO(n)$. It is parameterised by $(u^1, \dots, u^n) \in \mathbb{R}^n$:

$$\mathbb{H}^n = \{u \in \mathbb{R}^{n,1} \mid (u^1, \dots, u^n) \in \mathbb{R}^n, z = \sqrt{1 + (u^1)^2 + \dots (u^n)^2}\}. \tag{3.3.8}$$

In these coordinates, the $SO^+(n,1)$-invariant Haar measure on $\mathbb{H}^n$ can be written as

$$du = \frac{du^1 \dots du^n}{z(u)}, \quad z(u) \equiv \sqrt{1 + (u^1)^2 + \dots + (u^n)^2}. \tag{3.3.9}$$

Note that the Lorentz boost (3.3.4) maps $\mathbb{H}^n$ to $\mathbb{H}^n$, and that in the parameterization of $\mathbb{H}^n$ by $(u^1, \dots, u^n)$, the infinitesimal Lorentz boost in the xz-plane is given by

$$T \equiv z \frac{\partial}{\partial x}. \tag{3.3.10}$$

This is because T satisfies the defining equations (3.3.5): $Tz = x$, $Tx = z$, and $Tu^\alpha = 0$ for $\alpha \geqslant 2$. In the last calculation we have used the definition (3.3.8) of $z(u)$. The invariance of the measure du under Lorentz boosts implies that for differentiable $f \colon \mathbb{H}^n \to \mathbb{R}$ with sufficient decay,

$$\int_{\mathbb{H}^n} Tf \, du = 0. \tag{3.3.11}$$

Hyperbolic sigma model

Hyperbolic spin models are analogues of the Gaussian free field defined in terms of the Minkowski inner product instead of the Euclidean inner product. While it is possible to define a spin model associated to the entire causal future of Minkowski space, see Figure 3.1, for now we restrict ourselves to the *sigma model* version of this model in which spins are constrained to lie in $\mathbb{H}^n$. We will later consider (the supersymmetric version of) a spin model taking values in the causal future in Section 3.7.

In the $\mathbb{H}^n$ sigma model there is a spin $u_i \in \mathbb{H}^n$ for each $i \in \Lambda$. We again let β be a non-negative collection of edge weights and $\boldsymbol{h} \geqslant \boldsymbol{0}$ be a collection of non-negative vertex weights. For a spin configuration $\boldsymbol{u}$ we consider the energy

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) = \frac{1}{4}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2, \quad H_{\beta,h}(\boldsymbol{u}) = H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1}), \tag{3.3.12}$$

analogous to (3.2.6), except that the inner product in $(u_i - u_j)^2 = (u_i - u_j)\cdot(u_i - u_j)$ is now given by the Minkowski inner product. The mass term has also been replaced by the term $(\boldsymbol{h}, \boldsymbol{z} - \boldsymbol{1})$ since $z_i \geqslant 1$ for all i.

Note that $H_\beta(\boldsymbol{u})$ is invariant under the diagonal action of $SO^+(n,1)$, analogous to the invariance of (3.2.6) by the Euclidean group. Moreover, since $u_i \cdot u_i = -1$, we have $(u_i - u_j)^2 = -2 - 2u_i \cdot u_j$, we can thus rewrite $H_\beta(\boldsymbol{u})$ in terms of $\tilde{u} \equiv (u^1, \dots, u^n) \in \mathbb{R}^n$ as

$$H_\beta(\boldsymbol{u}) = -\frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}\left(\sum_{\alpha=1}^n u_i^\alpha u_j^\alpha - z_i z_j + 1\right), \tag{3.3.13}$$

where we recall that $z_i = z_i(\tilde{u}_i)$ is given by (3.3.8). Define an unnormalised expectation $[\cdot]_{\beta,h}$ on functions $F\colon \mathbb{H}^{n\Lambda} \to \mathbb{R}$ by

$$[F]_{\beta,h} \equiv \int_{\mathbb{H}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})}\, d\boldsymbol{u} = \int_{\mathbb{R}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} \prod_{i\in\Lambda} \frac{d\tilde{u}_i}{z(\tilde{u}_i)}, \tag{3.3.14}$$

where $d\boldsymbol{u}$ is the Λ-fold product of the invariant measure on $\mathbb{H}^n$. In the second equality we have written this integral using the parametrization by $\mathbb{R}^n$ in (3.3.9). When $\boldsymbol{h} = \boldsymbol{0}$ we set $[\cdot]_\beta \equiv [\cdot]_{\beta,h}$.

The $\mathbb{H}^n$*-model* is the probability measure on $\mathbb{H}^{n\Lambda}$ defined by the normalised expectation

$$\langle F\rangle_{\beta,h} \equiv \frac{1}{Z_{\beta,h}}[F]_{\beta,h}, \qquad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{3.3.15}$$

Note that for (3.3.15) to be well-defined we must have $Z_{\beta,h} < \infty$. This is the case if and only if $h_i > 0$ for some i due to the invariance of $H_\beta(\boldsymbol{u})$ under the non-compact boost symmetry of $\mathbb{H}^n$.

Remark 3.3.1. This model was studied in [99] as a toy model for some aspects of random band matrices. See Remark 3.5.8 below for further details on this connection.

Fundamental integration by parts identity

The statement of the following lemma is formally identical to that of Lemma 3.2.1. However, the objects in its statement are now hyperbolic versions: $\mathcal{L}$ is the generator of the VRJP, $[\cdot]_\beta$ is the unnormalised expectation from (3.3.14), T_j is the infinitesimal Lorentz boost in the xz-plane in the jth coordinate specified by (3.3.5), and $\frac{1}{2}|\boldsymbol{u}|^2$ is replaced by $\boldsymbol{z}$.

Lemma 3.3.2. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\ell}$ be the expectation of the VRJP. Let $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $\rho\colon \mathbb{H}^{n\Lambda} \to \mathbb{R}$ be smooth with moderate growth. Then:*

$$-\sum_{j\in\Lambda}[\rho(\boldsymbol{u}) x_j \mathcal{L} f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda}[(T_j\rho)(\boldsymbol{u}) f(j,\boldsymbol{z})]_\beta. \tag{3.3.16}$$

In particular, the following integrated version holds for all $f\colon \Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}[\rho(\boldsymbol{u})x_j f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\,dt\Big]_\beta. \tag{3.3.17}$$

Proof. The proof is again by integration by parts and closely follows that of Lemma 3.2.1. Indeed, using that $[\cdot]_\beta$ has density e^{-H_β} with respect to the Lorentz invariant measure on $\mathbb{H}^{n\Lambda}$, the identity (3.3.11) implies that for $f_1,f_2\colon\mathbb{H}^{n\Lambda}\to\mathbb{R}$ smooth and with sufficient decay,

$$[(T_if_1)f_2]_\beta = [f_1(T_i^\star f_2)]_\beta, \tag{3.3.18}$$

where

$$T_i^\star f(\boldsymbol{u}) = -T_if(\boldsymbol{u}) + (T_iH_\beta(\boldsymbol{u}))f(\boldsymbol{u}). \tag{3.3.19}$$

Using (3.3.13) and (3.3.5) yields

$$T_iH_\beta(\boldsymbol{u}) = -\frac{1}{2}\sum_{j,k\in\Lambda}\beta_{jk}T_i(x_jx_k - z_jz_k) = \sum_{j\in\Lambda}\beta_{ij}(x_iz_j - x_jz_i) \tag{3.3.20}$$

and hence, using (3.3.5) and the chain rule to compute T_if,

$$-T_i^\star f(\boldsymbol{z}) = \sum_{j\in\Lambda}\beta_{ij}(x_jz_i - x_iz_j)f(\boldsymbol{z}) + x_i\frac{\partial f(\boldsymbol{z})}{\partial \ell_i}. \tag{3.3.21}$$

Applying (3.3.21) to each function $f(i,\boldsymbol{z})$ and summing over i yields

$$-\sum_{i\in\Lambda}T_i^\star f(i,\boldsymbol{z}) = \sum_{i\in\Lambda}x_i\Big(\sum_{j\in\Lambda}\beta_{ij}z_j(f(j,\boldsymbol{z}) - f(i,\boldsymbol{z})) + \frac{\partial f(i,\boldsymbol{z})}{\partial\ell_i}\Big) = \sum_{i\in\Lambda}x_i(\mathcal{L}f)(i,\boldsymbol{z}) \tag{3.3.22}$$

by the formula (3.3.2) for $\mathcal{L}$. The remainder of the proof follows the proof of Lemma 3.2.1. □

Hyperbolic isomorphism theorems

The following theorems are analogues of the BFS–Dynkin, Ray–Knight, and Eisenbaum isomorphism theorems. Their proofs are analogous to those in Section 3.2, using Lemma 3.3.2 in place of Lemma 3.2.1, and using hyperbolic versions of ρ and f. We begin with the hyperbolic version of the BFS–Dynkin isomorphism, i.e., Theorem 3.2.3. It first appeared in [9] and was proven there using horospherical coordinates. Here we give a more intrinsic proof that avoids horospherical coordinates.

Theorem 3.3.3. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon\mathbb{R}^\Lambda\to\mathbb{R}$ have rapid decay, and let $a,b\in\Lambda$. Then*

$$[x_ax_bg(\boldsymbol{z})]_\beta = \Big[z_a\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt\Big]_\beta. \tag{3.3.23}$$

Proof. Apply Lemma 3.3.2 with $\rho(\boldsymbol{u}) = x_a$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and use $T_j\rho(\boldsymbol{u}) = 1_{j=a}z_j$. □

The next theorem is a hyperbolic version of the Ray–Knight isomorphism, i.e., Theorem 3.2.5. Recall the definition of a boost θ_s by $s\in\mathbb{R}$ in the xz-plane from (3.3.4). In what follows we let θ_s act diagonally on $\boldsymbol{u}\in\mathbb{H}^{n\Lambda}$, and we write $\theta_s\boldsymbol{z}$ to denote the first component of $\theta_s\boldsymbol{u}$. We also write $[f\delta_{u_0}(u_a)]_\beta$ for the expectation of the spin model in which the spin u_a is fixed at $u_0\in\mathbb{H}^n$.

Theorem 3.3.4. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon\mathbb{R}^\Lambda\to\mathbb{R}$ be a smooth compactly supported function, let $a\in\Lambda$, and let $s\in\mathbb{R}$. Then*

$$[g(\theta_s\boldsymbol{z})\delta_{u_0}(u_a)]_\beta = \big[\mathbb{E}_{a,\boldsymbol{z}}g(\boldsymbol{L}_{\tau(\cosh s)})\delta_{u_0}(u_a)\big]_\beta \tag{3.3.24}$$

where $\tau(\gamma) = \inf\{t\mid L_a^t\geq\gamma\}$ and $u_0 = (1,0,\dots,0)\in\mathbb{H}^n$.

Proof of Theorem 3.3.4. Since the identity is trivial if $s = 0$, assume $s \neq 0$. We begin by applying Lemma 3.3.2 with $\rho_\varepsilon(\boldsymbol{u}) = \rho_\varepsilon(u_a)$, $f(j, \boldsymbol{\ell}) = g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, with the functions $\rho_\varepsilon \colon \mathbb{H}^n \to \mathbb{R}$ and $\eta_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{\theta_s u_0}(u_a)$ and $\delta_{\cosh s}(\ell_a)$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a) = 0$ for all $u_a \in \mathbb{H}^n$. Since $s \neq 0$, these conditions can be shown to be satisfiable by explicit construction. Exactly as in the proof of Theorem 3.2.5 this yields

$$\left[T_a\rho_\varepsilon(u_a)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta = 0, \tag{3.3.25}$$

i.e.,

$$\left[\delta_{\theta_{s-\varepsilon}u_0,\varepsilon}(u_0)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt\right]_\beta = \left[\delta_{u_0,\varepsilon}(u_0)\int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \tag{3.3.26}$$

As in (3.2.28), by the continuity[2] of $s \mapsto \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\cosh s)})$ and the definition of η_ε,

$$\lim_{\varepsilon\to 0}\mathbb{E}_{a,\boldsymbol{\ell}}\int_0^\infty g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt = \lim_{\varepsilon\to 0}\int_0^\infty \mathbb{E}_{a,\boldsymbol{\ell}}(g(\boldsymbol{L}_{\tau(\gamma)})\eta_\varepsilon(\gamma)\,d\gamma = \mathbb{E}_{a,\boldsymbol{\ell}}g(\boldsymbol{L}_{\tau(\cosh s)}), \tag{3.3.27}$$

uniformly in $\boldsymbol{\ell}$ with $\ell_a \leqslant \cosh s$.

To conclude, we use (3.3.27) to take $\varepsilon \to 0$ in (3.3.26). More precisely, we use that $\delta_{\theta_s u_0}$ concentrates the u_a integral at $z_a = \cosh s$ on the left-hand side, and hence the time integral at $t = 0$. By the boost invariance of $[\cdot]_\beta$, this term produces the left-hand side of (3.3.24):

$$\left[\delta_{\theta_s u_0}(u_a)\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cosh s)}))\right]_\beta = \left[\delta_{u_0}(u_a)\mathbb{E}_{a,\theta_s\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cosh s)}))\right]_\beta = [\delta_{u_0}(u_a)g(\theta_s\boldsymbol{z})]_\beta. \tag{3.3.28}$$

Again by (3.3.27), the δ_{u_0} on the right-hand side of (3.3.26) concentrates the time integral at $\tau(\cosh s)$, which gives the right-hand side of (3.3.24). $\square$

Finally, we prove a hyperbolic version of the Eisenbaum isomorphism theorem, i.e., Theorem 3.2.6. This concerns a killed VRJP. The generator of this killed process $(X_t, \boldsymbol{L}_t)_{t\geqslant 0}$ acts on smooth functions $f \colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ as

$$(\mathcal{L}^h f)(i, \boldsymbol{\ell}) \equiv \mathcal{L}f(i, \boldsymbol{\ell}) - h_i f(i, \boldsymbol{\ell}), \quad \text{i.e.,} \quad \mathcal{L}^h = \mathcal{L} - \boldsymbol{h}, \tag{3.3.29}$$

where $\mathcal{L}$ is now the generator of the VRJP and h_i are the killing rates. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding deficient expectation. As for the SRW, the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)\big) = \mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j(L_t^j - \ell_j)}\big), \tag{3.3.30}$$

and we can thus write

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t)e^{-\sum_{j\in\Lambda}h_j(L_t^j-1)}) = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t))e^{-\sum_{j\in\Lambda}h_j(\ell_j-1)} = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t))e^{-(\boldsymbol{h},\boldsymbol{\ell}-\boldsymbol{1})}. \tag{3.3.31}$$

Theorem 3.3.5. *Let $\langle\cdot\rangle_{\beta,h}$ be the expectation of the $\mathbb{H}^n$ model, and let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ be the expectation of the killed VRJP with $\boldsymbol{h} \neq \boldsymbol{0}$. Let $g \colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be of moderate growth, and let $s \in \mathbb{R}$. Then*

$$\sum_{i\in\Lambda}\langle(\theta_s x_i)g(i, \theta_s\boldsymbol{z})\rangle_{\beta,h} = \sum_{i\in\Lambda}h_i\left\langle(\theta_s x_i - x_i)\int_0^\infty \mathbb{E}^h_{i,\theta_s\boldsymbol{z}}(g(X_t, \boldsymbol{L}_t))\,dt\right\rangle_{\beta,h}. \tag{3.3.32}$$

[2]Continuity can be proven by an argument similar to the one we gave for simple random walk near (3.2.28): after restricting to times at most T using compact support, the claim follow from the fact that $\mathbb{P}(J_\delta) = O_T(\delta)$ since the jump rates up to time T are bounded by $O(T)$.

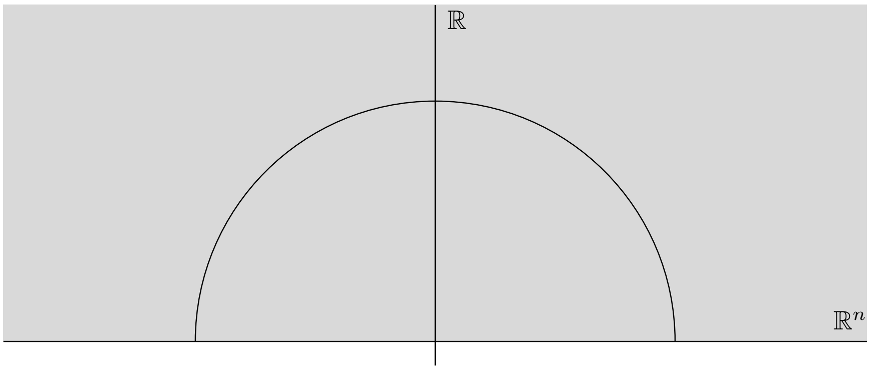

Figure 3.1. The upper half-plane in Euclidean space $\mathbb{R}^{n+1}$ (shaded) and the upper hemisphere $\mathbb{S}^n_+$.

Proof. Analogously to the proof of Theorem 3.2.6, we apply Lemma 3.3.2 with

$$\rho(\boldsymbol{u}) \equiv e^{(\boldsymbol{h},\boldsymbol{z}-\theta_{-s}\boldsymbol{z})} = e^{(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}(e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}) \tag{3.3.33}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{\ell}-\boldsymbol{1})}1_{j=a}, \tag{3.3.34}$$

and use that $(T_j\rho)(\boldsymbol{u}) = h_j(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})$ to obtain

$$\begin{aligned}\sum_{j\in\Lambda} h_j\left[(x_j-\theta_{-s}x_j)\rho(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{L_t}-\boldsymbol{1})})\,dt\right]_\beta \\ = \sum_{j\in\Lambda}\left[\rho(\boldsymbol{u})x_j g(\boldsymbol{z})1_{j=a}e^{-(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}\right]_\beta = \left[x_a^1 g(\boldsymbol{z})e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}\right]_\beta.\end{aligned} \tag{3.3.35}$$

Using (3.3.31) to substitute

$$\rho(\boldsymbol{u})\mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{L}_t-\boldsymbol{1})}) = \mathbb{E}^h_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})e^{-(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{1})}, \tag{3.3.36}$$

and the boost invariance of the spin expectation $[\theta_s\cdot]_\beta = [\cdot]_\beta$, we can rewrite (3.3.35) as

$$\sum_{j\in\Lambda} h_j\left[(\theta_s x_j - x_j)\int_0^\infty \mathbb{E}^h_{j,\theta_s\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})\,dt\right]_{\beta,h} = [(\theta_s x_a)g(\theta_s\boldsymbol{z})]_{\beta,h} \tag{3.3.37}$$

where we have absorbed the magnetic terms $e^{-(\boldsymbol{h},\boldsymbol{z}-\boldsymbol{1})}$ into the measures. Normalising gives (3.3.32). □

3.4 Isomorphism theorems for spherical geometry

In this section we describe analogues of the theorems of Sections 3.2 and 3.3 for spherical geometry.

The vertex-diminished jump process

The *vertex-diminished jump process* (VDJP) $(X_t, \boldsymbol{L}_t)$ with initial conditions $(v, \boldsymbol{L}_0) \in \Lambda \times (0,1]^\Lambda$ is the Markov process with conditional jump rates

$$\mathbb{P}_{v,\boldsymbol{L}_0}[X_{t+dt} = j \mid (X_s)_{s\leqslant t}, X_t = i] = \beta_{ij}L^j_t\,dt \tag{3.4.1}$$

that is stopped at the time $\zeta \equiv \inf\{s \mid \text{exists } j\in\Lambda \text{ s.t. } L^j_s \leqslant 0\}$. Here $\boldsymbol{L}_t$ is the collection of local times of X_t defined by

$$L^j_t \equiv L^j_0 - \int_0^t 1_{X_s=j}\,ds, \tag{3.4.2}$$

and $L_0^j > 0$ is the *initial local time* at j. It is straightforward to see that $(X_t, \boldsymbol{L}_t)$ is well-defined up to ζ by a step-by-step construction as is done for the VRJP in [31]. Note that (3.1.1) with $\varepsilon = -1$ describes the VDJP with $\boldsymbol{L}_0 = \boldsymbol{1}$.

The generator $\mathcal{L}$ of the VDJP acts on smooth functions $g\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}g)(i, \boldsymbol{\ell}) = \sum_{j\in\Lambda} \beta_{ij}\ell_j(g(j,\ell) - g(i,\ell)) - \frac{\partial g(i,\ell)}{\partial \ell_i}. \tag{3.4.3}$$

We write $\mathbb{P}_{i,\boldsymbol{L_0}}$ and $\mathbb{E}_{i,\boldsymbol{L_0}}$ for the law and expectation of the VDJP with initial condition $(i, \boldsymbol{L}_0)$.

Rotational symmetry

We consider the space $\mathbb{R}^{n+1}$ equipped with the Euclidean inner product $u \cdot v = u^0v^0 + \cdots + u^nv^n$, which is preserved by the orthogonal group $O(n+1)$. In the next section we will define an unnormalised expectation exactly as in Section 3.2, but we will investigate the consequences of rotational symmetries instead of translational symmetries.

The hemispherical spin model $\mathbb{S}^n_+$

Hemispherical space. In this section we discuss a spin system that takes values in $\mathbb{S}^n_+$, the open upper hemisphere of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. See Figure 3.1. For notational convenience we write $u = (u^0, \dots, u^n) \in \mathbb{R}^{n+1}$ and let $z = u^0$, and we will also often write $x = u^1$. Then

$$\mathbb{S}^n_+ \equiv \{u \in \mathbb{R}^{n+1} \mid u \cdot u = 1, z > 0\}, \tag{3.4.4}$$

where the inner product is Euclidean. $\mathbb{S}^n_+$ is parametrised by the open unit ball in $\mathbb{R}^n$, i.e., by

$$\mathbb{B}^n = \{(u^1, \dots, u^n) \in \mathbb{R}^n \mid (u^1)^2 + \cdots + (u^n)^2 < 1\}. \tag{3.4.5}$$

Symmetries. In the flat and hyperbolic settings we considered the Euclidean group $O(n) \ltimes \mathbb{R}^n$ and the restricted Lorentz group $SO^+(n,1)$. Unlike in these settings, the orthogonal group $O(n+1)$ does not preserve the hemisphere. Our results, however, were based on the *infinitesimal* symmetries of flat and hyperbolic space, and the hemisphere still possesses useful infinitesimal symmetries. This section briefly explains this; the key identity is (3.4.9).

The infinitesimal symmetries of the hemisphere form a representation of the Lie algebra $\mathfrak{so}(n+1)$, see Appendix 3.B. The associated invariant measure du on $\mathbb{S}^n_+$ can be written in coordinates as

$$du = \frac{du^1 \dots du^n}{z(u)}, \quad z(u) = \sqrt{1 - (u^1)^2 - \cdots - (u^n)^2}. \tag{3.4.6}$$

This is the invariant measure on the full sphere restricted to $\mathbb{S}^n_+$. We let θ_s denote a rotation by $s \in \mathbb{R}$ in the xz-plane. Note that in the coordinates $(x, u^2, \dots, u^n)$ the infinitesimal generator of rotations in the xz-plane is

$$T \equiv z\frac{\partial}{\partial x}, \tag{3.4.7}$$

which acts on the coordinate functions as

$$Tz = -x, \qquad Tx = z, \quad Tu^\alpha = 0, \quad (\alpha = 2, \dots, n). \tag{3.4.8}$$

A consequence of T being an infinitesimal symmetry of the hemisphere is that for compactly supported smooth $f\colon \mathbb{S}^n_+ \to \mathbb{R}$,

$$\int_{\mathbb{S}^n_+} Tf\, du = 0, \tag{3.4.9}$$

an identity which is also easily proven by rewriting the integral as an integral over $\mathbb{S}^n$ and using the rotational invariance of the full sphere.

The $\mathbb{S}^n_+$ model. By a by now familiar abuse of notation, we write $\mathbb{S}^{n\Lambda}_+$ in place of $(\mathbb{S}^n_+)^\Lambda$. Define, for $\boldsymbol{u} \in \mathbb{S}^{n\Lambda}_+$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \mathbf{1} - \boldsymbol{z}), \tag{3.4.10}$$

where as before β and $\boldsymbol{h}$ are collections of non-negative edge and vertex weights, respectively. For $F\colon \mathbb{S}^{n\Lambda}_+ \to \mathbb{R}$ define the unnormalised expectation

$$[F]_{\beta,h} \equiv \int_{\mathbb{S}^{n\Lambda}_+} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})}\, d\boldsymbol{u} = \int_{\mathbb{B}^{n\Lambda}} F(\boldsymbol{u}) e^{-H_{\beta,h}(\boldsymbol{u})} \prod_{i\in\Lambda} \frac{du_i^1 \dots du_i^n}{z(u_i)} \tag{3.4.11}$$

where $d\boldsymbol{u} \equiv \prod_{i\in\Lambda} du_i$, and each du_i is a copy of the invariant measure on $\mathbb{S}^n_+$. The $\mathbb{S}^n_+$ *model* is the probability measure defined by the normalised expectation

$$\langle F \rangle_{\beta,h} \equiv \frac{[F]_{\beta,h}}{Z_{\beta,h}}, \quad Z_{\beta,h} \equiv [1]_{\beta,h}. \tag{3.4.12}$$

Unlike the GFF and $\mathbb{H}^n$-model, the $\mathbb{S}^n_+$ model is well-defined if $\boldsymbol{h} = \mathbf{0}$, and we will omit the subscripts h to indicate $\boldsymbol{h} = \mathbf{0}$.

Remark 3.4.1. The *spherical $O(n)$ models* are obtained by removing the restriction that spins lie in the upper hemisphere in (3.4.11). See Remark 3.4.3 below.

Isomorphism theorems

The following isomorphism theorems are analogues of those in Section 3.2 and 3.3. We again start with a fundamental integration by parts identity, with the change that now $\mathcal{L}$ is the generator of the VDJP, $[\cdot]_\beta$ is the unnormalised expectation of (3.4.11), and T_j is the infinitesimal rotation in the xz-plane in the jth coordinate specified by (3.4.7).

Lemma 3.4.2. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function and let $\rho\colon \mathbb{S}^{n\Lambda}_+ \to \mathbb{R}$ be smooth. Then:*

$$-\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j \mathcal{L} f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} [(T_j\rho)(\boldsymbol{u}) f(j,\boldsymbol{z})]_\beta. \tag{3.4.13}$$

In particular, the following integrated version holds for compactly supported $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$:

$$\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j f(j,\boldsymbol{z})]_\beta = \sum_{j\in\Lambda} \left[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t))\, dt \right]_\beta. \tag{3.4.14}$$

Proof. By (3.4.9) we can integrate by parts. The proof is almost identical to that of Lemma 3.3.2, the only differences being $\mathbb{H}^{n\Lambda}$ is replaced $\mathbb{S}^{n\Lambda}$, and $T_i = z_i \frac{\partial}{\partial x_i}$ is the infinitesimal generator of a rotation in the xz-plane at i instead of a Lorentz boost. This introduces a sign, i.e.,

$$T_i f(\boldsymbol{z}) = -x_i \frac{\partial f(\boldsymbol{z})}{\partial \ell_i} \tag{3.4.15}$$

where the hyperbolic model had a factor of $+1$ in (3.3.21), producing the VDJP generator instead of the VRJP generator. The remainder of the proof is essentially unchanged. $\square$

Remark 3.4.3. Analytically, (3.4.13) holds for the spherical $O(n)$ model, although it is no longer obvious how to interpret $\mathcal{L}$ as the generator of a Markov process since 'jump rates' become negative. In particular, it is unclear how to obtain a formula like (3.4.14). A probabilistic interpretation of $\mathcal{L}$ for the $O(n)$ model, without restricting to the hemisphere, would be very interesting.

The hemispherical BFS–Dynkin isomorphism theorem for the VDJP is as follows:

Theorem 3.4.4. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Suppose $g\colon (0,1]^\Lambda \to \mathbb{R}$ is compactly supported. Then for $a, b \in \Lambda$,*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \left[z_a \int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t) 1_{X_t=b})\, dt \right]_\beta . \tag{3.4.16}$$

Proof. Apply Lemma 3.4.2 with $\rho(\boldsymbol{u}) = x_a$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell}) 1_{j=b}$, and use $T_j \rho(\boldsymbol{u}) = 1_{j=a} z_j$. □

The fact that finite symmetries do not preserve the hemisphere leads to slightly different formulations of the Eisenbaum and Ray–Knight isomorphism theorems as compared to the GFF and $\mathbb{H}^n$ models. We let $[F(\boldsymbol{u})\delta_{u_0}(u_a)]_\beta$ denote the unnormalised expectation for the spin model in which the spin at u_a is fixed to be $u_0 \in \mathbb{S}^n_+$.

Theorem 3.4.5. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\left[\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_{\tau(\cos s)}) 1_{\{\tau(\cos s) < \zeta\}}) \delta_{u_0}(u_a)\right]_\beta = [g(\boldsymbol{z}) \delta_{\theta_s u_0}(u_a)]_\beta \tag{3.4.17}$$

where $\tau(\gamma) = \inf\{t \mid L^a_t \le \gamma\}$ and $u_0 = (1, 0, \dots, 0) \in \mathbb{S}^n_+$.

Proof. The proof is analogous to the proof of Theorem 3.2.5. Since the identity is trivial if $s = 0$, assume $s \neq 0$. We begin by applying Lemma 3.4.2 with $\rho(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell}) \eta_\varepsilon(\ell_a) 1_{j=a}$, with the functions $\rho_\varepsilon\colon \mathbb{S}^n_+ \to \mathbb{R}$ and $\eta_\varepsilon\colon (0,1] \to \mathbb{R}$ chosen such that $T_a\rho$ and η are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{\theta_s u_0}(u_a)$ and $\delta_{\cos s}(\ell_a)$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a) = 0$ for all $u_a \in \mathbb{S}^n_+$. Since $s \neq 0$, these conditions can be shown to be satisfiable by explicit construction. Exactly as in the proof of Theorem 3.2.5 this yields

$$\left[T_a \rho_\varepsilon(u_a) \int_0^\infty \mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t) \eta_\varepsilon(L^a_t) 1_{X_t=a})\, dt \right]_\beta = 0. \tag{3.4.18}$$

To conclude, we use that $\theta_s u_0$ has z-coordinate $\cos s$, so the term with $\delta_{\theta_s u_0}(u_a)$ concentrates the u_a integral at $z_a = \cos s$, and hence the time integral at $t = 0$. This gives the right-hand side of (3.3.24). The term with $\delta_{u_0}(u_a)$ concentrates the time integral at $\tau(\cos s)$ and gives the left-hand side of (3.3.24) as the integrand is non-zero only if $\tau(\cos s) < \zeta$. □

The hemispherical Eisenbaum isomorphism theorem concerns a killed VDJP. The generator of this killed process $(X_t, \boldsymbol{L}_t)_{t \geqslant 0}$ acts on smooth compactly supported $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ by

$$(\mathcal{L}^h f)(i,\boldsymbol{\ell}) \equiv \mathcal{L} f(i,\boldsymbol{\ell}) - h_i f(i,\boldsymbol{\ell}), \quad \text{i.e.,} \quad \mathcal{L}^h = \mathcal{L} - \boldsymbol{h}, \tag{3.4.19}$$

where $\mathcal{L}$ is the generator of the VDJP and $h_i \geqslant 0$ are the killing rates. We let $\mathbb{E}^h_{i,\boldsymbol{\ell}}$ denote the corresponding deficient expectation. As for the SRW, the killing does not depend on the initial local times, i.e.,

$$\mathbb{E}^h_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t)\big) = \mathbb{E}_{i,\boldsymbol{\ell}}\big(g(X_t, \boldsymbol{L}_t) e^{-\sum_{j\in\Lambda} h_j(\ell_j - L^j_t)}\big). \tag{3.4.20}$$

Notice that the sign in the killing term $e^{-\sum_{j\in\Lambda} h_j(\ell_j - L^j_t)}$ is reversed: this because the local times of the VDJP are decreasing rather than increasing by (3.4.2). We can rewrite (3.4.20) as

$$\mathbb{E}_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t) e^{-\sum_{j\in\Lambda} h_j(1 - L^j_t)})) = \mathbb{E}^h_{i,\boldsymbol{\ell}}(g(X_t, \boldsymbol{L}_t)) e^{-\sum_{j\in\Lambda} h_j(1-\ell_j)}. \tag{3.4.21}$$

Theorem 3.4.6. *Let $[\cdot]_\beta$ be the unnormalised expectation of the $\mathbb{S}^n_+$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the killed VDJP. Suppose that $g\colon (0,1]^\Lambda \to \mathbb{R}$ is compactly supported, and $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$\left[x_a g(\boldsymbol{z}) e^{-(\boldsymbol{h}, \boldsymbol{1} - \theta_{-s}\boldsymbol{z})}\right]_\beta = \sum_{i\in\Lambda} h_i \left[(x_i - \theta_{-s} x_i) \int_0^\infty \mathbb{E}^h_{i,\boldsymbol{z}}(g(X_t, \boldsymbol{L}_t))\, dt\, e^{(\boldsymbol{h}, \boldsymbol{1} - \theta_{-s}\boldsymbol{z})} \right]_\beta . \tag{3.4.22}$$

Proof. We apply Lemma 3.4.2 with

$$\rho(\boldsymbol{u}) \equiv e^{(\boldsymbol{h},\theta_{-s}\boldsymbol{z}-\boldsymbol{z})} = e^{(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{z})}(e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})}) \tag{3.4.23}$$

$$f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{\ell})}1_{j=a}, \tag{3.4.24}$$

and use that $(T_j\rho)(\boldsymbol{u}) = h_j(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u})$ to obtain

$$\begin{aligned}\sum_{j\in\Lambda} h_j \Big[(x_j - \theta_{-s}x_j)\rho(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{L}_t)})\,dt\Big]_\beta &= \sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_j g(\boldsymbol{z})1_{j=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{z})}\Big]_\beta \\ &= \Big[x_a g(\boldsymbol{z})e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})}\Big]_\beta.\end{aligned} \tag{3.4.25}$$

Using (3.4.21) to substitute

$$\rho(\boldsymbol{u})\mathbb{E}_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a}e^{-(\boldsymbol{h},\boldsymbol{1}-\boldsymbol{L}_t)}) = \mathbb{E}^h_{j,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=a})e^{-(\boldsymbol{h},\boldsymbol{1}-\theta_{-s}\boldsymbol{z})} \tag{3.4.26}$$

on the left hand side of (3.4.25) gives the desired result. □

3.5 Isomorphism theorems for supersymmetric spin models

In this section we introduce the supersymmetric $\mathbb{R}^{2|2}$, $\mathbb{H}^{2|2}$, and $\mathbb{S}_+^{2|2}$ spin models and derive isomorphism theorems that relate them to the SRW, the VRJP, and the VDJP. Readers who are not familiar with the mathematics of supersymmetry may consult Appendix 3.A, which contains an introduction to supersymmetry as used in this article, before reading this section.

Supersymmetric Gaussian free field

Super-Euclidean space and the SUSY GFF. The *supersymmetric Gaussian free field* (*SUSY GFF* or $\mathbb{R}^{2|2}$ *model*) is defined in terms of the algebra of observables $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda}) \equiv \Omega^{2|\Lambda|}(\mathbb{R}^{2|\Lambda|})$, see Appendix 3.A. This algebra replaces the algebra of observables $C^\infty(\mathbb{R}^{n\Lambda})$ of the usual n-component Gaussian free field.

Concretely, let $(\xi_i)_{i\in\Lambda}$ and $(\eta_i)_{i\in\Lambda}$ be the generators of the Grassmann algebra $\Omega^{2\Lambda}$, let $(x_i, y_i)_{i\in\Lambda}$ be coordinates for $\mathbb{R}^{2\Lambda}$, and let $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ be the algebra with coefficients in $C^\infty(\mathbb{R}^{2\Lambda})$ generated by $(\xi_i)_{i\in\Lambda}$ and $(\eta_i)_{i\in\Lambda}$ as in Appendix 3.A. We call elements F of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ *forms*, and say that a form is smooth, rapidly decaying, compactly supported, etc., if all of its coefficient functions have this property.

We think of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ as the smooth functions on a putative superspace $(\mathbb{R}^{2|2})^\Lambda$, though $(\mathbb{R}^{2|2})^\Lambda$ has no formal meaning, i.e., we will only work with the algebra $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$. There are two ordinary (even) coordinates and two anticommuting (odd) coordinates for each element $i \in \Lambda$, and by analogy with the familiar representation of a vector $u_i \in \mathbb{R}^2$ in terms of its coordinate functions $u_i = (x_i, y_i)$, we will abuse notation and write $u_i \in \mathbb{R}^{2|2}$ to refer to a *supervector* $u_i = (x_i, y_i, \xi_i, \eta_i)$, i.e., a tuple of of even and odd coordinates. Further, we define the super-Euclidean 'inner product' on $\mathbb{R}^{2|2}$ by

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - \xi_i\eta_j + \eta_i\xi_j. \tag{3.5.1}$$

Note that the 'inner product' (3.5.1) defines a form, and is not an inner product in the standard sense of the term. Similarly, we write $\boldsymbol{u} = (u_i)_{i\in\Lambda}$ to denote the collection of the u_i, and define $(\boldsymbol{u}, -\Delta_\beta\boldsymbol{u})$ analogously, i.e., by

$$(\boldsymbol{u}, -\Delta_\beta\boldsymbol{u}) \equiv \sum_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij}(x_i(x_i - x_j) + y_i(y_i - y_j) - \xi_i(\eta_i - \eta_j) + \eta_i(\xi_i - \xi_j)) \tag{3.5.2}$$

$$= \frac{1}{2}\sum_{i,j\in\Lambda}\beta_{ij}(u_i\cdot u_i + u_j\cdot u_j - u_i\cdot u_j - u_j\cdot u_i), \tag{3.5.3}$$

where the second equality is a calculation. The formal rules introduced above imply the last quantity is $\frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}(u_i-u_j)^2$ if we interpret u_i-u_j as $(x_i-x_j, y_i-y_j, \xi_i-\xi_j, \eta_i-\eta_j)$ and use (3.5.1) to compute $(u_i-u_j)^2 \equiv (u_i-u_j)\cdot(u_i-u_j)$.

For $F \in \Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$, the normalised Berezin integral is denoted

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}}\int d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}\, F, \tag{3.5.4}$$

where $\partial_{\boldsymbol{\eta}}\partial_{\boldsymbol{\xi}}$ is defined by $\partial_{\eta_{|\Lambda|}}\partial_{\xi_{|\Lambda|}}\dots\partial_{\eta_1}\partial_{\xi_1}$, $d\boldsymbol{x} = dx_{|\Lambda|}\dots dx_1$, and $d\boldsymbol{y} = dy_{|\Lambda|}\dots dy_1$ for some fixed ordering of the $i\in\Lambda$ from 1 to $|\Lambda|$.

To define the supersymmetric GFF, suppose $\boldsymbol{h} \geqslant \boldsymbol{0}$ and let

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + \frac{1}{2}(\boldsymbol{h}, |\boldsymbol{u}|^2), \tag{3.5.5}$$

where $|\boldsymbol{u}|^2 \equiv (u_i\cdot u_i)_{i\in\Lambda}$, and hence $(\boldsymbol{h}, |\boldsymbol{u}|^2) = \sum_{i\in\Lambda} h_i u_i\cdot u_i$. Both H_β and $H_{\beta,h}$ are elements of $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$. The superexpectation of the *supersymmetric Gaussian free field* is the linear map that assigns to each $F\in\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ the value

$$[F]_{\beta,h} \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} F e^{-H_{\beta,h}}, \tag{3.5.6}$$

and we write $[F]_\beta$ when $\boldsymbol{h}=\boldsymbol{0}$. For $\boldsymbol{h}\neq\boldsymbol{0}$, this superexpectation is indeed normalised; see the paragraph below (3.5.13).

Symmetries. In this section we describe the two aspects of the symmetries of the SUSY GFF that we require. Further details about these symmetries, which form a Lie superalgebra, can be found in Appendix 3.B.

As for the GFF, the infinitesimal generator of translation in the x-direction at $i\in\Lambda$ is

$$T_i \equiv \frac{\partial}{\partial x_i}, \tag{3.5.7}$$

and T_i acts on coordinates as

$$T_i x_j = 1_{i=j}, \quad T_i y_j = 0, \quad T_i\eta_j = 0, \quad T_i\xi_j = 0, \qquad i,j\in\Lambda. \tag{3.5.8}$$

Thus it is analogous to the operators T_i for the ordinary GFF, and it leads to analogous Ward identities, i.e., for forms F with sufficient decay,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} (T_i F) = 0. \tag{3.5.9}$$

For $s\in\mathbb{R}$ the finite symmetry associated to $\sum_{i\in\Lambda} T_i$ will be denoted θ_s, which acts by

$$\theta_s x_i = x_i + s, \quad \theta_s y_i = y_i, \quad \theta_s \eta_i = \eta_i, \quad \theta_s\xi_i = \xi_i, \qquad i\in\Lambda. \tag{3.5.10}$$

The second symmetry of importance is the *supersymmetry generator*

$$Q \equiv \sum_{i\in\Lambda} Q_i \qquad Q_i \equiv \xi_i\frac{\partial}{\partial x_i} + \eta_i\frac{\partial}{\partial y_i} - x_i\frac{\partial}{\partial \eta_i} + y_i\frac{\partial}{\partial\xi_i}, \tag{3.5.11}$$

which acts on coordinates as

$$Qx_i = \xi_i, \quad Qy_i = \eta_i, \quad Q\xi_i = -y_i, \quad Q\eta_i = x_i, \qquad i\in\Lambda. \tag{3.5.12}$$

This supersymmetry generator is responsible for a very powerful Ward identity known as the *localisation lemma*: for any smooth function $f\colon \mathbb{R}^{\Lambda\times\Lambda}\to\mathbb{R}$ with sufficient decay,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} f(\boldsymbol{u}\boldsymbol{u}^T) = f(\boldsymbol{0}), \tag{3.5.13}$$

where $\boldsymbol{u}\boldsymbol{u}^T$ denotes the collection of forms $(u_i\cdot u_j)_{i,j\in\Lambda}$; see Theorem 3.A.8 and Corollary 3.A.10. In particular, the expectation (3.5.6) is normalised if $\boldsymbol{h}\neq\boldsymbol{0}$, i.e., $[1]_{\beta,h}=1$.

Isomorphism theorems for the SUSY GFF. This section presents isomorphism theorems for the SUSY GFF. The statement of the following fundamental Ward identity is formally identical to that of Lemma 3.2.1, but now the expectation $[\cdot]_\beta$ is that of a SUSY GFF.

Lemma 3.5.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $\rho \in \Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$ have moderate growth. Then:*

$$-\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_j\mathcal{L}f(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})f(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta. \tag{3.5.14}$$

In particular, the following integrated version holds for all smooth $f\colon \Lambda\times\mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}\Big[\rho(\boldsymbol{u})x_jf(j,\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \sum_{j\in\Lambda}\Big[(T_j\rho)(\boldsymbol{u})\int_0^\infty \mathbb{E}_{j,\frac{1}{2}|\boldsymbol{u}|^2}(f(X_t,\boldsymbol{L}_t))\,dt\Big]_\beta. \tag{3.5.15}$$

Proof. Starting from (3.5.9), the proof is identical to that of Lemma 3.2.1. □

As a consequence, we obtain the same isomorphism theorems for the supersymmetric GFF as for the non-supersymmetric one. However, for the supersymmetric model, we may in addition use *localisation* to greatly simplify the right-hand side of (3.5.15) when $T_j\rho(\boldsymbol{u})$ is supersymmetric.

Theorem 3.5.2. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay, and let $a,b\in\Lambda$. Then*

$$\Big[x_ax_bg(\frac{1}{2}|\boldsymbol{u}|^2)\Big]_\beta = \int_0^\infty \mathbb{E}_{a,0}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt. \tag{3.5.16}$$

Proof. Apply Lemma 3.5.1 with $\rho(\boldsymbol{u}) = x_a$, $f(j,\boldsymbol{\ell}) = g(\boldsymbol{\ell})1_{j=b}$, and note $T_j\rho(\boldsymbol{u}) = 1_{j=a}$. Thus the integrand on the right-hand side of (3.5.15) is a function of $|\boldsymbol{u}|^2$, and hence is supersymmetric. By applying localisation, i.e., (3.5.13), we conclude

$$\Big[\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(1_{X_t=b}g(\boldsymbol{L}_t))\,dt\Big]_\beta = \int_0^\infty \mathbb{E}_{a,0}(1_{X_t=b}g(\boldsymbol{L}_t))\,dt. \qquad \square$$

Remark 3.5.3. Theorem 3.5.2 has its origins in physics [66, 67, 72, 83]. A formulation similar to the one presented here was given in [22], see also [63].

The Ray–Knight isomorphism theorem applies to spin models in which the spin at vertex a is fixed; in the supersymmetric version the constraint is now $u_a = (0,0,0,0)$. We write the corresponding unnormalised expectation of an observable F as

$$[F\delta_{u_0}(u_a)]_\beta, \tag{3.5.17}$$

where we have defined the pinning using a formal delta function $\delta_{u_0}(u_a)$. The precise meaning of $\delta_{u_0}(u_a)$, and of its smooth approximations used below, may be found in Appendix 3.B.

Theorem 3.5.4. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{R}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the SRW. Let $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ be smooth and compactly supported, let $a\in\Lambda$, and let $s\in\mathbb{R}$. Then*

$$\Big[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\Big]_\beta = \mathbb{E}_{a,0}g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}) \tag{3.5.18}$$

where $\tau(\gamma) \equiv \inf\{t \mid L_a^t \geq \gamma\}$ and $u_0 = (0,0,0,0)$.

Proof. The proof is by applying Lemma 3.5.1 with $\rho(\boldsymbol{u}) \equiv \rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell}) \equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the form $\rho_\varepsilon \in \Omega^2(\mathbb{R}^2)$ and function $\eta_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a) - \delta_{u_0}(\theta_{-s}u_a)$ and $\delta_{\frac{1}{2}s^2}$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(\frac{1}{2}|u_a|^2) = 0$.

An argument identical to the one in the proof of Theorem 3.2.5 shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a}\,dt\right]_\beta$$

$$= \left[\delta_{u_0,\varepsilon}(u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|\boldsymbol{u}|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \qquad (3.5.19)$$

By choosing $\delta_{u_0,\varepsilon}(u_a)$ to be supersymmetric, i.e., $Q\delta_{u_0,\varepsilon} = 0$, the integrand on the right-hand side is a product of supersymmetric forms and is therefore supersymmetric. Applying supersymmetric localisation (i.e., (3.5.13)) hence shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty \mathbb{E}_{a,\frac{1}{2}|u|^2}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta = \int_0^\infty \mathbb{E}_{a,0}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt. \qquad (3.5.20)$$

Applying a global translation $\theta_{s-\varepsilon}$ on the left-hand side and then taking $\varepsilon \to 0$ as in the proof of Theorem 3.2.5 gives the desired result

$$\left[g(\frac{1}{2}|\theta_s\boldsymbol{u}|^2)\delta_{u_0}(u_a)\right]_\beta = \mathbb{E}_{a,0}\,g(\boldsymbol{L}_{\tau(\frac{s^2}{2})}). \qquad \square$$

The preceding two theorems are analogues of the BFS–Dynkin and Ray–Knight isomorphisms for the SUSY GFF. While calculations analogous to those leading to the Eisenbaum isomorphism can be carried out for the SUSY GFF, it is not possible to apply localisation, because the form $\frac{1}{2}|\theta_s\boldsymbol{u}|^2$ that arises (recall (3.2.34)) is not supersymmetric.

SUSY hyperbolic model $\mathbb{H}^{2|2}$

In this section we introduce the supersymmetric analogue of the $\mathbb{H}^2$ model, and then obtain the associated isomorphism theorems.

Super-Minkowski space $\mathbb{R}^{3|2}$ and the super-Minkowski model. Let $(\xi_i,\eta_i)_{i\in\Lambda}$ be the generators of the Grassmann algebra $\Omega^{2\Lambda}$. The algebra of observables $\Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ is the algebra generated by $(\xi_i,\eta_i)_{i\in\Lambda}$ with coefficients in $C^\infty(\mathbb{R}^{3\Lambda})$. Choosing orthonormal coordinates $(z_i,x_i,y_i)_{i\in\Lambda}$ for $\mathbb{R}^{3\Lambda}$, a supervector $u_i \in \mathbb{R}^{3|2}$ is a tuple of even and odd coordinates $u_i = (z_i,x_i,y_i,\xi_i,\eta_i)$, and we say that $\mathbb{R}^{3|2}$ is a *super-Minkowski space* when equipped with the 'inner product'

$$u_i\cdot u_j \equiv -z_iz_j + x_ix_j + y_iy_j - \xi_i\eta_j + \eta_i\xi_j. \qquad (3.5.21)$$

We have written 'inner product' to emphasise that $u_i\cdot u_j$ is a form, and hence this is not an inner product in the standard sense of the term.

$\mathbb{H}^{2|2}$ sigma model. To define a supersymmetric analogue of $\mathbb{H}^2$, define the even form

$$z = z(x,y,\xi,\eta) \equiv \sqrt{1+x^2+y^2-2\xi\eta} = \sqrt{1+x^2+y^2} - \frac{\xi\eta}{\sqrt{1+x^2+y^2}}. \qquad (3.5.22)$$

Using the definition (3.5.21), a short calculation shows that $u_i\cdot u_i = -1$, just as for $\mathbb{H}^2$. The algebra of forms $\Omega^2(\mathbb{H}^2)$ is the algebra over $C^\infty(\mathbb{H}^2)$ generated by two Grassmann generators ξ and η. In coordinates, we have $F(u) = F(z,x,y,\xi,\eta) = F(\sqrt{1+x^2+y^2-2\xi\eta},x,y,\xi,\eta)$, and hence every form $F \in \Omega^2(\mathbb{H}^2)$ can be identified with a form in $\Omega^2(\mathbb{R}^2)$. Using this correspondence we define the Berezin integral for $F \in \Omega^2(\mathbb{H}^2)$ as

$$\int_{\mathbb{H}^{2|2}} F \equiv \int_{\mathbb{R}^{2|2}} \frac{1}{z}F = \frac{1}{2\pi}\int dx\,dy\,\partial_\xi\,\partial_\eta\frac{1}{z}F \qquad (3.5.23)$$

where on the right-hand side we are viewing F as a form in $\Omega^2(\mathbb{R}^2)$. Similarly,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} \frac{1}{\prod_{i\in\Lambda} z_i} F = \frac{1}{(2\pi)^{|\Lambda|}} \int d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}\, \frac{1}{\prod_{i\in\Lambda} z_i} F \tag{3.5.24}$$

where we note there is no ambiguity in the product of the z_i as they are even forms.

Define, for $\boldsymbol{h} \geqslant \boldsymbol{0}$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z}-\boldsymbol{1}), \tag{3.5.25}$$

where

$$\begin{aligned}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) &\equiv \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(u_i\cdot u_i + u_j \cdot u_j - u_i\cdot u_j - u_j \cdot u_i) = \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(-2 - 2u_i\cdot u_j),\\ (\boldsymbol{h}, \boldsymbol{z}-\boldsymbol{1}) &\equiv \sum_{i\in\Lambda} h_i(z_i - 1),\end{aligned} \tag{3.5.26}$$

and each $u_i \cdot u_j$ is defined as in (3.5.21). The equality in the first line holds because $u_i\cdot u_i = -1$. We define the $\mathbb{H}^{2|2}$ *model* superexpectation for $F \in \Omega^{2\Lambda}(\mathbb{H}^{2\Lambda})$ by

$$[F]_{\beta,h} \equiv \int_{(\mathbb{H}^{2|2})^\Lambda} F e^{-H_{\beta,h}}, \tag{3.5.27}$$

and we write $[F]_\beta$ in the case $\boldsymbol{h} = \boldsymbol{0}$. For $\boldsymbol{h} \neq \boldsymbol{0}$, the superexpectation is normalised, i.e., $[1]_{\beta,h} = 1$. This is a consequence of supersymmetry, see (3.5.32) below.

Symmetries. There are two symmetries necessary for what follows, and we introduce them in this section. For a further discussion of the Lie superalgebra of symmetries associated to the $\mathbb{H}^{2|2}$ model see Appendix 3.B.

The first relevant symmetry is the infinitesimal Lorentz boost in the xz plane at $i \in \Lambda$:

$$T_i \equiv z_i \frac{\partial}{\partial x_i} = \sqrt{1 + x^2 + y^2 - 2\xi\eta}\,\frac{\partial}{\partial x_i}, \tag{3.5.28}$$

which acts on coordinates as

$$T_i z_j = x_j 1_{i=j}, \quad T_i x_j = z_j 1_{i=j}, \quad T_i y_j = 0, \quad T_i \xi_j = 0, \quad T_i \eta_j = 0 \qquad i,j\in\Lambda. \tag{3.5.29}$$

As for the SUSY GFF, this leads to a Ward identity for forms F with rapid decay:

$$\int_{(\mathbb{H}^{2|2})^\Lambda} (T_i F) = 0. \tag{3.5.30}$$

For $s \in \mathbb{R}$ the finite symmetry associated to $\sum_{i\in\Lambda} T_i$ will be denoted θ_s, and acts as (for $j \in \Lambda$)

$$\theta_s z_j = z_j \cosh s + x_j \sinh s, \;\; \theta_s x_j = x_j \cosh s + z_j \sinh s, \;\; \theta_s y_j = y_j, \;\; \theta_s \xi_j = \xi_j, \;\; \theta_s \eta_j = \eta_j. \tag{3.5.31}$$

The second relevant symmetry is the supersymmetry generator Q, which is defined by (3.5.11). Note that z_i can be written as $z_i = \sqrt{1+|\tilde{u}_i|^2}$, where $\tilde{u}_i \equiv (x_i, y_i, \xi_i, \eta_i) \in \mathbb{R}^{2|2}$. Thus, z_i is supersymmetric, i.e., $Qz_i = 0$. This implies the same localisation Ward identity applies for $\mathbb{H}^{2|2}$ as for $\mathbb{R}^{2|2}$, i.e., for smooth functions $f\colon \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda\times\Lambda} \to \mathbb{R}$ with sufficient decay,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} f(\boldsymbol{z}, \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T) = f(\boldsymbol{1}, \boldsymbol{0}) \tag{3.5.32}$$

where $\boldsymbol{0}$ is the matrix indexed by Λ with all entries 0, and we have written $\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T$ to denote the set of forms $(\tilde{u}_i \cdot \tilde{u}_j)_{i,j\in\Lambda}$.

Isomorphism theorems for the $\mathbb{H}^{2|2}$ model. Let $\mathbb{E}_{i,\boldsymbol{\ell}}$ denote the expectation for a VRJP started from initial conditions $(i,\boldsymbol{\ell})$. We begin with the SUSY analogue of Lemma 3.3.2.

Lemma 3.5.5. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $f\colon \Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ be a smooth function with rapid decay, and let $\rho\in\Omega^{2\Lambda}(\mathbb{H}^{2\Lambda})$ have moderate growth. Then:*

$$-\sum_{j\in\Lambda}[\rho(\boldsymbol{u})x_j\mathcal{L}f(j,\boldsymbol{z})]_\beta=\sum_{j\in\Lambda}[(T_j\rho)(\boldsymbol{u})f(j,\boldsymbol{z})]_\beta. \tag{3.5.33}$$

In particular, the following integrated version holds for all smooth $f\colon\Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ with rapid decay:

$$\sum_{j\in\Lambda}[\rho(\boldsymbol{u})x_jf(j,\boldsymbol{z})]_\beta=\sum_{j\in\Lambda}\left[(T_j\rho)(\boldsymbol{u})\int_0^\infty\mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\,dt\right]_\beta. \tag{3.5.34}$$

Proof. The proof is identical to that of Lemma 3.3.2. □

The SUSY analogue of Theorem 3.3.3 is the following.

Theorem 3.5.6. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon\Lambda\times\mathbb{R}^\Lambda\to\mathbb{R}$ be a smooth function with rapid decay, and let $a,b\in\Lambda$. Then*

$$[x_ax_bg(\boldsymbol{z})]_\beta=\int_0^\infty\mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt. \tag{3.5.35}$$

Proof. Apply Lemma 3.5.5 with $\rho(\boldsymbol{u})=x_a$ and $f(j,\boldsymbol{\ell})=g(\boldsymbol{\ell})1_{j=b}$. Thus $T_j\rho(\boldsymbol{u})=1_{j=a}z_a$. By applying localisation, i.e., (3.5.32), we obtain

$$[x_ax_bg(\boldsymbol{z})]_\beta=\left[z_a\int_0^\infty\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt\right]_\beta=\int_0^\infty\mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_t)1_{X_t=b})\,dt.$$ □

Theorem 3.5.7. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. Let $g\colon\mathbb{R}^\Lambda\to\mathbb{R}$ be a smooth compactly supported function, let $a\in\Lambda$, and let $s\in\mathbb{R}$. Then*

$$[g(\theta_s\boldsymbol{z})\delta_{u_0}(u_a)]_\beta=\mathbb{E}_{a,\mathbf{1}}g(\boldsymbol{L}_{\tau(\cosh s)}) \tag{3.5.36}$$

where $\tau(\gamma)=\inf\{t\,|\,L_a^t\geq\gamma\}$ and $u_0=(1,0,0,0,0)$.

Proof. Applying Lemma 3.5.5 with $\rho(\boldsymbol{u})\equiv\rho_\varepsilon(u_a)$, $f(j,\boldsymbol{\ell})\equiv g(\boldsymbol{\ell})\eta_\varepsilon(\ell_a)1_{j=a}$, and the form $\rho_\varepsilon\in\Omega^2(\mathbb{H}^2)$ and function $\eta_\varepsilon\colon\mathbb{R}_+\to\mathbb{R}$ chosen such that $T_a\rho_\varepsilon$ and η_ε are smooth compactly supported approximations to $\delta_{u_0}(u_a)-\delta_{\theta_su_0}(u_a)$ and $\delta_{\cosh s}$ subject to $\rho_\varepsilon(u_a)\eta_\varepsilon(z_a)=0$, an argument identical to the proof of Theorem 3.3.4 shows

$$\left[\delta_{u_0,\varepsilon}(\theta_{-(s-\varepsilon)}u_a)\int_0^\infty\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta=\left[\delta_{u_0,\varepsilon}(u_a)\int_0^\infty\mathbb{E}_{a,\boldsymbol{z}}(g(\boldsymbol{L}_t)\eta_\varepsilon(L_t^a)1_{X_t=a})\,dt\right]_\beta. \tag{3.5.37}$$

As in the proof of Theorem 3.5.4, $\delta_{u_0,\varepsilon}(u_a)$ is chosen to be supersymmetric. The claim follows by applying localisation to the right-hand side, boosting the left-hand side by $\theta_{s-\varepsilon}$, and then taking $\varepsilon\to0$ as in the proof of Theorem 3.3.4:

$$\left[\mathbb{E}_{a,\boldsymbol{z}}g(\boldsymbol{L}_{\tau(\cosh s)})\delta_{u_0}(u_a)\right]_\beta=\mathbb{E}_{a,\mathbf{1}}g(\boldsymbol{L}_{\tau(\cosh s)}).$$ □

Remark 3.5.8. The $\mathbb{H}^{2|2}$ model was introduced in [105]; it serves as a toy model for Efetov's supersymmetric approach to studying random band matrices [45]. The connection between random band matrices and hyperbolic symmetry goes back to Wegner and Schäfer [93, 103], and Efetov made use of supersymmetry to avoid the use of the replica trick. For further discussion see [40], and for other uses of supersymmetry in the study of random matrices see, e.g., [35, 36, 96].

Remark 3.5.9. Unlike the $\mathbb{H}^n$ models, the $\mathbb{H}^{2|2}$ model captures the phenomenology of a localisation/delocalisation transition [40, 97]. Probabilistically, this is captured by the recurrence/transience of the VRJP.

SUSY hemispherical model $\mathbb{S}_+^{2|2}$

In this section we introduce the supersymmetric analogue of the $\mathbb{S}_+^2$ model, and then obtain the associated isomorphism theorems.

Integrals over $\mathbb{S}_+^{2|2}$. In this subsection we work with smooth compactly supported forms in $\Omega^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$, which we denote $\Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$. Concretely, we will identify such forms with compactly supported forms in $\Omega^{2\Lambda}(\mathbb{B}^{2\Lambda})$, where $\mathbb{B}^2$ is the open unit ball, by setting

$$z = z(x, y, \xi, \eta) \equiv \sqrt{1 - x^2 - y^2 + 2\xi\eta} = \sqrt{1 - x^2 - y^2} + \frac{\xi\eta}{\sqrt{1 - x^2 - y^2}}, \tag{3.5.38}$$

By considering $\mathbb{B}^2$ as a subset of $\mathbb{R}^2$, a compactly supported form in $\Omega^{2\Lambda}(\mathbb{B}^{2\Lambda})$ can be trivially extended to a form in $\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda})$, and we may therefore apply the results of Appendix 3.A.

Similarly to the notation introduced in Section 3.5, let $u_i = (z_i, x_i, y_i, \xi_i, \eta_i)$, and let

$$u_i \cdot u_j \equiv z_i z_j + x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \qquad i, j \in \Lambda. \tag{3.5.39}$$

With these definitions, $u_i \cdot u_i = 1$, just as for $\mathbb{S}_+^2$. We define, for $F \in \Omega_c^2(\mathbb{S}_+^2)$,

$$\int_{\mathbb{S}_+^{2|2}} F \equiv \frac{1}{2\pi} \int dx\, dy\, \partial_\xi\, \partial_\eta \frac{1}{z} F, \tag{3.5.40}$$

and similarly, for $F \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$,

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}} \int d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} \frac{1}{\prod_{i\in\Lambda} z_i} F, \tag{3.5.41}$$

where we note there is no ambiguity in the product of the z_i as they are even forms.

$\mathbb{S}_+^{2|2}$ *model.* Define, for $\boldsymbol{h} \geqslant \boldsymbol{0}$,

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}), \quad H_{\beta,h}(\boldsymbol{u}) \equiv H_\beta(\boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{1} - \boldsymbol{z}), \tag{3.5.42}$$

where

$$\begin{aligned}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) &\equiv \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(u_i\cdot u_i + u_j \cdot u_j - u_i \cdot u_j - u_j \cdot u_i) = \frac{1}{2}\sum_{i,j\in\Lambda}\beta_{ij}(2 - 2u_i\cdot u_j),\\ (\boldsymbol{h}, \boldsymbol{1} - \boldsymbol{z}) &\equiv \sum_{i\in\Lambda} h_i(1 - z_i)\end{aligned} \tag{3.5.43}$$

and $u_i \cdot u_j$ is defined as in (3.5.39). We define the $\mathbb{S}_+^{2|2}$ *model* superexpectation of $F \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$ by

$$[F]_\beta \equiv \int_{(\mathbb{S}_+^{2|2})^\Lambda} F e^{-H_\beta}, \quad [F]_{\beta,h} \equiv \int_{(\mathbb{S}_+^{2|2})^\Lambda} F e^{-H_{\beta,h}}. \tag{3.5.44}$$

Symmetries. As in the previous sections, there are two symmetries of relevance to the following discussion. For details on the Lie superalgebra associated to $\mathbb{S}_+^{2|2}$, see Appendix 3.B. The first symmetry of relevance is an infinitesimal rotation in the xz-plane at $i \in \Lambda$, which has generator

$$T_i \equiv z_i \frac{\partial}{\partial x_i} = \sqrt{1 - x_i^2 - y_i^2 + 2\xi_i\eta_i}\frac{\partial}{\partial x_i}, \tag{3.5.45}$$

and acts on coordinates as

$$T_i z_j = -x_j 1_{i=j}, \quad T_i x_j = z_j 1_{i=j}, \quad T_i y_j = 0, \quad T_i \xi_j = 0, \quad T_i \eta_j = 0, \qquad i, j \in \Lambda. \tag{3.5.46}$$

As for the SUSY GFF, this leads to a Ward identity for all sufficiently rapidly decaying forms F:

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} (T_i F) = 0. \tag{3.5.47}$$

For $s \in \mathbb{R}$ the finite rotation associated to $\sum_{i\in\Lambda} T_i$ is denoted θ_s, and acts as, for $j \in \Lambda$,

$$\theta_s z_j = z_j \cos s - x_j \sin s, \quad \theta_s x_j = x_j \cos s + z_j \sin s, \quad \theta_s y_j = y_j, \quad \theta_s \xi_j = \xi_j, \quad \theta_s \eta_j = \eta_j. \tag{3.5.48}$$

The second symmetry of importance is the supersymmetry generator Q defined by (3.5.11). Note that z_i can be written as $z_i = \sqrt{1 - |\tilde{u}_i|^2}$, where $\tilde{u}_i \equiv (x_i, y_i, \xi_i, \eta_i) \in \mathbb{R}^{2|2}$. It follows that z_i is supersymmetric, i.e., $Qz_i = 0$. This implies the same localisation Ward identity applies for $\mathbb{S}_+^{2|2}$ as for $\mathbb{R}^{2|2}$, i.e., for $f\colon (0,1]^\Lambda \times [-1,1]^{\Lambda\times\Lambda} \to \mathbb{R}$ that are smooth and compactly supported,

$$\int_{(\mathbb{S}_+^{2|2})^\Lambda} f(\boldsymbol{z}, \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T) = f(\mathbf{1}, \mathbf{0}), \tag{3.5.49}$$

where $\mathbf{0}$ is the matrix indexed by Λ with all entries 0 and $\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}^T \equiv (\tilde{u}_i \cdot \tilde{u}_j)_{i,j\in\Lambda}$.

Isomorphism theorems for the $\mathbb{S}_+^{2|2}$ model. Let $\mathbb{E}_{i,\boldsymbol{\ell}}$ denote the expectation for a VDJP started from initial conditions $(i, \boldsymbol{\ell}) \in \Lambda \times (0,1]^\Lambda$.

Lemma 3.5.10. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function and let $\rho \in \Omega_c^{2\Lambda}(\mathbb{S}_+^{2\Lambda})$. Then:*

$$-\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j \mathcal{L} f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} [(T_j\rho)(\boldsymbol{u}) f(j, \boldsymbol{z})]_\beta. \tag{3.5.50}$$

In particular, the following integrated version holds for smooth and compactly supported $f\colon \Lambda \times (0,1]^\Lambda \to \mathbb{R}$:

$$\sum_{j\in\Lambda} [\rho(\boldsymbol{u}) x_j f(j, \boldsymbol{z})]_\beta = \sum_{j\in\Lambda} \left[(T_j\rho)(\boldsymbol{u}) \int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t, \boldsymbol{L}_t))\, dt \right]_\beta. \tag{3.5.51}$$

Proof. The proof is identical to that of Lemma 3.4.2. □

The SUSY analogue of Theorem 3.4.4 is the following.

Theorem 3.5.11. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, and let $a, b \in \Lambda$. Then*

$$[x_a x_b g(\boldsymbol{z})]_\beta = \int_0^\infty \mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_t) 1_{X_t=b})\, dt. \tag{3.5.52}$$

Proof. The proof is essentially identical to that of Theorem 3.5.6. □

Theorem 3.5.12. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{S}_+^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VDJP. Let $g\colon (0,1]^\Lambda \to \mathbb{R}$ be a smooth compactly supported function, let $a \in \Lambda$, and let $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then*

$$[g(\boldsymbol{z}) \delta_{\theta_s u_0}(u_a)]_\beta = \mathbb{E}_{a,\mathbf{1}}(g(\boldsymbol{L}_{\tau(\cos s)}) 1_{\tau(\cos s) < \zeta}) \tag{3.5.53}$$

where $\tau(\gamma) = \inf\{t \mid L_t^a \le \gamma\}$ and $\theta_s u_0 = (\cos s, \sin s, 0, 0, 0) \in \mathbb{S}_+^{2|2}$.

Proof. The proof is, *mutatis mutandis*, identical to that of Theorem 3.5.7. □

3.6 Application to limiting local times: the Sabot–Tarrès limit

In [89], Sabot and Tarrès established the first connection between the vertex-reinforced jump process and the SUSY hyperbolic sigma model. Their result relates the asymptotic local time distribution of a time-changed VRJP to a certain *horospherical* marginal of the $\mathbb{H}^{2|2}$ model. In this section we derive their result (as stated in [91, Appendix B]) from the Ray–Knight isomorphism for the $\mathbb{H}^{2|2}$ model. The essence of the result is the following corollary of Theorem 3.5.7. Recall that we write $(z, x, y, \xi, \eta) \in \mathbb{R}^{3|2}$.

Corollary 3.6.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $\mathbb{E}_{i,\boldsymbol{\ell}}$ be the expectation of the VRJP. For $g\colon \mathbb{R}^\Lambda \to \mathbb{R}$ smooth and compactly supported,*

$$\lim_{\gamma\to\infty} \mathbb{E}_{a,\mathbf{1}}\left(g(\frac{1}{\gamma}\boldsymbol{L}_{\tau(\gamma)})\right) = [g(\boldsymbol{z}+\boldsymbol{x})\delta_{u_0}(u_a)]_\beta \tag{3.6.1}$$

where $\tau(\gamma) = \inf\{t | L_a^t \geqslant \gamma\}$ and $u_0 = (1,0,0,0,0)$.

Proof. We write $\gamma = \cosh s$. Then by Theorem 3.5.7 applied to $g(\boldsymbol{L}_{\tau(\cosh s)}/\cosh s)$,

$$\begin{aligned}\mathbb{E}_{a,\mathbf{1}}\left(g(\frac{1}{\cosh s}\boldsymbol{L}_{\tau(\cosh s)})\right) &= \left[g(\frac{1}{\cosh s}\theta_s\boldsymbol{z})\delta_{u_0}(u_a)\right]_\beta \\ &= [g(\boldsymbol{z}+\boldsymbol{x}\tanh s)\delta_{u_0}(u_a)]_\beta,\end{aligned} \tag{3.6.2}$$

by using (3.5.31) to compute $\theta_s\boldsymbol{z} = \cosh s\boldsymbol{z} + \sinh s\boldsymbol{x}$. Since $\tanh s \to 1$ as $s\to\infty$, by dominated convergence we obtain

$$\lim_{s\to\infty} \mathbb{E}_{a,\mathbf{1}}\left(g(\frac{1}{\cosh s}\boldsymbol{L}_{\tau(\cosh s)})\right) = [g(\boldsymbol{z}+\boldsymbol{x})\delta_{u_0}(u_a)]_\beta. \qquad \square$$

We now recover [89, Theorem 2] as stated in [92, Theorem B]. Write $\log(\boldsymbol{v}) = (\log(v_i))_{i\in\Lambda}$. Applying Corollary 3.6.1 to a function $g\circ\log$ yields

$$\lim_{\gamma\to\infty} \mathbb{E}_{a,\mathbf{1}}\big(g(\log(\boldsymbol{L}_{\tau(\gamma)}) - \log\gamma)\big) = [g(\log(\boldsymbol{z}+\boldsymbol{x}))\delta_{u_0}(u_a)]_\beta \tag{3.6.3}$$

where $\log\gamma = (\log\gamma)_{i\in\Lambda}$. To recover [89, Theorem 2] we rewrite the right-hand side of (3.6.3). To do this, recall, e.g., from [40, Section 2.2], that horospherical coordinates for the $\mathbb{H}^{2|2}$ model are given by the change of generators from (x, y, ξ, η) to $(s, t, \psi, \bar\psi)$, where

$$\begin{gathered} x \equiv \sinh t - \frac{1}{2}(s^2 + 2\psi\bar\psi)e^t, \quad y \equiv se^t, \quad z \equiv \cosh t + \frac{1}{2}(s^2+2\psi\bar\psi)e^t, \\ \xi \equiv \psi e^t, \quad \eta \equiv \bar\psi e^t. \end{gathered} \tag{3.6.4}$$

Let

$$H_1(\boldsymbol{t}) \equiv \frac{1}{2}\sum_{i,j\in\Lambda} \beta_{ij}(\cosh(t_i - t_j) - 1). \tag{3.6.5}$$

The right-hand side of (3.6.3) can be written explicitly in horospherical coordinates as

$$[g(\log(\boldsymbol{z}+\boldsymbol{x}))\delta(u_a)]_\beta = \frac{1}{\sqrt{2\pi}^{|\Lambda|-1}} \int_{\mathbb{R}^{|\Lambda|-1}} g(\boldsymbol{t})e^{-H_1(\boldsymbol{t})}\sqrt{\det D(\beta,\boldsymbol{t})}\prod_{i\neq a} e^{-t_i}\,dt_i, \tag{3.6.6}$$

where $D(\beta,\boldsymbol{t})$ is the $(|\Lambda|-1)\times(|\Lambda|-1)$ matrix with entries

$$D_{ij}(\beta,\boldsymbol{t}) \equiv \begin{cases} -\beta_{ij}e^{t_i+t_j}, & i\neq j \\ \sum_{k\neq a}\beta_{ik}e^{t_i+t_k} + \beta_{ai}e^{t_i} & i = j \end{cases} \tag{3.6.7}$$

indexed by $i, j \in \Lambda\backslash\{a\}$. This is [89, Theorem 2] as stated in [92, Theorem B]. In obtaining this formula we have used Theorem 3.A.12 to perform the change of generators and then integrated out s, ψ and $\bar\psi$, which can be done explicitly as conditioned on the t-variables these are Gaussian integrals, see [40, Section 2.3].

Remark 3.6.2. Qualitatively, the appearance of horospherical coordinates can be explained as follows. The hyperbolic Ray–Knight isomorphism relates the time evolution of the VRJP by $\cosh s$ to the Lorentz boost by s in the xz-plane. Since the asymptotics of Lorentz boosts in the xz-plane are captured by the t marginal in horospherical coordinates, the formulation of the asymptotic local time distribution in terms of the t marginal is quite geometrically natural.

The Sabot–Tarrès limit formula [89, Theorem 2] can also be derived from the hyperbolic BFS–Dynkin isomorphism, see 3.C. More precisely, this can be done by using Corollary 3.7.2 below. In this derivation the role of horospherical coordinates can be seen even more explicitly.

For another explanation of the relation of horospherical coordinates to the VRJP, see [73].

3.7 Time changes and resolvent formulas

In this section we describe some useful variations and reformulations of our theorems. For the sake of simplicity we only consider the VRJP, but analogous results also hold for the SRW and the VDJP.

Time-changed and fixed-time formulas

In the literature on the VRJP time changes have played an important role; see, for example, [89]. For comparision with these references, this section briefly explains how isomorphism theorems can be translated to these time-changes.

For a Markov process $(X_s, \boldsymbol{L}_s)$ on $\Lambda \times \mathbb{R}^\Lambda$, let $V\colon [\min_{i\in\Lambda} L_0^i, \infty) \to [\min_{i\in\Lambda} V(L_0^i), \infty)$ be an increasing diffeomorphism and define a random function $A\colon [0,\infty) \to [0,\infty)$ by

$$A(s) \equiv \int_0^s V'(L_u^{X_u})\, du = \sum_{i\in\Lambda} V(L_s^i) - V(L_0^i). \tag{3.7.1}$$

We define $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$, the *time-change by V* of $(X_t, \boldsymbol{L}_t)$, by

$$\tilde{X}_t \equiv X_{A^{-1}(t)}, \quad \tilde{L}_t^i \equiv V(L^i_{A^{-1}(t)}) = V(L_0^i) + \int_0^t 1_{\tilde{X}_u = i}\, du\,. \tag{3.7.2}$$

Note that $A(0) = A^{-1}(0) = 0$, $\tilde{X}_0 = X_0$ and $\tilde{L}_0^i = V(L_0^i)$.

In this section we will write $V(\mathbf{1}) \equiv (V(1))_{i\in\Lambda}$. The next corollary is an example of an isomorphism theorem for a time-changed process.

Corollary 3.7.1. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$ be the time-change by V of the VRJP with expectation $\mathbb{E}_{i,\boldsymbol{\ell}}$. Then*

$$\int_0^\infty \mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{X}_t, \tilde{\boldsymbol{L}}_t))\, dt = \sum_{i\in\Lambda} [x_a x_i V'(z_i)\, g(i, V(\boldsymbol{z}))]_\beta. \tag{3.7.3}$$

Proof. By (3.7.2) and the change of variable $s = A^{-1}(t)$,

$$\begin{aligned}\int_0^\infty \mathbb{E}_{\tilde{X}_0,\tilde{\boldsymbol{L}}_0}(g(\tilde{X}_t, \tilde{\boldsymbol{L}}_t))\, dt &= \int_0^\infty \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_{A^{-1}(t)}, V(\boldsymbol{L}_{A^{-1}(t)}))\big)\, dt \\ &= \int_{A^{-1}(0)}^{A^{-1}(\infty)} \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_s, V(\boldsymbol{L}_s))A'(s)\big)\, ds \\ &= \int_0^\infty \mathbb{E}_{X_0,\boldsymbol{L}_0}\big(g(X_s, V(\boldsymbol{L}_s))V'\big(L_s^{X_s}\big)\big)\, ds.\end{aligned} \tag{3.7.4}$$

The claim now follows from Theorem 3.5.6 in the case that $g(i,\boldsymbol{\ell})$ is of the form $\delta_{i,j} f(\boldsymbol{\ell})$. The result for more general functions follows by summing (or by using the second part of Lemma 3.5.5). $\square$

The next corollary shows that supersymmetric isomorphism theorems also give formulas for the local time distribution at *fixed* times.

Corollary 3.7.2. *Let $[\cdot]_\beta$ be the superexpectation of the $\mathbb{H}^{2|2}$ model, and let $(\tilde{X}_t, \tilde{\boldsymbol{L}}_t)$ be the time-change by V of the VRJP with expectation $\mathbb{E}_{i,\boldsymbol{\ell}}$. Let $\delta_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ be a smooth and compactly supported approximation to δ_0. Then for $g \colon \mathbb{R}^\Lambda \to \mathbb{R}$ smooth and rapidly decaying and any $T > 0$,*

$$\mathbb{E}_{a,V(\mathbf{1})} g\Big(\tilde{\boldsymbol{L}}_T - \frac{T}{N}\Big) = \lim_{\varepsilon \to 0} \sum_{i \in \Lambda} \Bigg[x_a x_i V'(z_i) g\Big(V(\boldsymbol{z}) - \frac{T}{N}\Big) \delta_\varepsilon \Big(\sum_{i \in \Lambda} \Big(V(z_i) - V(1) - \frac{T}{N} \Big) \Big) \Bigg]_\beta .$$

Proof. The left-hand side can be written as

$$\begin{aligned}
\mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_T - \frac{T}{N})) &= \sum_{i \in \Lambda} \mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_T - \frac{T}{N}) 1_{X_T = i}) \\
&= \lim_{\varepsilon \to 0} \sum_{i \in \Lambda} \int_0^\infty dt \, \mathbb{E}_{a,V(\mathbf{1})} \Big(g(\tilde{\boldsymbol{L}}_t - \frac{T}{N}) 1_{X_t = i} \Big) \, \delta_\varepsilon(t - T) \\
&= \lim_{\varepsilon \to 0} \sum_{i \in \Lambda} \int_0^\infty dt \, \mathbb{E}_{a,V(\mathbf{1})} \Big(g(\tilde{\boldsymbol{L}}_t - \frac{T}{N}) 1_{X_t = i} \delta_\varepsilon (\sum_{i \in \Lambda} (\tilde{L}_t^i - V(1)) - T) \Big) \\
&= \lim_{\varepsilon \to 0} \sum_{i \in \Lambda} \Bigg[x_a x_i V'(z_i) g\Big(V(\boldsymbol{z}) - \frac{T}{N}\Big) \delta_\varepsilon \Big(\sum_{i \in \Lambda} (V(z_i) - V(1) - \frac{T}{N}) \Big) \Bigg]_\beta .
\end{aligned} \tag{3.7.5}$$

The second equality used that $t \mapsto \mathbb{E}_{a,V(\mathbf{1})}(g(\tilde{\boldsymbol{L}}_t - T/N) 1_{X_t = i})$ is continuous, the third equality used that $\sum_{i \in \Lambda} (L_t^i - V(1)) = t$ for any $t \geqslant 0$, and the fourth equality is Corollary 3.7.1. $\square$

By making use of an appropriate time-change, Corollary 3.7.2 is the starting point for an alternative derivation of the Sabot–Tarrès limit formula (3.6.6), see Remark 3.6.2. Similar results have also been used as the starting point for the study of large deviations of the local time of the SRW [17, Theorem 1].

Resolvent of the joint local time process

The supersymmetric isomorphism theorems for the VRJP in Section 3.5 concern fixed initial local times for the joint process $(X_t, \boldsymbol{L}_t)$, i.e., $\boldsymbol{L}_0 = \mathbf{1}$. This initial condition arises from supersymmetric localisation at $(z, x, y, \xi, \eta) = (1, 0, 0, 0, 0)$ due to the sigma model constraint $u \cdot u = -1$. A more general and geometrically instructive formulation can be obtained by considering the joint process $(X_t, \boldsymbol{L}_t)$ with a general initial condition. This formulation involves the super-Minkowski space from Section 3.5 as opposed to the space $\mathbb{H}^{2|2}$.

Super-Minkowski model. Recall super-Minkowski space $\mathbb{R}^{3|2}$ from Section 3.5. We define the Berezin integral for an observable $F \in \Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ by

$$\int_{(\mathbb{R}^{3|2})^\Lambda} F \equiv \frac{1}{(2\pi)^{|\Lambda|}} \int d\boldsymbol{x} \, d\boldsymbol{y} \, d\boldsymbol{z} \, \partial_{\boldsymbol{\eta}} \, \partial_{\boldsymbol{\xi}} \, F, \tag{3.7.6}$$

where $\partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}}$ is defined by $\partial_{\eta_{|\Lambda|}} \partial_{\xi_{|\Lambda|}} \dots \partial_{\eta_1} \partial_{\xi_1}$, $d\boldsymbol{x} = dx_{|\Lambda|} \dots dx_1$, $d\boldsymbol{y} = dy_{|\Lambda|} \dots dy_1$, and $d\boldsymbol{z} = dz_{|\Lambda|} \dots dz_1$ for some fixed ordering of the $i \in \Lambda$ from 1 to $|\Lambda|$.

For $u \in \mathbb{R}^{3|2}$, we write $u \cdot u < 0$ if the degree 0 part of the form $u \cdot u$ is negative, where here $u \cdot u$ denotes the super-Minkowski inner product (3.5.21). For a spin configuration $\boldsymbol{u} \in (\mathbb{R}^{3|2})^\Lambda$ we write $\boldsymbol{u} \cdot \boldsymbol{u} < 0$ if $u_i \cdot u_i < 0$ for all $i \in \Lambda$ and we then define

$$r_i \equiv \sqrt{-u_i \cdot u_i}, \tag{3.7.7}$$

and let $\boldsymbol{r} = (r_i)_{i\in\Lambda}$. For such a spin configuration we consider the Hamiltonian

$$H_\beta(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) + \frac{1}{2}(\boldsymbol{r}, -\Delta_\beta \boldsymbol{r}), \tag{3.7.8}$$

where the inner product for the u_i is the one from (3.5.21) and the r_i are forms that are multiplied in the ordinary way: $(\boldsymbol{r}, -\Delta_\beta \boldsymbol{r}) = \sum_{i\in\Lambda} r_i(-\Delta_\beta r)_i$. Let $F \in \Omega^{2\Lambda}(\mathbb{R}^{3\Lambda})$ be a smooth form compactly supported on $\{\boldsymbol{u}\cdot\boldsymbol{u} < 0, \boldsymbol{z} > 0\}$, i.e., whose coefficient functions vanish when the degree 0 part of any form $u_i \cdot u_i$ is non-negative or when $z_i \leqslant 0$ for any i. We define an unnormalised superexpectation by

$$[F]_\beta \equiv \int_{(\mathbb{R}^{3|2})^\Lambda} F(\boldsymbol{u}) e^{-H_\beta(\boldsymbol{u})} 1_{\boldsymbol{u}\cdot\boldsymbol{u}<0} 1_{\boldsymbol{z}>0}, \tag{3.7.9}$$

with $\boldsymbol{u}\cdot\boldsymbol{u} < 0$ as defined above. The assumption that F has compact support ensures the integrand is smooth. We call this the *super-Minkowski model*. Note that $\{\boldsymbol{u}\cdot\boldsymbol{u} < 0, \boldsymbol{z} > 0\}$ is a version of the causal future for super-Minkowski space; see Figure 3.1.

Symmetries and localisation. Let

$$T = x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}. \tag{3.7.10}$$

Then T represents an infinitesimal Lorentz boost in the xz-plane since

$$Tx = z, \qquad Tz = x, \tag{3.7.11}$$

and $Ty = T\xi = T\eta = 0$. Note also that $Tr = 0$.

The Hamiltonian H_β is invariant under T, i.e., $\sum_{i\in\Lambda} T_i H_\beta(\boldsymbol{u}) = 0$. Here we have written T_i for the version of T applying to the i-th coordinate. Moreover the integral (3.7.6) is invariant under T. In addition, the model is supersymmetric with supersymmetry generator Q as in (3.5.11), and the following localisation statement holds for all smooth $f\colon (0,\infty)^{2\Lambda} \to \mathbb{R}$ with compact support:

$$[f(\boldsymbol{z}, \boldsymbol{r})]_\beta = \int_{(0,\infty)^\Lambda} d\boldsymbol{z}\, f(\boldsymbol{z}, \boldsymbol{z}). \tag{3.7.12}$$

This can be seen by integrating over $\boldsymbol{z}$ last when computing the superexpectation, and using localisation for (x, y, η, ξ), i.e., Corollary 3.A.10.

Resolvent formula. The super-Minkowski model is related to the resolvent of the VRJP.

Theorem 3.7.3. *Let $[\cdot]_\beta$ be the superexpectation of the super-Minkowski model, and let $\pi = (\pi(i, \boldsymbol{r}))$ be a smooth compactly support probability measure on $\Lambda \times (0,\infty)^\Lambda$. For all smooth $f\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ with rapid decay,*

$$\int_0^\infty \mathbb{E}_{\boldsymbol{\pi}} f(X_t, \boldsymbol{L}_t)\, dt = \sum_{i,j\in\Lambda} \left[\frac{\pi(i, \boldsymbol{r})}{r_i} x_i x_j f(j, \boldsymbol{z})\right]_\beta \tag{3.7.13}$$

where we have written $\mathbb{E}_{\boldsymbol{\pi}}$ to denote the expectation of a VRJP with initial condition $(X_0, \boldsymbol{L}_0)$ distributed according to π.

Remark 3.7.4. In the notation of Remark 3.2.2, Theorem 3.7.3 can be compactly rewritten as

$$\int_0^\infty \mathbb{E}_{\boldsymbol{\pi}} f(X_t, \boldsymbol{L}_t)\, dt = \left[(\boldsymbol{x}, \frac{\boldsymbol{\pi}(\boldsymbol{r})}{\boldsymbol{r}})(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{z}))\right]_\beta. \tag{3.7.14}$$

The proof of Theorem 3.7.3 uses that Lemma 3.5.5 remains true if $[\cdot]_\beta$ is interpreted as the expectation of the super-Minkowski model, and then follows the standard route as follows.

Proof. Let $\rho(\boldsymbol{u}) \equiv \sum_{i\in\Lambda} \pi(i,\boldsymbol{r})x_i/r_i$, and let T_i be the infinitesimal boost given by (3.5.28). Since $T_i r_i = 0$ and $T_i x_i = z_i$ we have $T_j\rho = \pi(j,\boldsymbol{r})z_j/r_j$. Since Lemma 3.5.5 holds for the super-Minkowski model, we apply (3.5.34) to obtain

$$\sum_{i,j\in\Lambda}\left[\frac{\pi(i,\boldsymbol{r})}{r_i}x_i x_j f(j,\boldsymbol{z})\right]_\beta = \sum_{j\in\Lambda}\left[\frac{z_j}{r_j}\pi(j,\boldsymbol{r})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\,dt\right]_\beta. \tag{3.7.15}$$

By localisation, i.e., (3.7.12), the right-hand side equals

$$\int_{\mathbb{R}_+^\Lambda} d\boldsymbol{z}\sum_{j\in\Lambda}\pi(j,\boldsymbol{z})\int_0^\infty \mathbb{E}_{j,\boldsymbol{z}}(f(X_t,\boldsymbol{L}_t))\,dt = \int_0^\infty \mathbb{E}_{\boldsymbol{\pi}}(f(X_t,\boldsymbol{L}_t))\,dt.$$

□

3.8 Application to exponential decay of correlations in spin systems

In this section we prove theorems about the exponential decay of spin-spin correlations. Let $d(i,j)$ denote the graph distance between vertices i and j in the graph induced by the edge weights β; this distance is finite since the induced graph is finite and connected by assumption.

We first consider the $\mathbb{H}^{2|2}$ model with constant and non-zero external field.

Theorem 3.8.1. *Consider the $\mathbb{H}^{2|2}$ model with $\sup_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij} \leqslant \beta_*$ and $h_i = h > 0$ for all $i\in\Lambda$. Let $c(\beta_*,h) \equiv \log(1+h/\beta_*)$. Then for all $i,j\in\Lambda$,*

$$[x_i x_j]_{\beta,h} \leqslant \frac{1}{h}e^{-c(\beta_*,h)d(i,j)}. \tag{3.8.1}$$

Proof. Let τ_j be the hitting time of j, i.e., $\tau_j \equiv \inf\{s\geqslant 0 \mid X_s = j\}$. Then by choosing g an exponential in Theorem 3.5.6, and using the killed representation of (3.3.31),

$$\begin{aligned}
[x_i x_j]_{\beta,h} = \mathbb{E}^h_{i,\mathbf{1}}\int_0^\infty 1_{X_s=j}\,ds &= \mathbb{E}^h_{i,\mathbf{1}}\big((L^j_\infty - 1)1_{\tau_j<\infty}\big)\\
&= \mathbb{E}^h_{i,\mathbf{1}}\big((L^j_\infty-1)\mid 1_{\tau_j<\infty}\big)\mathbb{P}^h_{i,\mathbf{1}}(\tau_j<\infty)\\
&\le \frac{1}{h}\mathbb{P}^h_{i,\mathbf{1}}(\tau_j<\infty)
\end{aligned} \tag{3.8.2}$$

The inequality follows because the expected remaining lifetime of the conditioned walk after hitting j is h^{-1} (by memorylessness of the killing), and that $L^j_{\tau_j} - 1 = 0$.

If $\tau_j < \infty$ then there are at least $d(i,j)$ times at which a rate h exponential clock does *not* ring before a rate β_* clock, as there are at least $d(i,j)$ jumps to previously unvisited vertices on any path from i to j. The probability of a rate h clock ringing only after some rate β_{ij} clock is at most $\beta_*/(\beta_*+h)$. Each of these events are independent by the memorylessness of the exponential, and hence

$$\mathbb{P}^h_{i,\mathbf{1}}(\tau_j<\infty) \leqslant \left(\frac{\beta_*}{\beta_*+h}\right)^{d(i,j)} = e^{-c(\beta_*,h)d(i,j)} \tag{3.8.3}$$

Combined with (3.8.2) this proves the theorem. □

Remark 3.8.2. Theorem 3.8.1 gives a positive rate $\log(1+h/\beta_*) \sim ch$ of exponential decay for some $c>0$ for any value of β. For small β, i.e., high temperatures, it is known that the rate stays uniformly bounded away from 0 as $h\downarrow 0$ [5,39]. The rate is expected to be bounded away from 0 for any β when the graph Λ tends to $\mathbb{Z}^2$. On the other hand, for $\Lambda\uparrow\mathbb{Z}^d$ with $d\geqslant 3$ it is conjectured that the rate behaves asymptotically as $\sim c\sqrt{h}$ as $h\downarrow 0$.

It would be interesting to obtain an analogue of Theorem 3.8.1 for the $\mathbb{H}^n$ model by using Theorem 3.3.3. This would require an appropriate estimate on the z-field to control the initial local times of the VRJP. We do not pursue this direction here.

For the hemispherical spin models, the estimates on the z-field are trivial because $|z_i| \leqslant 1$, and we thus consider both the $\mathbb{S}_+^n$ model and the $\mathbb{S}_+^{2|2}$ model. For $\mathbb{S}_+^{2|2}$ we have only defined the superexpectation of compactly supported observables. To define the superexpectation of non-compactly supported observables requires a treatment of superintegrals with boundaries; since we do not need this general treatment we instead define the two-point function $[x_i x_j]_{\beta,h}$ for the $\mathbb{S}_+^{2|2}$ model by $[x_i x_j]_{\beta,h} \equiv \lim_{n\to\infty}[x_i x_j f_n(\boldsymbol{z})]_{\beta,h}$ where f_n is a sequence of smooth and bounded approximations to $1_{\boldsymbol{z}>0}$. The proof of the following theorem shows that this limit exists.

Theorem 3.8.3. *Consider the $\mathbb{S}_+^n$ model with $\sup_{i\in\Lambda}\sum_{j\in\Lambda}\beta_{ij} \leqslant \beta_*$, and let $c(\beta_*) = \log(1-e^{-\beta_*})$. Then for all $i,j \in \Lambda$,*

$$\langle x_i x_j\rangle_{\beta,h} \leqslant e^{-c(\beta_*)d(i,j)}. \tag{3.8.4}$$

The same result holds for the superexpectation $[x_i x_j]_{\beta,h}$ of the $\mathbb{S}_+^{2|2}$ model.

Proof. We first consider $\mathbb{S}_+^{2|2}$. Let f_n be a sequence of smooth and bounded approximations to $1_{\boldsymbol{z}>0}$. Letting $\mathbb{E}_{i,\mathbf{1}}$ be the expectation for a VDJP with initial local time $\mathbf{1}$, Theorem 3.5.11 implies

$$[x_i x_j]_{\beta,h} = \lim_{n\to\infty}[x_i x_j f_n(\boldsymbol{z})]_{\beta,h} = \lim_{n\to\infty}\mathbb{E}_{i,\mathbf{1}}\int_0^\infty f_n(\boldsymbol{L})1_{X_t=j}e^{-\sum_v h_v L_v^t}\,dt. \tag{3.8.5}$$

To obtain upper bounds we may assume, without loss of generality, that $\boldsymbol{h}=\mathbf{0}$. By definition, X_t dies once the local time at any vertex reaches 0. Since f_n is asymptotically bounded above by one, it therefore suffices to bound the probability that X_t reaches j.

By the definition of the VDJP, for each $r \in \Lambda$ the jump rate out of r is bounded above by β_*. Thus for each $k \in \mathbb{N}$ there is probability at least $e^{-\beta_*}$ the walk X_t dies after its kth jump and before its $(k+1)$st jump. The probability X_t reaches j is at most the probability that X_t does not die before taking $d(i,j)$ steps, and hence

$$[x_i x_j]_{\beta,h} \leqslant (1-e^{-\beta_*})^{d(i,j)} = e^{-c(\beta_*)d(i,j)}. \tag{3.8.6}$$

This completes the proof for $\mathbb{S}_+^{2|2}$. For $\mathbb{S}_+^n$, we use (the normalised form of) Theorem 3.4.4 in place of Theorem 3.5.11. The argument above applies pointwise in the initial local time, so using $0 \leqslant z_i \leqslant 1$ we obtain the same conclusion. □

Remark 3.8.4. A result closely related to Theorem 3.8.3 is given in [71, Theorem 2].

Appendices

3.A Introduction to supersymmetric integration

This appendix gives a self-contained introduction to the mathematics of supersymmetry that is relevant for this article. For complementary treatments, see in particular [13, 21, 79]. In Appendix 3.B we discuss some further aspects of supersymmetry that are relevant to this article, but that are not needed to understand the main text.

Integration of differential forms

We begin by reviewing the important example of integration of differential forms on Euclidean space $\mathbb{R}^N$. Let $x_1, \dots, x_N$ be coordinates on $\mathbb{R}^N$. A *differential form* on $\mathbb{R}^N$ can be written as

$$F = F_0 + \cdots + F_N \tag{3.A.1}$$

where $F_0 \in C^\infty(\mathbb{R}^N)$ is a 0*-form,* i.e., an ordinary function, and F_p is a p*-form*, i.e., a nonzero sum of terms of the form

$$f_{i_1,\dots,i_p}(x_1,\dots,x_N)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \qquad 1 \leqslant i_j \leqslant N, \;\; 1 \leqslant j \leqslant p, \tag{3.A.2}$$

where $f_{i_1,\dots,i_p} \in C^\infty(\mathbb{R}^N)$, the coordinates are viewed as functions $x_i \colon \mathbb{R}^N \to \mathbb{R}$ in $C^\infty(\mathbb{R}^N)$, and the differentials dx_i are the generators of a Grassmann algebra. This means that the dx_i are formal variables that are multiplied with the anti-commuting wedge product:

$$dx_i \wedge dx_j = -dx_j \wedge dx_i. \tag{3.A.3}$$

In particular, $dx_i \wedge dx_i = 0$. Later, the $\wedge$ will often be omitted. By extending the wedge product to differential forms by linearity, we obtain a unital associative algebra over $C^\infty(\mathbb{R}^N)$. This is the exterior algebra of differential forms on $\mathbb{R}^N$, which we denote $\Omega(\mathbb{R}^N)$.

Example 3.A.1 (Change of variables)**.** The differential notation and the use of the wedge product is consistent with, and motivated by, the following change of variable formula. Let $\Phi \colon \mathbb{R}^N \to \mathbb{R}^N$ be an orientation preserving diffeomorphism. Then by the change of variables formula from calculus

$$\begin{aligned} \int f(x_1,\dots,x_N)\, dx_1 \wedge \cdots \wedge dx_N &= \int f(\Phi_1(\boldsymbol{x}),\dots,\Phi_N(\boldsymbol{x}))(\det D\Phi)\, dx_1 \wedge \cdots \wedge dx_N \\ &= \int f(\Phi_1(\boldsymbol{x}),\dots,\Phi_N(\boldsymbol{x}))\, d\Phi_1(\boldsymbol{x}) \wedge \cdots \wedge d\Phi_N(x) \end{aligned} \tag{3.A.4}$$

where $D\Phi$ is the Jacobian matrix of Φ and the second equality has made use of the definition

$$d\Phi_i(\boldsymbol{x}) = \sum_{j=1}^N \frac{\partial \Phi_i(\boldsymbol{x})}{\partial x_j} dx_j, \tag{3.A.5}$$

which leads, by a calculation, to the identity

$$d\Phi_1(\boldsymbol{x}) \wedge \cdots \wedge d\Phi_N(\boldsymbol{x}) = (\det D\Phi)\; dx_1 \wedge \cdots \wedge dx_N. \tag{3.A.6}$$

Odd and even forms

A differential form is *even* if it is a sum of p-forms with all p even and it is *odd* if it is a sum of p-forms with all p odd. We say a form is *homogeneous* if it is either even or odd. We can decompose a general form F as

$$F = F_{\text{even}} + F_{\text{odd}}, \qquad \Omega(\mathbb{R}^N) = \Omega_{\text{even}}(\mathbb{R}^N) \oplus \Omega_{\text{odd}}(\mathbb{R}^N), \tag{3.A.7}$$

where F_{even} is the sum of the degree p parts of F with p even, and similarly for F_{odd}. As the wedge product of a p-form with a q-form is either 0 or a $(p+q)$-form, the exterior algebra equipped with the wedge product is a $\mathbb{Z}_2$-graded algebra. $\mathbb{Z}_2$-graded algebras are also called *superalgebras*. Formally, this means that if we define the parity of a homogeneous form as

$$\alpha(F) \equiv \begin{cases} 0 \in \mathbb{Z}_2, & F = F_{\text{even}} \\ 1 \in \mathbb{Z}_2, & F = F_{\text{odd}} \end{cases} \tag{3.A.8}$$

then $\alpha(F \wedge G) = \alpha(F) + \alpha(G)$ mod 2. A calculation shows that for homogeneous F, G

$$F \wedge G = (-1)^{\alpha(F)\alpha(G)} G \wedge F, \tag{3.A.9}$$

and in particular, even elements commute with all other elements by linearity.

Berezin integral

In this section we introduce Grassmann algebras and the Berezin integral. Integration of differential forms as introduced in the previous sections constitute a special case.

Grassmann algebras. Let Ω^M be a Grassmann algebra with generators $\xi_1, \dots, \xi_M$; as the subscripts suggest we will always assume there is a fixed (but arbitrary) order on the generators. Thus Ω^M is the unital associative algebra generated by the $(\xi_i)_{i=1}^M$ subject to the anticommutation relations

$$\xi_i \xi_j + \xi_j \xi_i = 0, \quad 1 \leqslant i \leqslant j \leqslant M. \tag{3.A.10}$$

Let $\Omega^M(\mathbb{R}^N)$ be the algebra over $C^\infty(\mathbb{R}^N)$ generated by the $(\xi_i)_{i=1}^M$. Elements of this algebra can be written as

$$\sum_{\substack{I \subset \{1,\dots,M\} \\ I = \{i_1,\dots,i_p\}}} f_I(\boldsymbol{x})\, \xi_{i_1} \cdots \xi_{i_p} \tag{3.A.11}$$

where $f_I \in C^\infty(\mathbb{R}^N)$ for each $I \subset \{1, \dots, M\}$, and we have arranged the product of generators according to the given fixed order: $i_1 < i_2 < \dots < i_p$.

Example 3.A.2. The differentials $\xi_i = dx_i$ are an instance of a Grassmann algebra, and the algebra of differential forms on $\mathbb{R}^N$ can be identified with $\Omega^N(\mathbb{R}^N)$.

We continue to use the term *form* for elements of $\Omega^M(\mathbb{R}^N)$ when $N \neq M$. The notion of the degree of a form and the $\mathbb{Z}_2$-grading that we defined for differential forms extends to this more general context.

Integration. For $i \in \{1, 2, \dots, M\}$ the *left-derivative* $\frac{\partial}{\partial \xi_i} \colon \Omega^M \to \Omega^M$ is the unique linear map determined by

$$\frac{\partial}{\partial \xi_i}(\xi_i F) = F \quad \text{if } \xi_i F \neq 0, \qquad \frac{\partial}{\partial \xi_i} 1 = 0. \tag{3.A.12}$$

We sometimes write $\partial_{\xi_i} = \frac{\partial}{\partial \xi_i}$. Note that ∂_{ξ_i} is an *anti-derivation*: if F is a homogeneous form, then

$$\partial_{\xi_i}(FG) = (\partial_{\xi_i} F)G + (-1)^{\alpha(F)} F(\partial_{\xi_i} G). \tag{3.A.13}$$

The left-derivative extends naturally to an anti-derivation on $\Omega^M(\mathbb{R}^N)$ by defining

$$\partial_{\xi_i}(f(\boldsymbol{x})\xi_{i_1} \dots \xi_{i_p}) = f(\boldsymbol{x}) \partial_{\xi_i}(\xi_{i_1} \dots \xi_{i_p}). \tag{3.A.14}$$

Example 3.A.3. The left-derivative gives a convenient formulation of the integral of a differential form. Let $F \in \Omega^N(\mathbb{R}^N)$ be a differential form and write $\xi_i = dx_i$. Then

$$\int F = \int_{\mathbb{R}^N} dx_1 \cdots dx_N \, \partial_{\xi_N} \cdots \partial_{\xi_1} F = \int_{\mathbb{R}^N} d\boldsymbol{x} \, \partial_{\boldsymbol{\xi}} F \tag{3.A.15}$$

where the left-hand side is the integral as a differential form in the sense of Section 3.A, and the last equality made use of the definition $\partial_{\boldsymbol{\xi}} \equiv \partial_{\xi_N} \dots \partial_{\xi_1}$. Note that the order used in defining $\partial_{\boldsymbol{\xi}}$ matters.

The notation on the right-hand side of (3.A.15) is called the *Berezin integral*. This is a useful notion because it is possible to change variables in $\boldsymbol{x}$ and $\boldsymbol{\xi}$ separately, as will be discussed below in Section 3.A. The Berezin integral generalises to $N \neq M$ as follows.

Definition 3.A.4. For $F \in \Omega^M(\mathbb{R}^N)$, the *Berezin integral of* F is

$$\int F \equiv \int_{\mathbb{R}^N} dx_1 \cdots dx_N \, \partial_{\xi_M} \cdots \partial_{\xi_1} F = \int_{\mathbb{R}^N} d\boldsymbol{x} \, \partial_{\boldsymbol{\xi}} F, \tag{3.A.16}$$

where the last equality is by the definitions $d\boldsymbol{x} = dx_1 \dots dx_N$ and $\partial_{\boldsymbol{\xi}} \equiv \partial_{\xi_M} \dots \partial_{\xi_1}$. We say a form F is *integrable* if it can be written as a finite sum of forms of the type $f(\boldsymbol{x}) \, \xi_{i_1} \dots \xi_{i_p}$ with f integrable on $\mathbb{R}^N$.

The expression $d\boldsymbol{x} \, \partial_{\boldsymbol{\xi}}$ on the right-hand side of (3.A.16) is an example of a *superintegration form*. More generally a superintegration form is given by $d\boldsymbol{x} \, \partial_{\boldsymbol{\xi}} F$ for F an even integrable form, and integration with respect to this superintegration form is defined by $\int G = \int_{\mathbb{R}^N} d\boldsymbol{x} \, \partial_{\boldsymbol{\xi}} FG$.

Functions of forms. Suppose $g \in C^\infty(\mathbb{R}^k)$. We will use $\alpha = (\alpha_1, \dots, \alpha_k)$ to denote multiindices, and we will also use the notation

$$g^{(\alpha)}(\boldsymbol{x}) \equiv \frac{\partial}{\partial x_1^{\alpha_1}} \cdots \frac{\partial}{\partial x_k^{\alpha_k}} g(x), \qquad x^\alpha \equiv x_1^{\alpha_1} \cdots x_k^{\alpha_k}.$$

Definition 3.A.5. Let $g \in C^\infty(\mathbb{R}^k)$ and $F^1, \dots F^k \in \Omega^M(\mathbb{R}^N)$ be even forms. Then $g(F^1, \dots, F^k) \in \Omega^M(\mathbb{R}^N)$ is defined by the following formula, where the sum runs over all multiindices α:

$$g(F^1, \dots, F^k) \equiv \sum_\alpha \frac{1}{\alpha!} g^{(\alpha)}(F_0^1, \dots, F_0^k)(F - F_0)^\alpha. \tag{3.A.17}$$

Note that the product defining $(F - F_0)^\alpha$ is the wedge product, i.e., this is shorthand for $(F^1 - F_0^1)^{\alpha_1} \wedge \dots \wedge (F^k - F_0^k)^{\alpha_k}$, and $(F^1 - F_0^1)^{\alpha_1}$ is the α_1-fold wedge product of this form with itself. There is no ambiguity in the ordering since all forms are assumed even. The formal Taylor expansion in (3.A.17) is finite because forms of degree greater than N do not exist. As a simple example of a function of a form, the reader may wish to verify that

$$e^{-x_1^2 - \xi_1 \xi_2} = e^{-x_1^2}(1 - \xi_1 \xi_2). \tag{3.A.18}$$

Gaussian integrals and localisation

Let $A \in \mathbb{R}^{N \times N}$ be positive definite. The $O(2)$-invariant Gaussian measure on $\mathbb{R}^{2N}$ associated to the matrix A has density

$$e^{-\frac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}) - \frac{1}{2}(\boldsymbol{y}, A\boldsymbol{y})} (\det A) \prod_{i=1}^N \frac{dx_i \, dy_i}{2\pi}. \tag{3.A.19}$$

Let $\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$ be generators of the Grassmann algebra Ω^{2N}, and define

$$\partial_{\boldsymbol{\eta}} \partial_{\boldsymbol{\xi}} \equiv \partial_{\eta_N} \partial_{\xi_N} \cdots \partial_{\eta_1} \partial_{\xi_1} \qquad (\boldsymbol{\xi}, A\boldsymbol{\eta}) \equiv \sum_{i=1}^N A_{ij} \xi_i \eta_j. \tag{3.A.20}$$

A computation shows that

$$\partial_{\boldsymbol{\eta}}\partial_{\boldsymbol{\xi}} e^{(\boldsymbol{\xi},A\boldsymbol{\eta})} = \partial_{\boldsymbol{\eta}}\partial_{\boldsymbol{\xi}} \frac{1}{N!}(\sum_{i=1}^{N} A_{ij}\xi_i\eta_j)^N = \det A. \tag{3.A.21}$$

Remark 3.A.6. The form $e^{(\boldsymbol{\xi},A\boldsymbol{\eta})} = e^{\frac{1}{2}(\boldsymbol{\xi},A\boldsymbol{\eta})-\frac{1}{2}(\boldsymbol{\eta},A\boldsymbol{\xi})} \in \Omega^{2N}$ is called a *Grassmann Gaussian*. The corresponding Grassmann Gaussian expectation $\langle F\rangle \equiv [F]/[1]$ where $[F] \equiv \partial_{\boldsymbol{\eta}}\partial_{\boldsymbol{\xi}}(e^{(\boldsymbol{\xi},A\boldsymbol{\eta})}F) \in \mathbb{R}$ for $F \in \Omega^{2N}$, and hence $[1] = \det A$ by (3.A.21), behaves in many ways like a Gaussian integral.

Using (3.A.21), the Gaussian density (3.A.19) can be written as

$$\prod_{i=1}^{N} \frac{dx_i\, dy_i\, \partial_{\eta_i}\partial_{\xi_i}}{2\pi} e^{-\frac{1}{2}(\boldsymbol{x},A\boldsymbol{x})-\frac{1}{2}(\boldsymbol{y},A\boldsymbol{y})+\frac{1}{2}(\boldsymbol{\xi},A\boldsymbol{\eta})-\frac{1}{2}(\boldsymbol{\eta},A\boldsymbol{\xi})}. \tag{3.A.22}$$

The form given by $(2\pi)^{-N}$ times the exponential in (3.A.22) is called the *super-Gaussian form*. Thus the Gaussian density is the coefficient of the top degree part of the super-Gaussian form.

To lighten the notation, we will now write $u_i \equiv (x_i, y_i, \xi_i, \eta_i)$ and call u_i a *supervector*. For supervectors u_i and u_j define a form

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - \xi_i\eta_j + \eta_i\xi_j. \tag{3.A.23}$$

We unite the supervectors u_i into $\boldsymbol{u} \equiv (u_i)_{i=1}^N$ and introduce the following shorthand notation for the form that occurs in the exponent of (3.A.22):

$$(\boldsymbol{u}, A\boldsymbol{u}) \equiv \sum_{i,j=1}^{N} A_{ij} u_i \cdot u_j. \tag{3.A.24}$$

For a form F we define *the superintegral of* F by

$$\int_{(\mathbb{R}^{2|2})^N} F \equiv \frac{1}{(2\pi)^N}\int_{\mathbb{R}^{2N}} d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, \partial_{\boldsymbol{\xi}}\, F, \tag{3.A.25}$$

where $d\boldsymbol{x} \equiv dx_N \dots dx_1$ and similarly for $d\boldsymbol{y}$. Then, since the coefficient of the top degree part of (3.A.22) is the density of a Gaussian,

$$\int_{(\mathbb{R}^{2|2})^N} e^{-\frac{1}{2}(\boldsymbol{u},A\boldsymbol{u})} = 1. \tag{3.A.26}$$

The fact that this superintegral is one is a simple example of *localisation* for superintegrals of supersymmetric forms. The rest of this section describes this phenomenon.

The *supersymmetry generator* $Q\colon \Omega^{2N}(\mathbb{R}^{2N}) \to \Omega^{2N}(\mathbb{R}^{2N})$ is defined as

$$Q \equiv \sum_{i=1}^{N} Q_i\,, \qquad Q_i \equiv \xi_i\frac{\partial}{\partial x_i} + \eta_i\frac{\partial}{\partial y_i} - x_i\frac{\partial}{\partial \eta_i} + y_i\frac{\partial}{\partial \xi_i}. \tag{3.A.27}$$

Thus Q formally exchanges the even and odd generators of $\Omega^{2N}(\mathbb{R}^{2N})$:

$$Qx_i = \xi_i, \quad Qy_i = \eta_i, \quad Q\xi_i = -y_i, \quad Q\eta_i = x_i. \tag{3.A.28}$$

A form $F \in \Omega^{2N}(\mathbb{R}^{2N})$ is defined to be *supersymmetric* if $QF = 0$. Note that Q is an anti-derivation, and hence $Q(F_1F_2) = 0$ if F_1 and F_2 are both supersymmetric forms.

Example 3.A.7. The following forms are supersymmetric:

$$u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i\eta_j + \eta_i\xi_j. \tag{3.A.29}$$

Much of the magic of supersymmetry is due to the fundamental *localisation theorem*:

Theorem 3.A.8. *Suppose $F \in \Omega^{2N}(\mathbb{R}^{2N})$ is supersymmetric and integrable. Then*

$$\int_{(\mathbb{R}^{2|2})^N} F = F_0(0) \tag{3.A.30}$$

where the right-hand side is the degree-0 part of F evaluated at 0.

To keep this introduction to supersymmetry self-contained, we provide the beautiful and instructive proof of this theorem in Appendix 3.B. To prove an important corollary of the theorem we need the following chain rule, proven in [79, p.59] or [7, Solution to Exercise 11.4.3].

Lemma 3.A.9. *The supersymmetry generator Q obeys the chain rule for even forms, in the sense that if $K = (K_j)_{j=1}^J$ is a finite collection of even forms, and if $f\colon \mathbb{R}^J \to \mathbb{C}$ is C^∞, then*

$$Q(f(K)) = \sum_{j=1}^J f_j(K) QK_j, \tag{3.A.31}$$

where f_j denotes the partial derivative of f with respect to the jth coordinate.

Let $\boldsymbol{u}\boldsymbol{u}^T$ denote the collection $(u_i \cdot u_j)_{i,j=1}^N$ of forms defined in (3.A.29).

Corollary 3.A.10. *For any smooth function $f\colon \mathbb{R}^{N\times N} \to \mathbb{R}$ with sufficient decay,*

$$\int_{(\mathbb{R}^{2|2})^N} f(\boldsymbol{u}\boldsymbol{u}^T) = f(\mathbf{0}). \tag{3.A.32}$$

Proof. Let $F = f(\boldsymbol{u}\boldsymbol{u}^T)$. Then $F_0(\mathbf{0}) = f(\mathbf{0})$ and $QF = \sum_{ij} f_{ij}(\boldsymbol{u}\boldsymbol{u}^T) Q(u_i \cdot u_j) = 0$ by the chain rule of Lemma 3.A.9, where f_{ij} denotes the partial derivative of f with respect to the ij-th coordinate. The claim follows from Theorem 3.A.8. □

Change of generators

Recall the general expression (3.A.11) for a form $F \in \Omega^M(\mathbb{R}^N)$. We will sometimes write $F(\boldsymbol{x}, \boldsymbol{\xi})$ or $F(x_1, \dots, x_N, \xi_1, \dots, \xi_M)$ to denote a form written in this way.

Definition 3.A.11. A collection of even elements $(x_i)_{i=1}^N$ and odd elements $(\xi_j)_{j=1}^M$ is a *set of generators* for $\Omega^M(\mathbb{R}^N)$ if every $F \in \Omega^M(\mathbb{R}^N)$ can be written in the form (3.A.11).

Note that Example 3.A.1 provided an example of a change of generators

$$y_i = \Phi_i(x_1, \dots, x_N), \quad \eta_i = dy_i = \sum_{j=1}^N \frac{\partial \Phi_i}{\partial x_j}(x_1, \dots, x_N)\, dx_j \tag{3.A.33}$$

along with a corresponding change of variables formula.

It is both possible and useful to change between sets of generators in the sense of Definition 3.A.11 without the even and odd generators changing together. Moreover, there is an extension of the usual change of variables formula that applies in this setting. This formula relies on the notion of *superdeterminant* (or Berezinian) of a *supermatrix* M:

$$\mathrm{sdet} M \equiv \det(A - BD^{-1}C) \det D^{-1} \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.A.34}$$

where the entries of M are elements of a Grassmann algebra, the entries of the blocks A and D are even, the entries of the blocks B and C are odd, and D is invertible. Invertibility means invertibility in the (commutative) algebra of even elements of the Grassmann algebra. The next result is [13, Theorem 2.1]. In the theorem rapid decay means each of the coefficient functions of F have rapid decay.

Theorem 3.A.12. *Suppose $y_i = y_i(\boldsymbol{x}, \boldsymbol{\xi})$ and $\eta_i = \eta_i(\boldsymbol{x}, \boldsymbol{\xi})$ are a set of generators. Then for any F with sufficiently rapid decay,*

$$\int d\boldsymbol{y}\, \partial_{\boldsymbol{\eta}}\, F(\boldsymbol{y}, \boldsymbol{\eta})\, \mathrm{sdet}(M) = \int d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, F(\boldsymbol{x}, \boldsymbol{\xi}), \tag{3.A.35}$$

where M is of the form in (3.A.34) *with entries $A_{ij} = \frac{\partial y_i}{\partial x_j}$, $B_{ij} = \frac{\partial y_i}{\partial \xi_j}$, $C_{ij} = \frac{\partial \eta_i}{\partial x_j}$, $D_{ij} = \frac{\partial \eta_i}{\partial \xi_j}$.*

Implicit in Theorem 3.A.12 is that a change of generators always results in an invertible D, so the superdeterminant is well-defined.

Example 3.A.13. Let x, ξ_1, ξ_2 be generators for $\Omega^2(\mathbb{R})$. Then the set of forms $\{x + g(x)\xi_1\xi_2, \xi_1, \xi_2\}$ is also a set of generators, and

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x, \xi_1, \xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2)(1 + g'(x)\xi_1\xi_2). \tag{3.A.36}$$

It is instructive to verify the claims of the previous example by hand, and we briefly do so. To see the claim that these forms are a set of generators, recall that by definition

$$F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = F(x, \xi_1, \xi_2) + F'(x, \xi_1, \xi_2)g(x)\xi_1\xi_2. \tag{3.A.37}$$

Letting $y \equiv g(x)\xi_1\xi_2$, a general form of $\{x + g(x)\xi_1\xi_2, \xi_1, \xi_2\}$ is thus, for some functions a, b, c, d,

$$a(x+y) + b(x+y)\xi_1 + c(x+y)\xi_2 + d(x+y)\xi_1\xi_2 = a(x) + b(x)\xi_1 + c(x)\xi_2 + (d(x) + a'(x)g(x))\xi_1\xi_2,$$

which clearly shows a general form in $\{x, \xi_1, \xi_2\}$ can be expressed as a form in $\{x+g(x)\xi_1\xi_2, \xi_1, \xi_2\}$.

To verify (3.A.36) integrate (3.A.37). Integrating the term containing F' by parts yields

$$\int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x + g(x)\xi_1\xi_2, \xi_1, \xi_2) = \int dx\, \partial_{\xi_1}\partial_{\xi_2}\, F(x, \xi_1, \xi_2)(1 - g'(x)\xi_1\xi_2). \tag{3.A.38}$$

Since $F(x+g(x)\xi_1\xi_2, \xi_1, \xi_2)g'(x)\xi_1\xi_2 = F(x, \xi_1, \xi_2)g'(x)\xi_1\xi_2$, (3.A.36) follows. This can alternately be verified by computing the superdeterminant of

$$M = \begin{pmatrix} 1 + g'(x)\xi_1\xi_2 & \xi_2 & -\xi_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.A.39}$$

3.B Further aspects of symmetries and supersymmetry

This appendix discusses some additional aspects of supersymmetry. First, we briefly introduce complex coordinates, which have often been used in the literature (see, e.g., [21]). Second, we prove Theorem 3.A.8. The remaining sections discuss symmetries and Ward identities, and in particular, highlight how Theorem 3.A.8 is an example of a Ward identity arising from an infinitesimal supersymmetry.

Complex coordinates

In Appendix 3.A we introduced Grassmann algebras over $\mathbb{R}$ and forms given by smooth functions with values in $\mathbb{R}$. Sometimes it is convenient to work with Grassmann algebras over $\mathbb{C}$ and complex-valued functions, and many discussions of supersymmetry do so, see [21] and references therein. To facilitate comparisons with the literature we briefly introduce complex coordinates and relate them to the presentation of Appendix 3.A.

To introduce complex coordinates we set

$$z = \frac{1}{\sqrt{2}}(x+iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x-iy), \quad \zeta = \frac{1}{\sqrt{2i}}(\xi + i\eta), \quad \bar{\zeta} = \frac{1}{\sqrt{2i}}(\xi - i\eta). \tag{3.B.1}$$

Correspondingly, define

$$\frac{\partial}{\partial z_i} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right), \tag{3.B.2}$$

and define ∂_{ζ_i} and $\partial_{\bar{\zeta}_i}$ to be the antiderivations on Ω^{2N} such that

$$\frac{\partial}{\partial \zeta_i}\zeta_j = \frac{\partial}{\partial \bar{\zeta}_i}\bar{\zeta}_j = \delta_{ij}, \qquad \frac{\partial}{\partial \zeta_i}\bar{\zeta}_j = \frac{\partial}{\partial \bar{\zeta}_i}\zeta_j = 0. \tag{3.B.3}$$

Up to an irrelevant factor of $\sqrt{i}$ (a constant factor plays no role in determining if a form is supersymmetric), the supersymmetry generator can be written in complex coordinates as

$$Q = \sum_{i=1}^{N} Q_i, \quad Q_i = \zeta_i \frac{\partial}{\partial z_i} + \bar{\zeta}_i \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial \zeta_i} + \bar{z}_i \frac{\partial}{\partial \bar{\zeta}_i}. \tag{3.B.4}$$

Hence it acts on the complex generators by

$$Qz_i = \zeta_i, \quad Q\bar{z}_i = \bar{\zeta}_i, \quad Q\zeta_i = -z_i, \quad Q\bar{\zeta}_i = \bar{z}_i. \tag{3.B.5}$$

Writing $u_i = (z_i, \zeta_i)$ for $i = 1, \dots, N$, the following forms are supersymmetric:

$$u_i \cdot \bar{u}_j \equiv z_i \bar{z}_j + \zeta_i \bar{\zeta}_j. \tag{3.B.6}$$

Realisation by differential forms. Complex coordinates can be realised in terms of differential forms as follows. Denote the coordinates of $\mathbb{R}^2$ by x and y with differentials dx and dy, and set

$$z = \frac{1}{\sqrt{2}}(x+iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x-iy), \quad dz = \frac{1}{\sqrt{2i}}(dx + idy), \quad d\bar{z} = \frac{1}{\sqrt{2i}}(dx - idy). \tag{3.B.7}$$

Proof of Theorem 3.A.8

The proof of Theorem 3.A.8 will use the complex coordinates introduced in Appendix 3.B, and will also make use of the following terminology and facts. A form is called *Q-closed* (*supersymmetric*) if $QF = 0$ and it is called *Q-exact* if $F = QG$ for some form $G \in \Omega^{2N}(\mathbb{R}^{2N})$. The Q-closed forms $u_i \cdot u_j$ from Example 3.A.7 are also Q-exact, as can be verified by checking

$$z_i\bar{z}_j + z_j\bar{z}_i + \zeta_i\bar{\zeta}_j - \bar{\zeta}_i\zeta_j = Q\lambda_{ij}, \quad \lambda_{ij} \equiv z_i\bar{\zeta}_j + z_j\bar{\zeta}_i. \tag{3.B.8}$$

Proof of Theorem 3.A.8. Any integrable form F can be written as $K = \sum_\alpha F^\alpha \zeta^\alpha$ with (i) ζ^α a monomial in $\{\zeta_i, \bar{\zeta}_i\}_{i=1}^N$ and (ii) F^α an integrable function of $\{z_i, \bar{z}_i\}_{i=1}^N$. To emphasise this, we write $K = K(\boldsymbol{z}, \bar{\boldsymbol{z}}, \boldsymbol{\zeta}, \bar{\boldsymbol{\zeta}})$. To simplify notation we write $\int$ in place of $\int_{(\mathbb{R}^{2|2})^N}$.

Step 1. Let $S = \sum_{i=1}^N (z_i\bar{z}_i + \zeta_i\bar{\zeta}_i)$. We prove the following version of Laplace's Principle:

$$\lim_{t\to\infty} \int e^{-tS} F = F_0(\mathbf{0}). \tag{3.B.9}$$

Let $t > 0$. We make the change of generators $z_i = \frac{1}{\sqrt{t}}z_i'$ and $\zeta_i = \frac{1}{\sqrt{t}}\zeta_i'$. This transformation has unit Berezinian. Let $\omega \equiv -\sum_{i=1}^N \zeta_i\bar{\zeta}_i$. After dropping the primes, we obtain

$$\int e^{-tS} F = \int e^{-\sum_{i=1}^N z_i\bar{z}_i + \omega} F(\tfrac{1}{\sqrt{t}}\boldsymbol{z}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}}\boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{\zeta}}), \tag{3.B.10}$$

where $\frac{1}{\sqrt{t}}\boldsymbol{z} \equiv \{\frac{1}{\sqrt{t}}z_i\}_{i=1}^N$, and similarly for the other generators. To evaluate the right-hand side, we expand e^ω and and obtain

$$\int e^{-tS}F = \sum_{n=0}^{N} \int e^{-\sum_{i=1}^N z_i\bar{z}_i} \frac{1}{n!}\omega^n F(\tfrac{1}{\sqrt{t}}\boldsymbol{z}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}}\boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{\zeta}}). \tag{3.B.11}$$

We write $K = K^0 + G$, where K^0 is the degree zero part of K. The contribution of K^0 to (3.B.11) involves only the $n = N$ term and equals

$$\int e^{-tS}F^0 = \int e^{-\sum_{i=1}^N z_i\bar{z}_i} \frac{1}{N!}\omega^N F^0(\tfrac{1}{\sqrt{t}}\boldsymbol{z}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{z}}), \tag{3.B.12}$$

so by the continuity of F_0,

$$\lim_{t\to\infty} \int e^{-tS}F_0 = F_0(\mathbf{0}) \int e^{-\sum_{i=1}^N z_i\bar{z}_i} \frac{1}{N!}\omega^N = F_0(\mathbf{0}) \int e^{-S}. \tag{3.B.13}$$

By (3.A.26) with A the identity matrix, this proves that

$$\lim_{t\to\infty} \int e^{-tS}F_0 = F_0(0). \tag{3.B.14}$$

To complete the proof of (3.B.9), it remains to show that $\lim_{t\to\infty} \int e^{-tS}G = 0$. As above,

$$\int e^{-tS}G = \sum_{n=0}^{N} \int e^{-\sum_{i=1}^N z_i\bar{z}_i} \frac{1}{n!}\omega^n\, G\Big(\tfrac{1}{\sqrt{t}}\boldsymbol{z}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{z}}, \tfrac{1}{\sqrt{t}}\boldsymbol{\zeta}, \tfrac{1}{\sqrt{t}}\bar{\boldsymbol{\zeta}}\Big). \tag{3.B.15}$$

Since G has no degree-zero part, the term with $n = N$ is zero. Terms with smaller values of n require factors $\zeta_i\bar{\zeta}_i$ for some i from G, and these factors carry inverse powers of t. They therefore vanish in the limit, and the proof of (3.B.9) is complete.

Step 2. The Laplace approximation is exact:

$$\int e^{-tS}F \quad \text{is independent of } t \geq 0. \tag{3.B.16}$$

To prove this, recall that $S = Q\lambda$. Also, $Qe^{-S} = 0$ by the chain rule of Lemma 3.A.9, and $QF = 0$ by assumption. Therefore,

$$\frac{d}{dt}\int e^{-tS}F = -\int e^{-tS}SF = -\int e^{-tS}(Q\lambda)F = -\int Q\big(e^{-tS}\lambda F\big) = 0, \tag{3.B.17}$$

since the integral of any Q-exact form is zero, because it can be written as a sum of derivatives (whose integral vanishes due to the assumption of rapid decay) and a form of degree lower than the top degree (whose integral vanishes by definition).

Step 3. Finally, we combine Laplace's Principle (3.B.9) and the exactness of the Laplace approximation (3.B.16), to obtain the desired result

$$\int F = \lim_{t\to\infty} \int e^{-tS}F = F_0(\mathbf{0}). \qquad \square$$

Symmetries

This appendix briefly reviews symmetries in the context of smooth manifolds, to prepare the way for a discussion of symmetries of superalgebras.

Infinitesimal symmetries. For a smooth manifold M, infinitesimal symmetries are described by the infinite-dimensional Lie algebra of smooth vector fields, $\mathrm{Vect}(M)$. Vector fields act on functions through the Lie derivative, which associates to every vector field $X \in \mathrm{Vect}(M)$ a *derivation* $T_X \colon C^\infty(M) \to C^\infty(M)$. We recall that a derivation is a linear map that obeys the Leibniz rule $T_X(fg) = T_X(f)g + fT_X(g)$. Concretely, if M is n-dimensional and X is represented in local coordinates as $X = \sum_{\alpha=1}^n g(u^1, \dots, u^n)\frac{\partial}{\partial u^\alpha}$, then $T_X(f) = \sum_{\alpha=1}^n g(u^1, \dots, u^n)\frac{\partial f}{\partial u^\alpha}$.

In fact, every derivation on $C^\infty(M)$ arises from a vector field, and hence there is an isomorphism $\mathrm{Vect}(M) \simeq \mathrm{Der}(C^\infty(M))$. Thus we can replace geometric objects (vector fields) with algebraic objects (derivations). The perspective will be useful for superspaces, as their definition is fundamentally algebraic rather than geometric.

Integral symmetries. Rather than examining the entire Lie algebra $\mathrm{Der}(C^\infty(M))$, it is often useful to consider subalgebras that respect additional structures on the manifold. We will be interested in the following case where M carries a measure μ. Let $\int_M f$ denote the integral of a function $f \colon M \to \mathbb{R}$ with respect to the measure μ. We call $\int_M$ an *integral on M*.

Definition 3.B.1. Let $\int_M$ be an integral on a smooth manifold M. A derivation $T \in \mathrm{Der}(C^\infty(M))$ is an *infinitesimal symmetry* of the integral if for all $f \in C^\infty(M)$ with rapid decay

$$\int_M Tf = 0. \tag{3.B.18}$$

Infinitesimal symmetries lead to integration by parts formulas, otherwise known as *Ward identities*: suppose T is a symmetry of $\int_M$, and that $f, g \in C^\infty(M)$ have rapid decay. Then

$$\int_M T(fg) = 0, \tag{3.B.19}$$

since fg has rapid decay. Since T acts as a derivation, we obtain the Ward identity

$$\int_M (Tf)g = -\int_M f(Tg). \tag{3.B.20}$$

For spin systems, different infinitesimal symmetries are obtained depending on whether we examine the Gibbs measure $e^{-H_\beta}\, d\boldsymbol{u}$ or the underlying measure $d\boldsymbol{u}$. Ward identities for one lead to (anomalous) Ward identities for the other. For instance, letting $[f]_\beta = \int_{M^\Lambda} f e^{-H_\beta}\, d\boldsymbol{u}$ denote an unnormalised expectation, and letting T be an infinitesimal symmetry of $d\boldsymbol{u}$,

$$\int_{M^\Lambda} T(fe^{-H_\beta}) = 0, \quad \text{i.e.,} \quad \int_{M^\Lambda} (Tf - f(TH_\beta))e^{-H_\beta} = 0 \tag{3.B.21}$$

and hence

$$[Tf]_\beta = [f(TH_\beta)]_\beta. \tag{3.B.22}$$

Global symmetries. For spin system Gibbs measures $[F]_\beta = \int_{M^\Lambda} F\, e^{-H_\beta} d\boldsymbol{u}$, an important role is played by derivations $T \in \mathrm{Der}(C^\infty(M^\Lambda))$ which can be written in the form

$$T \equiv \sum_{i\in\Lambda} T_i, \tag{3.B.23}$$

where each T_i is a copy of a single site derivation

$$T_i = \sum_{\alpha=1}^n f_\alpha(u_i)\frac{\partial}{\partial u_i^\alpha} \tag{3.B.24}$$

with f_α independent of $i \in \Lambda$. We call these *diagonal derivations*. If a diagonal derivation is an infinitesimal symmetry of the Gibbs measure, then we say that it is a *global symmetry*. The spin system Hamiltonians in this paper are of the form $H_\beta(\boldsymbol{u}) = \frac{1}{4}\sum_{i,j\in\Lambda}\beta_{ij}(u_i - u_j)^2$ with $(u_i - u_j)^2 \equiv (u_i - u_j)\cdot(u_i - u_j)$ for some inner product. Hence the global symmetries are equivalently those diagonal derivations which satisfy

$$T(u_i - u_j)^2 = 0 \tag{3.B.25}$$

for all $i, j \in \Lambda$. These correspond to the infinitesimal isometries of the target space, and form a representation of a finite dimensional Lie algebra.

For the GFF on $\mathbb{R}^n$, the global symmetries are of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{n} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{n} S_\gamma \frac{\partial}{\partial u_i^\gamma}, \tag{3.B.26}$$

where R is an $n\times n$ real skew-symmetric matrix and S is a real vector in $\mathbb{R}^n$. The global symmetries of $\mathbb{R}^n$ hence form a representation of the Euclidean Lie algebra $\mathfrak{so}(n) \ltimes \mathbb{R}^n$ under the Lie bracket of derivations. Global symmetries of Minkowski space $\mathbb{R}^{n,1}$ are of the same form as (3.B.26), but R is now skew-symmetric with respect to the Minkowski inner product, i.e.,

$$R^T J + JR = 0, \quad J = \mathrm{diag}(-1, 1, \dots, 1). \tag{3.B.27}$$

This gives a representation of the Poincare Lie algbera $\mathfrak{so}(n,1) \ltimes \mathbb{R}^{n,1}$.

Global symmetries of the $\mathbb{H}^n$ and $\mathbb{S}^n_+$ spin models are induced from Lorentz/orthogonal symmetries of $\mathbb{R}^{n,1}$ and $\mathbb{R}^{n+1}$ respectively, i.e., global symmetries have the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta}. \tag{3.B.28}$$

For the $\mathbb{H}^n$ model these form a representation of the Lorentzian Lie algebra $\mathfrak{so}(n,1)$, and for the $\mathbb{S}^n_+$ model these form a representation of the orthogonal Lie algebra $\mathfrak{so}(n+1)$. In coordinates, these symmetries can be written as

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{n} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{n} S_\gamma z_i \frac{\partial}{\partial u_i^\gamma} \tag{3.B.29}$$

where $S_\gamma = R_{0\gamma}$ and $z = \sqrt{1 + (u^1)^2 + \dots + (u^n)^2}$ for $\mathbb{H}^n$, while $S_\gamma = R_{(n+1)\gamma}$ and $z = \sqrt{1 - (u^1)^2 - \dots - (u^n)^2}$ for $\mathbb{S}^n_+$.

Symmetries of supersymmetric spaces

Infinitesimal symmetries of Berezin integrals and the global symmetries of supersymmetric spaces have descriptions similar to those of the previous section. The primary difference is that all objects are graded.

Superderivations and supersymmetries. Let A be a $\mathbb{Z}^2$-graded algebra (or superalgebra) such as $A = \Omega^n(\mathbb{R}^m)$. Thus $A = A_0 \oplus A_1$ where elements in A_0 are even and elements in A_1 are odd. Using this decomposition, a linear map $T\colon A \to A$ can be written in blocks as

$$Tf = \begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}. \tag{3.B.30}$$

A linear map is *even* if $T_{01} = T_{10} = 0$, and *odd* if $T_{00} = T_{11} = 0$. As for functions, a *homogeneous* linear map is one that is even or odd. We extend the parity function to homogeneous maps by

$$\alpha(T) = \begin{cases} 0 \in \mathbb{Z}_2, & T \text{ is even} \\ 1 \in \mathbb{Z}_2, & T \text{ is odd} \end{cases}, \tag{3.B.31}$$

and for homogeneous f we have $\alpha(Tf) = \alpha(T) + \alpha(f)$. A *homogeneous superderivation* is then defined as a homogeneous linear map $T\colon A \to A$ that obeys the super-Leibniz rule

$$T(fg) = (Tf)g + (-1)^{\alpha(T)\alpha(f)} f(Tg). \tag{3.B.32}$$

Thus even and odd superderivations are derivations and antiderivations, respectively. A general superderivation is a sum of an even and an odd superderivation. The collection of superderivations on A forms a Lie superalgebra $\mathrm{SDer}(A)$ with the supercommutator defined on homogeneous superderivations by

$$[T_1, T_2] = T_1 \circ T_2 - (-1)^{\alpha(T_1)\alpha(T_2)} T_2 \circ T_1, \tag{3.B.33}$$

and extended to all superderivations by linearity. If $A = \Omega^n(M)$ is a superalgebra of forms on an m-dimensional manifold M, then every superderivation $T \in \mathrm{SDer}(A)$ can be realised in coordinates $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ as

$$T = \sum_{\alpha=1}^{m} F_\alpha \frac{\partial}{\partial x^\alpha} + \sum_{\alpha=1}^{n} G_\alpha \frac{\partial}{\partial \xi^\alpha} \tag{3.B.34}$$

where $F_\alpha, G_\alpha \in A$. If T is an even/odd superderivation then F_α are even/odd forms and G_α are odd/even forms.

Berezin integral symmetries and global symmetries. We define a *Berezin integral* $\int_M$ on a superalgebra $\Omega^n(M)$ to be a linear map defined by integrating forms F against an even Berezin integral form $d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, \rho(\boldsymbol{x}, \boldsymbol{\xi})$, i.e.,

$$\int_M F \equiv \int_{\mathbb{R}^{m|n}} d\boldsymbol{x}\, \partial_{\boldsymbol{\xi}}\, \rho(\boldsymbol{x}, \boldsymbol{\xi}) F(\boldsymbol{x}, \boldsymbol{\xi}). \tag{3.B.35}$$

Definition 3.B.2. Let $\int_M$ be a Berezin integral on a superalgebra $\Omega^n(M)$. A superderivation $T \in \mathrm{SDer}(\Omega^n(M))$ is an *infinitesimal symmetry* of $\int_M$ if for all $F \in \Omega^n(M)$ with rapid decay

$$\int_M TF = 0. \tag{3.B.36}$$

This leads to Ward identities in the same manner as the non-supersymmetric case, the only difference coming from the super-Leibniz rule: for homogeneous superderivations $T \in \mathrm{SDer}(\Omega^n(M))$ and forms $F, G \in \Omega^n(M)$ we have

$$\int_M TF = (-1)^{\alpha(T)\alpha(F)+1} \int_M TG. \tag{3.B.37}$$

Global symmetries of supersymmetric spin systems are infinitesimal symmetries of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \tag{3.B.38}$$

i.e., they are diagonal infinitesimal symmetries. For the spin systems considered in this paper, which are defined in terms of quadratic Hamiltonians $\frac{1}{4}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2$, global symmetries are those that annihilate the appropriate super-Euclidean or super-Minkowski inner product

$$T(u_i - u_j)^2 = 0 \tag{3.B.39}$$

for all $i, j \in \Lambda$. Here we have written $(u_i - u_j)^2$ for the form $(u_i - u_j)\cdot(u_i - u_j)$. The following subsections briefly discuss this condition for the $\mathbb{R}^{2|2}, \mathbb{H}^{2|2}$, and $\mathbb{S}_+^{2|2}$ models.

$\mathbb{R}^{2|2}$ *model.* The inner product associated to the SUSY GFF is

$$u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \tag{3.B.40}$$

giving the global symmetries as diagonal superderivations $T \in \mathrm{SDer}(\Omega^{2\Lambda}(\mathbb{R}^{2\Lambda}))$ satisfying

$$T(u_i - u_j)^2 = T\big((x_i - x_j)^2 + (y_i - y_j)^2 - 2(\xi_i - \xi_j)(\eta_i - \eta_j)\big) = 0 \tag{3.B.41}$$

for all $i, j \in \Lambda$.

Concretely, letting $u_i = (u_i^1, \dots, u_i^4) = (x_i, y_i, \xi_i, \eta_i)$, these are derivations of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=1}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{4} S_\gamma \frac{\partial}{\partial u_i^\gamma} \tag{3.B.42}$$

where R is a real 4×4 matrix (independent of $i \in \Lambda$) such that

$$R^{ST} J + JR = 0, \tag{3.B.43}$$

where R^{ST}, the *supertranspose* of R, and J are given by

$$R^{ST} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{ST} = \begin{bmatrix} A^T & C^T \\ -B^T & D^T \end{bmatrix}, \qquad J \equiv \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right], \tag{3.B.44}$$

and S is a real vector. With the supercommutator of superderivations, these form a representation of the super-Euclidean Lie superalgebra $\mathfrak{osp}(2|2) \ltimes \mathbb{R}^{2|2}$. In particular, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \tag{3.B.45}$$

and the infinitesimal global translation

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} \frac{\partial}{\partial x_i} \tag{3.B.46}$$

are global symmetries.

A short computation shows that the individual T_i and Q_i are symmetries of the flat Berezin–Lebesgue measure $d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}}$. For instance, if F is a compactly supported form with top degree component $F_{2\Lambda}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\xi\eta}$,

$$\int_{(\mathbb{R}^{2|2})^\Lambda} (T_i F) = \int_{\mathbb{R}^{2\Lambda}} d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} (T_i F) = \int_{\mathbb{R}^{2\Lambda}} d\boldsymbol{x}\, d\boldsymbol{y} \frac{\partial}{\partial x_i} F_{2\Lambda}(\boldsymbol{x}, \boldsymbol{y}) = 0 \tag{3.B.47}$$

where in the last step we have used the translation invariance of the usual Lebesgue measure. A particular case of this is formula (3.5.9).

Super-Minkowski space $\mathbb{R}^{3|2}$. The inner product associated to the super-Minkowski model is the super-Minkowski inner product

$$u_i \cdot u_j = -z_i z_j + x_i x_j + y_i y_j - \xi_i \eta_j + \eta_i \xi_j, \tag{3.B.48}$$

giving the global symmetries as diagonal superderivations $T \in \mathrm{SDer}(\Omega^{2\Lambda}(\mathbb{R}^{3\Lambda}))$ satisfying

$$T(u_i - u_j)^2 = T\big(-(z_i - z_j)^2 + (x_i - x_j)^2 + (y_i - y_j)^2 - 2(\xi_i - \xi_j)(\eta_i - \eta_j)\big) = 0 \tag{3.B.49}$$

for all $i, j \in \Lambda$. Concretely, letting $u_i = (u_i^0, u_i^1, u_i^2, u_i^3, u_i^4) = (z_i, x_i, y_i, \xi_i, \eta_i)$, these are derivations of the form

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=0}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{5} S_\gamma \frac{\partial}{\partial u_i^\gamma} \tag{3.B.50}$$

where R is a real 5×5 matrix such that

$$R^{ST} J + JR = 0 \tag{3.B.51}$$

with J now the 5×5 matrix

$$J = \left[\begin{array}{ccc|cc} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right], \tag{3.B.52}$$

and S a real vector. These global symmetries form a representation of the super-Poincare Lie superalgebra $\mathfrak{osp}(2,1|2) \ltimes \mathbb{R}^{3|2}$ with the supercommutator of superderivations. In particular, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \left(\xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \right) \tag{3.B.53}$$

and the global Lorentz boost

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} \left(z_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial z_i} \right). \tag{3.B.54}$$

are global symmetries of the super-Minkowski spin model. As for the $\mathbb{R}^{2|2}$ model, the individual T_i and Q_i are symmetries of the Berezin–Lebesgue measure $d\boldsymbol{x}\, d\boldsymbol{y}\, d\boldsymbol{z}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}}$.

$\mathbb{S}_+^{2|2}$ *and* $\mathbb{H}^{2|2}$ *models.* As for their standard counterparts, the global symmetries of the $\mathbb{S}_+^{2|2}$ and $\mathbb{H}^{2|2}$ models are induced from the ambient super-Euclidean and super-Minkowski spaces. In both cases, the global symmetries in ambient coordinates are

$$T \equiv \sum_{i\in\Lambda} T_i, \quad T_i = \sum_{\alpha,\beta=0}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta}, \tag{3.B.55}$$

which form a representation of $\mathfrak{osp}(2,1|2)$ for the $\mathbb{H}^{2|2}$ model, and a representation of $\mathfrak{osp}(3|2)$ for $\mathbb{S}_+^{2|2}$. In coordinates, the T_i are written

$$T_i = \sum_{\alpha,\beta=1}^{4} R_{\alpha\beta} u_i^\alpha \frac{\partial}{\partial u_i^\beta} + \sum_{\gamma=1}^{4} S_\gamma z_i \frac{\partial}{\partial u_i^\gamma} \tag{3.B.56}$$

with $z_i = \sqrt{1 + x_i^2 + y_i^2 - 2\xi\eta}$ for $\mathbb{H}^{2|2}$ and $z_i = \sqrt{1 - x_i^2 - y_i^2 + 2\xi\eta}$ for $\mathbb{S}_+^{2|2}$ and $S_\gamma = R_{3\gamma}$ in both cases. As before, the supersymmetry generator

$$Q \equiv \sum_{i\in\Lambda} Q_i = \sum_{i\in\Lambda} \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i} \tag{3.B.57}$$

is a global symmetry of both the $\mathbb{H}^{2|2}$ and $\mathbb{S}_+^{2|2}$ models, as is the global Lorentz boost/rotation

$$T \equiv \sum_{i\in\Lambda} T_i = \sum_{i\in\Lambda} z_i \frac{\partial}{\partial x_i}. \tag{3.B.58}$$

A short computation also shows that the individual T_i and Q_i are symmetries of the Berezin–Haar measure $d\boldsymbol{x}\, d\boldsymbol{y}\, \partial_{\boldsymbol{\xi}}\, \partial_{\boldsymbol{\eta}} \frac{1}{\prod_{i\in\Lambda} z_i}$.

SUSY delta functions

We begin by defining Dirac delta functions to integrate against forms F in $\Omega^2(\mathbb{R}^2)$. We will assume F is given by a smooth function of an even form. Let $u_0 = (0,0,0,0) \in \mathbb{R}^{2|2}$, and let $G \in \Omega^2(\mathbb{R}^2)$ be a smooth compactly supported form with $\int_{\mathbb{R}^{2|2}} G = 1$. For $\varepsilon > 0$ define smooth forms

$$\delta_{u_0}^{(\varepsilon)}(u) \equiv G(\frac{1}{\varepsilon}u), \qquad \frac{1}{\varepsilon}u = (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{\xi}{\varepsilon}, \frac{\eta}{\varepsilon}). \tag{3.B.59}$$

We then define

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0} \equiv \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0}^{(\varepsilon)}(u). \tag{3.B.60}$$

The change of generators that rescales each generator by ε^{-1} has unit Berezinian, and hence

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2|2}} F(\varepsilon u)\delta_{u_0}^{(1)}(u) = F_0(0)\int_{\mathbb{R}^{2|2}} \delta_{u_0}^{(1)}(u) = F_0(0), \tag{3.B.61}$$

where we recall F_0 is the degree zero part of F. In the second equality we have used that the degree p parts of F for $p \geqslant 1$ carry factors of ε, and hence vanish in the limit. The last equality follows since $\int_{\mathbb{R}^{2|2}} \delta_{u_0}^{(1)} = \int_{\mathbb{R}^{2|2}} G = 1$.

Suppose $\theta_s \colon (x,y,\xi,\eta) \mapsto (\theta_s x, \theta_s y, \theta_s \xi, \theta_s \eta)$ is invertible with inverse θ_{-s}, and that $\theta_s u_0$ only has non-zero even components. In this setting we define $\delta_{\theta_s u_0}(u)$ by $\delta_{u_0}(\theta_{-s}u)$. If the transformation θ_s has unit Berezinian, then we obtain

$$\int_{\mathbb{R}^{2|2}} F(u)\delta_{\theta_s u_0}(u) = \int_{\mathbb{R}^{2|2}} F(u)\delta_{u_0}(\theta_{-s}u) = \int_{\mathbb{R}^{2|2}} F(\theta_s u)\delta_{u_0}(u) = F_0(\theta_s u_0). \tag{3.B.62}$$

It is often convenient to choose G as a supersymmetric form. For $\mathbb{R}^{2|2}$, this can be achieved by choosing any smooth compactly supported function $g\colon \mathbb{R} \to \mathbb{R}$ with $g(0) = 1$, and setting $G = g(|u|^2)$.

The definition of delta functions on $\Omega^{2N}(\mathbb{R}^{2N})$ is analogous, but now based on a smooth compact form $G \in \Omega^{2N}(\mathbb{R}^{2N})$.

For $\mathbb{H}^{2|2}$ and $\mathbb{S}_+^{2|2}$, we define delta functions by making using of the definition on $\mathbb{R}^{2|2}$. Namely, for $\mathbb{H}^{2|2}$ in the coordinates $\tilde{u} = (x,y,\xi,\eta)$ with $z(\tilde{u}) = \sqrt{1+x^2+y^2-2\xi\eta}$, we set

$$\delta_{u_0}^{(\varepsilon,\mathbb{H}^{2|2})}(u) = z(\tilde{u})\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u}) \tag{3.B.63}$$

where $u_0 = (1,0,0,0,0) \in \mathbb{H}^{2|2}$, $\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u})$ is a delta function for $\mathbb{R}^{2|2}$ as constructed above, and $\tilde{u}_0 = (0,0,0,0) \in \mathbb{R}^{2|2}$. Then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{H}^{2|2}} F\delta_{u_0}^{(\varepsilon,\mathbb{H}^{2|2})} = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^{2|2}} F(z(\tilde{u}),x,y,\xi,\eta)\delta_{\tilde{u}_0}^{(\varepsilon)}(\tilde{u}) = F_0(1,0,0), \tag{3.B.64}$$

i.e., the zero-degree part of F evaluated at the point $(z,x,y) = (1,0,0) \in \mathbb{H}^2$. The construction for $\mathbb{S}_+^{2|2}$ is analogous.

3.C Sabot–Tarrès limit and the BFS–Dynkin Isomorphism Theorem

Here we show how to derive the Sabot–Tarrès limit from the hyperbolic BFS–Dynkin isomorphism theorem. As will be evidenced by the length of the proof, using the BFS–Dynkin isomorphism as a starting point is somewhat unnatural compared to the proof beginning with Ray–Knight, but it nevertheless interesting to see how the initially complicated expression magically simplifies in the limit.

In this section, we will use the mean 0 version of the Sabot–Tarrés limit.

Theorem 3.C.1 (Sabot–Tarres [89, Theorem 2]). *Let (X_t, L_t) be a time-changed VRJP with $V(\ell) = \log(1+\ell)$. Then for any smooth compactly supported function $g : \mathbb{R}^{|\Lambda|} \to \mathbb{R}$*

$$\lim_{T\to\infty} \mathbb{E}_{a,0}\left(g\left(L_T - \frac{T}{|\Lambda|}\right)\right) = \frac{\sqrt{|\Lambda|}}{\sqrt{2\pi}^{|\Lambda|-1}} \int_{\sum t=0} e^{t_a} g(t) e^{-H_1(t)} \sqrt{\det D(\beta,t)} e^{-t}\, dt \tag{3.C.1}$$

where

$$H_1(t) = \frac{1}{2}\sum_{ij} \beta_{ij}(\cosh(t_i - t_j) - 1)$$

is the horospherical t marginal of the $\mathbb{H}^{2|2}$ Hamiltonian and $D(\beta,t)$ is any diagonal minor of the t-Laplacian $\tilde{\Delta}_{\beta,t}$ with entries

$$(\tilde{\Delta}_{\beta,t})_{ij} = \begin{cases} -\beta_{ij} e^{t_i+t_j}, & i \neq j \\ \sum_{k\neq i} \beta_{ik} e^{t_i+t_k} & i = j \end{cases} \tag{3.C.2}$$

indexed by $i, j \in \Lambda$.

Before we present the proof in detail, we would like to give some intuition as to how the horospherical-t marginal arises in the limit. The starting point of our proof is Corollary 3.7.2 of the hyperbolic BFS–Dynkin isomorphism; taking the $T \to \infty$ limit of (3.7.2) gives

$$\lim_{T\to\infty} \mathbb{E}_a\left(g\left(L_T - \frac{T}{|\Lambda|}\right)\right) = \lim_{T\to\infty}\lim_{\varepsilon\to 0} \sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} x_a \frac{x_b}{z_b} g\left(\log z - \frac{1}{|\Lambda|}\sum_i \log z_i\right) \delta_\varepsilon\left(\sum_i \log z_i - T\right) e^{-H} d\mu \tag{3.C.3}$$

provided the limits exist. It is in fact convenient to exchange the order of the limits on the right-hand side. This is justified by the following elementary lemma.

Lemma 3.C.2. *Let $G_{\varepsilon,T} = \int_0^\infty \mathbb{E}_{a,0}(g(L_t - \frac{t}{|\Lambda|}))\delta_\varepsilon(t-T)dt$, $\operatorname{supp}(\delta_\varepsilon) \subseteq [0,\varepsilon]$, and $g \in \mathcal{S}(\mathbb{R}^{|\Lambda|})$. Then*

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0} G_{\varepsilon,T} = \lim_{\varepsilon\to 0}\lim_{T\to\infty} G_{\varepsilon,T} \tag{3.C.4}$$

provided the limit on the right-hand side exists.

Proof. In order to exchange the limits, it suffices to show that i) $\lim_{T\to\infty} G_{\varepsilon,T}$ exists pointwise for all $\varepsilon \neq 0$ and ii) $\lim_{\varepsilon\to 0} G_{\varepsilon,T}$ converges uniformly. By assumption, i) holds. To show ii), we Taylor expand g as

$$g\left(L_t - \frac{t}{|\Lambda|}\right) = g\left(L_T - \frac{T}{|\Lambda|}\right) + \left(\sum_i \frac{\partial g}{\partial \ell_i}(L_T - \frac{T}{|\Lambda|})(1_{X_T=i} - \frac{1}{|\Lambda|})\right)(t-T) + o(t-T), \tag{3.C.5}$$

to give

$$\begin{aligned} G_{\varepsilon,T} = \mathbb{E}_{a,0} g\left(L_T - \frac{T}{|\Lambda|}\right) + \mathbb{E}_{a,0}\left(\sum_i \frac{\partial g}{\partial \ell_i}(L_T - \frac{T}{|\Lambda|})(1_{X_T=i} - \frac{1}{|\Lambda|})\right) \int_T^{T+\varepsilon} (t-T)\delta_\varepsilon(t-T)dt \\ + \int_T^{T+\varepsilon} o(t-T)\delta_\varepsilon(t-T)dt. \end{aligned} \tag{3.C.6}$$

Then, $G_{0,T} = g\Big(L_T - \frac{T}{|\Lambda|}\Big)$, and hence

$$\begin{aligned}
|G_{\varepsilon,T} - G_{0,T}| &\le \sup_T |G_{\varepsilon,T} - G_{0,T}| \\
&\le \left|\mathbb{E}_{a,0}\left(\sum_i \frac{\partial g}{\partial \ell_i}(L_T - \frac{T}{|\Lambda|})(1_{X_T=i} - \frac{1}{|\Lambda|})\right)\int_T^{T+\varepsilon}(t-T)\delta_\varepsilon(t-T)dt + \int_T^{T+\varepsilon} o(t-T)\delta_\varepsilon(t-T)dt\right| \\
&\le \sup_T |\nabla g|\varepsilon + o(\varepsilon) \\
&= O_g(\varepsilon) \to 0
\end{aligned} \tag{3.C.7}$$

gives the desired uniform convergence. □

To evaluate (3.C.3), the physical picture on the right hand side is that the spin 'centre of mass' is pinned by the delta function; sending $T \to \infty$ pulls the spins off to infinity. This suggests that we should adopt a *moving reference frame*, boosting all coordinates in the xz-plane by a hyperbolic angle σ. This is best done in horospherical coordinates

$$\begin{gathered}
x \equiv \sinh t - \frac{1}{2}(s^2 - 2\psi\bar{\psi})e^t, \quad y \equiv se^t, \quad z \equiv \cosh t + \frac{1}{2}(s^2 - 2\psi\bar{\psi})e^t, \\
\xi \equiv \psi e^t, \quad \eta \equiv \bar{\psi}e^t.
\end{gathered} \tag{3.C.8}$$

Then, setting

$$\nu_i^{\pm} = \frac{1 \pm e^{-\sigma}(e^{-2t_i} + s_i^2 - 2\psi_i\bar{\psi}_i)}{2} \tag{3.C.9}$$

and boosting by $\sigma = \frac{T}{|\Lambda|}$, we have

$$\begin{aligned}
\theta_\sigma x_a &= e^{t_a+\sigma}\nu_a^- \\
\theta_\sigma \frac{x_b}{z_b} &= \frac{\nu_b^-}{\nu_b^+} \\
\theta_\sigma \log z_i &= t_i + \sigma + \log \nu_i^+,
\end{aligned} \tag{3.C.10}$$

giving the integral in horospherical coordinates as

$$\begin{aligned}
\lim_{\sigma\to\infty}\lim_{\varepsilon\to 0}\sum_b \int e^{t_a+\sigma}\frac{\nu_a^-\nu_b^-}{\nu_b^+} g\left(t + \log \nu^+ - \frac{1}{|\Lambda|}\sum_i\big(t_i + \log \nu_i^+\big)\right) \\
\times \delta_\varepsilon\left(\sum_i \big(t_i + \log \nu_i^+\big)\right) e^{-H_1(t) - H_2(t,s,\psi,\bar{\psi})}e^{-t}dtDq,
\end{aligned} \tag{3.C.11}$$

where the Hamiltonian $H = H_1 + H_2$ is now

$$\begin{aligned}
H_1(t) &= \frac{1}{2}\sum_{ij}\beta_{ij}(\cosh(t_i - t_j) - 1), \\
H_2(t,s,\psi,\bar{\psi}) &= \frac{1}{2}\sum_{ij}\beta_{ij}e^{t_i+t_j}\big((s_i - s_j)^2 - 2(\psi_i - \psi_j)(\bar{\psi}_i - \bar{\psi}_j)\big),
\end{aligned} \tag{3.C.12}$$

and $q = (s, \psi, \bar{\psi})$ are the quadratic coordinates.

As currently stated, we cannot take the limit $\sigma \to \infty$ due to the $e^{t_a+\sigma}$ term. Let us ignore this for now, and consider what happens to the other terms. We would then have

$$\lim_{\sigma\to\infty} \nu_i^{\pm} = \frac{1}{2}, \tag{3.C.13}$$

and, after shifting $t_i \mapsto t_i + \log 2$, would obtain

$$\lim_{\varepsilon\to 0}\sum_b \int e^{t_a} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i\right)\delta_\varepsilon\left(\sum_i t_i\right) e^{-H_1(t)} e^{-t}\left[\lim_{\sigma\to\infty} 2^{-|\Lambda|} e^{\sigma}\int e^{-H_2(t+\log 2, s,\psi,\bar\psi)} Dq\right] dt. \tag{3.C.14}$$

The dependence on the quadratic coordinates is now only through H_2; integrating this out would give a square root determinant factor, albeit, one which includes the zero eigenvalue of the t-Laplacian. A more careful analysis used in the proof below will involve rescaling the corresponding eigenmode (i.e., the quadratic variable means), and will show that this zero eigenvalue exactly cancels the infinite prefactor; this leaves a matrix minor term, and hence, the Sabot–Tarres limit distribution.

There is one additional subtlety glossed over in the above discussion, namely, that dominant terms in the integral are contributed not just when $x \approx e^{\frac{T}{|\Lambda|}}$, but also when $x \approx -e^{\frac{T}{|\Lambda|}}$. Thus, if we simply boost in the positive x direction, a significant fraction of mass still escapes in the other direction. The problem is avoided by using the $x \mapsto -x$ symmetry to restrict ourselves to the positive side, rewriting (3.C.3) as

$$\begin{aligned}\lim_{T\to\infty}\mathbb{E}_a\left(g\left(L_T - \frac{T}{|\Lambda|}\right)\right) = \lim_{\varepsilon\to 0}\lim_{T\to\infty}\lim_{\varepsilon'\to 0}\sum_b 2\int \chi_{\varepsilon'}(x_a) x_a \frac{x_b}{z_b} g\left(\log z - \frac{1}{|\Lambda|}\sum_i \log z_i\right)\\ \times\delta_\varepsilon\left(\sum_i \log z_i - T\right) e^{-H} d\mu,\end{aligned} \tag{3.C.15}$$

where $\chi_{\varepsilon'}(x_a) = \frac{1}{2}(1 + \tanh \frac{x_a}{\varepsilon'})$ is a smooth step function with $\chi_{\varepsilon'}(x) + \chi_{\varepsilon'}(-x) = 1$, and we have used Lemma 3.C.2 to exchange the order of the outer two limits. This expression will be the starting point of our proof.

Proof. We proceed along the same lines as above, but now starting from the reformulation of Corollary 3.7.2 given in (3.C.15). Once again, we transform to horospherical coordinates and boost by $\sigma = \frac{T}{|\Lambda|}$, now giving the right hand side as (dropping sums and limits for now)

$$2\int \chi_{\varepsilon'}(e^{t_a+\sigma}\nu_a^-) e^{t_a+\sigma}\nu_a^- \frac{\nu_b^-}{\nu_b^+} g\left(t + \log\nu^+ - \frac{1}{|\Lambda|}\sum_i t_i + \log\nu_i^+\right)\delta_\varepsilon\left(\sum_i t_i + \log\nu_i^+\right) e^{-H_1(t) - H_2(t,s,\psi,\bar\psi)} e^{-t} d\mu. \tag{3.C.16}$$

Here, $d\mu$ is the is the Berezin-Lebesgue measure, normalised with a factor of $\frac{1}{\sqrt{2\pi}}$ for each bosonic coordinate:

$$d\mu = \prod_i \frac{dt_i}{\sqrt{2\pi}}\frac{ds_i}{\sqrt{2\pi}} d\psi_i d\bar\psi_i. \tag{3.C.17}$$

For readability, we will suppress these extra factors below.

As discussed, in order to take the limit $\sigma\to\infty$, we need to separate out and then rescale the zero eigenmode of the t-Laplacian. This is achieved by shifting to 'mean 0' coordinates in the quadratic variables $q = (s,\psi,\bar\psi)$. We now change variables as

$$s_i' = s_i - \frac{1}{|\Lambda|}\sum_i s_i, \qquad \psi_i' = \psi_i - \frac{1}{|\Lambda|}\sum_i \psi_i, \qquad \bar\psi_i' = \bar\psi_i - \frac{1}{|\Lambda|}\sum_i \bar\psi_i, \tag{3.C.18}$$

but further rescale the means as

$$s_0' = \frac{e^{-\sigma}}{|\Lambda|}\sum_i s_i, \qquad \psi_0' = \frac{e^{-\sigma}}{|\Lambda|}\sum_i \psi_i, \qquad \bar\psi_0' = \frac{e^{-\sigma}}{|\Lambda|}\sum_i \bar\psi_i. \tag{3.C.19}$$

This transforms the Berezin measure as

$$e^{-\sum_i t_i} d\mu \mapsto \frac{e^{-\sigma} e^{-\sum_i t_i}}{|\Lambda|} dt\, dq_0\, dq_\perp \tag{3.C.20}$$

and hence gives the integral as (dropping primes)

$$\begin{aligned}\frac{2}{|\Lambda|} \int \chi_{\varepsilon'}(e^{t_a+\sigma}\tilde{\nu}_a^-) e^{t_a} \tilde{\nu}_a^- \frac{\tilde{\nu}_b^-}{\tilde{\nu}_b^+} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i + \log \tilde{\nu}^+ - \frac{1}{|\Lambda|}\sum_i \log \tilde{\nu}_i^+\right) \\ \times \delta_\varepsilon\left(\sum_i t_i + \log \tilde{\nu}_i^+\right) e^{-H_1(t) - H_2(t,s,\psi,\bar{\psi})} e^{-t} dt\, dq_0\, dq_\perp\end{aligned} \tag{3.C.21}$$

where

$$\tilde{\nu}_i^\pm = \frac{1 \pm \left(e^{-2t_i - 2\sigma} + (s_i e^{-\sigma} + s_0)^2 - 2(\psi_i e^{-\sigma} + \psi_0)(\bar{\psi}_i e^{-\sigma} + \bar{\psi}_0)\right)}{2}. \tag{3.C.22}$$

We now Grassmann-Taylor expand the indicator function and take the limit $\varepsilon' \to 0$ to give

$$\begin{aligned}\lim_{\varepsilon' \to 0} \chi_{\varepsilon'}(e^{t_a+\sigma}\tilde{\nu}_a^-) &= \lim_{\varepsilon' \to 0} \chi_{\varepsilon'}(e^{t_a+\sigma}\mathbf{b}(\nu_a^-)) + \chi'_{\varepsilon'}(e^{t_a+\sigma}\mathbf{b}(\nu_a^-)) e^{t_a+\sigma}\mathbf{s}(\nu_a^-) \\ &= \chi(e^{t_a+\sigma}\mathbf{b}(\nu_a^-)) + \delta(e^{t_a+\sigma}\mathbf{b}(\nu_a^-)) e^{t_a+\sigma}\mathbf{s}(\nu_a^-) \\ &= \chi(\mathbf{b}(\nu_a^-)) + \delta(\mathbf{b}(\nu_a^-))\mathbf{s}(\nu_a^-),\end{aligned} \tag{3.C.23}$$

where in the third line we have made use of the indicator and delta function scaling properties, and

$$\begin{aligned}\mathbf{b}(\nu_a^-) &= \frac{1 - (e^{-2t_a - 2\sigma} + (s_a e^{-\sigma} + s_0)^2)}{2}, \\ \mathbf{s}(\nu_a^-) &= (\psi_a e^{-\sigma} + \psi_0)(\bar{\psi}_a e^{-\sigma} + \bar{\psi}_0)\end{aligned} \tag{3.C.24}$$

are the body and soul of $\tilde{\nu}_a^-$. This gives the integral as

$$\begin{aligned}\lim_{\varepsilon \to 0} \sum_b \frac{2}{|\Lambda|} \int \left(\chi(\mathbf{b}(\nu_a^-)) + \delta(\mathbf{b}(\nu_a^-))\mathbf{s}(\nu_a^-)\right) e^{t_a} \tilde{\nu}_a^- \frac{\tilde{\nu}_b^-}{\tilde{\nu}_b^+} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i + \log \tilde{\nu}^+ - \frac{1}{|\Lambda|}\sum_i \log \tilde{\nu}_i^+\right) \\ \times \delta_\varepsilon\left(\sum_i t_i + \log \tilde{\nu}_i^+\right) e^{-H_1(t) - H_2(t,s,\psi,\bar{\psi})} e^{-t} dt\, dq_0\, dq_\perp\end{aligned} \tag{3.C.25}$$

We are now free to take the limit $\sigma \to \infty$. Setting

$$\lim_{\sigma \to \infty} \tilde{\nu}_i^\pm = \frac{1 \pm (s_0^2 - 2\psi_0\bar{\psi}_0)}{2} =: \eta_0^\pm, \tag{3.C.26}$$

we obtain the $\sigma \to \infty$ limit as

$$\sum_b \frac{2}{|\Lambda|} \int \left(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\right) e^{t_a} \frac{(\eta_0^-)^2}{\eta_0^+} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i\right) \delta_\varepsilon\left(\sum_i t_i + \log \eta_0^+\right) e^{-H_1(t) - H_2(t,s,\psi,\bar{\psi})} e^{-t} dt\, dq_0 \tag{3.C.27}$$

Noting that the $s_\perp, \psi_\perp, \bar{\psi}_\perp$ variables only enter through H_2, we can write

$$\begin{aligned}\sum_b \frac{2}{|\Lambda|} \int \left(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\right) e^{t_a} \frac{(\eta_0^-)^2}{\eta_0^+} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i\right) \\ \times\, \delta_\varepsilon\left(\sum_i t_i + \log \eta_0^+\right) e^{-H_1(t)} \left[\int e^{-H_2(t,s,\psi,\bar{\psi})}\, dq_\perp\right] e^{-t} dt\, dq_0,\end{aligned} \tag{3.C.28}$$

and integrate them out directly as Gaussian integral

$$\int e^{-H_2(t,s,\psi,\bar{\psi})}dq_\perp = \sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_\perp\right)}. \tag{3.C.29}$$

As all terms in the sum are the same, we are thus left with

$$\lim_{\varepsilon\to 0} 2\int \big(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\big)e^{t_a}\frac{(\eta_0^-)^2}{\eta_0^+} g\left(t - \frac{1}{|\Lambda|}\sum_i t_i\right)\delta_\varepsilon\left(\sum_i t_i + \log\eta_0^+\right)e^{-H_1(t)}\sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_\perp\right)}e^{-t}dt\, d \tag{3.C.30}$$

We now shift to mean 0 coordinates for the t field,

$$t_i' = t_i - \frac{1}{|\Lambda|}\sum_i t_i \tag{3.C.31}$$

rescaling and shifting the mean as

$$t_0 = \sum_i (t_i + \log\eta_0^+). \tag{3.C.32}$$

This transforms the Berezin measure as

$$e^{-\sum_i t_i}dt \mapsto (\eta_0^+)^{|\Lambda|}e^{-t_0}dt_0\, dt_\perp, \tag{3.C.33}$$

the e^{t_a} factor as

$$e^{t_a} \mapsto (\eta_0^+)^{-1}e^{\frac{t_0}{|\Lambda|}}e^{t_a'}, \tag{3.C.34}$$

the t-Laplacian as

$$(\tilde{\Delta}_{\beta,t})_{ij} \mapsto (\eta_0^+)^{-2}e^{\frac{2t_0}{|\Lambda|}}(\tilde{\Delta}_{\beta,t'})_{ij}, \tag{3.C.35}$$

and hence the determinant factor as

$$\sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_\perp\right)} \mapsto (\eta_0^+)^{-(|\Lambda|-1)}e^{\frac{(|\Lambda|-1)t_0}{|\Lambda|}}\sqrt{\det\left(\tilde{\Delta}_{\beta,t'}\big|_\perp\right)} \tag{3.C.36}$$

by multilinearity. These additional terms cancel out, giving the integral as (dropping primes)

$$\lim_{\varepsilon\to 0} 2\int \big(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\big)e^{t_a}\frac{(\eta_0^-)^2}{\eta_0^+}g(t)\delta_\varepsilon(t_0)e^{-H_1(t)}\sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_\perp\right)}dt_0\, dt_\perp dq_0, \tag{3.C.37}$$

splitting into three terms as

$$\lim_{\varepsilon\to 0} 2\int \big(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\big)\frac{(\eta_0^-)^2}{\eta_0^+}\, dq_0 \int \delta_\varepsilon(t_0)\, dt_0 \int e^{t_a}g(t)e^{-H_1(t)}\sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_\perp\right)}dt_\perp. \tag{3.C.38}$$

As there are implicit factors of $\frac{1}{\sqrt{2\pi}}$ in the measure for all bosonic coordinates (which we now explicitly write), the integral of the second term is $\int \delta_\varepsilon(t_0)\,\frac{dt_0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$, and a short calculation shows

$$\begin{aligned}
&2\int \big(\chi(\mathbf{b}(\eta_0^-)) + \delta(\mathbf{b}(\eta_0^-))\mathbf{s}(\eta_0^-)\big)\frac{(\eta_0^-)^2}{\eta_0^+}\, dq_0 \\
&= \int \left(\mathbf{1}\left(\frac{1-s_0^2}{2} > 0\right) - \delta\left(\frac{1-s_0^2}{2}\right)\psi_0\bar{\psi}_0\right)\frac{(1-s_0^2-2\psi_0\bar{\psi}_0)^2}{1+s_0^2+2\psi_0\bar{\psi}_0}\,\frac{ds_0}{\sqrt{2\pi}}\, d\psi_0\, d\bar{\psi}_0 \\
&= \sqrt{2\pi},
\end{aligned} \tag{3.C.39}$$

and so the product of the first and second terms is identically 1. The remaining term is the Sabot–Tarrès limit distribution

$$\int e^{t_a} g(t) e^{-H_1(t)} \sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_{\perp}\right)} \frac{dt_\perp}{\sqrt{2\pi}^{|\Lambda|-1}}. \tag{3.C.40}$$

Applying the matrix tree theorem

$$\sqrt{\det\left(\tilde{\Delta}_{\beta,t}\big|_{\perp}\right)} = \sqrt{|\Lambda|}\sqrt{D(\beta,t)}, \tag{3.C.41}$$

with $D(\beta,t)$ any diagonal matrix minor of $\tilde{\Delta}_{\beta,t}$, (3.C.40) can also be written as

$$\frac{\sqrt{|\Lambda|}}{\sqrt{2\pi}^{|\Lambda|-1}} \int e^{t_a} g(t) e^{-H_1(t)} \sqrt{D(\beta,t)} dt_\perp \tag{3.C.42}$$

which is the claim. □

Chapter 4

Random spanning forests and hyperbolic symmetry

4.1 The arboreal gas and uniform forest model

Definition and main results

Let $\mathbb{G} = (\Lambda, E)$ be a finite (undirected) graph. A *forest* is a subgraph $F = (\Lambda, E')$ that does not contain any cycles. We write $\mathcal{F}$ for the set of all forests. For $\beta > 0$ the *arboreal gas* (or *weighted uniform forest model*) is the measure on forests F defined by

$$\mathbb{P}_\beta[F] \equiv \frac{1}{Z_\beta}\beta^{|F|}, \qquad Z_\beta \equiv \sum_{F \in \mathcal{F}} \beta^{|F|}, \tag{4.1.1}$$

where $|F|$ denotes the number of edges in F. It is an elementary observation that the arboreal gas with parameter β is precisely Bernoulli bond percolation with parameter $p_\beta = \beta/(1+\beta)$ conditioned to be acyclic:

$$\mathbb{P}^{\mathrm{perc}}_{p_\beta}[F \mid \text{acyclic}] \equiv \frac{p_\beta^{|F|}(1-p_\beta)^{|E|-|F|}}{\sum_F p_\beta^{|F|}(1-p_\beta)^{|E|-|F|}} = \frac{\beta^{|F|}}{\sum_F \beta^{|F|}} = \mathbb{P}_\beta[F]. \tag{4.1.2}$$

The arboreal gas model is also the limit, as $q \to 0$ with $p = \beta q$, of the q-state random cluster model, see [84]. The particular case $\beta = 1$ is the *uniform forest model* mentioned in, e.g., [52, 53, 59, 84]. We emphasize that the uniform forest model is *not* the weak limit of a uniformly chosen spanning tree; emphasis is needed since the latter model is called the 'uniform spanning forest' (USF) in the probability literature. We will shortly see that the arboreal gas has a richer phenomenology than the USF. In fact, in finite volume, the uniform spanning tree is the $\beta \to \infty$ limit of the arboreal gas.

Given that the arboreal gas arises from bond percolation, it is natural to ask about the percolative properties of the arboreal gas. It is straightforward to rule out the occurrence of percolation for small values of β via the following proposition, see Appendix 4.A.

Proposition 4.1.1. *On any finite graph, the arboreal gas with parameter β is stochastically dominated by Bernoulli bond percolation with parameter p_β.*

In particular, all subgraphs of $\mathbb{Z}^d$, all trees have uniformly bounded expectation if $p_\beta < p_c(d)$ where $p_c(d)$ is the critical parameter for Bernoulli bond percolation on $\mathbb{Z}^d$.

In the infinite-volume limit, the arboreal gas is a singular conditioning of bond percolation, and hence the existence of a percolation transition as β varies is non-obvious. However, on the complete graph it is known that there is a phase transition, see [11, 65, 70]. To illustrate some of our methods we will give a new proof of the existence of a transition.

Proposition 4.1.2. *Let $\mathbb{E}_{N,\alpha}$ denote the expectation of the arboreal gas on the complete graph K_N with $\beta = \alpha/N$, and let T_0 be the tree containing a fixed vertex 0. Then*

$$\mathbb{E}_{N,\alpha}|T_0| = (1+o(1))\begin{cases} \frac{\alpha}{1-\alpha} & \alpha < 1 \\ cN^{1/3} & \alpha = 1 \\ (\frac{\alpha-1}{\alpha})^2 N & \alpha > 1. \end{cases} \tag{4.1.3}$$

where $c = 3^{2/3}\Gamma(4/3)/\Gamma(2/3)$ and Γ denotes the Euler Gamma function.

Thus there is a transition for the arboreal gas exactly as for the Erdős–Rényi random graph with edge probability α/N. To compare the arboreal gas directly with the Erdős–Rényi graph, recall that Proposition 4.1.1 shows the arboreal gas is stochastically dominated by the Erdős–Rényi graph with edge probability $p_\beta = \beta - \beta^2/(1+\beta)$. The fact that the Erdős–Rényi graph asymptotically has all components trees in the subcritical regime $\alpha < 1$ makes the behaviour of the arboreal gas when $\alpha < 1$ unsurprising. On the other hand, the conditioning plays a role when $\alpha > 1$, as can be seen at the level of the expected tree size. For the supercritical Erdős–Rényi graph the expected size is $4(\alpha-1)^2 N$ as $\alpha \downarrow 1$ — this follows from the fact that the largest component for the Erdős–Rényi graph with $\alpha > 1$ has size yN where y solves $e^{-\alpha y} = 1 - y$, see, e.g., [4]. For further discussion, see Section 4.1.

On $\mathbb{Z}^2$, the singular conditioning that defines the arboreal gas has a profound effect. In the next theorem statement and henceforth, for finite subgraphs Λ of $\mathbb{Z}^2$ we write $\mathbb{P}_{\Lambda,\beta}$ for the arboreal gas on Λ.

Theorem 4.1.3. *For all $\beta > 0$ there is a universal constant $c_\beta > 0$ such that the connection probabilities satisfy*

$$\mathbb{P}_{\Lambda,\beta}[0 \leftrightarrow j] \leqslant |j|^{-c_\beta} \quad \textit{for } j \in \Lambda \subset \mathbb{Z}^2, \tag{4.1.4}$$

for all $\Lambda \subset \mathbb{Z}^2$, where '$i \leftrightarrow j$' denotes the event that the vertices i and j are in the same tree.

This theorem, together with classical techniques from percolation theory, imply the following corollary for the infinite volume limit, see Appendix 4.A.

Corollary 4.1.4. *Suppose $\mathbb{P}_\beta$ is a translation-invariant weak limit of $\mathbb{P}_{\Lambda_n,\beta}$ for an increasing exhaustion of finite volumes $\Lambda_n \uparrow \mathbb{Z}^2$. Then all trees are finite $\mathbb{P}_\beta$-almost surely.*

Thus on $\mathbb{Z}^2$ the behaviour of the arboreal gas is completely different from that of Bernoulli percolation. The absence of a phase transition can be non-rigorously predicted from the representation of the arboreal gas as the $q \to 0$ limit (with $p = \beta q$ fixed) of the random cluster model with $q > 0$ [33]. We briefly describe how this prediction can be made. The critical point of the random cluster model for $q \geqslant 1$ on $\mathbb{Z}^2$ is known to be $p_c(q) = \sqrt{q}/(1+\sqrt{q})$ [12]. Conjecturally, this formula holds for $q > 0$. Thus $p_c(q) \sim \sqrt{q}$ as $q \downarrow 0$, and by assuming continuity in q one obtains $\beta_c = \infty$ for the arboreal gas. This heuristic applies also to the triangular and hexagonal lattices. Our proof is in fact quite robust, and applies to much more general recurrent two-dimensional graphs. We have focused on $\mathbb{Z}^2$ for the sake of concreteness.

This absence of percolation is not believed to persist in dimensions $d \geqslant 3$: we expect that there is a percolative transition on $\mathbb{Z}^d$ with $d \geqslant 3$. In the next section we will discuss the conjectural behaviour of the arboreal gas on $\mathbb{Z}^d$ for all $d \geqslant 2$. Before this, we outline how we obtain the above results. Our starting point is an alternate formulation of the arboreal gas. Namely, in [25, 26, 28] it was noticed that the arboreal gas can be represented in terms of a model of fermions, and that this fermionic model can be extended to a sigma model with values in the superhemisphere. We also use this fermionic representation, but our results rely in an essential way on the new observation that this model is most naturally connected to a sigma model taking values in a *hyperbolic* superspace. Similar sigma models have recently received a great deal of attention due to their relationship with random band matrices and reinforced random walks [9, 40, 88, 89]. We will discuss the connection between our techniques and these papers after introducing the sigma models relevant to the present paper. A key step in our proof is the following integral formula for connection probabilities in the arboreal gas (see Corollary 4.2.14 for a version with general edge weights):

$$\mathbb{P}_{\Lambda,\beta}[0 \leftrightarrow j] = \frac{1}{Z_\beta}\int_{\mathbb{R}^\Lambda} e^{t_j} e^{-\sum_{i\sim j}\beta(\cosh(t_i-t_j)-1)}\left(e^{-2\sum_i t_i}\det(-\Delta_{\beta(t)})\right)^{3/2}\delta_0(dt_0)\prod_{i\neq 0}\frac{dt_i}{\sqrt{2\pi}} \tag{4.1.5}$$

where $\Delta_{\beta(t)}$ is the graph Laplacian with edge weights $\beta e^{t_i+t_j}$, understood as acting on $\Lambda \setminus 0$. This formula is a consequence of the hyperbolic sigma model representation of the arboreal gas.

Surprisingly, if the exponent $3/2$ in (4.1.5) is replaced by $1/2$, then the integrand on the right-hand side is the mixing measure of the vertex-reinforced jump process found by Sabot and Tarrès [89]. The Sabot–Tarrès formula (along with a closely related version for the edge-reinforced random walk) is known as the *magic formula* [61]. It seems even more magical to us that the same formula, with only a change of exponent, describes the arboreal gas. We will explain in Section 4.2 that there are in fact three ingredients to this magic: a 'non-linear' version of the matrix-tree theorem, supersymmetric localisation, and horospherical coordinates for (super-)hyperbolic space.

We remark that the whole family of sigma models taking values in hyperbolic superspaces has interesting behaviour, but for the present paper we restrict our attention to those related to the arboreal gas. A more general discussion of such models can be found in [29] by the second author.

Context and conjectured behaviour

Recall that '$i \leftrightarrow j$' denotes the event that the vertices i and j are in the same tree. We also write $\mathbb{P}_\beta[ij]$ for the probability an edge ij is in the forest.

The following conjecture asserts that the arboreal gas has a phase transition in dimensions $d \geqslant 3$, just as in mean-field theory (Proposition 4.1.2). Numerical evidence for this transition can be found in [33].

Conjecture 4.1.5. *For $d \geqslant 3$ there exists $\beta_c > 0$ such that*

$$\lim_{n\to\infty}\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathbb{E}_{\Lambda,\beta}\frac{|T_0\cap B_n|}{|B_n|}\begin{cases} = 0 & (\beta<\beta_c)\\ > 0 & (\beta>\beta_c)\end{cases} \tag{4.1.6}$$

where T_0 is the tree containing 0 and B_n is the ball of radius n centred at 0. Moreover, when $\beta < \beta_c$ there is a universal constant $c_\beta > 0$ such that

$$\mathbb{P}_{\Lambda,\beta}[i \leftrightarrow j] \leqslant Ce^{-c_\beta|i-j|}, \qquad (i,j\in\mathbb{Z}^d). \tag{4.1.7}$$

When $\beta > \beta_c$ there is a universal constant $c'_\beta > 0$ such that

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\mathbb{P}_{\Lambda,\beta}[i \leftrightarrow j] \geqslant c'_\beta. \tag{4.1.8}$$

As indicated in the previous section, it is straightforward to prove the first equality of (4.1.6) when β is sufficiently small. The existence of a transition, i.e., a percolating phase for β large, is open. However, a promising approach to proving the existence of a percolation transition when $d \geqslant 3$ and $\beta \gg 1$ is to adapt the methods of [40]; we are currently pursuing this direction. Obviously, the existence of a sharp transition, i.e., a precise β_c separating the two behaviours in (4.1.6) is also open. The next conjecture distinguishes the supercritical behaviour of the arboreal gas from that of percolation for which the (centered) connection probabilities have exponential decay.

Conjecture 4.1.6. *For $d \geqslant 3$, when $\beta > \beta_c$*

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}_{\Lambda,\beta}[i \leftrightarrow j] - c'_\beta \approx |i-j|^{-(d-2)}, \qquad \text{as } |i-j| \to \infty, \tag{4.1.9}$$

where c'_β is the optimal constant for which (4.1.8) *holds.*

Assuming the existence of a phase transition, one can also ask about the *critical* behaviour of the arboreal gas. One intriguing aspect of this question is that the upper critical dimension is not clear, even heuristically. There is some evidence that the critical dimension of the arboreal gas should be $d = 6$, as for percolation, and opposed to $d = 4$ for the Heisenberg model. For further details, and for other related conjectures, see [28, Section 12].

Theorem 4.1.3 shows that the behaviour of the arboreal gas in two dimensions is different from that of percolation. This difference would be considerably strengthened by the following conjecture, which first appeared in [25].

Conjecture 4.1.7. *For $\Lambda \subset \mathbb{Z}^2$, for any $\beta > 0$ there exists a universal constant $c_\beta > 0$ such that*

$$\lim_{\Lambda \uparrow \mathbb{Z}^2} \mathbb{P}_{\Lambda,\beta}[i \leftrightarrow j] \approx e^{-c_\beta |i-j|}, \qquad (i,j \in \mathbb{Z}^2). \tag{4.1.10}$$

As $\beta \to \infty$, the constant c_β is exponentially small in β:

$$c_\beta \approx e^{-c\beta}. \tag{4.1.11}$$

In particular, $\mathbb{E}_\beta|T_0| \approx e^{c\beta} < \infty$ (with a different c) where T_0 is the tree containing 0.

This conjecture is *much* stronger than the main result of the present paper, Theorem 4.1.3, which establishes only that all trees are finite almost surely, a significantly weaker property than having finite expectation.

Conjecture 4.1.7 is a version of the mass gap conjecture for ultraviolet asymptotically free field theories. The conjecture is based on the field theory representation discussed in Section 4.2, and supporting heuristics can be found in, e.g., [25]. Other models with the same conjectural feature include the two-dimensional Heisenberg model [85], the two-dimensional vertex-reinforced jump process [40] (and other $\mathbb{H}^{n|2m}$ models with $2m - n \leqslant 0$, see [29]), the two-dimensional Anderson model [2], and most prominently four-dimensional Yang–Mills Theories [56, 85].

Let us briefly indicate discuss why Conjecture 4.1.7 seems challenging. Note that in finite volume the (properly normalized) arboreal gas converges weakly to the uniform spanning tree as $1/\beta \to 0$, see Appendix 4.B. For the uniform spanning tree it is a triviality that $c_\beta = 0$, and this is consistent with the conjecture $c_\beta \approx e^{-c\beta}$ as $\beta \to \infty$. On the other hand $c_\beta \approx e^{-c\beta}$ suggests a subtle effect, not approachable via perturbative methods such as using $1/\beta > 0$ as a small parameter for a low-temperature expansion as can be done for, e.g., the Ising model. Indeed, since $t \mapsto e^{-c/t}$ has an essential singularity at $t = 0$, its behaviour as $t = 1/\beta \to 0$ cannot be detected at any finite order in $t = 1/\beta$. The same difficulty applies to the other models mentioned above for which analogous behaviour is conjectured.

The last conjecture we mention is the negative correlation conjecture stated in [53, 59, 84] and recently in [14, 54]. This conjecture is also expected to hold true for general (positive) edge weights, see Section 4.2.

Conjecture 4.1.8. *For any finite graph and any $\beta > 0$ negative correlation holds: for distinct edges ij and kl,*

$$\mathbb{P}_\beta[ij, kl] \leqslant \mathbb{P}_\beta[ij]\mathbb{P}_\beta[kl]. \tag{4.1.12}$$

More generally, for all distinct edges $i_1j_1, \dots, i_nj_n$ and $m < n$,

$$\mathbb{P}_\beta[i_1j_1, \dots, i_nj_n] \leqslant \mathbb{P}_\beta[i_1j_1, \dots, i_mj_m]\mathbb{P}_\beta[i_{m+1}j_{m+1}, \dots, i_nj_n]. \tag{4.1.13}$$

The weaker inequality $\mathbb{P}_\beta[ij,kl] \leqslant 2\mathbb{P}_\beta[ij]\mathbb{P}_\beta[kl]$ was recently proved in [14]. It is intriguing that the Lorentzian signature plays an important role in both [14] and the present work, but we are not aware of a direct relation. An important consequence of the full conjecture (with factor 1) is the existence of translation invariant arboreal gas measures on $\mathbb{Z}^d$; we prove this in Appendix 4.A.

Proposition 4.1.9. *Assume Conjecture 4.1.8 is true. Suppose Λ_n is an increasing family of subgraphs such that $\Lambda_n \uparrow \mathbb{Z}^d$, and let $\mathbb{P}_{\beta,n}$ be the arboreal gas on the finite graph Λ_n. Then the weak limit $\lim_n \mathbb{P}_{\beta,n}$ exists and is translation invariant.*

Remark 4.1.10. The conjectured inequality (4.1.12) can be recast as a reversed second Griffiths inequality. More precisely, (4.1.12) can be rewritten in terms of the $\mathbb{H}^{0|2}$ spin model introduced below in Section 4.2 as

$$\langle (u_i \cdot u_j)(u_k \cdot u_l)\rangle_\beta - \langle u_i \cdot u_j\rangle_\beta \, \langle u_k \cdot u_l\rangle_\beta \leqslant 0. \tag{4.1.14}$$

This equivalence follows immediately from the results in Section 4.2.

Related literature

The arboreal gas has received attention under various names. An important reference for our work is [25], along with subsequent works by subsets of these authors and collaborators [10, 11, 26–28, 55]. These authors considered the connection of the arboreal gas with the *antiferromagnetic* $\mathbb{S}^{0|2}$ model.

Our results are in part based on a re-interpretation of the $\mathbb{S}^{0|2}$ formulation in terms of the hyperbolic $\mathbb{H}^{0|2}$ model. At the level of infinitesimal symmetries these models are equivalent. The power behind the hyperbolic language is that it allows for a further reformulation in terms of the $\mathbb{H}^{2|4}$ model, which is analytically useful. The $\mathbb{H}^{2|4}$ representation arises from a dimensional reduction formula, which in turn is a consequence of supersymmetric localization [3,20,83]. Much of Section 4.2 is devoted to explaining this. The upshot is that this representation allows us to make use of techniques originally developed for the non-linear $\mathbb{H}^{2|2}$ sigma model [39,40,104–106] and the vertex-reinforced jump process [5, 89]. In particular, our proof of Theorem 4.1.3 makes use of an adaptation of a Mermin–Wagner argument for the $\mathbb{H}^{2|2}$ model [9, 62, 88]; the particular argument we adapt is due to Sabot [88]. For more on the connections between these models, see [9, 89].

Conjecture 4.1.8 seems to have first appeared in print in [58]. Subsequent related works, including proofs for some special subclasses of graphs, include [14, 53, 95, 100].

As mentioned before, considerably stronger results are known for the arboreal gas on the complete graph. The first result in this direction concerned forests with a fixed number of edges [65], and later a fixed number of trees was considered [11]. Later in [70] the arboreal gas itself was considered, in the guise of the Erdős–Rényi graph conditioned to be acyclic. In [65] it was understood that the scaling window is of size $N^{-1/3}$, and results on the behaviour of the ordered component sizes when $\alpha = 1 + \lambda N^{-1/3}$ were obtained. In particular, the large components in the scaling window are of size $N^{2/3}$. A very complete description of the component sizes in the critical window was obtained in [70].

We remark on an interesting aspect of the arboreal gas that was first observed in [65] and is consistent with Conjecture 4.1.6. Namely, in the supercritical regime, the component sizes of the k largest non-giant components are of order $N^{2/3}$ [65, Theorem 5.2]. This is in contrast to the Erdős–Rényi graph, where the non-giant components are of logarithmic size. The critical size of the non-giant components is reminiscent of self-organised criticality, see [86] for example. A clearer understanding of the mechanism behind this behaviour for the arboreal gas would be interesting.

Outline

In the next section we introduce the $\mathbb{H}^{0|2}$ and $\mathbb{H}^{2|4}$ sigma models, relate them to the arboreal gas, and derive several useful facts. In Section 4.3 we use the $\mathbb{H}^{0|2}$ representation and Hubbard–Stratonovich type transformations to prove Theorem 4.3.1 by a stationary phase argument. In Section 4.4 we prove the quantitative part of Theorem 4.1.3, i.e., (4.1.4). The deduction that all trees are finite almost surely follows from adaptions of well-known arguments and is given in Appendix 4.A. For the convenience of readers, we briefly discuss the fermionic representation of *rooted* spanning forests and spanning trees in Appendix 4.B.

4.2 Hyperbolic sigma model representation

In [25], it was noticed that the arboreal gas has a formulation in terms of fermionic variables, which in turn can be related to a supersymmetric spin model with values in the superhemisphere and negative (i.e., antiferromagnetic) spin couplings. In Section 4.2, we reinterpret this fermionic model as the $\mathbb{H}^{0|2}$ model (defined there) with positive (i.e., ferromagnetic) spin couplings. This reinterpretation has important consequences: in Section 4.2, we relate the $\mathbb{H}^{0|2}$ model to the $\mathbb{H}^{2|4}$ model (defined there) by a form of dimensional reduction applied to the target space. Technically this amounts to exploiting supersymmetric localisation associated to an additional set of fields. The $\mathbb{H}^{2|4}$ model allows the introduction of horospherical coordinates, which leads to an analytically useful *probabilistic* representation of the model as a gradient model with a non-local and non-convex potential. This gradient model is very similar to gradient models that arise in the study of linearly-reinforced random walks. In fact, up to the power of a determinant, this representation is in terms of a measure that is identical to the *magic formula* describing the mixing measure of the vertex-reinforced jump process, see (4.1.5).

$\mathbb{H}^{0|2}$ model and arboreal gas

Let Λ be a finite set, let $\boldsymbol{\beta} = (\beta_{ij})_{i,j\in\Lambda}$ be real-valued symmetric edge weights, and let $\boldsymbol{h} = (h_i)_{i\in\Lambda}$ be real-valued vertex weights. Throughout we will use this bold notation to denote tuples indexed by vertices or edges. For $f\colon \Lambda \to \mathbb{R}$, we define the Laplacian associated with the edge weights by

$$\Delta_\beta f(i) \equiv \sum_{j\in\Lambda} \beta_{ij}(f(j) - f(i)). \tag{4.2.1}$$

The non-zero edge weights induce a graph $\mathbb{G} = (\Lambda, E)$, i.e., $ij \in E$ if and only if $\beta_{ij} \neq 0$.

Let $\Omega^{2\Lambda}$ be a (real) Grassmann algebra (or exterior algebra) with generators $(\xi_i, \eta_i)_{i\in\Lambda}$, i.e., all of the ξ_i and η_i anticommute with each other. For $i, j \in \Lambda$, define the even elements

$$z_i \equiv \sqrt{1 - 2\xi_i\eta_i} \equiv 1 - \xi_i\eta_i \tag{4.2.2}$$

$$u_i \cdot u_j \equiv -\xi_i\eta_j - \xi_j\eta_i - z_iz_j = -1 - \xi_i\eta_j - \xi_j\eta_i + \xi_i\eta_i + \xi_j\eta_j - \xi_i\eta_i\xi_j\eta_j. \tag{4.2.3}$$

Note that $u_i \cdot u_i = -1$ which we formally interpret as meaning that $u_i = (\xi, \eta, z) \in \mathbb{H}^{0|2}$ by analogy with the hyperboloid model for hyperbolic space. However, we emphasize that '$\in \mathbb{H}^{0|2}$' does not have any literal sense. Similarly we write $\boldsymbol{u} = (u_i)_{i\in\Lambda} \in (\mathbb{H}^{0|2})^\Lambda$. The fermionic derivative ∂_{ξ_i} is defined in the natural way, i.e., as the odd derivation on that acts on $\Omega^{2\Lambda}$ by

$$\partial_{\xi_i}(\xi_i F) \equiv F, \quad \partial_{\xi_i} F \equiv 0 \tag{4.2.4}$$

for any form F that does not contain ξ_i. An analogous definition applies to ∂_{η_i}. The hyperbolic fermionic integral is defined in terms of the fermionic derivative by

$$[F]_0 \equiv \int_{(\mathbb{H}^{0|2})^\Lambda} F \equiv \prod_{i\in\Lambda} \left(\partial_{\eta_i}\partial_{\xi_i}\frac{1}{z_i}\right) F = \partial_{\eta_N}\partial_{\xi_N}\cdots\partial_{\eta_1}\partial_{\xi_1}\left(\frac{1}{z_1\cdots z_N}F\right) \in \mathbb{R} \tag{4.2.5}$$

if $\Lambda = \{1, \dots, N\}$. It is well-known that while the fermionic integral is formally equivalent to a fermionic derivative, it behaves in many ways like an ordinary integral. The factors of $1/z$ make the hyperbolic fermionic integral invariant under a fermionic version of the Lorentz group; see (4.2.18).

The $\mathbb{H}^{0|2}$ sigma model action is the even form $H_{\beta,h}(\boldsymbol{u})$ in $\Omega^{2\Lambda}$ given by

$$H_{\beta,h}(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z} - 1) = \frac{1}{4}\sum_{i,j} \beta_{ij}(u_i - u_j)^2 + \sum_i h_i(z_i - 1) \tag{4.2.6}$$

where $(a, b) \equiv \sum_i a_i \cdot b_i$, with $a_i \cdot b_i$ interpreted as the $\mathbb{H}^{0|2}$ inner product defined by (4.2.3). The corresponding unnormalised expectation $[\cdot]_{\beta,h}$ and normalised expectation $\langle \cdot \rangle_{\beta,h}$ are defined by

$$[F]_{\beta,h} \equiv [F e^{-H_{\beta,h}}]_0, \qquad \langle F \rangle_{\beta,h} \equiv \frac{[F]_{\beta,h}}{[1]_{\beta,h}}, \tag{4.2.7}$$

the latter definition holding when $[1]_{\beta,h} \neq 0$. In (4.2.7) the exponential of the even form $H_{\beta,h}$ is defined by the formal power series expansion, which truncates at finite order since Λ is finite. For an introduction to Grassmann algebras and integration as used in this paper, see [8, Appendix A].

Note that the unnormalised expectation $[\cdot]_{\beta,h}$ is well-defined for all real values of the β_{ij} and h_i, including negative values, and in particular $\boldsymbol{h} = \boldsymbol{0}$, $\boldsymbol{\beta} = \boldsymbol{0}$, or both, are permitted. We will use the abbreviations $[\cdot]_\beta \equiv [\cdot]_{\beta,0}$ and $\langle \cdot \rangle_\beta \equiv \langle \cdot \rangle_{\beta,0}$.

The following theorem shows that the partition function $[1]_{\beta,h}$ of the $\mathbb{H}^{0|2}$ model is exactly the partition function of the arboreal gas Z_β defined in (4.1.1) when $\boldsymbol{h} = \boldsymbol{0}$, and that it is a generalization the partition function when $\boldsymbol{h} \neq \boldsymbol{0}$ which we will subsequently denote by $Z_{\beta,h}$. This connection between spanning forests and the antiferromagnetic $\mathbb{S}^{0|2}$ model, which is equivalent to our ferromagnetic $\mathbb{H}^{0|2}$ model, was previously observed in [25]. As mentioned earlier, our hyperbolic interpretation will have important consequences in what follows.

Theorem 4.2.1. *For any real-valued weights $\boldsymbol{\beta}$ and $\boldsymbol{h}$,*

$$[1]_{\beta,h} = \sum_{F \in \mathcal{F}} \prod_{ij \in F} \beta_{ij} \prod_{T \in F} (1 + \sum_{i \in T} h_i) \tag{4.2.8}$$

where the inner product runs over the trees T that make up the forest F.

For the reader's convenience and to keep our exposition self contained, we provide a concise proof of Theorem 4.2.1 below. The interested reader may consult the original paper [25], where they can also find generalizations to hyperforests. The $\boldsymbol{h} = \boldsymbol{0}$ case of Theorem 4.2.1 also implies the following useful representations of probabilities for the arboreal gas.

Corollary 4.2.2. *Let $\boldsymbol{h} = \boldsymbol{0}$ and assume the edge weights $\boldsymbol{\beta}$ are non-negative. Then for all edges ab,*

$$\mathbb{P}_\beta[ab] = \beta_{ab} \langle u_a \cdot u_b + 1 \rangle_\beta, \tag{4.2.9}$$

and more generally, for all sets of edges S,

$$\mathbb{P}_\beta[S] = \langle \prod_{ij \in S} \beta_{ij}(u_i \cdot u_j + 1) \rangle_\beta. \tag{4.2.10}$$

Moreover, for all vertices $a, b \in \Lambda$,

$$\mathbb{P}_\beta[a \leftrightarrow b] = -\langle z_a z_b \rangle_\beta = -\langle u_a \cdot u_b \rangle_\beta = \langle \xi_a \eta_b \rangle_\beta = 1 - \langle \eta_a \xi_a \eta_b \xi_b \rangle_\beta, \tag{4.2.11}$$

and also

$$\langle z_a \rangle_\beta = 0. \tag{4.2.12}$$

We will prove Theorem 4.2.1 and Corollary 4.2.2 in Section 4.2, but first we establish some integration identities associated with the symmetries of $\mathbb{H}^{0|2}$.

Ward Identities for $\mathbb{H}^{0|2}$

Define the operators

$$L \equiv \sum_{i\in\Lambda} L_i \equiv \sum_{i\in\Lambda} z_i\partial_{\xi_i}, \qquad \bar{L} \equiv \sum_{i\in\Lambda} \bar{L}_i \equiv \sum_{i\in\Lambda} z_i\partial_{\eta_i}, \qquad S \equiv \sum_{i\in\Lambda} S_i \equiv \sum_{i\in\Lambda}(\eta_i\partial_{\xi_i} + \xi_i\partial_{\eta_i}). \quad (4.2.13)$$

Using (4.2.2), one computes that these act on coordinates as

$$L\xi_a = z_a, \qquad L\eta_a = 0, \qquad Lz_a = -\eta_a, \quad (4.2.14)$$
$$\bar{L}\xi_a = 0, \qquad \bar{L}\eta_a = z_a, \qquad \bar{L}z_a = \xi_a, \quad (4.2.15)$$
$$S\xi_a = \eta_a, \qquad S\eta_a = \xi_a, \qquad Sz_a = 0. \quad (4.2.16)$$

The operator S is an even derivation on $\Omega^{2\Lambda}$, meaning that it obeys the usual Leibniz rule $S(FG) = S(F)G + FS(G)$ for any forms F, G. On the other hand, the operators L and $\bar{L}$ are odd derivations on $\Omega^{2\Lambda}$, also called *supersymmetries*. This means that if F is an even or odd form, then $L(FG) = (LF)G \pm F(TG)$, with '$+$' for F even and '$-$' for F odd. We remark that L and $\bar{L}$ can be regarded as analogues of the infinitesimal Lorentz boost symmetries of $\mathbb{H}^n$, while S is an infinitesimal symplectic symmetry. In particular, the inner product (4.2.3) is invariant with respect to these symmetries, in the sense that

$$L(u_a \cdot u_b) = \bar{L}(u_a \cdot u_b) = S(u_a \cdot u_b) = 0. \quad (4.2.17)$$

For L, this follows from $L(u_a \cdot u_b) = L(-\xi_a\eta_b - \xi_b\eta_a - z_az_b) = -z_a\eta_b - z_b\eta_a + \eta_az_b + \eta_bz_a = 0$ since the z_i are even. Analogous computations apply to $\bar{L}$ and S.

A complete description of the infinitesimal symmetries of $\mathbb{H}^{0|2}$ is given by the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, which is spanned by the three operators described above, together with a further two symplectic symmetries; see [25, Section 7] for details.

Lemma 4.2.3. *For any $a \in \Lambda$, the operators L_a, $\bar{L}_a$ and S are symmetries of the non-interacting expectation $[\cdot]_0$ in the sense that, for any form F,*

$$[L_aF]_0 = [\bar{L}_aF]_0 = [S_aF]_0 = 0. \quad (4.2.18)$$

Moreover, for any $\boldsymbol{\beta} = (\beta_{ij})$ and $\boldsymbol{h} = \mathbf{0}$, also $L = \sum_{i\in\Lambda} L_i$ and $\bar{L} = \sum_{i\in\Lambda} \bar{L}_i$ are symmetries of the interacting expectation $[\cdot]_\beta$:

$$[LF]_\beta = [\bar{L}F]_\beta = 0, \quad (4.2.19)$$

and similarly $S = \sum_{i\in\Lambda} S_i$ is a symmetry of $[\cdot]_{\beta,h}$ for any $\boldsymbol{\beta}$ and $\boldsymbol{h}$.

Proof. First assume that $\boldsymbol{\beta} = \mathbf{0}$. Then by (4.2.13),

$$[L_aF]_0 = \int \prod_i \partial_{\eta_i}\partial_{\xi_i}\frac{1}{z_i}(L_aF) = \int \left(\prod_{i\neq a} \partial_{\eta_i}\partial_{\xi_i}\frac{1}{z_i}\right)\partial_{\eta_a}\partial_{\xi_a}\partial_{\xi_a}F = 0 \quad (4.2.20)$$

since $(\partial_{\xi_a})^2$ acts as 0 since any form can have at most one factor of ξ_a. The same argument applies to $\bar{T}$, and a similar argument applies to S.

We now show that this implies L and $\bar{L}$ are also symmetries of $[\cdot]_\beta$. Indeed, for any form F that is even (respectively odd), the fact that L is an odd derivation and the fact that $[\cdot]_0$ is invariant implies the integration by parts formula

$$[LF]_\beta = \pm[F(LH_\beta)]_\beta, \qquad H_\beta = H_{\beta,0} = \frac{1}{4}\sum_{i,j\in\Lambda} \beta_{ij}(u_i - u_j)^2. \quad (4.2.21)$$

For any $\boldsymbol{\beta}$ the right-hand side vanishes since $LH_\beta = 0$ by (4.2.17). A similar argument applies for $\bar{L}$. Since every form F can be written as a sum of an even and an odd form, (4.2.19) follows.

The argument for S being a symmetry of $[\cdot]_{\beta,h}$ is similar. □

To illustrate the use of these operators, we give a proof of the identities on the right-hand side of (4.2.11) and a proof of (4.2.12). Define

$$\lambda_{ab} \equiv z_b \xi_a, \qquad \bar{\lambda}_{ab} \equiv z_b \eta_a, \tag{4.2.22}$$

and note $L\lambda_{ab} = \xi_a \eta_b + z_a z_b$ and $\bar{L}\bar{\lambda}_{ab} = \xi_b \eta_a + z_a z_b$. Hence

$$\langle u_a \cdot u_b \rangle_\beta = \langle z_a z_b - L\lambda_{ab} - \bar{L}\bar{\lambda}_{ab} \rangle_\beta = \langle z_a z_b \rangle_\beta, \tag{4.2.23}$$

where the final equality is by linearity and Lemma 4.2.3. In particular, $\langle z_a^2 \rangle_\beta = -1$. Reasoning similarly, we obtain

$$\langle z_a \rangle_\beta = \langle L\xi_a \rangle_\beta = 0, \tag{4.2.24}$$

$$\langle z_a z_b \rangle_\beta = \langle L\lambda_{ab} \rangle_\beta - \langle \xi_a \eta_b \rangle_\beta = -\langle \xi_a \eta_b \rangle_\beta, \tag{4.2.25}$$

which proves (4.2.12), and implies $\langle \xi_a \eta_a \rangle_\beta = 1$. Since $z_a z_b = (1 - \xi_a \eta_a)(1 - \xi_b \eta_b) = 1 - \xi_a \eta_a - \xi_b \eta_b + \xi_a \eta_a \xi_b \eta_b$ this also gives

$$-\langle z_a z_b \rangle_\beta = 1 - \langle \xi_a \eta_a \xi_b \eta_b \rangle_\beta. \tag{4.2.26}$$

Finally, we note that the symplectic symmetry and $S(\xi_a \xi_b) = \xi_a \eta_b - \xi_b \eta_a$ imply

$$\langle \xi_a \eta_b \rangle_{\beta,h} = \langle \xi_b \eta_a \rangle_{\beta,h}. \tag{4.2.27}$$

Proofs of Theorem 4.2.1 and Corollary 4.2.2

Our first lemma relies on the identities of the previous section.

Lemma 4.2.4. *For any forest F,*

$$\left[\prod_{ij \in F} (u_i \cdot u_j + 1) \right]_0 = 1. \tag{4.2.28}$$

Proof. By factorization for fermionic integrals, it suffices to prove (4.2.28) when F is in fact a tree, which we call T. We recall the definition of the non-interacting expectation of a form G,

$$[G]_0 = \prod_i \partial_{\eta_i} \partial_{\xi_i} \frac{1}{z_i} G = \prod_i \partial_{\eta_i} \partial_{\xi_i} (1 + \xi_i \eta_i) G. \tag{4.2.29}$$

Hence, if T contains no edges then we have $[1]_0 = 1$. We complete the proof by induction, with the inductive assumption that the claim holds for all trees on k or fewer vertices. To advance the induction, let T be a tree on $k+1 \geqslant 2$ vertices and choose a leaf edge $\{a, b\}$ of T. We will advance the induction by considering the sum of the integrals that result from expanding $(u_a \cdot u_b + 1)$ in (4.2.28).

Note that by Lemma 4.2.3, if G_1 is even (resp. odd), then

$$[(LG_1)G]_0 = \mp [G_1(LG)]_0 \tag{4.2.30}$$

and similarly for $\bar{L}$. Recalling the definition (4.2.22) of λ_{ab} and $\bar{\lambda}_{ab}$, and furthermore suppose that $LG = \bar{L}G = 0$, then

$$\begin{aligned}[(u_a \cdot u_b)G]_0 = [(z_a z_b - L\lambda_{ab} - \bar{L}\bar{\lambda}_{ab})G]_0 = [z_a z_b G]_0 &= \frac{1}{2}[((L\xi_a) z_b + (\bar{L}\eta_a) z_b)G]_0 \\ &= \frac{1}{2}[(-\xi_a \eta_b + \eta_a \xi_b)G]_0, \end{aligned} \tag{4.2.31}$$

where we have used the assumption that $LG = \bar{L}G = 0$ in the second and final equalities. Applying this identity with $G = \prod_{ij\in T\setminus\{a,b\}}(u_i \cdot u_j + 1)$, the right-hand side is 0 since the product does not contain the missing generator at a to give a non-vanishing expectation. The inductive assumption and factorization for fermionic integrals implies $[G]_0 = 1$, and thus

$$[\prod_{ij\in T}(u_i \cdot u_j + 1)]_0 = [(u_a \cdot u_b + 1)G]_0 = [G]_0 = 1, \tag{4.2.32}$$

advancing the induction. $\square$

Lemma 4.2.5. *For any $i, j \in \Lambda$ we have $(u_i \cdot u_j + 1)^2 = 0$, and for any graph C that contains a cycle,*

$$\prod_{ij\in C}(u_i \cdot u_j + 1) = 0. \tag{4.2.33}$$

Proof. It suffices to consider when C is a cycle or doubled edge. Orienting C, the oriented edges of C are $(1,2), \dots, (k-1,k), (k,1)$ for some $k \geqslant 2$. Then, with the convention $k+1 = 1$,

$$\begin{aligned}\prod_{i=1}^{k}(u_i \cdot u_{i+1} + 1) &= \prod_{i=1}^{k}(-\xi_i\eta_{i+1} + \eta_i\xi_{i+1} + \xi_i\eta_i + \xi_{i+1}\eta_{i+1} - \xi_i\eta_i\xi_{i+1}\eta_{i+1}) \\ &= \prod_{i=1}^{k}(-\xi_i\eta_{i+1} + \eta_i\xi_{i+1} + \xi_i\eta_i + \xi_{i+1}\eta_{i+1}),\end{aligned} \tag{4.2.34}$$

the second equality by nilpotency of the generators and $k \geqslant 2$. To complete the proof of the claim we consider which terms are non-zero in the expansion of this product. First consider the term that arises when choosing $\xi_1\eta_1$ in the first term in the product: then for the second term any choice other than $\xi_2\eta_2$ results in zero. Continuing in this manner, the only non-zero contribution is $\prod_{i=1}^{k}\xi_i\eta_i$. Similar arguments apply to the other three choices possible in the first product, leading to

$$\begin{aligned}\prod_{i=1}^{k}(-\xi_i\eta_{i+1} + \eta_i\xi_{i+1} + \xi_i\eta_i + \xi_{i+1}\eta_{i+1}) &= \prod_{i=1}^{k}\xi_i\eta_i + \prod_{i=1}^{k}\xi_{i+1}\eta_{i+1} + \prod_{i=1}^{k}(-\xi_i\eta_{i+1}) + \prod_{i=1}^{k}\eta_i\xi_{i+1} \\ &= (1 + (-1)^k + (-1)^{2k-1} + (-1)^{k-1})\prod_{i=1}^{k}\xi_i\eta_i\end{aligned} \tag{4.2.35}$$

which is zero for all k. The signs arise from re-ordering the generators. We have used that C is a cycle for the third and fourth terms. $\square$

Proof of Theorem 4.2.1 when $\boldsymbol{h} = \mathbf{0}$. By Lemma 4.2.5,

$$e^{\frac{1}{2}(\boldsymbol{u},\Delta_\beta \boldsymbol{u})} = \sum_{S}\prod_{ij\in S}\beta_{ij}(u_i \cdot u_j + 1) = \sum_{F}\prod_{ij\in F}\beta_{ij}(u_i \cdot u_j + 1), \tag{4.2.36}$$

where the sum runs over sets S of edges and that over F is over forests. By taking the unnormalised expectation $[\cdot]_0$ we conclude from Lemma 4.2.4 that

$$Z_{\beta,0} = [e^{\frac{1}{2}(\boldsymbol{u},\Delta_\beta \boldsymbol{u})}]_0 = \sum_{F}\prod_{ij\in F}\beta_{ij}. \tag{4.2.37}$$

$\square$

To establish the theorem for $\boldsymbol{h} \neq \mathbf{0}$ requires one further preliminary, which uses the idea of *pinning* the spin u_0 at a chosen vertex $0 \in \Lambda$. Informally, this means that u_0 always evaluates to $(\xi, \eta, z) = (0, 0, 1)$. Formally, this means the following. To compute the pinned expectation of a function F of the forms $(u_i \cdot u_j)_{i,j\in\Lambda}$, we replace Λ by $\Lambda_0 = \Lambda \setminus \{0\}$, set

$$h_j = \beta_{0j}, \tag{4.2.38}$$

in H_β, and replace all instances of $u_0 \cdot u_j$ by $-z_j$ in both F and e^{-H_β}. The pinned expectation of F is the hyperbolic fermionic integral (4.2.5) of this form with respect to the generators $(\xi_i, \eta_i)_{i\in\Lambda_0}$. We denote this expectation by

$$[\cdot]^0_\beta, \quad \langle\cdot\rangle^0_\beta. \tag{4.2.39}$$

This procedure gives a way to identify any function of the forms $(u_i \cdot u_j)_{i,j\in\Lambda}$ with a function of the forms $(u_i \cdot u_j)_{i,j\in\Lambda_0}$ and $(z_i)_{i\in\Lambda_0}$. To minimize the notation, we will implicitly identify $u_0 \cdot u_j$ with $-z_j$ when taking pinned expectations of functions F of the $(u_i \cdot u_j)$.

The following proposition relates the pinned and unpinned models.

Proposition 4.2.6. *For any polynomial F in $(u_i \cdot u_j)_{i,j\in\Lambda}$,*

$$[F]^0_\beta = [(1-z_0)F]_\beta, \qquad \langle F\rangle^0_\beta = \langle(1-z_0)F\rangle_\beta. \tag{4.2.40}$$

Proof. It suffices to prove the first equation of (4.2.40), as this implies $[1]^0_\beta = [1-z_0]_\beta = [1]_\beta$ since $[z_0]_\beta = 0$ by (4.2.24).

Since $1 - z_0 = \xi_0\eta_0$, for any form F that contains a factor of ξ_0 or η_0, we have $(1-z_0)F = 0$. Thus the expectation $[(1-z_0)F]_\beta$ amounts to the expectation with respect to $[\cdot]_0$ of Fe^{-H_β} with all terms containing factors ξ_0 and η_0 removed. The claim thus follows from by computing the right-hand side using the observations that (i) removing all terms with factors of ξ_0 and η_0 from $u_0 \cdot u_i$ yields $-z_i$, and (ii) $\partial_{\eta_0}\partial_{\xi_0}\xi_0\eta_0 z_0^{-1} = 1$. □

There is a correspondence between pinning and external fields. If one first chooses Λ and then pins at $0 \in \Lambda$, the result is that there is an external field h_j for all $j \in \Lambda \setminus 0$. One can also view this the other way around, by beginning with Λ and an external field h_j for all $j \in \Lambda$, and then realizing this as due to pinning at an 'external' vertex $\delta \notin \Lambda$. This idea shows that Theorem 4.2.1 with $\boldsymbol{h} \neq \boldsymbol{0}$ follows from the case $\boldsymbol{h} = \boldsymbol{0}$; for the reader who is not familiar with arguments of this type, we provide the details below.

Proof of Theorem 4.2.1 when $\boldsymbol{h} \neq \boldsymbol{0}$. The partition function of the arboreal gas with $\boldsymbol{h} \neq \boldsymbol{0}$ can be interpreted as that of the arboreal gas with $\boldsymbol{h} \equiv \boldsymbol{0}$ on a graph $\tilde{G}$ augmented by an additional vertex δ and with weights $\tilde{\beta}$ given by $\tilde{\beta}_{ij} = \beta_{ij}$ for all $i, j \in G$ and $\tilde{\beta}_{i\delta} = \tilde{\beta}_{\delta i} = h_i$. Each $F' \in \mathcal{F}(\tilde{G})$ is a union of $F \in \mathcal{F}(G)$ with a collection of edges $\{i_r\delta\}_{r\in R}$ for some $R \subset V(G)$. Since F' is a forest, $|T \cap R| \leqslant 1$ for each tree T in F. Moreover, for any $F \in \mathcal{F}(G)$ and any $R \subset V(G)$ satisfying $|V(T) \cap R| \leqslant 1$ for each T in F, $F \cup \{i_r\delta\}_{r\in R} \in \mathcal{F}(\tilde{G})$. Thus

$$Z^{\tilde{G}}_{\tilde{\beta},0} = \sum_{F\in\mathcal{F}(G_\delta)} \prod_{ij\in F'} \beta_{ij} = \sum_{F\in\mathcal{F}(G)} \prod_{ij\in F'} \beta_{ij} \prod_{T\in F} (1 + \sum_{i\in T} h_i) = Z^G_{\beta,h}. \tag{4.2.41}$$

To conclude, note that $[(1-z_\delta)F]_{\tilde{\beta}} = [F]_{\tilde{\beta}}$ for any function F with $TF = 0$; this follows from $[z_a F] = [(T\xi_a)F] = -[\xi_a(TF)] = 0$. The conclusion now follows from Proposition 4.2.6 (where δ takes the role of 0 in that proposition), which shows $[(1-z_\delta)F]_{\tilde{\beta}} = [F]_{\beta,h}$. □

Proof of Corollary 4.2.2. Since $\mathbb{P}_\beta[ab] = \beta_{ab}\frac{d}{d\beta_{ab}}\log Z$, we have

$$\mathbb{P}_\beta[ab] = -\frac{1}{2}\beta_{ab}\langle(u_a - u_b)^2\rangle_\beta, \tag{4.2.42}$$

and expanding the right-hand side yields (4.2.9). Alternatively, multiplying (4.2.36) by $\beta_{ij}(1+u_i \cdot u_j)$, using Lemma 4.2.5, and then applying Lemma 4.2.4 yields the result. Similar considerations yield (4.2.10), and also show that

$$\mathbb{P}_\beta[i \nleftrightarrow j] = \langle 1 + u_i \cdot u_j\rangle_\beta. \tag{4.2.43}$$

Therefore $\mathbb{P}_\beta[i \leftrightarrow j] = -\langle u_i \cdot u_j\rangle_\beta$. Together with the identities (4.2.23)–(4.2.26), this proves (4.2.11). We already established (4.2.12) in Section 4.2. □

$\mathbb{H}^{2|4}$ model and dimensional reduction

In this section we define the $\mathbb{H}^{2|4}$ model, and show that for a class of 'supersymmetric observables' expectations with respect to the $\mathbb{H}^{2|4}$ model can be reduced to expectations with respect to the $\mathbb{H}^{0|2}$ model. To study the arboreal gas we will use this reduction in reverse: first we express arboreal gas quantities as $\mathbb{H}^{0|2}$ expectations, and in turn as $\mathbb{H}^{2|4}$ expectations. The utility of this rewriting will be explained in the next section, but in short, $\mathbb{H}^{2|4}$ expectations can be rewritten as ordinary integrals, and this carries analytic advantages.

The $\mathbb{H}^{2|4}$ model is a special case of the following more general $\mathbb{H}^{n|2m}$ model. These models originate with Zirnbauer's $\mathbb{H}^{2|2}$ model [40, 106], but makes sense for all $n, m \in \mathbb{N}$. For fixed n and m with $n + m > 0$, the $\mathbb{H}^{n|2m}$ model is defined as follows.

Let $\phi^1, \dots, \phi^n$ be n real variables, and let $\xi^1, \eta^1, \dots, \xi^m, \eta^m$ be $2m$ generators of a Grassmann algebra (i.e., they anticommute pairwise and are nilpotent of order 2). Note that we are using superscripts to distinguish variables. *Forms*, sometimes called *superfunctions*, are elements of $\Omega^{2m}(\mathbb{R}^n)$, where $\Omega^{2m}(\mathbb{R}^n)$ is the Grassmann algebra generated by $(\xi^k, \eta^k)_{k=1}^m$ over $C^\infty(\mathbb{R}^n)$. See [8, Appendix A] for details. We define a distinguished even element z of $\Omega^{2m}(\mathbb{R}^n)$ by

$$z \equiv \sqrt{1 + \sum_{\ell=1}^{n} (\phi^\ell)^2 + \sum_{\ell=1}^{m} (-2\xi^\ell \eta^\ell)} \tag{4.2.44}$$

and let $u = (\phi, \xi, \eta, z)$. Given a finite set Λ, we write $\boldsymbol{u} = (u_i)_{i\in\Lambda}$, where $u_i = (\phi_i, \xi_i, \eta_i, z_i)$ with $\phi_i \in \mathbb{R}^n$ and $\xi_i = (\xi_i^1, \dots, \xi_i^m)$ and $\eta_i = (\eta_i^1, \dots, \eta_i^m)$, each ξ_i^j (resp. η_i^j) a generator of $\Omega^{2m\Lambda}(\mathbb{R}^{n\Lambda})$. We define the 'inner product'

$$u_i \cdot u_j \equiv \sum_{\ell=1}^{n} \phi_i^\ell \phi_j^\ell + \sum_{\ell=1}^{m} (\eta_i^\ell \xi_j^\ell - \xi_i^\ell \eta_j^\ell) - z_i z_j. \tag{4.2.45}$$

Note that these definitions imply $u_i \cdot u_i = -1$. If $m = 0$, the constraint $u_i \cdot u_i = -1$ defines the hyperboloid model for hyperbolic space $\mathbb{H}^n$, as in this case $u_i \cdot u_j$ reduces to the Minkowski inner product on $\mathbb{R}^{n+1}$. For this reason we write $u_i \in \mathbb{H}^{n|2m}$ and $\boldsymbol{u} \in (\mathbb{H}^{n|2m})^\Lambda$ and think of $\mathbb{H}^{n|2m}$ as a hyperbolic supermanifold. As we do not need to enter into the details of this mathematical object, we shall not discuss it further (see [106] for further details). We remark, however, that the expression $\sum_{\ell=1}^m (-\xi_i^\ell \eta_j^\ell + \eta_i^\ell \xi_j^\ell)$ is the natural fermionic analogue of the Euclidean inner product $\sum_{\ell=1}^n \phi_i^\ell \phi_j^\ell$ and motivates the supermanifold terminology.

The general class of models of interest are defined analogously to the $\mathbb{H}^{0|2}$ model by the action

$$H_{\beta,h}(\boldsymbol{u}) \equiv \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) + (\boldsymbol{h}, \boldsymbol{z} - 1), \tag{4.2.46}$$

where we now require $\boldsymbol{\beta} \geqslant 0$ and $\boldsymbol{h} \geqslant 0$, i.e., $\boldsymbol{\beta} = (\beta_{ij})_{i,j\in\Lambda}$ and $\boldsymbol{h} = (h_i)_{i\in\Lambda}$ satisfy $\beta_{ij} \geqslant 0$ and $h_i \geqslant 0$ for all $i, j \in \Lambda$. We have again used the notation $(a, b) = \sum_{i\in\Lambda} a_i \cdot b_i$ but where $\cdot$ now refers to (4.2.45). For a form $F \in \Omega^{2m\Lambda}(\mathbb{H}^n)$, the corresponding unnormalised expectation is

$$[F]^{\mathbb{H}^{n|2m}} \equiv \int_{(\mathbb{H}^{n|2m})^\Lambda} F e^{-H_{\beta,h}} \tag{4.2.47}$$

where the superintegral of a form G is

$$\int_{(\mathbb{H}^{n|2m})^\Lambda} G \equiv \int_{\mathbb{R}^{n\Lambda}} \prod_{i\in\Lambda} \frac{d\phi_i^1 \dots d\phi_i^n}{(2\pi)^{n/2}} \partial_{\eta_i^1} \partial_{\xi_i^1} \cdots \partial_{\eta_i^m} \partial_{\xi_i^m} \left(\prod_{i\in\Lambda} \frac{1}{z_i} \right) G, \tag{4.2.48}$$

where the z_i are defined by (4.2.44).

Henceforth we will only consider the $\mathbb{H}^{0|2}$ and $\mathbb{H}^{2|4}$ models, and hence we will write $x_i = \phi_i^1$ and $y_i = \phi_i^2$ for notational convenience. We will also assume $\boldsymbol{\beta} \geqslant 0$ and $\boldsymbol{h} \geqslant 0$ to ensure both models are well-defined.

Dimensional reduction. The following proposition shows that, due to an internal supersymmetry, all observables F that are functions of $u_i \cdot u_j$ have the same expectations under the $\mathbb{H}^{0|2}$ and the $\mathbb{H}^{2|4}$ expectation. Here $u_i \cdot u_j$ is defined as in (4.2.3) for $\mathbb{H}^{0|2}$, respectively as in (4.2.45) for $\mathbb{H}^{2|4}$. In this section and henceforth we work under the convention that $z_i = u_\delta \cdot u_i$ with $u_\delta = (0, \dots, 0, 1)$, and that $(u_i \cdot u_j)_{i,j}$ refers to the collection of forms indexed by $i, j \in \tilde{\Lambda} \equiv \Lambda \cup \{\delta\}$. In other words, functions of $(u_i \cdot u_j)_{i,j}$ are also permitted to depend on $(z_i)_i$.

Proposition 4.2.7. *For any $F\colon \mathbb{R}^{\tilde{\Lambda}\times\tilde{\Lambda}} \to \mathbb{R}$ smooth with enough decay that the integrals exist,*

$$[F((u_i \cdot u_j)_{i,j})]_{\beta,h}^{\mathbb{H}^{0|2}} = [F((u_i \cdot u_j)_{i,j})]_{\beta,h}^{\mathbb{H}^{2|4}}. \tag{4.2.49}$$

In view of this proposition we will subsequently drop the superscript $\mathbb{H}^{n|2m}$ for expectations of observables F that are functions of $(u_i \cdot u_j)_{i,j}$. That is, we will simply write $[F]_{\beta,h}$ for

$$[F]_{\beta,h} = [F]_{\beta,h}^{\mathbb{H}^{0|2}} = [F]_{\beta,h}^{\mathbb{H}^{2|4}}. \tag{4.2.50}$$

We will similarly write $\langle F\rangle_{\beta,h} = \langle F\rangle_{\beta,h}^{\mathbb{H}^{0|2}} = \langle F\rangle_{\beta,h}^{\mathbb{H}^{2|4}}$ whenever $[1]_{\beta,h}^{\mathbb{H}^{2|4}}$ positive and finite.

The proof of Proposition 4.2.7 uses the following fundamental *localisation theorem*. To state the theorem, consider forms in $\Omega^{2N}(\mathbb{R}^{2N})$ and denote the even generators of this algebra by (x_i, y_i) and the odd generators by (ξ_i, η_i). Then we define

$$Q \equiv \sum_{i=1}^{N} Q_i\,, \qquad Q_i \equiv \xi_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial \eta_i} + y_i \frac{\partial}{\partial \xi_i}. \tag{4.2.51}$$

Theorem 4.2.8. *Suppose $F \in \Omega^{2N}(\mathbb{R}^{2N})$ is integrable and satisfies $QF = 0$. Then*

$$\int_{\mathbb{R}^{2N}} \frac{dx\, dy\, \partial_\eta\, \partial_\xi}{2\pi} F = F_0(0) \tag{4.2.52}$$

where the right-hand side is the degree-0 part of F evaluated at 0.

A proof of this theorem can be found, for example, in [8, Appendix B].

Proof of Proposition 4.2.7. To distinguish $\mathbb{H}^{0|2}$ and $\mathbb{H}^{2|4}$ variables, we write the latter as u_i', i.e.,

$$u_i \cdot u_j = -\xi_i^1 \eta_j^1 - \xi_j^1 \eta_i^1 - z_i z_j \tag{4.2.53}$$

$$u_i' \cdot u_j' = x_i x_j + y_i y_j - \xi_i^1 \eta_j^1 - \xi_j^1 \eta_i^1 - \xi_i^2 \eta_j^2 - \xi_j^2 \eta_i^2 - z_i' z_j'. \tag{4.2.54}$$

We begin by considering the case $N = 1$, i.e., a graph with a single vertex. Since $e^{-H_{\beta,h}(\boldsymbol{u})}$ is a function of $(u_i \cdot u_j)_{i,j}$, we will absorb the factor of $e^{-H_{\beta,h}(\boldsymbol{u})}$ into the observable F to ease the notation. The $\mathbb{H}^{2|4}$ integral can be written as

$$\int_{\mathbb{H}^{2|4}} F = \int_{\mathbb{R}^2} \frac{dx\, dy}{2\pi}\, \partial_{\eta^1} \partial_{\xi^1}\, \partial_{\eta^2} \partial_{\xi^2} \frac{1}{z'} F = \partial_{\eta^1} \partial_{\xi^1} \int_{\mathbb{R}^2} \frac{dx\, dy}{2\pi}\, \partial_{\eta^2} \partial_{\xi^2} \frac{1}{z'} F \tag{4.2.55}$$

where

$$z' = \sqrt{1 + x^2 + y^2 - 2\xi^1\eta^1 - 2\xi^2\eta^2} \tag{4.2.56}$$

and $\int_{\mathbb{R}^2} dx\, dy\, \partial_{\eta^2} \partial_{\xi^2} \frac{1}{z'} F$ is the form in (ξ^1, η^1) obtained by integrating the coefficient functions term-by-term. Applying the localisation theorem (Theorem 4.2.8) to the variables (x, y, ξ^2, η^2) gives, after noting z' localises to $z = \sqrt{1 - 2\xi^1\eta^1}$,

$$\int_{\mathbb{R}^2} \frac{dx\, dy}{2\pi}\, \partial_{\eta^2} \partial_{\xi^2} \frac{1}{z'} F((u_i' \cdot u_j')) = \frac{1}{z} F((u_i \cdot u_j)_{i,j}). \tag{4.2.57}$$

Therefore

$$\int_{\mathbb{H}^{2|4}} F((u_i' \cdot u_j')_{i,j}) = \int_{\mathbb{H}^{0|2}} F((u_i \cdot u_j)_{i,j}) \tag{4.2.58}$$

which is the claim. The argument for the case of general N is exactly analogous. $\square$

Horospherical coordinates

Proposition 4.2.7 showed that 'supersymmetric observables' have the same expectations in the $\mathbb{H}^{0|2}$ and the $\mathbb{H}^{2|4}$ model. This is useful because the richer structure of the $\mathbb{H}^{2|4}$ model allows the introduction of *horospherical coordinates*, whose importance was recognised in [40, 99]. We will shortly define horospherical coordinates, but before doing this we state the result that we will deduce using them.

For the statement of the proposition, we require the following definitions. Let $-\Delta_{\beta(t),h(t)}$ be the matrix with (i,j)th element $\beta_{ij}e^{t_i+t_j}$ for $i \neq j$ and ith diagonal element $-\sum_{j\in\Lambda}\beta_{ij}e^{t_i+t_j} - h_ie^{t_i}$. Let

$$\tilde{H}_{\beta,h}(t,s) \equiv \sum_{ij}\beta_{ij}(\cosh(t_i-t_j)+\frac{1}{2}e^{t_i+t_j}(s_i-s_j)^2-1) + \sum_i h_i(\cosh(t_i)+\frac{1}{2}e^{t_i}s_i-1) - 2\log\det(-\Delta_{\beta(t),h(t)}) + 3\sum_i t_i \tag{4.2.59}$$

$$\tilde{H}_{\beta,h}(t) \equiv \sum_{ij}\beta_{ij}(\cosh(t_i-t_j)-1) + \sum_i h_i(\cosh(t_i)-1) - \frac{3}{2}\log\det(-\Delta_{\beta(t),h(t)}) + 3\sum_i t_i \tag{4.2.60}$$

where we abuse notation by using the symbol $\tilde{H}_{\beta,h}$ both for the function $\tilde{H}_{\beta,h}(t,s)$ and $\tilde{H}_{\beta,h}(t)$. Below we will assume that $\boldsymbol{\beta}$ is irreducible, by which we mean that $\boldsymbol{\beta}$ induces a connected graph.

Proposition 4.2.9. *Assume $\boldsymbol{\beta} \geqslant 0$ and $\boldsymbol{h} \geqslant 0$ with $\boldsymbol{\beta}$ irreducible and $h_i > 0$ for at least one $i \in \Lambda$. For all smooth functions $F\colon \mathbb{R}^{2\Lambda} \to \mathbb{R}$, respectively $F\colon \mathbb{R}^{\Lambda} \to \mathbb{R}$, such that the integrals on the left- and right-hand sides converge absolutely,*

$$[F((x_i+z_i)_i,(y_i)_i)]^{\mathbb{H}^{2|4}}_{\beta,h} = \int_{\mathbb{R}^{2\Lambda}} F((e^{t_i})_i,(e^{t_i}s_i)_i)e^{-\tilde{H}_{\beta,h}(t,s)}\prod_i \frac{dt_i\,ds_i}{2\pi} \tag{4.2.61}$$

$$[F((x_i+z_i)_i)]^{\mathbb{H}^{2|4}}_{\beta,h} = \int_{\mathbb{R}^{\Lambda}} F((e^{t_i})_i)e^{-\tilde{H}_{\beta,h}(t)}\prod_i \frac{dt_i}{\sqrt{2\pi}}. \tag{4.2.62}$$

In particular, the normalising constant $[1]^{\mathbb{H}^{2|4}}_{\beta,h}$ is the partition function $Z_{\beta,h}$ of the arboreal gas.

Abusing notation further, we will denote either of the expectations on the right-hand sides of (4.2.61) and (4.2.62) by $[\cdot]_{\beta,h}$, and we will write $\langle\cdot\rangle_{\beta,h}$ for the normalised versions. Before giving the proof of the proposition, which is essentially standard, we collect some resulting identities that will be used later.

Corollary 4.2.10. *For all $\boldsymbol{\beta}$ and $\boldsymbol{h}$ as in Proposition 4.2.9,*

$$\langle e^{t_i}\rangle_{\beta,h} = \langle e^{2t_i}\rangle_{\beta,h} = \langle z_i\rangle_{\beta,h}, \quad \langle e^{3t_i}\rangle_{\beta,h} = 1 \tag{4.2.63}$$

and

$$\langle s_is_je^{t_i+t_j}\rangle_{\beta,h} = \langle \xi_i\eta_j\rangle_{\beta,h}, \tag{4.2.64}$$

where the left-hand sides are evaluated as on the right-hand side of (4.2.61)*, and the right-hand sides are given by the $\mathbb{H}^{0|2}$ expectation* (4.2.7).

Proof. To lighten notation, we write $\langle\cdot\rangle \equiv \langle\cdot\rangle_{\beta,h}$. For the $\mathbb{H}^{2|4}$ expectation (4.2.47), we have $\langle x_i^q z_i^p\rangle = 0$ whenever $q > 0$ is an odd integer by the symmetry $x \mapsto -x$ (recall that $x = \phi^1$). Also note that

$$\langle x_i^2\rangle = \langle y_i^2\rangle = \langle \xi_i^1\eta_i^1\rangle = \langle \xi_i^2\eta_i^2\rangle, \tag{4.2.65}$$

where we emphasize that the superscript of x_i^2 denotes the square and the superscript of ξ_i^2 denotes the second component. These identies follow from the $x \leftrightarrow y$ and $\xi_i^1\eta_i^1 \leftrightarrow \xi_i^2\eta_i^2$ symmetries of the $\mathbb{H}^{2|4}$ model and $\langle x_i^2 + y_i^2 - 2\xi_i^1\eta_i^1\rangle = 0$ by supersymmetric localisation, i.e., Theorem 4.2.8. Since

$$\langle z_i^2\rangle = 1 - 2\langle \xi_i\eta_i\rangle \qquad \text{in } \mathbb{H}^{0|2}, \tag{4.2.66}$$

$$\langle z_i^2\rangle = 1 + \langle x_i^2 + y_i^2 - 2\xi_i^1\eta_i^1 - 2\xi_i^2\xi_i^2\rangle = 1 - 2\langle \xi_i^2\eta_i^2\rangle \qquad \text{in } \mathbb{H}^{2|4}, \tag{4.2.67}$$

and since the left-hand sides are equal by Proposition 4.2.7, we further see that the $\mathbb{H}^{2|4}$ expectation (4.2.65) equals the $\mathbb{H}^{0|2}$ expectation $\langle \xi_i\eta_i\rangle$. Similarly, $\langle x_i^2 z_i\rangle = \langle y_i^2 z_i\rangle = \langle \xi_i^1\eta_i^1 z_i\rangle = \langle \xi_i^2\eta_i^2 z_i\rangle$. By using the preceding equalities and by expanding $\langle(-1 + z_i^2)z_i\rangle = \langle(u_i \cdot u_i + z_i^2)z_i\rangle$ in both $\mathbb{H}^{0|2}$ and $\mathbb{H}^{2|4}$, one obtains

$$-2\langle x_i^2 z_i\rangle = -\langle z_i\rangle + \langle z_i^3\rangle = -2\langle \xi_i\eta_i\rangle, \tag{4.2.68}$$

where the first expectation is with respect to $\mathbb{H}^{2|4}$ and the others are with respect to $\mathbb{H}^{0|2}$. Using these identities and (4.2.61), we then find

$$\langle e^{t_i}\rangle = \langle x_i + z_i\rangle = \langle z_i\rangle \tag{4.2.69}$$

$$\langle e^{2t_i}\rangle = \langle (x_i + z_i)^2\rangle = \langle x_i^2\rangle + \langle z_i^2\rangle = \langle \xi_i\eta_i\rangle + \langle 1 - 2\xi_i\eta_i\rangle = \langle 1 - \xi_i\eta_i\rangle = \langle z_i\rangle \tag{4.2.70}$$

$$\langle e^{3t_i}\rangle = \langle (x_i + z_i)^3\rangle = \langle 3x_i^2 z_i\rangle + \langle z_i^3\rangle = 3\langle \xi_i\eta_i\rangle + \langle 1 - 3\xi_i\eta_i\rangle = 1. \tag{4.2.71}$$

The identity (4.2.64) follows analogously:

$$\langle s_i s_j e^{t_i + t_j}\rangle = \langle y_i y_j\rangle = \frac{1}{2}\langle \xi_i\eta_j + \xi_j\eta_i\rangle = \langle \xi_i\eta_j\rangle \tag{4.2.72}$$

where we used the generalisation of (4.2.65) for the mixed expectation $\langle x_i x_j\rangle$ and that $\langle \xi_i\eta_j\rangle = \langle \xi_j\eta_i\rangle$, see (4.2.27). □

To describe the proof of Proposition 4.2.9 we now define horospherical coordinates for $\mathbb{H}^{2|4}$. These are a change of generators from the variables $(x, y, \xi^\gamma, \eta^\gamma)$ with $\gamma = 1, 2$ to $(t, s, \psi^\gamma, \bar\psi^\gamma)$, where

$$x = \sinh t - e^t(\frac{1}{2}s^2 + \bar\psi^1\psi^1 + \bar\psi^2\psi^2), \quad y = e^t s, \quad \eta^i = e^t\bar\psi^i, \quad \xi^i = e^t\psi^i. \tag{4.2.73}$$

We note that $\bar\psi_i$ is simply notation to indicate a generator distinct from ψ_i, i.e., the bar does not denote complex conjugation, which would not make sense. In these coordinates the action is quadratic in $s, \bar\psi^1, \psi^1, \bar\psi^2, \psi^2$. This leads to a proof of Proposition 4.2.9 by explicitly integrating out these variables when t is fixed via the following standard lemma, whose proof we omit.

Lemma 4.2.11. *For any $N \times N$ matrix A,*

$$\left(\prod_i \partial_{\eta_i}\partial_{\xi_i}\right) e^{(\xi, A\eta)} = \det A, \tag{4.2.74}$$

and, for a positive definite $N \times N$ matrix A,

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(s, As)} \frac{ds}{\sqrt{2\pi}} = (\det A)^{-1/2}. \tag{4.2.75}$$

Proof of Proposition 4.2.9. The first step is to compute the Berezinian for the horospherical change of coordinates. This can be done as in [9, Appendix A]. There is an e^t for the s-variables and an e^{-t} for each fermionic variable, leading to a Berezinian ze^{-3t}, i.e.,

$$[F]_{\beta,h}^{\mathbb{H}^{2|4}} = \int \left(\prod_i ds_i dt_i \partial_{\psi_i^1}\partial_{\bar\psi_i^1}\partial_{\psi_i^2}\partial_{\bar\psi_i^2}\right) F e^{-\tilde H_{\beta,h}(s,t,\psi,\bar\psi)} \prod_i \frac{e^{-3t_i}}{2\pi}. \tag{4.2.76}$$

where $\bar{H}_{\beta,h}(s,t,\psi,\bar{\psi})$ is $H_{\beta,h}$ expressed in horospherical coordinates.

The second step is to apply Lemma 4.2.11 repeatedly. To prove (4.2.62), we apply it twice, once for $(\bar{\psi}^1,\psi^1)$ and once for $(\bar{\psi}^2,\psi^2)$. The lemma applies since F does not depend on $\psi^1,\bar{\psi}^1,\psi^2,\bar{\psi}^2$ by assumption. To prove (4.2.62), we apply it three times, once for $(\bar{\psi}^1,\psi^1)$, once for $(\bar{\psi}^2,\psi^2)$, and once for s. Each integral contributes a power of $\det(-\Delta_{\beta(t),h(t)})$, namely $-1/2$ for the Gaussian and $+1$ for each fermionic Gaussian. This explains the coefficient 2 in (4.2.61) and the coefficient $3/2 = 2-1/2$ in (4.2.62).

The final claim follows as the conditions that $\boldsymbol{\beta}$ induces a connected graph and some $h_i > 0$ implies $[1]^{\mathbb{H}^{2|4}}_{\beta,h}$ is finite. The claim thus follows from Theorems 4.2.7 and 4.2.1. □

Pinned measure for the $\mathbb{H}^{2|4}$ model

This section introduces a pinned version of the $\mathbb{H}^{2|4}$ model and relates it to the pinned $\mathbb{H}^{0|2}$ model that was introduced in Section 4.2. For the $\mathbb{H}^{2|4}$ pinning means u_0 always evaluates to $(x,y,\xi^1,\eta^1,\xi^2,\eta^2,z) = (0,0,0,0,0,0,1)$. As before, we implement this by replacing Λ by $\Lambda_0 = \Lambda\setminus\{0\}$ and setting

$$h_j = \beta_{0j}, \tag{4.2.77}$$

and replacing $u_0\cdot u_j$ by $-z_j$. We denote the corresponding expectations by

$$[\cdot]^0_\beta, \quad \langle\cdot\rangle^0_\beta. \tag{4.2.78}$$

We can relate the pinned and unpinned measures exactly as for the $\mathbb{H}^{0|2}$ model.

Proposition 4.2.12. *For any polynomial F in $(u_i\cdot u_j)_{i,j\in\Lambda}$,*

$$[F]^0_\beta = [(1-z_0)F]_\beta, \qquad \langle F\rangle^0_\beta = \langle(1-z_0)F\rangle_\beta. \tag{4.2.79}$$

Moreover, $[1]^0_\beta = [1]_\beta$ and hence for any pairs of vertices i_kj_k,

$$\langle\prod_k(u_{i_k}\cdot u_{j_k}+1)\rangle^0_\beta = \langle\prod_k(u_{i_k}\cdot u_{j_k}+1)\rangle_\beta. \tag{4.2.80}$$

Proof. The first equality in (4.2.79) follows by reducing the $\mathbb{H}^{2|4}$ expectation to a $\mathbb{H}^{0|2}$ expectation by Proposition 4.2.7 (recall the convention that $z_0 = u_\delta\cdot u_0$), then applying Proposition 4.2.6 for the $\mathbb{H}^{0|2}$ expectation, and finally applying Proposition 4.2.7 again (in reverse). The second equality in (4.2.79) then follows by normalising using that $[1]^0_\beta = [1-z_0]_\beta = [1]_\beta$ (as in Proposition 4.2.6). The equalities (4.2.80) follow from $[1]^0_\beta = [1]_\beta$ by differentiating with respect to the $\beta_{i_kj_k}$. □

The next corollary expresses the pinned model in horospherical coordinates. For $i,j\in\Lambda$, set

$$\beta_{ij}(t) \equiv \beta_{ij}e^{t_i+t_j}, \tag{4.2.81}$$

and let $\tilde{D}_\beta(t)$ be the determinant of $-\Delta_{\beta(t)}$ restricted to $\Lambda_0 = \Lambda\setminus\{0\}$, i.e., the determinant of submatrix of $-\Delta_{\beta(t)}$ indexed by Λ_0. When $\boldsymbol{\beta}$ induces a connected graph, this determinant is non-zero, and by the matrix-tree theorem it can be written as

$$\tilde{D}_\beta(t) = \sum_T\prod_{ij}\beta_{ij}e^{t_i+t_j} \tag{4.2.82}$$

where the sum is over all spanning trees on Λ. For $t\in\mathbb{R}^\Lambda$, then define

$$\tilde{H}^0_\beta(t) \equiv \frac{1}{2}\sum_{i,j}\beta_{ij}(\cosh(t_i-t_j)-1) - \frac{3}{2}\log\tilde{D}_\beta(t) - 3\sum_i t_i. \tag{4.2.83}$$

By combining Proposition 4.2.12 with Proposition 4.2.9, we have the following representation of the pinned measure in horospherical coordinates:

Corollary 4.2.13. *For any smooth function $F\colon \mathbb{R}^\Lambda \to \mathbb{R}$ with sufficient decay,*

$$[F((x+z)_i)]^0_\beta = \int F((e^{t_i})_i) e^{-\tilde{H}^0_\beta(t)}\, \delta_0(dt_0) \prod_{i\neq 0} \frac{dt_i}{\sqrt{2\pi}}. \tag{4.2.84}$$

Proof. We recall the definition of the left-hand side, i.e., that the expectation $[\cdot]^0_\beta$ is defined in (4.2.77)–(4.2.78) as the expectation on Λ_0 given by $[\cdot]^0_\beta = [\cdot]_{\tilde{\beta},\tilde{h}}$ with $\tilde{\beta}_{ij} = \beta_{ij}$ and $\tilde{h}_i = \beta_{0i}$ for $i,j \in \Lambda_0$. The equality now follows from (4.2.62), together with the observation that $\Delta_{\beta(t)}|_{\Lambda_0}$ is $\Delta_{\tilde{\beta}(t),\tilde{h}(t)}$ if $t_0 = 0$. □

In view of (4.2.84) and since $[1]^0_\beta = Z_\beta$ by Proposition 4.2.12, we again abuse notation somewhat and write the normalised expectation of a function of $t = (t_i)_{i\in\Lambda}$ as

$$\langle F\rangle^0_\beta = \frac{1}{Z_\beta} \int_{\mathbb{R}^\Lambda} F((t_i)_i) e^{-\tilde{H}^0_\beta(t)} \delta_0(dt_0) \prod_{i\neq 0} \frac{dt_i}{\sqrt{2\pi}}. \tag{4.2.85}$$

Corollary 4.2.14. *The connection probabilities can be written as in terms of the pinned $\mathbb{H}^{2|4}$ measure:*

$$\mathbb{P}_\beta[0 \leftrightarrow i] = \langle e^{t_i}\rangle^0_\beta. \tag{4.2.86}$$

Moreover, for any vertex i,

$$\langle e^{3t_i}\rangle^0_\beta = 1. \tag{4.2.87}$$

Proof. (4.2.86) follows by applying first (4.2.11), then (4.2.80), then using the fact that $u_0 \cdot u_i = -z_i$ under $\langle\cdot\rangle^0_\beta$, then using that $\langle x_i\rangle_\beta = 0$ by symmetry, and finally applying (4.2.84):

$$\mathbb{P}_\beta[0 \leftrightarrow i] = -\langle u_0 \cdot u_i\rangle_\beta = \langle z_i\rangle^0_\beta = \langle z_i + x_i\rangle^0_\beta = \langle e^{t_i}\rangle^0_\beta. \tag{4.2.88}$$

The argument that $\langle e^{3t_i}\rangle^0_\beta = 1$ is identical to (4.2.71) with $\langle\cdot\rangle_\beta$ replaced by $\langle\cdot\rangle^0_\beta$. □

4.3 Phase transition on the complete graph

The following theorem shows that on the complete graph the arboreal gas undergoes a transition very similar to the percolation transition, i.e., the Erdős–Rényi graph. As mentioned in the introduction, this result has been obtained previously [11, 65, 70]. We have included a proof only to illustrate the utility of the $\mathbb{H}^{0|2}$ representation. The study of spanning forests of the complete graph goes back to (at least) Rényi [87] who obtained a formula which can be seen to imply that their asymptotic number grows like $\sqrt{e}n^{n-2}$, see [80].

Throughout this section we consider $\mathbb{G} = K_N$, the complete graph on N vertices with vertex set $\{0, 1, 2, \dots, N-1\}$, and we choose $\beta_{ij} = \alpha/N$ with $\alpha > 0$ fixed for all edges ij. For notational simplicity we write Z_β and $\mathbb{P}_\beta$, i.e., we leave the dependence on N implicit.

Theorem 4.3.1. *In the high temperature phase $\alpha < 1$,*

$$Z_\beta \sim e^{(N+1)\alpha/2}\sqrt{1-\alpha}, \qquad \mathbb{P}_\beta[0 \leftrightarrow 1] \sim \left[\frac{\alpha}{1-\alpha}\right]\frac{1}{N}. \tag{4.3.1}$$

In the low temperature phase $\alpha > 1$,

$$Z_\beta \sim \frac{a^{N+3/2} e^{(a^2+N)/(2a)}}{(a-1)^{5/2} N}, \qquad \mathbb{P}_\beta[0 \leftrightarrow 1] \sim \left[\frac{\alpha-1}{\alpha}\right]^2. \tag{4.3.2}$$

In the critical case $\alpha = 1$,

$$Z_\beta \sim \frac{3^{1/6}\Gamma(\frac{2}{3}) e^{(N+1)/2}}{N^{1/6}\sqrt{2\pi}}, \qquad \mathbb{P}_\beta[0 \leftrightarrow 1] \sim \left[\frac{3^{2/3}\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})}\right]\frac{1}{N^{2/3}}. \tag{4.3.3}$$

Integral representation

The first step in the proof of the theorem is the following integral representation that follows from a transformation of the fermionic field theory representation from Section 4.2. We introduce the effective potential

$$V(\tilde{z}) \equiv -P(i\alpha\tilde{z}), \qquad P(w) \equiv \frac{w^2}{2\alpha} + w + \log(1-w) \tag{4.3.4}$$

and set

$$F(w) \equiv 1 - \frac{\alpha}{1-w}, \qquad F_{01}(w) \equiv -\left(\frac{w}{1-w}\right)^2 \left(F(w) - \frac{2\alpha}{N(-w)(1-w)}\right). \tag{4.3.5}$$

Proposition 4.3.2. *For all $\alpha > 0$ and all positive integers N,*

$$Z_\beta = e^{(N+1)\alpha/2}\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})} F(i\alpha\tilde{z}) \tag{4.3.6}$$

$$Z_\beta[0 \leftrightarrow 1] = e^{(N+1)\alpha/2}\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})} F_{01}(i\alpha\tilde{z}), \tag{4.3.7}$$

where $Z_\beta[0 \leftrightarrow 1] \equiv \mathbb{P}_\beta[0 \leftrightarrow 1] Z_\beta$.

Proof. We start from the representations of the partition functions in terms of the $\mathbb{H}^{0|2}$ model, i.e., Theorem 4.2.1 and Corollary 4.2.2, which we simplify using the assumption that the graph is the complete graph. Let $(\Delta_\beta f)_i = \frac{\alpha}{N}\sum_{j=0}^{N-1}(f_i - f_j)$ be the mean-field Laplacian and $\boldsymbol{h} = (h_i)_i$. Then

$$\begin{aligned} \frac{1}{2}(\boldsymbol{u}, -\Delta_\beta \boldsymbol{u}) &= -(\boldsymbol{\xi}, -\Delta_\beta \boldsymbol{\eta}) - \frac{1}{2}(\boldsymbol{z}, -\Delta_\beta \boldsymbol{z}) \\ &= -(\boldsymbol{\xi}, -\Delta_\beta \boldsymbol{\eta}) + \alpha \sum_{i=0}^{N-1} \xi_i\eta_i + \frac{\alpha}{2N}\left(\sum_{i=0}^{N-1} z_i\right)^2 - \frac{\alpha N}{2} \end{aligned} \tag{4.3.8}$$

$$(\boldsymbol{h}, \boldsymbol{z} - \mathbf{1}) = -\sum_{i=0}^{N-1} h_i \xi_i \eta_i. \tag{4.3.9}$$

In the sequel we will omit the range of sums and products when there is no risk of ambiguity.

To decouple the two terms that are not diagonal sums we use the following Hubbard–Stratonovich-type transforms in terms of auxiliary variables $\tilde{\xi}, \tilde{\eta}$ (fermionic) and $\tilde{z}$ (real). Let $\mathbf{1}$ be the vector such that $\mathbf{1}_i = 1$ for all $0 \leqslant i \leqslant N-1$.

$$e^{+(\boldsymbol{\xi}, -\Delta_\beta \boldsymbol{\eta})} = \frac{1}{N\alpha}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}} e^{\alpha(\tilde{\xi}\mathbf{1} - \boldsymbol{\xi}, \tilde{\eta}\mathbf{1} - \boldsymbol{\eta})} = \frac{1}{N\alpha}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\left[e^{N\alpha\tilde{\xi}\tilde{\eta}} \prod_i e^{\alpha(\xi_i\eta_i - \tilde{\xi}\eta_i - \xi_i\tilde{\eta})}\right] \tag{4.3.10}$$

$$e^{-\frac{\alpha}{2N}(\sum_i z_i)^2} = \sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} d\tilde{z}\, e^{-\frac{1}{2}N\alpha\tilde{z}^2} e^{i\alpha\tilde{z}\sum_i z_i}. \tag{4.3.11}$$

The second formula is the formula for the Fourier transform of a Gaussian measure. The first formula can be seen by making use of the following identity. Write $Af \equiv \frac{1}{N}\sum_i f_i$ for the average of f, so that

$$\begin{aligned} \alpha(\tilde{\xi}\mathbf{1} - \boldsymbol{\xi}, \tilde{\eta}\mathbf{1} - \boldsymbol{\eta}) &= \alpha([\tilde{\xi} - A\boldsymbol{\xi}]\mathbf{1} - [\boldsymbol{\xi} - (A\boldsymbol{\xi})\mathbf{1}], [\tilde{\eta} - A\boldsymbol{\eta}]\mathbf{1} - [\boldsymbol{\eta} - (A\boldsymbol{\eta})\mathbf{1}]) \\ &= \alpha([\tilde{\xi} - A\boldsymbol{\xi}]\mathbf{1}, [\tilde{\eta} - A\boldsymbol{\eta}]\mathbf{1}) + \alpha(\boldsymbol{\xi} - (A\boldsymbol{\xi})\mathbf{1}, \boldsymbol{\eta} - (A\boldsymbol{\eta})\mathbf{1}) \\ &= N\alpha(\tilde{\xi} - A\boldsymbol{\xi})(\tilde{\eta} - A\boldsymbol{\eta}) + (\boldsymbol{\xi}, -\Delta_\beta \boldsymbol{\eta}). \end{aligned} \tag{4.3.12}$$

Using this identity the first equality in (4.3.10) is readily obtained by computing the fermionic derivatives, while the second equality follows by expanding the exponent. In the second line of

(4.3.12) we used the orthogonality of constant functions with the mean 0 function $\boldsymbol{\xi} - (A\boldsymbol{\xi})\mathbf{1}$. Finally, on the last line of (4.3.12), we used that $[\tilde{\eta} - A\boldsymbol{\eta}]\mathbf{1}$ is a constant to write the ℓ^2 inner product as a product multiplied by a factor N, and the factor α in the second term was absorbed into Δ_β.

Substituting (4.3.10)–(4.3.11) into (4.2.8) gives

$$\begin{aligned} Z_{\beta,h} &= \prod_i \partial_{\eta_i}\partial_{\xi_i} \frac{1}{z_i} e^{-\frac{1}{2}(\boldsymbol{u},-\Delta_\beta \boldsymbol{u})-(\boldsymbol{h},\boldsymbol{z}-\mathbf{1})} \\ &= \frac{e^{N\alpha/2}}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\, e^{-\frac{1}{2}N\alpha\tilde{z}^2+N\alpha\tilde{\xi}\tilde{\eta}+\alpha/2} \\ &\qquad \prod_{i=1}^N \Big[\partial_{\eta_i}\partial_{\xi_i}\Big(\exp\Big(\alpha(\xi_i\eta_i - \tilde{\xi}\eta_i - \xi_i\tilde{\eta}) + i\alpha\tilde{z}(1-\xi_i\eta_i) - \alpha\xi_i\eta_i + (1+h_i)\xi_i\eta_i\Big)\Big)\Big] \end{aligned} \tag{4.3.13}$$

Simplifying the term inside the exponential gives

$$\begin{aligned} Z_{\beta,h} = \frac{e^{N\alpha/2}}{\sqrt{2\pi N\alpha}} &\int_{\mathbb{R}} d\tilde{z}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\, e^{-\frac{1}{2}N\alpha\tilde{z}^2+N\alpha\tilde{\xi}\tilde{\eta}+N\alpha i\tilde{z}+\alpha/2} \\ &\prod_{i=1}^N \Big[\partial_{\eta_i}\partial_{\xi_i}\Big(\exp\Big((1+h_i-i\alpha\tilde{z})(\xi_i\eta_i) - \alpha(\tilde{\xi}\eta_i+\xi_i\tilde{\eta})\Big)\Big)\Big]. \end{aligned} \tag{4.3.14}$$

Since $(\tilde{\xi}\tilde{\eta})^2 = 0$ and $(\tilde{\xi}\eta_i + \xi_i\tilde{\eta})^3 = 0$, the exponential can be replaced by its third-order Taylor expansion, giving

$$\begin{aligned} Z_{\beta,h} &= \frac{e^{(N+1)\alpha/2}}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\, e^{-N\alpha[\frac{1}{2}\tilde{z}^2-\tilde{\xi}\tilde{\eta}-i\tilde{z}]} \prod_i \Big[(1+h_i-i\alpha\tilde{z}) - \alpha^2\tilde{\xi}\tilde{\eta}\Big]. \\ &= \frac{e^{(N+1)\alpha/2}}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\, e^{-N\alpha[\frac{1}{2}\tilde{z}^2-\tilde{\xi}\tilde{\eta}-i\tilde{z}]} \prod_i (1+h_i-i\alpha\tilde{z}) \prod_i [1 - \frac{\alpha^2}{1+h_i-i\alpha\tilde{z}}\tilde{\xi}\tilde{\eta}] \end{aligned} \tag{4.3.15}$$

Using again nilpotency of $\tilde{\xi}\tilde{\eta}$ this may be rewritten as

$$Z_{\beta,h} = \frac{e^{(N+1)\alpha/2}}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\partial_{\tilde{\eta}}\partial_{\tilde{\xi}}\, e^{-N\alpha[\frac{1}{2}\tilde{z}^2-i\tilde{z}]} \prod_i (1+h_i-i\alpha\tilde{z}) \left[1 + \left(N\alpha - \sum_i \frac{\alpha^2}{1+h_i-i\alpha\tilde{z}}\right)\tilde{\xi}\tilde{\eta}\right]. \tag{4.3.16}$$

Evaluating the fermionic derivatives gives the identity

$$Z_{\beta,h} = \frac{e^{(N+1)\alpha/2}\alpha N}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\, e^{-N\alpha[\frac{1}{2}\tilde{z}^2-i\tilde{z}]} \prod_{i=1}^N (1+h_i-i\alpha\tilde{z}) \left[1 - \frac{\alpha}{N}\sum_i (1+h_i-i\alpha\tilde{z})^{-1}\right]. \tag{4.3.17}$$

To show (4.3.6)–(4.3.7) we now take $\boldsymbol{h} = 0$. By definition the last bracket in (4.3.17) is then $F(i\alpha\tilde{z})$ and the remaining integrand defines $e^{-NV(\tilde{z})}$, proving (4.3.6). For (4.3.7) we use that $z_i = e^{z_i - 1}$, and hence that $[z_0 z_1]_\beta = Z_{\beta,-1_0-1_1}$. Therefore (4.3.17) implies

$$[z_0 z_1]_\beta = \frac{e^{(N+1)\alpha/2}\alpha N}{\sqrt{2\pi N\alpha}} \int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})} \left(\frac{-i\alpha\tilde{z}}{1-i\alpha\tilde{z}}\right)^2 \left[F(i\alpha\tilde{z}) + \frac{2\alpha}{N}\left[\frac{1}{1-i\alpha\tilde{z}} - \frac{1}{-i\alpha\tilde{z}}\right]\right]. \tag{4.3.18}$$

By definition, the integrand equals $-F_{01}(i\alpha\tilde{z})$, so together with the relation $Z_\beta[0 \leftrightarrow 1] = -[z_0 z_1]_\beta$, which holds by (4.2.11), the claim (4.3.7) follows. $\square$

Asymptotic analysis

To apply the method of stationary phase to evaluate the asymptotics of the integrals, we need the stationary points of V, and asymptotic expansions for V and F. The first two derivatives of P are

$$P'(w) = \frac{w}{\alpha} + 1 - \frac{1}{1-w}, \qquad P''(w) = \frac{1}{\alpha} - \frac{1}{(1-w)^2}. \tag{4.3.19}$$

The stationary points are those $w = i\alpha\tilde{z}$ such that $P'(w) = 0$. This equation can be rewritten as

$$w^2 - w(1-\alpha) = 0, \tag{4.3.20}$$

which has solutions $w = 0$ and $w = 1-\alpha$. We call a root w_0 *stable* if $P''(w_0) > 0$ and *unstable* if $P''(w_0) < 0$. For $\alpha < 1$ the root 0 is stable whereas $1-\alpha$ is unstable; for $\alpha > 1$ the root $1-\alpha$ is stable whereas 0 is unstable; for $\alpha = 0$ the two roots collide at 0 and $P''(0) = 0$.

For the asymptotic analysis, we start with the nondegenerate case $\alpha \neq 1$. First observe that we can view the right-hand sides of (4.3.6)–(4.3.7) as contour integrals and can, due to analyticity of the integrand and the decay of $e^{-N\alpha\tilde{z}^2/2}$ when $\operatorname{Re}\tilde{z}$ is large, shift this contour to the horizontal line $\mathbb{R} + iw$ for any $w \in \mathbb{R}$. We will then apply Laplace's method in the version given by the next theorem, which is a simplified formulation of [82, Theorem 7, p.127].

Theorem 4.3.3. *Let I be a horizontal line in $\mathbb{C}$. Suppose that $V, G\colon U \to \mathbb{R}$ are analytic in a neighbourhood U of the contour I, that $t_0 \in I$ is such that V' has a simple root at t_0, and that $\operatorname{Re}(V(t) - V(t_0))$ is positive and bounded away from 0 for t away from t_0. Then*

$$\int_I e^{-NV(t)} G(t)\, dt \sim 2e^{-NV(t_0)} \sum_{s=0}^{\infty} \Gamma(s+1/2) \frac{b_s}{N^{s+1/2}}, \tag{4.3.21}$$

where the notation $\sim$ means that the right-hand side is an asymptotic expansion for the left-hand side, and the coefficients are given by (with all functions evaluated at t_0):

$$b_0 = \frac{G}{(2V'')^{1/2}}, \qquad b_1 = \left(2G'' - \frac{2V'''G'}{V''} + \left[\frac{5V'''^2}{6V''^2} - \frac{V''''}{2V''}\right] G\right) \frac{1}{(2V'')^{3/2}}, \tag{4.3.22}$$

and with b_s as given in [82] for $s \geq 2$. (Also recall that $\Gamma(1/2) = \sqrt{\pi}$ and that $\Gamma(s+1) = s\Gamma(s)$.)

For $\alpha \neq 1$, denote by w_0 the unique stable root. As discussed in the previous paragraph, we can shift the contour to the line $\mathbb{R} - i\frac{w_0}{\alpha}$, and the previous theorem implies that

$$\begin{aligned}\sqrt{\frac{N\alpha}{2\pi}} &\int_{\mathbb{R}} e^{-NV(\tilde{z})} G(\tilde{z}) d\tilde{z} \\ &= \sqrt{\frac{1}{\alpha P''}} e^{NP} \left[F - \frac{1}{4NP''}\left(2F'' - \frac{2P'''F'}{P''} + \left[\frac{5P'''^2}{6P''^2} - \frac{P''''}{2P''}\right] F\right) + O\left(\frac{1}{N^2}\right)\right],\end{aligned} \tag{4.3.23}$$

with all functions on the right-hand side are evaluated at w_0. From this the proof of Theorem 4.3.1 for $\alpha \neq 1$ is an elementary (albeit somewhat tedious) computation of the derivatives of P and F and F_{01} at w_0.

Proof of Theorem 4.3.1, $\alpha < 1$. The stable root is $w_0 = 0$. By (4.3.23) and elementary computations for the derivatives of P and F and F_{01}, we find

$$\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} e^{-NV(\tilde{z})} F(i\alpha\tilde{z}) d\tilde{z} \sim \sqrt{1-\alpha} \tag{4.3.24}$$

$$\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} e^{-NV(\tilde{z})} F_{01}(i\alpha\tilde{z}) d\tilde{z} \sim \frac{\alpha^2}{\sqrt{1-\alpha}}. \tag{4.3.25}$$

Recalling the definitions (4.3.6)–(4.3.7), this implies the claims. □

Proof of Theorem 4.3.1, $\alpha > 1$. The stable root is $w_0 = 1 - \alpha$. Again (4.3.23) and elementary computations for the derivatives of P and F and F_{01} lead to

$$\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} e^{-NV(\tilde{z})} F(i\alpha\tilde{z}) d\tilde{z} \sim e^{NP} \frac{\alpha^{3/2}}{N(\alpha-1)^{5/2}} \tag{4.3.26}$$

$$\sqrt{\frac{N\alpha}{2\pi}} \int_{\mathbb{R}} e^{-NV(\tilde{z})} F_{01}(i\alpha\tilde{z}) d\tilde{z} \sim e^{NP} \frac{1}{N(\alpha-1)^{1/2}\alpha^{1/2}}, \tag{4.3.27}$$

and $P = P(w_0) = P(1-\alpha)$. Again the claims follow from (4.3.6)–(4.3.7). □

At the critical point $\alpha = 1$, the two roots collide at 0 and $P''(0) = 0$. We analyse the integral as follows.

Proof of Theorem 4.3.1, $\alpha = 1$. We begin by using the conjugate flip symmetry to write

$$N^{\frac{2}{3}} \int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})} F(i\tilde{z}) = 2N^{\frac{2}{3}} \operatorname{Re} \int_0^\infty d\tilde{z}\, e^{-NV(\tilde{z})} F(i\tilde{z}). \tag{4.3.28}$$

Using analyticity of the integrand, we then deform the contour from $[0,\infty)$ to $[0, e^{i\pi/6}\infty)$; the contribution of the boundary arc vanishes due to the decay of $e^{-N\alpha\tilde{z}^2/2}$ on this arc. We now split the contour into two intervals $I_1 = [0, e^{i\pi/6}N^{-3/10})$ and $I_2 = [e^{i\pi/6}N^{-3/10}, e^{i\pi/6}\infty)$, and denote the integrals over these regions as J_1 and J_2 respectively.

Over the first interval I_1, we introduce the new real variable $s = N^{\frac{1}{3}} e^{-i\pi/6}\tilde{z}$, in terms of which

$$J_1 = 2N^{\frac{2}{3}} \operatorname{Re} \int_{I_1} d\tilde{z}\, e^{-NV(\tilde{z})} F(i\tilde{z}) = 2\operatorname{Re} \int_0^{N^{\frac{1}{30}}} ds\, e^{-NV(e^{\frac{i\pi}{6}} N^{-\frac{1}{3}} s)} N^{\frac{1}{3}} e^{\frac{i\pi}{6}} F(e^{\frac{2i\pi}{3}} N^{-\frac{1}{3}} s). \tag{4.3.29}$$

We then approximate the arguments as

$$NV(e^{\frac{i\pi}{6}} N^{-\frac{1}{3}} s) = \frac{1}{3}s^3 + O(N^{-\frac{1}{3}} s^4) = \frac{1}{3}s^3 + O(N^{-\frac{6}{30}}) \tag{4.3.30}$$

$$N^{\frac{1}{3}} e^{\frac{i\pi}{6}} F(e^{\frac{2i\pi}{3}} N^{-\frac{1}{3}} s) = e^{-\frac{i\pi}{6}} s + O(N^{-\frac{1}{3}} s^2) = e^{-\frac{i\pi}{6}} s + O(N^{-\frac{8}{30}}), \tag{4.3.31}$$

where the last error bounds hold uniformly for $s \in [0, N^{1/30}]$. This gives

$$J_1 = 2\operatorname{Re} \int_0^{N^{\frac{1}{30}}} ds\, e^{-\frac{i\pi}{6}} s e^{-\frac{1}{3}s^3} + O(N^{-\frac{4}{30}}) = 2\operatorname{Re} \int_0^\infty ds\, e^{-\frac{i\pi}{6}} s e^{-\frac{1}{3}s^3} + o(1) = 3^{\frac{1}{6}}\Gamma(\tfrac{2}{3}) + o(1). \tag{4.3.32}$$

The second term J_2 is asymptotically negligible. To see this, we bound $|F(i\tilde{z})| \le 1$, introduce the real variable $s = e^{-\frac{i\pi}{6}}\tilde{z}$, and split the resulting domain as $[N^{-3/10}, 2) \cup [2, \infty) = I_2' \cup I_2''$:

$$J_2 = 2N^{\frac{2}{3}} \operatorname{Re} \int_{I_2} d\tilde{z}\, e^{-NV(\tilde{z})} F(i\tilde{z}) \le 2N^{\frac{2}{3}} \operatorname{Re} \int_{I_2'} ds\, e^{-NV(\frac{i\pi}{6}s)} + 2N^{\frac{2}{3}} \operatorname{Re} \int_{I_2''} ds\, e^{-NV(\frac{i\pi}{6}s)}. \tag{4.3.33}$$

Over I_2', we use that $|I_2'| \le 2$ and bound the integral in terms of the supremum of the integrand:

$$2N^{\frac{2}{3}} \operatorname{Re} \int_{I_2'} ds\, e^{-NV(\frac{i\pi}{6}s)} e^{\frac{i\pi}{6}} F(e^{\frac{2i\pi}{3}} s) \le 2N^{\frac{2}{3}} \operatorname{Re} \int_{I_2'} ds\, e^{-NV(\frac{i\pi}{6}s)} \le 4N^{\frac{2}{3}} \sup_{s\in I_2'} e^{-\operatorname{Re}[NV(\frac{i\pi}{6}s)]}, \tag{4.3.34}$$

and as $\operatorname{Re} NV(\frac{i\pi}{6}s)$ is decreasing, this supremum is attained on the boundary $s = N^{-3/10}$. Taylor expanding as before gives us

$$4N^{\frac{2}{3}} \sup_{s\in I_2'} e^{-\operatorname{Re} NV(\frac{i\pi}{6}s)} = 4N^{\frac{2}{3}} e^{-\operatorname{Re} NV(\frac{i\pi}{6}N^{-\frac{3}{10}})} = e^{-(\frac{1}{3}+o(1))N^{\frac{1}{10}}} \tag{4.3.35}$$

Over I_2'', we use that $\mathrm{Re}[NV(\frac{i\pi}{6}s)] \geq \frac{Ns^2}{4}$ for all $s \geq 2$ to bound the second term as

$$2N^{\frac{2}{3}}\,\mathrm{Re}\int_{I_2'} ds\, e^{-NV(\frac{i\pi}{6}s)} \leq 2N^{\frac{2}{3}}\int_{I_2'} ds\, e^{-\frac{Ns^2}{4}} \leq e^{-(1+o(1))N}. \tag{4.3.36}$$

Putting together the estimates for J_1 and J_2, we therefore find

$$N^{\frac{2}{3}}\int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})}F(i\tilde{z}) = J_1 + J_2 = 3^{\frac{1}{6}}\Gamma(\tfrac{2}{3}) + o(1) \tag{4.3.37}$$

and hence the first asymptotic relation in (4.3.3) follows from (4.3.6), i.e.,

$$Z_\beta \sim \frac{3^{\frac{1}{6}}\Gamma(\frac{2}{3})e^{\frac{(N+1)}{2}}}{N^{\frac{1}{6}}\sqrt{2\pi}}. \tag{4.3.38}$$

Using the same procedure, we can compute $\mathbb{P}_\beta[0 \leftrightarrow 1]$. We again split the (conveniently scaled) integral into two terms as

$$\begin{aligned} N^{\frac{4}{3}}\int_{\mathbb{R}} d\tilde{z}\, e^{-NV(\tilde{z})}F_{01}(i\tilde{z}) &= 2\,\mathrm{Re}\int_0^{N^{\frac{1}{30}}} ds\, e^{-NV(e^{\frac{i\pi}{6}}N^{-\frac{1}{3}}s)}Ne^{\frac{i\pi}{6}}F_{01}(e^{\frac{2i\pi}{3}}N^{-\frac{1}{3}}s) \\ &\quad + 2\,\mathrm{Re}\int_{N^{\frac{1}{30}}}^{\infty} ds\, e^{-NV(e^{\frac{i\pi}{6}}N^{-\frac{1}{3}}s)}Ne^{\frac{i\pi}{6}}F_{01}(e^{\frac{2i\pi}{3}}N^{-\frac{1}{3}}s) = J_1 + J_2. \end{aligned} \tag{4.3.39}$$

As before J_2 is asymptotically negligible. For J_1, we approximate the F_{01} term as

$$Ne^{\frac{i\pi}{6}}F_{01}(e^{\frac{2i\pi}{3}}N^{-\frac{1}{3}}s) = e^{\frac{i\pi}{6}}s^3 + O(N^{-\frac{1}{3}}s^4) = e^{\frac{i\pi}{6}}s^3 + O(N^{-\frac{6}{30}}), \tag{4.3.40}$$

uniformly for $s \in [0, N^{1/30}]$, to obtain the asymptotic relation

$$J_1 = 2\,\mathrm{Re}\int_0^{N^{\frac{1}{30}}} ds\, e^{-NV(e^{\frac{i\pi}{6}}N^{-\frac{1}{3}}s)}Ne^{\frac{i\pi}{6}}F_{01}(e^{\frac{2i\pi}{3}}N^{-\frac{1}{3}}s) \sim 2\,\mathrm{Re}\int_0^{\infty} ds\, e^{\frac{i\pi}{6}}s^3e^{-\frac{1}{3}s^3} = 3^{\frac{5}{6}}\Gamma(\tfrac{4}{3}). \tag{4.3.41}$$

From (4.3.7), we therefore find

$$Z_\beta[0 \leftrightarrow 1] \sim \frac{3^{\frac{5}{6}}\Gamma(\frac{4}{3})e^{\frac{(N+1)}{2}}}{N^{\frac{5}{6}}\sqrt{2\pi}} \tag{4.3.42}$$

which after dividing by Z_β shows the second asymptotic relation in (4.3.3). □

4.4 No percolation in two dimensions

In this section, we consider the arboreal gas on (finite approximations of) $\mathbb{Z}^2$ with constant nearest neighbour weights, i.e., with $\beta_{ij} = \beta > 0$ for all edges ij and vertex weights $h_i = h$ for all vertices i. As such we write β instead of $\boldsymbol{\beta}$ in this section. Constant weights are merely a convenient choice; everything in this section also applies to translation-invariant finite range weights, for example. In contrast with the case of the complete graph, we show that on $\mathbb{Z}^2$ the tree containing a fixed vertex always has finite density. Our arguments are closely based on estimates developed for the vertex-reinforced jump process [9, 62, 88]. The main new idea is to use these bounds in combination with dimensional reduction from Section 4.2.

Two-point function decay in two dimensions

The proof of Theorem 4.1.3 makes use of the representation from Section 4.2, and closely follows [88]; an alternative proof could likely be obtained by adapting instead [62].

To lighten the notation, for a finite subgraph $\Lambda \subset \mathbb{Z}^2$ we write $\mathbb{P}_\beta$ in place of $\mathbb{P}_{\Lambda,\beta}$. By (4.2.86), the connection probability can be written in the horospherical coordinates of the $\mathbb{H}^{2|4}$ model as

$$\mathbb{P}_\beta[0 \leftrightarrow j] = \langle e^{t_j} \rangle_\beta^0 \tag{4.4.1}$$

where $\langle \cdot \rangle_\beta^0$ denotes the expectation with pinning at vertex 0. Explicitly, by (4.2.85), the measure $\langle \cdot \rangle_\beta^0$ on the right-hand side can be written as the $a = 3/2$ case of

$$Q_{\beta,a}(dt) \equiv \frac{1}{Z_{\beta,a}} \exp\left(-\frac{1}{2} \sum_{i,j} \beta_{ij} (\cosh(t_i - t_j) - 1) \right) D(\beta,t)^a \prod_{i \neq 0} \frac{dt_i}{\sqrt{2\pi}}, \tag{4.4.2}$$

where

$$D(\beta,t) \equiv \tilde{D}_\beta(t) \prod_i e^{-2t_i}, \tag{4.4.3}$$

and where $\tilde{D}_\beta(t)$ was given explicitly in (4.2.82) and $Z_{\beta,a}$ is a normalising constant. We have made the parameter a explicit as our argument adapts that of [88], which concerned the case $a = 1/2$. When $a = 1/2$ supersymmetry implies that $Z_{\beta,1/2} = 1$ and $\mathbb{E}_{Q_{\beta,1/2}}(e^{t_k}) = 1$ for all $\boldsymbol{\beta} = (\beta_{ij})$ and all $k \in \Lambda$. These identities require the following replacement when $a \neq 1/2$:

$$Z_{\beta,a} \text{ is increasing in all of the } \beta_{ij}, \qquad \mathbb{E}_{Q_{\beta,a}}(e^{2at_k}) = 1 \quad \text{for all } (\beta_{ij}) \text{ and all } k \in \Lambda. \tag{4.4.4}$$

When $a = 3/2$ the first of these facts follow from the forest representation for the partition function, see Proposition 4.2.9, and the second is (4.2.87) of Corollary 4.2.13. Proof that (4.4.4) holds for general half-integer $a \geqslant 0$ appears in [29], and we conjecture that these assumptions are true for any $a \geqslant 0$.

With (4.4.4) given, it is straightforward to adapt [88, Lemma 1] to obtain the following lemma. In the next lemma we assume $0, i \in \Lambda$, but we make no further assumptions beyond that $\boldsymbol{\beta}$ induces a connected graph.

Lemma 4.4.1 (Sabot [88, Lemma 1] for $a = 1/2$). *Let $a \geqslant 0$, $s \in (0,1)$, and $\gamma > 0$. Assume* (4.4.4) *holds. Then for any $v \in \mathbb{R}^\Lambda$ with $v_j = 1$, $v_0 = 0$, and*

$$\gamma |v_i - v_k| \leqslant \frac{1}{2}(1-s)^2 \quad \textit{for all } i \sim k, \tag{4.4.5}$$

one has, with $q = 1/(1-s)$,

$$\mathbb{E}_{Q_{\beta,a}}(e^{2ast_j}) \leqslant e^{-2as\gamma} e^{\frac{1}{2}\gamma^2 q^2 \sum_{i,k} (\beta_{ik} + 2a)(v_i - v_k)^2}. \tag{4.4.6}$$

Proof. As mentioned, our proof is an adaptation of [88, Lemma 1], and hence we indicate the main steps but will be somewhat brief. In this reference $a = 1/2$, $Q_{\beta,a}$ is denoted Q, β_{ij} is denoted W_{ij}, and t is denoted by u. Let $Q_{\beta,a}^\gamma$ denote the distribution of $t - \gamma v$. Since the partition function does not change under translation of the underlying measure, by following [88, Prop. 1] we obtain,

$$\frac{dQ_{\beta,a}}{dQ_{\beta,a}^\gamma}(t) = \exp\left(-\frac{1}{2} \sum_{i,k} \beta_{ik} (\cosh(t_i - t_k) - \cosh(t_i - t_k + \gamma(v_i - v_k)) \right) \frac{D(\beta,t)^a}{D(\beta + \gamma v, t)^a}. \tag{4.4.7}$$

With e^t replaced by e^{2at} but otherwise exactly as in the argument leading to [88, (2)], by using that s^{-1} and q are Hölder conjugate and using the second part of (4.4.4),

$$\begin{aligned}\mathbb{E}_{Q_{\beta,a}}(e^{2ast_k}) = \mathbb{E}_{Q^\gamma_{\beta,a}}\left(\frac{dQ_{\beta,a}}{dQ^\gamma_{\beta,a}}e^{2ast_k}\right) &\leqslant \mathbb{E}_{Q^\gamma_{\beta,a}}\left(\left(\frac{dQ_{\beta,a}}{dQ^\gamma_{\beta,a}}\right)^q\right)^{1/q}\left(\mathbb{E}_{Q^\gamma_{\beta,a}}(e^{2at_k})\right)^s\\ &\leqslant \mathbb{E}_{Q^\gamma_{\beta,a}}\left(\left(\frac{dQ_{\beta,a}}{dQ^\gamma_{\beta,a}}\right)^q\right)^{1/q} e^{-2as\gamma}. \end{aligned} \tag{4.4.8}$$

The expectation on the right-hand side is estimated as in [88], with the only change that $\sqrt{D(\beta,t)}$ is replaced by $D(\beta,t)^a$ in all expressions, and that the change of measure from $Q_{\beta,a}$ to $Q_{\tilde\beta,a}$ involves the normalisation constants, i.e., a factor $Z_{\tilde\beta,a}/Z_{\beta,a}$. Setting $\gamma' = \gamma(q-1)$, we obtain

$$\begin{aligned}\mathbb{E}_{Q^\gamma_{\beta,a}}\left(\left(\frac{dQ_{\beta,a}}{dQ^\gamma_{\beta,a}}\right)^q\right) &= \mathbb{E}_{Q^{\gamma'}_{\beta,a}}\left(\left(\frac{dQ_{\beta,a}}{dQ^\gamma_{\beta,a}}\right)^{q-1}\frac{dQ_{\beta,a}}{dQ^{\gamma'}_{\beta,a}}\right)\\ &\leqslant \mathbb{E}_{Q^{\gamma'}_{\beta,a}}\left(\frac{q}{2}\sum_{i,k}\beta_{ik}\cosh(t_i-t_k+\gamma'(v_i-v_k))(2q^2\gamma^2(v_i-v_k)^2)\right)\\ &= e^{\frac12\sum_{i,k}\beta_{ik}q^3\gamma^2(v_i-v_k)^2}\,\frac{Z_{\tilde\beta,a}}{Z_{\beta,a}}\mathbb{E}_{Q_{\tilde\beta,a}}\left(\left(\frac{D(\beta,t)}{D(\tilde\beta,t)}\right)^a\right)\end{aligned} \tag{4.4.9}$$

where

$$\tilde\beta_{ik} = \beta_{ik}(1-2q^3\gamma^2(v_i-v_k)^2) \in [\frac12\beta_{ik},\beta_{ik}]. \tag{4.4.10}$$

The ratio of determinants is bounded using the matrix-tree theorem as done on [88, p.7], and we use that $Z_{\tilde\beta,a} \leqslant Z_{\beta,a}$, by (4.4.4). The result is (4.4.6). □

Proof of Theorem 4.1.3. We may choose $s = 1/(2a) = 1/3 \in (0,1)$ in Lemma 4.4.1. We then combine (4.4.1) and (4.4.6) and choose v as a difference of Green functions (exactly as in [88, Section 2.2]) to find that,

$$\mathbb{P}_\beta[0\leftrightarrow j] = \mathbb{E}_{Q_{\beta,a}}(e^{t_j}) = \mathbb{E}_{Q_{\beta,a}}(e^{2ast_j}) \leqslant |j|^{-c_\beta} \tag{4.4.11}$$

as needed. □

Mermin–Wagner theorem

We now show that the vanishing of the density of the cluster containing a fixed vertex on the torus also follows from a version of the classical Mermin–Wagner theorem. We first derive an expression for a quantity closely related to the mean tree size. For constant h, Theorem 4.2.1 implies that

$$[z_a]_{\beta,h} = \sum_{F\in\mathcal{F}}\prod_{ij\in F}\beta_{ij}\prod_{T\in F}(1+\sum_{k\in T}(h-1_{a=k})), \tag{4.4.12}$$

which leads to

$$\langle z_i\rangle_{\beta,h} = \mathbb{E}_{\beta,h}\frac{h|T_i|}{1+h|T_i|}, \tag{4.4.13}$$

where T_i is the (random) tree containing the vertex i.

Let Λ be a d-dimensional discrete torus, and let $\lambda(p)$ by the Fourier multiplier of the corresponding discrete Laplacian:

$$\lambda(p) \equiv \sum_{j\in\Lambda}\beta_{0j}(1-\cos(p\cdot j)), \qquad p\in\Lambda^\star \tag{4.4.14}$$

where $\cdot$ is the Euclidean inner product on $\mathbb{R}^d$ and $\Lambda^\star$ is the Fourier dual of the discrete torus Λ.

Theorem 4.4.2. *Let $d \geqslant 1$, and let Λ be a d-dimensional discrete torus of side length L. Then*

$$\frac{1}{\langle z_0\rangle_{\beta,h}} \geq 1 + \frac{1}{(2\pi L)^d}\sum_{p\in\Lambda^\star}\frac{1}{\lambda(p)+h}. \tag{4.4.15}$$

Proof. The proof is analogous to [9, Theorem 1.5]. We write the $\mathbb{H}^{0|2}$ expectations $\langle \xi_i\eta_j\rangle_{\beta,h}$ and $\langle z_i\rangle_{\beta,h}$ in horospherical coordinates using Corollary 4.2.10:

$$\langle \xi_i\eta_j\rangle_{\beta,h} = \langle s_is_je^{t_i+t_j}\rangle_{\beta,h}, \quad \langle z_i\rangle_{\beta,h} = \langle e^{t_i}\rangle_{\beta,h} = \langle e^{2t_i}\rangle_{\beta,h}. \tag{4.4.16}$$

Set

$$S(p) = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{i(p\cdot j)}e^{t_j}s_j, \quad D = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{-i(p\cdot j)}\frac{\partial}{\partial s_j}. \tag{4.4.17}$$

Since the expectation of functions depending only on (s,t) in horospherical coordinates is an expectation with respect to a probability measure, denoted $\langle\cdot\rangle$ from hereon, the Cauchy–Schwarz inequality implies

$$\langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle}. \tag{4.4.18}$$

Since the density in horospherical coordinates is $e^{-\widetilde{H}(s,t)}$, the probability measure $\langle\cdot\rangle$ obeys the integration by parts $\langle FD\widetilde{H}\rangle = \langle DF\rangle$ identity for any function $F = F(s,t)$ that does not grow too fast. Therefore by translation invariance, with $y_i = s_ie^{t_i}$,

$$\langle |S(p)|^2\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_jy_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0y_{j-l}\rangle = \sum_j e^{i(p\cdot j)}\langle y_0y_j\rangle, \tag{4.4.19}$$

$$\langle S(p)D\widetilde{H}\rangle = \langle DS(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle\frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j \langle e^{t_j}\rangle = \langle z_0\rangle. \tag{4.4.20}$$

By Cauchy–Schwarz, translation invariance, and (4.4.16) we also have

$$\langle e^{t_j+t_l}\rangle \leqslant \langle e^{2t_0}\rangle = \langle z_0\rangle. \tag{4.4.21}$$

Using (5.4.13) and the integration by parts identity it follows that

$$\langle |D\widetilde{H}|^2\rangle = \langle D\bar{D}\widetilde{H}\rangle = \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle e^{t_j+t_l}\rangle(1-\cos(p\cdot(j-l))) + \frac{h}{|\Lambda|}\sum_j\langle e^{t_j}\rangle \leqslant \langle z_0\rangle(\lambda(p)+h). \tag{4.4.22}$$

In summary, we have proved

$$\sum_j e^{i(p\cdot j)}\langle \xi_0\eta_j\rangle = \sum_j e^{i(p\cdot j)}\langle y_0y_j\rangle = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle} \geqslant \frac{\langle z_0\rangle}{\lambda(p)+h} \tag{4.4.23}$$

Summing over $p \in \Lambda^\star$ in the Fourier dual of Λ (with the sum correctly normalized), the left-hand side becomes $\langle \xi_0\eta_0\rangle$. Using $\langle z_0\rangle = 1 - \langle \xi_0\eta_0\rangle$ this then gives the claim:

$$\frac{1}{\langle z_0\rangle} - 1 \geq \frac{1}{(2\pi L)^d}\sum_{p\in\Lambda^*}\frac{1}{\lambda(p)+h}. \tag{4.4.24}$$

□

From the Mermin–Wagner theorem we obtain that on a finite torus of side length L the density of the tree containing 0 tends to 0 as $L \to \infty$. We write $\lesssim$ for inequalities that hold up to universal constants.

Corollary 4.4.3. *Let Λ be the 2-dimensional discrete torus of side length L. Then*

$$\mathbb{E}_{\beta,0}\frac{|T_0|}{|\Lambda|} \lesssim \frac{1}{\sqrt{\log L}}. \tag{4.4.25}$$

Proof. For any $h \leqslant 1/|\Lambda|$ we have $h|T_0| \leqslant 1$. By Theorem 4.4.2, for $d = 2$ thus

$$\mathbb{E}_{\beta,h}\frac{|T_0|}{|\Lambda|} = \frac{1}{|\Lambda|h}\mathbb{E}_{\beta,h}h|T_0| \leqslant \frac{2}{|\Lambda|h}\mathbb{E}_{\beta,h}\frac{h|T_0|}{1+h|T_0|} = \frac{2}{|\Lambda|h}\langle z_0\rangle_{\beta,h} \lesssim \frac{1}{hL^2\log L} \tag{4.4.26}$$

where we used that, for all $h \geqslant 0$, the Green's function of the discrete torus satisfies

$$\frac{1}{(2\pi L)^2}\sum_{p\in\Lambda^\star}\frac{1}{\lambda(p)+h} \gtrsim \log(h^{-1}\wedge L). \tag{4.4.27}$$

Directly following the conclusion of the present proof, we shall show that if X is a random variable with $|X| \leqslant 1$, and if $h \ll 1/|\Lambda|$,

$$|\mathbb{E}_{\beta,h}X - \mathbb{E}_{\beta,0}X| = O(h|\Lambda|). \tag{4.4.28}$$

Applying this estimate with $X = |T_0|/|\Lambda|$, for $h \ll 1/|\Lambda|$ we have

$$\left|\mathbb{E}_{\beta,h}\frac{|T_0|}{|\Lambda|} - \mathbb{E}_{\beta,0}\frac{|T_0|}{|\Lambda|}\right| = O(hL^2). \tag{4.4.29}$$

With $h = L^{-2}(\log L)^{-1/2}$, combining both estimates gives

$$\mathbb{E}_{\beta,0}\frac{|T_0|}{|\Lambda|} \lesssim \frac{1}{hL^2\log L} + hL^2 \lesssim \frac{1}{\sqrt{\log L}}. \tag{4.4.30}$$

□

Lemma 4.4.4. *Let Λ be any finite graph with $|\Lambda|$ vertices. Let X be a random variable with $|X| \leqslant 1$. Then for $h \ll 1/|\Lambda|$,*

$$|\mathbb{E}_{\beta,h}X - \mathbb{E}_{\beta,0}X| = O(h|\Lambda|). \tag{4.4.31}$$

Proof. By definition,

$$\mathbb{E}_{\beta,h}X = \frac{\mathbb{E}_{\beta,0}(X\prod_{T\in F}(1+h|T|))}{\mathbb{E}_{\beta,0}(\prod_{T\in F}(1+h|T|))}. \tag{4.4.32}$$

With $A'/(1+\varepsilon) - A = (A'-A) - A'(\varepsilon/(1+\varepsilon)) = (A'-A) + (A'/(1+\varepsilon))\varepsilon$ we get

$$\mathbb{E}_{\beta,h}X - \mathbb{E}_{\beta,0}X = \mathbb{E}_{\beta,0}(X(\prod_T(1+h|T|)-1)) - \mathbb{E}_{\beta,h}(X)\mathbb{E}_{\beta,0}(\prod_T(1+h|T|)-1). \tag{4.4.33}$$

Since $|X| \leqslant 1$ it suffices to bound

$$\prod_{T\in F}(1+h|T|) - 1 = \sum_{F'\subset F}\prod_{T\in F'}h|T| \tag{4.4.34}$$

where the sum runs over subforests F' of F, i.e., unions of the disjoint trees in F. Since $\sum_i |T_i| \leqslant |\Lambda|$,

$$\sum_{F'\subset F}\prod_{T\in F'}h|T| \leqslant \sum_{n\geqslant 1}\sum_{i_1,\dots,i_n}\prod_{i=1}^n(h|T_i|) \leqslant \sum_{n\geqslant 1}\left(h\sum_i|T_i|\right)^n \leqslant \sum_{n\geqslant 1}(h|\Lambda|)^n = O(h|\Lambda|) \tag{4.4.35}$$

whenever $h|\Lambda| \ll 1$. □

Appendices

4.A Percolation properties

In this appendix we indicate how to deduce Theorem 4.1.3 from our results in Section 4.4. We also give proofs of the other unproven claims from Section 4.1. While we are unaware of any references for these results, it is likely that they have been independently discovered in the past. In particular, we thank G. Grimmett for pointing out Proposition 4.1.1.

Stochastic domination

The proof of Proposition 4.1.1 is an application of Holley's inequality, and we begin by recalling the set-up and result. For a finite set X and probability measures $\mu_i : 2^X \to [0, \infty)$, μ_1 *convexly dominates* μ_2 if for all $A, B \subset 2^X$

$$\mu_1(A \cup B)\mu_2(A \cap B) \geqslant \mu_1(A)\mu_2(B). \tag{4.A.1}$$

Holley's inequality, as stated in [32], says that μ_1 convexly dominating μ_2 is a sufficient condition for μ_1 to stochastically dominate μ_2.

Proof of Proposition 4.1.1. To prove the proposition, we verify the condition (4.A.1) when μ_1 is p_β bond percolation and μ_2 is the arboreal gas with parameter β. This is straightforward: if B is not a forest the inequality is trivial because the right-hand side is 0, whereas if B is a forest then both sides are actually equal. $\square$

Remark 4.A.1. Proposition 4.1.1 implies a monotone coupling between the arboreal gas with parameter β and p_β-bond percolation exists. An explicit construction of such a coupling would be interesting.

The arboreal gas in infinite volume

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of vertices such that the subgraph $\mathbb{G}_\Lambda = (\Lambda, E(\Lambda))$ induced by Λ is connected. Write $\mathbb{P}_{\Lambda,\beta}$ for the arboreal graph measure on $\mathbb{G}_\Lambda$. In this section we prove Proposition 4.1.9, i.e., we show how Conjecture 4.1.8 implies the existence of the infinite-volume limit $\lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}_{\Lambda,\beta}$, where $\Lambda_n \uparrow \mathbb{Z}^d$ means that Λ_n is increasing and for any finite set $A \subset \mathbb{Z}^d$, there is an n_A such that $A \subset \Lambda_n$ for $n \geqslant n_A$.

Proof of Proposition 4.1.9. We consider the case of general non-negative weights $\beta = (\beta_{ij})$. We first claim it suffices to prove that for any finite graph $\mathbb{G} = (V, E)$, any set $\tilde{E}$ of edges and any $e \notin \tilde{E}$, that

$$\mathbb{P}_{\mathbb{G},\beta}[\tilde{E} \cup \{e\}] \leqslant \mathbb{P}_{\mathbb{G},\beta}[\tilde{E}]\mathbb{P}_{\mathbb{G},\beta}[e]. \tag{4.A.2}$$

Note that this implies $\mathbb{P}_{\mathbb{G},\beta}[\tilde{E}]$ is (weakly) monotone decreasing in β_{ij} for all edges $ij \notin \tilde{E}$. The sufficiency of this claim is a standard argument, but we provide it for completeness.

Observe that monotonicity and probabilities being bounded below by zero implies that for any finite collection of edges $\tilde{E}$ in $\mathbb{Z}^d$, $\lim_{n\to\infty} \mathbb{P}_{\mathbb{G}_n,\beta}[\tilde{E}]$ exists. This is because the transition from $\mathbb{G}_n$

to $\mathbb{G}_{n+1}$ can be viewed as a limit when $\beta_{ij}^{(n)}$ (weakly) increases to $\beta_{ij}^{(n+1)}$ – the increase is in fact no change for $ij \in E(\mathbb{G}_n)$ and is positive for $ij \notin E(\mathbb{G}_n)$. Moreover, the limit is independent of the sequence $\mathbb{G}_n$, as can be seen by interlacing any two sequence $\mathbb{G}_n^{(i)}$ that increase to $\mathbb{Z}^d$. By inclusion-exclusion the probability of any cylinder event depending on edges $\tilde{E}$ can be expressed in terms of the occurrence of finite subsets of edges in $\tilde{E}$, and hence every cylinder event has a well-defined limiting probability. Since all cylinder probabilities converge, there is a well-defined probability measure $\mathbb{P}_\beta$ on $\{0,1\}^{E(\mathbb{Z}^d)}$ that is the weak limit of the $\mathbb{P}_{\mathbb{G}_n,\beta}$. Moreover, $\mathbb{P}_\beta$ is translation invariant by the interlacing argument used above.

What remains is to prove (4.A.2). This is obvious if $\tilde{E}$ is the empty set of edges, so we may assume $\tilde{E}$ is non-empty. We use an argument of Feder–Mihail [48]. In the proof of [48, Lemma 3.2] it is shown that (4.A.2) follows if one knows, for all finite graphs $\mathbb{G} = (V, E)$, that

(i) $\mathbb{P}_{\mathbb{G},\beta}[e, f] \leqslant \mathbb{P}_{\mathbb{G},\beta}[e]\mathbb{P}_{\mathbb{G},\beta}[f]$ for all distinct $e, f \in E$, and

(ii) For any $\tilde{E} \subset E$ and $e \notin \tilde{E}$, there is an $f \in E$ such that $\mathbb{P}_{\mathbb{G},\beta}[\tilde{E} \mid e, f] \geqslant \mathbb{P}_{\mathbb{G},\beta}[\tilde{E} \mid e, \bar{f}]$, where $\bar{f}$ means f is not present.

The first of these conditions is precisely Conjecture 4.1.8. The second is obvious: choose $f \in \tilde{E}$, for which the right-hand side is zero. □

Proof of Corollary 4.1.4

In this section we show how to deduce Corollary 4.1.4 from the quantitative estimate of Theorem 4.1.3; we thank Tom Hutchcroft for suggesting this proof. The proof crucially exploits planar duality and the resulting connected subgraph model that is dual to the arboreal gas. The precise definitions are as follows.

Given a set $\omega \in \{0,1\}^{E(\mathbb{Z}^2)}$, we write $\omega^\star$ for the dual set of edges, i.e., if $e^\star$ is the edge dual to e, then $\omega^\star_{e^\star} \equiv 1 - \omega_e$. In what follows we will identify $\mathbb{Z}^2$ with its dual; with this identification $\omega \mapsto \omega^\star$ is an involution on the set of edge configurations $\{0,1\}^{E(\mathbb{Z}^2)}$.

Suppose $\mathbb{P}_\beta$ is an arboreal gas measure, either on a finite graph, or a weak limit of measures on finite graphs. We define the *connected subgraph measure* $\mathbb{P}^\star_\beta$ by $\mathbb{P}^\star_\beta(A^\star) = \mathbb{P}_\beta(A)$ for all edge configurations A. The name arises as for finite-volume measures $\mathbb{P}^\star_\beta$ is supported on connected subgraphs of $\mathbb{Z}^2$ since $\mathbb{P}_\beta$ is supported on forests with finite components, see, e.g., [53, Theorem 2.1]. It is important to note, however, that this is not necessarily true for infinite-volume measures: in this case it may be that $\mathbb{P}^\star_\beta$ has disconnected graphs in its support.

Remark 4.A.2. The connected subgraph measure as defined above is a special case of a more general construction that occurs in the context of $q \to 0$ limits of the q-state random cluster model, see [53].

Given an event $A \subset \{0,1\}^{E(\mathbb{Z}^2)}$, we write $A_e = \{\omega \cup \{e\} \mid \omega \in A\}$ and $A^e = \{\omega \setminus \{e\} \mid \omega \in A\}$ for the events in which we add or remove the edge e, respectively.

Lemma 4.A.3. *For any arboreal gas measure $\mathbb{P}_\beta$, the dual measure $\mathbb{P}^\star_\beta$ is* insertion tolerant*, i.e., for $A \subset \{0,1\}^{E(\mathbb{Z}^2)}$ and any edge e,*

$$\mathbb{P}^\star_\beta[A_e] > 0 \qquad \textit{if } \mathbb{P}^\star_\beta[A] > 0. \tag{4.A.3}$$

Proof. This is equivalent to proving that the arboreal gas is deletion tolerant, i.e., that $\mathbb{P}_\beta[A^e] > 0$ if $\mathbb{P}_\beta[A] > 0$. We will need a standard notion of boundary conditions [41, Section 1.2.1]. In brief, for a finite-volume Λ, a *boundary condition* ω is a partition of the boundary vertices of Λ. Configurations are valid for a given boundary condition if they are forests after identifying each set of the partition together. For any finite-volume Λ, any boundary condition ω, and any forest F,

$$\mathbb{P}^\omega_{\Lambda,\beta}[F^e] \geqslant \min(1/\beta, 1)\mathbb{P}^\omega_{\Lambda,\beta}[F],$$

and hence the same inequality holds true for all events. Following a standard argument (e.g., [52, Theorem 4.17 (b)]) implies this inequality transfers to the infinite volume limit. $\square$

Recall that a *ray* is a semi-infinite self-avoiding walk. Two rays γ_1 and γ_2 are *equivalent* if there is no finite set of vertices X that separates infinitely many vertices of γ_1 from infinitely many vertices of γ_2. This is an equivalence relation, and equivalence classes are called *ends*.

Proposition 4.A.4. *For any translation invariant connected subgraph measure $\mathbb{P}^{\star}_{\beta}$ on $\mathbb{Z}^2$, the number of components is $\mathbb{P}^{\star}_{\beta}$-a.s. one. Further, the number of ends of the random subgraph with law $\mathbb{P}^{\star}_{\beta}$ is almost surely in $\{1,2\}$.*

Proof. Since translations act transitively on $\mathbb{Z}^2$, [68, Theorem 7.9] implies that there is at most one infinite component under $\mathbb{P}^{\star}_{\beta}$. To complete the proof of the first conclusion, note that for any fixed $K \in N$, for all sufficiently large volumes the finite-volume connected subgraph measures give probability zero to the existence of a cluster of size at most K.

The second claim is well known, see, e.g., [68, Exercise 7.24]. $\square$

Lemma 4.A.5. *For any infinite-volume translation invariant arboreal gas measure $\mathbb{P}_{\beta}$ there are at most two infinite trees.*

Proof. Note first that translation invariance of $\mathbb{P}_{\beta}$ implies translation invariance of $\mathbb{P}^{*}_{\beta}$. Next, we note that almost surely all infinite trees in the arboreal gas are one-ended: if not, there is a positive probability of the arboreal gas containing a bi-infinite path. The dual of this bi-infinite path is an edge cut of $\mathbb{Z}^2$, contradicting the almost sure connectedness of the dual of the arboreal gas from Proposition 4.A.4.

If the arboreal gas contains three infinite trees with positive probability, then there exist three disjoint semi-infinite paths γ_i with initial vertex x_i, $i = 1, 2, 3$. Fix a ball B containing the x_i, and note that the dual of the edges in $B \cup \bigcup_{i=1}^{3} \gamma_i$ divides $\mathbb{Z}^2$ into three connected components. Since the dual to the arboreal gas is connected, it contains an infinite path in each of these components, which implies it has at least three ends. By Proposition 4.A.4 this is a contradiction. $\square$

Proof of Corollary 4.1.4. Let T_0 denote the tree containing the origin. By translation invariance and ergodic decomposition, $\mathbb{P}_{\beta}[T_0$ is infinite$]$ is the density of the vertices in infinite trees. Moreover, by an adaptation of [23, Theorem 1], each individual infinite tree has a well-defined density. We now argue by contradiction. Suppose that $\mathbb{P}_{\beta}[0$ is in an infinite tree$] = p > 0$. By Lemma 4.A.5, this implies the existence of an infinite tree with a positive density, and hence of a $p' > 0$ such that

$$\mathbb{P}_{\beta}[T_0 \text{ has positive density}] = p'. \tag{4.A.4}$$

This is a contradiction, as Theorem 4.1.3 implies that the expected density of T_0 is zero in any infinite-volume limit. $\square$

4.B Rooted spanning forests and the uniform spanning tree

For the reader's convenience, we include a short summary of the well-known representation of *rooted* spanning forests and uniform spanning *trees* in the terms of the fermionic Gaussian free field (fGFF). We follow the notation of Section 4.2. The fGFF is the unnormalised expectation on $\Omega^{2\Lambda}$ defined by

$$[F]^{\mathrm{fGFF}}_{\beta,h} \equiv \left(\prod_{i\in\Lambda} \partial_{\eta_i}\partial_{\xi_i}\right) \exp\Big[(\boldsymbol{\xi}, \Delta_{\beta}\boldsymbol{\eta}) + (\boldsymbol{h}, \boldsymbol{\xi\eta})\Big] F. \tag{4.B.1}$$

where $\boldsymbol{\xi\eta} \equiv (\xi_i\eta_i)_i$. The normalised version is again denoted by $\langle\cdot\rangle^{\mathrm{fGFF}}_{\beta,h}$ if $[1]^{\mathrm{fGFF}}_{\beta,h} > 0$; see Section 4.2. It is straightforward that the fGFF is the properly normalised $\beta \to \infty$ limit of the $\mathbb{H}^{0|2}$ model as stated in the following fact; we omit the details.

Fact. For all weights $\boldsymbol{\beta}$ and $\boldsymbol{h}$,

$$[F(\boldsymbol{\xi},\boldsymbol{\eta})]^{\mathrm{fGFF}}_{\beta,h} = \lim_{\alpha\to\infty} \frac{1}{\alpha^{|\Lambda|}} \left[F(\sqrt{\alpha}\boldsymbol{\xi},\sqrt{\alpha}\boldsymbol{\eta})\right]_{\alpha\beta,\alpha h}, \tag{4.B.2}$$

where the unnormalised expectation on the right-hand side is that of the $\mathbb{H}^{0|2}$ model.

As a consequence of this fact and Theorem 4.2.1, the partition function of the fGFF can be expressed in terms of weighted *rooted* spanning forests. Let $\mathcal{F}_{\mathrm{root}}$ denote the set of all spanning forests together with a choice of root vertex in each tree of the forest.

Corollary 4.B.1. *For all weights $\boldsymbol{\beta}$ and $\boldsymbol{h}$,*

$$[1]^{\mathrm{fGFF}}_{\beta,h} = \sum_{F\in\mathcal{F}_{\mathrm{root}}} \prod_{(T,r)\in F} \left(\prod_{ij\in T} \beta_{ij}\right) h_r. \tag{4.B.3}$$

Corollary 4.B.1 also has an elementary proof: it can be seen as a consequence of the matrix-tree theorem.

The case of the uniform spanning tree (UST) is obtained by pinning the fGFF at a single arbitrary vertex which we denote 0. This corresponds to taking $h_j = 1_{j=0}$, or equivalently to adding a factor $\xi_0\eta_0$ inside the expectation. In analogy to Section 4.2, we denote the pinned expectation by an additional superscript 0, i.e.,

$$[F]^{\mathrm{fGFF},0}_{\beta} = [\xi_0\eta_0 F]^{\mathrm{fGFF}}_{\beta}. \tag{4.B.4}$$

The following corollary is then immediate from the previous one.

Corollary 4.B.2. *For all sets of edges S,*

$$\mathbb{P}^{\mathrm{UST}}_{\beta}[S] = \left[\prod_{ij\in S} \beta_{ij}(\xi_i-\xi_j)(\eta_i-\eta_j)\right]^{\mathrm{fGFF},0}_{\beta}. \tag{4.B.5}$$

For the UST, it is well-known that negative association holds, i.e., that the occurrence of disjoint edges ij, kl are negatively correlated. Various proofs exist, see e.g. [48, 53]. We include a new proof that mimics the proof of the Ginibre inequality [51].

Proposition 4.B.3. *For the uniform spanning tree, negative association holds: for all distinct ij and kl,*

$$\mathbb{P}^{\mathrm{UST}}_{\beta}[ij,kl] \leqslant \mathbb{P}^{\mathrm{UST}}_{\beta}[ij]\mathbb{P}^{\mathrm{UST}}_{\beta}[kl]. \tag{4.B.6}$$

Proof. Consider the doubled Grassman algebra $\Omega^{4\Lambda}$ with generators $\xi_i,\eta_i,\xi_i',\eta_i'$ where $i\in\Lambda'$. Abusing notation, we write $\langle\cdot\rangle$ for the product of the two fGFF expectations, i.e.,

$$\langle F(\xi,\eta)G(\xi',\eta')\rangle = \langle F(\xi,\eta)\rangle^{\mathrm{fGFF},0}\langle G(\xi,\eta)\rangle^{\mathrm{fGFF},0}. \tag{4.B.7}$$

Set $\chi_{ij} = (\xi_i-\xi_j)(\eta_i-\eta_j)$ and define χ_{ij}' analogously. Then

$$\mathbb{P}^{\mathrm{UST}}_{\beta}[ij,kl] - \mathbb{P}^{\mathrm{UST}}_{\beta}[ij]\mathbb{P}^{\mathrm{UST}}_{\beta}[kl] = \frac{1}{2}\beta_{ij}\beta_{kl}\langle(\chi_{ij}-\chi_{ij}')(\chi_{kl}-\chi_{kl}')\rangle. \tag{4.B.8}$$

Mimicking Ginibre [51], we change generators in $\Omega^{4\Lambda}$ according to

$$\xi_i \mapsto \frac{1}{\sqrt{2}}(\xi_i+\xi_i'), \quad \eta_i \mapsto \frac{1}{\sqrt{2}}(\eta_i+\eta_i'), \quad \xi_i' \mapsto \frac{1}{\sqrt{2}}(\xi_i-\xi_i'), \quad \eta_i' \mapsto \frac{1}{\sqrt{2}}(\eta_i-\eta_i'). \tag{4.B.9}$$

The action defining the product of two fGFFs is invariant under this change of generator and the integrand of the RHS of (4.B.8) transforms as

$$\begin{aligned}(\chi_{ij}-\chi'_{ij})(\chi_{kl}-\chi'_{kl}) \mapsto &-(\xi_i-\xi_j)(\xi_k-\xi_l)(\eta'_i-\eta'_j)(\eta'_k-\eta'_l)\\ &-(\eta_i-\eta_j)(\eta_k-\eta_l)(\xi'_i-\xi'_j)(\xi'_k-\xi'_l)\\ &-(\xi_i-\xi_j)(\eta_k-\eta_l)(\xi'_k-\xi'_l)(\eta'_i-\eta'_j)\\ &-(\xi_k-\xi_l)(\eta_i-\eta_j)(\xi'_i-\xi'_j)(\eta'_k-\eta'_l).\end{aligned} \tag{4.B.10}$$

Taking the expectation, only the last two terms contribute since only monomials with the same number of factors of ξ as η have non-vanishing expectation, e.g., $\langle \xi_i\xi_j\rangle^{\mathrm{fGFF}} = 0$. These last two terms give the same expectation:

$$\mathbb{P}^{\mathrm{UST}}_\beta[ij,kl]-\mathbb{P}^{\mathrm{UST}}_\beta[ij]\mathbb{P}^{\mathrm{UST}}_\beta[kl] = -\beta_{ij}\beta_{kl}\langle(\xi_i-\xi_j)(\eta_k-\eta_l)\rangle^{\mathrm{fGFF},0}\langle(\xi_k-\xi_l)(\eta_i-\eta_j)\rangle^{\mathrm{fGFF},0}. \tag{4.B.11}$$

By (4.2.27) the two terms in the product on the right-hand side are equal, and hence the right-hand side is non-positive. □

Remark 4.B.4. The right-hand side in (4.B.11) gives an alternate expression for the deficit $\Delta^2_{ij,kl}$ that occurs in [48, Theorem 2.1].

Chapter 5

Dynkin isomorphism and Mermin–Wagner theorems for hyperbolic sigma models and recurrence of the two-dimensional vertex-reinforced jump process

5.1 Introduction and results

Introduction

Our results have motivation from two different perspectives, that of sigma models with hyperbolic symmetry and their relevance for the Anderson transition, and that of a model of reinforced random walks known as the *vertex-reinforced jump process* (VRJP).

The VRJP was originally introduced by Werner and has attracted a great deal of attention recently [30, 37, 38, 89, 92]. The VRJP on a vertex set Λ is a continuous-time random walk that jumps from a vertex i to a neighbouring vertex j at time t with rate $\beta_{ij}(1 + L_t^j)$, where L_t^j is the local time of j at time t and $\beta_{ij} \geqslant 0$ are the initial rates. One should view Λ as the vertex set of an undirected graph with edge set $E = \{\langle ij \rangle \mid \beta_{ij} > 0\}$. The dependence of the jump rates on the local time leads the VRJP to be attracted to itself.

One of our new results is the following theorem.

Theorem 5.1.1. *Consider a vertex-reinforced jump process (X_t) on the vertex set $\mathbb{Z}^d$ with initial rates β that are finite-range and translation invariant. If $d = 1, 2$ then (X_t) is recurrent in the sense that the expected time (X_t) spends at the origin is infinite.*

As the VRJP is not a Markov process, different notions of recurrence are not *a priori* equivalent. For example, another natural notion of recurrence would be to ask if the VRJP visits the origin infinitely often almost surely. For non-Markovian processes neither of these definitions of recurrence implies the other: there may be infinitely many visits to the origin with the increments of the local time being summable. To the best of our knowledge, neither implication is known for the VRJP.

For sufficiently small initial rates recurrence results for the VRJP have previously been established [5, 39, 89]. These results are for recurrence in the sense of visiting the origin infinitely often almost surely. See [5] for a discussion and precise statements. It has also been shown that the linearly edge-reinforced random walk (ERRW) with constant initial weights is recurrent in two dimensions [75, 92], but the recurrence of the VRJP for all initial rates was an open problem until the present work. The relation between the ERRW and VRJP is discussed below.

Theorem 5.1.1 is in fact a consequence of our proof of a Mermin–Wagner theorem for hyperbolic sigma models and a new and very direct relation between VRJPs and hyperbolic sigma models that parallels the well-known relationship between simple random walks and Gaussian free fields (the BFS–Dynkin isomorphism theorem).

Before giving precise definitions of our models and stating our results, we briefly indicate the motivations behind hyperbolic sigma models, and their relations with reinforced random walks. We also explain some consequences of our results for hyperbolic sigma models. Readers primarily interested in the VRJP may wish to skip ahead to Section 5.1.

Hyperbolic sigma models were introduced as effective models to understand the Anderson transition [40, 97–99, 105]. In Efetov's supersymmetric method [45] the expected absolute value squared of the resolvent of random band matrices, i.e., $\mathbb{E}|(H-z)^{-1}(i,j)|^2$ where $z \in \mathbb{C}_+$ and H is a random band matrix, can be expressed as a correlation function of a supersymmetric spin model. The spins of this model are invariant under the hyperbolic symmetry $OSp(2,1|2)$. Extended states correspond to spontaneous breaking of this non-compact symmetry. The supersymmetric hyperbolic sigma model, or $\mathbb{H}^{2|2}$ model, was introduced by Zirnbauer [105] and first studied by Disertori, Spencer and Zirnbauer [40]. It is an approximation of the random band matrix model above where radial fluctuations are neglected. This is similar to how the $O(n)$ model is an approximation of models of $\mathbb{R}^n$-valued spins with rotational symmetry such as $|\varphi|^4$-theories. More detailed motivation for hyperbolic spin models is given in [97, 99].

The $\mathbb{H}^{2|2}$ model is believed to capture the physics of the Anderson transition. As is expected for the Anderson model, it was proved in [40] that the $OSp(2,1|2)$ symmetry of the $\mathbb{H}^{2|2}$ model is spontaneously broken in $d \geqslant 3$ for sufficiently small disorder — consistent with the existence of extended states. Furthermore, it was proved [39] that for sufficiently large disorder this is not the case — consistent with Anderson localisation. In dimension $d \leqslant 2$, it is conjectured that extended states do not exist for any disorder strength. Equation (5.1.16) below is the corresponding statement for the $\mathbb{H}^{2|2}$ model, and we have thus completed the expected qualitative picture for the phase diagram of the $\mathbb{H}^{2|2}$ model; see Remark 5.1.9 for a discussion of the conjectured optimal bounds. Equation (5.1.16) can be considered as a version of the Mermin–Wagner theorem. For recent and extremely precise results in dimension one, see [96].

Based on the similarity of certain explicit formulas, it was suggested that there is a connection between the $\mathbb{H}^{2|2}$ model and linearly edge-reinforced random walks [40]. This connection was first confirmed in [89] by relating marginals of the $\mathbb{H}^{2|2}$ model to the limiting local time profile of a time change of the VRJP. It was also shown there that the linearly edge reinforced walk is obtained from the VRJP when averaging over random initial rates. Further marginals of the $\mathbb{H}^{2|2}$ model were explored in [37]. For a discussion of the history of the VRJP, see [89].

Our hyperbolic analogue of the BFS–Dynkin isomorphism theorem, Theorem 5.1.2 below, is a different relation between the $\mathbb{H}^{2|2}$ model and the VRJP than was found in [89], and it provides a more direct relation between the correlation structures of the models. Moreover, our statement also applies without supersymmetry, i.e., when the spins take values in $\mathbb{H}^n$. We will explain further extensions of Theorem 5.1.2 in the case of $\mathbb{H}^n$, e.g., to multipoint correlations, in a forthcoming publication.

Model definitions

We now define the VRJP and the hyperbolic sigma models. The walk and the sigma models are both defined in terms of a set Λ of vertices and non-negative edge weights $\beta = (\beta_{ij})_{i,j\in\Lambda}$, where by edge weights we mean that $\beta_{ij} = \beta_{ji}$. For our Mermin–Wagner theorem we will make use of two assumptions on β. We call β *finite-range* if for each $i \in \Lambda$ we have $\beta_{ij} = 0$ for all but finitely many j. If $\Lambda = \mathbb{Z}^d$ we call β *translation invariant* if $\beta_{ij} = \beta_{T(i)T(j)}$ for all translations T of $\mathbb{Z}^d$.

Vertex-reinforced jump process. Let Λ be a finite or countable set. The VRJP is a history-dependent continuous-time random walk (X_t) on Λ that takes jumps from vertex i to vertex j with rate

$\beta_{ij}(1+L_t^j)$, where

$$L_t^j \equiv \int_0^t 1_{X_s=j}\, ds. \tag{5.1.1}$$

L_t^j is called the *local time* of the walk at vertex j up to time t. We will write $L_t \equiv (L_t^i)_{i\in\Lambda}$ for the collection of local times. It will also be useful to consider the joint process (X_t, L_t), which is a Markov process with generator $\mathcal{L}$ acting on sufficiently nice functions $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ by

$$\mathcal{L}^\beta g(i,\ell) = \sum_j \beta_{ij}(1+\ell_j)(g(j,\ell)-g(i,\ell)) + \frac{\partial}{\partial \ell_i} g(i,\ell), \quad i\in\Lambda,\ \ \ell \in \mathbb{R}^\Lambda. \tag{5.1.2}$$

We denote by $\mathbb{E}^\beta_{i,\ell}$ the expectation of the process (X_t, L_t) with initial condition $X_0 = i$ and $L_0 = \ell$. The VRJP is the marginal of X_t in the special case $L_0 = 0$; by a slight abuse of terminology we call (X_t, L_t) the VRJP as well.

Hyperbolic sigma models. Let $\mathbb{R}^{n,1}$ denote $(n+1)$-dimensional Minkowski space. Its elements are vectors $u = (x, y^1, \dots, y^{n-1}, z)$, and it is equipped with the indefinite inner product $u\cdot u = x^2 + (y^1)^2 + \cdots + (y^{n-1})^2 - z^2$. Note that although x plays the same role as the y^i, we distinguish it in our notation for later convenience. Recall that n-dimensional hyperbolic space $\mathbb{H}^n$ can be realized as

$$\mathbb{H}^n \equiv \{u \in \mathbb{R}^{n,1} \mid u\cdot u = -1, z>0\}. \tag{5.1.3}$$

Suppose Λ is finite and $h>0$. To each vertex $i\in\Lambda$ we associate a spin $u_i \in \mathbb{H}^n$. The energy of a spin configuration $u = (u_i)_{i\in\Lambda} \in (\mathbb{H}^n)^\Lambda$ is

$$H(u) = H_{\beta,h}(u) \equiv \sum_{\langle ij\rangle} \beta_{ij}(-u_i\cdot u_j - 1) + h\sum_j (z_j - 1), \tag{5.1.4}$$

where the sum is over edges $\langle ij\rangle$; since the summands are symmetric in i and j this notation will not cause any confusion. The $\mathbb{H}^n$ sigma model is the measure with density proportional to $e^{-H(u)}$ with respect to the $|\Lambda|$-fold product of the measure μ on $\mathbb{H}^n$ induced by the Minkowski metric, see (5.2.2) and (5.2.4) for explicit expressions, and we let $\langle\cdot\rangle_{\mathbb{H}^n}$ denote the expectation associated to this model:

$$\langle F(u)\rangle_{\mathbb{H}^n} \equiv \frac{\int_{(\mathbb{H}^n)^\Lambda} F(u)\, e^{-H(u)}\, \mu^{\otimes\Lambda}(du)}{\int_{(\mathbb{H}^n)^\Lambda} e^{-H(u)}\, \mu^{\otimes\Lambda}(du)}. \tag{5.1.5}$$

The energy (5.1.4) favours spin alignment because $u\cdot v \leqslant -1$ for $u, v\in\mathbb{H}^n$ with equality if and only if $u = v$.

Supersymmetric hyperbolic sigma model. In this section we will introduce a probability measure which enables the computation of a special class of observables of the full supersymmetric $\mathbb{H}^{2|2}$ model. These restricted observables will suffice for a description of a special, but interesting, case of our results. Our most general results use the full supersymmetric formalism.

As will be explained further in Section 5.2, at each vertex $i\in\Lambda$ there is a superspin $u_i = (x_i, y_i, z_i, \xi_i, \eta_i) \in \mathbb{H}^{2|2}$ where ξ_i and η_i are Grassmann variables. For the moment all that is needed is that the expectation of a function $F(y)$ of the $y \equiv (y_i)_{i\in\Lambda}$ coordinates can be written as

$$\langle F(y)\rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{R}^2)^\Lambda} F(e^t s)\, e^{-\widetilde{H}(s,t)}\, dt\, ds, \tag{5.1.6}$$

where $dt\,ds \equiv \prod_i dt_i\, ds_i$, $e^t s \equiv (e^{t_i} s_i)_{i\in\Lambda}$,

$$\begin{aligned}\widetilde{H}(s,t) = \widetilde{H}_{\beta,h}(s,t) &\equiv \sum_{\langle ij\rangle} \beta_{ij}\Big(\cosh(t_i - t_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{t_i+t_j}\Big)\\ &\quad + h\sum_i \Big(\cosh(t_i) - 1 + \frac{1}{2}s_i^2 e^{t_i}\Big) + \sum_i (t_i + \log(2\pi)) - \log\det D_{\beta,h}(t),\end{aligned} \tag{5.1.7}$$

and the matrix $D_{\beta,h}(t)$ on $\mathbb{R}^\Lambda$ is defined by the quadratic form

$$(v, D_{\beta,h}(t)v) \equiv \sum_{\langle ij \rangle} \beta_{ij} e^{t_i+t_j}(v_i - v_j)^2 + h \sum_i e^{t_i} v_i^2, \quad v \in \mathbb{R}^\Lambda. \tag{5.1.8}$$

The determinant $\det D_{\beta,h}(t)$ does not depend on the s variables and it is positive since $D_{\beta,h}(t)$ is positive definite. Thus $e^{-\widetilde{H}(s,t)} dt\, ds$ is a positive measure, and we will show in Section 5.2 that it is in fact a probability measure, i.e., $\langle 1 \rangle_{\mathbb{H}^{2|2}} = 1$.

Results

We now state our main results and show how Theorem 5.1.1 is a consequence.

Hyperbolic BFS–Dynkin Isomorphism. The following theorem is a hyperbolic analogue of the Dynkin isomorphism theorem, which relates the local times of a simple random walk to the square of a Gaussian free field. As the Dynkin isomorphism theorem was proved by Brydges–Fröhlich–Spencer in [16, Theorem 2.2], and later expressed in a better form by Dynkin [43], we prefer to call it the BFS–Dynkin isomorphism. The general idea of relating Gaussian fields to simple random walks is due to Symanzik [101]. For recent discussions of these ideas see [64, 102]. Supersymmetric versions of these results for simple random walks go back to Luttinger and Le Jan [63, 67].

Note that while we have not yet defined the meaning of $\langle g \rangle_{\mathbb{H}^{2|2}}$ for a general function g, we have given a meaning in the case that g is identically one by (5.1.6). It is this case of g identically one that will be most relevant for the VRJP.

Theorem 5.1.2. *Suppose Λ is finite and β is a collection of non-negative edge weights. Let $h > 0$, let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be any bounded smooth function, and let $a, b \in \Lambda$. Consider the $\mathbb{H}^n$ model, $n \geqslant 2$, let $y = (y_i)_{i\in\Lambda} = (y_i^r)_{i\in\Lambda}$ for some $r = 1, \dots, n-1$, and $z = (z_i)_{i\in\Lambda}$. Then*

$$\sum_b \langle y_a y_b g(b, z-1) \rangle_{\mathbb{H}^n} = \langle z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t, L_t))\, e^{-ht}\, dt \rangle_{\mathbb{H}^n}. \tag{5.1.9}$$

For the $\mathbb{H}^{2|2}$ model, we have

$$\sum_b \langle y_a y_b g(b, z-1) \rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t, L_t))\, e^{-ht}\, dt. \tag{5.1.10}$$

Remark 5.1.3. Theorem 5.1.2 also holds for the $\mathbb{H}^1$ model, but as the proof requires slightly different considerations we have not included it here.

Taking the function g to be identically one in (5.1.10) implies that

$$\langle y_a y_b \rangle_{\mathbb{H}^{2|2}} = \int_0^\infty \mathbb{E}^\beta_{a,0}(1_{X_t = b})\, e^{-ht}\, dt. \tag{5.1.11}$$

The right-hand side can be interpreted as the two-point function of the VRJP with a uniform killing rate h.

Remark 5.1.4. Theorem 5.1.2 can be extended in a straightforward way to the case in which $h = (h_i)_{i\in\Lambda}$ is non-constant, provided $h_i \geqslant 0$ and at least one value is strictly positive.

Hyperbolic Mermin–Wagner Theorem. In this section we assume that $\Lambda = \Lambda_L$ is the discrete d-dimensional torus $\mathbb{Z}^d/(L\mathbb{Z})^d$ of side length $L \in \mathbb{N}$, and that β is translation invariant and finite-range. We will write $\langle\cdot\rangle = \langle\cdot\rangle_{\beta,h}$ in place of $\langle\cdot\rangle_{\mathbb{H}^n}$ and $\langle\cdot\rangle_{\mathbb{H}^{2|2}}$. Denote

$$\lambda(p) \equiv \sum_{j\in\Lambda} \beta_{0j}(1-\cos(p\cdot j)), \qquad p \in \Lambda^\star, \tag{5.1.12}$$

where here $\cdot$ is the Euclidean inner product on $\mathbb{R}^d$ and $\Lambda^\star$ is the Fourier dual of the discrete torus Λ. Denote the two-point function and its Fourier transform by

$$G_{\beta,h}(j) = G^L_{\beta,h}(j) \equiv \langle y_0 y_j\rangle_{\beta,h}, \qquad \hat{G}_{\beta,h}(p) = \hat{G}^L_{\beta,h}(p) = \sum_{j\in\Lambda} G_{\beta,h}(j)e^{i(p\cdot j)}. \tag{5.1.13}$$

The following theorem is an analogue of the Mermin–Wagner Theorem for the $O(n)$ model, in the form presented in [50].

Theorem 5.1.5. *Let $\Lambda = \mathbb{Z}^d/(L\mathbb{Z})^d$, $L \in \mathbb{N}$. For the $\mathbb{H}^n$ model, $n \geqslant 2$, with magnetic field $h > 0$,*

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1+(n+1)G_{\beta,h}(0))\lambda(p)+h}. \tag{5.1.14}$$

Similarly, for the $\mathbb{H}^{2|2}$ model with $h > 0$,

$$\hat{G}_{\beta,h}(p) \geqslant \frac{1}{(1+G_{\beta,h}(0))\lambda(p)+h}. \tag{5.1.15}$$

Remark 5.1.6. By (5.1.11) the two-point function $G_{\beta,h}$ equals that of the VRJP in the case of the $\mathbb{H}^{2|2}$ model, and hence the two-point function of the VRJP satisfies (5.1.15) as well.

Remark 5.1.7. For $d \geqslant 3$, the bound (5.1.15) shows that $\tilde{f}$ can be replaced by f in [40, Theorem 3] using the upper bound proved there for $G_{\beta,h}(0)$.

Corollary 5.1.8. *Under the assumptions of Theorem 5.1.5, for $d = 1, 2$,*

$$\lim_{h\downarrow 0}\lim_{L\to\infty} G_{\beta,h}(0) = \infty. \tag{5.1.16}$$

Proof. Since $(2\pi L)^{-d}\sum_{p\in\Lambda^\star} e^{i(p\cdot j)} = 1_{j=0}$, summing the bounds (5.1.14) and (5.1.15) over $p \in \Lambda^\star$ and interchanging sums implies (with $n = 0$ for $\mathbb{H}^{2|2}$)

$$G_{\beta,h}(0) \geqslant \frac{1}{(2\pi L)^d}\sum_{p\in\Lambda^\star}\frac{1}{(1+(n+1)G_{\beta,h}(0))\lambda(p)+h}. \tag{5.1.17}$$

The assumption of β being finite-range and non-negative implies $\lambda(p) \leqslant C(\beta)|p|^2$. If $d \leqslant 2$ it follows that

$$\lim_{L\to\infty}\frac{1}{(2\pi L)^d}\sum_{p\in\Lambda^\star}\frac{1}{\lambda(p)+h} \uparrow \infty \quad \text{as } h \downarrow 0, \tag{5.1.18}$$

and, as $G_{\beta,h} \geqslant 0$, this implies (5.1.16). □

Remark 5.1.9. In fact, the proof shows $G_{\beta,h}(0) \geqslant c_\beta/\sqrt{\log h}$ with $c_\beta > 0$ when $h > 0$ is small. For the $\mathbb{H}^{2|2}$ model, we conjecture that the optimal bound is $G_{\beta,h}(0) \asymp c_\beta/h$ for h small, with $c_\beta > 0$ exponentially small as β becomes large. This is consistent with Anderson localisation. On the other hand, for the $\mathbb{H}^n$ model with $n \geqslant 2$, localisation is not expected, i.e., $G_{\beta,h}(0) \ll 1/h$.

Consequences for the vertex-reinforced jump process. In contrast to Corollary 5.1.8, it has been proven [40, 99] that when $d \geqslant 3$ and $\beta_{ij} = \beta 1_{|i-j|=1}$,

$$\lim_{h \downarrow 0} \lim_{L \to \infty} G_{\beta,h}(0) < \infty \tag{5.1.19}$$

for all $\beta > 0$ in the case of $\mathbb{H}^2$ and for all sufficiently large $\beta > 0$ for $\mathbb{H}^{2|2}$. In the $\mathbb{H}^{2|2}$ case (5.1.19) corresponds to transience of the VRJP (in the sense of bounded expected local time, see Corollary 5.1.10 below) and to the uniform boundedness (in the spectral parameter $z \in \mathbb{C}_+$) of the expected square of the absolute value of the resolvent for random band matrices in the sigma model approximation [97] (recall Section 5.1). It also implies that the hyperbolic symmetry is spontaneously broken.

Due to the non-amenability of hyperbolic group actions, the question of spontaneous symmetry breaking for hyperbolic sigma models is, in general, subtle. The usual formulations of the Mermin–Wagner theorem for models with compact symmetries cannot hold in the non-amenable case [94], and, in fact, spontaneous symmetry breaking appears to occur in all dimensions [42, 81]. Nonetheless, (5.1.16) and (5.1.19) show that the two-point function — the observable of interest for the VRJP and the random matrix problem — does undergo a transition analogous to that occurring in systems with compact symmetries.

Proof of Theorem 5.1.1. We must prove that for any translation invariant finite-range β

$$\int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, dt = \infty, \tag{5.1.20}$$

where the expectation refers to that of the VRJP on $\mathbb{Z}^d$ and $d = 1, 2$. This is true since, for any finite-range β, one has

$$\begin{aligned} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, dt &= \lim_{h \downarrow 0} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\mathbb{Z}^d}(1_{X_t=0})\, e^{-ht}\, dt \\ &= \lim_{h \downarrow 0} \lim_{L \to \infty} \int_0^\infty \mathbb{E}_{0,0}^{\beta,\Lambda_L}(1_{X_t=0})\, e^{-ht}\, dt = \infty. \end{aligned} \tag{5.1.21}$$

The first equality is by monotone convergence, and the final equality is obtained by combining (5.1.16) for the $\mathbb{H}^{2|2}$ model and (5.1.11).

For the second equality it suffices, by using the tail of the exponential e^{-ht}, to verify that the integrand converges for $t \leqslant T$ for any bounded T. Since the jump rate $1 + L_t^i$ is bounded by $1 + T$, the walk is exponentially unlikely to take more than $O(T^3)$ jumps to new vertices up to time T. VRJPs on Λ_L and $\mathbb{Z}^d$ can be coupled to be the same until they exit a ball of radius less than $\frac{1}{2}L$, an event which requires at least L/R jumps to occur, where R is the radius of the finite-range step distribution. This completes the proof. □

The analogue of Theorem 5.1.1 for the ERRW with constant initial weights was established in [75, 92], but not for the VRJP. Mermin–Wagner type theorems have also been proven for the ERRW in one and two dimensions [74, 75]. The techniques used deal directly with ERRWs, and hence are rather different from those employed in this paper.

Our relation between the two-point functions of the $\mathbb{H}^{2|2}$ model and the VRJP also yields a transience result.

Corollary 5.1.10. *The vertex-reinforced jump process (X_t) on $\mathbb{Z}^d$, $d \geqslant 3$, with initial rates $\beta_{ij} = \beta 1_{|i-j|=1}$ and β sufficiently large is transient, in the sense that the expected time (X_t) spends at the origin is finite.*

Proof. The argument mirrors the proof of Theorem 5.1.1, using (5.1.19) in place of (5.1.16). □

Transience in the sense of visiting the origin finitely often almost surely when β is sufficiently large was established in [89, Corollary 4]; this result also makes use of [40]. As with recurrence, see the discussion following the statement of Theorem 5.1.1, there is in general no relation between the two notions of transience.

5.2 Supersymmetry and horospherical coordinates

In this section we define horospherical coordinates for $\mathbb{H}^n$ and then define the supersymmetric $\mathbb{H}^{2|2}$ model precisely. We also collect Ward identities and relations between derivatives that will be used in the proofs of Theorems 5.1.2 and 5.1.5.

Horospherical coordinates

As observed in [99, 105], the hyperbolic spaces $\mathbb{H}^n$ are naturally parametrised by horospherical coordinates that are useful for the analysis of the corresponding sigma models. For $\mathbb{H}^n$, these are global coordinates $t \in \mathbb{R}$, $\tilde{s} \in \mathbb{R}^{n-1}$, in terms of which

$$x = \sinh t - \frac{1}{2}|\tilde{s}|^2 e^t, \quad y^i = e^t s^i \quad (i = 1, \dots, n-1), \quad z = \cosh t + \frac{1}{2}|\tilde{s}|^2 e^t. \tag{5.2.1}$$

Both x, z are scalars while $\tilde{y} = (y^1, \dots, y^{n-1})$ and $\tilde{s} = (s^1, \dots, s^{n-1}) \in \mathbb{R}^{n-1}$ are $n-1$ dimensional vectors and $|\tilde{s}|^2 = \sum_{i=1}^{n-1}(s^i)^2$. By this change of variables one has (see Appendix 5.A),

$$\int_{(\mathbb{H}^n)^\Lambda} F(u)\, \mu^{\otimes\Lambda}(du) = \int_{(\mathbb{R}^n)^\Lambda} F(u(\tilde{s}, t)) \prod_i e^{(n-1)t_i}\, dt_i\, d\tilde{s}_i. \tag{5.2.2}$$

By a short calculation,

$$-u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j}, \qquad z_i = \cosh t_i + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i}. \tag{5.2.3}$$

Thus in horospherical coordinates,

$$H(\tilde{s}, t) = \sum_{\langle ij \rangle} \beta_{ij} \left(\cosh(t_i - t_j) - 1 + \frac{1}{2}|\tilde{s}_i - \tilde{s}_j|^2 e^{t_i + t_j} \right) + h \sum_i \left(\cosh(t_i) - 1 + \frac{1}{2}|\tilde{s}_i|^2 e^{t_i} \right), \tag{5.2.4}$$

where by a slight abuse of notation we have re-used the symbol H. Moreover, the following relations, in which we set $s_i = s_i^r$ and $y_i = y_i^r$ for some fixed $r = 1, \dots, n-1$, hold:

$$\frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j). \tag{5.2.5}$$

Furthermore,

$$\begin{gathered} \frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j, \\ \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j + t_l} = -(x_j + z_j)(x_l + z_l), & i = j, \\ +e^{t_j + t_l} = +(x_j + z_j)(x_l + z_l), & i = l, \\ 0, & \text{else.} \end{cases} \end{gathered} \tag{5.2.6}$$

Supersymmetry

Let Λ be a finite set. We will define an algebra Ω_Λ of forms (which generalise random variables) that constitute the observables on the super-space $(\mathbb{R}^{2|2})^\Lambda$. The super-space itself only has meaning through this algebra of observables. We also define an integral associated to this algebra. We then introduce the supersymmetry generator and the localisation lemma. For a more detailed introduction to the mathematics of supersymmetry, see, e.g., [15, 21, 40].

Supersymmetric integration. For each vertex $i \in \Lambda$, let x_i, y_i be real variables and ξ_i, η_i be two Grassmann variables. Thus by definition all of the x_i and y_i commute with each other and with all of the ξ_i and η_i and all of the ξ_i and η_i anticommute. The way in which the anticommutation relations are realized is unimportant, but concretely, we can define an algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices ξ_i and η_i realising the required anticommutation relations for the Grassmann variables. To fix signs in forthcoming expressions, fix an arbitrary order $i_1, \dots, i_{|\Lambda|}$ of the vertices in Λ.

We define the algebra Ω_Λ to be the algebra of smooth functions on $(\mathbb{R}^2)^\Lambda$ with values in the algebra of $4^{|\Lambda|} \times 4^{|\Lambda|}$ matrices that have the form

$$F = \sum_{I,J \subset \Lambda} F_{I,J}(x,y)(\eta\xi)_{I,J}, \tag{5.2.7}$$

where the coefficients $F_{I,J}$ are smooth functions on $(\mathbb{R}^2)^\Lambda$, and $(\eta\xi)_{I,J}$ is given by the ordered product $\prod_{i\in I\cap J} \eta_i\xi_i \prod_{i\in I\setminus J} \xi_i \prod_{j \in J\setminus I} \eta_j$. This ordering has been chosen so that $(\eta\xi)_{\Lambda,\Lambda}$ is $\eta_1\xi_1 \dots \eta_\Lambda\xi_\Lambda$. We call elements of Ω_Λ *forms* because the forms of differential geometry are instances [21, 63]. The integral (sometimes called a superintegral) of a form $F \in \Omega_\Lambda$ is defined by

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^2)^\Lambda} F_{\Lambda,\Lambda}(x,y) \prod_{i\in\Lambda} \frac{dx_i\, dy_i}{2\pi}, \tag{5.2.8}$$

where $\mathbb{R}^{2|2}$ refers to the number of commuting and anticommuting variables.

The *degree* of a coefficient $F_{I,J}$ is $|I| + |J|$. Thus the integral of a form F is a constant multiple of the usual Lebesgue integral of the top degree part of F. A form $F \in \Omega_\Lambda$ is *even* if the degree of all non-vanishing coefficients $F_{I,J}$ is even in (5.2.7). Even forms commute. For even forms $F^1, \dots, F^p$ and a smooth function $g \in C^\infty(\mathbb{R}^p)$, the form $g(F^1, \dots, F^p) \in \Omega_\Lambda$ is defined by formally Taylor expanding g about the degree-0 part $(F^1_{\varnothing,\varnothing}(x,y), \dots, F^p_{\varnothing,\varnothing}(x,y))$. This is well-defined as there is no ambiguity in the ordering if the F^i are all even, and the anticommutation relations satisfied by the ξ_i and η_i imply the expansion is finite.

Localisation. Temporarily set $x = x_i, y = y_i, \xi = \xi_i$, and $\eta = \eta_i$. Define an operator $\partial_\eta \colon \Omega_\Lambda \to \Omega_\Lambda$ by linearity, $\partial_\eta(\eta F) = F$, and $\partial_\eta F = 0$ if F does not contain a factor η. Define ∂_ξ in the same manner. Define Q_i by its action on forms F by

$$Q_i F \equiv \xi\partial_x F + \eta\partial_y F + x\partial_\eta F - y\partial_\xi F. \tag{5.2.9}$$

The *supersymmetry generator* Q acts on a form $F \in \Omega_\Lambda$ by $QF \equiv \sum_{i\in\Lambda} Q_i F$.

Definition 5.2.1. $F \in \Omega_\Lambda$ is *supersymmetric* if $QF = 0$.

The supersymmetry generator acts as an anti-derivation on the algebra of forms, see, e.g., [21, Section 6]. This implies that the forms

$$\tau_{ji} = \tau_{ij} \equiv x_i x_j + y_i y_j + \xi_i\eta_j - \eta_i\xi_j, \qquad i,j \in \Lambda, \tag{5.2.10}$$

are supersymmetric. Moreover, any smooth function of the τ_{ij} is supersymmetric as Q obeys a chain rule, see [21, Equation (6.5)]. The following *localisation lemma* is fundamental. For a proof, see [40, Lemma 16].

Lemma 5.2.2 (Localisation lemma)**.** *Let $F \in \Omega_\Lambda$ be a smooth form with sufficient decay that is supersymmetric, i.e., satisfies $QF = 0$. Then*

$$\int_{(\mathbb{R}^{2|2})^\Lambda} F = F_{\varnothing,\varnothing}(0,0). \tag{5.2.11}$$

The $\mathbb{H}^{2|2}$ model

We can now define the $\mathbb{H}^{2|2}$ sigma model and justify our earlier claim that its y marginal is the probability measure (5.1.6). Given $(x_i, y_i, \xi_i, \eta_i)$ as above define an even variable z_i by

$$z_i \equiv \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i} = \sqrt{1 + x_i^2 + y_i^2} + \frac{\xi_i\eta_i}{\sqrt{1 + x_i^2 + y_i^2}}, \tag{5.2.12}$$

where the equality is by the definition of a function of a form. We will write $u_i = (x_i, y_i, z_i, \xi_i, \eta_i)$. Define the "inner product"

$$u_i \cdot u_j \equiv x_i x_j + y_i y_j - z_i z_j + \xi_i \eta_j - \eta_i \xi_j, \tag{5.2.13}$$

generalising the Minkowski inner product above (5.1.3); we have written "inner product" as this is only terminology, since (5.2.13) is not a quadratic form in the classical sense. Then by a short calculation

$$u_i \cdot u_i = -1, \tag{5.2.14}$$

which we interpret as meaning that u_i is in the supermanifold $\mathbb{H}^{2|2}$. Since $z_i = \sqrt{1+\tau_{ii}}$ and $u_i \cdot u_j = \tau_{ij} - z_i z_j$, the forms $u_i \cdot u_j$ and z_i are supersymmetric for all $i, j \in \Lambda$.

The $\mathbb{H}^{2|2}$ integral of a form $F \in \Omega_\Lambda$ is defined by

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F \equiv \int_{(\mathbb{R}^{2|2})^\Lambda} F \prod_{i\in\Lambda} \frac{1}{z_i}, \tag{5.2.15}$$

and the $\mathbb{H}^{2|2}$ model is defined by the following action (which is now a form in Ω_Λ)

$$H \equiv H_{\beta,h} = \sum_{\langle ij\rangle} \beta_{ij}(-u_i \cdot u_j - 1) + h\sum_i (z_i - 1) \in \Omega_\Lambda. \tag{5.2.16}$$

Lastly, we define the super-expectation of an observable $F \in \Omega_\Lambda$ in the $\mathbb{H}^{2|2}$ model by

$$\langle F\rangle_{\mathbb{H}^{2|2}} \equiv \int_{(\mathbb{H}^{2|2})^\Lambda} F e^{-H}. \tag{5.2.17}$$

Lemma 5.2.2 implies that $\langle 1\rangle_{\mathbb{H}^{2|2}} = 1$, as promised in Section 5.1.

Supersymmetric horospherical coordinates

The $\mathbb{H}^{2|2}$ model can also be reparametrised in a supersymmetric version of horospherical coordinates [40, Sec. 2.2]. For the convenience of the reader, the explicit change of variables is computed in Appendix 5.A. In this parametrisation, t and s are two real variables and $\bar\psi$ and ψ are two Grassmann variables. As in the previous section, we denote the algebra of such forms by $\widetilde\Omega_\Lambda$. The tilde refers to horospherical coordinates. We write

$$\begin{gathered} x = \sinh t - e^t(\tfrac{1}{2}s^2 + \bar\psi\psi), \quad y = e^t s, \quad z = \cosh t + e^t(\tfrac{1}{2}s^2 + \bar\psi\psi), \\ \xi = e^t\bar\psi, \quad \eta = e^t\psi. \end{gathered} \tag{5.2.18}$$

There is a generalisation of the change of variables formula from standard integration to super-integration. We only require the following special case given in [40, Sec. 2.2] and Appendix 5.A. Forms $F \in \Omega_\Lambda$ are in correspondence with forms $\widetilde F \in \widetilde\Omega_\Lambda$ obtained by substituting the relations (5.2.18) into (5.2.7) using the definition of functions of forms. Moreover, expanding

$$\widetilde F = \sum_{I,J\subset\Lambda} \widetilde F_{I,J}(t,s)(\psi\bar\psi)_{I,J} \tag{5.2.19}$$

the superintegral over F can expressed as

$$\int_{(\mathbb{H}^{2|2})^\Lambda} F = \int_{(\mathbb{R}^2)^\Lambda} \widetilde{F}_{\Lambda,\Lambda}(t,s) \prod_i e^{-t_i} \frac{dt_i\, ds_i}{2\pi}. \tag{5.2.20}$$

If a function $F(y)$ depends only on the y coordinates then F has degree 0, and a computation (see [40, Sec. 2.2] and Appendix 5.A) shows that

$$\begin{aligned}\langle F(y)\rangle_{\mathbb{H}^{2|2}} = \int_{(\mathbb{H}^{2|2})^\Lambda} F(y)e^{-H} &= \int_{(\mathbb{R}^2)^\Lambda} F(e^t s)(e^{-H})_{\Lambda,\Lambda} \prod_i e^{-t_i} \frac{dt_i\, ds_i}{2\pi} \\ &= \int_{(\mathbb{R}^2)^\Lambda} F(e^t s) e^{-\widetilde{H}(t,s)} \prod_i dt_i\, ds_i, \end{aligned} \tag{5.2.21}$$

with the function $\widetilde{H}$ given by (5.1.6).

Analogously to (5.2.3) a calculation gives the expressions

$$-u_i \cdot u_j = \cosh(t_i - t_j) + \frac{1}{2}(s_i - s_j)^2 e^{t_i+t_j} + (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j)e^{t_i+t_j} \tag{5.2.22}$$

$$z_i = \cosh t_i + (\frac{1}{2}s_i^2 + \bar{\psi}_i\psi_i)e^{t_i}. \tag{5.2.23}$$

We again check that

$$\frac{\partial z_i}{\partial s_i} = y_i, \qquad \frac{\partial y_i}{\partial s_i} = x_i + z_i, \qquad \frac{\partial (u_i \cdot u_j)}{\partial s_i} = y_j(x_i + z_i) - y_i(x_j + z_j) \tag{5.2.24}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial s_j^2} z_j = e^{t_j} = x_j + z_j, \\ \frac{\partial^2}{\partial s_i \partial s_l}(-1 - u_j \cdot u_l) = \begin{cases} -e^{t_j+t_l} = -(x_j+z_j)(x_l+z_l), & i = j, \\ +e^{t_j+t_l} = +(x_j+z_j)(x_l+z_l), & i = l, \\ 0, & \text{else.} \end{cases} \end{aligned} \tag{5.2.25}$$

Ward identities

In this section we establish some useful Ward identities. These Ward identities are a reflection of the underlying symmetries of the target spaces $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$, see [40, Appendix B]. Note that these identities are most easily seen in the ambient coordinates $(x, y^1, \dots, y^{n-1}, z)$.

$\mathbb{H}^n$. For the $\mathbb{H}^n$ model we have the identities

$$\langle x_j g(z)\rangle_{\mathbb{H}^n} = 0. \tag{5.2.26}$$

for any smooth function g. This identity follows simply from the invariance of the measure under $x \mapsto -x$ (see (5.1.4)–(5.1.5)). Moreover, by rotational symmetry, we have $\langle g(y^r)\rangle_{\mathbb{H}^n} = \langle g(x)\rangle_{\mathbb{H}^n}$ for $r = 1, \dots, n-1$.

$\mathbb{H}^{2|2}$. For the $\mathbb{H}^{2|2}$ model we have identities analogous to (5.2.26):

$$\langle x_j g(z)\rangle_{\mathbb{H}^{2|2}} = 0 \tag{5.2.27}$$

for any smooth function g. This identity again follows from the symmetry $x \mapsto -x$ (see (5.2.16)–(5.2.17)). We also have $\langle g(x)\rangle_{\mathbb{H}^{2|2}} = \langle g(y)\rangle_{\mathbb{H}^{2|2}}$ by rotational symmetry. The following identities arise from (5.2.27):

$$\begin{aligned} \langle e^{t_j+t_l}\rangle_{\mathbb{H}^{2|2}} &= \langle (x_j+z_j)(x_l+z_l)\rangle_{\mathbb{H}^{2|2}} = \langle x_j x_l + z_j z_l\rangle_{\mathbb{H}^{2|2}} \\ \langle e^{t_j}\rangle_{\mathbb{H}^{2|2}} &= \langle x_j + z_j\rangle_{\mathbb{H}^{2|2}} \end{aligned} \tag{5.2.28}$$

and hence by supersymmetry and rotational invariance

$$\begin{aligned} \langle e^{t_j+t_l}\rangle_{\mathbb{H}^{2|2}} &= 1+\langle y_j y_l\rangle_{\mathbb{H}^{2|2}}, \\ \langle e^{t_j}\rangle_{\mathbb{H}^{2|2}} &= 1. \end{aligned} \tag{5.2.29}$$

Indeed, the evaluations $\langle z_i z_j\rangle_{\mathbb{H}^{2|2}} = \langle z_i\rangle_{\mathbb{H}^{2|2}} = 1$ are by Lemma 5.2.2, which implies more generally that for any smooth function g with rapid decay,

$$\int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} g(z) = g(1). \tag{5.2.30}$$

5.3 Proof of Theorem 5.1.2

In this section, for the $\mathbb{H}^n$ model, we will let y_a denote the component y_a^1 of $u_a \in \mathbb{H}^n$ and s_a the corresponding component s_a^1 in horospherical coordinates. By symmetry (recall Section 5.2), the results of this section are valid if we replace y_a^1 by any of the first $n-1$ components of u_a.

We will prove that for the $\mathbb{H}^n$ model, $n \geqslant 2$,

$$\begin{aligned} \sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1) = \\ \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,h}} z_a \int_0^\infty \mathbb{E}^\beta_{a,z-1}(g(X_t,L_t))\, e^{-ht}\, dt. \end{aligned} \tag{5.3.1}$$

In (5.3.1), and in the rest of this section, we omit the measure $\mu^{\otimes\Lambda}(du)$ for integrals over $(\mathbb{H}^n)^\Lambda$ from the notation. For the $\mathbb{H}^{2|2}$ model we prove that

$$\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,h}} y_a y_b g(b, z-1) = \int_0^\infty \mathbb{E}^\beta_{a,0}(g(X_t,L_t))\, e^{-ht}\, dt. \tag{5.3.2}$$

Theorem 5.1.2 in the case of $\mathbb{H}^{2|2}$ is precisely (5.3.2), and Theorem 5.1.2 in the case of $\mathbb{H}^n$ follows by normalising (5.3.1). The identities (5.3.1) and (5.3.2) are a result of the following integration by parts formulas. Recall that $\mathcal{L}^\beta$ denotes the generator (5.1.2) of the joint position and local time process (X_t, L_t) of the VRJP.

Lemma 5.3.1. *Let Λ be finite, let $a \in \Lambda$, and let $g\colon \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ be a smooth function with rapid decay. For the $\mathbb{H}^n$ model, $n \geqslant 2$,*

$$-\sum_b \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = \int_{(\mathbb{H}^n)^\Lambda} e^{-H_{\beta,0}} z_a g(a, z-1). \tag{5.3.3}$$

For the $\mathbb{H}^{2|2}$ model,

$$-\sum_b \int_{(\mathbb{H}^{2|2})^\Lambda} e^{-H_{\beta,0}} y_a y_b \mathcal{L}^\beta g(b, z-1) = g(a, 0). \tag{5.3.4}$$

Proof. The proofs are essentially the same for $\mathbb{H}^n$ and $\mathbb{H}^{2|2}$, so we carry them out in parallel.

We write $\mathcal{L}$ for $\mathcal{L}^\beta$, H for $H_{\beta,0}$, and the integral $\int$ for $\int_{(\mathbb{H}^n)^\Lambda}$ and, respectively, $\int_{(\mathbb{H}^{2|2})^\Lambda}$. By (5.2.5) (resp. (5.2.24)) we have $y_b \frac{\partial}{\partial \ell_b} g(b, z-1) = \frac{\partial}{\partial s_b} g(b, z-1)$ where $\frac{\partial}{\partial \ell_b}$ denotes the derivative with respect to the b-th component of the second argument. Therefore

$$\begin{aligned} &\sum_b \int e^{-H} y_a y_b \mathcal{L} g(b, z-1) \\ &\qquad = \int e^{-H} y_a \Big(\sum_{b,c} \beta_{bc} y_b z_c (g(c, z-1) - g(b, z-1)) + \sum_b \frac{\partial}{\partial s_b} g(b, z-1) \Big). \end{aligned} \tag{5.3.5}$$

Recall (5.2.2) (resp. (5.2.20)) and integrate the second term in the equation above by parts. This produces two terms; by the rapid decay of g there are no boundary terms. For the first term produced by the integration by parts, using (5.2.5) (resp. (5.2.24)) again,

$$\begin{aligned}
&\sum_b \int e^{-H} y_a \Big(-\frac{\partial H}{\partial s_b}\Big) g(b,z-1) \\
&\qquad = \sum_b \int e^{-H} y_a \Bigg(\sum_c \beta_{bc} \frac{\partial (u_b \cdot u_c)}{\partial s_b}\Bigg) g(b,z-1) \\
&\qquad = \sum_{b,c} \int e^{-H} y_a \beta_{bc} y_b z_c (g(c,z-1) - g(b,z-1)). \qquad (5.3.6)
\end{aligned}$$

This term cancels the first term on the right-hand side of (5.3.5). For the second term produced by the integration by parts, we use that $\int x_a e^{-H} g(b,z) = 0$ by (5.2.26) (resp. (5.2.27)):

$$\begin{aligned}
\int e^{-H} \frac{\partial y_a}{\partial s_b} g(b,z-1) &= \delta_{ab} \int e^{-H} (x_a + z_a) g(b,z-1) \\
&= \delta_{ab} \int e^{-H} z_a g(a,z-1). \qquad (5.3.7)
\end{aligned}$$

In the supersymmetric case, the localisation lemma in the special case (5.2.30) further implies that the last right-hand side can be evaluated as

$$\delta_{ab} \int e^{-H} z_a g(a,z-1) = \delta_{ab} g(a,0). \qquad (5.3.8)$$

Altogether, we have shown (5.3.3) (resp. (5.3.4)). □

Proof of Theorem 5.1.2. It suffices to show (5.3.1) and (5.3.2) with $h = 0$, by replacing $g(b,z-1)$ by $g(b,z-1)e^{-h(z-1)}$. Therefore from now on assume $h = 0$. To get (5.3.2) from (5.3.4), we apply (5.3.4) with $g(i,\ell)$ replaced by $g_t(i,\ell) = \mathbb{E}_{i,\ell}(g(X_t, L_t))$. By the definition of the generator we have $\mathcal{L} g_t(i,\ell) = \frac{\partial}{\partial t} g_t(i,\ell)$, so (5.3.4) gives

$$\mathbb{E}_{a,0}(g(X_t,L_t)) = -\frac{\partial}{\partial t}\Bigg(\sum_b \int e^{-H} y_a y_b g_t(b,z-1)\Bigg). \qquad (5.3.9)$$

Note that the process (X_t, L_t) is transient even if the marginal (X_t) is recurrent because $\sum_i L_t^i \to \infty$ as $t \to \infty$. Therefore, integrating both sides over t and using that $g_t(x,\ell) \to 0$ as $t \to \infty$, which follows from the transience of (X_t, L_t) and the rapid decay of $g = g_0$, we get

$$\int_0^\infty \mathbb{E}_{a,0}(g(X_t,L_t))\, dt = \sum_b \int e^{-H} y_a y_b g(b,z-1). \qquad (5.3.10)$$

The proof of (5.3.1) from (5.3.3) is entirely analogous. □

5.4 Proof of Theorem 5.1.5

The proof of the hyperbolic Mermin–Wagner follows that of the usual Mermin–Wagner theorem closely [77, 78]; see also the presentation in [50]. We begin with the non-supersymmetric case. Due to the non-compact target space, differences occur in the bound of the term $\langle |DH|^2 \rangle$ and in the role of the coordinate in the direction of the magnetic field. As in the previous section we write H for $H_{\beta,h}$. We will write $\bar{A}$ to denote the complex conjugate of A.

Proof of (5.1.14). As in the previous section we write y_j for y_j^1. We also write $\langle\cdot\rangle$ for $\langle\cdot\rangle_{\mathbb{H}^n}$, and we use horospherical coordinates throughout the proof. Throughout the proof H will denote the energy of a spin configuration in horospherical coordinates, recall (5.2.4).

Let

$$S(p) = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{i(p\cdot j)}y_j, \quad D = \frac{1}{\sqrt{|\Lambda|}}\sum_j e^{-i(p\cdot j)}\frac{\partial}{\partial s_j}. \tag{5.4.1}$$

By the Cauchy–Schwarz inequality,

$$\langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)DH\rangle|^2}{\langle |DH|^2\rangle}. \tag{5.4.2}$$

In the following, we compute the terms on the left- and right-hand sides of the above inequality. Note that we have the integration by parts identity $\langle FDH\rangle = \langle DF\rangle$ for any smooth $F\colon (\mathbb{H}^n)^\Lambda \to \mathbb{R}$ that does not grow too fast; the vanishing of boundary terms can be seen by looking at the expression for H (i.e., by (5.2.4)).

By the assumed translation invariance of β,

$$\begin{aligned}\langle |S(p)|^2\rangle &= \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_j y_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0 y_{j-l}\rangle \\ &= \sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle,\end{aligned} \tag{5.4.3}$$

$$\begin{aligned}\langle S(p)DH\rangle &= \langle DS(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle \frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j \langle x_j + z_j\rangle \\ &= \langle z_0\rangle,\end{aligned} \tag{5.4.4}$$

$$\langle |DH|^2\rangle = \langle D\bar{D}H\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\left\langle \frac{\partial^2 H}{\partial s_j\partial s_l}\right\rangle. \tag{5.4.5}$$

In (5.4.4) we have used $\langle x_j\rangle = 0$; recall Section 5.2. By $\langle x_j z_k\rangle = 0$, Cauchy–Schwarz, translation invariance, that $\langle x_0^2\rangle = \langle y_0^2\rangle$ (recall the symmetries from Section 5.2), and the constraint $u_0\cdot u_0 = -1$, observe that

$$\langle (x_j+z_j)(x_l+z_l)\rangle = \langle x_j x_l + z_j z_l\rangle \leqslant \langle x_0^2\rangle + \langle z_0^2\rangle = 1 + (n+1)\langle y_0^2\rangle. \tag{5.4.6}$$

Thus, using (5.2.6) and $\langle x_j\rangle = 0$ once more, (5.4.5) can be rewritten and bounded above by

$$\begin{aligned}\langle |DH|^2\rangle &= \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle (x_j+z_j)(x_l+z_l)\rangle(1-e^{ip\cdot(j-l)}) + \frac{h}{|\Lambda|}\sum_j\langle x_j+z_j\rangle \\ &\leqslant \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}(1+(n+1)\langle y_0^2\rangle)(1-\cos(p\cdot(j-l))) + h\langle z_0\rangle.\end{aligned} \tag{5.4.7}$$

In summary, we have shown (recall (5.1.12))

$$\langle |DH|^2\rangle \leqslant (1+(n+1)\langle y_0^2\rangle)\lambda(p) + h\langle z_0\rangle. \tag{5.4.8}$$

Using (5.4.3) and substituting the above bounds into (5.4.2) gives

$$\begin{aligned}\sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle \geqslant \frac{|\langle S(p)DH\rangle|^2}{\langle |DH|^2\rangle} &\geqslant \frac{\langle z_0\rangle^2}{(1+(n+1)\langle y_0^2\rangle)\lambda(p) + h\langle z_0\rangle} \\ &\geqslant \frac{1}{(1+(n+1)\langle y_0^2\rangle)\lambda(p) + h}.\end{aligned} \tag{5.4.9}$$

The last inequality follows from $h \geqslant 0$ and $1 \leqslant \langle z_0\rangle$, which holds by the definition of $\mathbb{H}^n$. □

Proof of (5.1.15). We use that the expectation of a function $F(y)$ can be written using horospherical coordinates in terms of the *probability measure* (5.1.6). Throughout this proof, we denote the expectation with respect to this probability measure by $\langle\cdot\rangle$. By the Cauchy–Schwarz inequality, and since $S(p)$ is a function of the y,

$$\langle |S(p)|^2\rangle_{\mathbb{H}^{2|2}} = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle}. \tag{5.4.10}$$

The probability measure $\langle\cdot\rangle$ obeys the integration by parts $\langle FD\widetilde{H}\rangle = \langle DF\rangle$ identity for any function $F = F(s,t)$ that does not grow too fast. Therefore by translation invariance we find that, as in the case of $\mathbb{H}^n$,

$$\begin{aligned}\langle |S(p)|^2\rangle &= \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_j y_l\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle y_0 y_{j-l}\rangle \\ &= \sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle,\end{aligned} \tag{5.4.11}$$

$$\langle S(p)D\widetilde{H}\rangle = \langle DS(p)\rangle = \frac{1}{|\Lambda|}\sum_{j,l} e^{ip\cdot(j-l)}\langle\frac{\partial y_j}{\partial s_l}\rangle = \frac{1}{|\Lambda|}\sum_j \langle e^{t_j}\rangle = 1, \tag{5.4.12}$$

where the last identity uses (5.2.29). By (5.2.29), Cauchy–Schwarz, and translation invariance we have

$$\langle e^{t_j+t_l}\rangle = 1 + \langle y_j y_l\rangle \leqslant 1 + \langle y_0^2\rangle. \tag{5.4.13}$$

Using (5.4.13) and the integration by parts identity it follows that

$$\begin{aligned}\langle |D\widetilde{H}|^2\rangle = \langle D\bar{D}\widetilde{H}\rangle &= \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}\langle e^{t_j+t_l}\rangle(1-\cos(p\cdot(j-l))) + \frac{h}{|\Lambda|}\sum_j\langle e^{t_j}\rangle \\ &\leqslant \frac{1}{|\Lambda|}\sum_{j,l}\beta_{jl}(1+\langle y_0^2\rangle)(1-\cos(p\cdot(j-l))) + h \\ &= (1+\langle y_0^2\rangle)\lambda(p) + h.\end{aligned} \tag{5.4.14}$$

In summary, we have proved

$$\sum_j e^{i(p\cdot j)}\langle y_0 y_j\rangle = \langle |S(p)|^2\rangle \geqslant \frac{|\langle S(p)D\widetilde{H}\rangle|^2}{\langle |D\widetilde{H}|^2\rangle} \geqslant \frac{1}{(1+\langle y_0^2\rangle)\lambda(p)+h} \tag{5.4.15}$$

as claimed. □

Appendices

5.A Horospherical coordinates

$\mathbb{H}^n$

Under the change of variables

$$x = \sinh t - \frac{1}{2}|\tilde{s}|^2 e^t, \quad y^i = e^t s^i, \quad z = \cosh t + \frac{1}{2}|\tilde{s}|^2 e^t, \tag{5.A.1}$$

the measure transforms as

$$\frac{1}{z}\, dx \wedge dy^1 \wedge \cdots \wedge dy^{n-1} \mapsto \frac{\det J}{\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t} dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1}, \tag{5.A.2}$$

where the Jacobian matrix in block form is

$$J = \begin{bmatrix} A_{1\times 1} & B_{1\times n-1} \\ C_{n-1\times 1} & D_{n-1\times n-1} \end{bmatrix} \tag{5.A.3}$$

with

$$A = \frac{\partial x}{\partial t} = \cosh t - \frac{1}{2}|\tilde{s}|^2 e^t, \qquad B_j = \frac{\partial x}{\partial s^j} = -s_j e^t, \tag{5.A.4}$$

$$C_i = \frac{\partial y^i}{\partial t} = s^i e^t, \qquad D_{ij} = \frac{\partial y^i}{\partial s^j} = \delta_{ij} e^t. \tag{5.A.5}$$

Noting that $D = e^t I$, the determinant is easily computed using the Schur complement formula,

$$\begin{aligned} \det J &= (\det D) \det (A - BD^{-1}C) \\ &= e^{(n-1)t} \left(\cosh t - \frac{1}{2}|\tilde{s}^2| e^t - \sum_{i=1}^{n-1} (-s_i e^t) e^{-t} (s_i e^t) \right) \\ &= e^{(n-1)t} (\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t), \end{aligned} \tag{5.A.6}$$

giving the transformed measure as

$$\frac{\det J}{\cosh t + \frac{1}{2}|\tilde{s}|^2 e^t}\, dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1} = e^{(n-1)t}\, dt \wedge ds^1 \wedge \cdots \wedge ds^{n-1}. \tag{5.A.7}$$

$\mathbb{H}^{2|2}$

The calculation for $\mathbb{H}^{2|2}$ is similar to the previous case, but the Jacobian is replaced by the Berezinian. The notation in (5.2.8) corresponds to the following notation in [40] resp. [13]:

$$\int_{\mathbb{R}^{2|2}} F = \int dx \wedge dy \circ \partial_\xi\, \partial_\eta\, F = \int F\, d_\eta\, d_\xi\, dx\, dy. \tag{5.A.8}$$

Applying [13, Theorem 2.1] to the change of variables

$$x = \sinh t - \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t, \quad y = se^t, \quad z = \cosh t + \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t, \\ \eta = \psi e^t, \quad \xi = \bar{\psi}e^t, \tag{5.A.9}$$

the Berezin measure transforms as

$$\frac{1}{z}\, d_\eta\, d_\xi\, dx\, dy \mapsto \frac{\mathrm{sdet} M}{\cosh t + \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t}\, d_\psi\, d_{\bar{\psi}}\, dt\, ds, \tag{5.A.10}$$

where M is the Berezinian supermatrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial \eta}{\partial t} & \frac{\partial \xi}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial \eta}{\partial s} & \frac{\partial \xi}{\partial s} \\ \frac{\partial x}{\partial \psi} & \frac{\partial y}{\partial \psi} & \frac{\partial \eta}{\partial \psi} & \frac{\partial \xi}{\partial \psi} \\ \frac{\partial x}{\partial \bar{\psi}} & \frac{\partial y}{\partial \bar{\psi}} & \frac{\partial \eta}{\partial \bar{\psi}} & \frac{\partial \xi}{\partial \bar{\psi}} \end{bmatrix}, \tag{5.A.11}$$

and $\mathrm{sdet} M = (\det D)^{-1} \det(A - BD^{-1}C)$ is its Berezinian (superdeterminant). The four blocks are then

$$A = \begin{bmatrix} \cosh t - \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t & se^t \\ -se^t & e^t \end{bmatrix}, \qquad B = \begin{bmatrix} \psi e^t & \bar{\psi}e^t \\ 0 & 0 \end{bmatrix}, \tag{5.A.12}$$

$$C = \begin{bmatrix} \bar{\psi}e^t & 0 \\ -\psi e^t & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}. \tag{5.A.13}$$

The first term in the Berezinian is simply $(\det D)^{-1} = e^{-2t}$, whilst the second is

$$\begin{aligned} \det(A - BD^{-1}C) &= \det\left(\begin{bmatrix} \cosh t - \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t & se^t \\ -se^t & e^t \end{bmatrix} + \begin{bmatrix} 2\bar{\psi}\psi e^t & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= e^t \left(\cosh t + \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t \right), \end{aligned} \tag{5.A.14}$$

giving the transformed Berezin measure as

$$\frac{\mathrm{sdet} M}{\cosh t + \frac{1}{2}(s^2 + 2\bar{\psi}\psi)e^t}\, d_\psi\, d_{\bar{\psi}}\, dt\, ds = e^{-t}\, d_\psi\, d_{\bar{\psi}}\, dt\, ds, \tag{5.A.15}$$

which corresponds to (5.2.20).

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