A Tropical Approach to the Brill-Noether Theory Over Hurwitz Spaces

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Digital Object Identifier: https://doi.org/10.13023/etd.2021.231

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DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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2021

ABSTRACT OF DISSERTATION

A Tropical Approach to the Brill-Noether Theory Over Hurwitz Spaces

The geometry of a curve can be analyzed in many ways. One way of doing this is to study the set of all divisors on a curve of prescribed rank and degree, known as a Brill-Noether variety. A sequence of results, starting in the 1980s, answered several fundamental questions about these varieties for general curves. However, many of these questions are still unanswered if we restrict to special families of curves. This dissertation has three main goals. First, we examine Brill-Noether varieties for these special families and provide combinatorial descriptions of their irreducible components. Second, we provide a natural generalization of Brill-Noether varieties, known as Splitting-Type varieties, that parameterize this decomposition. Lastly, we provide purely combinatorial descriptions of these Splitting-Type varieties and explore the geometric consequences of these descriptions. These results are based upon and extend tools and techniques from tropical geometry.

KEYWORDS: curves, divisors, geometry, tableaux, tropical

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June 29, 2021

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ACKNOWLEDGMENTS

I don't think it's possible for me to accurately acknowledge everything that has gone into the development of this thesis. Over the course of 10 years I've received assistance, encouragement, and support from more people than I can shake 2 sticks at, but I will try to do what I can with the space I have available.

First, I want to thank my advisor, Dave, for taking me on as a graduate student when he did and keeping me focused on the long-term goal of research for this dissertation when I wasn't sure I would make it through my preliminary exams. Thank you for how often you had your door open and being willing to talk about more than just math. You've helped me realize what it means to be a mathematician doing research and have done more than I realize to guide me down my current path. You were a great advisor and I know your future students will also appreciate the time, effort, and energy you put into your students.

Next, I want to thank everyone in the Math Department at UKY for their support and help over the years. The department has built a community of grad students, facutly, and staff that are friendly, warm, and welcoming. They helped me find my bearings as an academic, taught me what it means to be a teacher who cares for their students, and made me feel welcome in a new state. I will miss you all dearly.

Thanks to all of the professors I had in my undergraduate career as well, most notably Professors Clinkscale, Ganjizadeh, and Villareal for your support. You believed in me before I really believed in myself and I still look back to your classes for encouragement and inspiration.

Thanks to my family and friends down home in Texas who supported my decision to go back to college and supported me from afar when I up and moved away. You've been excited for me for the accomplishments that didn't make much sense and you've

been supportive through the tough times that you didn't understand. You even took the time to ask about what I do as a mathematician even though it's hard enough to explain to other mathematicians.

I want to express my heartfelt thanks to Center Point Church for doing everything you can to personally welcome and connect with everyone who comes through your doors. Thank you for all the years that you have invested into my life through worship, connect group, summer studies, and more. Even though I'm sad that I have to leave I'm excited that God let me spend a season with you.

Finally, I want to thank a few special friends I made at Kentucky. Camille, Hunter, Kalila, and Margaret, you're a bunch of wonderful nerds who have shown me love and friendship that I couldn't repay with all of the time in the world. I don't think I would have made it without your support. You've born through my bad (but amazing) jokes, forgiven the times I've been grumpy or unreasonable, and helped push me to improve myself with your examples.

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Chapter 1 Background and Motivation

This chapter will provide a list of definitions, methods, and questions underlying chapters 2, 3, and 4. While not self-contained, hopefully there is enough background material here to give the non-expert a starting point for everything that follows. References will be provided in each corresponding section.

1.1 Algebraic Geometry

Algebraic geometry is the study of geometric objects using algebraic methods. I suggest chapter 1 of [21] for a thorough introduction. The central geometric objects of interest are *algebraic varieties*, simultaneous solutions to a set of polynomials. Some simple examples occur with only one equation, known as hypersurfaces. For example, consider the polynomial $y - x$. The associated variety $V(y - x)$, depicted in 1.1, is the set of all pairs (x, y) such that $y - x = 0$.

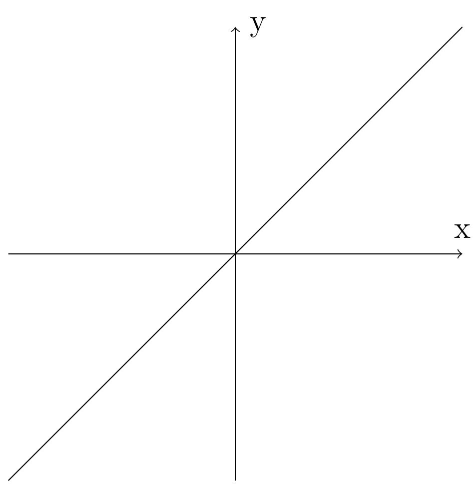


Figure 1.1: The variety $V(y - x)$

For an example with more than one equation we can look to linear algebra. Consider the system of equations

$$f_1 = 2x - y - x - 2,$$

$$f_2 = -x + 2y - 1,$$
$$f_3 = -x + 2z - 1.$$

The variety $X = V(f_1, f_2, f_3)$ is the set of all triples (x, y, z) such that $f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z) = 0$. By solving this system of equations we can see that X is just the single point $(3, 2, 2)$.

When you encounter a variety X in the wild there are several fundamental questions that you might want to answer.

1. How "large" is X?
2. Can X be decomposed into "smaller" pieces?
3. Is X "smooth?"

The term we use to describe the size of a variety is its *dimension.* For example, a finite collection of points, like $V(f_1, f_2, f_3)$ in the previous example, is 0-dimensional, a 1-dimensional variety, like $V(y - x)$, is known as a curve, and a 2-dimensional variety is known as a surface. In chapter 3 we focus on computing the dimensions of certain varieties associated to a curve. Then in chapter 4 we provide a method to count the number of points they contain in the 0-dimensional case.

Decomposing an object is a common theme in many mathematical fields. Algebraic geometry uses a notion known as reducibility. Given a variety X there is a suitable topology one can place on it, known as the *Zariski topology*, where the closed subsets of X coincide with subvarieties of X. X is called *reducible* if it can be decomposed into a finite union $X = X_1 \cup X_2$ of distinct subvarieties $X_1, X_2 \not\subseteq X$. If no such decomposition exists, then we say X is *irreducible.* For example, the variety $V(xy) \subset \mathbb{R}^2$ is a reducible variety, and can be decomposed as the union of the x and y-axis, which are the varieties defined by $V(y)$ and $V(x)$, respectively.

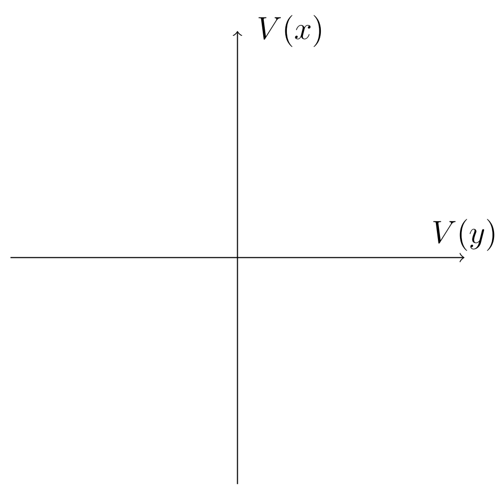


Figure 1.2: The reducible variety $V(xy)$

The line $V(y)$, on the other hand, is irreducible since the only subvarieties of $V(y)$ are finite unions of points. In chapter 3 we produce decompositions of certain varieties into irredicuble components.

Finally, smoothness appears in many contexts. Algebraic geometers measure smoothness using tangent spaces. Every variety X is defined by a set of polynomial equations. Apriori, this set may be infinite. However the *Hilbert Basis Theorem* says that there always exists a finite set $f_1, ..., f_r$ of polynomials that cuts out the variety $X = V(f_1, ..., f_r)$. The importance of this fact here is that we can look at the Jacobian matrix of first-order partial derivatives of $f_1, ..., f_r$ and evaluate this at a point $p \in X$. We say p is a *singular point* if the rank of this evaluated Jacobian matrix is lower than the rank of another point $p' \in X$. A singular point is a point with no well-defined tangency. In the case of curves this says the tangent-line at the point p does not exist. Otherwise we say that p is a *non-singular* or *smooth* point of X. A variety with no singular points is called *smooth.* For example, the unit circle $V(x^2 + y^2 - 1)$ is smooth, but the cuspidal cubic $V(y^2 - x^3)$ is not.

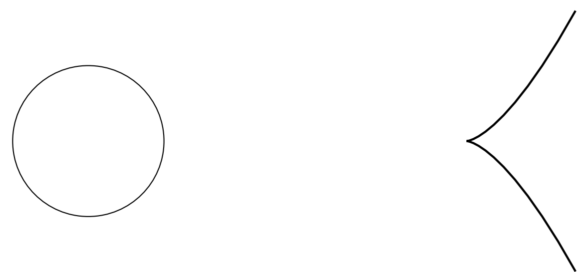


Figure 1.3: The smooth variety $V(x^2 + y^2 - 1)$ and the singular variety $V(y^2 - x^3)$

An important development in algebraic geometry has been the study of *moduli spaces.* A moduli space is a space parameterizing all varieties, or associated objects, with a given set of properties. An important feature of this development is that these parameter spaces are also varieties that convey information about the objects they parameterize. The moduli space $\overline{M}_g$ parameterizing all smooth, projective curves of genus g is of particular importance in the study of algebraic curves. Chapters 3-4 focus on questions arising from curves in another moduli space $H_{g,k}$ which parameterizes a particular family of genus g curves, known as *k-gonal curves.* When we say a curve is *general* we'll mean that it is contained in a dense open subset of one of these two moduli spaces.

1.2 Brill-Noether Theory

For this section I, again, refer the reader to [21], in particular Chapter 4. An important question that arises when one studies varieties is the following. Suppose I have two varieties X_1, X_2, how I do tell them apart? One way to distinguish them is if they have different dimensions, but what if their dimensions are equal? The next step might be to consider the reducibility or smoothness of these varieties.

After some consideration, though, one might notice that these types of questions also affect the functions defined on the varieties. This functional perspective is a modern approach to the classification problem. We instead try to understand which functions encode which geometric properties and then set about trying to understand all functions defined on an object. Brill-Noethery theory is concerned with studying maps of curves into projective space $f : C \hookrightarrow \mathbb{P}^r$.

If a curve has an embedding into projective space $\mathbb{P}^r$ we call it *projective.* If it is also smooth we can equivalently phrase the study of all embeddings as the study of *divisors* on our curve. A divisor D on a curve C is a finite, linear combination of the points of C using integer coefficients. The *degree* of a divisor is the sum of the coefficients appearing in this linear combination, and it is *effective* if the coefficients are all non-negative.

A *rational function* f on a variety X is a function that can be expressed as a quotient of polynomials $f = h/g$. One can associate a divisor to f in canonical way by defining $(f) = \sum_{p \in X} \text{ord}_p(f)p$, where $\text{ord}_p(f)$ is the order of vanishing of f at p. If f has a zero at p, then this coefficient is positive, if it has a pole it's negative, and otherwise it's zero. Two divisors are called *linearly equivalent* if their difference is the divisor of a rational function.

The *rank* of a divisor D is defined to be the largest integer r such that $D - E$ is linearly equivalent to an effective divisor, for all effective divisors E of degree r. If D is not equivalent to any effective divisor it has rank -1.

Given a curve C, the *Picard group* of C, $Pic(C)$, is the group of all divisors on C modulo this linear equivalence. Further, given a pair (r, d) of non-negative integers, one can define a space $W^r_d(C) \subset Pic(C)$ consisting of all divisor classes on C of degree d and rank at least r. This parameter space is known as a *Brill-Noether variety.* These varieties stratify the Picard group of C, so developing an understanding of Brill-Noether varieties is key to understanding all divisors on a curve. A sequence of results in the 1980s answered the fundamental questions listed in the previous section in the case that C is a general curve in $\overline{M}_g$. Specifically, when C is a general curve in $\overline{M}_g$:

- (Griffiths-Harris, [18]) $\dim W^r_d(C) = \rho(g, r, d) = g - (r + 1)(g - d + r)$, and $W^r_d(C)$ is empty when this value is negative,
- (Fulton-Lazarsfeld, [16]) $W^r_d(C)$ is irreducible when $\rho(g, r, d) > 0$, and
- (Gieseker, [17]) $W^r_d(C)$ is smooth away from $W^{r+1}_d(C)$.

A key technique to these results is known as degeneration. Instead of studying divisors on smooth curves one studies divisors on mildly singular curves. If you can show that you have a complete understanding of divisors on these singular curves and can regenerate to divisors on smooth curves, then you also have a complete understanding of divisors on smooth curves. One reason to focus on the divisor theory of these singular curves is that this analysis can be done purely combinatorially.

An important distinction to emphasize is that the preceding results concern *general* curves, not *all* curves. To generalize to *all* curves one first needs another way of distinguishing curves. Since we first classify divisors by rank and degree, we can start by restricting to divisors of a given rank. Given a fixed rank r one can ask for the minimum degree d such that the Brill-Noether variety $W^r_d(C)$ is non-empty. A special case occurs when we fix $r = 1$, which we refer to as the *gonality* of C. Returning to general curves, you can use the Brill-Noether number

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

to deduce that the gonality of a general curve is $\lfloor\frac{g+3}{2}\rfloor$. From here on, when we refer to a k-gonal curve we mean a curve with gonality k smaller than $\lfloor\frac{g+3}{2}\rfloor$. The moduli space of all curves of genus g and gonality k is the Hurwitz space $H_{g,k}$ mentioned in the preceding section.

The purpose of this dissertation is to develop tools and techniques to understand the Brill-Noether varieties of k-gonal curves. We do this after a degeneration, where instead of focusing on the divisors of smooth curves, or even mildly singular curves, we focus on the divisor theory of a particular family of tropical curves. The divisor theory on these curves may again be described completely combinatorially, as we show, and introduces the field to rich, well-studied areas of combinatorics that may now have unknown geometric implications.

1.3 Tropical Geometry

Tropical geometry is sometimes referred to as a combinatorial shadow of algebraic geometry. Most of the definitions and language one sees in papers about tropical geometry are natural combinatorial analogues of those found in algebraic geometry. There also tend to be two distinct approaches one finds when reading papers, articles, or books about the subject. While they may seem incomparable at first, there is an important connection between them. In the end, one may view the difference between these approaches as analogous to the difference between studying a variety with a choice of embedding or an abstract variety.

The first approach involves the study of varieties defined over the *tropical semiring*, $(\mathbb{R}\cup\{\infty\},\oplus,\otimes)$. For an appropriate introduction I suggest [28]. As a word of caution, the symbols $\oplus,\otimes$ are used differently in this section than any other section of this dissertation. In the following sections $\oplus$ is the direct sum of line bundles and $\otimes$ is their tensor product.

Given two elements $x, y \in \mathbb{R}\cup\{\infty\}$ we define

- $x \oplus y = \min\{x, y\}$, and
- $x \otimes y = x + y$.

All of the usual properties one desires of an addition and a multiplication hold *except* for the existence of subtraction. That is to say, something like $x\oplus 3 = 3$ does not have a unique solution. It is for this reason that $(\mathbb{R}\cup\{\infty\},\oplus,\otimes)$ is a semiring and not a ring, or field. *Tropical varieties* are defined to be the solutions to sets of polynomials in the tropical semiring, which is to say one must use the definitions of $\oplus$ and $\otimes$. A first step to understanding tropical varieties is to start with tropical hypersurfaces. In this case we have

$$V(f) = \{\boldsymbol{w} \in \mathbb{R}^n | \text{the minimum in } f(\boldsymbol{w}) \text{ is achieved at least twice}\}.$$

For example, consider $f = x \oplus y \oplus 0 = \min\{x, y, 0\}$. $V(f)$, a *tropical line*, is

$$V(f) = \{x = y \leq 0\} \cup \{x = 0 \leq y\} \cup \{y = 0 \leq x\}.$$

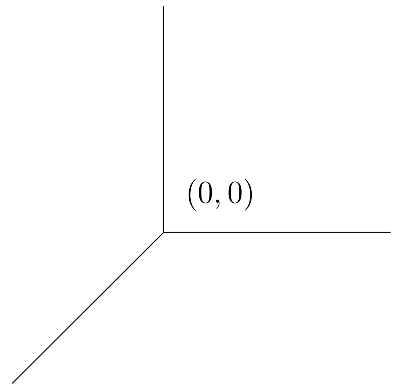


Figure 1.4: A tropical line

The transition from algebraic to tropical geometry involves valuations. If K is a field, a *valuation* on K is a function $\nu : K^* \to \mathbb{R} \cup \{\infty\}$ satisfying the following[1]:

- $\nu(a) = \infty$ if and only if $a = 0$
- $\nu(a + b) \geq \min\{a, b\}$, and
- $\nu(ab) = \nu(a) + \nu(b)$.

The way we travel from algebraic geometry to tropical geometry is by picking our favorite field K, a valuation ν on K, and our favorite subvariety X of the torus $(K^*)^n$. The *tropicalization* of X, $\text{trop}(X)$, is

$$\text{trop}(X) = \{(\nu(x_1), ..., \nu(x_n)) | (x_1, ..., x_n) \in X\}.$$

Tropical varieties are combinatorial objects known as polyhedral complexes, a collection Σ of polyhedra satisfying the following:

- If P is in Σ, then so is any face of P, and
- if P and Q are in Σ, then $P \cap Q$ is either empty or a face of both P and Q.

A polyhedron P is the intersection of finitely many closed half-spaces, that is $P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$, where A is a $d \times n$ matrix and $\mathbf{b} \in \mathbb{R}^d$.

In particular, when X is an irreducible variety of dimension d trop(X) is of pure dimension d and connected through codimension 1 ([28]). A polyhedral complex is *pure of dimension* d if every polyhedron of Σ that isn't a face of another polyhedron has dimension d. It is additionally *connectd through codimension 1* if for any two d-dimensional cells P, P' of Σ there is a chain $P = P_1, P_2, ..., P_s = P'$ such that P_i and P_{i+1} share a common facet F_i.

[1] There are two conventions for valuations one encounters, known as the "max" and "min" conventions. The one introduced here is the min convention. The other uses the max by replacing ∞ with $-\infty$ and flipping the necessary inequality, but there's no substantial difference between the two.

It is important to be aware that the converse of this statement is not true in general. Not every polyhedral complex of pure dimension d that is connected through codimension 1 is the tropicalization of a variety. This suggests the usage of the phrase combinatorial shadow; what you see may not come from what you expect.

Establishing when a polyhedral complex satisfying these criteria is the tropicalization of a variety is an important question in this field. In order to accomplish this you most commonly develop what are called lifting techniques, which are the tropical analogs of regeneration theorems from Brill-Noether theory and other problems using degeneration arguments. In chapter 4 we establish that the tropical varieties we study are of pure dimension d and connected through codimension 1. This suggest, with other evidence, that they are the tropicalizations of their algebraic analogues.

Now for a brief discussion on the other flavor of tropical geometry one finds that references Berkovich analytic spaces. For a much better description I highly recommend [31]. First, let K be a nonarchimedean field, which is to say that K is a field with a norm $|-|$ that does not satisfy the archimedean property. Given a variety $X = V(f_1, ..., f_r)$ one can associate a seminorm on the ring $K[X] = K[x_1, ..., x_n]/(f_1, ..., f_r)$ to a point $x \in X$. A *multiplicative seminorm* is a function $|-| : K[x] \to \mathbb{R}_{\geq 0}$ that satisfies

- $|fg| = |f||g|$, and
- $|f + g| \leq |f| + |g|$.

The seminorm associated to $x \in X$ is $|f|_x = |f(x)|$. For simplicity we restrict to the cases when $|-|_x$ is actually the given norm $|-|$ when restricting to K, in which case we say $|-|_x$ extends the norm on K. Since the norm on K is nonarchimedean it holds that $|-|_x$ also satisfies the *ultrametric inequality*

$$|f + g| \leq \max\{|f|, |g|\}, \text{ with equality if and only if } |f| \neq |g|.$$

We then define the *Berkovich analytification* of X, X^{an}, to be the space of all seminorms on $K[X]$ that extend the norm on K.

The connections between these two approaches was made precise in [30] and [15], which show that the analytification of a variety X is the limit, in a technical sense, of all tropicalizations of X. We focus on something known as the skeleton of X^{an}. When X is a curve the analytification is a metric graph Γ. A metric graph can either be viewed as a union of intervals of varying length, or a discrete graph with weights associated to the edges.

One can form a theory of divisors on metric graphs analogous to that of divisors on algebraic curves. Here rational functions on Γ are piecewise linear functions $\phi : \Gamma \to \mathbb{R}$ with integer slopes and $ord_v(\phi)$ is the sum of the incoming slopes of ϕ at v. While these definitions are for arbitrary metric graphs, we are interested in the case when Γ is a chain of loops, as depicted in Figure 2.2. The reason we focus on this case is that the chain of loops behaves much like the mildly singular curves discussed in the preceding chapter.

Chapter 2 Preliminaries

The material for this preliminary chapter is drawn from the papers *Components of Brill-Noether Loci for Curves with Fixed Gonality* and *Tropical Methods in Hurwitz-Brill-Noether Theory*, both authored by Cook-Powell-Jensen. The first has been accepted for publication in the *Michigan Math Journal* and the second is a preprint.

2.1 Splitting Types

In this section, we review the definition of splitting types and discuss some of their basic properties. Let $\pi : C \to \mathbb{P}^1$ be a branched cover of degree k and genus g, and let L be a line bundle on C. The pushforward $\pi_* L$ is a vector bundle of rank k on $\mathbb{P}^1$, and every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles

$$\pi_* L \cong \mathcal{O}(\mu_1) \oplus \cdots \oplus \mathcal{O}(\mu_k).$$

The integers $\mu_1, \ldots, \mu_k$ are unique up to permutation. We will assume throughout that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k.$$

The vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ is known as the *splitting type* of the vector bundle, and we write $\pi_*(L) \cong \mathcal{O}(\boldsymbol{\mu})$ for ease of notation. It is helpful to think of a splitting type $\boldsymbol{\mu}$ as a partition with possibly negative parts. This is because, for any ℓ, the sum of the ℓ smallest entries of $\boldsymbol{\mu}$ is a lower semicontinuous invariant. It is therefore natural to endow the set of splitting types with a partial order, extending the dominance order on partitions.

Definition 2.1.1. *We define the* dominance order *on splitting types as follows. Let $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ be splitting types satisfying $\sum_{i=1}^k \mu_i = \sum_{i=1}^k \lambda_i$. We say that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ if and only if*

$$\mu_1 + \cdots + \mu_\ell \leq \lambda_1 + \cdots + \lambda_\ell \qquad \text{for all } \ell \leq k.$$

The splitting type of $\pi_* L$ determines the rank and degree of the line bundle L, as well as the rank of all its twists by line bundles pulled back from the $\mathbb{P}^1$. This can be seen by the Projection Formula, as follows:

$$\begin{aligned} h^0(C, L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(m)) &= h^0(\mathbb{P}^1, \pi_* L \otimes \mathcal{O}_{\mathbb{P}^1}(m)) \qquad (\star) \\ &= \sum_{i=1}^k h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mu_i + m)) \\ &= \sum_{i=1}^k \max\{0, \mu_i + m + 1\}. \end{aligned}$$

In particular, we have

$$h^0(L) = \sum_{i=1}^{k} \max\{0, \mu_i + 1\} \qquad \text{and}$$

$$\deg L = g + k - 1 + \sum_{i=1}^{k} \mu_i.$$

For ease of notation, we write these expressions as:

$$\begin{aligned} h^0(C, L \otimes \pi^*\mathcal{O}(m)) &= x_m(\boldsymbol{\mu}) := \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\} \qquad (2.1) \\ h^1(C, L \otimes \pi^*\mathcal{O}(m)) &= y_m(\boldsymbol{\mu}) := \sum_{i=1}^{k} \max\{0, -\mu_i - m - 1\} \\ \deg L = d(\boldsymbol{\mu}) &:= g - 1 + \sum_{i=1}^{k} (\mu_i + 1). \end{aligned}$$

This data suggests the following definition.

Definition 2.1.2. *Let C be a k-gonal curve and $\boldsymbol{\mu}$ a splitting type, then the splitting type loci are the strata*

$$W^{\boldsymbol{\mu}}(C) = \{L \in \mathrm{Pic}(C) \mid \pi_* L \cong \mathcal{O}(\boldsymbol{\mu})\}$$
$$\overline{W}^{\boldsymbol{\mu}}(C) = \left\{L \in \mathrm{Pic}^{d(\boldsymbol{\mu})}(C) \mid h^0(C, L \otimes \pi^*\mathcal{O}(m)) \geq x_m(\boldsymbol{\mu}) \text{ for all } m\right\}.$$

Example 2.1.3. Let C be a trigonal curve of genus 5. We will show that $W^1_4(C)$ has 2 irreducible components, both isomorphic to C. First, there is a 1-dimensional family of rank 1 divisor classes obtained by adding a basepoint to the g^1_3. If $D \in W^1_4(C)$ is not in this 1-dimensional family, then $D - g^1_3$ is not effective. It follows from the basepoint free pencil trick that the multiplication map

$$\nu : H^0(D) \otimes H^0(g^1_3) \to H^0(D + g^1_3)$$

is injective. The divisor class $D + g^1_3$ is therefore special. From this we see that the Serre dual $K_C - D$ is a divisor class in $W^1_4(C)$ with the property that $(K_C - D) - g^1_3$ is effective.

We therefore see that $W^1_4(C)$ has two components, both isomorphic to C, as pictured in Figure 2.1. One of these components consists of divisor classes D such that $D - g^1_3$ is effective, and the other component consists of the Serre duals of classes in the first component. Since $K_C - 2g^1_3$ is effective of degree 2, we see that these two components intersect in 2 points.

Alternatively, this analysis can be carried out by examining the splitting type stratification of $W^1_4(C)$. By ($\star$), we see that line bundles in the first component, in the complement of the two intersection points, have splitting type $(-2, -2, 1)$. Similarly, line bundles in the second component, in the complement of the two intersection points, have splitting type $(-3,0,0)$. Finally, the two line bundles in the intersection have splitting type $(-3,-1,1)$. Notice that this third splitting type is smaller than each of the previous two in the dominance order, and that the codimension of each stratum in $\mathrm{Pic}^4(C)$ is the magnitude of the splitting type.

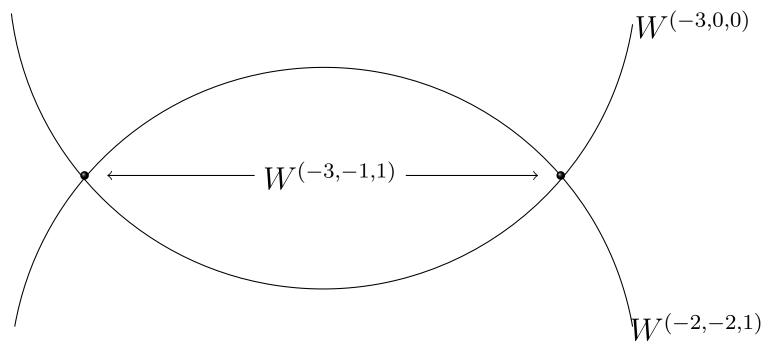


Figure 2.1: Stratification of W^1_4 for a general curve of genus 5 and gonality 3.

Equation (2.1) above show that $W^{\boldsymbol{\mu}}(C)$ is contained in $\overline{W}^{\boldsymbol{\mu}}(C)$. The strata $\overline{W}^{\boldsymbol{\mu}}(C)$ are closed, whereas the strata $W^{\boldsymbol{\mu}}(C)$ are locally closed. It is not necessarily the case that $\overline{W}^{\boldsymbol{\mu}}(C)$ is the closure of $W^{\boldsymbol{\mu}}(C)$. This is the case, however, when all splitting type loci have the expected dimension. (See [25, Lemma 2.1].)

The expected codimension of $\overline{W}^{\boldsymbol{\mu}}(C)$ in $\mathrm{Pic}^{d(\boldsymbol{\mu})}(C)$ is given by the *magnitude*

Definition 2.1.4. *The* magnitude *of a splitting type* $\boldsymbol{\mu}$ *is*

$$|\boldsymbol{\mu}| := \sum_{i<j} \max\{0, \mu_j - \mu_i - 1\}.$$

A consequence of (2.1) is that the sum of the ℓ largest entries of $\boldsymbol{\mu}$ is an upper semicontinuous invariant. This defines a natural partial order on splitting types. Specifically, given two splitting types $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ such that $d(\boldsymbol{\mu}) = d(\boldsymbol{\lambda})$, we say that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ if

$$\mu_1 + \cdots + \mu_\ell \leq \lambda_1 + \cdots + \lambda_\ell \text{ for all } \ell \leq k.$$

If one considers a splitting type to be a partition of $d(\boldsymbol{\mu})$ with possibly negative parts, then this partial order is the usual dominance order on partitions. This partial order has the following interpretation.

Lemma 2.1.5. *If* $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$, *then* $x_m(\boldsymbol{\mu}) \geq x_m(\boldsymbol{\lambda})$ *for all* m, *hence* $\overline{W}^{\boldsymbol{\mu}}(C) \subseteq \overline{W}^{\boldsymbol{\lambda}}(C)$.

Proof. Let m be an integer and J the minimal index such that $\lambda_J + m + 1 \geq 0$. Since $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$, we have

$$\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$$
$$\mu_1 + \cdots + \mu_{J-1} \leq \lambda_1 + \cdots + \lambda_{J-1},$$

which together imply that

$$\mu_J + \cdots + \mu_k \geq \lambda_J + \cdots + \lambda_k.$$

Hence

$$\sum_{i=1}^{k} \max\{0, \lambda_i + m + 1\} = (\lambda_J + m + 1) + \cdots + (\lambda_k + m + 1)$$
$$\leq (\mu_J + m + 1) + \cdots + (\mu_k + m + 1) \leq \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\}.$$

□

Maximal Splitting Types

For the remainder of this section, we fix positive integers g, r, d, and k such that $r > d - g$. Among the possible splitting types of line bundles of degree d and rank at least r on a k-gonal curve of genus g, we identify those that are maximal with respect to the dominance order.

Definition 2.1.6. *Let $\alpha \leq \min\{r+1, k-1\}$ be a positive integer. By the division algorithm, there exists a unique pair of integers q, β such that*

$$r + 1 = q\alpha + \beta, \quad 0 \leq \beta < \alpha.$$

Similarly, there exists a unique pair of integers q', β' such that

$$g - d + r = q'(k - \alpha) + \beta', \quad 0 \leq \beta' < k - \alpha.$$

We define the splitting type $\boldsymbol{\mu}_\alpha$ as follows:

$$\mu_{\alpha,i} := \begin{cases} -q' - 2 & \text{if } 0 < i \leq \beta' \\ -q' - 1 & \text{if } \beta' < i \leq k - \alpha \\ q - 1 & \text{if } k - \alpha < i \leq k - \beta \\ q & \text{if } k - \beta < i \leq k. \end{cases}$$

Heuristically, $\boldsymbol{\mu}_\alpha$ is the "most balanced" splitting type of degree d and rank r, subject to the constraint that precisely α of its entries are nonnegative. We show that the expected codimension of $W^{\boldsymbol{\mu}_\alpha}(C)$ coincides with the dimensions of irreducible components of $W^r_d(C)$ predicted by [33, Question 1.12].

Lemma 2.1.7. *For any integer α, we have*

$$g - |\boldsymbol{\mu}_\alpha| = \rho(g, \alpha - 1, d) - (r + 1 - \alpha)k.$$

Proof. First, recall that

$$|\boldsymbol{\mu}_\alpha| = \sum_{i<j} \max\{0, \mu_{\alpha,j} - \mu_{\alpha,i} - 1\}.$$

If $i < j \leq k - \alpha$, then $\mu_{\alpha,j} - \mu_{\alpha,i} \leq 1$, so the pair (i, j) does not contribute to the sum above. Similarly, if $k - \alpha < i < j$, then $\mu_{\alpha,j} - \mu_{\alpha,i} \leq 1$, so again the pair (i, j) does not contribute to the sum above.

On the other hand, if $i \leq k - \alpha$ and $j > k - \alpha$, then the pair (i, j) does contribute to the sum. There are precisely $(k - \alpha)\alpha$ such pairs, each $\mu_{\alpha,i}$ with $i \leq k - \alpha$ appears in exactly α of these pairs, and each $\mu_{\alpha,j}$ with $j > k - \alpha$ appears in exactly $k - \alpha$ of these pairs. It follows that we may rewrite the sum above as

$$\begin{aligned} |\boldsymbol{\mu}| &= (k-\alpha) \sum_{j=k-\alpha+1}^{k} \mu_j - \alpha \sum_{i=1}^{k-\alpha} \mu_i - (k-\alpha)\alpha \\ &= (k-\alpha)(r+1-\alpha) + \alpha(g-d+r+k-\alpha) - (k-\alpha)\alpha \\ &= \alpha(g-d+\alpha-1) + (r+1-\alpha)k. \end{aligned}$$

Subtracting both sides from g yields the result. □

Recall that the integers g, r, d, and k are fixed. We will say that a splitting type is *maximal* if it is maximal with respect to the dominance order among all splitting types satisfying

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} \geq r + 1.$$

In the rest of this section, we show that the maximal splitting types are precisely the splitting types $\boldsymbol{\mu}_\alpha$, when either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$. We first prove the following reduction step.

Lemma 2.1.8. *A maximal splitting type* $\boldsymbol{\mu}$ *satisfies*

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Proof. For the purposes of this argument, we define

$$h(\boldsymbol{\mu}) = \sum_{i=1}^{k} \max\{0, \mu_i + 1\}.$$

Let $\boldsymbol{\mu}$ be a splitting type satisfying

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and $h(\boldsymbol{\mu}) \geq r + 1$. We will show, by induction on $h(\boldsymbol{\mu})$, that there exists a splitting type $\boldsymbol{\lambda}$ such that $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ and $h(\boldsymbol{\lambda}) = r + 1$.

Since $r \geq 0$, we see that $\mu_k \geq 0$, and since $h(\boldsymbol{\mu}) \geq r+1 > d-g+1$, we see that $\mu_1 < -1$. There therefore exists an integer i such that $\mu_i > \mu_{i-1}$. Let j be the smallest such integer and j' the largest such integer. Since $\mu_1 < -1$ and $\mu_k \geq 0$, either $j < j'$, or $j = j'$ and $\mu_{j-1} < \mu_j - 1$. It follows that the vector $\boldsymbol{\mu}'$ obtained from $\boldsymbol{\mu}$ by adding 1 to μ_{j-1} and subtracting 1 from $\mu_{j'}$ is nondecreasing, and therefore a valid splitting type. Moreover, we have $\boldsymbol{\mu} < \boldsymbol{\mu}'$. Since $\mu_{j-1} < -1$ and $\mu_{j'} \geq 0$, we see that $h(\boldsymbol{\mu}') = h(\boldsymbol{\mu}) - 1$, and the result follows by induction. □

We now show that every maximal splitting type is of the form $\boldsymbol{\mu}_\alpha$ for some α.

Lemma 2.1.9. *Let $\boldsymbol{\mu}$ be a splitting type satisfying*

$$\sum_{i=1}^{k} \mu_i = d + 1 - g - k$$

and

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Let α denote the number of nonnegative entries of $\boldsymbol{\mu}$. Then $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$.

Proof. By assumption, we have

$$\sum_{i=k-\alpha+1}^{k} \mu_i = r + 1 - \alpha = \sum_{i=k-\alpha+1}^{k} \mu_{\alpha,i}.$$

It follows that

$$\sum_{i=1}^{k-\alpha} \mu_i = -(g-d+r) - (k-\alpha) = \sum_{i=1}^{k-\alpha} \mu_{\alpha,i}.$$

Because the entries of $\boldsymbol{\mu}$ are ordered from smallest to largest, for any $\ell \leq k - \alpha$, we see that

$$\sum_{i=1}^{\ell} \mu_i \leq \frac{\ell}{k-\alpha} \sum_{i=1}^{k-\alpha} \mu_i = \frac{-\ell(g-d+r)}{k-\alpha} - \ell.$$

Similarly, for any $\ell \leq \alpha$, we see that

$$\sum_{i=k-\alpha+1}^{k-\alpha+\ell} \mu_i \leq \frac{\ell}{\alpha} \sum_{i=k-\alpha+1}^{k} \mu_i = \frac{\ell(r+1)}{\alpha} - \ell.$$

By definition of $\boldsymbol{\mu}_\alpha$, therefore, we have $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$. □

Corollary 2.1.10. *If $\boldsymbol{\mu}$ is a maximal splitting type, then $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ for some integer α.*

Proof. Let $\boldsymbol{\mu}$ be a maximal splitting type. By Lemma 2.1.8, we see that

$$\sum_{i=1}^{k} \max\{0, \mu_i + 1\} = r + 1.$$

Let α denote the number of nonnegative entries of $\boldsymbol{\mu}$. By Lemma 2.1.9, we have $\boldsymbol{\mu} \leq \boldsymbol{\mu}_\alpha$, but since $\boldsymbol{\mu}$ is maximal, it follows that $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$. □

We now show that, if $\alpha < \min\{k - (g - d + r), r + 1\}$, then $\boldsymbol{\mu}_\alpha$ is not maximal.

Lemma 2.1.11. *If $\alpha < \min\{k - (g - d + r), r + 1\}$, then $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}_{\alpha+1}$.*

Proof. Since $g - d + r < k - \alpha$, by definition we have $\mu_{\alpha,k-\alpha} = -1$. If $r + 1$ is not divisible by α, consider the splitting type $\boldsymbol{\mu}$ obtained from $\boldsymbol{\mu}_\alpha$ by adding 1 to $\mu_{\alpha,k-\alpha}$ and subtracting 1 from $\mu_{\alpha,k-\beta+1}$. On the other hand, if $r + 1$ is divisible by α, then since $\alpha < r + 1$, we must have $\mu_{\alpha,k-\alpha+1} > 0$. In this case, consider the splitting type $\boldsymbol{\mu}$ obtained from $\boldsymbol{\mu}_\alpha$ by adding 1 to $\mu_{\alpha,k-\alpha}$ and subtracting 1 from $\mu_{\alpha,k-\alpha+1}$. In either case, we see that $\boldsymbol{\mu}$ is a splitting type with $\alpha + 1$ nonnegative entries, satisfying $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}$. By Lemma 2.1.9, we have $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu} \leq \boldsymbol{\mu}_{\alpha+1}$. □

Finally, we see that the remaining splitting types $\boldsymbol{\mu}_\alpha$ are maximal.

Proposition 2.1.12. *The splitting type $\boldsymbol{\mu}$ is maximal if and only if $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$ for some integer α satisfying either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$.*

Proof. By Corollary 2.1.10, every maximal splitting type is of the form $\boldsymbol{\mu}_\alpha$ for some integer α. By Lemma 2.1.11, if $\alpha < k - (g - d + r)$ and $\alpha \neq r + 1$, then $\boldsymbol{\mu}_\alpha$ is not maximal. It therefore suffices to show that, if $\alpha \neq \gamma$ are both greater than or equal to $k - (g - d + r)$, then $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are incomparable.

Without loss of generality, assume that $\alpha < \gamma$. We write

$$\begin{aligned} r + 1 &= q_\alpha \alpha + \beta_\alpha & \qquad 0 \leq \beta_\alpha < \alpha \\ &= q_\gamma \gamma + \beta_\gamma & \qquad 0 \leq \beta_\gamma < \gamma. \end{aligned}$$

Since $\alpha < \gamma$, we see that $q_\alpha \geq q_\gamma$. Moreover, since $\gamma \leq r + 1$, we see that both q_α and q_γ are positive. It follows that, if $q_\alpha = q_\gamma$, then $\beta_\alpha > \beta_\gamma$. Thus, if j is the largest integer such that $\mu_{\alpha,j} \neq \mu_{\gamma,j}$, then $\mu_{\alpha,j} > \mu_{\gamma,j}$. If $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are comparable, then we see that $\boldsymbol{\mu}_\alpha < \boldsymbol{\mu}_\gamma$.

Since $k - \alpha \leq g - d + r$, we see by a similar argument that if j' is the smallest integer such that $\mu_{\alpha,j'} \neq \mu_{\gamma,j'}$, then $\mu_{\alpha,j'} > \mu_{\gamma,j'}$. It follows that if $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are comparable, then $\boldsymbol{\mu}_\alpha > \boldsymbol{\mu}_\gamma$. Combining these two observations, we see that $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{\mu}_\gamma$ are incomparable. □

2.2 Divisor Theory of Chains of Loops

In this section, we survey the theory of special divisors on chains of loops, as discussed in [33, 34, 22]. We refer the reader to those papers for more details. For a more general overview of divisors on tropical curves, we refer the reader to [3, 4]. For the uninitiated, we will not require most of the material of these papers; we will use only the classification of special divisors on chains of loops from [33, 34].

Chains of Loops and Torsion Profiles

Let Γ be a chain of g loops with bridges, as pictured in Figure 2.2. Each of the g loops consists of two edges. We denote the lengths of the top and bottom edge of the jth loop by ℓ_j and m_j, respectively. The Brill-Noether theory of chains of loops is governed by the *torsion orders* of the loops.

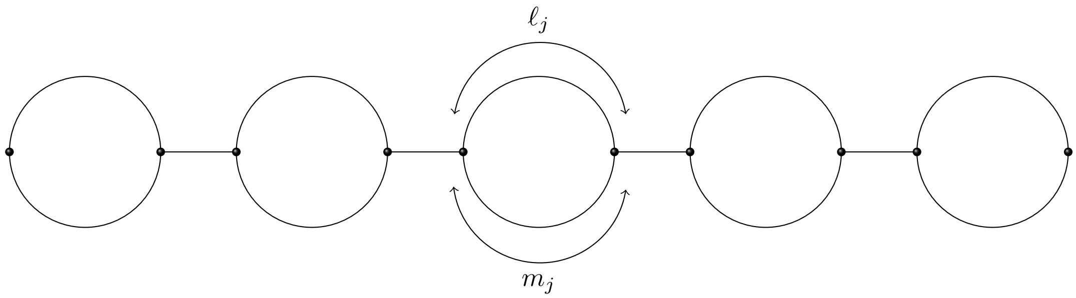


Figure 2.2: The chain of loops Γ.

Definition 2.2.1. *[34, Definition 1.9] If $\ell_j + m_j$ is an irrational multiple of m_j, then the jth* torsion order *τ_j of Γ is 0. Otherwise, we define τ_j to be the minimum positive integer such that $\tau_j m_j$ is an integer multiple of $\ell_j + m_j$. The sequence $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_g)$ is called the* torsion profile *of Γ.*

For the remainder of this paper, we assume that the torsion profile of Γ is given by

$$\tau_i := \begin{cases} 0 & \text{if } i < k \text{ or } i > g - k + 1 \\ k & \text{otherwise.} \end{cases}$$

This chain of loops with this torsion profile possesses a distinguished divisor class of rank 1 and degree k, given by $g^1_k = kv_k$, where v_k is the lefthand vertex of the kth loop.

Remark 2.2.2. Note that, unlike [33, Definition 2.1], we do *not* require the first $k-1$ loops or the last $k-1$ loops to have torsion order k. This choice does not affect the gonality, or more generally the Brill-Noether theory, of this metric graph. A primary reason for this choice is that the space of such metric graphs has dimension equal to that of the Hurwitz space, namely $2g + 2k - 5$.

In [34], Pflueger classifies the special divisor classes on chains of loops. This classification generalizes that of special divisor classes on generic chains of loops in [11]. Specifically, Pflueger shows that $W^r_d(\Gamma)$ is a union of tori, where the tori are indexed by certain types of tableaux. While Pflueger's analysis applies to chains of loops with arbitrary torsion profiles, we record it only for the torsion profile above. For ease of notation, given a positive integer a we write $[a]$ for the finite set $\{1, \ldots, a\}$.

Definition 2.2.3. *[33, Definition 2.5] Let λ be a partition. Recall that a* tableau *on λ with alphabet $[g]$ is a function $t : \lambda \to [g]$ satisfying:*

$$t(x, y) < t(x, y + 1) \text{ and } t(x, y) < t(x + 1, y) \text{ for all } (x, y).$$

A tableau t is standard *if t is injective. A tableau t on a partition λ is called a k-*uniform displacement tableau *if, whenever $t(x, y) = t(x', y')$, we have $y - x \equiv y' - x' \pmod{k}$.*

It is standard to depict a tableau on λ as the diagram of boxes $(x, y) \in \lambda$, where the box in position (x, y) is filled with the symbol $t(x, y)$. We draw our tableaux according to the English convention, so that the box $(1, 1)$ appears in the upper lefthand corner.

Coordinates on $\mathrm{Pic}(\Gamma)$

A nice feature of the chain of loops is that its Picard group has a natural system of coordinates. On the jth loop, let $\langle \xi \rangle_j$ denote the point located ξm_j units from the righthand vertex in the counterclockwise direction. Note that

$$\langle \xi \rangle_j = \langle \eta \rangle_j \text{ if and only if } \xi = \eta \pmod{\tau_j}.$$

By the tropical Abel-Jacobi theorem [5], every divisor class D of degree d on Γ has a *unique* representative of the form

$$(d - g)\langle 0 \rangle_g + \sum_{j=1}^{g} \langle \xi_j(D) \rangle_j,$$

for some real numbers $\xi_j(D)$. Because this expression is unique, the functions ξ_j form a system of coordinates on $\mathrm{Pic}^d(\Gamma)$. This representative of the divisor class D is known as the *break divisor* representative [29, 1].

Definition 2.2.4. *[34, Definition 3.5] Given a degree d and a k-uniform displacement tableau t with alphabet $[g]$, we define the coordinate subtorus $\mathbb{T}(t)$ as follows.*

$$\mathbb{T}(t) := \{D \in \mathrm{Pic}^d(\Gamma) | \xi_{t(x,y)}(D) = y - x \pmod{k}\}.$$

Note that the coordinate $\xi_j(D)$ of a divisor class D in $\mathbb{T}(t)$ is determined if and only if j is in the image of t. It follows that the codimension of $\mathbb{T}(t)$ in $\mathrm{Pic}^d(\Gamma)$ is the number of distinct symbols in t. The main combinatorial result of [34] is a classification of special divisors on Γ.

2.3 Partitions and Tableaux

Throughout, we use the convention that $\mathbb{N}$ denotes the positive integers. By a slight abuse of terminology, we use the term *partition* to refer to the Ferrers diagram of a partition.

Definition 2.3.1. *A* partition *is a finite subset* $\lambda \subset \mathbb{N}^2$ *with the property that, if* $(x, y) \in \lambda$, *then*

1. *either* $x = 1$ *or* $(x - 1, y) \in \lambda$, *and*
2. *either* $y = 1$ *or* $(x, y - 1) \in \lambda$.

It is standard to depict a partition as a set of boxes, with a box in position (x, y) if $(x, y) \in \lambda$. We follow the English convention, so that the box $(1, 1)$ appears in the upper lefthand corner. Given a partition λ, we define its *transpose* to be

$$\lambda^T := \{(x, y) \in \mathbb{N}^2 \mid (y, x) \in \lambda\}.$$

The corners of a partition will play an important role in our discussion.

Definition 2.3.2. *Let* λ *be a partition. A box* $(x, y) \in \lambda$ *is called an* inside corner *if* $(x + 1, y) \notin \lambda$ *and* $(x, y + 1) \notin \lambda$. *A box* $(x, y) \notin \lambda$ *is called an* outside corner *if*

1. *either* $x = 1$ *or* $(x - 1, y) \in \lambda$, *and*
2. *either* $y = 1$ *or* $(x, y - 1) \in \lambda$.

In other words, a box $(x, y) \in \lambda$ is an inside corner if $\lambda \setminus (x, y)$ is a partition, and a box $(x, y) \notin \lambda$ is an outside corner if $\lambda \cup (x, y)$ is a partition.

Given a positive integer g, we write $[g]$ for the finite set $\{1, 2, \ldots, g\}$, and let $\binom{[g]}{n}$ denote the set of size-n subsets of $[g]$. A *tableau* on a partition λ with alphabet $[g]$ is a function $t : \lambda \to [g]$ satisfying:

$$t(x, y) > t(x, y - 1) \text{ for all } (x, y) \in \lambda \text{ with } y > 1, \text{ and}$$
$$t(x, y) > t(x - 1, y) \text{ for all } (x, y) \in \lambda \text{ with } x > 1.$$

We depict a tableau by filling each box of λ with an element of $[g]$. The tableau condition is satisfied if the symbols in each row are increasing and the symbols in each column are increasing. We write $YT(\lambda)$ for the set of tableaux on the partition λ. Given a tableau t on λ, we define its *transpose* to be the tableau t^T on λ^T given by

$$t^T(x, y) = t(y, x) \text{ for all } (x, y) \in \lambda^T.$$

We write $YT_k(\lambda)$ for the set of k-uniform displacement tableaux on the partition λ. The k-uniform displacement condition is satisfied if the lattice distance (or taxicab distance) between any two boxes containing the same symbol is a multiple of k. For example, Figure 2.3 depicts a 3-uniform displacement tableau with alphabet $[5]$. Note that the two boxes containing the symbol 3 have lattice distance 3, and any two of the three boxes containing the symbol 5 have lattice distance a multiple of 3.

1	3	4	5
2	5		
3			
5			

Figure 2.3: A 3-uniform displacement tableau with alphabet [5].

The Jacobian of Γ has two natural systems of coordinates. The first uses the theory of break divisors from [29, 1]. On the jth loop, define $\langle \xi \rangle_j$ to be the point of distance ξm_j from the righthand vertex in the counterclockwise direction. Every divisor class D of degree d has a unique *break divisor* representative of the form

$$(d-g)\langle 0 \rangle_g + \sum_{j=1}^{g} \langle \xi_j(D) \rangle_j.$$

Because this representative is unique, the functions $\xi_j \colon \mathrm{Pic}^d(\Gamma) \to \mathbb{R}/\left(\frac{m_j+\ell_j}{m_j}\right)\mathbb{Z}$ act as a system of coordinates on $\mathrm{Pic}^d(\Gamma)$.

Alternatively, define an orientation on Γ by orienting each of the loops counterclockwise, and let ω_j be the harmonic 1-form supported on the jth loop with weight 1. Given a divisor class D on Γ, define

$$\widetilde{\xi}_j(D) := \frac{1}{m_j} \int_{\langle 0 \rangle_g}^{D} \omega_j.$$

By the tropical Abel-Jacobi theorem [5], since the set of 1-forms $\omega_1, \ldots, \omega_g$ is a basis for $\Omega(\Gamma)$, the functions $\widetilde{\xi}_j \in \Omega(\Gamma)^*/H_1(\Gamma, \mathbb{Z})$ form a system of coordinates on $\mathrm{Jac}(\Gamma)$. In our combinatorial arguments, we tend to use the functions ξ_j more often that $\widetilde{\xi}_j$, but the latter are useful due to their linearity. That is, $\widetilde{\xi}_j(D_1+D_2) = \widetilde{\xi}_j(D_1)+\widetilde{\xi}_j(D_2)$.

It is straightforward to translate between the two systems of coordinates. Specifically, we have $\widetilde{\xi}_j(D) = \xi_j(D) - (j-1)$. Since $\widetilde{\xi}_j$ is linear, it follows that

$$\xi_j(D_1 + D_2) = \xi_j(D_1) + \widetilde{\xi}_j(D_2). \tag{2.2}$$

In [34], Pflueger classifies the special divisor classes on Γ. This classification specializes to the "generic" case where $k = \lfloor \frac{g+3}{2} \rfloor$, studied in [11].

Definition 2.3.3. *[34, Definition 3.5] Let a and b be positive integers and let λ be the rectangular partition*

$$\lambda = \{(x,y) \in \mathbb{N}^2 \mid x \leq a,\ y \leq b\}.$$

Given a k-uniform displacement tableau t on λ with alphabet $[g]$, we define $\mathbb{T}(t)$ as follows.

$$\mathbb{T}(t) := \{D \in \mathrm{Pic}^{g+a-b-1}(\Gamma) \mid \xi_{t(x,y)}(D) = y - x \pmod{k}\}.$$

In the system of coordinates ξ_j, $\mathbb{T}(t)$ is a coordinate subtorus, where the coordinate ξ_j is fixed if and only if the symbol j is in the image of t. The codimension of $\mathbb{T}(t)$ is therefore equal to the number of distinct symbols in t. If the symbol j appears in multiple boxes of the tableau t, then the k-uniform displacement condition guarantees that the two boxes impose the same condition on ξ_j.

Theorem 2.3.4. *[34, Theorem 1.4] For any positive integers r and d satisfying $r > d - g$, we have*

$$W^r_d(\Gamma) = \bigcup \mathbb{T}(t),$$

where the union is over k-uniform displacement tableaux on $[r+1] \times [g-d+r]$ with alphabet $[g]$.

A consequence of Theorem 2.3.4 is that Γ has a unique divisor class of degree k and rank 1, which we denote by g^1_k. This justifies the terminology that Γ is a k-gonal chain of loops. Specifically, the unique k-uniform displacement tableau on $[2] \times [g-k+1]$ with alphabet $[g]$ contains the symbols $1, 2, \ldots, g-k+1$ in the first column and the symbols $k, k+1, \ldots, g$ in the second column. In particular, we have

$$\widetilde{\xi}_j(g^1_k) = \begin{cases} 0 & \text{if } j \leq g-k+1 \\ k & \text{if } j > g-k+1. \end{cases}$$

Chapter 3 Irreducible Components of Brill-Noether Loci

This chapter is taken from the paper *Components of Brill-Noether Loci for Curves with Fixed Gonality* [10], authored by Cook-Powell-Jensen, which has been accepted for publication in the *Michigan Math Journal.*

3.1 Introduction

We describe a conjectural stratification of the Brill-Noether variety for general curves of fixed genus and gonality. As evidence for this conjecture, we show that this Brill-Noether variety has at least as many irreducible components as predicted by the conjecture, and that each of these components has the expected dimension. Our proof uses combinatorial and tropical techniques. Specifically, we analyze containment relations between the various strata of tropical Brill-Noether loci identified by Pflueger in his classification of special divisors on chains of loops.

Recall from section2 that, given a curve C over the complex numbers, the *Brill-Noether variety* $W^r_d(C)$ parameterizes line bundles of degree d and rank at least r on C. Brill-Noether varieties encode a significant amount of geometric information, and consequently are among the most well-studied objects in the theory of algebraic curves. A series of results in the eighties concern the geometry of $W^r_d(C)$ when C is general in the moduli space $\mathcal{M}_g$. In this case, the locally closed stratum $W^r_d(C) \smallsetminus W^{r+1}_d(C)$ is smooth [17] of dimension

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \qquad [18],$$

and irreducible when $\rho(g, r, d)$ is positive [16].

More recent work has focused on the situation where C is general in the Hurwitz space $\mathcal{H}_{k,g}$ parameterizing branched covers of the projective line of degree k and genus g. The Hurwitz space $\mathcal{H}_{k,g}$ admits a natural map to the moduli space $\mathcal{M}_g$, given by forgetting the data of the map to $\mathbb{P}^1$. When $k \geq \lfloor \frac{g+3}{2} \rfloor$, this map is dominant and there is nothing new to show, so we restrict our attention to the case where k is smaller than $\lfloor \frac{g+3}{2} \rfloor$. We refer to a general point in the Hurwitz space $\mathcal{H}_{k,g}$ as a *general curve of genus g and gonality k*. Our main result is the following.

Theorem 3.1.1. *Let C be a general curve of genus g and gonality $k \geq 2$. Then there exists an irreducible component of $W^r_d(C)$ of dimension*

$$\rho(g, \alpha - 1, d) - (r + 1 - \alpha)k,$$

as long as this number is nonnegative, for every positive integer $\alpha \leq \min\{r+1, k-1\}$ satisfying either $\alpha \geq k - (g - d + r)$ or $\alpha = r + 1$.

We strongly suspect that Theorem 3.1.1 identifies *all* of the irreducible components of $W^r_d(C)$, for a reason that we will explain in Section 3.1. Theorem 3.1.1 is a generalization of several previous results. In [33], Pflueger shows that the dimension of $W^r_d(C)$ is at most

$$\rho_k(g,r,d) := \max_{\alpha} \rho(g,\alpha-1,d) - (r+1-\alpha)k,$$

and asks whether every component has dimension $\rho(g,\alpha-1,d)-(r+1-\alpha)k$ for some value of α. In [22], Ranganathan and the second author show that the maximal dimensional component has dimension exactly $\rho_k(g,r,d)$. In [12], Coppens and Martens exhibit components of dimension $\rho(g,\alpha-1,d)-(r+1-\alpha)k$ for α equal to $1, r$, and $r+1$. They further expand on this result in [13], constructing components of dimension $\rho(g,\alpha-1,d)-(r+1-\alpha)k$ for all α dividing r or $r+1$.

The Splitting Type Stratification

The splitting type of $\pi_* L$ determines not only the degree and rank of the line bundle L, but also the rank of $L \otimes \pi^* \mathcal{O}(m)$ for all integers m (see Section 2.1). In this way, the varieties $W^{\boldsymbol{\mu}}(C)$ stratify $W^r_d(C)$. The number of irreducible components of $W^r_d(C)$, as well as the dimensions of these components, are predicted by theorem 3.1.2. We refer the reader to Definition 2.1.1 for the definition of the partial order on splitting types, and to Definition 2.1.4 for the definition of the magnitude of a splitting type.

For a given rank r and degree d, the maximal elements of the poset of splitting types are in correspondence with positive integers $\alpha \leq \min\{r+1, k-1\}$ satisfying either $\alpha \geq k-(g-d+r)$ or $\alpha = r+1$. (See Definition 2.1.6 and Proposition 2.1.12 for details.) Let $\boldsymbol{\mu}_\alpha$ denote the splitting type corresponding to the integer α. Theorem 3.1.2 predicts that the irreducible components of $W^r_d(C)$ are precisely the closures of the strata $W^{\boldsymbol{\mu}_\alpha}(C)$. We prove the following stronger version of Theorem 3.1.1.

Theorem 3.1.2. *Let C be a general curve of genus g and gonality $k \geq 2$. If $g \geq |\boldsymbol{\mu}_\alpha|$, then $W^{\boldsymbol{\mu}_\alpha}(C)$ has an irreducible component of dimension $g - |\boldsymbol{\mu}_\alpha|$. The closure of this component is an irreducible component of $W^r_d(C)$.*

Approach and Techniques

Our approach is based on tropical techniques developed in [11, 33, 34, 22]. Each of these papers establishes results about Brill-Noether varieties by studying the divisor theory of a particular family of metric graphs, known as the chains of loops. The first of these papers [11] provides a new proof of the Brill-Noether Theorem. Key to this argument is the classification of special divisors on chains of loops Γ with generic edge lengths. Specifically, [11] shows that $W^r_d(\Gamma)$ is a union of tori $\mathbb{T}(t)$, where the tori are indexed by standard Young tableaux t.

In [34], Pflueger generalizes this result to chains of loops with arbitrary edge lengths. In this case, $W^r_d(\Gamma)$ is still a union of tori, but here the tori are indexed by a more general type of tableaux, known as *displacement tableaux*. (See Definition 2.2.3 and Theorem 2.3.4.) In [33], Pflueger computes the dimension of the largest of these tori, and thus obtains his bound on the dimensions of Brill-Noether loci for general k-gonal curves.

Instead of studying the tori of maximum dimension, we study the tori that are maximal with respect to containment. The tableaux corresponding to maximal-dimensional tori belong to a larger family, known as *scrollar tableaux*. (See Definition 3.2.3.) There is a natural partition of scrollar tableaux into types, where the types are indexed by positive integers $\alpha \leq \min\{r+1, k-1\}$ satisfying either $\alpha \geq k-(g-d+r)$ or $\alpha = r+1$. It is shown in [22] that, under certain mild hypotheses, divisor classes corresponding to scrollar tableaux lift to divisor classes on k-gonal curves in families of the expected dimension.

Our main combinatorial result is the following.

Theorem 3.1.3. *Let Γ be a k-gonal chain of loops of genus g, and let t be a k-uniform displacement tableau on $[r+1] \times [g-d+r]$. The torus $\mathbb{T}(t)$ is maximal with respect to containment in $W^r_d(\Gamma)$ if and only if t is scrollar. In other words,*

$$W^r_d(\Gamma) = \bigcup_{t \text{ scrollar}} \mathbb{T}(t).$$

Outline of the Chapter

In Section 3.3 we discuss the relation between our combinatorial and geometric results, and in particular show that Theorem 3.1.3 implies Theorem 3.1.2. In the final two sections, which are purely combinatorial, we prove Theorem 3.1.3. In Section 3.4, we show that if t is a scrollar tableau, then $\mathbb{T}(t)$ is maximal, and in Section 3.5, we establish the converse.

3.2 Divisor Theory of Chains of Loops

In [34], Pflueger provides a description of $W^r_d(\Gamma)$ into a union of tori, where the union is indexed by tableaux. Notably, Pflueger does not consider the containment relations between the various tori $\mathbb{T}(t)$. These containment relations are the primary concern of Sections 3.4 and 3.5. We note the following, which will be explored in more detail in these later sections.

Lemma 3.2.1. *Let t and t' be k-uniform displacement tableaux on $[a] \times [b]$. Then $\mathbb{T}(t) \subseteq \mathbb{T}(t')$ if and only if*

1. *every symbol in t' is a symbol in t, and*

2. *if $t(x,y) = t'(x',y')$, then $x - y = x' - y' \pmod{k}$.*

Under Pflueger's classification of special divisors, there is a natural interpretation of Serre duality. Given a tableau t on $[a] \times [b]$, define the *transpose tableau* to be the tableau t^T on $[b] \times [a]$ given by $t^T(x,y) = t(y,x)$.

Lemma 3.2.2. *[34, Remark 3.6] Let t be a k-uniform displacement tableau on $[r+1] \times [g-d+r]$ with alphabet $[g]$, and let $D \in \mathbb{T}(t)$ be a divisor class. Then the Serre dual $K_\Gamma - D$ is contained in $\mathbb{T}(t^T)$.*

Scrollar Tableaux

In [22], Ranganathan and the second author consider a special type of k-uniform displacement tableaux, known as *scrollar tableaux*. Throughout this section, we fix positive integers a and b, and a positive integer $\alpha \leq \{a, k-1\}$, satisfying either $\alpha \geq k - b$ or $\alpha = a$. As in Definition 2.1.6, we write

$$a = q\alpha + \beta, \quad 0 \leq \beta < \alpha$$

and

$$b = q'(k - \alpha) + \beta', \quad 0 \leq \beta' < k - \alpha.$$

Definition 3.2.3. *Let t be a tableau on $[a] \times [b]$. We define t to be* scrollar of type α *if it satisfies the following three conditions.*

1. *$t(x, y) = t(x', y')$ if and only if there exists an integer ℓ such that both*

$$x' - x = \ell\alpha \text{ and } y' - y = \ell(\alpha - k).$$

2. *If $\alpha = a$, then $t(1, y) > t(a, y + a - k)$ for all $y > k - a$.*
3. *If $\alpha = k - b$, then $t(x, 1) > t(x + b - k, b)$ for all $x > k - b$.*

Remark 3.2.4. When $k-b < \alpha < a$, Definition 3.2.3 agrees with [22, Definition 7.1], but in the edge cases the two definitions disagree. This is because, when α is equal to a or $k-b$, every standard tableau satisfies [22, Definition 7.1] trivially. In Sections 3.4 and 3.5, however, we will see that $\mathbb{T}(t)$ is maximal only for tableaux satisfying Definition 3.2.3. We note that when $\alpha < a$, condition (1) implies an inequality analogous to that of condition (2), because

$$t(1, y) = t(\alpha + 1, y + \alpha - k) > t(\alpha, y + \alpha - k).$$

Similarly, when $\alpha > k - b$, condition (1) implies an inequality analogous to that of condition (3).

For the reader interested in comparing the definitions in the two papers, we provide a brief dictionary. The integer α appearing here is the same as n in [22]. The integer β agrees with b in [22], and q is equal to $\lfloor \frac{a}{\alpha} \rfloor = \lfloor \frac{r+1}{n} \rfloor$.

Example 3.2.5. A typical example of a scrollar tableau appears in Figure 3.1. Note that the boxes in the first α columns necessarily contain distinct symbols, as do the boxes in the last $k - \alpha$ rows. The symbols in the remaining boxes are obtained by repeatedly translating the symbols in this L-shaped region α boxes rightward and $k - \alpha$ boxes upward.

1	2	4	5	10	11	12
3	7	8	9	13	16	18
5	10	11	12	15	17	20
9	13	16	18	19	22	23
12	15	17	20	21	24	26

Figure 3.1: A scrollar tableau of type 3, where $k = 5$.

Example 3.2.6. Figure 3.2 depicts three different 3-uniform displacement tableaux on $[3] \times [2]$. The first tableau t is scrollar of type 2. To see this, note that there is only one pair of boxes whose x coordinates differ by a multiple of 2 and whose y coordinates differ by the same multiple of -1, and these boxes contain the same symbol. The second tableau t' is scrollar of type 1, because it is standard, $t'(2,1) > t'(1,2)$, and $t'(3,1) > t'(2,2)$. The final tableau t^* is not scrollar of either type. Specifically, it is not scrollar of type 1 because $t^*(2,1) < t^*(1,2)$, and it is not scrollar of type 2 because $t^*(3,1) \neq t^*(1,2)$. By Lemma 3.2.1, we see that $\mathbb{T}(t^*) \subset \mathbb{T}(t)$.

$t =$

1	2	4
4	5	6

$t' =$

1	3	5
2	4	6

$t^* =$

1	2	3
4	5	6

Figure 3.2: Three different 3-uniform displacement tableaux. The first two are scrollar of different types, and the third is not scrollar.

The following observation from [22] is central to our argument.

Proposition 3.2.7. *Let t be a scrollar tableau of type α on $[r+1] \times [g-d+r]$ with alphabet $[g]$. Then $g \geq |\boldsymbol{\mu}_\alpha|$ and*

$$\dim \mathbb{T}(t) = g - |\boldsymbol{\mu}_\alpha|.$$

Proof. By [22, Proposition 7.4], we have

$$\dim \mathbb{T}(t) = \rho(g, \alpha - 1, d) - (r + 1 - \alpha)k.$$

The result then follows from Lemma 2.1.7. □

Proposition 3.2.7 suggests a connection between scrollar tableaux of type α and the splitting type $\boldsymbol{\mu}_\alpha$. This connection will be established in Proposition 3.3.1 below. The following lemma is key to the proof of Proposition 3.3.1.

Lemma 3.2.8. *[22, Corollary 7.3] Let t be a scrollar tableau of type α, and let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. Then*

1. $\operatorname{rk}(D - qg^1_k) = \beta - 1$, *and*

2. $\text{rk}(D - (q+1)g^1_k) = -1$.

Remark 3.2.9. In Lemma 3.2.8, when we say that the divisor class $D \in \mathbb{T}(t)$ is "sufficiently general", we mean that D lies in the complement of finitely many coordinate subtori of codimension at least 1 in $\mathbb{T}(t)$. In particular, the set of divisor classes in $\mathbb{T}(t)$ satisfying the conclusion of Lemma 3.2.8 is open and dense in $\mathbb{T}(t)$.

Much of [22] is devoted to a lifting result for divisor classes in $\mathbb{T}(t)$ when t is a scrollar tableau. Unfortunately, [22] does not establish this lifting result for all scrollar tableaux, but only for those that satisfy the following condition.

Definition 3.2.10. *We say that a tableau* t has no vertical steps *if*

$$t(x, y+1) \neq t(x, y) + 1 \text{ for all } x, y.$$

We note that if $g \geq |\boldsymbol{\mu}_\alpha|$ and $\alpha > 1$, then there exists a scrollar tableau of type α with no vertical steps. For example, the transpose of the tableau defined in the proof of [33, Lemma 3.5] has no vertical steps. Another example of such a tableau appears in Figure 3.3.

1	2	3	7	8	9	13
4	5	6	10	11	12	16
7	8	9	13	14	15	19
10	11	12	16	17	18	20
13	14	15	19	21	22	23

Figure 3.3: A scrollar tableau of type 3, where $k = 5$, with no vertical steps.

The following proposition is one of the main technical results of [22]. In this proposition and throughout Section 3.3, we let K be an algebraically closed, non-archimedean valued field of equicharacteristic zero.

Proposition 3.2.11. *[22, Proposition 9.2] Let t be a scrollar tableau of type α with no vertical steps, and let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. Then there exists a curve C of genus g and gonality k over K with skeleton Γ, and a divisor class $\mathcal{D} \in W^r_d(C)$ specializing to D.*

3.3 Connections Between Combinatorics and Algebraic Geometry

In this section, we demonstrate the connection between our combinatorial and geometric results. Specifically, we show that Theorem 3.1.3 implies Theorem 3.1.2. To begin, we establish the connection between scrollar tableaux of type α and the splitting types $\boldsymbol{\mu}_\alpha$.

Proposition 3.3.1. *Let C be a curve of genus g and gonality k over K with skeleton Γ. Let t be a scrollar tableau of type α, let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class, and let $\mathcal{D} \in W^r_d(C)$ be a divisor that specializes to D. Then $\mathcal{D} \in W^{\boldsymbol{\mu}_\alpha}(C)$.*

Proof. Let $\boldsymbol{\mu}$ denote the splitting type of $\pi_*\mathcal{O}(\mathcal{D})$. By Lemma 3.2.8, we have

$$\operatorname{rk}(D - qg^1_k) = \beta - 1,$$
$$\operatorname{rk}(D - (q+1)g^1_k) = -1.$$

By Baker's Specialization Lemma [3], it follows that

$$h^0(\mathcal{D} - qg^1_k) \leq \beta, \tag{3.1}$$
$$h^0(\mathcal{D} - (q+1)g^1_k) = 0. \tag{3.2}$$

Recall that, if t^T denotes the transpose of t, then the Serre dual $K_\Gamma - D$ is contained in $\mathbb{T}(t^T)$. Note that t^T is also a scrollar tableau. By Lemma 3.2.8, therefore, since $K_\Gamma - D$ is sufficiently general, we see that

$$\operatorname{rk}(K_\Gamma - D - q'g^1_k) = \beta' - 1,$$
$$\operatorname{rk}(K_\Gamma - D - (q'+1)g^1_k) = -1.$$

By Baker's Specialization Lemma, it follows that

$$h^0(K_C - \mathcal{D} - q'g^1_k) \leq \beta', \tag{3.3}$$
$$h^0(K_C - \mathcal{D} - (q'+1)g^1_k) = 0. \tag{3.4}$$

By ($\star$), (3.2) implies that $\mu_k \leq q$ and (3.1) implies that $\mu_{k-\beta} \leq q-1$. It follows that

$$\mu_{k-\alpha+1} + \cdots + \mu_{k-\alpha+\ell} \leq \mu_{\alpha,k-\alpha+1} + \cdots + \mu_{\alpha,k-\alpha+\ell} \qquad \text{for all } \ell \leq \alpha.$$

Similarly, (3.4) implies that $\mu_1 \geq -q' - 2$, and (3.3) implies that $\mu_{\beta'+1} \geq -q' - 1$. It follows that

$$\mu_1 + \cdots + \mu_\ell \geq \mu_{\alpha,1} + \cdots + \mu_{\alpha,\ell} \qquad \text{for all } \ell \leq k - \alpha.$$

Putting these together, we see that $\boldsymbol{\mu} \geq \boldsymbol{\mu}_\alpha$. By Proposition 2.1.12, however, $\boldsymbol{\mu}_\alpha$ is maximal, hence $\boldsymbol{\mu} = \boldsymbol{\mu}_\alpha$. □

Corollary 3.3.2. *Let t be a scrollar tableau of type α with no vertical steps, and let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. Then there exists a curve C of genus g and gonality k over K with skeleton Γ, and a divisor class $\mathcal{D} \in W^{\boldsymbol{\mu}_\alpha}(C)$ specializing to D.*

Proof. By Proposition 3.2.11, there exists a curve C of genus g and gonality k over K with skeleton Γ, and a divisor class $\mathcal{D} \in W^r_d(C)$ specializing to D. By Proposition 3.3.1, the divisor class $\mathcal{D}$ is in $W^{\boldsymbol{\mu}_\alpha}(C)$. □

We now show that Theorem 3.1.3 implies Theorem 3.1.2. We do this in two steps. First, we obtain an upper bound on a particular component of $W^r_d(C)$.

Proposition 3.3.3. *Let C and $\mathcal{D}$ be as in Corollary 3.3.2, and let $\mathcal{Y}$ be any irreducible component of $W^r_d(C)$ containing $\mathcal{D}$. Then*

$$\dim \mathcal{Y} \leq g - |\boldsymbol{\mu}_\alpha|.$$

Proof. By [19, Theorem 6.9],

$$\dim \mathcal{Y} = \dim \operatorname{Trop} \mathcal{Y}.$$

By Baker's Specialization Lemma, we see that $\operatorname{Trop} \mathcal{Y} \subseteq W^r_d(\Gamma)$. It follows that $\dim \mathcal{Y}$ cannot exceed the local dimension of $W^r_d(\Gamma)$ in a neighborhood of D. By Theorem 3.1.3, $\mathbb{T}(t)$ is maximal with respect to containment in $W^r_d(\Gamma)$, and since $D \in \mathbb{T}(t)$ is sufficiently general, the local dimension of $W^r_d(\Gamma)$ in a neighborhood of D is equal to that of $\mathbb{T}(t)$. Finally, by Proposition 3.2.7, we have

$$\dim \mathcal{Y} \leq \dim \mathbb{T}(t) = g - |\boldsymbol{\mu}_\alpha|.$$

□

Proof that Theorem 3.1.3 implies Theorem 3.1.2. The case $k = 2$ is classical, so we assume that $k \geq 3$. Let $\alpha \leq \min\{r+1, k-1\}$ be a positive integer satisfying either $\alpha \geq k - (g-d+r)$ or $\alpha = r+1$. If $\alpha \geq k - (g-d+r)$, then applying Serre duality exchanges α with $k - \alpha$, so we may assume that $\alpha > 1$.

Since $|\boldsymbol{\mu}| \leq g$ and $\alpha > 1$, there exists a scrollar tableau t of type α with no vertical steps. Let $D \in \mathbb{T}(t)$ be a sufficiently general divisor class. By Corollary 3.3.2, there exists a curve C of genus g and gonality k over K with skeleton Γ, and a divisor class $\mathcal{D} \in W^{\mu_\alpha}(C)$ specializing to D. If $\mathcal{Y}$ is an irreducible component of $W^{\mu_\alpha}(C)$ containing $\mathcal{D}$, then by Proposition 3.3.3, we have

$$\dim \mathcal{Y} \leq g - |\boldsymbol{\mu}_\alpha|.$$

It therefore suffices to prove the reverse inequality.

The rest of the proof is identical to that of [22, Theorem 9.3], which we reproduce here for the sake of completeness. Let $\mathcal{M}^k_g$ be the moduli space of curves of genus g that admit a degree k map to $\mathbb{P}^1$, let $\mathcal{C}_k$ be the universal curve, and let $\mathcal{W}^{\boldsymbol{\mu}_\alpha}$ be the universal splitting-type locus over $\mathcal{M}^k_g$. Let $\widetilde{\mathcal{W}}^{\boldsymbol{\mu}_\alpha}$ be the locus in the symmetric dth fiber power of $\mathcal{C}_k$ parameterizing divisors $\mathcal{D}$ such that $\pi_*\mathcal{O}(\mathcal{D})$ has splitting type $\boldsymbol{\mu}_\alpha$.

We work in the Berkovich analytic domain of k-gonal curves whose skeleton is a k-gonal chain of loops. By Corollary 3.3.2, the tropicalization of $\widetilde{\mathcal{W}}^{\boldsymbol{\mu}_\alpha}$, has dimension at least

$$3g - 5 + 2k - |\boldsymbol{\mu}_\alpha| + r.$$

If $\pi_*\mathcal{O}(\mathcal{D}) \cong \mathcal{O}(\boldsymbol{\mu}_\alpha)$, then $\mathcal{D}$ has rank exactly r. It follows that $\mathcal{W}^{\boldsymbol{\mu}_\alpha}$ has dimension at least

$$3g - 5 + 2k - |\boldsymbol{\mu}_\alpha|.$$

By Corollary 3.3.2, there is an irreducible component of $\mathcal{W}^{\boldsymbol{\mu}_\alpha}$ whose tropicalization contains pairs of the form $(\Gamma, \mathcal{D})$ where Γ is a k-gonal chain of loops and $\mathcal{D} \in \mathbb{T}(t)$ is sufficiently general. The image of this component in $\mathcal{M}_g^{k,\mathrm{trop}}$ has dimension $2g-5+2k$. It follows that this component dominates $\mathcal{M}_g^k$, and the fibers have dimension at least $g - |\boldsymbol{\mu}_\alpha|$.

Combining the two bounds, we see that there exists an irreducible component $\mathcal{Y}$ of $W^{\boldsymbol{\mu}_\alpha}(C)$, containing $\mathcal{D}$, of dimension $g - |\boldsymbol{\mu}_\alpha|$. If $\mathcal{Z}$ is a component of $W_d^r(C)$ containing $\mathcal{Y}$, then by Proposition 3.3.3, we see that

$$\dim \mathcal{Z} = \dim \mathcal{Y}.$$

It follows that $\mathcal{Z}$ is the closure of $\mathcal{Y}$. ☐

3.4 Maximality of Scrollar Tableaux

Having established that Theorem 3.1.2 follows from our combinatorial results, it remains to prove the combinatorial results. The goal of this section is to prove the following.

Theorem 3.4.1. *Let t be a scrollar tableau of type α on $[a] \times [b]$. Then $\mathbb{T}(t)$ is maximal with respect to containment.*

Before proving Theorem 3.4.1, we first make two simple observations. These will be useful because, if $\mathbb{T}(t) \subseteq \mathbb{T}(t')$, then by Lemma 3.2.1, for every box (n,m) in $[a] \times [b]$, there exists a box (x,y) such that $t'(n,m) = t(x,y)$. Our argument will break into cases, depending on the location of (x,y) relative to that of (n,m).

Lemma 3.4.2. *Let α be a positive integer and (n,m) any box in $[a] \times [b]$. For any box (x,y) in $[a] \times [b]$, there exists an integer ℓ such that one of the following holds:*

1. $x \leq n - \ell\alpha$ *and* $y \leq m + \ell(k-\alpha)$,
2. $x \geq n - \ell\alpha$ *and* $y \geq m + \ell(k-\alpha)$, *or*
3. $n - (\ell+1)\alpha < x < n - \ell\alpha$ *and* $m + \ell(k-\alpha) < y < m + (\ell+1)(k-\alpha)$.

Proof. By the division algorithm, there exists an integer ℓ such that

$$n - (\ell+1)\alpha < x \leq n - \ell\alpha.$$

If $y \leq m + \ell(k-\alpha)$, then case (1) holds. If $y \geq m + (\ell+1)(k-\alpha)$, or if $x = n - \ell\alpha$ and $y \geq m + \ell(k-\alpha)$, then case (2) holds. Otherwise, $x \neq n - \ell\alpha$, and case (3) holds. ☐

Lemma 3.4.2 is illustrated in Figure 3.4. Boxes of the form $(n - \ell\alpha, m + \ell(k-\alpha))$ are labeled with stars, and the three cases of Lemma 3.4.2 are depicted in gray. Note that every box is contained in one of the three gray regions.

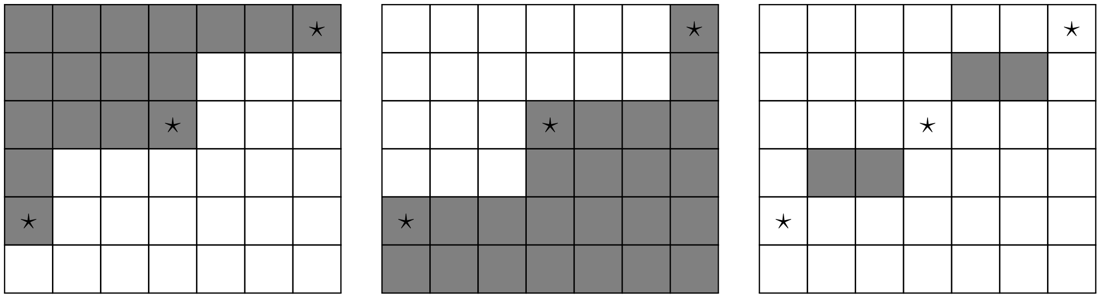


Figure 3.4: The three regions described in Lemma 3.4.2.

Remark 3.4.3. If α is equal to either a or $k-b$, then the integer ℓ in Lemma 3.4.2 can be taken to be one of -1, 0, or 1, as illustrated in Figure 3.5. If $\ell = \pm 1$, then the box $(n - \ell\alpha, m + \ell(k-\alpha))$ is not contained in $[a] \times [b]$.

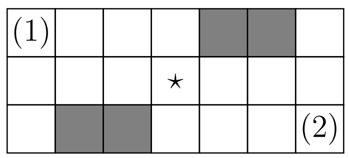


Figure 3.5: When $\alpha = k - b$, the integer ℓ can be taken to be one of -1, 0, or 1.

The following simple lemma is key to our argument.

Lemma 3.4.4. *Let α be a positive integer, let (n,m) be any box in $[a] \times [b]$, and (x,y) a box satisfying condition (3) of Lemma 3.4.2. Then*

$$x - y \not\equiv n - m \pmod{k}.$$

Proof. Since

$$n - (\ell+1)a < x < n - \ell a$$

and

$$m + \ell(k-a) < y < m + (\ell+1)(k-a),$$

we have

$$(n-m) - (\ell+1)k < x - y < (n-m) - \ell k.$$

Hence $x - y \not\equiv n - m \pmod{k}$. □

We are now prepared to prove the main result of this section, the maximality of scrollar tableaux.

Proof of Theorem 3.4.1. Let t' be a k-uniform displacement tableau such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. We will show that $t = t'$. We first demonstrate, by induction, that $t'(n,m) \geq t(n,m)$ for all $(n,m) \in [a] \times [b]$. The base case $t'(1,1) \geq t(1,1)$ holds because, by Lemma 3.2.1, $t'(1,1)$ must be a symbol in t, and $t(1,1)$ is the smallest symbol in t.

For our inductive hypothesis, suppose that $t'(x,y) \geq t(x,y)$ for all (x,y) such that $x \leq n$ and $y \leq m$, not both equal. We will show that $t'(n,m) \geq t(n,m)$. By Lemma 3.2.1, there exists $(x,y) \in [a] \times [b]$ such that $t'(n,m) = t(x,y)$. By Lemma 3.4.2, there exists an integer ℓ such that one of the following holds:

1. $x \leq n - \ell\alpha$ and $y \leq m + \ell(k-\alpha)$,
2. $x \geq n - \ell\alpha$ and $y \geq m + \ell(k-\alpha)$, or
3. $n - (\ell+1)\alpha < x < n - \ell\alpha$ and $m + \ell(k-\alpha) < y < m + (\ell+1)(k-\alpha)$.

If (x,y) satisfies (3), then by Lemma 3.4.4, $x - y \not\equiv n - m \pmod{k}$, a contradiction to Lemma 3.2.1. Hence (x,y) must satisfy either (1) or (2).

There are now two cases to consider – the case where the box $(n-\ell\alpha, m+\ell(k-\alpha))$ is contained in $[a] \times [b]$, and the case where it is not. We first consider the case where $(n-\ell\alpha, m+\ell(k-\alpha))$ is contained in $[a] \times [b]$. Notice that, if α is equal to a or $k-b$, then in this case we must have $\ell = 0$. If (x,y) satisfies (2), then

$$t(x,y) \geq t(n-\ell\alpha, m+\ell(k-\alpha)) = t(n,m),$$

hence $t'(n,m) \geq t(n,m)$, as desired. If (x,y) satisfies (1) and $(x,y) \neq (n-\ell\alpha, m+\ell(k-\alpha))$, we have either

$$t'(n,m) = t(x,y) \leq t(n-\ell\alpha-1, m+\ell(k-\alpha)), \text{ or}$$
$$t'(n,m) = t(x,y) \leq t(n-\ell\alpha, m+\ell(k-\alpha)-1).$$

First, assume that $n, m > 1$. Since t is scrollar, we have

$$t(n-\ell\alpha-1, m+\ell(k-\alpha)) = t(n-1,m) \text{ and}$$
$$t(n-\ell\alpha, m+\ell(k-\alpha)-1) = t(n,m-1).$$

By our inductive hypothesis, however, we have $t(n-1,m) \leq t'(n-1,m)$ and $t(n,m-1) \leq t'(n,m-1)$. This guarantees that either $t'(n,m) \leq t'(n-1,m)$ or $t'(n,m) \leq t'(n,m-1)$, a contradiction. It follows that

$$t'(n,m) = t(x,y) = t(n-\ell\alpha, m+\ell(k-\alpha)) = t(n,m).$$

Now, suppose that $m = 1$ and $n > 1$. The case where $n = 1$ will follow from a similar argument. Without loss of generality, let ℓ be the smallest integer such that (x,y) is above and to the right of $(n-\ell\alpha, m+\ell(k-\alpha))$. If $x < n-\ell\alpha$, then the conclusion follows from the argument above. On the other hand, if $x = n-\ell\alpha$, then since

$$m + (\ell-1)(k-\alpha) < y < m + \ell(k-\alpha),$$

we see that $x - y \neq n - m \pmod{k}$, a contradiction to Lemma 3.2.1.

We now turn to the case where $(n-\ell\alpha, m+\ell(k-\alpha))$ is not contained in $[a] \times [b]$. First, suppose that (x,y) satisfies (2). In this case, either $n-\ell\alpha \leq 0$ or $m+\ell(k-\alpha) \leq 0$, but not both. We will assume that $n-\ell\alpha \leq 0$; the other case follows by a similar argument. If ℓ is any integer satisfying $n - \ell\alpha \leq 0$, then (x, y) is below and to the right of $(1, m + \ell(k - \alpha))$. We may therefore assume without loss of generality that ℓ is the minimal integer such that $n - \ell\alpha \leq 0$. If α is equal to a, then $\ell = 1$. Because t is scrollar, we observe that

$$\begin{aligned} t'(n, m) = t(x, y) \geq t(1, m + \ell(k - \alpha)) &> t(\alpha, m + (\ell - 1)(k - \alpha)) \\ &\geq t(n - (\ell - 1)\alpha, m + (\ell - 1)(k - \alpha)) = t(n, m). \end{aligned}$$

Now, suppose that (x, y) satisfies (1). In this case, either $b < m + \ell(k - \alpha)$ or $a < n - \ell\alpha$, but not both. We will assume that $b < m + \ell(k - \alpha)$. The other case follows by a similar argument. Without loss of generality, assume that ℓ is the minimal integer such that $b < m + \ell(k - \alpha)$. As above, if $\alpha = k - b$, then $\ell = 1$. If $y \leq m + (\ell - 1)(k - \alpha)$, then by replacing ℓ with $\ell - 1$, we may reduce to the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is in $[a] \times [b]$. We may therefore assume that

$$m + (\ell - 1)(k - \alpha) < y \leq b < m + \ell(k - \alpha).$$

This situation is illustrated in Figure 3.6. The boxes $(n - \ell\alpha, m + \ell(k - \alpha))$ and $(n - (\ell - 1)\alpha, m + (\ell - 1)(k - \alpha))$ are labeled with stars, the box $(n - \ell\alpha, b)$ is labeled with a diamond, and the box (x, y) is located somewhere in the shaded region.

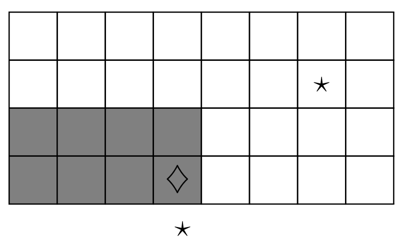


Figure 3.6: An illustration of the case where $(n - \ell\alpha, m + \ell(k - \alpha))$ is not contained in $[a] \times [b]$.

If $x = n - \ell\alpha$, then since

$$m + (\ell - 1)(k - \alpha) < y < m + \ell(k - \alpha),$$

we see that $x - y \not\equiv n - m \pmod{k}$, a contradiction to Lemma 3.2.1. We may therefore assume that $x < n - \ell\alpha$. Because t is scrollar, we have

$$\begin{aligned} t'(n, m) = t(x, y) \leq t(n - \ell\alpha - 1, b) &< t(n - (\ell - 1)\alpha - 1, b + 1 - (k - \alpha)) \\ &\leq t(n - (\ell - 1)\alpha - 1, m + (\ell - 1)(k - \alpha)) = t(n - 1, m). \end{aligned}$$

By induction, however, we have $t(n-1, m) \leq t'(n-1, m)$, hence $t'(n, m) \leq t'(n-1, m)$, a contradiction.

Thus, in every case we see that $t'(n, m) \geq t(n, m)$. We now show that $t'(n, m) \leq t(n, m)$ for all $(n, m) \in [a] \times [b]$. Combining the two inequalities, we see that $t' = t$. Given a tableau t, define the "rotated" tableau t_R as follows:

$$t_R(x, y) = g + 1 - t(a + 1 - x, b + 1 - y).$$

(See Figure 3.7 for an example.) Returning to our tableaux t and t', we see that by definition, both t_R and t'_R are k-uniform displacement tableaux, the tableau t_R is scrollar, and $\mathbb{T}(t_R) \subseteq \mathbb{T}(t'_R)$. By the argument above, we see that $t'_R(n,m) \geq t_R(n,m)$ for all $(n,m) \in [a] \times [b]$, hence $t'(n,m) \leq t(n,m)$ for all $(n,m) \in [a] \times [b]$, and the conclusion follows. $\square$

1	2	4	5	10	11	12
3	7	8	9	13	16	18
5	10	11	12	15	17	20
9	13	16	18	19	22	23
12	15	17	20	21	24	26

26	24	21	20	17	15	12
23	22	19	18	16	13	9
20	17	15	12	11	10	5
18	16	13	9	8	7	3
12	11	10	5	4	2	1

1	3	6	7	10	12	15
4	5	8	9	11	14	18
7	10	12	15	16	17	22
9	11	14	18	19	20	24
15	16	17	22	23	25	26

Figure 3.7: To obtain the "rotation" of the tableau on the left, first rotate 180 degrees, and then subtract each entry from $g+1$.

3.5 Non-Existence of Other Maximal Tableaux

In this section, we prove the following.

Theorem 3.5.1. *Let t be a k-uniform displacement tableau on $[a] \times [b]$. Then there exists a scrollar tableau t' on $[a] \times [b]$ such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$.*

Together with Theorem 3.4.1, this establishes Theorem 3.1.3.

To prove Theorem 3.5.1, we will describe an algorithm that, starting with t, produces a scrollar tableau t' by replacing certain symbols in t with other symbols in t. We first introduce a statistic on the boxes in a k-uniform displacement tableau.

Definition 3.5.2. *Let t be a k-uniform displacement tableau on $[a] \times [b]$. Given a box (x,y) such that $x+y \geq k$, we define a statistic $S_t(x,y)$ as follows. Consider the symbols appearing above (x,y) in column x and to the left of (x,y) in row y. Among these symbols, the $k-1$ largest ones form a hook of width α and height $k-\alpha$. We define $S_t(x,y)$ to be α.*

Remark 3.5.3. Note that if $x+y < k$, then $S_t(x,y)$ is undefined. In this case the box (x,y) is left empty. Additionally, the statistic α cannot appear in any box (x,y) with $x < \alpha$ or $y < k-\alpha$. In particular, for any k-uniform displacement tableau t, we have $S_t(\alpha, k-\alpha) = \alpha$.

Note also that S_t is well-defined. To see this let i be the smallest positive integer such that $t(x-i,y) = t(x,y-j)$ for some positive integer j. By the definition of a k-uniform displacement tableau, $i+j$ must be a multiple of k. It follows that the hook from $(x-i,y)$ to $(x,y-j)$ contains at least $k-1$ distinct symbols, all greater than $t(x-i,y)$.

Example 3.5.4. Figure 3.8 depicts an example of a 5-uniform displacement tableau t on $[4] \times [4]$. The first figure is t, the second is S_t, and the last two depict example hooks of width 2 and 3, respectively.

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

			4
		3	3
	2	3	2
1	2	3	3

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

Figure 3.8: A 5-uniform displacement tableau, its associated statistics, and some example hooks.

Before proceeding further, we will first need the following property of the statistic S_t.

Lemma 3.5.5. *Let t be a k-uniform displacement tableau. We have the following inequalities on statistics:*

$$S_t(x+1, y) \leq S_t(x, y) + 1$$
$$S_t(x, y-1) \leq S_t(x, y) + 1$$
$$S_t(x+1, y-1) \leq S_t(x, y) + 1.$$

Proof. Let H be the hook containing the $k-1$ largest symbols appearing above (x, y) in column x and to the left of (x, y) in row y. By the definition of S_t, H contains the boxes

$$(x, y+1-k+S_t(x, y)) \text{ and } (x+1-S_t(x, y), y),$$

but not the boxes

$$(x, y-k+S_t(x, y)) \text{ or } (x-S_t(x, y), y).$$

It follows that

$$t(x-S_t(x, y), y) < t(x, y+1-k+S_t(x, y)) \text{ and}$$

$$t(x+1-S_t(x, y), y) > t(x, y-k+S_t(x, y)).$$

If $S_t(x+1, y) > S_t(x, y) + 1$, then

$$\begin{aligned} t(x-S_t(x, y), y) &\geq t(x+2-S_t(x+1, y), y) \\ > t(x+1, y-k+S_t(x+1, y)) &> t(x, y+1-k+S_t(x, y)), \end{aligned}$$

a contradiction.

Similarly, if $S_t(x, y-1) > S_t(x, y) + 1$, then

$$\begin{aligned} t(x+1-S_t(x, y), y) &< t(x-S_t(x, y-1), y-1) \\ < t(x, y-k+S_t(x, y-1)) &\leq t(x, y-k+S_t(x, y)), \end{aligned}$$

a contradiction.

Finally, if $S_t(x+1, y-1) > S_t(x, y) + 1$, then

$$\begin{aligned} t(x - S_t(x,y), y) &> t(x+2-S_t(x+1,y-1), y-1) \\ > t(x+1, y-1-k+S_t(x+1,y-1)) &> t(x, y-k+S_t(x,y)), \end{aligned}$$

another contradiction. □

Definition 3.5.6. *Let t be a k-uniform displacement tableau on $[a] \times [b]$, and suppose that $a + b \geq k$. An* admissible path P *of type α in t is a sequence of boxes*

$$P = (x_0, y_0), (x_1, y_1), \ldots, (x_{a+b-k}, y_{a+b-k})$$

satisfying the following conditions:

1. $(x_0, y_0) = (\alpha, k - \alpha)$ *and* $(x_{a+b-k}, y_{a+b-k}) = (a, b)$.
2. *For all i, (x_i, y_i) is equal to either $(x_{i-1}+1, y_{i-1})$ or $(x_{i-1}, y_{i-1}+1)$.*
3. *If $(x_i, y_i) = (x_{i-1}+1, y_{i-1})$, then $S_t(x_i, y_i) \leq \alpha$.*
4. *If $(x_i, y_i) = (x_{i-1}, y_{i-1}+1)$, then $S_t(x_i, y_i) \geq \alpha$.*

In other words, an admissible path is a sequence of pairwise adjacent boxes starting at $(\alpha, k - \alpha)$ and ending in the bottom right corner of the tableau. Every time the path moves right, the statistic in the new box must be at most α, and every time the path moves down, the statistic in the new box must be at least α.

Example 3.5.7. Figure 3.9 depicts the statistics S_t for the tableau t from Example 3.5.4, together with two admissible paths of type 3 shaded. Note that the first path is admissible because the box labeled 2 is to the right of the previous box in the path.

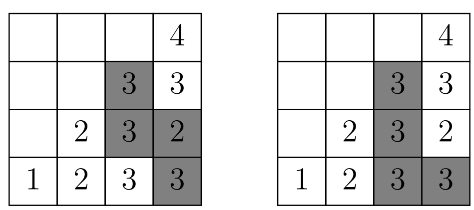


Figure 3.9: Two admissible paths of type 3

Note that an admissible path of type a is completely vertical, and an admissible path of type $k - b$ is completely horizontal. If there is an admissible path of type a in a tableau t, then $S_t(a, y) = a$ for all $y \geq k - a$. It follows that $t(1, y + k - a) > t(a, y)$ for all $y > k - a$, so t is scrollar of type a. Similarly, if there is an admissible path of type $k - b$ in a tableau t, then t is scrollar of type $k - b$.

The first main goal of this section is to prove the existence of admissible paths. That is, given a k-uniform tableau t on $[a] \times [b]$, we show that there exists an integer α and a admissible path P of type α. Our argument will require the following lemma.

Lemma 3.5.8. *Let t be a k-uniform displacement tableau. If P_1 and P_2 are two admissible paths in t of types α_1 and α_2, respectively, then $\alpha_1 = \alpha_2$.*

Proof. First, note that the last box in any admissible path is (a, b), so any two admissible paths intersect. Let (x, y) be the box in the intersection that minimizes $x + y$. Without loss of generality, assume that $\alpha_1 > \alpha_2$. Note that P_1 starts at $(\alpha_1, k - \alpha_1)$, which is above and to the right of $(\alpha_2, k - \alpha_2)$. Because (x, y) is the first box at which the two paths cross, we see that P_1 must contain the box $(x, y - 1)$ and P_2 must contain the box $(x - 1, y)$. By the definition of admissible paths, we have

$$\alpha_1 \leq S_t(x, y) \leq \alpha_2,$$

contradicting our assumption that $\alpha_1 > \alpha_2$. □

We now prove that admissible paths exist.

Proposition 3.5.9. *Let t be a k-uniform displacement tableau on $[a] \times [b]$, and suppose that $a + b \geq k$. Then there exists an admissible path in t.*

Proof. We proceed by induction on $a + b$. In the base case $b = k - a$, the admissible path consists of the single box (a, b).

If $a + b > k$, then by induction the tableau t_1 obtained by deleting the last row of t contains an admissible path P_1 of type α_1. Similarly, the tableau t_2 obtained by deleting the last column of t contains an admissible path P_2 of type α_2. We will show that either the path P_1' obtained by appending (a, b) to P_1 or the path P_2' obtained by appending (a, b) to P_2 is admissible. Note that P_1' is admissible if and only if $S_t(a, b) \geq \alpha_1$ and P_2' is admissible if and only if $S_t(a, b) \leq \alpha_2$.

If $S_t(a, b) < S_t(a, b - 1)$, then by Lemma 3.5.5, we have

$$S_t(a, b - 1) = S_t(a, b) + 1 \text{ and}$$

$$S_t(a - 1, b) \geq S_t(a, b - 1) - 1 = S_t(a, b).$$

It follows that either $S_t(a, b) \geq S_t(a, b - 1)$ or $S_t(a, b) \leq S_t(a - 1, b)$. We assume that $S_t(a, b) \geq S_t(a, b - 1)$; the case where $S_t(a, b) \leq S_t(a - 1, b)$ follows by a similar argument. If $S_t(a, b) \geq \alpha_1$, then P_1' is an admissible path of type α_1, and we are done. If P_1 contains the box $(a, b - 2)$, then $S_t(a, b) \geq S_t(a, b - 1) \geq \alpha_1$, by the definition of an admissible path. We may therefore assume that P_1 contains the box $(a - 1, b - 1)$, and $S_t(a, b) < \alpha_1$.

Now consider the path P_2. If the paths P_1 and P_2 intersect, let (x, y) be a box in the intersection, and let t_3 be the tableau obtained by restricting t to $[x] \times [y]$. The restrictions of P_1 and P_2 to t_3 are both admissible, and it follows from Lemma 3.5.8 that $\alpha_1 = \alpha_2$. Since $S_t(a, b) < \alpha_1$, we see that P_2' is an admissible path.

If P_1 and P_2 do not intersect, then P_1 lies entirely above and to the right of P_2, so $\alpha_1 > \alpha_2$. Let $(x, b - 1)$ be the leftmost box of P_1 in row $b - 1$. Because the two paths do not intersect, the boxes $(x - 1, b)$ and (x, b) must be contained in P_2. By the definition of admissible paths, we have $\alpha_1 \leq S_t(x, b-1)$ and $\alpha_2 \geq S_t(x, b)$. By Lemma 3.5.5, however, we have

$$\alpha_1 \leq S_t(x, b-1) \leq S_t(x, b) + 1 \leq \alpha_2 + 1.$$

It follows that $\alpha_1 = \alpha_2 + 1$. Since $S_t(a, b) < \alpha_1$, we see that $S_t(a, b) \leq \alpha_1 + 1 = \alpha_2$, hence P_2' is an admissible path. $\square$

Now that we know admissible paths exist, we can use them to construct a scrollar tableau from an arbitrary tableau.

Example 3.5.10. Before proving Theorem 3.5.1, we first illustrate the idea with an example. Figure 3.10 depicts the example of a 5-uniform displacement tableau t and an admissible path of type $\alpha = 3$ from Example 3.5.7. The proof of Theorem 3.5.1 provides us with an iterative procedure for constructing a scrollar tableau t' of type 3 such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. This procedure begins with the subtableau on $[\alpha] \times [k - \alpha] = [3] \times [2]$. It then follows the admissible path, extending the tableau one row or one column at a time. Every time we extend the tableau by a column, we replace each symbol in the new column with the symbol appearing α boxes to the left and $k - \alpha$ boxes below in the previous tableau. Similarly, every time we extend the tableau by a row, we replace each symbol in the new row with the symbol appearing α boxes to the right and $k - \alpha$ boxes above in the previous tableau. The definition of admissible paths guarantees that this construction yields a tableau.

1	2	3	9
4	6	7	10
5	8	11	13
12	14	15	16

1	2	3
4	6	7

1	2	3
4	6	7
5	8	11

1	2	3	5
4	6	7	10
5	8	11	13

1	2	3	5
4	6	7	10
5	8	11	13
10	14	15	16

Figure 3.10: Construction of a scrollar tableau from a given k-uniform displacement tableau and admissible path.

Proof of Theorem 3.5.1. First, note that if $a + b \leq k$, then t is scrollar of type a for trivial reasons. We therefore assume that $a + b > k$. By Proposition 3.5.9, there exists an admissible path P in t of type α. We will prove, by induction on $a + b$, that there exists a scrollar tableau t' of type α on $[a] \times [b]$ such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. In addition, we will see that $t'(a - i, b) = t(a - i, b)$ for all $i < \alpha$ and $t'(a, b - j) = t(a, b - j)$ for all $j < k - \alpha$. We assume that P contains the box $(a - 1, b)$; the case where P contains the box $(a, b-1)$ follows by a similar argument. By the definition of admissible paths, this implies that $S_t(a, b) \leq \alpha$.

Let t_1 be the tableau obtained by deleting the last column from t. The restriction of P to t_1 is an admissible path of type α in t_1. By induction, therefore, there exists a scrollar tableau t_1' on $[a - 1] \times [b]$ such that $\mathbb{T}(t_1) \subseteq \mathbb{T}(t_1')$. Moreover, we have $t_1'(a - 1 - i, b) = t_1(a - 1 - i, b)$ for all $i < \alpha$, and $t_1'(a - 1, b - j) = t_1(a - 1, b - j)$

for all $j < k - \alpha$. By Lemma 3.2.1, every symbol in t_1' is a symbol in t_1, and if $t_1(x,y) = t_1'(x',y')$, then $x - y = x' - y' \pmod{k}$.

We now define a tableau t' on $[a] \times [b]$.

$$t'(x,y) = \begin{cases} t_1'(x,y) & \text{if } x < a \\ t_1'(x-\alpha, y+k-\alpha) & \text{if } x = a \text{ and } y \leq b-k+\alpha \\ t(x,y) & \text{if } x = a \text{ and } y > b-k+\alpha. \end{cases}$$

We first show that t' is a tableau. Let $(x,y) \in [a] \times [b]$. If $x < a$, then since t_1' is a tableau, we see that $t'(x,y) > t'(x-1,y)$ and $t'(x,y) > t'(x,y-1)$. If $y \leq b-k+\alpha$, then because t_1' is a tableau, we have $t'(a,y) > t'(a,y-1)$, and because t_1' is scrollar of type α, we have

$$t'(a-1,y) < t'(a-\alpha, y+k-\alpha) = t'(a,y).$$

If $y > b-k+\alpha$, then since t is a tableau and $t_1'(a-1,y) = t(a-1,y)$, we have $t'(a-1,y) < t'(a,y)$. If $y > b+1-k+\alpha$, then since t is a tableau, we have $t'(a,y-1) < t'(a,y)$. Finally, since $S_t(a,b) \leq \alpha$, we have

$$t'(a, b-k+\alpha) = t(a-\alpha, b) < t(a, b+1-k+\alpha) = t'(a, b+1-k+\alpha).$$

To see that t' is scrollar, we show that if $b > k-\alpha$, then $t'(x,y) = t'(x+\alpha, y-k+\alpha)$ for all pairs (x,y). This is clear if $x+\alpha < a$, because t_1' is scrollar of type α. On the other hand, if $x+\alpha = a$, then this holds by construction. If $\alpha = k-b$, then $t'(x,1) > t'(x+b-k,b)$ for all $x < a$ because t_1' is scrollar, and $t'(a,1) > t'(a+b-k,b)$ because $S_t(a,b) \leq \alpha = k-b$.

Finally, we show that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. Note that the symbol $t'(x,y)$ is also a symbol in t_1 if and only if $x < a$ or $y \leq b-k+\alpha$. By construction, every symbol in t_1 is also a symbol in t, and if

$$t'(x,y) = t_1(x',y') = t(x'',y''),$$

then

$$x - y = x' - y' = x'' - y'' \pmod{k}.$$

On the other hand, if $y > b-k+\alpha$, then the symbol $t'(a,y) = t(a,y)$ appears only in one box, and there is nothing to prove. By Lemma 3.2.1, it follows that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. □

Chapter 4 Tropical Splitting Loci

This chapter is taken from the preprint *Tropical Methods in Hurwitz-Brill-Noether Theory* [9], authored by Cook-Powell-Jensen.

4.1 Introduction

Recall from Chapter 2 that the Picard variety of a curve C is stratified by the subschemes $W^r_d(C)$, parameterizing line bundles of degree d and rank at least r. The study of these subschemes, known as Brill-Noether theory, is a central area of research in algebraic geometry. The celebrated Brill-Noether Theorem of Griffiths and Harris says that, if $C \in \mathcal{M}_g$ is general, then the varieties $W^r_d(C)$ are equidimensional of the expected dimension, with the convention that a variety of negative dimension is empty [18].

If C is not general, what can we say about its Brill-Noether theory? The *gonality* of C is the smallest integer k such that $W^1_k(C)$ is nonempty, and a consequence of the Brill-Noether Theorem is that the gonality of a general curve is $\lfloor \frac{g+3}{2} \rfloor$. If we assume that C has smaller gonality than this, what effect does this assumption have on the dimensions of $W^r_d(C)$ for other values of r and d? Along these lines, several recent papers have focused on the Brill-Noether theory of curves that are general in the Hurwitz space $\mathcal{H}_{g,k}$, rather than the moduli space $\mathcal{M}_g$ [12, 13, 33, 22, 25, 10, 27, 14]. The Hurwitz space $\mathcal{H}_{g,k}$ parameterizes degree k branched covers of $\mathbb{P}^1$, where the source has genus g. If $k < \lfloor \frac{g+3}{2} \rfloor$ and $(C, \pi) \in \mathcal{H}_{g,k}$ is general, then the varieties $W^r_d(C)$ can have multiple components of varying dimensions, prohibiting a naive generalization of the Brill-Noether Theorem.

In this setting, however, the Picard variety of C admits a more refined stratification. We say that a line bundle $L \in \mathrm{Pic}(C)$ has *splitting type* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ if $\pi_* L \cong \oplus_{i=1}^k \mathcal{O}(\mu_i)$. (See Section 2.1.) Since the splitting type of a line bundle determines that line bundle's rank and degree, it is a more refined invariant. The *splitting type locus* $W^{\boldsymbol{\mu}}(C) \subseteq \mathrm{Pic}(C)$ parameterizing line bundles of splitting type $\boldsymbol{\mu}$ is locally closed, of expected codimension

$$|\boldsymbol{\mu}| := \sum_{i<j} \max\{0, \mu_j - \mu_i - 1\}.$$

In [25], H. Larson proves an analogue of the Brill-Noether Theorem for the strata $W^{\boldsymbol{\mu}}(C)$.

Theorem 4.1.1. *[25] Let $(C, \pi) \in \mathcal{H}_{g,k}$ be general. If $g \geq |\boldsymbol{\mu}|$, then*

$$\dim W^{\boldsymbol{\mu}}(C) = g - |\boldsymbol{\mu}|.$$

If $g < |\boldsymbol{\mu}|$, then $W^{\boldsymbol{\mu}}(C)$ is empty.

Theorem 4.1.1 is proven by considering analogous closed strata $\overline{W}^{\boldsymbol{\mu}}(C)$ containing $W^{\boldsymbol{\mu}}(C)$. We refer the reader to Section 2.1 for a precise definition. As in the original Brill-Noether Theorem, the fact that the dimension of $\overline{W}^{\boldsymbol{\mu}}(C)$ is at least $g - |\boldsymbol{\mu}|$ holds for all $(C, \pi) \in \mathcal{H}_{g,k}$. This follows from standard results about degeneracy loci, provided that $\overline{W}^{\boldsymbol{\mu}}(C)$ is nonempty. Larson demonstrates the nonemptiness of $\overline{W}^{\boldsymbol{\mu}}(C)$ by showing that a certain intersection number is nonzero.

The fact that the dimension of $\overline{W}^{\boldsymbol{\mu}}(C)$ is at most $g - |\boldsymbol{\mu}|$ is much deeper. Here, we give a new proof of this result using tropical and combinatorial techniques. Our approach builds on earlier work exploring the divisor theory of a certain family of tropical curves known as *chains of loops* [11, 33, 22, 10]. Theorem 4.1.1 is a consequence of the following result.

Theorem 4.1.2. *Let Γ be a k-gonal chain of loops of genus g. If $g \geq |\boldsymbol{\mu}|$, then $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is equidimensional and*

$$\dim \overline{W}^{\boldsymbol{\mu}}(\Gamma) = g - |\boldsymbol{\mu}|.$$

If $g < |\boldsymbol{\mu}|$, then $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is empty.

Tropical Splitting Type Loci

In her proof of Theorem 4.1.1, Larson uses the theory of limit linear series on a chain of elliptic curves. Remarkably, however, her proof does not require a description of splitting type loci on this degenerate curve. That is, it is not necessary for her to classify those limit linear series that are limits of line bundles with a given splitting type $\boldsymbol{\mu}$. In contrast, our proof of Theorem 4.1.2 follows from an explicit description of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$. This description is used to prove new results and formulate Conjectures 4.1.6 and 4.1.7.

Our description of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ builds on the earlier work of [11, 33, 22, 10] mentioned above. The main technical result of [11] is a classification of special divisor classes on chains of loops, when the lengths of the edges are sufficiently general. Specifically, if Γ is such a chain of loops, then $W^r_d(\Gamma)$ is union of tori $\mathbb{T}(t)$, where each torus corresponds to a standard Young tableau t on a certain rectangular partition. This result was generalized in [34, 33] to chains of loops with arbitrary edge lengths. If Γ is the k-gonal chain of loops referred to in Theorem 4.1.2, then $W^r_d(\Gamma)$ is again a union of tori $\mathbb{T}(t)$ indexed by rectangular tableaux, but here the tableaux are non-standard. Instead, the tableaux are required to satisfy an arithmetic condition known as *k-uniform displacement* (see Definition 2.2.3).

Given a splitting type $\boldsymbol{\mu} \in \mathbb{Z}^k$, we define a partition $\lambda(\boldsymbol{\mu})$ in Definition 4.2.2. We call a partition of this type a *k-staircase*. Our description of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is analogous to that of $W^r_d(\Gamma)$ mentioned above.

Theorem 4.1.3. *Let Γ be a k-gonal chain of loops of genus g. Then*

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) = \bigcup \mathbb{T}(t),$$

where the union is over all k-uniform displacement tableau t on $\lambda(\boldsymbol{\mu})$ with alphabet $[g]$.

We prove Theorem 4.1.3 in Section 4.2. The remainder of the paper uses this classification to establish various geometric properties of the tropical splitting type loci $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$. For example, we compute the dimension of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ in Section 4.4, proving Theorem 4.1.2. In Section 4.5, we study the connectedness of tropical splitting type loci.

Theorem 4.1.4. *Let Γ be a k-gonal chain of loops of genus g. If $g > |\boldsymbol{\mu}|$, then $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is connected in codimension 1.*

Unfortunately, the connectedness of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ does not imply that of $\overline{W}^{\boldsymbol{\mu}}(C)$ for a general $(C, \pi) \in \mathcal{H}_{g,k}$. Theorem 4.1.4 is nevertheless interesting for at least two reasons. First, by [25, Theorem 1.2], we know that the locally closed stratum $W^{\boldsymbol{\mu}}(C)$ is smooth for a general $(C, \pi) \in \mathcal{H}_{g,k}$, so it is irreducible if and only if it is connected. Theorem 4.1.4 therefore suggests that $W^{\boldsymbol{\mu}}(C)$ is irreducible if it is positive dimensional, as predicted in [10, Conjecture 1.2]. Second, by [8, Theorem 1], the tropicalization of a variety is equidimensional and connected in codimension one, so Theorems 4.1.2 and 4.1.4 can be seen as evidence that $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is the tropicalization of $\overline{W}^{\boldsymbol{\mu}}(C)$ (see Conjecture 4.1.7 below).

Numerical Classes

These geometric results follow from a careful study of k-staircases and k-uniform displacement tableaux. These combinatorial objects are explored in Section 4.3. Staircases belong to a wider class of partitions, known as *k-cores*, which have been studied extensively in other contexts. (See, for example, [23].) The set $\mathcal{P}_k$ of k-cores is a ranked poset (Corollary 4.3.18), with cover relations given by *upward displacements* in the sense of [32, Definition 6.1]. We write $\mathcal{P}_k(\lambda)$ for the interval below $\lambda \in \mathcal{P}_k$. In Section 4.6, we use these observations to compute the cardinality of zero-dimensional tropical splitting type loci.

Theorem 4.1.5. *Let Γ be a k-gonal chain of loops of genus g. If $g = |\boldsymbol{\mu}|$, then $|\overline{W}^{\boldsymbol{\mu}}(\Gamma)|$ is equal to the number of maximal chains in $\mathcal{P}_k(\lambda(\boldsymbol{\mu}))$.*

The number of maximal chains in $\mathcal{P}_k(\lambda(\boldsymbol{\mu}))$ has received significant interest in the combinatorics and representation theory literature, and has connections to the affine symmetric group. More precisely, there is a bijection between such maximal chains and reduced words in the affine symmetric group [24]. For this reason, several of our results have equivalent formulations in terms of these groups (see Remarks 4.3.20 and 4.5.1). There is currently no known closed form expression for these numbers, but they satisfy a simple recurrence (Lemma 4.6.3) that allows one to compute a given number in polynomial time (Algorithm 4.6.2).

Theorem 4.1.5 has implications beyond the zero-dimensional case. In [25, Lemma 5.4], Larson shows that the numerical class of $\overline{W}^{\boldsymbol{\mu}}(C)$ in $\text{Pic}(C)$ is of the form $a_{\boldsymbol{\mu}}\Theta^{|\boldsymbol{\mu}|}$, where the coefficient $a_{\boldsymbol{\mu}}$ is independent of the genus. To compute the coefficient $a_{\boldsymbol{\mu}}$, therefore, it suffices to compute the cardinality of $\overline{W}^{\boldsymbol{\mu}}(C)$ in the case where $g = |\boldsymbol{\mu}|$. In this way, Theorem 4.1.5 suggests the following conjecture.

Conjecture 4.1.6. *Let $(C, \pi) \in \mathcal{H}_{g,k}$ be general. The numerical class of $\overline{W}^{\boldsymbol{\mu}}(C)$ in $\mathrm{Pic}^{d(\boldsymbol{\mu})}(C)$ is*

$$\left[\overline{W}^{\boldsymbol{\mu}}(C)\right] = \frac{1}{|\boldsymbol{\mu}|!} \cdot \alpha(\mathcal{P}_k(\lambda(\boldsymbol{\mu}))) \cdot \Theta^{|\boldsymbol{\mu}|},$$

where $\alpha(\mathcal{P})$ denotes the number of maximal chains in the poset $\mathcal{P}$.

At the end of Section 4.6, we provide evidence for Conjecture 4.1.6, in the form of numerous examples where it holds. We also compute the number of maximal chains in $\mathcal{P}_k(\lambda(\boldsymbol{\mu}))$ for some infinite families of splitting types where the class of $\overline{W}^{\boldsymbol{\mu}}(C)$ is unknown. For such families, these numbers form well-known integer sequences, including binomial coefficients (Examples 4.6.7 and 4.6.13), geometric sequences (Example 4.6.8), Catalan numbers (Example 4.6.6), and Fibonacci numbers (Example 4.6.9). Conjecture 4.1.6 would be implied by the following.

Conjecture 4.1.7. *Let Γ be a k-gonal chain of loops, and let C be a curve of genus g and gonality k over a nonarchimedean field K with skeleton Γ. Then the tropicalization map*

$$\mathrm{Trop} : \overline{W}^{\boldsymbol{\mu}}(C) \to \overline{W}^{\boldsymbol{\mu}}(\Gamma)$$

is surjective. Moreover, if $g = |\boldsymbol{\mu}|$, then it is a bijection.

Conjecture 4.1.7 is known to hold in several important cases. It is the main result of [7] in the case where Γ has generic edge lengths (or equivalently, when $k = \lfloor \frac{g+3}{2} \rfloor$). The main results of [10] and [22] combined show that the tropicalization map is surjective for the "maximal" splitting types $\boldsymbol{\mu}_\alpha$ of [10, Definition 2.5]. We do not, however, know that it is bijective in the zero-dimensional case. Conjecture 4.1.7 remains open in many cases where Conjecture 4.1.6 is known to hold.

4.2 Tropical Splitting Loci

Given a splitting type $\boldsymbol{\mu} \in \mathbb{Z}^k$, we define the tropical splitting type locus

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) = \left\{ D \in \mathrm{Pic}^{d(\boldsymbol{\mu})}(\Gamma) \mid \mathrm{rk}(D + m g^1_k) \geq x_m(\boldsymbol{\mu}) - 1 \text{ for all } m \right\}.$$

Note that the tropical splitting type locus can be defined in this way for *any* tropical curve Γ with a distinguished g^1_k. By Lemma 2.1.5, if $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$, then $\overline{W}^{\boldsymbol{\mu}}(\Gamma) \subseteq \overline{W}^{\boldsymbol{\lambda}}(\Gamma)$. The following is a straightforward consequence of Baker's Specialization Lemma.

Proposition 4.2.1. *Let C be a curve of genus g and gonality k over a nonarchimedean field K with skeleton Γ. Then*

$$\mathrm{Trop}\left(\overline{W}^{\boldsymbol{\mu}}(C)\right) \subseteq \overline{W}^{\boldsymbol{\mu}}(\Gamma).$$

Proof. Since the divisor of degree k and rank 1 on Γ is unique, it must be the tropicalization of the g^1_k on C by Baker's Specialization Lemma. If $D \in \overline{W}^{\boldsymbol{\mu}}(C)$, then by definition we have

$$h^0(C, D + m g^1_k) \geq x_m(\boldsymbol{\mu}) \text{ for all } m.$$

By Baker's Specialization Lemma, we have

$$\mathrm{rk}(\mathrm{Trop}(D + mg^1_k)) \geq h^0(C, D + mg^1_k) - 1 \geq x_m(\boldsymbol{\mu}) - 1 \text{ for all } m.$$

Thus, $\mathrm{Trop}(D) \in \overline{W}^{\boldsymbol{\mu}}(\Gamma)$. □

In this section, we prove Theorem 4.1.3, which gives an explicit description of splitting type loci on a k-gonal chain of loops. Before proving Theorem 4.1.3, we first define a partition $\lambda(\boldsymbol{\mu})$ associated to each splitting type $\boldsymbol{\mu}$.

Staircases

Definition 4.2.2. *Given a splitting type $\boldsymbol{\mu} \in \mathbb{Z}^k$ and an integer m, we define the rectangular partition*

$$\lambda_m(\boldsymbol{\mu}) := \left\{(x, y) \in \mathbb{N}^2 \mid x \leq x_m(\boldsymbol{\mu}),\ y \leq y_m(\boldsymbol{\mu})\right\}.$$

We further define

$$\begin{aligned}\lambda(\boldsymbol{\mu}) &= \bigcup_{m \in \mathbb{Z}} \lambda_m(\boldsymbol{\mu}) \\ &= \left\{(x, y) \in \mathbb{N}^2 \mid \exists m \in \mathbb{Z} \text{ s.t. } x \leq x_m(\boldsymbol{\mu}),\ y \leq y_m(\boldsymbol{\mu})\right\}.\end{aligned}$$

*We call a partition of the form $\lambda(\boldsymbol{\mu})$ a k-*staircase*.*

Example 4.2.3. Let $\boldsymbol{\mu} = (-3, -1, 1)$. Figure 4.1 depicts the rectangular partitions $\lambda_{-1}(\boldsymbol{\mu})$, $\lambda_0(\boldsymbol{\mu})$, and $\lambda_1(\boldsymbol{\mu})$, together with $\lambda(\boldsymbol{\mu})$. Note that $\lambda_m(\boldsymbol{\mu})$ is empty for all m other than -1, 0, or 1.

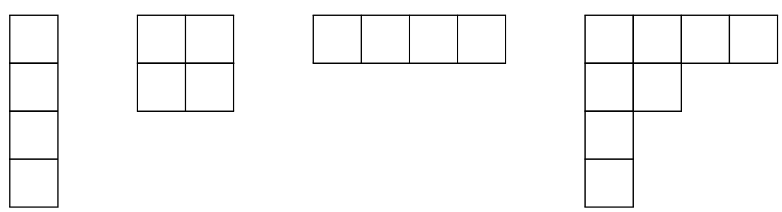


Figure 4.1: The partitions $\lambda_{-1}(\boldsymbol{\mu})$, $\lambda_0(\boldsymbol{\mu})$, $\lambda_1(\boldsymbol{\mu})$, and $\lambda(\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (-3, -1, 1)$.

Remark 4.2.4. If $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ and $\boldsymbol{\mu}' = (\mu_1 + m, \ldots, \mu_k + m)$ for some $m \in \mathbb{Z}$, then there is an isomorphism between $\overline{W}^{\boldsymbol{\mu}}(C)$ and $\overline{W}^{\boldsymbol{\mu}'}(C)$, given by twisting by $\pi^*\mathcal{O}(m)$. Correspondingly, we have $\lambda(\boldsymbol{\mu}) = \lambda(\boldsymbol{\mu}')$.

Remark 4.2.5. If $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, then by Lemma 2.1.5, we have $\lambda(\boldsymbol{\mu}') \subseteq \lambda(\boldsymbol{\mu})$.

If both $x_m(\boldsymbol{\mu})$ and $y_m(\boldsymbol{\mu})$ are positive, then the box $(x_m(\boldsymbol{\mu}), y_m(\boldsymbol{\mu}))$ is the unique inside corner of the rectangular partition $\lambda_m(\boldsymbol{\mu})$, and one of the inside corners of $\lambda(\boldsymbol{\mu})$. We define

$$\alpha_m(\boldsymbol{\mu}) = x_m(\boldsymbol{\mu}) - x_{m-1}(\boldsymbol{\mu}).$$

Note that $\alpha_m(\boldsymbol{\mu}) \leq \alpha_{m+1}(\boldsymbol{\mu})$ for all m, and $y_{m-1}(\boldsymbol{\mu}) - y_m(\boldsymbol{\mu}) = k - \alpha_m(\boldsymbol{\mu})$. We say that an integer α is a *rank jump* in $\lambda(\boldsymbol{\mu})$ if $\alpha = \alpha_m(\boldsymbol{\mu})$ for some integer m. We say that α is a *strict rank jump* in $\lambda(\boldsymbol{\mu})$ if $\alpha = \alpha_m(\boldsymbol{\mu})$ for some integer m such that both $x_{m-1}(\boldsymbol{\mu})$ and $y_m(\boldsymbol{\mu})$ are positive. In other words, the strict rank jumps are $\alpha_{1-\mu_k}(\boldsymbol{\mu}), \alpha_{2-\mu_k}(\boldsymbol{\mu}), \ldots, \alpha_{-2-\mu_1}(\boldsymbol{\mu})$.

Tropical Splitting Type Loci

We now define the analogue of the coordinate tori from [34].

Definition 4.2.6. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type. Given an integer m and a k-uniform displacement tableau t on $\lambda(\boldsymbol{\mu})$ with alphabet $[g]$, let t_m denote the restriction of t to the rectangular subpartition $\lambda_m(\boldsymbol{\mu})$. We define the coordinate subtorus $\mathbb{T}(t)$ as follows.*

$$\mathbb{T}(t) = \left\{ D \in \mathrm{Pic}^{d(\boldsymbol{\mu})}(\Gamma) \mid D + mg^1_k \in \mathbb{T}(t_m) \textit{ for all } m \right\}.$$

From the definition it appears that, if one wants to determine whether a divisor class D is contained in $\mathbb{T}(t)$, one has to compute $\xi_j(D+mg^1_k)$ for all integers m. Using (2.2), however, we can simplify Definition 4.2.6 as follows.

Lemma 4.2.7. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type, and let t be a k-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$ with alphabet $[g]$. Define the function*

$$Z(x,y) = \begin{cases} y - x & \text{if } t(x,y) \leq g-k+1 \\ y - x + mk & \text{if } t(x,y) > g-k+1 \text{ and } x_{m-1}(\boldsymbol{\mu}) < x \leq x_m(\boldsymbol{\mu}). \end{cases}$$

Then

$$\mathbb{T}(t) := \{ D \in \mathrm{Pic}^{d(\boldsymbol{\mu})}(\Gamma) \mid \xi_{t(x,y)}(D) = Z(x,y) \}.$$

Proof. Let m be an integer and let $t_m(x,y) = j$. If $j \leq g-k+1$, then $\widetilde{\xi_j}(g^1_k) = 0$, and by (2.2) we see that for any divisor class D we have

$$\xi_j(D) = \xi_j(D + mg^1_k).$$

It follows that $\xi_j(D) = y - x$ if and only if $\xi_j(D+mg^1_k) = y - x$.

On the other hand, if $j > g-k+1$, then we must first show that $x_{m-1}(\boldsymbol{\mu}) < x \leq x_m(\boldsymbol{\mu})$. The second inequality follows from the fact that $(x,y) \in \lambda_m(\boldsymbol{\mu})$. If $x \leq x_{m-1}(\boldsymbol{\mu})$, then the $k+1$ boxes in the hook

$$H_m = \{(x,y) \in \lambda(\boldsymbol{\mu}) \mid x \geq x_{m-1}(\boldsymbol{\mu}), y \geq y_m(\boldsymbol{\mu})\}$$

are all below and to the right of (x,y). The two inside corners $(x_{m-1}(\boldsymbol{\mu}), y_{m-1}(\boldsymbol{\mu}))$ and $(x_m(\boldsymbol{\mu}), y_m(\boldsymbol{\mu}))$ have lattice distance k, so they are the only two boxes of H_m that can contain the same symbol. It follows that H_m contains at least k distinct symbols greater than or equal to j. Since $j > g-k+1$, this is impossible, hence $x > x_{m-1}(\boldsymbol{\mu})$. Now, since $\widetilde{\xi_j}(g^1_k) = k$, by (2.2) we see that for any divisor class D we have

$$\xi_j(D) = \xi_j(D + mg^1_k) - mk.$$

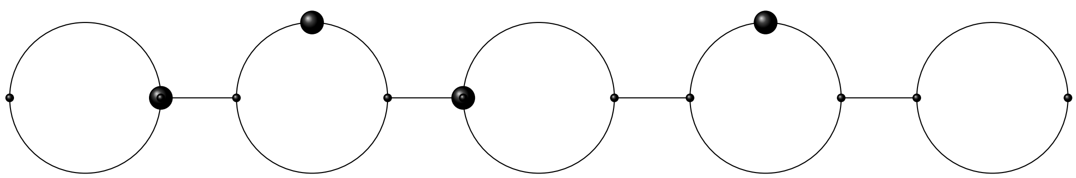
It follows that $\xi_j(D) = y - x + mk$ if and only if $\xi_j(D+mg^1_k) = y - x$. $\square$

As in Definition 2.3.3, the k-uniform displacement condition guarantees that, if the symbol j appears in more than one box, then the boxes impose the same condition on ξ_j. In particular, if $j > g-k+1$ and $t_m(x,y) = t_{m'}(x',y') = j$, then the k-uniform displacement condition guarantees that

$$(y'-x') - (y-x) = (m-m')k,$$

so $Z(x,y) = Z(x',y')$. As a consequence, we see that the codimension of $\mathbb{T}(t)$ is equal to the number of distinct symbols in t.

Example 4.2.8. Figure 4.2 depicts a 3-uniform displacement tableau t on $\lambda(\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (-3,-1,1)$. Since the tableau contains $g = 5$ distinct symbols, $\mathbb{T}(t)$ is a zero-dimensional torus. In other words, it consists of a single divisor class D, also depicted in Figure 4.2. In this picture, the chips on loops 2 and 4 are located at the points $\langle 1 \rangle_2$ and $\langle 1 \rangle_4$. By Theorem 4.1.3, the divisor class D is in $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$. That is, $D - g^1_3$ has rank 0, D has rank 1, and $D + g^1_3$ has rank 3.



1	3	4	5
2	5		
3			
5			

Figure 4.2: A 3-uniform displacement tableau on $\lambda(-3,-1,1)$ and the corresponding divisor class.

Lemma 4.2.7 allows us to formulate the following analogue of [10, Lemma 3.6].

Lemma 4.2.9. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type, and let t, t' be k-uniform displacement tableaux on $\lambda(\boldsymbol{\mu})$. Then $\mathbb{T}(t) \subseteq \mathbb{T}(t')$ if and only if*

1. *every symbol in t' is a symbol in t, and*
2. *if $t(x,y) = t'(x',y')$, then $y - x \equiv y' - x' \pmod{k}$.*

We now prove Theorem 4.1.3.

Proof of Theorem 4.1.3. We first show that

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) \supseteq \bigcup \mathbb{T}(t).$$

Let t be a k-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$, and let $D \in \mathbb{T}(t)$. By definition, $D + mg^1_k \in \mathbb{T}(t_m)$ for all m. It follows from Theorem 2.3.4 that $D + mg^1_k$ has degree $d(\boldsymbol{\mu})+mk$ and rank at least $x_m(\boldsymbol{\mu})-1$ for all m. By definition, we see that $D \in \overline{W}^{\boldsymbol{\mu}}(\Gamma)$.

We now show that

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) \subseteq \bigcup \mathbb{T}(t).$$

Let $D \in \overline{W}^{\boldsymbol{\mu}}(\Gamma)$. By definition, $D + mg^1_k$ has degree $d(\boldsymbol{\mu}) + mk$ and rank at least $x_m(\boldsymbol{\mu})-1$ for all m. By Theorem 2.3.4, there exists a k-uniform displacement tableau t_m on the rectangular partition $\lambda_m(\boldsymbol{\mu})$ such that $D + mg^1_k \in \mathbb{T}(t_m)$. We construct a tableau t on $\lambda(\boldsymbol{\mu})$ as follows. For each box (x, y) in $\lambda(\boldsymbol{\mu})$, define

$$t(x, y) = \min_{\substack{m\in\mathbb{Z} \text{ s.t.} \\ (x,y)\in\lambda_m(\boldsymbol{\mu})}} t_m(x, y).$$

We first show that t is a tableau on $\lambda(\boldsymbol{\mu})$. To see that t is strictly increasing across rows, suppose that $x > 1$ and $t(x, y) = t_m(x, y)$. Since $(x, y) \in \lambda_m(\boldsymbol{\mu})$, we see that $(x-1, y) \in \lambda_m(\boldsymbol{\mu})$ as well. It follows that

$$t(x-1, y) \le t_m(x-1, y) < t_m(x, y) = t(x, y).$$

The same argument shows that t is strictly increasing down the columns.

We now show that the tableau t satisfies the k-uniform displacement condition. Suppose that $t(x, y) = t(x', y')$. By construction, there exist integers m and m' such that $t(x, y) = t_m(x, y)$ and $t(x', y') = t_{m'}(x', y')$. Since $D + mg^1_k \in \mathbb{T}(t_m)$ and $D + m'g^1_k \in \mathbb{T}(t_{m'})$, we see that

$$\begin{aligned} \xi_{t(x,y)}(D + mg^1_k) &\equiv y - x \pmod k \\ \xi_{t(x,y)}(D + m'g^1_k) &\equiv y' - x' \pmod k. \end{aligned}$$

It therefore suffices to show that

$$\xi_j(D + mg^1_k) \equiv \xi_j(D + m'g^1_k) \text{ for all } j.$$

This follows from (2.2) and the fact that $\widetilde{\xi}_j(g^1_k) \equiv 0 \pmod k$ for all j.

Finally, we show that $D \in \mathbb{T}(t)$. For every box $(x, y) \in \lambda(\boldsymbol{\mu})$, there is an integer m such that $\xi_{t(x,y)}(D + mg^1_k) = y - x$. By Lemma 4.2.7, we have $\xi_{t(x,y)}(D) = Z(x, y)$. Since this holds for all $(x, y) \in \lambda(\boldsymbol{\mu})$, we see that $D \in \mathbb{T}(t)$ by Lemma 4.2.7. $\square$

Operations on Splitting Types

Several operations on splitting types have simple interpretations in terms of the corresponding partitions. The first of these corresponds to Serre duality.

Lemma 4.2.10. *Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ be a splitting type, and let $\boldsymbol{\mu}^T = (-\mu_k, \ldots, -\mu_1)$. Then $\lambda(\boldsymbol{\mu}^T) = \lambda(\boldsymbol{\mu})^T$.*

Proof. Since both operations are involutions, it suffices to show that $\lambda(\boldsymbol{\mu})^T \subseteq \lambda(\boldsymbol{\mu}^T)$. Let $(x, y) \in \lambda(\boldsymbol{\mu})$. Then there exists an integer $m \in \mathbb{Z}$ such that

$$x \leq \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\},$$

$$y \leq \sum_{i=1}^{k} \max\{0, -\mu_i - m - 1\}.$$

Setting $m' = -2 - m$, we see that

$$y \leq \sum_{i=1}^{k} \max\{0, -\mu_i - m - 1\} = \sum_{i=1}^{k} \max\{0, -\mu_i + m' + 1\}$$
$$x \leq \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\} = \sum_{i=1}^{k} \max\{0, \mu_i - m' - 1\}.$$

Thus, $(y, x) \in \lambda(\boldsymbol{\mu}^T)$. □

As a consequence, we see that the set of partitions of the form $\lambda(\boldsymbol{\mu})$ is closed under transpose. We now show that it is also closed under the operations of deleting the top row or the leftmost column.

Lemma 4.2.11. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type, let s be the minimal index such that $\mu_s < \mu_{s+1}$, and let*

$$\boldsymbol{\mu}^+ = (\mu_1, \ldots, \mu_{s-1}, \mu_s + 1, \mu_{s+1}, \ldots, \mu_k).$$

Then $\lambda(\boldsymbol{\mu}^+)$ is the partition obtained from $\lambda(\boldsymbol{\mu})$ by deleting the first row. Moreover, $|\boldsymbol{\mu}| - |\boldsymbol{\mu}^+|$ is equal to the largest strict rank jump in $\lambda(\boldsymbol{\mu})$.

Similarly, let s' be the maximal index such that $\mu_{s'} > \mu_{s'-1}$, and let

$$\boldsymbol{\mu}^- = (\mu_1, \ldots, \mu_{s'-1}, \mu_{s'} - 1, \mu_{s'+1}, \ldots, \mu_k).$$

Then $\lambda(\boldsymbol{\mu}^-)$ is the partition obtained from $\lambda(\boldsymbol{\mu})$ by deleting the leftmost column. Moreover, $|\boldsymbol{\mu}| - |\boldsymbol{\mu}^+|$ is equal to $k - \alpha$, where α is the smallest strict rank jump in $\lambda(\boldsymbol{\mu})$.

Proof. We prove the statements about $\boldsymbol{\mu}^+$. The statements about $\boldsymbol{\mu}^-$ follow from Lemma 4.2.10, together with the observation that $\boldsymbol{\mu}^- = (\boldsymbol{\mu}^{T+})^T$. Let $(x, y) \in \lambda(\boldsymbol{\mu}^+)$. Then there exists an integer m such that

$$x \leq \sum_{i=1}^{k} \max\{0, \mu_i^+ + m + 1\} \text{ and}$$
$$y \leq \sum_{i=1}^{k} \max\{0, -\mu_i^+ - m - 1\}.$$

Since y is positive and μ_s is minimal, we see that $m \leq -2 - \mu_s$. It follows that $\mu_s^+ + m + 1 \leq 0$, so

$$x \leq \sum_{i=1}^{k} \max\{0, \mu_i^+ + m + 1\} = \sum_{i=1}^{k} \max\{0, \mu_i + m + 1\}$$

$$y + 1 \leq 1 + \sum_{i=1}^{k} \max\{0, -\mu_i^+ - m - 1\} = \sum_{i=1}^{k} \max\{0, -\mu_i - m - 1\}.$$

So $(x, y+1) \in \lambda(\boldsymbol{\mu})$. An analogous argument shows that, if $(x, y) \in \lambda(\boldsymbol{\mu})$, then either $y = 1$ or $(x, y-1) \in \lambda(\boldsymbol{\mu}^+)$.

We now compute $|\boldsymbol{\mu}| - |\boldsymbol{\mu}^+|$. If $i, j \neq s$, then $\mu_j^+ - \mu_i^+ = \mu_j - \mu_i$. If $i < s$, then $\mu_s^+ - \mu_i^+ - 1 = 0$. Finally, if $j > s$, then $\mu_j^+ - \mu_s^+ = \mu_j - \mu_s - 1$. Thus,

$$|\boldsymbol{\mu}| - |\boldsymbol{\mu}^+| = \sum_{i<j} \Big(\max\{0, \mu_j - \mu_i\} - \max\{0, \mu_j^+ - \mu_i^+ - 1\} \Big)$$

$$= \sum_{j=1}^{k} \Big(\max\{0, \mu_j - \mu_s - 1\} - \max\{0, \mu_j - \mu_s - 2\} \Big).$$

On the other hand, the largest rank jump in $\lambda(\boldsymbol{\mu})$ is

$$\alpha_{-\mu_s - 2}(\boldsymbol{\mu}) = \sum_{j=1}^{k} \Big(\max\{0, \mu_j - \mu_s - 1\} - \max\{0, \mu_j - \mu_s - 2\} \Big).$$

□

4.3 Cores and Displacement

This section contains the main combinatorial arguments that will be used in our examination of tropical splitting type loci. We study an operation on partitions known as *displacement*, and a certain class of partitions known in the combinatorics literature as k-*cores*, which includes the k-staircases. Because of Theorem 4.1.3, we are interested in k-uniform displacement tableaux on partitions of this type. A tableau t on a partition λ can be thought of as a chain of partitions

$$\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_n = \lambda,$$

where $\lambda_j = \{(x, y) \in \lambda | t(x, y) \leq j\}$. This observation naturally leads us to study posets of partitions, where the cover relations guarantee that the resulting tableaux satisfy k-uniform displacement.

Diagonals and Displacement

Following [14], given $a \in \mathbb{Z}/k\mathbb{Z}$, we define the corresponding *diagonal (mod k)* to be

$$D_a := \{(x, y) \in \mathbb{N}^2 \mid y - x \equiv a \pmod{k}\}.$$

Definition 4.3.1. *[32, Definition 6.1] Let λ be a partition. The* upward displacement[1] *of λ with respect to $a \in \mathbb{Z}/k\mathbb{Z}$ is the partition λ_a^+ obtained from λ by adding all outside corners in D_a.*

Similarly, the downward displacement *of λ with respect to $a \in \mathbb{Z}/k\mathbb{Z}$ is the partition λ_a^- obtained from λ by deleting all inside corners in D_a.*

Example 4.3.2. The operations of upward displacement and downward displacement are not inverses. For example, consider the partition λ on the left in Figure 4.3, where each box has been decorated with its diagonal (mod 4). The second partition in the figure is λ_2^+, the upward displacement with respect to 2 (mod 4), and the third partition is $(\lambda_2^+)_2^-$, the downward displacement of the second partition, again with respect to 2 (mod 4). Note that the first partition and the third partition do not agree.

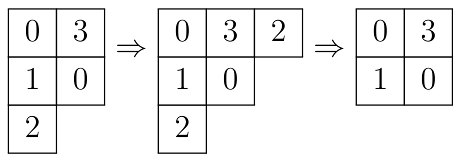


Figure 4.3: Upward displacement followed by downward displacement does not necessarily yield the original partition.

There are important examples of partitions for which the concatenation of an upward and a downward displacement *is* the identity.

Definition 4.3.3. *A partition λ is called a k-*core *if it can be obtained from the empty partition by a sequence of upward displacements with respect to congruence classes in $\mathbb{Z}/k\mathbb{Z}$.*

We write $\mathcal{P}_k$ for the poset of k-cores, where $\lambda' \leq \lambda$ if λ can be obtained from λ' by a sequence of upward displacements with respect to congruence classes in $\mathbb{Z}/k\mathbb{Z}$. If $\lambda \in \mathcal{P}_k$, we write $\mathcal{P}_k(\lambda)$ for the interval (or principal order ideal) below λ in $\mathcal{P}_k$.

Example 4.3.4. Figure 4.4 depicts a Hasse diagram for $\mathcal{P}_3(\lambda(\boldsymbol{\mu}))$, where $\boldsymbol{\mu} = (-3, -1, 1)$. The diagram is drawn from left to right, rather than bottom to top, to preserve space on the page. Note that $\lambda(\boldsymbol{\mu})$ is a 3-core, and that every maximal chain in the interval below $\lambda(\boldsymbol{\mu})$ has the same length. As we shall see, the fact that the length of a maximal chain is 5 corresponds to the fact that any 3-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$ has at least 5 symbols. The fact that there are 2 maximal chains corresponds to the fact that there are 2 such tableaux with alphabet [5].

Remark 4.3.5. Recall that, if $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$, then $\lambda(\boldsymbol{\mu}') \subseteq \lambda(\boldsymbol{\mu})$. It is not necessarily true, however, that $\lambda(\boldsymbol{\mu}') \leq \lambda(\boldsymbol{\mu})$ in the poset $\mathcal{P}_k$. For example, if $\boldsymbol{\mu} = (-3, -1, 1)$ and

[1]This terminology is consistent with [32]. In that paper, partitions are depicted according to the French convention, whereas ours are in the English style. Because of this, the *upward* displacement adds boxes *below* the partition.

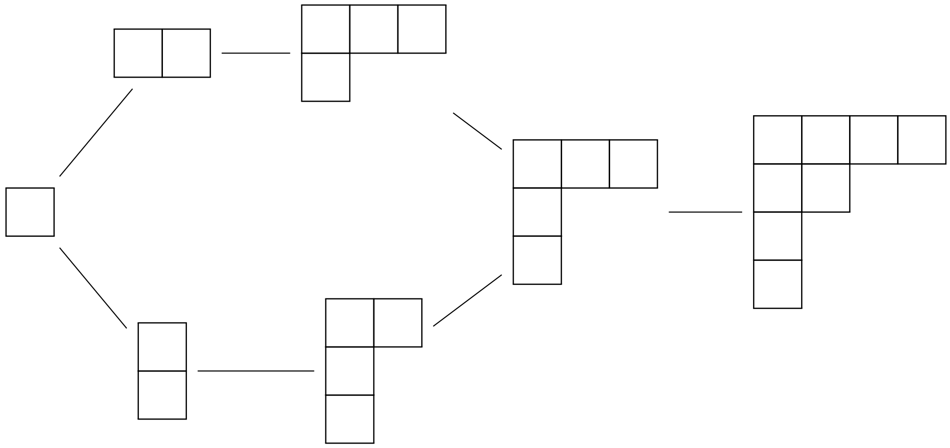


Figure 4.4: A principal order ideal in $\mathcal{P}_3$.

$\boldsymbol{\mu}' = (-3, 0, 0)$, then $\boldsymbol{\mu} \leq \boldsymbol{\mu}'$ but the partition $\lambda(\boldsymbol{\mu}')$, pictured in Figure 4.5, is not contained in $\mathcal{P}_3(\lambda(\boldsymbol{\mu}))$, pictured in Figure 4.4.

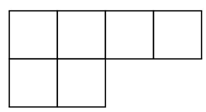


Figure 4.5: The partition $\lambda(\boldsymbol{\mu}')$ is not in the principal order ideal of Figure 4.4.

We note the following simple observation.

Lemma 4.3.6. *The transpose of a k-core is a k-core.*

Proof. This follows directly from the fact that $(\lambda_a^+)^T = (\lambda^T)_{-a}^+$. □

We now define some invariants of partitions. Let λ be a partition and $a \in \mathbb{Z}/k\mathbb{Z}$ a congruence class. We define

$$C_a(\lambda) := \max \{y \mid \exists (x, y) \in \lambda \cap D_a \text{ with } (x, y+1) \notin \lambda\}.$$

In other words, $C_a(\lambda)$ is the height of the tallest column whose last box is in D_a. If no such column exists, we define $C_a(\lambda)$ to be zero. We write

$$\mathbf{C}(\lambda) = (C_0(\lambda), C_1(\lambda), \ldots, C_{k-1}(\lambda)),$$

and further define

$$\rho_k(\lambda) := \sum_{a \in \mathbb{Z}/k\mathbb{Z}} C_a(\lambda).$$

Example 4.3.7. Figure 4.6 again depicts the partition $\lambda(\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (-3, -1, 1)$. Each column is labeled by the diagonal (mod 3) containing its last box. The tallest column whose last box is in D_0 has height 4, the tallest column whose last box is in D_1 has height 1, and there is no column whose last box is in D_2. Therefore, $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (4, 1, 0)$, and

$$\rho_3(\lambda(\boldsymbol{\mu})) = 4 + 1 + 0 = 5 = |\boldsymbol{\mu}|.$$

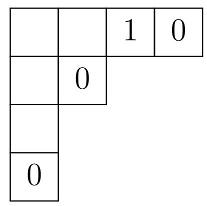


Figure 4.6: The partition $\lambda(-3, -1, 1)$, with each column labeled by the diagonal (mod 3) containing its last box.

Descent

We now provide an alternate characterization of k-cores. Most of the material in this and the next subsection has appeared previously in the literature on k-cores. (See, for example, [24, 23].) We nevertheless include these arguments here, as they are fairly short and we wish to advertise these ideas.

Definition 4.3.8. *We say that a partition λ satisfies k-*descent *if the following condition holds for every congruence class $a \in \mathbb{Z}/k\mathbb{Z}$. Whenever $(x, y) \in \lambda \cap D_a$ and $(x + 1, y) \notin \lambda$, then $C_{a-1}(\lambda) < y$.*

Example 4.3.9. The partition λ pictured on the left in Figure 4.3 does not satisfy 4-descent, because the last box in the first row is in D_3, and there exists a column whose last box is in D_2. In other words, $(2, 1) \in \lambda \cap D_3$ and $(3, 1) \notin \lambda$, but $C_2(\lambda) = 3 \geq 1$.

On the other hand, the partition $\lambda(\boldsymbol{\mu})$ pictured in Figure 4.6 does satisfy 3-descent. There is no row whose last box is in D_1. The last box in the first row is in D_0, and there is no column whose last box is in D_2. The last box in the third row is in D_2, and $C_1(\lambda(\boldsymbol{\mu})) = 1 < 3$.

Remark 4.3.10. If λ satisfies k-descent, then there is a congruence class $a \in \mathbb{Z}/k\mathbb{Z}$ such that $C_a(\lambda) = 0$. Specifically, if $(x, 1)$ is the last box in the first row, then by definition $C_{-x}(\lambda) = 0$.

Our goal for this subsection is to prove the following.

Proposition 4.3.11. *A partition λ is a k-core if and only if both λ and λ^T satisfy k-descent.*

To prove Proposition 4.3.11, we will need a few preliminary results. First, we examine the behavior of inside corners in partitions that satisfy k-descent.

Lemma 4.3.12. *Let λ be a partition that satisfies k-descent, and let $a \in \mathbb{Z}/k\mathbb{Z}$ be a congruence class. If λ has an inside corner in D_a, then the tallest column whose last box is in D_a contains an inside corner.*

Proof. Let $(x, y) \in \lambda \cap D_a$ be an inside corner, and consider the tallest column whose last box is in D_a. If it doesn't contain an inside corner, then the column immediately to the right has the same height, and its last box is in D_{a-1}. But the height of this column is greater than y, contradicting the definition of k-descent. $\square$

Lemma 4.3.13. *Let λ be a partition that satisfies k-descent. Then λ has an inside corner in D_a if and only if $C_{a-1}(\lambda) < C_a(\lambda)$.*

Proof. First, suppose that λ has an inside corner in D_a. By Lemma 4.3.12, the tallest column of λ whose last box is in D_a ends in an inside corner. In other words, there is an x such that $(x, C_a(\lambda)) \in \lambda \cap D_a$ and $(x + 1, C_a(\lambda)) \notin \lambda$. Thus, by the definition of k-descent, we see that $C_{a-1}(\lambda) < C_a(\lambda)$.

Conversely, suppose that $C_{a-1}(\lambda) < C_a(\lambda)$, and consider the tallest column of λ whose last box is in D_a. Let $(x, C_a(\lambda))$ be the last box in this column. By definition, $(x, C_a(\lambda) + 1) \notin \lambda$. Since $C_{a-1}(\lambda) < C_a(\lambda)$ and $(x + 1, C_a(\lambda)) \in D_{a-1}$, we see that $(x + 1, C_a(\lambda)) \notin \lambda$. Thus, $(x, C_a(\lambda)) \in D_a$ is an inside corner. $\square$

Lemma 4.3.14. *Let λ be a partition, and suppose that both λ and λ^T satisfy k-descent. For any congruence class $a \in \mathbb{Z}/k\mathbb{Z}$, λ cannot have both an inside corner and an outside corner in D_a.*

Proof. Suppose that $(x, y) \in D_a$ is an inside corner and $(x', y') \in D_a$ is an outside corner. By definition, either $y' = 1$ or $(x', y' - 1) \in \lambda \cap D_{a-1}$, hence $C_{a-1}(\lambda) \geq y' - 1$. Since λ satisfies k-descent, we see that $y' - 1 < y$. Similarly, since λ^T satisfies k-descent, we see that $x' - 1 < x$. Together, these inequalities imply that $(x', y') \in \lambda$, contradicting our assumption that (x', y') is an outside corner. $\square$

Lemma 4.3.14 implies that, when restricted to partitions satisfying k-descent, the operations of upward and downward displacement are inverses.

Lemma 4.3.15. *Let λ be a partition, and suppose that both λ and λ^T satisfy k-descent. If λ has an inside corner in D_a, then $\lambda = (\lambda_a^-)_a^+$. Similarly, if λ has an outside corner in D_a, then $\lambda = (\lambda_a^+)_a^-$.*

Proof. We show the first equality above. The second equality follows from an analogous argument. Note that $\lambda \subseteq (\lambda_a^-)_a^+$. To see the reverse containment, let $(x, y) \in (\lambda_a^-)_a^+$. If $(x, y) \notin D_a$ or (x, y) is not an inside corner of $(\lambda_a^-)_a^+$, then $(x, y) \in \lambda_a^- \subset \lambda$. On the other hand, if $(x, y) \in D_a$ is an inside corner of $(\lambda_a^-)_a^+$, then neither $(x - 1, y)$ nor $(x, y - 1)$ are in D_a, so either $x = 1$ or $(x - 1, y) \in \lambda$, and either $y = 1$ or $(x, y - 1) \in \lambda$. It follows that either $(x, y) \in \lambda$ or (x, y) is an outside corner of λ. By Lemma 4.3.14, however, λ cannot have an outside corner in D_a. Thus, $(x, y) \in \lambda$, and $(\lambda_a^-)_a^+ \subseteq \lambda$. $\square$

Crucially, the k-descent property is preserved by upward and downward displacements.

Lemma 4.3.16. *Let λ be a partition that satisfies k-descent. Then, for any $a \in \mathbb{Z}/k\mathbb{Z}$, λ_a^+ and λ_a^- also satisfy k-descent.*

Proof. We prove the statement about λ_a^+. The statement about λ_a^- holds by an analogous argument. Suppose that $(x, y) \in \lambda_a^+$ and $(x+1, y) \notin \lambda_a^+$. By the definition of k-descent, either $(x, y) \notin \lambda$, or $(x, y) \in \lambda$ and $C_{y-x-1}(\lambda) < y$. We first consider the case where $(x, y) \notin \lambda$. Since $(x, y) \in \lambda_a^+$, this implies that $(x, y) \in D_a$. Note that $(x-1, y) \in \lambda \cap D_{a+1}$ and $(x, y) \notin \lambda$. By the definition of k-descent, we see that $C_a(\lambda) < y$. It follows that, if $(x', y') \in \lambda \cap D_{a-1}$ with $y' \geq y$ and $(x', y'+1) \notin \lambda$, then $(x', y'+1)$ is an outside corner, and thus in λ_a^+. From this we obtain $C_{a-1}(\lambda_a^+) < y$.

On the other hand, if $(x, y) \in \lambda$, then $C_{y-x-1}(\lambda) < y$. We may assume that $(x, y) \in D_{a+1}$, because otherwise we have $C_{y-x-1}(\lambda_a^+) \leq C_{y-x-1}(\lambda)$. Then, since $(x+1, y) \notin \lambda_a^+$, we must have $(x+1, y-1) \notin \lambda$. Since λ satisfies k-descent and $(x, y-1) \in \lambda \cap D_a$, we see that $C_{a-1}(\lambda) < y-1$. Since $C_a(\lambda) < y$ and $C_{a-1}(\lambda) < y-1$, we see that $C_a(\lambda_a^+) < y$. □

We now establish that this is an alternate characterization of k-cores.

Proof of Proposition 4.3.11. First, let λ be a k-core. By Lemma 4.3.6, λ^T is a k-core. It therefore suffices to prove that λ satisfies k-descent. By definition, λ is obtained from the empty partition by a sequence of upward displacements with respect to congruence classes in $\mathbb{Z}/k\mathbb{Z}$. We prove that λ satisfies k-descent by induction on the number of upward displacements in this sequence. The base case is the empty partition, which satisfies k-descent trivially. The inductive step follows from Lemma 4.3.16, which says that the upward displacement of a partition satisfying k-descent also satisfies k-descent.

Now, let λ be a partition such that both λ and λ^T satisfy k-descent. We prove that λ is a k-core by induction on the number of boxes in λ. The base case is the empty partition, which is a k-core. If λ is non-empty, then there is an inside corner $(x, y) \in \lambda$. By Lemma 4.3.16, the downward displacements λ_{y-x}^- and $(\lambda_{y-x}^-)^T = (\lambda^T)_{x-y}^-$ satisfy k-descent. By induction, λ_{y-x}^- is therefore a k-core, hence by definition, $(\lambda_{y-x}^-)_{y-x}^+$ is a k-core as well. By Lemma 4.3.15, however, $\lambda = (\lambda_{y-x}^-)_{y-x}^+$, so λ is a k-core. □

Behavior of Invariants Under Displacement

A consequence of this characterization is that $\mathcal{P}_k$ is a graded poset. To see this, given a vector $\mathbf{C} = (C_0, C_1, \ldots, C_{k-1})$ and a congruence class $a \in \mathbb{Z}/k\mathbb{Z}$, define the vector $\mathbf{C}_a^- = (C_{0a}^-, C_{1a}^-, \ldots, C_{k-1a}^-)$ by

$$C_{ba}^- = \begin{cases} C_a - 1 & \text{if } b = a-1 \\ C_{a-1} & \text{if } b = a \\ C_b & \text{otherwise.} \end{cases}$$

The notation is justified by the following proposition.

Proposition 4.3.17. *If $\lambda \in \mathcal{P}_k$ has an inside corner in D_a, then $\mathbf{C}(\lambda_a^-) = \mathbf{C}(\lambda)_a^-$.*

Proof. It is straightforward to see that, if $b \neq a, a-1$, then $C_b(\lambda_a^-) = C_b(\lambda)$. By Lemma 4.3.12, the tallest column of λ whose last box is in D_a contains an inside corner, and by Lemma 4.3.13, $C_{a-1}(\lambda) < C_a(\lambda)$. It follows that $C_{a-1}(\lambda_a^-) = C_a(\lambda) - 1$.

Now, suppose that $(x, y) \in \lambda \cap D_a$ is the last box of a column. If $y > C_{a-1}(\lambda)$, then (x, y) is an inside corner of λ, because $(x+1, y) \in D_{a-1}$ cannot be in λ by definition. It follows that $(x, y) \notin \lambda_a^-$, and thus that $C_a(\lambda_a^-) \leq C_{a-1}(\lambda)$. We now show that equality holds. If $C_{a-1}(\lambda) = 0$, then there is nothing to show. Otherwise, suppose that column x is the tallest column whose last box is in D_{a-1}. By Lemma 4.3.14, $(x, C_{a-1}(\lambda) + 1)$ cannot be an outside corner of λ, hence $x > 1$ and $(x-1, C_{a-1}(\lambda)+1) \notin \lambda$. It follows that $(x-1, C_{a-1}(\lambda)) \in D_a$ is the last box in its column. Since $(x-1, C_{a-1}(\lambda))$ is not an inside corner, it is contained in λ_a^-, so $C_a(\lambda_a^-) \geq C_{a-1}(\lambda)$. □

Corollary 4.3.18. *The set $\mathcal{P}_k$ is a graded poset with rank function ρ_k.*

Proof. Let $\lambda \in \mathcal{P}_k$, and suppose that λ has an inside corner in D_a. It suffices to show that

$$\rho_k(\lambda) = \rho_k(\lambda_a^-) + 1.$$

This follows from Proposition 4.3.17 by summing over all $b \in \mathbb{Z}/k\mathbb{Z}$. □

Saturated Tableaux

Corollary 4.3.18 provides a natural interpretation for the function ρ_k. As we shall see in Corollary 4.3.22, if $\lambda \in \mathcal{P}_k$, then $\rho_k(\lambda)$ is the minimal number of symbols in a k-uniform displacement tableau on λ. Let $\mathscr{C}(\mathcal{P})$ denote the set of maximal chains in a poset $\mathcal{P}$. Given a partition $\lambda \in \mathcal{P}_k$, we define a map

$$\Phi_\lambda : \binom{[g]}{\rho_k(\lambda)} \times \mathscr{C}(\mathcal{P}_k(\lambda)) \to YT_k(\lambda)$$

as follows. Let

$$s_1 < s_2 < \cdots < s_{\rho_k(\lambda)}$$

be the elements of $S \subseteq [g]$, and let

$$\emptyset = \lambda_0 < \lambda_1 < \cdots < \lambda_{\rho_k(\lambda)} = \lambda$$

be a maximal chain in $\mathcal{P}_k(\lambda)$. Define the tableau $t = \Phi_\lambda(S, \vec{\lambda})$ by setting

$$t(x, y) = s_j \text{ if } (x, y) \in \lambda_j \smallsetminus \lambda_{j-1}.$$

For each j, every symbol in λ_{j-1} is smaller than s_j, so t is a tableau. Moreover, every box containing the symbol s_j is in the same diagonal (mod k), so t satisfies k-uniform displacement. We say that a tableau t on λ is *k-saturated* if it is in the image of Φ_λ. Note that every k-saturated tableau contains exactly $\rho_k(\lambda)$ distinct symbols.

Theorem 4.3.19. *Let λ be a k-core, and let t be a k-uniform displacement tableau on λ. Then there exists a k-saturated tableau t' on λ such that:*

1. *every symbol in t' is a symbol in t, and*

2. *if $t(x,y) = t'(x',y')$, then $y - x \equiv y' - x' \pmod{k}$.*

Proof. We prove this by induction on $\rho_k(\lambda)$. The base case is when $\rho_k(\lambda) = 0$, in which case λ is the empty partition, and the result is trivial.

For the inductive step, suppose that h is the largest symbol in t. Note that any box containing h must be an inside corner of λ, and every such box is contained in the same diagonal D_a. In particular, the symbol h does not appear in the restriction $t|_{\lambda_a^-}$. By induction, there exists a k-saturated tableau t'' on λ_a^- such that every symbol in t'' is a symbol in $t|_{\lambda_a^-}$, and if $t(x,y) = t''(x',y')$, then $y - x \equiv y' - x' \pmod{k}$.

By Corollary 4.3.18, $\rho_k(\lambda_a^-) = \rho_k(\lambda) - 1$, so the set S of symbols in t'' has size $\rho_k(\lambda) - 1$. By definition, there is a maximal chain

$$\emptyset = \lambda_0 < \lambda_1 < \cdots < \lambda_{\rho_k(\lambda)-1} = \lambda_a^-$$

such that $t'' = \Phi_{\lambda_a^-}(S, \vec{\lambda})$. Let $S' = S \cup \{h\}$, let $\vec{\lambda}'$ be the chain obtained by appending λ to the end of $\vec{\lambda}$, and let $t' = \Phi_\lambda(S', \vec{\lambda}')$. In other words,

$$t'(x,y) = \begin{cases} t''(x,y) & \text{if } (x,y) \in \lambda_a^- \\ h & \text{if } (x,y) \notin \lambda_a^-. \end{cases}$$

Clearly, every symbol in t' is a symbol in t. Since h is larger than every symbol appearing in $t|_{\lambda_a^-}$, we see that t' is a tableau. Finally, since every box containing h is in D_a, we see that if $t(x,y) = h$, then $y - x \equiv a \pmod{k}$. □

Remark 4.3.20. Under the bijection between k-uniform displacement tableaux on k-cores and words in the affine symmetric group, Theorem 4.3.19 is equivalent to the statement that every word is equivalent to a reduced word.

Example 4.3.21. Given a k-uniform displacement tableau t on λ, the proof of Theorem 4.3.19 provides an explicit algorithm for producing the k-saturated tableau t'. At each step, find the diagonal D_a containing the largest symbol in t. Replace every inside corner in D_a with this symbol, then downward displace with respect to a, and iterate the procedure.

Figure 4.7 illustrates this procedure for a 3-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$, where $\boldsymbol{\mu} = (-3, -1, 1)$. The tableau on the left uses 8 symbols. At each step, we highlight in gray the downward displacement of the previous partition in the sequence, replacing symbols as we go until we arrive at a tableau with $\rho_3(\lambda(\boldsymbol{\mu})) = 5$ symbols.

Corollary 4.3.22. *Let λ be a k-core. The minimum number of symbols in a k-uniform displacement tableau on λ is $\rho_k(\lambda)$.*

Proof. Let t be a k-uniform displacement tableau on λ. By Theorem 4.3.19, there exists a k-uniform displacement tableau t' on λ such that every symbol in t' is a symbol in t, and t' has exactly $\rho_k(\lambda)$ symbols. It follows that t has at least $\rho_k(\lambda)$ symbols. □

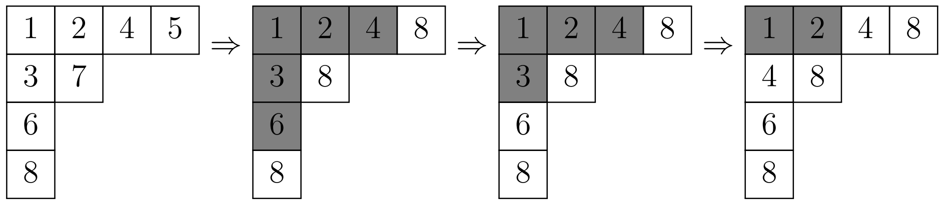


Figure 4.7: Starting with the tableau on the left, we produce a 3-uniform displacement tableau with only 5 symbols.

4.4 Dimensions of Tropical Splitting Type Loci

In this section, we compute the dimension of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$, proving Theorem 4.1.2. In order to do this, we first apply the results of Section 4.3 to k-staircases.

Lemma 4.4.1. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type, and let $c(\boldsymbol{\mu}) = -\sum_{i=1}^k \mu_i$. Then every inside corner of $\lambda(\boldsymbol{\mu})$ is in $D_{c(\boldsymbol{\mu})}$.*

Proof. Recall that the inside corners of $\lambda(\boldsymbol{\mu})$ are the boxes $(x_m(\boldsymbol{\mu}), y_m(\boldsymbol{\mu}))$. By definition, we have

$$\begin{aligned} y_m(\boldsymbol{\mu}) - x_m(\boldsymbol{\mu}) &= \sum_{i=1}^k \Big(\max\{0, -\mu_i - m - 1\} - \max\{0, \mu_i + m + 1\} \Big) \\ &= \sum_{i=1}^k \Big(\max\{0, -\mu_i - m - 1\} + \min\{0, -\mu_i - m - 1\} \Big) \\ &= \sum_{i=1}^k (-\mu_i - m - 1) \\ &\equiv -\sum_{i=1}^k \mu_i \pmod{k}. \end{aligned}$$

□

If λ is a k-staircase, then there is a simple expression for the invariants $C_a(\lambda)$.

Lemma 4.4.2. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type. Then*

$$C_{c(\boldsymbol{\mu})+i}(\lambda(\boldsymbol{\mu})) = y_{-\mu_{k-i}}(\boldsymbol{\mu}) = \sum_{j=1}^{k-1-i} \max\{0, \mu_{k-i} - \mu_j - 1\} \text{ for all } 0 \le i \le k-1.$$

Proof. We first identify the congruence classes $a \in \mathbb{Z}/k\mathbb{Z}$ such that $C_a(\lambda(\boldsymbol{\mu})) = 0$. Let (x, y) be the last box in a column of $\lambda(\boldsymbol{\mu})$. Then there exists an integer m such that $y = y_m(\boldsymbol{\mu})$ and $x_{m-1}(\boldsymbol{\mu}) < x \le x_m(\boldsymbol{\mu})$. Since $(x_m(\boldsymbol{\mu}), y_m(\boldsymbol{\mu})) \in D_{c(\boldsymbol{\mu})}$, we see that $(x, y) \in D_{c(\boldsymbol{\mu})+i}$ for some i in the range $0 \le i < \alpha_m(\boldsymbol{\mu})$. Since $\alpha_m(\boldsymbol{\mu}) \le \alpha_{m+1}(\boldsymbol{\mu})$ for all m, we may reduce to the case where $m = -2 - \mu_1$ is maximal. We see that

$C_{c(\boldsymbol{\mu})+i}(\lambda(\boldsymbol{\mu}))$ is nonzero for i in the range $0 \leq i < \alpha_{-2-\mu_1}(\boldsymbol{\mu})$ and zero for i in the range $\alpha_{-2-\mu_1}(\boldsymbol{\mu}) \leq i \leq k-1$. Note that $\alpha_{-2-\mu_1}(\boldsymbol{\mu})$ is the minimal index j such that $\mu_{j+1} \geq \mu_1 + 2$.

To establish the formula when $C_a(\lambda(\boldsymbol{\mu}))$ is nonzero, we proceed by induction on the number of rows of $\lambda(\boldsymbol{\mu})$. The base case is when $\mu_j - \mu_i \leq 1$ for all $i < j$, in which case $\lambda(\boldsymbol{\mu})$ is the empty partition. In this case, $C_i(\lambda(\boldsymbol{\mu})) = y_{-\mu_{k-i}}(\boldsymbol{\mu}) = 0$ for all $0 \leq i \leq k-1$.

For the inductive step, recall from Lemma 4.2.11 that $\lambda(\boldsymbol{\mu}^+)$ is the partition obtained by deleting the first row of $\lambda(\boldsymbol{\mu})$. It follows that

$$C_{a+1}(\lambda(\boldsymbol{\mu}^+)) = \begin{cases} C_a(\lambda(\boldsymbol{\mu})) - 1 & \text{if } C_a(\lambda(\boldsymbol{\mu})) \neq 0 \\ 0 & \text{if } C_a(\lambda(\boldsymbol{\mu})) = 0. \end{cases}$$

Note that $c(\boldsymbol{\mu}^+) = c(\boldsymbol{\mu})+1$. If $\mu_{k-i} \leq \mu_1+1$, then $C_{c(\boldsymbol{\mu}^+)+i}(\lambda(\boldsymbol{\mu}^+)) = y_{-\mu^+_{k-i}}(\boldsymbol{\mu}^+) = 0$. By induction, if $\mu_{k-i} \geq \mu_1 + 2$, then

$$C_{c(\boldsymbol{\mu}^+)+i}(\lambda(\boldsymbol{\mu}^+)) = \sum_{j=1}^{k-1-i} \max\{0, \mu_{k-i} - \mu_j^+ - 1\} = \sum_{j=1}^{k-1-i} \max\{0, \mu_{k-i} - \mu_j - 1\} - 1,$$

and the result follows. □

Corollary 4.4.3. *Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type. Then $\rho_k(\lambda(\boldsymbol{\mu})) = |\boldsymbol{\mu}|$.*

Proof. By Lemma 4.4.2, we have

$$\begin{aligned} \rho_k(\lambda(\boldsymbol{\mu})) &= \sum_{i=0}^{k-1} C_{c(\boldsymbol{\mu})+i}(\lambda(\boldsymbol{\mu})) \\ &= \sum_{i=0}^{k-1} \sum_{j=1}^{k-1-i} \max\{0, \mu_{k-i} - \mu_j - 1\} \\ &= \sum_{j<i} \max\{0, \mu_i - \mu_j - 1\} = |\boldsymbol{\mu}|. \end{aligned}$$

□

In order to use the results of Section 4.3, we must show that k-staircases are in $\mathcal{P}_k$.

Proposition 4.4.4. *Every k-staircase is a k-core.*

Proof. Let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type. By Proposition 4.3.11, we must show that $\lambda(\boldsymbol{\mu})$ and $\lambda(\boldsymbol{\mu})^T$ satisfy k-descent. By Lemma 4.2.10, it suffices to show that $\lambda(\boldsymbol{\mu})$ satisfies k-descent. Let $(x, y) \in \lambda(\boldsymbol{\mu}) \cap D_a$ and suppose that $(x+1, y) \notin \lambda(\boldsymbol{\mu})$. We will show that $C_{a-1}(\lambda(\boldsymbol{\mu})) < y$. By assumption, there is an integer m such that $x = x_m(\boldsymbol{\mu})$ and $y_{m+1}(\boldsymbol{\mu}) < y \leq y_m(\boldsymbol{\mu})$. Since $(x_{m+1}(\boldsymbol{\mu}), y_{m+1}(\boldsymbol{\mu})) \in D_{c(\boldsymbol{\mu})}$, we see that $(x, y) \in D_{c(\boldsymbol{\mu})+i}$ for some i in the range $\alpha_{m+1}(\boldsymbol{\mu}) < i \leq k$. By Lemma 4.4.2, we have

$$C_{c(\boldsymbol{\mu})-i-1}(\lambda(\boldsymbol{\mu})) = y_{-\mu_{k-i+1}}(\boldsymbol{\mu}).$$

If $m+1 \geq -\mu_{k-i+1}(\boldsymbol{\mu})$, then $\alpha_{m+1}(\boldsymbol{\mu}) \geq i$, a contradiction. It follows that

$$y_{-\mu_{k-i+1}}(\boldsymbol{\mu}) < y_{m+1}(\boldsymbol{\mu}) < y.$$

□

We now prove the main theorem.

Theorem 4.4.5. *Let Γ be a k-gonal chain of loops of genus g, and let $\boldsymbol{\mu} \in \mathbb{Z}^k$ be a splitting type. Then*

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) = \bigcup \mathbb{T}(t),$$

where the union is over all k-saturated tableaux on $\lambda(\boldsymbol{\mu})$ with alphabet $[g]$.

Proof. Let t be a k-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$. By Theorem 4.1.3, it suffices to show that there is a k-saturated tableau t' on $\lambda(\boldsymbol{\mu})$ such that $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. By Proposition 4.4.4, $\lambda(\boldsymbol{\mu})$ is a k-core. Thus, by Theorem 4.3.19, there is a k-saturated tableau t' such that every symbol in t' is a symbol in t and, if $t(x,y) = t(x',y')$, then $y - x \equiv y' - x' \pmod{k}$. By Lemma 4.2.9, we have $\mathbb{T}(t) \subseteq \mathbb{T}(t')$. □

Proof of Theorem 4.1.2. Recall that the codimension of $\mathbb{T}(t)$ is equal to the number of symbols in t. The result then follows from Theorem 4.4.5 because every k-saturated tableau on λ contains exactly $\rho_k(\lambda)$ symbols, and by Corollary 4.4.3, $\rho_k(\lambda(\boldsymbol{\mu})) = |\boldsymbol{\mu}|$. □

We now explain the connection between the tropical geometry and classical algebraic geometry. The following has become a standard argument in tropical geometry, for instance in [11, 33, 22, 10]. Recall that, if $\overline{W}^{\boldsymbol{\mu}}(C)$ is nonempty, then $\dim \overline{W}^{\boldsymbol{\mu}}(C) \geq g - |\boldsymbol{\mu}|$. We show the reverse inequality.

Proof of Theorem 4.1.1. By [33, Lemma 2.4], there exists a curve C of genus g and gonality k over a nonarchimedean field K with skeleton Γ. By Proposition 4.2.1, we have

$$\operatorname{Trop}\left(\overline{W}^{\boldsymbol{\mu}}(C)\right) \subseteq \overline{W}^{\boldsymbol{\mu}}(\Gamma).$$

By [19, Theorem 6.9], we have

$$\dim \overline{W}^{\boldsymbol{\mu}}(C) = \dim \operatorname{Trop}\left(\overline{W}^{\boldsymbol{\mu}}(C)\right) \leq \dim \overline{W}^{\boldsymbol{\mu}}(\Gamma) = g - |\boldsymbol{\mu}|,$$

where the last equality comes from Theorem 4.1.2. □

4.5 Connectedness of Tropical Splitting Type Loci

In this section, we prove Theorem 4.1.4, which says that $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$ is connected in codimension one. We borrow the ideas and terminology from [14, Section 4.2].

Let t be a k-uniform displacement tableau, let a be a symbol that is not in t, and let b be either the smallest symbol in t that is greater than a or the largest symbol in t that is smaller than a. If we take a proper subset of the boxes containing b and replace them with a, then we obtain a k-uniform displacement tableau t', with

$\mathbb{T}(t') \subset \mathbb{T}(t)$ and $\dim \mathbb{T}(t') = \dim \mathbb{T}(t) - 1$. If we instead replace every instance of the symbol b in t with the symbol a, then we obtain a k-uniform displacement tableau t', with $\dim \mathbb{T}(t') = \dim \mathbb{T}(t)$, such that $\mathbb{T}(t)$ and $\mathbb{T}(t')$ intersect in codimension one. This procedure is called *swapping* in a for b.

Given a symbol b in t, we obtain a k-uniform displacement tableau t' without the symbol b, by iterating the procedure above. If there is a symbol $a < b$ that is not in t, then the resulting tableau can be described explicitly:

$$t'(x,y) = \begin{cases} t(x,y) - 1 & \text{if } a < t(x,y) \leq b \\ t(x,y) & \text{otherwise.} \end{cases}$$

If there is a symbol $a > b$ that is not in t, then t' is obtained instead by increasing by 1 every symbol in t between b and a. Because t' is obtained by a sequence of swaps, we see that there is a chain of tori from $\mathbb{T}(t)$ to $\mathbb{T}(t')$, such that each consecutive pair of tori in the chain intersect in codimension one. This procedure is called *cycling* out b.

Proof of Theorem 4.1.4. Let t, t' be k-saturated tableaux on $\lambda(\boldsymbol{\mu})$. By Theorem 4.4.5, it suffices to construct a sequence

$$t = t_0, t_1, \ldots, t_m = t'$$

of k-saturated tableaux, where $\mathbb{T}(t_i)$ and $\mathbb{T}(t_{i+1})$ intersect in codimension one for all i. Both t and t' contain precisely $|\boldsymbol{\mu}|$ symbols. By cycling out all symbols greater than $|\boldsymbol{\mu}|$, we may assume that the symbols in t and t' are precisely those in $[|\boldsymbol{\mu}|]$. In other words, there exist maximal chains

$$\emptyset = \lambda_0 < \lambda_1 < \cdots < \lambda_{|\boldsymbol{\mu}|} = \lambda(\boldsymbol{\mu}),$$
$$\emptyset = \lambda'_0 < \lambda'_1 < \cdots < \lambda'_{|\boldsymbol{\mu}|} = \lambda(\boldsymbol{\mu})$$

such that $t = \Phi([|\boldsymbol{\mu}|], \vec{\lambda})$ and $t' = \Phi([|\boldsymbol{\mu}|], \vec{\lambda}')$. If $\vec{\lambda}$ and $\vec{\lambda}'$ coincide, then $t = t'$, and we are done.

We prove the remaining cases by induction, having just completed the base case. Let j be the largest symbol such that $\lambda_{j-1} \neq \lambda'_{j-1}$. Equivalently, the symbols $j+1, \ldots, |\boldsymbol{\mu}|$ appear in the same set of boxes of t and t'. We will construct a sequence

$$t = t'_0, t'_1, \ldots, t'_n = t''$$

of k-saturated tableaux, where $\mathbb{T}(t'_i)$ and $\mathbb{T}(t'_{i+1})$ intersect in codimension one for all i, and where each of the symbols $j, \ldots, |\boldsymbol{\mu}|$ appears in the same set of boxes of t' and t''.

Since $g > |\boldsymbol{\mu}|$, either $g = j+1$ or there exists a symbol in $[g]$ that is greater than $j+1$. We let $\widehat{t}$ be the tableau obtained by cycling $j+1$ out of t. In other words,

$$\widehat{t}(x,y) = \begin{cases} t(x,y) & \text{if } t(x,y) \leq j \\ t(x,y) + 1 & \text{if } t(x,y) > j. \end{cases}$$

We define

$$\widetilde{t}(x,y) = \begin{cases} j+1 & \text{if } (x,y) \in \lambda'_j \smallsetminus \lambda'_{j-1} \\ \widehat{t}(x,y) & \text{otherwise.} \end{cases}$$

To see that $\widetilde{t}$ is a tableau, note that

$$\lambda_j = \lambda'_j = \{(x,y) \in \lambda(\boldsymbol{\mu}) \mid \widehat{t}(x,y) \le j\},$$

and $\widehat{t}$ does not contain the symbol $j+1$, so every box in $\lambda(\boldsymbol{\mu}) \smallsetminus \lambda'_j$ contains a symbol that is greater than $j+1$, and every box in λ'_{j-1} contains a symbol that is smaller than $j+1$. Note that $\widetilde{t}$ contains one more symbol than $\widehat{t}$, so $\mathbb{T}(\widetilde{t}) \subset \mathbb{T}(\widehat{t})$ has codimension 1. Applying the procedure of Example 4.3.21, we obtain a k-saturated tableau $\widetilde{t}'$ such that $\mathbb{T}(\widetilde{t}) \subset \mathbb{T}(\widetilde{t}')$. Since $i+1$ is the largest symbol in λ'_i for all $i \ge j$, we see that $\widetilde{t}'(x,y) = \widetilde{t}(x,y)$ for all $(x,y) \in \lambda(\boldsymbol{\mu}) \smallsetminus \lambda'_{j-1}$. Finally, we let t'' be the tableau obtained by cycling out all symbols greater than $|\boldsymbol{\mu}|$ from $\widetilde{t}'$. By construction, each of the symbols $j, \ldots, |\boldsymbol{\mu}|$ appears in the same set of boxes of t' and t''. □

Remark 4.5.1. Under the bijection with words in the affine symmetric group, Theorem 4.1.4 is equivalent to the statement that any two reduced expressions for the same word can be connected via a sequence of "braid moves" (see [6, Theorem 3.3.1]).

Example 4.5.2. Figure 4.8 illustrates the procedure in the proof of Theorem 4.1.4. The two tableaux t, t' on the ends correspond to two maximal-dimension tori in $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$, where $\boldsymbol{\mu} = (-3,-1,1)$. If $g \ge 6$, we construct a chain of tori from $\mathbb{T}(t)$ to $\mathbb{T}(t')$ in this tropical splitting type locus, where each torus intersects the preceding torus in codimension one. The largest symbol where t and t' disagree is 4. We therefore begin by cycling out 5, to obtain the second tableau in the chain. We then place a 5 in each box where a 4 appears in t', to obtain the third tableau in the chain, using all 6 symbols. Applying the procedure of Example 4.3.21, we obtain the fourth tableau. Finally, by cycling out 6, we arrive at t'.

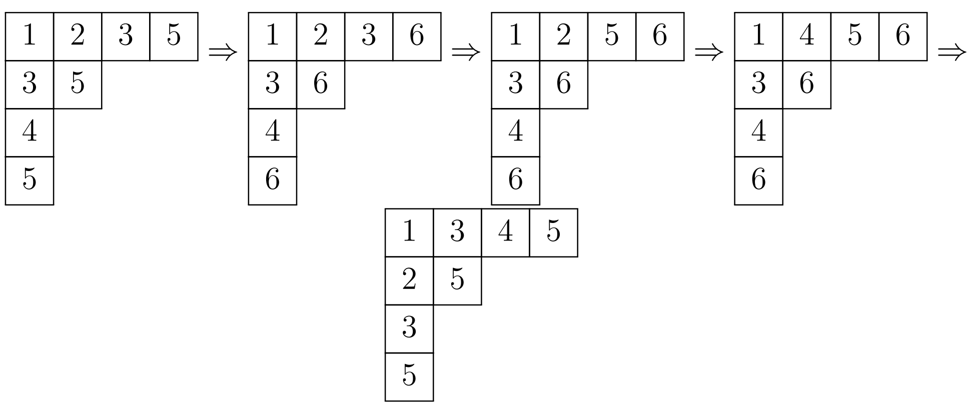


Figure 4.8: If $g \ge 6$, then $\overline{W}^{(-3,-1,1)}(\Gamma)$ is connected in codimension 1.

4.6 Cardinality of Tropical Splitting Type Loci

We begin this section by proving Theorem 4.1.5.

Proof of Theorem 4.1.5. By Theorem 4.4.5,

$$\overline{W}^{\boldsymbol{\mu}}(\Gamma) = \bigcup \mathbb{T}(t),$$

where the union is over all k-saturated tableaux on $\lambda(\boldsymbol{\mu})$ with alphabet $[g]$. Since $g = |\boldsymbol{\mu}|$, each torus $\mathbb{T}(t)$ in this union is 0-dimensional, and therefore consists of a single divisor class. Consider the composition of $\Phi_{\lambda(\boldsymbol{\mu})}$ with the map sending a tableau t to the unique divisor class in $\mathbb{T}(t)$. By the above, this composition surjects onto $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$, and it suffices to show that it is injective. Let

$$\emptyset = \lambda_0 < \lambda_1 < \cdots < \lambda_g = \lambda(\boldsymbol{\mu})$$

$$\emptyset = \lambda'_0 < \lambda'_1 < \cdots < \lambda'_g = \lambda(\boldsymbol{\mu})$$

be distinct maximal chains in $\mathcal{P}_k(\lambda(\boldsymbol{\mu}))$, and let j be the minimal index such that $\lambda'_j \neq \lambda_j$. By definition, $\lambda_j = \lambda^+_{j-1,a}$ and $\lambda'_j = \lambda^+_{j-1,b}$ for some $a \not\equiv b \pmod{k}$. It follows that, if $\mathbb{T}(t) = \{D\}$, then $\xi_j(D) \equiv a \not\equiv b \pmod{k}$, so $D \notin \mathbb{T}(t')$. Therefore, every maximal chain in $\mathcal{P}_k$ corresponds to a distinct divisor class in $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$. □

Algorithm for Computing Maximal Chains

The number of maximal chains in $\mathcal{P}_k(\lambda)$ is an important invariant of a partition $\lambda \in \mathcal{P}_k$, not only because of Theorem 4.1.5, but also because of its connection to the affine symmetric group [24]. We would therefore like to compute this invariant in examples. In order to simplify our arguments, we first show that a partition $\lambda \in \mathcal{P}_k$ is uniquely determined by the vector $\mathbf{C}(\lambda)$.

Lemma 4.6.1. *Let $\lambda, \lambda' \in \mathcal{P}_k$. If there exists a permutation $\sigma \in S_k$ such that $C_a(\lambda) = C_{\sigma(a)}(\lambda')$ for all $a \in \mathbb{Z}/k\mathbb{Z}$, then $\lambda = \lambda'$.*

Proof. We prove this by induction on $\rho_k(\lambda) = \rho_k(\lambda')$. The base case is when $\rho_k(\lambda) = 0$, in which case $\lambda = \lambda'$ is the empty partition. For the inductive step, let

$$y = \max_{a \in \mathbb{Z}/k\mathbb{Z}} C_a(\lambda) = \max_{a \in \mathbb{Z}/k\mathbb{Z}} C_a(\lambda'),$$

and let x be the number of congruence classes $a \in \mathbb{Z}/k\mathbb{Z}$ such that $C_a(\lambda) = y$. By definition, the first x columns of both λ and λ' must all have height y. If λ is nonempty then it has an inside corner. This implies that $x \leq k-1$ by Lemma 4.3.13. It follows that column $x+1$ of both λ and λ' has height less than y, so (x, y) is an inside corner of both partitions, and $y = C_{y-x}(\lambda) = C_{y-x}(\lambda')$. By Proposition 4.3.17, there exists a permutation $\pi \in S_k$ such that

$$C_a(\lambda^-_{y-x}) = C_{\pi(a)}(\lambda'^-_{y-x}) \text{ for all } a \in \mathbb{Z}/k\mathbb{Z}.$$

By Lemma 4.3.16, $\lambda^-_{y-x}, \lambda'^-_{y-x} \in \mathcal{P}_k$, hence by induction, $\lambda^-_{y-x} = \lambda'^-_{y-x}$. Finally, by Lemma 4.3.15, we have

$$\lambda = (\lambda^-_{y-x})^+_{y-x} = (\lambda'^-_{y-x})^+_{y-x} = \lambda'.$$

□

Lemma 4.6.1 allows us to simplify arguments by focusing on the vectors $\mathbf{C}(\lambda)$, rather than the partitions λ. For example, Figure 4.9 depicts the Hasse diagram of a principal order ideal in $\mathcal{P}_6$, where each partition λ is represented by the vector $\mathbf{C}(\lambda)$.

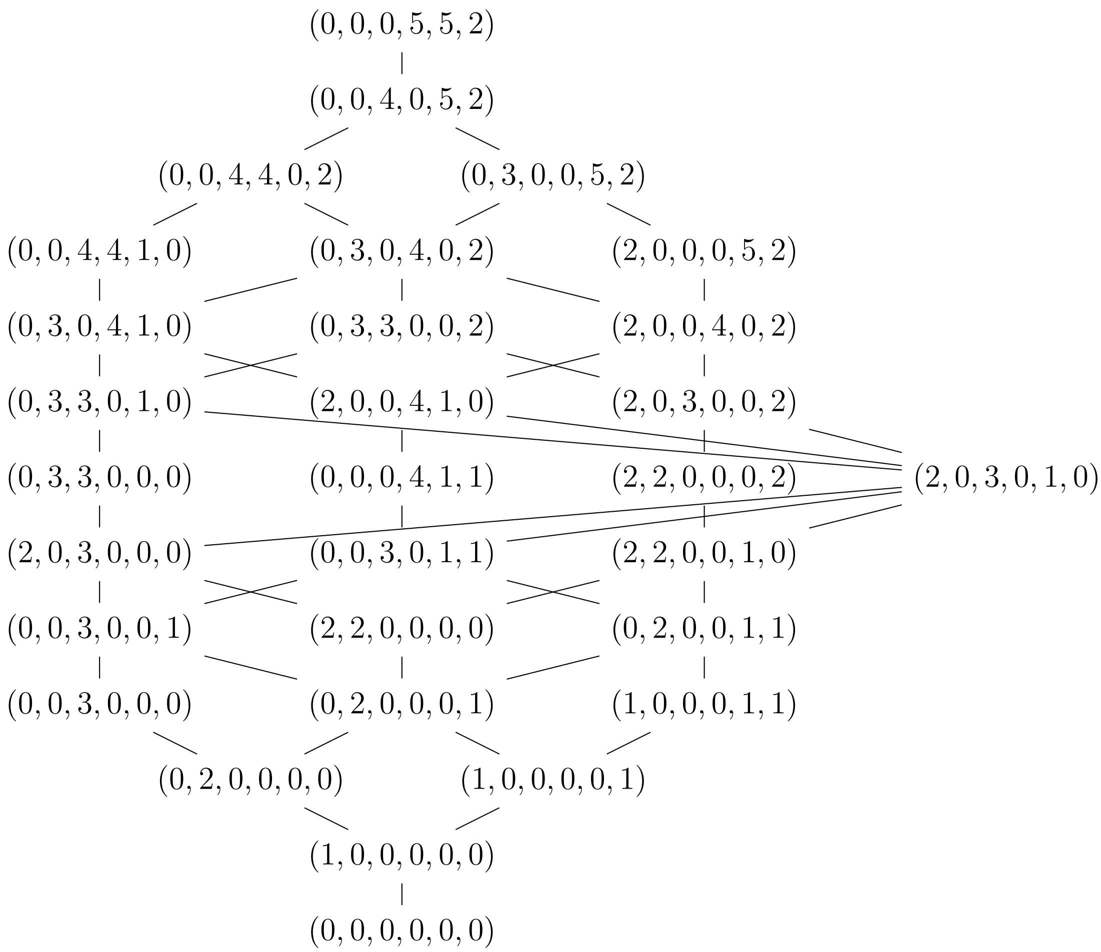


Figure 4.9: A principal order ideal in $\mathcal{P}_6$.

Given a partition $\lambda \in \mathcal{P}_k$, we provide an algorithm for producing the Hasse diagram $\mathcal{P}_k(\lambda)$, as in Figure 4.9.

Algorithm 4.6.2. Step 1: Initialize with the vector $\mathbf{C}(\lambda)$.

Step 2: For each vector $\mathbf{C}$, write below it the vectors $\mathbf{C}^-_a$, for all a such that $C_{a-1} < C_a$.

Step 3: Iterate Step 2 for each vector that is written down, until exhaustion.

By Lemma 4.6.1, the number of partitions in $\mathcal{P}_k$ or rank ρ is less than or equal to the number of partitions of ρ with at most $k-1$ parts. (In fact, these numbers are equal, see [23, Proposition 1.3].) Together with the fact that each partition covers at most $k-1$ others, this implies that the algorithm terminates in polynomial time for fixed k.

We introduce notation that will simplify our examples. Given $\lambda \in \mathcal{P}_k$, we define $\alpha(\mathbf{C}(\lambda))$ to be the number of maximal chains in $\mathcal{P}_k(\lambda)$. By Lemma 4.6.1, this is well-defined. We further define α up to cyclic permutation; that is,

$$\alpha\Big(C_i(\lambda), C_{i+1}(\lambda), \ldots, C_{i-1}(\lambda)\Big) = \alpha\Big(C_0(\lambda), C_1(\lambda), \ldots, C_{k-1}(\lambda)\Big).$$

Again, by Lemma 4.6.1, α is well-defined. Indeed, by Lemma 4.6.1, α could be defined up to arbitrary permutation, but in practice it is important to keep track of which values C_a are consecutive. This is because α satisfies the following recurrence.

Lemma 4.6.3. *For any $\lambda \in \mathcal{P}_k$, we have*

$$\alpha(\mathbf{C}(\lambda)) = \sum_{\substack{a \in \mathbb{Z}/k\mathbb{Z} \text{ s.t.} \\ C_{a-1}(\lambda) < C_a(\lambda)}} \alpha(\mathbf{C}(\lambda)_a^-).$$

Proof. The number of maximal chains in $\mathcal{P}_k(\lambda)$ is equal to the sum, over $\lambda' \in \mathcal{P}_k$ covered by λ, of the number of maximal chains in $\mathcal{P}_k(\lambda')$. By definition, $\lambda' \in \mathcal{P}_k$ is covered by λ if and only if $\lambda' = \lambda_a^-$ and λ has an inside corner in D_a. By Lemma 4.3.13, λ has an inside corner in D_a if and only if $C_{a-1}(\lambda) < C_a(\lambda)$. The result then follows from Proposition 4.3.17. □

Using Algorithm 4.6.2 and Lemma 4.6.3, one can compute $\alpha(\mathbf{C}(\lambda))$ recursively. Start at the bottom of the Hasse diagram, note that $\alpha(\vec{0}) = 1$, and then proceed upwards, summing the numbers that appear directly below each vector. These numbers appear in the circles in Figure 4.9.

Examples

The remainder of the paper consists of examples, using Lemma 4.6.3 to compute the number of maximal chains in $\mathcal{P}_k(\lambda(\boldsymbol{\mu}))$ for various splitting types $\boldsymbol{\mu}$. In many cases, we will see that this number agrees with the cardinality of $\overline{W}^{\boldsymbol{\mu}}(C)$ for general $(C, \pi) \in \mathcal{H}_{g,k}$. In each case, we assume that $g = |\boldsymbol{\mu}|$. By Theorem 4.1.1, this implies that $W^{\boldsymbol{\mu}}(C) = \overline{W}^{\boldsymbol{\mu}}(C)$.

Example 4.6.4. If $-2 \leq \mu_j \leq 0$ for all j, then $\lambda(\boldsymbol{\mu}) = \lambda_0(\boldsymbol{\mu})$ is a rectangle, and every k-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$ is a standard Young tableau. The number of such tableaux is counted by the standard hook-length formula:

$$|\overline{W}^{\boldsymbol{\mu}}(\Gamma)| = |\boldsymbol{\mu}|! \prod_{j=0}^{x_0(\boldsymbol{\mu})-1} \frac{j!}{(y_0(\boldsymbol{\mu}) + j)!}.$$

It is a classical result, due to Castelnuovo, that this formula also yields the number of g^r_d's on a general curve of genus $|\boldsymbol{\mu}|$, where $r = x_0(\boldsymbol{\mu}) - 1$, and $d = d(\boldsymbol{\mu})$ [2, p.211].

Example 4.6.5. If μ_j is equal to either μ_1 or $\mu_1 + 1$ for each $j < k$, then $d(\boldsymbol{\mu}) = k\mu_k$ and up to cyclic permutation we have $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (|\boldsymbol{\mu}|, 0, 0, \ldots, 0)$. For ease of notation, we write this as $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (|\boldsymbol{\mu}|, 0^{(k-1)})$. We show that $\alpha(z, 0^{(k-1)}) = 1$. This is easy to see by induction on z. It is clear that $\alpha(1, 0^{(k-1)}) = 1$, and by Lemma 4.6.3, we have

$$\alpha(z, 0^{(k-1)}) = \alpha(0^{(k-1)}, z-1) = \alpha(z-1, 0^{(k-1)}).$$

Now, if $D \in \overline{W}^{\boldsymbol{\mu}}(C)$, then by definition, $\deg D = k\mu_k$ and $D - \mu_k g^1_k$ is effective. It follows that $\overline{W}^{\boldsymbol{\mu}}(C) = \{\mu_k g^1_k\}$. This splitting type locus therefore has cardinality 1, equal to that of $\overline{W}^{\boldsymbol{\mu}}(\Gamma)$.

We note that Serre duality induces a bijection between $\overline{W}^{\boldsymbol{\mu}}(C)$ and $\overline{W}^{\boldsymbol{\mu}^T}(C)$. Tropically, this corresponds to the fact that the number of maximal chains in $\mathcal{P}_k(\lambda)$ is equal to the number of maximal chains in $\mathcal{P}_k(\lambda^T)$. If we apply this observation to Example 4.6.5, we see that if μ_j is equal to either μ_k or $\mu_k - 1$ for each $j > 1$, then

$$|\overline{W}^{\boldsymbol{\mu}}(C)| = |\overline{W}^{\boldsymbol{\mu}}(\Gamma)| = 1.$$

A similar remark applies to each of the examples below.

Example 4.6.6. Let $\boldsymbol{\mu} = (-3, -2, \ldots, -2, 0, 0)$. Then $g = 2k - 2$, and $\lambda(\boldsymbol{\mu})$ is the partition depicted in Figure 4.10.

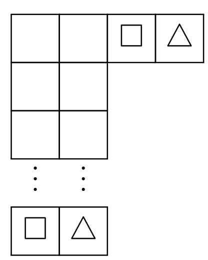


Figure 4.10: The partition $\lambda(\boldsymbol{\mu})$ of Example 4.6.6.

If t is a k-uniform displacement tableau on $\lambda(\boldsymbol{\mu})$, then the restriction of t to the first two columns is a standard Young tableau. If t has precisely $2k-2$ symbols, then we must have $t(3,1) = t(1,k-1)$ and $t(4,1) = t(2,k-1)$. (These are the boxes labeled with a square and a triangle, respectively, in Figure 4.10.) It follows that $t(2,1) < t(1,k-1)$. Since the number of standard Young tableaux on the first two columns is the $(k-1)$st Catalan number C_{k-1}, and since there is a unique such standard Young tableau t with $t(2,1) > t(1,k-1)$, we see that the number of k-uniform displacement tableaux on $\lambda(\boldsymbol{\mu})$ with precisely $2k-2$ symbols is $C_{k-1} - 1$.

A general curve C of genus $2k-2$ has gonality k, and by Example 4.6.4, the number of gonality pencils is precisely C_{k-1}. Such a pencil is in $\overline{W}^{\boldsymbol{\mu}}(C)$ if and only if it is not equal to the distinguished g^1_k. It follows that $|\overline{W}^{\boldsymbol{\mu}}(C)| = C_{k-1} - 1$, confirming Conjecture 4.1.6 in this case.

Example 4.6.7. If $k = 2$, then every splitting type $\boldsymbol{\mu}$ satisfies the hypotheses of Example 4.6.5. The first interesting examples, therefore, occur when k is equal to 3. Let $k = 3$, and suppose that $\boldsymbol{\mu}$ is not of the type considered in Example 4.6.5. In other words, $\mu_3 > \mu_2 + 1$, and $\mu_2 > \mu_1 + 1$. Then $g = 2(\mu_3 - \mu_1) - 3$ is odd, and up to cyclic permutation, we have

$$\mathbf{C}(\lambda(\boldsymbol{\mu})) = (2\mu_3 - \mu_2 - \mu_1 - 2, \mu_2 - \mu_1 - 1, 0).$$

We show that

$$\alpha(2\mu_3 - \mu_2 - \mu_1 - 2, \mu_2 - \mu_1 - 1, 0) = \binom{\mu_3 - \mu_1 - 2}{\mu_2 - \mu_1 - 1}.$$

One way to see that this formula is invariant under transposition is to note that $\mu_2 - \mu_1 - 1$ is equal to the number of strict rank jumps of size 2, whereas $\mu_3 - \mu_2 - 1$ is equal to the number of strict rank jumps of size 1. As in Example 4.6.5, we prove this by induction. When $\mu_2 - \mu_1 - 1 = 0$, the result follows from Example 4.6.5, and when $\mu_3 - \mu_2 - 1 = 0$, the result follows from the same example applied to $\lambda(\boldsymbol{\mu})^T$. If $z_1 - 1 > z_2 > 0$, then by Lemma 4.6.3, we have

$$\begin{aligned}\alpha(z_1, z_2, 0) &= \alpha(0, z_2, z_1 - 1)\\ &= \alpha(z_2 - 1, 0, z_1 - 1) + \alpha(0, z_1 - 2, z_2).\end{aligned}$$

This expression has the following interpretation. If $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (z_1, z_2, 0)$, then $\mathbf{C}(\lambda(\boldsymbol{\mu}^+)) = (z_2 - 1, 0, z_1 - 1)$ and $\mathbf{C}(\lambda(\boldsymbol{\mu}^-)) = (0, z_1 - 2, z_2)$. In other words, the number of k-saturated tableaux on $\lambda(\boldsymbol{\mu})$ is the sum of the number on a partition with one fewer row and the number on a partition with one fewer column. Evaluating this expression and applying induction, we see that

$$\alpha(2\mu_3 - \mu_2 - \mu_1 - 2, \mu_2 - \mu_1 - 1, 0) = \binom{\mu_3 - \mu_1 - 3}{\mu_2 - \mu_1 - 2} + \binom{\mu_3 - \mu_1 - 3}{\mu_2 - \mu_1 - 1} = \binom{\mu_3 - \mu_1 - 2}{\mu_2 - \mu_1 - 1}.$$

In [26, Theorem 1.1], Larson computes the cardinality of $\overline{W}^{\boldsymbol{\mu}}(C)$ for a general trigonal curve C of Maroni invariant n. Since g is odd, if $(C, \pi) \in \mathcal{H}_{g,3}$ is general, it has Maroni invariant 1. Larson's formula then yields the binomial coefficient above, confirming Conjecture 4.1.6 for $k = 3$.

Example 4.6.7 can be generalized to the case where k is arbitrary and

$$\mu_2 = \mu_3 = \cdots = \mu_{k-1}.$$

This is done in Example 4.6.13 below.

We now consider examples where k is equal to 4, 5, or 6. We do not consider every splitting type in these cases, considering only the "maximal" splitting types in which every strict rank jump has the same size α. If all strict rank jumps of $\boldsymbol{\mu}$ have size α, then all strict rank jumps of $\boldsymbol{\mu}^T$ have size $k - \alpha$, so it suffices to consider the case where $\alpha \leq \frac{k}{2}$. Since Example 4.6.5 is the case where $\alpha = 1$, the first interesting case occurs when k is equal to 4. We do not know if Conjecture 4.1.6 holds for these splitting types, proving it in only a small number of cases.

Example 4.6.8. Let $k = 4$, and suppose that $\alpha = 2$. In other words, μ_2 is equal to either μ_1 or $\mu_1 + 1$, and μ_3 is equal to either μ_4 or $\mu_4 - 1$. In this case we see that, up to cyclic permutation, $\mathbf{C}(\lambda(\boldsymbol{\mu}))$ is either of the form $(z, z, 0, 0)$ or $(z+1, z-1, 0, 0)$. We show, by induction on z, that

$$\alpha(z, z, 0, 0) = \alpha(z+1, z-1, 0, 0) = 2^{z-1}.$$

The base case, when z is equal to 1, is covered by Example 4.6.5. For the inductive step, by Lemma 4.6.3, we see that

$$\begin{aligned}\alpha(z, z, 0, 0) = \alpha(0, z, 0, z-1) &= \alpha(z-1, 0, 0, z-1) + \alpha(0, z, z-2, 0)\\ &= 2^{z-2} + 2^{z-2} = 2^{z-1}\\ \alpha(z+1, z-1, 0, 0) = \alpha(0, z-1, 0, z) &= \alpha(z-2, 0, 0, z) + \alpha(0, z-1, z-1, 0)\\ &= 2^{z-2} + 2^{z-2} = 2^{z-1}.\end{aligned}$$

As in Example 4.6.7, the expressions on the right are equal to $\alpha(\mathbf{C}(\lambda(\boldsymbol{\mu}^+))) + \alpha(\mathbf{C}(\lambda(\boldsymbol{\mu}^-)))$.

In general, we do not know if Conjecture 4.1.6 holds in this case. It holds for $z \leq 2$ by Example 4.6.4, and for $z = 3$ by Example 4.6.6. We will see in Example 4.6.12 below that it also holds for the splitting type $\boldsymbol{\mu} = (-3, -3, 0, 0)$, in which case $z = 4$.

Example 4.6.9. Let $k = 5$, and suppose that $\alpha = 2$. In other words, μ_2 and μ_3 are equal to either μ_1 or $\mu_1 + 1$, and μ_4 is equal to either μ_5 or $\mu_5 - 1$. Up to cyclic permutation, $\mathbf{C}(\lambda(\boldsymbol{\mu}))$ is either of the form $(z, z, 0, 0, 0)$ or $(z+2, z-1, 0, 0, 0)$. We show, by induction on z, that

$$\begin{aligned}\alpha(z, z, 0, 0, 0) &= F_{2z-2}\\ \alpha(z+2, z-1, 0, 0, 0) &= F_{2z-1},\end{aligned}$$

where F_n denotes the nth Fibonacci number. The base case, where $z = 1$, follows from Example 4.6.5. For the inductive step, by Lemma 4.6.3, we have

$$\begin{aligned}\alpha(z, z, 0, 0, 0) &= \alpha(0, z, 0, 0, z-1) \text{ and}\\ \alpha(z+2, z-1, 0, 0, 0) &= \alpha(0, z-1, 0, 0, z+1),\end{aligned}$$

so we will also show by induction that $\alpha(0, z, 0, 0, z-1) = F_{2z-2}$ and $\alpha(0, z-1, 0, 0, z+1) = F_{2z-1}$. Again, the base cases follow from Example 4.6.5. Together with the inductive hypothesis, by Lemma 4.6.3, we have

$$\begin{aligned}\alpha(0, z, 0, 0, z-1) &= \alpha(z-1, 0, 0, 0, z-1) + \alpha(0, z, 0, z-2, 0)\\ &= F_{2z-4} + F_{2z-3} = F_{2z-2}\\ \alpha(0, z-1, 0, 0, z+1) &= \alpha(z-2, 0, 0, 0, z+1) + \alpha(0, z-1, 0, z, 0)\\ &= F_{2z-3} + F_{2z-2} = F_{2z-1}.\end{aligned}$$

Conjecture 4.1.6 holds when $-2 \leq \mu_j \leq 0$ for all j by Example 4.6.4, and when $\boldsymbol{\mu} = (-3, -2, -2, 0, 0)$ by Example 4.6.6. We will see in Example 4.6.12 below that it also holds when $\boldsymbol{\mu} = (-3,-3,-2,0,0)$. We now show that it holds when $\boldsymbol{\mu} = (-3,-3,-2,-1,0)$.

In this case, $g = 7$, $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (5,2,0,0,0)$, and $\alpha(5,2,0,0,0) = F_5 = 8$. For $(C,\pi) \in \mathcal{H}_{7,5}$, we see that $D \in \overline{W}^{\boldsymbol{\mu}}(C)$ if and only if D is effective of degree 2 and $K_C - g^1_5 - D$ has rank at least 1. By Riemann-Roch, the divisor class $K_C - g^1_5$ has degree 7 and rank 2. The image of C under the complete linear series $|K_C - g^1_5|$ is a plane curve of degree 7, with $\binom{7-1}{2} - 7 = 8$ nodes. An effective divisor D satisfies $\mathrm{rk}(K_C - g^1_5 - D) \geq 1$ if and only if the image of D under this map is a single point. It follows that the divisor classes in $\overline{W}^{\boldsymbol{\mu}}(C)$ are precisely the preimages of the nodes, and thus that $|\overline{W}^{\boldsymbol{\mu}}(C)| = 8$.

Example 4.6.10. Let $k = 6$, and suppose that $\alpha = 2$. Up to cyclic permutation, $\mathbf{C}(\lambda(\boldsymbol{\mu}))$ is either of the form $(z,z,0,0,0,0)$ or $(z+2,z-2,0,0,0,0)$. We show, by induction on z, the following formulas:

$$\begin{aligned}
\alpha(z,z,0,0,0,0) = \alpha(0,z,0,0,0,z-1) &= \frac{3^{z-1}+1}{2} \\
\alpha(z+2,z-2,0,0,0,0) = \alpha(0,z-2,0,0,0,z+1) &= \frac{3^{z-1}-1}{2} \\
\alpha(z+1,0,0,z-1,0,0) &= 3^{z-1}.
\end{aligned}$$

The base cases, when $z = 1$ on the first and third line, or when $z = 2$ on the second line, follow from Example 4.6.5. The first equality on each of the first two lines above follows directly from Lemma 4.6.3. For the inductive step, by induction together with Lemma 4.6.3, we have

$$\begin{aligned}
\alpha(0,z,0,0,0,z-1) &= \alpha(z-1,0,0,0,0,z-1) + \alpha(0,z,0,0,z-2,0) \\
&= \frac{3^{z-2}+1}{2} + 3^{z-2} = \frac{3^{z-1}+1}{2} \\
\alpha(0,z-2,0,0,0,z+1) &= \alpha(z-3,0,0,0,0,z+1) + \alpha(0,z-2,0,0,z,0) \\
&= \frac{3^{z-2}-1}{2} + 3^{z-2} = \frac{3^{z-1}-1}{2} \\
\alpha(z+1,0,0,z-1,0,0) &= \alpha(0,0,0,z-1,0,z) + \alpha(z+1,0,z-2,0,0,0) \\
&= \frac{3^{z-1}+1}{2} + \frac{3^{z-1}-1}{2} = 3^{z-1}.
\end{aligned}$$

Conjecture 4.1.6 holds when $z \leq 3$, and for the splitting type $\boldsymbol{\mu} = (-2,-2,-2,-2,0,0)$ by Example 4.6.4. It also holds for the splitting type $\boldsymbol{\mu} = (-3,-2,-2,-2,0,0)$ by Example 4.6.6. The splitting types $\boldsymbol{\mu} = (-3,-3,-2,-2,0,0)$ and $\boldsymbol{\mu}^T = (-3,-3,-1,-1,0,0)$ will make an appearance in Example 4.6.12 below.

Example 4.6.11. Let $k = 6$, and suppose that $\alpha = 3$. Up to cyclic permutation, $\mathbf{C}(\lambda(\boldsymbol{\mu}))$ is of the form $(z,z,z,0,0,0)$, $(z+1,z+1,z-2,0,0,0)$, or $(z+2,z-1,z-1,0,0,0)$. To formulate expressions in these cases, we first introduce the function

$$\beta(z) := \begin{cases} 2 & \text{if } z \equiv 0 \pmod 3 \\ -1 & \text{otherwise.} \end{cases}$$

Note that $\beta(z-1)+\beta(z) = -\beta(z+1)$. By a similar argument to Examples 4.6.8, 4.6.9, and 4.6.10, we obtain the following formulas.

$$\alpha(z,z,z,0,0,0) = \alpha(0,z,z,0,0,z-1) = \frac{2^{3z-2}+(-1)^z\beta(z)}{3}$$
$$\alpha(z+1,z+1,z-2,0,0,0) = \alpha(0,z+1,z-2,0,0,z) = \frac{2^{3z-2}+(-1)^z\beta(z-1)}{3}$$
$$\alpha(z+2,z-1,z-1,0,0,0) = \alpha(0,z-1,z-1,0,0,z+1) = \frac{2^{3z-2}+(-1)^z\beta(z+1)}{3}$$
$$\alpha(z-1,0,z,0,z+1,0) = 2^{3z-2}.$$

We will consider the splitting type $\boldsymbol{\mu} = (-3,-3,-2,-1,0,0)$ in Example 4.6.12 below. The Hasse diagram pictured in Figure 4.9 is that of $\mathcal{P}_6(\lambda(\boldsymbol{\mu}))$.

Example 4.6.12. Let $(C,\pi) \in \mathcal{H}_{2k,k}$ be general, and let $L = K_C - g^1_k$. By Riemann-Roch, $h^0(C,L) = k+1$, and we consider the image of C in $\mathbb{P}^k$ under the complete linear series $|L|$. We have

$$\begin{aligned}\operatorname{expdim} H^0(\mathbb{P}^k, \mathcal{I}_C(2)) &= \dim \operatorname{Sym}^2 H^0(C,L) - \dim H^0(C,2L)\\ &= \binom{k+2}{2} - (4k-3).\end{aligned}$$

The variety X_4 parameterizing quadrics of rank at most 4 in $\mathbb{P}^k$ has dimension $4k-2$, so one expects the curve C to be contained in a finite number of rank 4 quadrics. The expected number of rank 4 quadrics in $H^0(\mathbb{P}^k, \mathcal{I}_C(2))$ is

$$\deg X_4 = \frac{\binom{k+1}{k-3}\binom{k+2}{k-4}\cdots\binom{2k-3}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2k-7}{k-4}} \text{ [20].}$$

Each rank 4 quadric is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$, and the pullback of $\mathcal{O}(1)$ from each of the two factors yields a pair of line bundles on C, each of rank 1, whose tensor product is L.

Conversely, given a pair of divisor classes D, D', each of rank 1, such that $D + D' = L$, we obtain a rank 4 quadric in $\mathbb{P}^k$ containing C. To see this, let s_0, s_1 be a basis for $H^0(C,D)$ and t_0, t_1 be a basis for $H^0(C,D')$. Then the entries of the 2×2 matrix $M_{ij} = (s_i \otimes t_j)$ are linear forms in $\mathbb{P}^k$, and the determinant of this matrix is a rank 4 quadric that vanishes on C. In other words, each rank 4 quadric corresponds to a pair of divisors in the set

$$\begin{aligned}&\left\{D \in \operatorname{Pic}(C) \mid h^0(C,D) = h^0(C,L-D) = 2\right\}\\ &= \left(\bigcup_{i=0}^{k-4} W^{(-3^{(2)},-2^{(i)},-1^{(k-4-i)},0^{(2)})}(C)\right) \cup \{g^1_k\} \cup \{L - g^1_k\}.\end{aligned}$$

Since (C,π) is general, the splitting type loci in the union above are all smooth of dimension zero, and we see that

$$2 + \sum_{i=0}^{k-4} \left|W^{(-3^{(2)},-2^{(i)},-1^{(k-4-i)},0^{(2)})}(C)\right| = 2\frac{\binom{k+1}{k-3}\binom{k+2}{k-4}\cdots\binom{2k-3}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{2}\cdots\binom{2k-7}{k-4}}.$$

We now show that this expression holds for Γ when $k \leq 6$. By Example 4.6.8, when $k = 4$, we have

$$2 + \left|W^{(-3,-3,0,0)}(\Gamma)\right| = 2 + 2^3 = 10 = 2\binom{5}{1}.$$

By Example 4.6.9, when $k = 5$, we have

$$\begin{aligned}&2 + \left|W^{(-3,-3,-2,0,0)}(\Gamma)\right| + \left|W^{(-3,-3,-1,0,0)}(\Gamma)\right| \\ &= 2 + F_8 + F_8 = 2 + 34 + 34 = 70 = 2\frac{\binom{6}{2}\binom{7}{1}}{\binom{1}{0}\binom{3}{1}}.\end{aligned}$$

By Examples 4.6.10 and 4.6.11, when $k = 6$, we have

$$\begin{aligned}&2 + \left|W^{(-3,-3,-2,-2,0,0)}(\Gamma)\right| + \left|W^{(-3,-3,-2,-1,0,0)}(\Gamma)\right| + \left|W^{(-3,-3,-1,-1,0,0)}(\Gamma)\right| \\ &= 2 + \frac{3^5+1}{2} + \frac{2^{10}+2}{3} + \frac{3^5+1}{2} = 2 + 122 + 342 + 122 = 588 = 2\frac{\binom{7}{3}\binom{8}{2}\binom{9}{1}}{\binom{1}{0}\binom{3}{1}\binom{5}{3}}.\end{aligned}$$

Example 4.6.13. We now consider the case where k is arbitrary and

$$\mu_2 = \mu_3 = \cdots = \mu_{k-1}.$$

The cases where $\mu_k \leq \mu_{k-1} + 1$ or $\mu_1 \geq \mu_2 - 1$ are covered in Example 4.6.5, so we assume otherwise. For ease of notation, we write $z_1 = (k-1)(\mu_k - 1) - (k-2)\mu_2 - \mu_1$ and $z_2 = \mu_2 - \mu_1 - 1$. Then $\mathbf{C}(\lambda(\boldsymbol{\mu})) = (z_1, z_2^{(k-2)}, 0)$, and we will show in Lemma 4.6.14 below that

$$\alpha(z_1, z_2^{(k-2)}, 0) = \binom{(k-2)(\mu_k - \mu_1 - 2)}{(k-2)(\mu_2 - \mu_1 - 1)}.$$

This expression matches the cardinality of $\overline{W}^{\boldsymbol{\mu}}(C)$ for general $(C, \pi) \in \mathcal{H}_{g,k}$. To see this, following [26, Lemma 2.2], we see that

$$W^{\boldsymbol{\mu}}(C) = \left\{D \in \operatorname{Pic}^{d(\boldsymbol{\mu})}(C) \mid h^0(D - \mu_k g^1_k) = h^0(K_C - D + (\mu_1 + 2)g^1_k) = 1\right\}.$$

In other words, $D \in W^{\boldsymbol{\mu}}(C)$ if and only if $D = \mu_k g^1_k + E$, where E is an effective divisor of degree $(k-2)(\mu_2 - \mu_1 - 1)$, such that $K_C - (\mu_k - \mu_1 - 2)g^1_k - E$ is also effective. Note that

$$\deg\left(K_C - (\mu_k - \mu_1 - 2)g^1_k\right) = (k-2)(\mu_k - \mu_1 - 2).$$

Since C is general, $K_C - (\mu_k - \mu_1 - 2)g^1_k$ is equivalent to a unique effective divisor. If this divisor is a sum of distinct points, then the set of divisor classes E satisfying the conditions above is simply the set of subsets of these points of size $(k-2)(\mu_2 - \mu_1 - 1)$. We therefore see that $|W^{\boldsymbol{\mu}}(C)|$ is equal to the binomial coefficient above.

Lemma 4.6.14. *Let $z_1 \geq z_2 \geq 0$ be integers, let $\vec{z}_i(z_2) = (z_2^{(k-2-i)}, (z_2 - 1)^{(i)}, 0)$, and let $\vec{z}_{ij}(z_1, z_2)$ be the vector obtained from $\vec{z}_i(z_2)$ by inserting z_1 between entries j and $j+1$. Then*

$$\alpha(\vec{z}_{ij}(z_1, z_2)) = \binom{\left\lfloor \frac{k-2}{k-1}\left(z_1 + (k-2)z_2\right)\right\rfloor - i}{(k-2)z_2 - i}.$$

Proof. Note that the expression $(z_1 + (k-2)z_2)$ is divisible by $k-1$ if and only if $z_1 \equiv z_2 \pmod{k-1}$. If $\mathbf{C}(\lambda) = \vec{z}_{ij}(z_1, z_2)$, then this congruence holds if and only if the partition λ', obtained by deleting all columns of λ that are taller than z_2, has an outside corner in D_{z_1}. Since $\mathbf{C}(\lambda') = \vec{z}_{ij}(z_2, z_2)$, this holds if and only if $j = k-2$.

We establish the above formula by induction. The base cases, where $z_1 = z_2$, or $z_2 = i = 0$, both follow from Example 4.6.5. If $j = k-1$, then by Lemma 4.6.3, we have

$$\alpha(\vec{z}_{i(k-1)}(z_1, z_2)) = \alpha(\vec{z}_{i(k-2)}(z_1 - 1, z_2)).$$

By induction, the expression on the right is equal to

$$\binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2 - 1\Big) \right\rfloor - i}{(k-2)z_2 - i}.$$

Since $j = k-1$, by the above we see that $z_1 \equiv z_2 + 1 \pmod{k-1}$, so the term $(z_1 + (k-2)z_2 - 1)$ is divisible by $k-1$. The expression above is therefore equal to

$$\binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2\Big) \right\rfloor - i}{(k-2)z_2 - i}.$$

Otherwise, if $j < k-1$, then by Lemma 4.6.3, we have

$$\alpha(\vec{z}_{ij}(z_1, z_2)) = \alpha(\vec{z}_{i(j-1)}(z_1 - 1, z_2)) + \alpha(\vec{z}_{(i-1)j}(z_1, z_2)).$$

By induction, the expression on the right is equal to

$$\begin{aligned}
&\binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2\Big) \right\rfloor - (i+1)}{(k-2)z_2 - (i+1)} + \binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2 - 1\Big) \right\rfloor - i}{(k-2)z_2 - i} \\
=&\binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2\Big) \right\rfloor - (i+1)}{(k-2)z_2 - (i+1)} + \binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2\Big) \right\rfloor - (i+1)}{(k-2)z_2 - i} \\
=&\binom{\left\lfloor \frac{k-2}{k-1}\Big(z_1 + (k-2)z_2\Big) \right\rfloor - i}{(k-2)z_2 - i},
\end{aligned}$$

where the second line holds because $(z_1 + (k-2)z_2 - 1)$ is not divisible by $k-1$. □

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