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PROPERTIES OF THE QUANTUM STATE ARISING AFTER THE L-PHOTON STATE HAS PASSED TROUGH A LINEAR QUANTUM AMPLIFIER

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Abstract. We consider the system of N two-level atoms, of which N_0 atoms are unexcited and N_1 are excited. This system of N two-level atoms, which forms a linear quantum amplifier, interacts with a single-mode electromagnetic field. The problem of amplification of the L-photon states using such an amplifier is studied. The evolution of the electromagnetic field density matrix is described by the master equation for the field under amplification. The dynamics of this process is such that it can be described as the transformation of the scale of the phase space. The exact solution of the master equation is expressed using the transformed Husimi function of the L-quantum state of the harmonic oscillator. The properties of this function are studied and using it the average photon number and its fluctuations in the amplified state are found.

Key words: linear quantum amplifier, Husimi function, phase space, scale transformation, stretched states, density matrix, number operator, mean value.

1. INTRODUCTION

Real quantum systems are not isolated. When we consider a certain quantum system, it inevitably interacts with its environment. Systems that interact with their environment are called open quantum systems. In the last years great interest arose about the dynamics of open quantum systems. If we want to operate within the framework of quantum mechanics, we must describe the state not only of our system, but also of its entire environment. The whole large association of the environment and our system must be described together. In other words, we need to take into

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account the impact of everything on our system. But in reality it is impossible to do this, because the environment has an infinite number of degrees of freedom. There are general theoretical results on how to describe the dynamics of open quantum systems [1] - [9]. They are described using the density operator. The density operator can be considered as some matrix, satisfying some dynamic equation. This equation is obtained by reduction, that is, by throwing out the degrees of freedom of the environment from the entire large combination of the system and the environment. In this paper, we consider the case when the environment affects the system, but the system itself does not affect the environment. The quantum description of amplification, absorption and dissipation is important in many different areas of physics. In quantum optics such problems exist in the quantum theory of lasers and photon detection. We will study the interaction of an electromagnetic field with a linear quantum amplifier. For this purpose, we apply an original method based on the use of Q-functions of Husimi-Kano, which was developed in the papers [10] - [13].

Several formulations of quantum mechanics, that are equivalent to each other, exist. In the standard formulation the quantum state is represented by a vector in Hilbert space and the physical observables by operators acting in this space.

In the phase space formulation of quantum mechanics the quantum state is represented by the corresponding quasi-probability distribution $D(q,p)$ which completely describes the state.

While quantum observable is described by operator $\hat{A}$ in standard formulation, in the phase space formulation it is described by the function $A_D(q,p)$, defined on the whole phase space so that the average value $\langle\hat{A}\rangle$ of the operator $\hat{A}$ in standard formulation, in phase space formulation is represented by the integral

$$\langle\hat{A}\rangle = \int A_D(q,p) D_\rho(q,p) dq dp. \tag{1}$$

The function $D(q,p)$ is called a quasi-probability distribution, which corresponds to this quantum state, and the function $A_D(q,p)$ is called the symbol of the operator $\hat{A}$, corresponding to the quasi-probability distribution $D(q,p)$.

In phase space formulation of quantum mechanics, a number of quasi-probability distributions are used. The most common of them are the Wigner $W(q,p)$-function [14], Glauber-Sudarshan $P(q,p)$-function [15, 16] and Husimi-Kano $Q(q,p)$-function [17, 18]. In this way, each $\hat{A}$ operator has several symbols depending on the quasi-probability used. The phase space formulation is mainly used in quantum optics. The applications of quasi-probability distributions are presented in books [19]-[21].

Apart from the mentioned quasi-probability distributions the proper density probability distribution, called the symplectic tomogram of the quantum state [22] is also used. The properties of tomograms are discussed in the review [23].

In this paper, we will work with the Husimi-Kano functions $Q(q,p)$, which for brevity we will call simply Husimi functions.

In [10, 11] we studied the relation of the Husimi function to the Wigner and Glauber-Sudarshan functions, as well as to symplectic tomograms.

The main idea of our approach is the following. There are a number of quantum states for which the exact analytical form of their Husimi functions is known. We can consider some transformation of the phase space (q,p), and the corresponding transformation of the Husimi functions. This leads to new functions that depend on transformation parameters. One can choose the transformation of phase space so that these new functions are also Husimi functions of some quantum state.

In some cases transformation of a phase space can be related to a specific physical process. In such a case, the transformed Husimi function corresponds to quantum state that arises as a result of this physical process. The density matrix of this new state can be reconstructed from the corresponding Husimi function.

In our papers [10, 11] we considered the transformation of the scale of the phase space

$$(q,p) \to (\lambda q, \lambda p); \;\; |\lambda|^2 \leq 1, \tag{2}$$

and the corresponding transformation of the functions defined on this phase space

$$Q(q,p) \to \tilde{Q}(q,p) = \lambda^2 Q(\lambda q, \lambda p). \tag{3}$$

We have shown that if $Q(q,p)$ is a Husimi function of a quantum state, then $\lambda^2 Q(\lambda q, \lambda p)$ is also a Husimi function of some quantum state, provided that $|\lambda|^2 \leq 1$.

The phase space scale transformation can be related to the dynamics of the linear quantum amplifier [19, 21]. This amplifier is a system of N two-level atoms, of which N_0 of atoms are in the ground state, and N_1 of atoms are in the excited state, $N_0 < N_1$, $N_0 + N_1 = N$. This system of atoms interacts with a single-mode quantum field. The state of the field is described using the corresponding Husimi function. The states of the field that occur after the passage through the quantum amplifier, we call amplified or stretched states.

In [10, 11] we considered the case of a fully inverted environment, which corresponds to $N_0 = 0$. In this case, the interaction of electromagnetic field with environment can be described using a scale transformation (2) of the phase space and corresponding Husimi functions transformation (3). For the stretched L-photon state of electromagnetic field, an explicit form of the density matrix was found. In [12, 13], we considered a more general situation, namely, when a state, which enters the amplifier, is an arbitrary superposition of number states. For this state the density matrix of the state at the output of the amplifier was found. The average number of photons, its fluctuation, and the form of Heisenberg and Robertson-Schroedinger uncertainty relations were found. Recall that there it was assumed that $N_0 = 0$.

In this paper we consider the case where the population of the lower level is $N_0 > 0$, provided that the relation $N_0 < N_1$ is satisfied. An explicit form of the state density matrix that occurs after the L-photon state has passed through such an amplifier is found and studied. The derivations and results will be presented in detail in the following sections.

2. AMPLIFICATION OF THE L-PHOTON STATES IN A QUANTUM AMPLIFIER

We use a linear quantum amplifier model, which is described in the monograph [19]. We consider a system of N two-level atoms, N_1 of which are in the excited and N_0 in the ground state, with $N_0 < N_1$. These atoms are interacting with a single-mode quantum field. We consider an eigenmode of a free field with frequency resonant with the atomic frequency. We suppose also that the populations N_1 and N_0 are kept constant in time by some pump and loss mechanism.

Let $\hat{\rho}$ be a density operator of the electromagnetic field. The master equation for $\hat{\rho}$ is [19, 24]

$$\frac{\partial \hat{\rho}}{\partial t} = -\gamma N_1(\hat{a}\hat{a}^\dagger \hat{\rho} - 2\hat{a}^\dagger \hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger) - \gamma N_0(\hat{a}^\dagger \hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger \hat{a}). \quad (4)$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the electromagnetic field, N_1 and N_0 are the populations of the upper and lower levels of a two-level atoms, and γ is the amplification coefficient.

Using relation between the density matrix and the Husimi function, one can pass from operator equation (4) to the corresponding ordinary differential equation for the Husimi function. Using this equation, the expression for the Husimi function for the amplified state was obtained in [24]:

$$Q(\alpha,t) = \int d^2\beta Q(\beta) \frac{1}{\pi m} \exp\left[-\frac{|\alpha - \beta G|^2}{m}\right]. \quad (5)$$

Here

$$G(t) = \exp[2(N_1 - N_0)\gamma t], \;\; m = \frac{N_0}{N_1 - N_0}(G^2 - 1). \quad (6)$$

In [10, 11] we considered the case where all atoms are in excited state and the population of the lower level is zero, $N_0 = 0$. In this case the parameter m is equal to zero, $m = 0$, and the expression for the Husimi function of the amplified state is simpler and takes the form (3)

$$Q_{out}(\alpha,t) = \frac{1}{G^2} Q_{in}(\alpha/G) = \left\langle \frac{\alpha}{G} |\hat{\rho}_{in}| \frac{\alpha}{G} \right\rangle. \quad (7)$$

We identify L-photon state of the electromagnetic field with the number state $|L\rangle$ of the harmonic oscillator.

The Husimi function of the number state $|L\rangle$ is

$$Q_L(\beta) = \langle\beta|L\rangle\langle L|\beta\rangle = e^{-|\beta|^2}\frac{|\beta|^{2L}}{L!}. \tag{8}$$

In [10, 11] we considered the situation when a pure L-photon state enters the amplifier and found the explicit form of the density matrix of the state at the output of the amplifier. This density matrix is

$$\hat{\rho}_L^{\lambda} = \sum_{j=0}^{\infty}\lambda^{2L+2}\frac{(1-\lambda^2)^j(L+j)!}{j!L!}|L+j\rangle\langle L+j|. \tag{9}$$

Here $\lambda = G^{-1}$.

The density matrix (9) is diagonal. Its first L diagonal elements with labels $0,1,\ldots,L-1$ are equal to zero, and then on its main diagonal are the values

$$F_{L+j}^L = \frac{(1-\lambda^2)^j(L+j)!}{j!L!}\lambda^{2L+2};\; j=0,1,\ldots. \tag{10}$$

The values (10) form a negative binomial distribution. The elements of this distribution are given by [25]

$$f(k,r,p) = \binom{r+k-1}{k}p^r q^k = \frac{(r+k-1)!}{(r-1)!(k)!}p^r(1-p)^k; \tag{11}$$
$$p+q=1;\; k=0,1,2,\ldots$$

These elements are defined by two parameters r and p, and the value k is the element label in the distribution. In our case $r = L+1,\ k=j,\ p=\lambda^2$. So we have

$$F_{L+j}^L = f(j, L+1, \lambda^2). \tag{12}$$

The density matrix (9) of the amplified L-photon state was found in our papers [10, 11]. Now we consider the case where the populations of levels $N_1, N_0;\ N_1 > N_0$ can take arbitrary values. In this case the evolution of Husimi function is defined by the equation (5). Here we again consider the situation when a pure L-photon state $|L\rangle$ is entering the amplifier. The Husimi function of this state has the form (8). We substitute this Husimi function in the integral (5), so that the Husimi function of the state at the output of the amplifier becomes

$$\tilde{Q}_L^{\lambda}(\alpha) = \frac{1}{\pi m L!}\int d^2\beta\, e^{-|\beta|^2}\,|\beta|^{2L}\exp\left[-\frac{|\alpha-\beta G|^2}{m}\right]. \tag{13}$$

Calculating this integral, we obtain

$$\tilde{Q}_L^{\lambda}(\alpha) = \frac{1}{L![m+G^2]}\frac{m^L}{[m+G^2]^N}\mathcal{L}_L\left(-\frac{|\alpha|^2G^2}{m(m+G^2)}\right)\exp\left(-\frac{|\alpha|^2}{m+G^2}\right). \tag{14}$$

Here $\mathcal{L}_L(x)$ is the Laguerre polynomial.

$$\mathcal{L}_L(x) = L!\sum_{k=0}^{L}(-1)^k\binom{L}{k}\frac{x^k}{k!}. \tag{15}$$

In [10, 11] we developed a method that allows one to reconstruct the density matrix from the Husimi function of the form similar to that in (14). Applying this method, we obtain that the density matrix $\tilde{\hat{\rho}}_L^{\lambda}$ of the amplified L-photon state reads

$$\begin{aligned}\tilde{\hat{\rho}}_L^{\lambda} &= \frac{m^L}{(m+G^2)^{L+1}}\sum_{k=0}^{\infty}\sum_{l=0}^{L}\binom{L}{L-l}\binom{k+L-l}{L-l}\cdot \\ &\cdot \left(\frac{G^2}{m(m+G^2)}\right)^{L-l}\left(1-\frac{1}{m+G^2}\right)^k|k+L-l\rangle\langle k+L-l|; \end{aligned}\tag{16}$$

$$\lambda^{-2} = m+G^2.$$

When $m=0$ all the terms in the sum vanish except the term for $l=0$, so that the density matrix reduces to

$$\tilde{\hat{\rho}}_L^{\lambda}(m\to 0) = \frac{1}{G^{2(L+1)}}\sum_{k=0}^{\infty}\binom{k+L}{L}\left(1-\frac{1}{G^2}\right)^k|k+L\rangle\langle k+L|. \tag{17}$$

The density matrix (16) can be presented in the form

$$\begin{aligned}\tilde{\hat{\rho}}_L^{\lambda} &= m^L\sum_{l=0}^{L}\binom{L}{l}\lambda^{2L+2}\left(\frac{G^2\lambda^2}{m}\right)^{L-l}\times \\ &\sum_{k=0}^{\infty}(1-\lambda^2)^k\binom{L-l+k}{k}|L-l+k\rangle\langle L-l+k| = \\ &= m^L\lambda^{2L}\sum_{l=0}^{L}\frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l}\hat{\rho}_{L-l}^{\lambda}. \end{aligned}\tag{18}$$

We see that the density matrix (18) of the state, which is obtained from the L-photon state, after it passes through the amplifier with arbitrary values of the occupation numbers N_1 and N_0, has the form of the sum of the density matrices $\hat{\rho}_{L-l}^{\lambda}$ of $(L-l)$-photon states passed through the amplifier, whose occupation number of the ground level is $N_0=0$, *i.e.* the medium is completely inverted. We studied the properties of such states in [13]. In the present paper they are given by (9). These density matrices are diagonal, with either zeros or values from the negative binomial distribution on the main diagonal. So, in the density matrix $\hat{\rho}_{L-l}^{\lambda}$ the first $(L-l-1)$ numbers on the diagonal are zeros, followed by values (10) F_{L+j}^{L} $j=0,1,\ldots$. The density matrix (18) is the sum of such matrices with coefficients that are elements of

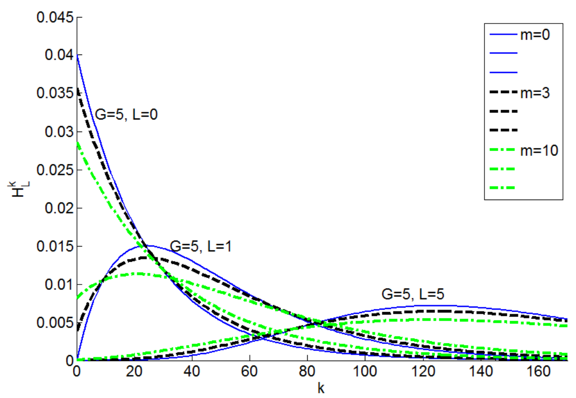

Fig. 1 – Families of curves representing diagonal elements of the density matrix of the amplified state for three different values of number of photons entering the amplifier L=0,1,5

the binomial distribution $\left(1+\frac{G^2}{m}\right)^L$. Thus, the elements of the density matrix (18) have the form of sums of the products of the elements of the binomial distribution on the elements of the negative binomial distribution. Note one important fact. In the case of the completely inverted medium, when the population $N_0 = 0$, the L-photon amplified state contains only states with indexes greater than L: $|L+1\rangle, |L+2\rangle, \ldots$. But if $N_0 > 0$, then at the output of the amplifier there are all k-photon states, starting with vacuum state (k=0). This also can be seen from Figs.1 and 2.

Let us find the form of diagonal elements of the matrix (18). To do this let us present the expression (18) as

$$\tilde{\hat{\rho}}^{\lambda}_L = \sum_{p=0}^{L} S^L_p \hat{\rho}^{\lambda}_p. \tag{19}$$

The coefficients S^L_p read

$$S^L_p = m^L \lambda^{2L} \left(\frac{G^2}{m}\right)^p \frac{L!}{p!(L-p)!} = m^{(L-p)} \lambda^{2L} G^{2p} \binom{L}{p}. \tag{20}$$

The density operator $\hat{\rho}^{\lambda}_p$ has the form

$$\hat{\rho}^{\lambda}_p = \sum_{j=0}^{\infty} F^p_{p+j} |p+j\rangle\langle p+j|, \tag{21}$$

where F^p_{p+j} are elements of the negative binomial distribution (10).

Now we can write the density operator (18) as

$$\tilde{\hat{\rho}}_L^\lambda = \sum_{k=0}^{\infty} H_k^L |k\rangle\langle k|, \tag{22}$$

here the coefficients H_k^L have the form

$$H_k^L = \sum_{j=0}^{k} S_j^L F_k^j, \quad k = 0, 1, \ldots, L; \tag{23}$$

$$H_k^L = \sum_{j=0}^{N} S_j^L F_k^j, \quad k = L+1, L+2, \ldots \infty.$$

Taking in mind (10) and (20) H_k^L, after some simple algebraic transformation, may be written in the form

$$H_k^L = (m\lambda^2)^L \lambda^2 (1-\lambda^2)^k \sum_{j=0}^{\min(L,k)} \left(\frac{\lambda^2 G^2}{m(1-\lambda^2)}\right)^j \binom{L}{j}\binom{k}{j}. \tag{24}$$

Using the following identity [26]

$$\sum_{k=0}^{\min(L,j)} \binom{L}{k}\binom{j}{k}\left(\frac{x-1}{2}\right)^k = P_{\min(L,j)}(x), \tag{25}$$

where $P_{\min(L,j)}(x)$ are Legendre polynomials, after some straightforward transformation we obtain

$$H_k^L = (m\lambda^2)^L \lambda^2 (1-\lambda^2)^k P_{\min(L,k)}\Big(1 + \frac{2\lambda^2 G^2}{m(1-\lambda^2)}\Big). \tag{26}$$

Let us evaluate the trace of the density operator (18).

$$\begin{aligned}
\mathrm{Tr}(\hat{\tilde{\rho}}_L^\lambda) &= \mathrm{Tr}\left(m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l} \rho_{L-l}^\lambda\right) \\
&= m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l} \mathrm{Tr}\left(\sum_{j=0}^{\infty} F_{p+j}^p |p+j\rangle\langle p+j|\right) \\
&= m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l} \sum_{j=0}^{\infty} F_{p+j}^p \\
&= \left(\frac{m}{m+G^2}\right)^L \sum_{l=0}^{L} \binom{L}{l}\left(\frac{G^2}{m}\right)^{L-l} \\
&= 1.
\end{aligned} \tag{27}$$

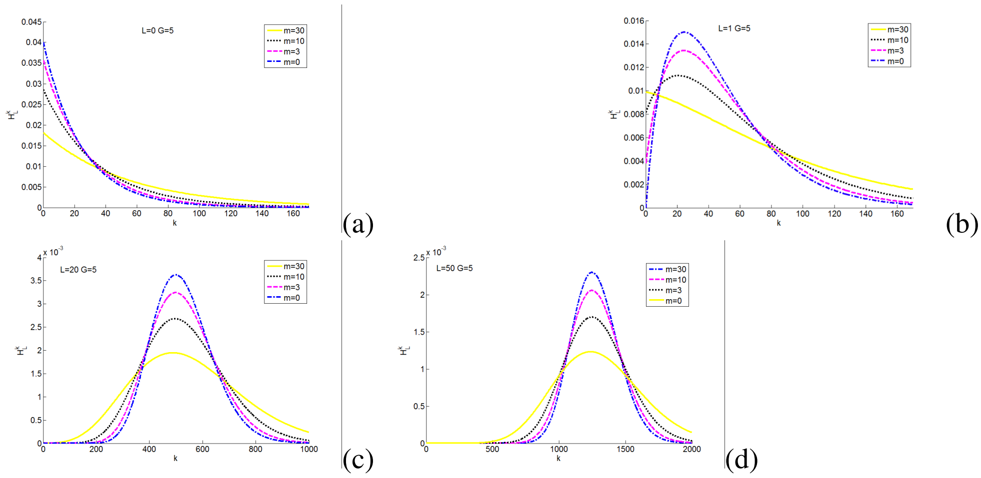

Fig. 2 – Diagonal elements of the density matrix of the amplified state for various values of parameter $m = 0, 3, 10, 30$ in the case that number of photons entering the amplifier (a) $L = 0$ (b) $L = 3$ (c)$L = 20$ (d) $L = 50$.

By direct calculation we proved that the trace of the operator (18) is equal to 1. The operator (18) is diagonal operator and the sum of its diagonal elements $\sum_{k=0}^{\infty} H_k^L = 1$. The trace of the operator $(\tilde{\hat{\rho}}_L^{\lambda})^2$

$$\text{Tr}((\tilde{\hat{\rho}}_L^{\lambda})^2) = \sum_{k=0}^{\infty} (H_k^L)^2 \le 1. \tag{28}$$

Therefore the operator (18) is indeed the density operator of a quantum state.

In the next section we will analyze some properties of the stretched states, starting from the state (9).

3. THE AVERAGE PHOTON NUMBER AND ITS FLUCTUATIONS FOR STRETCHED STATES

The average photon number for the state, given by the density matrix $\hat{\rho}$, is calculated as

$$\langle \hat{n} \rangle = Tr(\hat{n}\hat{\rho}). \tag{29}$$

Here $\hat{n} = \hat{a}^{\dagger}\hat{a}$ - is the number operator.

For the state (9) the average photon number is

$$\langle \hat{n}_L \rangle^{\lambda} = \sum_{j=0}^{\infty} \frac{\lambda^2 (1-\lambda^2)^j}{j!} \left(\frac{(L+j)!}{L!} \lambda^{2L} (L+j) \right), \tag{30}$$

with labels L and λ indicating the state for which the average photon number is calculated. In order to calculate this sum, we will use the following auxiliary sums

$$S_L^0(x) = \sum_{j=0}^{\infty} \frac{(L+j)!}{j!} x^j = \frac{L!}{(1-x)^{L+1}}, \;\; |x| < 1, \tag{31}$$

by substituting $x = 1-\lambda^2$, we get

$$\sum_{j=0}^{\infty} \lambda^{2L+2} \frac{(L+j)!}{L!j!} (1-\lambda^2)^j = 1. \tag{32}$$

Further we have

$$S_L^1(x) = \sum_{j=0}^{\infty} j \frac{(L+j)!}{j!} x^j = x \frac{d}{dx} S_L^0(x) = \frac{x(L+1)!}{(1-x)^{L+2}}. \tag{33}$$

Using the relations (32), (33), we obtain the expression for the average number of photons in the stretched state (9)

$$\langle \hat{n}_L \rangle^\lambda = L + (S_L^0)^{-1} S_L^1 = L + \frac{(L+1)(1-\lambda^2)}{\lambda^2} = \frac{L+1}{\lambda^2} - 1. \tag{34}$$

Let us now find the expression for the fluctuation of the average photon number in the state (9). This quantity is defined as

$$\sigma_n = \langle \hat{n}^2 \rangle - (\langle \hat{n} \rangle)^2. \tag{35}$$

To calculate its value it is necessary to know the quantity $\langle \hat{n}_L^2 \rangle$. Let us find its value.

$$\begin{aligned} \langle \hat{n}_L^2 \rangle^\lambda &= \sum_{j=0}^{\infty} \lambda^{2L+2} \frac{(1-\lambda^2)^j}{j!} \frac{(L+j)!}{L!} (L+j)^2 \\ &= L^2 + 2L \langle \hat{n}_L \rangle^\lambda + \sum_{j=0}^{\infty} \lambda^{2L+2} \frac{(1-\lambda^2)^j}{j!} \frac{(L+j)!}{L!} j^2. \end{aligned} \tag{36}$$

Using the auxiliary sum

$$\begin{aligned} S_L^2(x) &= \sum_{i=0}^{\infty} i^2 \frac{(L+i)!}{i!} x^i = x \frac{d}{dx} S_1 = \left(x \frac{d}{dx} \right)^2 S_0 \\ &= \frac{x(L+1)}{1-x} + \frac{x^2(L+1)(L+2)}{(1-x)^2}, \end{aligned} \tag{37}$$

we obtain

$$\begin{aligned}\langle \hat{n}_L^2 \rangle^\lambda &= L^2 + \frac{1-\lambda^2}{\lambda^2}(L+1)(2L+1) + \frac{(1-\lambda^2)^2}{\lambda^4}(L+1)(L+2) \\ &= \frac{(1-\lambda^2)(2-\lambda^2)}{\lambda^4} + 3\frac{1-\lambda^2}{\lambda^4}L + \frac{1}{\lambda^4}L^2. \end{aligned} \tag{38}$$

Finally, using the relations (34) and (38), we obtain the expression for the fluctuation of the number of photons in the state (9)

$$\sigma_n^\lambda = \langle \hat{n}_L^2 \rangle^\lambda - (\langle \hat{n}_L \rangle^\lambda)^2 = \frac{x(L+1)}{(1-x)^2} = \frac{(L+1)(1-\lambda^2)}{\lambda^4}. \tag{39}$$

Consider now the amplified L-photon state (18) obtained using a quantum amplifier with $m \neq 0$. We must calculate the average number of particles in the state (18). To do this, we use, as before (9). The expression for the density matrix now looks like (18). Substituting it into (29), we obtain

$$\langle \tilde{\hat{n}}_L \rangle^\lambda = Tr(\hat{n}\tilde{\hat{\rho}}_L^\lambda) = m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!} \left(\frac{G^2}{m}\right)^{L-l} \langle \hat{n}_{L-l} \rangle^\lambda. \tag{40}$$

The value $\langle \hat{n}_{L-l} \rangle^\lambda$ we have already calculated. Substituting the expression (34) into (40), we obtain

$$\begin{aligned}\langle \tilde{\hat{n}}_L \rangle^\lambda &= m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!} \left(\frac{G^2}{m}\right)^{L-l} \left(\frac{L-l+1}{\lambda^2} - 1\right) \\ &= m^L \lambda^{2L-2} \left(1 + \frac{G^2}{m}\right)^{L-1} \left(1 + \frac{G^2}{m} + L\frac{G^2}{m}\right) - \lambda^{2L}(m+G^2)^L \\ &= (m + G^2 + LG^2) - 1. \end{aligned} \tag{41}$$

Here we have taken into account that $\lambda^{-2} = m + G^2$.

When $m = 0$ the expression (41) reduces to the formula (34). Let us find now the expression for the fluctuation of the average photon number in the state (18). In order to do this, one must find the value of

$$\langle \tilde{\hat{n}}_L^2 \rangle^\lambda = m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!} \left(\frac{G^2}{m}\right)^{L-l} \langle \hat{n}_{L-l}^2 \rangle^\lambda. \tag{42}$$

We have calculated $\langle \hat{n}_L^2 \rangle^\lambda$ in (38), using it one obtains

$$\begin{aligned}\langle \tilde{\hat{n}}_L^2 \rangle^\lambda &= m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!} \left(\frac{G^2}{m}\right)^{L-l} \Big((L-l)^2 \\ &+ \frac{1-\lambda^2}{\lambda^2}(L-l+1)(2L-2l+1) + \frac{(1-\lambda^2)^2}{\lambda^4}(L-l+1)(L-l+2)\Big). \end{aligned} \tag{43}$$

First we calculate separately three sums P_0, P_1, P_2.

$$\begin{aligned} P_0 &= m^L\lambda^{2L}\sum_{l=0}^{L}\frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l}\left(\frac{1-\lambda^2}{\lambda^2}+2\frac{(1-\lambda^2)^2}{\lambda^4}\right) \\ &= \frac{(1-\lambda^2)(2-\lambda^2)}{\lambda^4}. \end{aligned} \quad (44)$$

$$\begin{aligned} P_1 &= m^L\lambda^{2L}\sum_{l=0}^{L}\frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l}(L-l)3\left(\frac{1-\lambda^2}{\lambda^2}+\frac{(1-\lambda^2)^2}{\lambda^4}\right) \\ &= 3LG^2\frac{1-\lambda^2}{\lambda^2}. \end{aligned} \quad (45)$$

$$\begin{aligned} P_2 &= m^L\lambda^{2L}\sum_{l=0}^{L}\frac{L!}{l!(L-l)!}\left(\frac{G^2}{m}\right)^{L-l}(L-l)^2\left(1+2\frac{1-\lambda^2}{\lambda^2}+\frac{(1-\lambda^2)^2}{\lambda^4}\right) \\ &= m^L\lambda^{2L-4}\left(L(L-1)\left(\frac{G^2}{m}\right)^2\left(1+\frac{G^2}{m}\right)^{L-2}+L\left(\frac{G^2}{m}\right)\left(1+\frac{G^2}{m}\right)^{L-1}\right) \\ &= LG^2(LG^2+m). \end{aligned} \quad (46)$$

Finaly we obtain

$$\langle\tilde{\hat{n}}_L^2\rangle^\lambda = P_0+P_1+P_2 = \frac{(1-\lambda^2)(2-\lambda^2)}{\lambda^4}+3LG^2\frac{1-\lambda^2}{\lambda^2}+LG^2(LG^2+m). \quad (47)$$

When $m=0$ the expression (47) reduces to (38).

Now, using the relations (41) and (47), we obtain the expression for the dispersion of the number of photons in the state (18)

$$\begin{aligned} \sigma_n^\lambda &= \langle\tilde{\hat{n}}_L^2\rangle^\lambda-(\langle\tilde{\hat{n}}_L\rangle^\lambda)^2 \\ &= \frac{(1-\lambda^2)(2-\lambda^2)}{\lambda^4}+3LG^2\frac{1-\lambda^2}{\lambda^2} \\ &+ LG^2(LG^2+m)-(m+G^2+LG^2-1)^2. \end{aligned} \quad (48)$$

When $m=0$ the expression (48) reduces to (39).

4. AVERAGES OF THE PHOTON NUMBER OPERATOR POWERS

We have found the quantities $\langle\tilde{\hat{n}}_L\rangle^\lambda$ and $\langle\tilde{\hat{n}}_L^2\rangle^\lambda$. Now we want to derive expression for the $\langle\tilde{\hat{n}}_L^l\rangle^\lambda$ in the state (18). To do this, first calculate the average value $\langle\hat{n}_L^l\rangle^\lambda$

in the state (9). In this case we have

$$\begin{aligned}\langle \hat{n}_L^l\rangle^\lambda &= \sum_{j=0}^{\infty}\lambda^{2L+2}\frac{(1-\lambda^2)^j}{j!}\frac{(L+j)!}{L!}(L+j)^l \\ &= \sum_{j=0}^{\infty}F_{L+j}^L\sum_{k=0}^{l}\binom{l}{k}j^kL^{l-k} \\ &= \sum_{k=0}^{l}G_L^k(\lambda)\binom{l}{k}L^{l-k}. \qquad (49)\end{aligned}$$

Here $\binom{l}{k}=\frac{l!}{(l-k)!k!}$ - are the binomial coefficients, and the quantities $G_L^k(\lambda)$ read

$$G_L^k(\lambda)=\sum_{j=0}^{\infty}\lambda^{2L+2}\frac{(1-\lambda^2)^j}{j!}\frac{(L+j)!}{L!}j^k. \qquad (50)$$

Let us calculate these sums. As we have shown, there exist equalities

$$S_L^0(x)=\sum_{i=0}^{\infty}\frac{(L+i)!}{i!}x^i=\frac{L!}{(1-x)^{L+1}}, \qquad (51)$$

$$\begin{aligned}S_L^1(x) &= \sum_{i=0}^{\infty}i\frac{(L+i)!}{i!}x^i=x\frac{d}{dx}S_0, \\ S_L^2(x) &= \sum_{i=0}^{\infty}i^2\frac{(L+i)!}{i!}x^i=x\frac{d}{dx}S_1=\left(x\frac{d}{dx}\right)^2S_0, \\ &\vdots \\ S_L^k(x) &= \sum_{i=0}^{\infty}i^k\frac{(L+i)!}{i!}x^i=\left(x\frac{d}{dx}\right)^kS_0, \\ &\vdots\end{aligned}$$

Using these relations and the definition (50), we obtain expressions for the quantities $G_L^k(\lambda)$

$$G_L^k(\lambda)=(S_L^0(1-\lambda^2))^{-1}S_L^k(1-\lambda^2). \qquad (52)$$

Thus, for $\langle \hat{n}_L^l\rangle^\lambda$ we obtain the following expression

$$\langle \hat{n}_l^l\rangle^\lambda=\sum_{k=0}^{l}(S_L^0(1-\lambda^2))^{-1}S_L^k(1-\lambda^2)\binom{l}{k}L^{l-k}. \qquad (53)$$

Note the fact that when constructing the value $\langle \hat{n}_L^l \rangle^\lambda$ both binomial and negative binomial coefficients are used.

Now one can get the expression for $\langle \tilde{\hat{n}}_L^l \rangle^\lambda$ in the state (18)

$$\langle \tilde{\hat{n}}_L^l \rangle^\lambda = m^L \lambda^{2L} \sum_{l=0}^{L} \frac{L!}{l!(L-l)!} \left(\frac{G^2}{m} \right)^{L-l} \langle \hat{n}^l \rangle. \tag{54}$$

For $l = 1$ formula (54) gives the expression (41), and for $l = 2$ formula (54) gives the expression (47).

5. CONCLUSION

In this work we investigated the relationship between the dynamics of physical processes and the transformations of the phase space and the functions defined on it. As an example we considered a linear quantum amplifier. For this system it is possible to describe all these relations explicitly. It is found that depending on the quantum amplifier condition, the transformation of the phase have fundamentally different form.

If all atoms of the quantum amplifier are excited, *i.e.*, $N_0 = 0$, the transformations of the phase space and Husimi functions are defined by the formulas (2), (3). But if both levels of the quantum amplifier, ground and excited, have non-zero population, then transformation of Husimi functions has a more complex form. It is determined by the formula (7). In our particular case this transform takes the form (18). There is the fundamental difference between transformations (13) and (7). The transformation (7) is local, and (13) is a non-local transformation. This difference of transformations generates a difference in the structure of the amplified states.

This approach can also be used to study dissipative systems [27]-[30]. In the monograph [21], an expression for the Q-function of a damped harmonic oscillator is given, which has the form similar to the formula (5).

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