Die boxes, workstations, graph theory and die charts

Numismatic handbooks often explain die studies with an image similar to that in Figure 1.[1] The neat die chart, with no crossing lines, is explained by the fact that dies were used to exhaustion and then replaced, in one single sequence. Anyone with a little experience of die studies, however, knows that die charts very rarely occur in this form. It is far more common that they involve crossed lines, which are normally explained either by the operation of multiple workstations in parallel, or the existence of a die box, by which multiple dies were made available for use at any one time. Which of these explanations is accepted has potentially important ramifications for our understanding of the coinage in question, particularly regarding the intensity of production. Multiple workstations allow for the production of the same volume of coinage in a shorter time period, a benefit that does not accrue from the use of a die box.

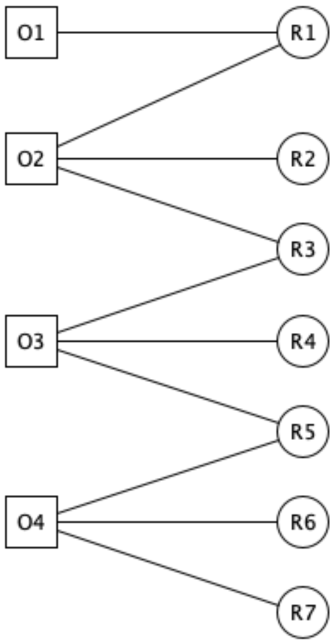

Figure 1: Hypothetical die chart

It would therefore be of interest if we were able to draw conclusions about minting process from more complex die charts. Unfortunately this is not very easy to do, as stated by Warren Esty in his conclusions to the seminal article on the topic:[2]

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[1] E.g., R.-Alföldi 1978, p. 47; Crawford 1983, p. 209.

[2] Esty 1990, p. 221.

Chronological inferences from linkage alone require strong and often unverifiable hypotheses about mint operation. Additional information from sequence marks and observations of the die-states is important. In the absence of such information, the links themselves can provide only limited information about the mint operation.

The additional information from which Esty suggests one could draw chronological inferences is often hard to come by, particularly in large die studies, where the number of comparisons is enormous and indications of die wear hard to come by.[3] For this reason, interpretation of die studies has tended to focus far more on the number of dies used and what this can tell us about the volume of production, rather than on the pattern of linkage.[4]

Nevertheless, a number of scholars have more recently attempted to use die charts to draw conclusions about processes within the mint and about the intensity of production.[5] This author believes that these attempts are misguided and wishes to re-state Esty's position: it is not possible to differentiate between the use of a die box and the operation of multiple workstations on the basis of the pattern of linkage alone. Moreover, I wish to highlight that Esty's mathematical approach to die charts remains the only possible means of interpreting them correctly. Die charts, as a category, are what mathematicians call graphs, and there is a whole body of literature on graph theory, with which numismatists have almost completely failed to engage. The only way that numismatists will be able to understand what deductions can and can't be drawn from the pattern of linkage in a die chart is to understand more of the mathematical theory behind the properties of graphs. To that end, I make use here of a simple computer program that simulates possible minting scenarios and outputs die charts; by analysing the properties of those charts, it will become clear which correlate to particular minting practices and which do not.

It will be useful at this point to introduce some basic terms in graph theory that may be unfamiliar to numismatists.[6] More specialist terms will be introduced at various points later in the paper. I have attempted, here and throughout the paper, to relate the mathematical terms to their equivalents in terms of real-world die studies. A **graph** is a representation of a set of points and the connections between them (for our purposes, a die chart); a **node** is a

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[3] It has been noted, for example, that Roman gold coins very rarely show any indications of die wear, see e.g. Beckmann 2014, p. 7.

[4] For an overview, see Callataÿ 2011.

[5] E.g., Carroccio 2011; Bracey 2012; Bracey 2017; Nurpetlian 2018.

[6] An accessible introduction to graph theory and its terminology is provided by Wilson 1996.

point within the graph (i.e. an obverse or reverse die); **vertex** is a synonym for node; an **edge** is a connection between two nodes (i.e. a die link).

Before coming to the computer simulation and the attendant mathematical analysis, however, we must clarify our numismatic terms of reference, and I therefore begin with a discussion of the terms “die box” and “workstation”. Real world examples in the discussion that follows tend to be drawn from the ancient world, since that is my own area of expertise, but the theoretical elements of the discussion are relevant to any coin series for which a die study may be carried out.

Die boxes and workstations

By “workstation” I mean a single unit of coin production, where one obverse and one reverse were utilised in tandem; in some literature this is referred to as an anvil.[7] At the mint of Rome, each workstation seems to have been manned by three operators: one to place the flan between the dies (the *suppostor*), one to hold the reverse die in place (the *signator*), and one to execute the hammer strike (the *malleator*).[8] Whether this was also the case at other mints in the Roman empire, or at other times and places, is unknown. I do not propose to equate workstation with *officina*, a term that appears in the Latin and seems to denote a subdivision of the mint; recent investigation has shown that precise definition of the term *officina* is not possible on the current state of the evidence, but that the equation *officina* = anvil is almost certainly wrong.[9]

Multiple workstations within the same mint are only relevant to the interpretation of die charts when they share reverse dies; if this is not the case, the die study will result in a series of disconnected die charts that each resemble the simple sequences represented by Figure 1.[10] This sharing could be either regular or exceptional. It is possible that even in mints where each workstation was intended to have its own supply of dies, sharing still

[7] Nurpetlian 2018 has suggested that some coin series may have been struck from two obverses set in the same anvil, but I am unconvinced that this was a frequent occurrence, and in any case it would make no difference to the die charts analysed here.

[8] Woytek 2013, pp. 255–66.

[9] Woytek 2012, p. 115.

[10] It may still be possible to identify separate workstations when there are no die links between them (see e.g., Beckmann 2011, p. 177), but this precedes from *a priori* knowledge of the context of production, rather than from an interpretation of the die chart itself.

occurs because of some emergency, or an accident in the logistics of the mint. Such instances are likely to have been rare, and I therefore term them exceptional. But sharing of reverse dies between workstations could also have been part of the *modus operandi* of the mint, resulting in far more frequent sharing. Both regular and exceptional sharing of dies between multiple workstations would result in die charts that cannot be drawn as a simple sequence like Figure 1, but must involve crossed lines.

The common pool of reverse dies just described in my explanation of regular sharing can be designated a "die box". This is the set of available dies, from which one must be chosen for each workstation at the commencement of striking. Die boxes are not, however, limited to use in conjunction with multiple workstations. A mint with one workstation could easily have multiple dies available, from which just one was selected at the start of work, and replaced at the end. It is this replacement that is key, since it allows couplings out of sequence and thus creates die charts that do not resemble Figure 1. One little discussed factor is how often this replacement took place. Did mint workers take a mid-morning break and replace their dies at this point? Or were dies only replaced at the end of the day? Or the end of the week or some longer period? We simply have no information by which to determine this, but clearly the more frequently dies were returned to the die box, the more opportunity there would be for their being used out of sequence.

As with the case of multiple workstations, it is important to distinguish between the regular and exceptional use of dies out of sequence. The former might properly be called a die box, as just described. The latter may have occurred when, for example, on nearing the end of production, the mint finds that it has broken its final reverse die before producing the required amount of coin; it therefore re-employs an old reverse that had been removed from use because it had become worn, though was not completely exhausted. Such occurrences I do not deem to be instances of a die box, but would nonetheless produce die charts with unavoidably crossed lines.[11]

Previous discussion of die boxes has tended to focus on their use for reverse dies. It is often assumed that obverse dies were fixed in an anvil, meaning that they could not be

[11] Note that the description of a die box by Esty 1990, pp. 207–10, as well as the die charts with which he illustrates it, are better explained by my exceptional circumstances, rather than a true die box. This distinction does not, however, impact upon his theoretical discussion.

removed and returned to an obverse die box.[12] Yet we have plentiful evidence that obverse dies could be—and were—removed from the anvil in antiquity, thus making obverse die boxes a possibility. Since obverse die boxes are so often dismissed out of hand, it is worth rehearsing that evidence here. A small number of Roman republican and imperial coins appear to have been struck by two obverse dies, and it has plausibly been suggested that some of these might have resulted from the re-deployment of an obverse die in the position of a reverse, whether as an emergency measure or otherwise.[13] This would necessarily imply that obverse dies could be removed from the anvil. In the Roman provincial coinage of the second and third centuries AD, the same obverse die is sometimes to be found utilised for the coinage of two or more cities; although the precise explanation for this is disputed, most scholars reject the idea of a centralised mint and hypothesise that the obverse die—presumably not still fixed in the anvil—was transferred between cities.[14] Finally, the late Roman coinage furnishes a number of examples of the same obverse die being used with reverses displaying different *officina* marks.[15] Since workstations can only belong to one *officina*, the use of an obverse die in multiple *officinae* would necessarily involve the transfer of that obverse from one anvil to another.

It must be granted that plausible explanations can be offered of all of these occurrences that do not involve the removal of obverse dies from the anvil: the deliberate engraving of an obverse design onto a reverse die in the first case, a centralised mint in the second, and the accidental use of a reverse die in the wrong *officina* for the third; these are not, however, the explanations currently favoured by the majority of scholarship. Indeed, we require only one of these instances actually to have involved the removal of obverse dies from the anvil for this to be admitted as a possibility in antiquity. Once we allow that obverse dies were not fixed in the anvil, an obverse die box becomes a possibility that we must entertain.

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[12] The possibility of an obverse die box is rejected out of hand by e.g. Esty 1990, pp. 207–08; Carter and Nord 1992, p. 149; Kraay 1956, p. 12

[13] Stannard 1987 for the republican coins, with a brief look at later coins. For an alternative explanation of Hadrianic double-headed coins, see Abdy 2015.

[14] Kraft 1972; Johnston 1982; Spoerri Butcher 2006; Watson 2019, pp. 111–26 For a dissenting viewpoint that favours a centralised mint, see Kellner 1973, p. 30.

[15] Some of these were the subject of a discussion by Bastien 1960; Grierson 1961; Sutherland 1962.

All of the minting scenarios discussed above, viz. multiple workstations with dies shared either regularly or exceptionally, the exceptional re-use of dies out of sequence, and die boxes for both obverse and reverse dies, would result in die charts that cannot be drawn without crossed lines. To answer the question of whether there are any other properties of these die charts that allow us to distinguish between these different scenarios, I turn now a computer simulation of an ancient mint that generates die charts and analyses their properties.

The simulation

The computer program to simulate different minting scenarios was written in Python programming language, and outputs results to a spreadsheet for easy analysis. Before the program can be run, the following parameters must be set:

d_{max}	The total number of obverses to be generated. This is simply a means of setting a limit to program's operation. For all of the simulations in this paper, $d_{max} = 20$.
T_r	The average lifetime, in minutes, of reverse dies. Carter and Nord estimated an average die lifetime of between 5 and 10 hours for Crepusius denarii.[16] For all simulations in this paper, $T_r = 450$ (= 7.5 hours).
a	A scale factor for how much longer obverses should last than reverses. This can be estimated by looking at ratios of reverses to obverses (r/o) in completed die studies, and de Callataÿ has shown that the figure is heavily dependent upon the size of coin being struck.[17] His survey of Hellenistic coinages suggests that the r/o ratio is generally around or a little over 1 for small and light pieces, between 1 and 3 for coins of medium weight, and can stretch up to 5 or 6 for the largest coins. A survey of Roman die studies suggests that the same holds true there: r/o for Roman precious metal coinages is generally in the range 1–1.5, while bronze coins, including Roman provincial coins, which were in general larger, have r/o ratios nearer 3 or 4.[18]

[16] Carter and Nord 1992, pp. 154–9.

[17] Callataÿ 1999.

[18] Data from Roman precious metal die studies is helpfully summarised by Bland 2013, pp. 279–80. Figures for bronze are from my own brief survey of the following publications: Kraay 1956; Woodward 1957; Kaenel 1986; Klose 1987; Delrieux 2008; Dalaison, Remy and Amandry 2009; Watson 2019.

t	The length, in minutes, of a working period, after which dies are returned to the die box.[19]
W	The number of workstations.
odb	The number of dies in the obverse die box. It is necessarily true that $odb \geq W$, so that all workstations can operate simultaneously; if $odb = W$, it can be presumed that the obverse die box does not exist, as each obverse is fixed to a particular workstation.
rdb	The number of dies in the reverse die box. As with obverse dies, it is necessary that $rdb \geq W$, to allow all workstations to operate simultaneously.[20]
i	The number of iterations to be completed, or in other words, the number of die charts to be produced with the inputted parameters. For all simulations in this paper, $i = 10{,}000$.

At the start of the simulation, the obverse and reverse die boxes are populated with dies. Each is assigned a lifetime, in minutes, according to the geometric model proposed by Esty.[21] A die pair is chosen for each workstation at random, and a time step, equivalent to one minute, advances until the length of the working period (t) has been reached. If, in this period, any of the dies exceed their assigned lifetime, a new die is selected from the die box and employed at that workstation. When there are no available dies in the die box—either because they are in use at other workstations or because the box is empty—the die box is repopulated with new dies.[22] This process is repeated until d_{max} is reached. The properties of the resultant die chart, which need never actually be drawn, are recorded in the spreadsheet and the process is repeated for the number of iterations specified by i.

The program as described above produces die charts that directly reflect activity in the mint, but this is not the case for die charts that arise from actual die studies. In the latter case, the vagaries of archaeological survival mean that the charts only reflect a sample of the

[19] Carter and Nord 1992 assume that reverse dies were returned to the die box at the end of a 12-hour working day.

[20] The same restriction is envisaged by Bracey 2017, p. 84.

[21] Esty 2011. The program can also assign die lifetimes according to a gamma distribution, as proposed by Carter 1983, but this does not significantly affect the results presented here.

[22] The moment at which the die box might have been replenished has not been much considered but has a significant impact upon the resultant die charts; for the impact of continuously keeping the die boxes at maximum capacity, see below note 33.

original material. Although this paper only deals with die charts that reflect the complete output of the mint, the simulation can also be made to produce die charts that reflect what numismatists actually deal with, that is to say, using evidence that is disrupted by chance survival. Three further parameters are required:

o_loss	The percentage of obverse dies to be deleted. This can be thought of as the inverse of the *obverse coverage* of a die study, as defined by Esty;[23] for example, to simulate a die study with obverse coverage of 95%, *o_loss* = 5.
r_loss	The percentage of reverse dies to be deleted. Existing die studies indicate that reverse coverage is normally slightly worse than obverse coverage, and *r_loss* should therefore be slightly higher than *o_loss*.
e_loss	The percentage of links to be deleted. This simulates the fact that links can be unattested in a die study even when all of the relevant dies are attested. It is impossible to calculate how many such links are missing from die studies, but a rough estimates can be made based on die studies that have been revisited after an interval of time. The relevant figure is the number of new die combinations in the revised study that involve dies already attested in the original dataset. While this does not, of course, reflect the number of original links that were unattested, it our best means of estimating how many links might be missing, even when the dies are known. A survey of three such revisited die studies suggests that a figure for *e_loss* of between 5 and 10% is reasonable.[24]

The program deletes both dies and links in inverse probability to their length of use. Dies with short lifetimes will have produced fewer coins, and are therefore less likely to appear in a modern sample; the chance of these dies being deleted in the simulation is therefore higher. The opposite is true of dies with long lifetimes, and the same principles apply to links or die combinations.

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[23] Esty 2006.

[24] Beckmann 2007 (revisiting Beckmann 2000) shows 17 new links between dies that were already known, out of a total of 179 die combinations in the final publication. Bland forthcoming (revisiting Bland 1991) shows 33 new die combinations out of 853. A 2021 revision of the die study in Bland 1993 showed 4 new die combinations out of 33. I am grateful to Roger Bland for sharing his data from these latter two studies with me.

The Python code for the mint simulation is available at: https://github.com/gcw30/mint-simulation. Since the simulation is based on pseudo-random numbers, it is possible for any reader to run the code and obtain *exactly* the same results as those presented in this paper. Readers are, of course, also encouraged to experiment with the code and to run their own simulations.

Planarity

Planarity is a property of graphs that has received increasing attention in recent numismatic literature.[25] A planar graph is defined as one "that can be drawn in the plane without crossings – that is, so that no two edges intersect geometrically, except at a [node] to which both are incident."[26] All graphs, including die charts, can, of course, be drawn in an infinite number of ways, but for some graphs all of these possible drawings necessarily involve crossed lines; these graphs are described as non-planar. Graphs that have even just one possible drawing without crossed lines are planar. Note that the concept of "crossed lines" differs here from that used above. Above, we were concerned only with die charts in which obverses and reverses were arranged separately on two parallel lines; in testing for planarity there can be no restriction on the placement of nodes in the graph.[27]

Planarity is interesting in the context of die charts, since it has been claimed that non-planar die charts are indicative of a particular minting process, specifically that "if a chart is non-planar … the original mint must have employed at least three workstations."[28] Bracey has gone further and suggested a semi-formal method for identifying the number of workstations that lay behind a non-planar chart.[29] In both instances, the analysis proceeds entirely from the pattern of linkage in the die chart, without any external information regarding e.g. die wear.

The central premise of this strand of scholarship, that non-planar die charts necessarily reflect a minting scenario in which at least three workstations were employed, can

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[25] E.g. Bracey 2012; Bracey 2017; Nurpetlian 2018.

[26] Wilson 1996, p. 60.

[27] These two concepts are confused by Ramskold 2016, pp. 177–8 The die chart he discusses cannot be drawn without crossed lines when obverses and reverses are arranged separately on parallel lines, but it is not planar, as Ramskold claims.

[28] Bracey 2012, p. 74.

[29] Bracey 2017.

be tested experimentally using the mint simulation described above. For the sake of simplicity, obverse die boxes have been discounted at first (i.e. *odb* = *W*), and the program was run for different values of *W* and *rdb*; other variables remained constant throughout the different runs.[30] Table 1 presents the number of non-planar die charts that were generated by each run of the program for different values of *W* and *rdb*; the total number of die charts generated in each run of the program was 10,000.[31]

W \ *rdb*	1	2	3	4
1	0	0	1	6
2	–	0	25	104
3	–	–	2072	2444
4	–	–	–	4393

Table 1: Number of non-planar graphs (out of 10,000) for different values of rdb *and* W *(*odb=W*,* t = 600, a*=1.5)*

It is clear from Table 1 that it is quite possible for a non-planar die chart to result from the operation of just one workstation with three or more dies in the reverse die box. The same is true of an obverse die box: non-planar die charts can result from one workstation and an obverse die box containing three or more dies.[32] It is true that non-planar die charts occur much more frequently when there are three or more workstations, but there are a number of other factors that impact upon the likelihood of a non-planar die chart arising: increasing d_{max} and decreasing *a* or *t* would all result in more non-planar charts, whatever the values of *W* and *rdb*.[33] It is therefore not even possible to determine which minting scenario is more likely

[30] The value of *a* = 1.5 reflects a plausible value for Roman precious metal coinage; *t* = 600 simulates dies being returned to the die box at the end of a ten hour working day: both seem reasonable assumptions.

[31] There are numerous approaches to testing for planarity. The simulation uses the NetworkX Python package (https://networkx.org), which implements the so-called Left-Right algorithm described by De Fraysseix, De Mendez and Rosenstiehl 2006.

[32] A simulation with the same parameters as in Table 1, but *W* = 1, *odb* = 3 and *rdb* = 1 resulted in three non-planar die charts out of 10,000.

[33] Modifying the program so that the die boxes are continuously kept at maximum capacity, rather than only being replenished when empty, also results in many more non-planar die charts. While the latter seems a more probable manner of operation, the former is possible, particularly in large mints that retained the services of their own engravers.

to have produced a non-planar die chart, let alone to definitively link such a chart to one and only one minting scenario.

The only condition upon the appearance of non-planar die charts is that one of *W*, *odb* or *rdb* must be greater than or equal to three; if all are less than three, non-planar die charts will not occur.[34] In other words, a non-planar die chart demonstrates that there were either at least three dies in the reverse die box, at least three dies in the obverse die box, or at least three workstations; it does not, however, allow us to determine which of these actually reflects the original production process. As a corollary we should also note that the reverse is not true: a planar die chart does not preclude the operation of multiple workstations and/or die boxes.

Bipartite crossing number

Although planarity has received a good deal of attention in recent numismatic scholarship, it is a rather unintuitive property for which to analyse a die chart. Tests for planarity assume that, in drawing the graph, nodes can be placed anywhere on the plane, but we rarely want to employ this freedom in drawing die charts. The drawing of die charts is intended to aid our understanding of a die study, and drawings that place no restrictions on the placement of nodes are unlikely to do this. As an example, Figure 2 shows a section of a die chart generated by the simulation described above, drawn in such a way as to demonstrate its planarity. No numismatist is likely to be able to draw any useful conclusions from it. To make it useful we must, at the very least, group together obverses and reverses; doing so would necessarily lead to crossed lines. I therefore turn to a property of die charts not

[34] For the maths of why this must be true, see Wilson 1996, pp. 62–3; for a more numismatically focussed explanation, see Bracey 2012, pp. 72–4.

previously discussed in the numismatic literature that explicitly relies on grouping obverses and reverses together.

Die charts are all examples of a particular category of graphs known as *bipartite graphs*. These are defined as graphs containing two distinct sets of nodes, where each node is connected only to nodes of the other set and not to nodes of its own set.[35] Since in a die chart

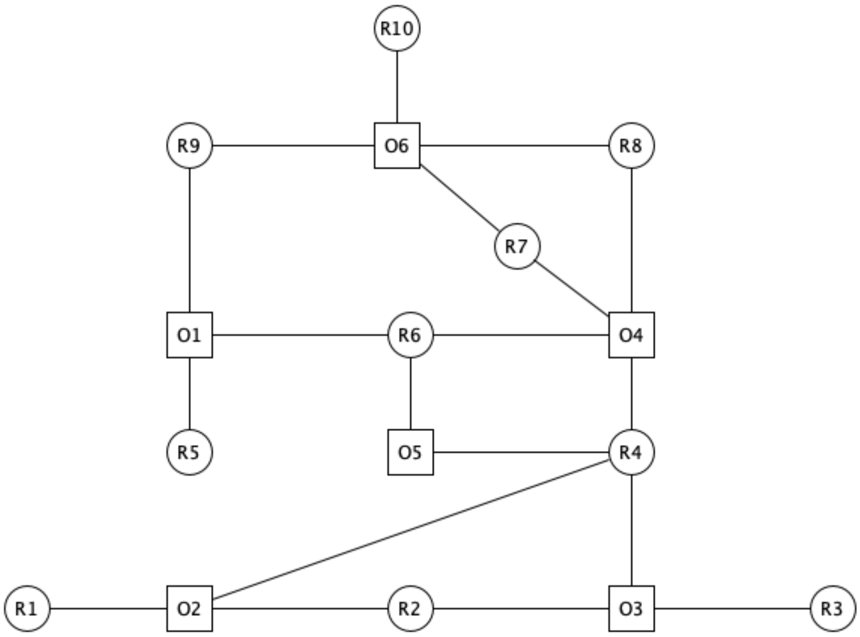

Figure 2: Non-sensical drawing of a hypothetical die chart, arranged to illustrate planarity

obverses cannot, in normal circumstances, be connected to other obverses, and reverses cannot be connected to other reverses, die charts are necessarily bipartite. One interesting property of bipartite graphs is the *bipartite crossing number* (*bcr*), which can be defined as the smallest number of crossings in a drawing of the graph such that the two sets of nodes lie on parallel lines.[36] In other words, for a die chart, we would place all obverses on one line and all reverses on a separate parallel line and then reposition the dies to minimize the number of crossings. The number of crossings in this optimal arrangement is the *bcr*.

Knowing the *bcr* of a die chart allows us to select an optimal drawing thereof. For example, when *bcr* = 1, it would be foolish to publish a die chart with five crossings, unless

[35] Wilson 1996, p. 18 defines a bipartite graph thus: "If a vertex set of a graph G can be split into two disjoint sets A and B so that each edge of G joins a vertex of A and a vertex of B, then G is a bipartite graph. Alternatively, a bipartite graph is one whose vertices can be coloured black and white in such a way that each edge joins a black vertex (in A) with a white vertex (in B)."

[36] This property is discussed under different names and notations in the mathematical literature. I follow the naming, definition and symbology of Schaefer 2020, p. 32.

there was a great deal of external evidence to suggest this was the correct interpretation. But beyond selecting an optimal drawing, knowledge of the *bcr* could also be an aid in interpreting the die charts. Die charts for which $bcr = 0$ will allow for drawing as in Figure 1, and must result from one workstation and no die boxes. Low values of *bcr* are likely to reflect the kind of exceptional patterns of die use discussed above, whereas regular sharing of dies between multiple workstations or true die boxes are likely to result in higher values of *bcr*.[37] This is not a hard and fast rule—the simulation confirms that the use of die boxes can result in low values of *bcr*—but it does offer a rough guide.[38]

The problem arises in calculating *bcr*. There are currently no known algorithms to calculate *bcr* for any bipartite graph and moreover, the problem is classified as NP-complete, a class of problems that have been extensively studied by mathematicians and for which it is questionable whether an efficient algorithm can ever be found.[39] Some values of *bcr* may be calculated by brute force. Die charts for which $bcr \neq 0$ are easy to detect manually; all that is required is for any two obverses to be connected, either directly or indirectly, to the same two reverses. When this feature is not present, $bcr = 0$. Similarly, experience suggests that when $bcr = 1$, it is easy enough to determine this visually by manipulating the die chart drawing. More complex die charts would require an algorithmic calculation, which, as stated, is not currently possible. NP-complete problems are generally approached via heuristics and approximations, and while various approximations of *bcr* exist for particular types of graph, none are applicable to die charts. Given that this remains an active field of mathematical research, it may be that an appropriate approximation is discovered, and numismatists would do well to be alert to such a discovery.

Other properties

Scholars have suggested that a number of other properties of die charts might be related to the original conditions of production. I analyse each of these in turn here and show

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[37] The dividing line between a low value and a high value must of course be left to the judgement of the individual researcher.

[38] A key factor is the length of the die chain: low values of *bcr* are only likely to be produced by multiple workstations and/or die boxes if there are few dies in the chain; low values in long chains are almost certainly the result of exceptional die use.

[39] Schaefer 2020, pp. 32–3.

that, while there is some correlation in some instances, it is not possible to tie any particular property to a particular production scenario.

Carter and Nord invented a parameter that they dubbed the *die combination ratio*, which they defined as:[40]

> $D_{dc} \div (D_o + D_r)$, where D_{dc} is the original number of die combinations, D_o is the original number of obverse dies, and D_r is the original number of reverse dies. When dies are kept at a given anvil until they fail, it is reasonable that for each die failure there is a new die combination, and hence $D_o + D_r$ should equal D_{dc}. The die combination ratio should actually be slightly less than one because occasionally both dies may have been replaced simultaneously. When dies are returned to the die boxes (or die box, if only the reverse dies are returned) at night, then the number of die combinations should be greater than $D_o + D_r$. The die combination ratio is greater than one since new die combinations can be achieved without the failure of a die.

The simulation reveals that the die combination ratio is indeed always less than 1 when minting occurs at one workstation without the use of die boxes. The inverse, however, does not hold true: die combination ratios of less than one can also occur when the minting scenario includes multiple workstations and/or die boxes. A die combination ratio of less than 1 is not therefore diagnostic of any particular minting scenario, and a die combination ratio of more than one simply indicates a more complex minting arrangement, without determining which one. Indeed, the die combination ratio is more strongly correlated with the length of the working period (shorter working periods result in higher ratios), than with the presence or absence of multiple workstations and/or die boxes.

Bracey has suggested that numismatists may be interested in what he terms the "largest reverse cluster", defined as "the largest number of obverses occurring in combination with a single reverse"; this is suggested to be indicative of the maximum number of workstations from which a die chain might have been produced.[41] This metric is more properly termed the *maximum reverse degree*: in graph theory the degree of a node is the number of edges connected to it, and the maximum degree of a graph is therefore the maximum number of edges connected to any one node.[42] Maximum reverse degree is, however, a meaningless metric with regard to die charts. The simulation reveals that, under reasonable assumptions

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[40] Carter and Nord 1992, p. 155.

[41] Bracey 2017, pp. 86–7.

[42] Wilson 1996, p. 12.

for other parameters, one workstation and no die boxes can produce die charts with a maximum reverse degree of higher than 10. This would occur when a particularly long-lived reverse die happened to be coupled with a number of short-lived obverse dies; the relevant factor is therefore the relative lifetimes of obverse and reverse dies (i.e. the value of *a* in the simulation), with longer lasting reverses (i.e. lower values of *a*) more likely to record larger maximum reverse degrees.

Bracey has also introduced a metric that he termed "length", defined as "the minimum number of die combinations necessary to connect any two dies in the chart".[43] Graph theory labels this measure the *diameter* of a graph, and defines it as "the largest geodesic between any pair of nodes", where a *geodesic* is defined as the shortest path between two nodes.[44] Bracey gives no indication as to what he believes this metric is supposed to indicate about die charts, but diameter is generally used as a measure of how far apart the farthest two nodes in a graph are, which can be interesting when the graph represents, for example, a communications network. There doesn't seem to be an obvious real-world interpretation of diameter in the context of die charts, but there is nevertheless some form of correlation between the diameter of the die chart and the original minting scenario. Complex minting scenarios (i.e. higher values of *W* and/or *odb* and *rdb*) result in lower diameters. This relationship cannot, however, be modelled mathematically, meaning that it is impossible to work backwards from a given die chart to the original values of *W*, *odb* and *rdb* on the basis of the chart's diameter.[45] Another measure of graph distance, the *average shortest path length*,[46] shows a similar correlation (higher values of *W*, *odb* and *rdb* result in lower average shortest path lengths) but this cannot be used as diagnostic of the original minting scenario.

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[43] Bracey 2017, p. 87.

[44] Iacobucci 1994, pp. 110–12 A useful explanation of this concept in an archaeological context is given in the glossary provided by Collar *et al.* 2015, pp. 17–25.

[45] For simulations where *W*, *odb* and *rdb* were varied between 1 and 4 and other variables kept constant ($a = 1.5$, $t = 600$), an Ordinary Least Squares regression analysis gives an r^2 value of 0.51, meaning that *W*, *odb* and *rdb* account for 51% of the variation in the diameter of die charts produced by the simulation; the other 49% will be accounted for by other input variables (e.g., T_r, a, t) that are not necessarily measurable in real world examples. Even this value is likely to be optimistic, however, since the analysis doesn't account for the collinearity of *W*, *odb* and *rdb*, and assumes that their relationship with the diameter of a die chart is linear, which is not necessarily true.

[46] Calculated by evaluating the shortest path between every pair of nodes (geodesics) and then taking the mean. See the glossary in Collar *et al.* 2015, s.v. "geodesic".

Graph theory also provides a number of metrics for measuring how closely connected a graph is. The simulation calculates two of these, namely density and clustering, for all die charts it generates. Density is defined as the number of links present as a fraction of the total possible links; clustering measures to what extent nodes that share a common neighbour are likely to have other neighbours in common.[47] Neither metric has any correlation with the number of workstations or the presence or absence of die boxes.

Conclusion

The preceding discussion has made it clear that there are currently no mathematical properties of dies charts that are indicative of the use of multiple workstations as opposed to die boxes, or vice versa. If a die chart cannot be drawn so that no lines cross when obverses and reverse are arranged separately on parallel lines—or, to use the language introduced in this paper, for die charts where $bcr \neq 0$—there is no means of determining on the basis of the pattern of linkage alone whether the chart resulted from the use of multiple workstations or of die boxes, let alone how many workstations or how many dies in the die box. Recent interest in the concept of planarity should be abandoned, as this reveals nothing about the original minting scenario; it would be more fruitful to replace this with the concept of the bipartite crossing number, although a full implementation of this will need to await the derivation of a suitable approximation.

As stated in the introduction, knowing whether a coin series was produced at multiple workstations, and if so, at how many, could be useful in determining the intensity of production. If two series use roughly the same number of dies (and can therefore be presumed to have produced roughly the same number of coins), but one was minted using one workstation and the other three, the latter series is likely to have been produced in a much shorter time period. This could be interesting in relation to the purpose of the coinage, or the historical context in which it was produced. However, in the absence of a mathematical means of determining the original minting scenario, we must rely instead on our contextual

[47] Both are defined in the glossary of Collar *et al.* 2015, but must be adapted for bipartite networks, see Latapy, Magnien and Vecchio 2008.

knowledge to determine whether the complexity of a die chart arose from multiple workstations or the use of die boxes.[48]

The kind of contextual knowledge that might facilitate this kind of judgement includes considerations such as the size of the mint and its usual operation. For example, if the complex die chart arises for only a short period of the mint's operation, when its normal processes seem to have been much simpler, the multiple workstation explanation seems implausible; it seems more likely that the mint would have temporarily employed a die box than that it would have invested in the extra equipment and facilities needed for extra workstations for only a short period of operation. We might also consider the supply of dies. In the Roman provincial coinage, for example, it is generally accepted that many cities received their dies from travelling workshops of engravers. It seems very plausible that a city would purchase a large bunch of dies at one time, not knowing when the workshop would next be available. A die box, where those dies—purchased *en masse*—were not necessarily used in sequence, would therefore seem the most logical explanation in this scenario.

If our assessment of the original minting arrangements is to be based on such qualitative judgements rather than on a mathematical calculation arising from the die chart, it is crucial that we are clear about the terminology that we use. As discussed above, the term "die box" should only be used for the systematic use of dies in no particular order; die boxes could operate for reverse dies, for obverse dies, or for both in tandem. The one-off or exceptional use of a die out of sequence should not be considered a die box. The difference between these two scenarios could indicate very different operating principles at the mint. Similarly, if multiple workstations are thought to have been in use, we must consider whether the sharing of dies between them was standard procedure or the result of exceptional usage; these two different scenarios would suggest very different conclusions about the organisation, operation and goals of the mint.

While the conclusions given here advocate a more subjective approach to determining the original process employed in the mint, this is not to say that the mathematical approach should be abandoned. Indeed, this paper has attempted to show that it is crucial that die charts be considered as mathematical graphs in order that their properties be fully understood. Only by engaging properly with graph theory will numismatists be able to understand what can and

[48] Note the danger of circularity here: if we use contextual knowledge to determine the minting scenario, we shouldn't then use our conclusions about the minting process to shed light on the historical context.

can't be concluded from the structure of a die chart. Mathematics is not a static discipline, and as advances are made in understanding the properties of bipartite graphs, it may be that there are new insights that can bring about a better understanding of die charts. It is crucial therefore that numismatists continue to engage with the mathematics of graph theory and to think critically about how it can be applied to the field of die studies.

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