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Combinatorial Invariants of Rational Polytopes

Andrés R. Vindas Meléndez
University of Kentucky, vindas1992@gmail.com
Author ORCID Identifier:
https://orcid.org/0000-0002-7437-3745
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Andrés R. Vindas Meléndez, Student

Dr. Benjamin Braun, Major Professor

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Combinatorial Invariants of Rational Polytopes

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Andrés R. Vindas Meléndez
Lexington, Kentucky

Director: Dr. Benjamin Braun, Professor of Mathematics
Lexington, Kentucky
2021

https://orcid.org/0000-0002-7437-3745

ABSTRACT OF DISSERTATION

Combinatorial Invariants of Rational Polytopes

The first part of this dissertation deals with the equivariant Ehrhart theory of the permutahedron. As a starting point to determining the equivariant Ehrhart theory of the permutahedron, Ardila, Schindler, and I obtain a volume formula for the rational polytopes that are fixed by acting on the permutahedron by a permutation, which generalizes a result of Stanley's for the volume for the standard permutahedron. Building from the aforementioned work, Ardila, Supina, and I determine the equivariant Ehrhart theory of the permutahedron, thereby resolving an open problem posed by Stapledon. We provide combinatorial descriptions of the Ehrhart quasipolynomial and Ehrhart series of the fixed polytopes of the permutahedron. Additionally, we answer questions regarding the polynomiality of the equivariant analogue of the h^*-polynomial.

The second part of this dissertation deals with decompositions of the h^*-polynomial for rational polytopes. An open problem in Ehrhart theory is to classify all Ehrhart quasipolynomials. Toward this classification problem, one may ask for necessary inequalities among the coefficients of an h^*-polynomial. Beck, Braun, and I contribute such inequalities when P is a rational polytope. Additionally, we provide two decompositions of the h^*-polynomial for rational polytopes, thereby generalizing results of Betke and McMullen and Stapledon. We use our rational Betke–McMullen formula to provide a novel proof of Stanley's Monotonicity Theorem for rational polytopes.

KEYWORDS: rational polytope, Ehrhart quasipolynomial, lattice points

Andrés R. Vindas Meléndez

May 24, 2021

Combinatorial Invariants of Rational Polytopes

By
Andrés R. Vindas Meléndez

Dr. Benjamin Braun
Director of Dissertation

Dr. Benjamin Braun
Director of Graduate Studies

May 24, 2021

Para mis creadores, Mami (Sara) y Papi (Rodolfo). Para mis hermanas, Alejandra y Sarah. Para mis padrinos, Ana y Julio. Para mi familia con la que el universo me bendició (mis tíos, tías, primos, y sobrinos) y para mis amigos que se han convertido en familia. Para mis maestros que me enseñaron a apreciar la belleza de la matemática.

For my creators, Mami (Sara) and Papi (Rodolfo). For my sisters, Alejandra and Sarah. For my godparents, Ana and Julio. For my family with whom the universe blessed me with (my uncles, aunts, cousins, and nephews) and for my friends who have become family. For my teachers who taught me to appreciate the beauty of math.

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¡Pura vida!

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Chapter 1 Introduction

Combinatorics is a versatile area of mathematics and has been described in many ways by different mathematicians. I share with you three of my favorite descriptions of the field. Combinatorics is:

- "the study of possibilities." - Federico Ardila (San Francisco State University)
- "the branch of mathematics concerned with selecting, arranging, constructing, classifying, and counting or listing things." - Igor Pak (UCLA)
- "the art and science of distilling a complex mathematical structure into simple attributes and developing from this a deeper understanding of the original structure." - Josephine Yu (Georgia Tech)

My research interests are in algebraic and geometric combinatorics, where we consider algebraic and geometric objects and study their underlying combinatorial structure. Some geometric objects that are of interest are polytopes. H.S.M. Coxeter wrote that "a polytope is a geometrical figure bounded by portions of lines, planes, or hyperplanes; e.g., in two dimensions it is a *polygon*, in three a *polyhedron*"[21]. Arguably, the foundations for the study of polytopes were laid by the Greeks over two millennia ago, as many can recall their study of what are regarded as Platonic solids.

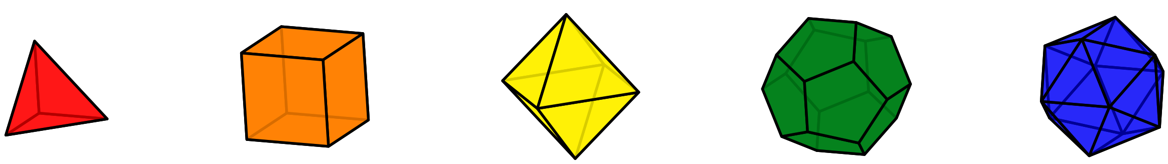

Figure 1.1: The platonic solids (from left to right): tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

In the 1700's the study of polytopes heightened with Leonhard Euler's polyhedral formula, which gave birth to the enumerative combinatorics on the number of faces of polytopes. The polyhedral formula is $v - e + f = 2$, where v denotes the number of vertices, e denotes the number of edges and f denotes the number of faces.

Example 1.0.1. Consider the cube in Figure 1.1, it has 8 vertices, 12 edges, and 6 square faces. We can check that the cube satisfies Euler's polyhedral formula: $v - e + f = 8 - 12 + 6 = 2$.

Almost a hundred and fifty years later, Georg Pick discovered a formula, which calculates the area of a lattice polygon using only its discrete information; this started the study of lattice point enumeration of polytopes [54]. Pick's formula is $A = b/2 + c - 1$, where A denotes the area, b is the number of boundary lattice points, and c denotes the number of interior lattice points.

Example 1.0.2. Consider the triangle T formed by connecting the vertices $(0,0)$, $(0,1)$, and $(1,0)$ as seen in Figure 1.2. Recall that the area of a triangle can be determined using the formula $A = 1/2 \cdot \text{base} \cdot \text{height}$. Hence, we determine that the area of T is $A = 1/2 \cdot 1 \cdot 1 = 1/2$. From Pick's formula we see that $A = b/2 + c - 1 = 3/2 + 0 - 1 = 1/2$, matching our previous calculation.

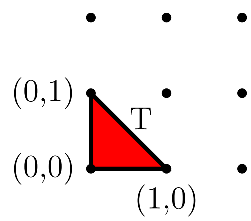

Figure 1.2: The triangle T.

In the 1960s, in an attempt to generalize Pick's formula to higher dimensions, Eugène Ehrhart started the study of counting the numbers of lattice points in dilations of polytopes, which established the theory of Ehrhart polynomials [25]. The essential question is, if we dilate a polytope by integer factor t, is there a polynomial that counts the number of lattice points? The answer to this is yes, and this is Ehrhart's theorem.

Example 1.0.3. Take T to be the triangle in Example 1.0.2 and now dilate it by integer dilations as seen in Figure 1.3. One can check that the polynomial that counts the number of lattice points for each integral dilation t is $\frac{1}{2}t^2 + \frac{3}{2}t + 1$. At $t = 1$, we simply have T and there are 3 lattice points, namely its vertices. At the second dilation of T, we can count in the figure that there are 6 lattice points and it is confirmed by substituting $t = 2$ in the polynomial: $\frac{1}{2}(2)^2 + \frac{3}{2}(2) + 1 = 6$. Similarly, we can count the number of lattice points for the other integer dilations.

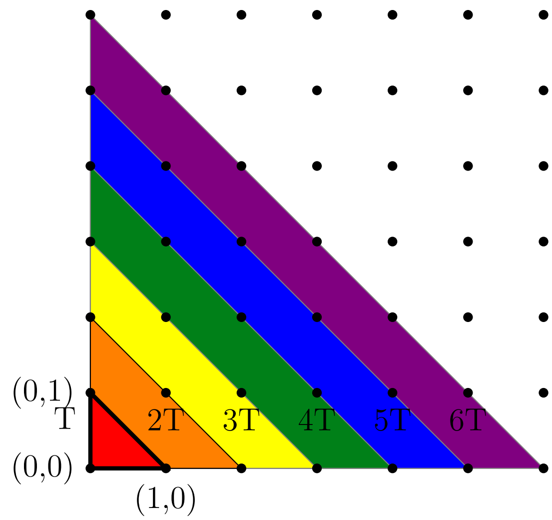

Figure 1.3: The triangle T with some of its integer dilates.

A rational convex polytope, i.e., the convex hull of finitely many rational points in $\mathbb{Q}^d$, its lattice point count, and its associated combinatorial invariants provide information on quantities of geometric and algebraic interest, such as algebraic varieties, and appear throughout mathematics. In algebraic geometry, a polytope P corresponds to a projective toric variety X_P and an ample line bundle L, whose Hilbert polynomial enumerates the lattice points in P. The lattice point enumerator of P can be expressed in terms of the Todd class of X_P [51, 55]. In commutative algebra, lattice point enumeration appears in the guise of Hilbert series of graded rings [20, 27]. In the representation theory of semisimple Lie algebras, the Kostant partition function enumerates the lattice points in flow polytopes [4, 50]. Recently, born out of representation theory and algebraic geometry, when P is symmetric, one can measure P with respect to its group of symmetries in relation to its lattice point count; this is the goal of equivariant Ehrhart theory [2, 3, 74].

Fundamental data of a polytope P is given by its (relative) volume. Computing volume, even for these simple to describe objects, is surprisingly difficult. One approach is to compute the *discrete volume* of the polytope, i.e., the number of lattice points in each integral dilate of the polytope. Ehrhart theory measures a polytope P by counting the number $L_P(t) := |tP \cap \mathbb{Z}^d|$ of lattice points in its dilations tP for $t \in \mathbb{Z}_{\geq 0}$. One can recover the continuous volume of P when it is full dimensional as $\lim_{t\to\infty} L_P(t)/t^d$. A fundamental result by Ehrhart [25] is that $L_P(t)$ is of the form

$$\mathrm{vol}(P)t^d + k_{d-1}(t)t^{d-1} + \cdots + k_1(t)t + k_0(t),$$

where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in t and we call this the *Ehrhart quasipolynomial* of P. Note that when P has integral vertices, then $L_P(t)$ has periods all 1 and is the well-studied Ehrhart polynomial [7, 22, 33, 34]. Polytopes can be alternatively described as bounded intersections of halfspaces, and if these halfspaces are given by linear systems with integral coefficients, the polytope is *rational*. Such linear systems appear throughout mathematics, and thus many counting functions can be interpreted as Ehrhart quasipolynomials. The generating series $\mathrm{Ehr}(P;z)$ can be written in the form

$$\mathrm{Ehr}(P;z) := \sum_{t\geq 0} L_P(t)z^t = \frac{h^*(P;z)}{(1-z^q)^{d+1}},$$

where $h^*(P;z)$ is a polynomial of degree less than $q(d+1)$. We call the polynomial

$$h^*(P;z) := h^*_{q(d+1)-1}z^{q(d+1)-1} + \cdots + h^*_1 z + h^*_0$$

encoding the h^*-coefficients the *h^*-polynomial* of P and its coefficients form the *h^*-vector* of P. While the literature on lattice polytopes is extensive, there are numerous open questions in the study of rational polytopes making it an active research area. In the next chapter, I will present further technical background and examples.

This dissertation is a compilation of two projects, which contribute to the limited literature on rational polytopes, namely,

1. the equivariant Ehrhart theory of the permutahedron, and

2. decompositions of the Ehrhart h^*-polynomial for rational polytopes.

These projects develop the theory of generating polynomials and other combinatorial invariants associated to polytopes and their lattice-point enumeration. They also follow three fundamental principles of combinatorial investigation: symmetry, decomposition, and classification.

The starting point to Project 1 dates back to my time as a master's student at San Francisco State, in joint work with Anna Schindler and under the co-advising of Federico Ardila, we were able to prove results about the fixed polytopes of the permutahedron. In particular, we computed their dimension, showed that they are combinatorially equivalent to permutahedra, provided hyperplane and vertex descriptions, and proved that they are zonotopes [60, 77]. During my time at the University of Kentucky, Federico Ardila, Anna Schindler, and I continued to work on this project where we obtained a formula for the volume of these fixed polytopes [2], which is a generalization of Richard Stanley's result of the volume for the standard permutahedron [66, 71]. I only include proofs for the results that were proven during my time as a PhD student at the University of Kentucky. In particular, I include our proofs regarding the volumes of the fixed polytopes of the permutahedron, statements in connection to the equivariant triangulations of the simplicial prism, and a result regarding the slices that are fixed by subgroups of the symmetric group.

The aforementioned work was the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Alan Stapledon [74]. In joint work with Federico Ardila and Mariel Supina, we provide an answer to Stapledon's question: we compute the equivariant Ehrhart theory of the permutahedron and verify his Effectiveness Conjecture in this special case [3]. Our proofs combine tools from discrete geometry, combinatorics, number theory, algebraic geometry, and representation theory.

An open problem in Ehrhart theory is to classify all Ehrhart quasipolynomials; that is, given a quasipolynomial $p(t)$, construct a rational polytope P whose Ehrhart quasipolynomial is $p(t)$, or explain why no such P exists. An equivalent question is, if we fix dimension, can we classify all h^*-vectors that arise from Ehrhart quasipolynomials? This question is open for Ehrhart polynomials arising from lattice polytopes in dimension 3 and higher, which illustrates that this a difficult problem. Toward this classification problem, one may ask for necessary inequalities among the coefficients of an h^*-polynomial. Matthias Beck, Benjamin Braun, and I contribute such inequalities when P is a rational polytope. We provide two decompositions of the h^*-polynomial for rational polytopes. The first decomposition generalizes a result of Betke and McMullen [11] and shows that the h^*-polynomial of a rational polytope can be decomposed in a way that brings together arithmetic data from the simplices of a triangulation and combinatorial information from the face structure of the triangulation. We use our rational Betke–McMullen formula to provide a novel proof of Stanley's Monotonicity Theorem for rational polytopes [68], which asserts that for $P \subseteq Q$, where P and Q are rational polytopes, every coefficient of $h^*(Q, z)$ dominates the corresponding coefficient of $h^*(P, z)$. The second decomposition generalizes a result of Stapledon, which we use to provide rational extensions of the Stanley and Hibi inequalities satisfied by the coefficients of the h^*-polynomial for lattice polytopes. Lastly, we apply our results to rational polytopes containing the origin whose duals are lattice polytopes [6].

The rest of this dissertation is structured as follows. In chapter 2, we provide notation, background, and examples on Ehrhart theory and related topics. In chapter 3, we show the results and proofs related to Project 1: the equivariant Ehrhart theory of the permutahedron. In chapter 4, we show the results and proofs related to Project 2: decompositions of the Ehrhart h^*-polynomial for rational polytopes.

Chapter 2 Background

This chapter presents the foundational topics and objects of study.

2.1 Lattice and Rational Polytopes

Convex polytopes are fundamental geometric objects that have been studied for millennia. Two main topics for the study of convex polytopes has been their combinatorial properties and their geometric properties. Our emphasis in studying polytopes is on connections to combinatorics, both in the sense of enumeration (e.g., Ehrhart theory) and combinatorial structures (e.g., graphs and permutations). For our interests, we focus only on convex polytopes and will simply refer to them as polytopes for the rest of this document. Some recommended background reading for polytopes are [9, 30, 78].

Definition 2.1.1. We now define the main characters of our story, namely, convex polytopes.

- A *polytope* P is the convex hull of finitely many points, $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$:

  $$P = \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) := \left\{ \sum_i \lambda_i \mathbf{v}_i : \ \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

  This is known as the V-description of a polytope and a polytope described in such a way is referred to as a V-polytope.

- Alternatively, we could define a *polytope* as the bounded solution set of a finite system of linear inequalities:

  $$P := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq b\},$$

  where $A \in \mathbb{R}^{m \times d}$ is a real matrix and $b \in \mathbb{R}^m$ is a real vector. This is known as the H-description of a polytope and a polytope described in such a way is referred to as an H-polytope.

One may think of polytopes as higher-dimensional generalizations of polygons.

Theorem 2.1.2 (Main Theorem of Polytope Theory)**.** *The definitions of V-polytopes and of H-polytopes are equivalent. That is, every V-polytope has a description by a finite system of inequalities, and every H-polytope can be obtained as a convex hull of finite set of points.*

Since both definitions of polytopes are equivalent, we need not specify whether a polytope is a V-polytope or H-polytope.

Figure 2.1: A hexagon as a bounded intersection of 6 half-spaces (left) and a hexagon as the convex hull of 6 vertices (right).

Definition 2.1.3. The *dimension* of a polytope P is the dimension of its affine hull: $\dim(P) := \dim(\text{aff}(P))$. Recall that $\text{aff}(P)$, the affine hull of a polytope, is $\{\sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \mathbf{v}_i \in P, \sum_{i=1}^{n} \lambda_i = 1\}$, the smallest affine subspace of $\mathbb{R}^d$ containing P. We call a d-dimensional polytope a *d-polytope*.

The following are several useful definitions regarding polytopes.

Definition 2.1.4. The integer points $\mathbb{Z}^d$ form a lattice in $\mathbb{R}^d$ and we refer to integer points as *lattice points*. Let P be a polytope.

- P is called an *integral* or *lattice polytope* if all its vertices have integer coordinates, i.e., P is the convex hull of finitely many points in $\mathbb{Z}^d$.
- P is called a *rational polytope* if all its vertices have rational coordinates, i.e., P is the convex hull of finitely many points in $\mathbb{Q}^d$.
- The intersection of P with a supporting hyperplane H is called a *k-dimensional face* or k-face if the dimension of $\text{aff}(P \cap H)$ is k. Each face itself is a polytope.
- If $\dim(P) = d$, the $(d-1)$-faces are called *facets* and the 0-faces are called *vertices* and are the extreme points of P.
- We call tP the t^{th} dilate of P, where $tP := \{t\mathbf{x} : \mathbf{x} \in P\}$ for $t \in \mathbb{Z}_{>0}$.

Definition 2.1.5. Polytopes are *combinatorially isomorphic* if their face lattices are isomorphic as abstract (unlabelled) partially ordered sets/lattices. Equivalently, two polytopes P and P' are *combinatorially equivalent* if their facet-vertex incidence matrices differ only by column and row operations. The equivalence class of polytopes under combinatorial equivalence is known as the *combinatorial type*.

We now move on to present some special groups of polytopes. One particular group of polytopes that will be useful are simplices, which we now define. We will denote a simplex by Δ for the rest of this chapter.

Definition 2.1.6. A d-polytope with exactly $d+1$ vertices is called a d-simplex.

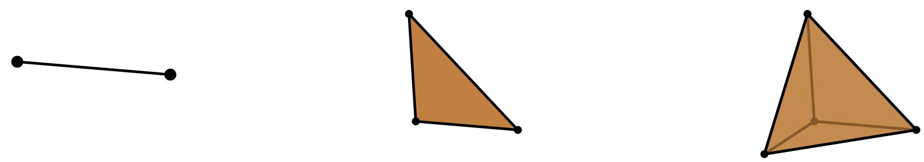

Figure 2.2: (Left to right): 1-simplex (line segment), 2-simplex (triangle), 3-simplex (tetrahedron).

Definition 2.1.7. A *triangulation* of a d-polytope P is a finite collection T of simplices satisfying the following properties:

- $P = \cup_{\Delta \in T} \Delta$, and
- for every $\Delta_1, \Delta_2 \in T$, $\Delta_1 \cap \Delta_2$ is a face common to both Δ_1 and Δ_2.

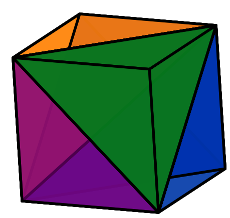

Figure 2.3: A triangulation of the 3-cube into four 3-simplices.

When extracting combinatorial information from geometric objects, a natural approach is to subdivide the combinatorial object into smaller, more accessible pieces as we shall see in later chapters. In the case of polytopes a natural approach is to triangulate a polytope into simplices. This idea is motivated by the following theorem.

Theorem 2.1.8. *[9] Every convex polytope P can be triangulated using no new vertices, i.e., there exists a triangulation T such that the vertices of every $\Delta \in T$ are vertices of P.*

Next, we introduce a special family of polytopes known as zonotopes. These are a rich and fascinating family of polytopes, with connections to many areas of mathematics. For a brief introduction to zonotopes, we refer you to an expository survey paper by Benjamin Braun and the author [16], from which the following background on zonotopes is borrowed from.

Zonotopes can be defined using either Minkowski sums or projections of cubes.

Definition 2.1.9. Consider polytopes, $P_1, P_2, \ldots, P_m \subset \mathbb{R}^n$. We define the *Minkowski sum* of the m polytopes as

$$P_1 + P_2 + \cdots + P_m := \{x_1 + x_2 + \cdots + x_m : x_j \in P_j \text{ for } 1 \leq j \leq m\}.$$

Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we write $[\mathbf{v}, \mathbf{w}]$ for the line segment from $\mathbf{v}$ to $\mathbf{w}$.

Example 2.1.10. Consider the Minkowski sum of $[(0,0),(1,0)]$, $[(0,0),(0,1)]$, and $[(0,0),(1,1)]$. The Minkowski sum of the first two segments is a unit square. Taking the Minkowski sum of this square with the line segment $[(0,0),(1,1)]$ can be visualized as sliding the square up and to the right along the line segment, with the resulting polytope consisting of all points touched by the square during the sliding movement, see Figure 2.4.

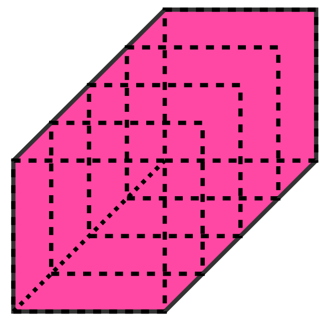

Figure 2.4: The Minkowski sum of $[(0,0),(1,0)]$, $[(0,0),(0,1)]$, and $[(0,0),(1,1)]$.

Definition 2.1.11. Consider m vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ in $\mathbb{R}^d$ and their corresponding line segments $[\mathbf{0},\mathbf{v}_j]$. The *zonotope* corresponding to $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is defined to be the Minkowski sum of these line segments:

$$Z(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m) := \{\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2+\cdots+\lambda_m\mathbf{v}_m : 0\le\lambda_j\le 1\} = [\mathbf{0},\mathbf{v}_1]+\cdots+[\mathbf{0},\mathbf{v}_m]\,.$$

We call any polytope that is translation-equivalent to such a polytope of this type a *zonotope*. When each $\mathbf{v}_j\in\mathbb{Z}^n$, we say that $Z(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m)$ is a *lattice zonotope*. Similarly, when each $\mathbf{v}_j\in\mathbb{Q}^n$, we say that $Z(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m)$ is a *rational zonotope*.

Example 2.1.12. The Minkowski sum in Figure 2.4 is $Z((1,0),(0,1),(1,1))$. The Minkowski sum in Figure 2.5 is $Z((1,0,0),(0,1,0),(0,0,1),(1,1,1))$.

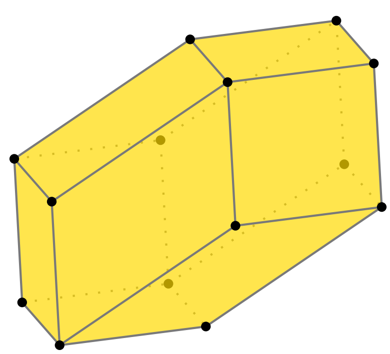

Figure 2.5: $Z((1,0,0),(0,1,0),(0,0,1),(1,1,1))$.

Definition 2.1.13. The zonotope $Z_0(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m) := Z(\pm\mathbf{v}_1, \pm\mathbf{v}_2, \ldots, \pm\mathbf{v}_m)$ is symmetric about the origin, that is, it has the property that $\mathbf{x} \in Z_0$ if and only if $-\mathbf{x} \in Z_0$; we call Z_0 a *centrally symmetric zonotope* defined by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Note that Z_0 can be obtained as $Z(2\mathbf{v}_1, 2\mathbf{v}_2, \ldots, 2\mathbf{v}_m) - (\mathbf{v}_1 + \cdots + \mathbf{v}_m)$.

An alternative definition of a zonotope is as a projection of the unit cube. In this case, let A denote the matrix with columns given by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then by definition it is immediate that $Z(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ is equal to $A \cdot [0,1]^m$. In some circumstances it is more convenient to work with this projection-based definition of zonotopes.

Remark 2.1.14. Zonotopes are deceptively simple to define, yet even the most elementary zonotopes are mathematically rich. For example, the unit cube $[0,1]^n$ is itself a zonotope, and the survey paper by Zong [79] shows that the mathematical properties of this object are both broad and deep.

Let V be a finite set of vectors in $\mathbb{R}^n$. We will denote the zonotope generated by V as $Z(V)$. As mentioned before, it is typically useful when objects can be decomposed as unions of simpler objects, e.g. the theory of triangulations. Zonotopes admit a particularly nice decomposition into parallelepipeds; parts of the boundaries of these parallelepipeds can be removed resulting in a disjoint decomposition. The combinatorial decomposition is useful when calculating volumes and counting lattice points. The following result is due to Shephard.

Theorem 2.1.15 (Shephard 1974, Theorem 54 [62])**.** *A zonotope $Z(V)$ can be subdivided into (half-open) parallelepipeds that are in bijection with the linearly independent subsets of V.*

Figure 2.6 is an illustration of the decomposition of the zonotope $Z((0,1),(1,0),(1,1)) \subset \mathbb{Z}^2$ as suggested by Shephard's theorem.

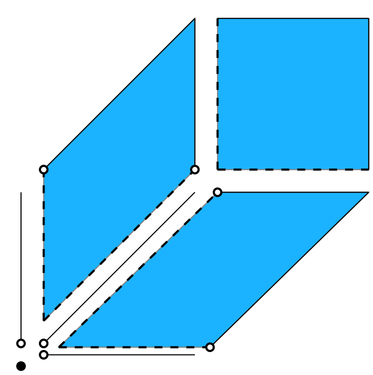

Figure 2.6: A zonotopal decomposition of $Z((0,1),(1,0),(1,1)$.

There are several interesting zonotopes associated with a finite simple graph G. We discuss one such construction, which is related to the number of spanning trees of G. For the first construction, we recall that for a polynomial $f = \sum_{\mathbf{a}\in\mathbb{Z}^n} \beta_{\mathbf{a}} t_1^{a_1} \cdots t_n^{a_n}$, the *Newton polytope* Newton(f) is the convex hull of integer points $\mathbf{a} \in \mathbb{Z}^n$ such that $\beta_{\mathbf{a}} \neq 0$. It is known that Newton$(f \cdot g)$ is the Minkowski sum Newton$(f)+$Newton(g), which is the fundamental ingredient for the following definition.

Definition 2.1.16. For a graph G on the vertex set $[n]$, the *graphical zonotope* Z_G is defined to be

$$Z_G := \sum_{(i,j)\in G} [e_i, e_j] = \text{Newton}\left(\prod_{(i,j)\in G} (t_i - t_j)\right),$$

where the Minkowski sum and the product are over edges (i,j), $i<j$, of the graph G, and $e_1, \ldots, e_n$ are the coordinate vectors in $\mathbb{R}^n$.

Example 2.1.17. The *n-permutahedron* is the polytope in $\mathbb{R}^n$ whose vertices are the $n!$ permutations of $[n]$:

$$\Pi_n := \text{conv}\{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \in \mathfrak{S}_n\}.$$

Using the Vandermonde determinant, one can show that Π_n is the graphical zonotope Z_{K_n} for the complete graph K_n. Figure 2.7 shows Π_4.

Proposition/Definition 2.1.18. [78] The permutahedron Π_n can be described in the following three ways:

1. (Inequalities) It is the set of points $x \in \mathbb{R}^n$ satisfying
   a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, and
   b) $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k$ for any subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$.
2. (Vertices) It is the convex hull of the points $(\pi(1), \ldots, \pi(n))$ as π ranges over the permutations of $[n]$.
3. (Minkowski sum) It is the Minkowski sum $\sum_{1\leq j<k\leq n} [e_k, e_j] + \sum_{1\leq k\leq n} e_k$.

The n-permutahedron is an $(n-1)$-dimensional zonotope and every permutation of $[n]$ is indeed a vertex.

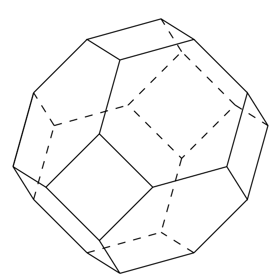

Figure 2.7: The permutahedron Π_4.

The following beautiful theorem shows that the volume and lattice points of Z_G encode the number of spanning trees and forests in G.

Theorem 2.1.19 (Stanley [71], Exercise 4.32; Postnikov [56], Proposition 2.4)**.** *For a connected graph G on n vertices, the volume of the graphical zonotope Z_G equals the number of spanning trees of G. The number of lattice points of Z_G equals the number of forests in the graph G.*

Remark 2.1.20. The number of spanning trees of the connected graph K_n is n^{n-2} and by Theorem 2.1.19 it is also the volume of Π_n. Furthermore, the number of lattice points of Π_n equals the number of forests on n labeled vertices. Richard Stanley first proved the volume of Π_n equals n^{n-2}, the number of spanning trees on n labeled vertices, in [66].

Ultimately, we are interested in the lattice-point enumeration of polytopes. This leads us to the following section on Ehrhart theory, the study of lattice points in dilations of polytopes.

2.2 Ehrhart theory

Life in 2 dimensions

We illustrate the intricacies of Ehrhart theory by examples in 2 dimensions.

Example 2.2.1. Let $\square$ be the lattice square depicted in Figure 2.8. The $\square$ can be described as the $\mathrm{conv}\{(x_1, x_2) :\ \text{all } x_k = 0, 1, \text{ or } 2\}$ or equivalently as the solution set of the bounded halfspaces $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_k \leq 2 \text{ for } k = 1, 2\}$.

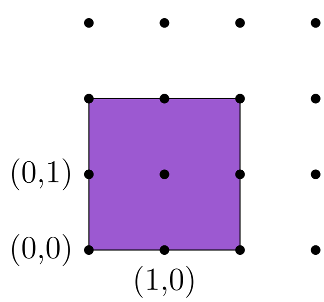

Figure 2.8: The square $\square \subset \mathbb{Z}^2$.

The following properties can be verified from Figure 2.8:

- $\square$ has area 4.
- $\square$ has 8 boundary lattice points.
- $\square$ has 1 interior lattice points.

These three properties are tied together by the expression

$$9 = 4 + \frac{8}{2} + 1.$$

In other words, the number of lattice points in $\square$ equals the sum of the area of $\square$, one-half the number of boundary points, and one. In 1899, Georg Pick[1] proved that this relationship holds for any convex lattice polygon [54].

Theorem 2.2.2 (Pick 1899, [54])**.** *For any convex lattice polygon P, the number of lattice points in the polygon is given by*

$$L = A + \frac{B}{2} + 1,$$

where A, B, and L denote the area of P, the number of lattice points on the boundary of P, and the total number of lattice points of P, respectively.

Now, what if we dilated the polygon by an integral factor of t? Is there a polynomial that counts the number of lattice points? The answer is yes, and is a result due to Eugène Ehrhart[2].

Theorem 2.2.3. *Let P be a convex lattice polygon and let t be a positive integer. Then the lattice-point enumerator, $L(t) = |tP \cap \mathbb{Z}^2|$, is given by the equation*

$$L(t) = At^2 + \frac{B}{2}t + 1.$$

Example 2.2.4. Taking $\square$ as defined in the previous example, we dilate by a factor of t, as illustrated in Figure 2.9, to obtain

$$L(t) = 4t^2 + 4t + 1 = (2t+1)^2.$$

It is quite remarkable that the counting function is in fact given by a polynomial. When we evaluate the polynomial for a nonnegative integer t, that number matches with the number of lattice points in the t^{th} dilate of $\square$. Take, for example $t = 2$, we get that $L(2) = 4(2)^2 + 4(2) + 1 = 25$, which we see from Figure 2.9 that $2\square$ has 25 lattice points.

---

[1] Georg Pick (1859-1942) was an Austrian mathematician. An interesting fact is that Pick was the driving force behind the appointment of Albert Einstein as chair of mathematical physics at the German University of Prague. Later in life, when the Nazis came to power, Pick was sent to Theresienstadt concentration camp where he died at age 82. While Pick worked in several areas of mathematics, he is most famously remembered for Pick's formula mentioned above in this dissertation [53].

[2] Eugène Ehrhart (1906-2000) was a French mathematician. He was a high school mathematics teacher for many years. During his time as a teacher he performed research for personal enjoyment. The fact that he was pursuing research led his colleagues to urge him to pursue a PhD. He completed his PhD thesis at the age of 60 [19].

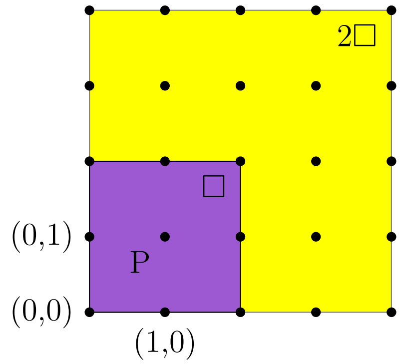

Figure 2.9: The square □ and its second dilate 2□.

Ehrhart theory in general dimensions

The finitely many lattice points in $\mathbb{Z}^n$ of a *lattice polytope* $P \subset \mathbb{R}^n$ of dimension d affinely span a d-dimensional hyperplane. For $t \in \mathbb{Z}_{>0}$, set $tP := \{tp : p \in P\}$, and let $L_P(t) = |tP \cap \mathbb{Z}^n|$. Ehrhart [25] proved a statement equivalent to the following. Recall that the set

$$\left\{ \binom{t+d-i}{d} : i = 0, 1, \ldots, d \right\}$$

is a basis for polynomials of degree d, where

$$\binom{t+d-i}{d} = (1/d!)(t+d-i)(t+d-i-1)(t+d-i-2)\cdots(t-i+1)$$

is clearly a polynomial in t of degree d. For any lattice polytope P, there exist rational values $c_0, c_1, \ldots, c_d$ and $h_0^*, h_1^*, \ldots, h_d^*$ such that

$$L_P(t) = \sum_{i=0}^{d} h_i^* \binom{t+d-i}{d} = \sum_{i=0}^{d} c_i t^i \,.$$

The polynomial $L_P(t)$ is called the *Ehrhart polynomial of* P and has connections to commutative algebra, algebraic geometry, combinatorics, and discrete and convex geometry. Equivalently, the Ehrhart polynomial is a polynomial of the form $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$. Finding geometric or combinatorial interpretations of the coefficients of the Ehrhart polynomial still remains a leading problem in Ehrhart theory. As it stands today, we know that the leading coefficient is the relative volume of P, the second leading coefficient is half the surface volume of P, and the constant term is always 1. We call the polynomial $h^*(P; z) := h_0^* + h_1^* z + \cdots + h_d^* z^d$ encoding the h^*-coefficients the *h^*-polynomial* (or *δ-polynomial*) of P. The coefficients of $h^*(P; z)$ form the *h^*-vector of* P.

Example 2.2.5. The unit square as depicted in Figure 2.10 has Ehrhart polynomial $L_{[0,1]^2}(t) = t^2 + 2t + 1$ and $h^*([0,1]^2; z) = 1 + z$.

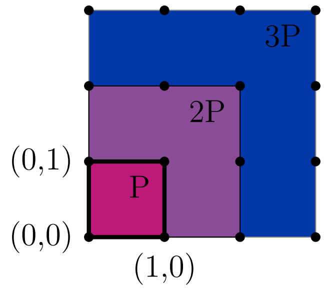

Figure 2.10: The unit square $P = [0,1]^2 \subset \mathbb{Z}^2$ and some of its dilates.

Various properties of P are reflected in its h^*-polynomial. For example, $\mathrm{EVol}(P) = (\sum_i h_i^*)/d!$, where $\mathrm{EVol}(P)$ denotes the Euclidean volume of P with respect to the integer lattice contained in the hyperplane spanned by P. We therefore define the *normalized volume* of P to be $\mathrm{Vol}(P) = \sum_i h_i^*$. Further, it is known that $h_0^* = 1$ for all P, and that h_d^* is equal to the number of lattice points in the relative (topological) interior of P within the affine span of P. Another interesting combinatorial property displayed by $h^*(P;x)$ for some lattice polytopes is *unimodality*. A polynomial $a_0 + a_1x + \cdots + a_dx^d$ is called *unimodal* if there exists an index j, $0 \le j \le d$, such that $a_{i-1} \le a_i$ for $i \le j$, and $a_i \ge a_{i+1}$ for $i \ge j$. Unimodality of h^*-polynomials is an area of active research [14].

Note that lattice polytopes are a subset of the more general rational polytopes. We now state fundamental results in Ehrhart theory for rational polytopes.

Definition 2.2.6. A *quasipolynomial* $f(t)$ is a function $f : \mathbb{Z} \to \mathbb{R}$ of the form

$$f(t) = c_d(t)t^d + \cdots + c_1(t)t + c_0(t),$$

where $c_0, c_1, \ldots, c_d$ are periodic functions in the integer variable t.

Alternatively, for a quasipolynomial f, there exists a positive integer k and polynomials $f_0, f_1, \ldots, f_{k-1}$, such that

$$f(t) = \begin{cases} f_0(t) & \text{if } t \equiv 0 \mod k \\ f_1(t) & \text{if } t \equiv 1 \mod k \\ \vdots & \\ f_{k-1}(t) & \text{if } t \equiv k-1 \mod k. \end{cases}$$

The minimal such k is the period of f, and for this minimal k, the polynomials $f_0, f_1, \ldots, f_{k-1}$ are are the *constituents* of f.

Definition 2.2.7. We define the *denominator* q of a rational polytope P to be the least common multiple of all the vertex coordinate denominators of P.

Theorem 2.2.8 (Ehrhart 1962, [25])**.** *Given a rational polytope in $\mathbb{R}^n$ the counting function $L(t) := |tP \cap \mathbb{Z}^n|$ is a quasipolynomial of degree $d := \dim P$ with period dividing the denominator q.*

Definition 2.2.9. Encoding the Ehrhart quasipolynomial in a generating function, we define the *Ehrhart series* of P to be

$$\mathrm{Ehr}_P(z) := \sum_{t \geq 0} L_P(t) z^t = L_P(0)z^0 + L_P(1)z^1 + L_P(2)z^2 + \dots .$$

Corollary 2.2.10. *The generating function* $\mathrm{Ehr}_P(z)$ *is a rational function of the form*

$$\frac{h^*(P;z)}{(1-z^q)^{d+1}},$$

where $h^*(P;z)$ *is a polynomial of degree less than* $q(d+1)$*, the* h^*-polynomial *of* P*.*

Note that our choice for the h^*-polynomial depends not only on q (though that is implicitly determined by P), but also on our choice of representing the rational function $\mathrm{Ehr}(P;z)$, which in our form will not be in lowest terms.

Example 2.2.11. Consider the rational rectangle $P = [0, \frac{1}{2}]^2$. This is the convex hull of the four vertices $(0,0), (0, \frac{1}{2}), (\frac{1}{2}, 0)$, and $(\frac{1}{2}, \frac{1}{2})$ as shown in Figure 2.11.

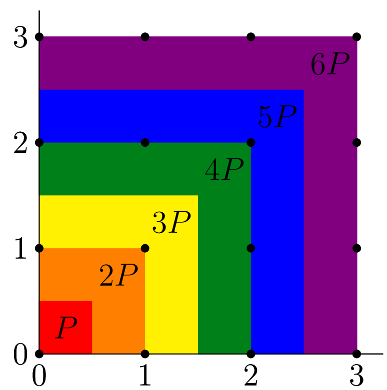

Figure 2.11: The rational rectangle $P = [0, \frac{1}{2}]^2 \subset \mathbb{Z}^2$ and some of its dilates.

The Ehrhart quasipolynomial of P is

$$L_P(t) = \begin{cases} \left(\frac{t+2}{2}\right)^2, & t \text{ even} \\ \left(\frac{t+1}{2}\right)^2, & t \text{ odd} \end{cases}.$$

Its corresponding Ehrhart series is

$$
\begin{aligned}
\mathrm{Ehr}_P(z) &= \sum_{m\geq 0} L_P(t) z^t \\
&= \sum_{\substack{t\geq 0\\ t \text{ even}}} \left(\frac{t^2}{4} + t + 1\right) z^t + \sum_{\substack{t\geq 1\\ t \text{ odd}}} \left(\frac{t^2}{4} + \frac{t}{2} + \frac{1}{4}\right) z^t \\
&= \sum_{k\geq 0} \left[\frac{(2k)^2}{4} + (2k) + 1\right] z^{2k} + \sum_{k\geq 0} \left[\frac{(2k+1)^2}{4} + \frac{2k+1}{2} + \frac{1}{4}\right] z^{2k+1} \\
&= \frac{z^2+1}{(1-z^2)^3} + \frac{z^3+z}{(1-z^2)^3} \\
&= \frac{z^3+z^2+z+1}{(1-z^2)^3}.
\end{aligned}
$$

The following two theorems are due to Richard Stanley. The first is known as Stanley's non-negativity theorem and states that the coefficients of the h^*-polynomial are nonnegative. The second is known as Stanley's monotonicity result, which asserts that for $P \subseteq Q$, where P and Q are rational polytopes, every coefficient of $h^*(Q, z)$ dominates the corresponding coefficient of $h^*(P, z)$.

Theorem 2.2.12 (Stanley 1980, [63])**.** *Let P be a rational d-polytope with Ehrhart series of the form*

$$
\mathrm{Ehr}_P(z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}.
$$

The coefficients of the h^*-polynomial are nonnegative integers, i.e., $h_j^* \in \mathbb{Z}_{\geq 0}$.

Theorem 2.2.13 (Stanley 1993, [68])**.** *Suppose that $P \subseteq Q$ are rational polytopes with qP and qQ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the h^*-polynomials via*

$$
\mathrm{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{\dim(P)+1}} \qquad \textit{and} \qquad \mathrm{Ehr}(Q;z) = \frac{h^*(Q;z)}{(1-z^q)^{\dim(Q)+1}}\,.
$$

Then $h_i^*(P;z) \leq h_i^*(Q;z)$ coefficient-wise.

The following theorem due in part to Ehrhart and to Ian Macdonald as is known as Ehrhart–Macdonald reciprocity; it belongs to a class of *reciprocity theorems*. Beck and Robins [9] write that,

> A common theme in combinatorics is to begin with an interesting object P, and
>
> 1. define a counting function $f(t)$ attached to P that makes physical sense for positive integer values of t;
> 2. recognize the function f as a polynomial in t;

3. substitute negative integral values of t into the counting function f, and recognize $f(-t)$ as a counting function of a new mathematical object Q.

Theorem 2.2.14 (Ehrhart–Macdonald 1971, [25, 47])**.** *Let P be a rational d-polytope. Then $L_P(-t) = (-1^d)L_{P^\circ}(t)$, where P° is the relative interior of P. Similarly,* $\mathrm{Ehr}_P(\frac{1}{z}) = (-1)^{d+1}\mathrm{Ehr}_{P^\circ}(z)$.

Reciprocity

I now take a slight detour and mention a reflection about abstract mathematics and the concept of *reciprocity*. Back in 2017, Dr. Rochelle Gutiérrez[3], whose work I admire, asked me to read a preliminary version of her article, "Living Mathematx" [31]. I am forever grateful to her for that opportunity. As a young master's student, I was nervous to read the article and provide feedback on my own. I asked her if it was okay for me to share it with some colleagues; I thank her for allowing me to share it with my colleagues at San Francisco State University's Math Education Research Group for Equity (MERGE) who agreed to help give Rochelle comments on that preliminary version. Regarding reciprocity, Gutiérrez writes that

> [t]he concept of reciprocity highlights the idea that different persons have different strengths and needs, and thus must rely on others for what they lack. More than simply recognizing that reciprocity enables persons to do things they could not otherwise do alone, it underscores a kind of ethic that is valued in maintaining harmony of the cosmos. In this sense, reciprocity is not only the productive thing to do, it is the right thing to do. Whereas In Lak'ech acknowledges the nature of the relationship between self and others, reciprocity highlights the actions that should result.

Gutiérrez further reflects on how Western mathematicians can begin to embrace the joy, emotions, reciprocity, and the interstitial space between worlds (e.g., the mathematics classroom and the lived realities of students). Reading her preliminary draft, I commented to her that the concept of reciprocity she was mentioning reminded me of the combinatorial concept of reciprocity. In her final version she reflects about the the intervention of reality for reciprocity and writes that

> [i]n fact, Beck and Sanyal (2017) ascribe animacy to the process by referring to it as moving from "your world" to "my world." The new counting

[3]Dr. Rochelle Gutiérrez is a mathematics education scholar at the University of Illinois at Urbana-Champaign (UIUC). I first met Rochelle at my very first Society for Advancement of Chicanos/Hispanics and Native Americans (SACNAS) conference in 2011. I also extend my thanks to my friend Gabriela E. Vargas, who also studied mathematics at UC Berkeley with me as undergraduates; she managed to get us to chat with Rochelle at that SACNAS conference. To exemplify the power of connections, Gabriela finished her master's in 2018 under Rochelle's guidance. Gabriela will soon finish her PhD in mathematics education at UIUC.

function has offered something that the original counting function could not. Is the mathematician grateful for the offering of this new counting function? Is there some joy in noting that functions can give back to each other? How might that starting point extend to other forms of reciprocity in doing mathematics with other persons?

I find this reflection quite powerful because it allows us to consider how we view and connect with mathematics. To what extent can ascribing animacy to mathematics or mathematical concepts influence mathematical research? It also motivates me to ask and reflect on, how far can our mathematics go and how impactful could it be if we allow and prioritize our mathematics to be driven by joy and connections/relationships? What do we have to offer mathematics and what does mathematics have to offer us for mutual growth?

Ehrhart Polynomials for Zonotopes

For a lattice zonotope Z, Stanley proved the following description of the coefficients of $L_Z(t)$.

Theorem 2.2.15 (Stanley 1991, Theorem 2.2 [66],)**.** *Let $Z := Z(\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a zonotope generated by the integer vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then the Ehrhart polynomial of Z is given by*

$$L_Z(t) = \sum_S m(S) t^{|S|},$$

where S ranges over all linearly independent subsets of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, and $m(S)$ is the greatest common divisor of all minors of size $|S|$ of the matrix whose columns are the elements of S.

The proof of Theorem 2.2.15 relies on Theorem 2.1.15, and is our first example of the usefulness of half-open decompositions of zonotopes. In fact, for a finite set of vectors V, a linearly independent subset $S \subseteq V$ corresponds under the bijection in Shephard's Theorem (Theorem 2.1.15) to the half-open parallelepiped

$$\square S := \sum_{\mathbf{v} \in S} [\mathbf{0}, \mathbf{v}).$$

With this in mind, we can geometrically reformulate Stanley' formula for the Ehrhart polynomial of a lattice zonotope as follows:

Theorem 2.2.16 ([66])**.** *Let $Z(V)$ be a lattice zonotope generated by V. Then*

$$L_{Z(V)}(t) = \sum_{\substack{S \subseteq V \\ \text{lin. indep.}}} \operatorname{Vol}(\square S) \cdot t^{|S|}. \tag{2.1}$$

Example 2.2.17. Consider the zonotope $Z((0,1),(1,0),(1,1),(1,-1)) \subset \mathbb{Z}^2$ depicted in Figure 2.12. We compute

$$\begin{aligned} L_{Z((0,1),(1,0),(1,1),(1,-1))}(t) = &\left|\det\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right| t^2 + \left|\det\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right| t^2 + \left|\det\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\right| t^2 + \\ &\left|\det\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right| t^2 + \left|\det\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right| t^2 + \left|\det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right| t^2 + \\ &\gcd(0,1)t + \gcd(1,0)t + \gcd(1,1)t + \gcd(1,-1)t + 1 \\ = &\, 7t^2 + 4t + 1 \end{aligned}$$

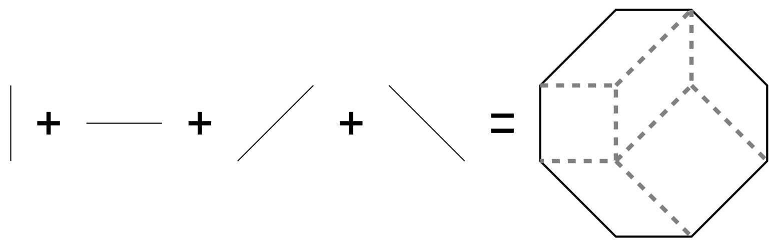

Figure 2.12: The zonotope $Z((0,1),(1,0),(1,1),(1,-1))$ is a Minkowski sum of four line segments.

Let's also see the nice combinatorial properties of the Ehrhart polynomial of the special zonotope mentioned earlier: the permutahedron.

Definition 2.2.18. A *graph* is a pair of sets (V, E), where V is the set of nodes and E is the set of edges formed by pairs of vertices. Furthermore, a *forest* is a graph with no cycles, i.e., a disjoint collection of trees.

Theorem 2.2.19. *The coefficient c_k of the Ehrhart polynomial*

$$L_{\Pi_n} = c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$$

of the permutahedron Π_n equals the number of labeled forests on n nodes with k edges.

For more on this theorem including its proof, refer to [9, 71].

Example 2.2.20. Let's compute the Ehrhart polynomial of the 3-permutahedron. We see that there are 3 labeled forests with 2 edges, 3 with 1 edge, and 1 with 0 edges. Hence, Theorem 2.2.19 suggests the Ehrhart polynomial of Π_3 is $3t^2 + 3t + 1$. In fact, we can see a bit more when invoking Theorem 2.1.15 and Theorem 2.2.16 because it turns out that each half-open parallepiped in the zonotopal decomposition of the permutahedron is in bijection with the linearly independent subsets of finite set generating Π_3 and are also in bijection with labeled forests, as we see suggested by Figure 2.13.

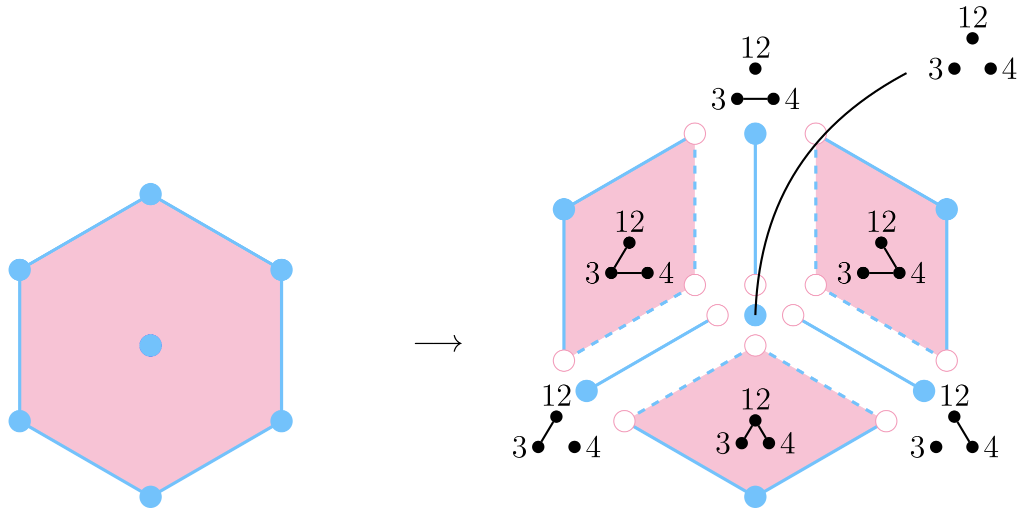

Figure 2.13: A zonotopal decomposition of Π_3 with corresponding labeled forests.

Chapter 3 The Equivariant Ehrhart Theory of the Permutahedron

Motivated by mirror symmetry, Stapledon [74, 75] introduced *equivariant Ehrhart theory* as a refinement of Ehrhart theory that takes into account the symmetries of the polytope P. Let G be a finite group acting linearly on a lattice polytope P. Combinatorially, the goal of equivariant Ehrhart theory is to understand, for each $g \in G$, the lattice point enumerator $L_{P^g}(t)$ of the polytope $P^g \subseteq P$ fixed by g. These quantities can be assembled into a sequence of virtual characters $\varphi_0, \varphi_1, \varphi_2, \ldots$ of G, which one wishes to understand representation-theoretically. Geometrically, these virtual characters arise naturally when one studies the action of G on the cohomology of a G-invariant hypersurface in the toric variety associated to P. Stapledon showed that if (X_P, L) admits a non-degenerate G-invariant hypersurface, then these virtual characters are effective and polynomial; in particular, they correspond to an actual representation of G. His Effectiveness Conjecture [74, Conjecture 12.1] states that the converse statement also holds.

To date, few examples of equivariant Ehrhart theory are understood. Stapledon computed it for regular simplices, hypercubes, and centrally symmetric polytopes. Finally, if Δ is the Coxeter fan associated to a root system and P is the convex hull of the primitive integer vectors of the rays of Δ, he used the equivariant Ehrhart theory of P recovers Procesi, Dolgachev–Lunts, and Stembridge's formula [57, 24, 76] for the character of the action of the Weyl group on the cohomology of the toric variety X_Δ. In [74], Stapledon asked for the computation of the next natural example: the permutahedron under the action of the symmetric group. The corresponding toric variety is the permutahedral variety, which is the subject of great interest. For example, Huh used it in his Ph.D. thesis [39] to prove Rota's conjecture on the log-concavity of the coefficients of chromatic polynomials. In algebraic geometry, it arises as the Losev-Manin moduli space of curves [46].

The goal of this chapter is to answer Stapledon's question: we compute the equivariant Ehrhart theory of the permutahedron and verify his Effectiveness Conjecture in this special case. Our proofs combine tools from discrete geometry, combinatorics, number theory, algebraic geometry, and representation theory.

A significant new challenge that arises is that the fixed polytopes of the permutahedron are not integral. Thus, the equivariant Ehrhart theory of the permutahedron requires surprisingly subtle arithmetic considerations – which are absent from the ordinary Ehrhart theory of lattice polytopes.

3.1 Equivariant Volumes of the Permutahedron

The first step towards describing the equivariant Ehrhart theory of the permutahedron is to determine the volume of the fixed polytopes of the permutahedron.

Definition 3.1.1. The *fixed polytope* or *slice of the permutahedron* Π_n *fixed by a* *permutation σ of* $[n]$ is

$$\Pi_n^\sigma = \{x \in \Pi_n \,:\, \sigma \cdot x = x\}.$$

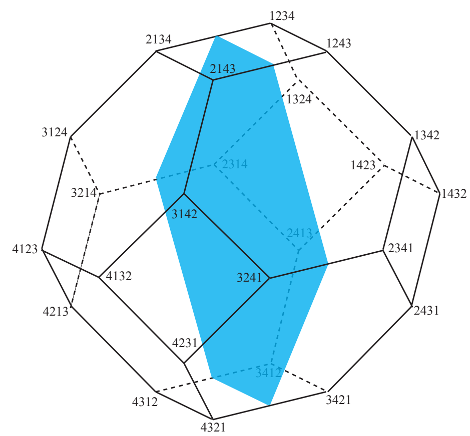

Figure 3.1: The slice $\Pi_4^{(12)}$ of the permutahedron Π_4 fixed by $(12) \in S_4$ is a hexagon.

Normalizing the volume

The permutahedron and its fixed polytopes are not full-dimensional; we must define their volumes carefully. We normalize volumes so that every primitive parallelepiped has volume 1. This is the normalization under which the volume of Π_n equals n^{n-2}.

More precisely, let P be a d-dimensional polytope on an affine d-plane $L \subset \mathbb{Z}^n$. Assume L is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a d-dimensional lattice Λ. We call a lattice d-parallelepiped in L *primitive* if its edges generate the lattice Λ; all primitive parallelepipeds have the same volume. Then we define the volume of a d-polytope P in L to be $\mathrm{Vol}(P) := \mathrm{EVol}(P)/\,\mathrm{EVol}(\square)$ for any primitive parallelepiped $\square$ in L, where EVol denotes Euclidean volume. By convention, the normalized volume of a point is 1.

The definition of $\mathrm{Vol}(P)$ makes sense even when P is not an integral polytope. This is important for us because the fixed polytopes of the permutahedron are not necessarily integral.

Notation

We identify each permutation $\pi \in S_n$ with the point $(\pi(1), \ldots, \pi(n))$ in $\mathbb{R}^n$. When we write permutations in cycle notation, we do not use commas to separate the entries of each cycle. For example, we identify the permutation 246513 in S_6 with the point $(2, 4, 6, 5, 1, 3) \in \mathbb{R}^6$, and write it as $(1245)(36)$ in cycle notation.

Our main object of study is the fixed polytope Π_n^σ for a permutation $\sigma \in S_n$. We assume that σ has m cycles $\sigma_1, \ldots, \sigma_m$ of lengths $l_1 \geq \cdots \geq l_m$. We let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and $e_S := e_{s_1} + \cdots + e_{s_k}$ for $S = \{s_1, \ldots, s_k\} \subseteq [n]$.

Standardizing the permutation

We define the *cycle type* of a permutation σ to be the partition of n consisting of the lengths $l_1 \geq \cdots \geq l_m$ of the cycles $\sigma_1, \ldots, \sigma_m$ of σ.

Lemma 3.1.2. *The volume of Π_n^σ only depends on the cycle type of σ.*

Proof. Two permutations of S_n have the same cycle type if and only if they are conjugate [59]. For any two conjugate permutations σ and $\tau\sigma\tau^{-1}$ (where $\sigma, \tau \in S_n$) we have

$$\Pi_n^{\tau\sigma\tau^{-1}} = \tau \cdot \Pi_n^\sigma. \tag{3.1}$$

Every permutation $\tau \in S_n$ acts isometrically on $\mathbb{R}^n$ because S_n is generated by the transpositions $(i \;\; i+1)$ for $1 \leq i \leq n-1$, which act as reflections across the hyperplanes $x_i = x_{i+1}$. It follows from (3.1) that the fixed polytopes $\Pi_n^{\tau\sigma\tau^{-1}}$ and Π_n^σ have the same volume, as desired. □

The three descriptions of the fixed polytope of Π_n

We begin by describing a set $\mathrm{Vert}(\sigma)$ of $m!$ points associated to a permutation σ of S_n.

Definition 3.1.3. The set $\mathrm{Vert}(\sigma)$ of σ-vertices consists of the $m!$ points

$$\overline{v_\prec} := \sum_{k=1}^{m} \left(\frac{l_k+1}{2} + \sum_{j\,:\,\sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}$$

as $\prec$ ranges over the $m!$ possible linear orderings of $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Definition 3.1.4. Let M_σ denote the Minkowski sum

$$\begin{aligned} M_\sigma \;&:=\; \sum_{1 \leq j < k \leq m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k} \\ &=\; \sum_{1 \leq j < k \leq m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m} \left(\frac{l_k+1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \end{aligned} \tag{3.2}$$

Theorem 3.1.5 (Ardila, Schindler, Vindas-Meléndez 2021, [2, 60, 77])**.** *Let σ be a permutation of $[n]$ whose cycles $\sigma_1, \ldots, \sigma_m$ have respective lengths $l_1, \ldots, l_m$. The fixed polytope Π_n^σ can be described in the following four ways:*

0. *It is the set of points x in the permutahedron Π_n such that $\sigma \cdot x = x$.*

1. *It is the set of points $x \in \mathbb{R}^n$ satisfying*

   a) $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$,

   b) $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k$ *for any subset* $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, *and* *c) $x_i = x_j$ for any i and j which are in the same cycle of σ.*

2. *It is the convex hull of the set* $\mathrm{Vert}(\sigma)$ *of σ-vertices described in Definition 3.1.3.*

3. *It is the Minkowski sum* $M_\sigma = \displaystyle\sum_{1\le j<k\le m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k+1}{2} e_{\sigma_k}$ *of Definition 3.1.4.*

Consequently, the fixed polytope Π_n^σ is a zonotope that is combinatorially isomorphic to the permutahedron Π_m. It is $(m-1)$-dimensional and every σ-vertex is indeed a vertex of Π_n^σ.

The volumes of the fixed polytopes of Π_n

To compute the volume of the fixed polytope Π_n^σ we will use its description as a zonotope, recalling that a zonotope can be tiled by parallelepipeds as follows. If A is a set of vectors, then $B \subseteq A$ is called a *basis* for A if B is linearly independent and $\mathrm{rank}(B) = \mathrm{rank}(A)$. We define the parallelepiped $\Box B$ to be the Minkowski sum of the segments in B, that is,

$$\Box B := \Big\{ \sum_{b\in B} \lambda_b b \,:\, 0 \le \lambda_b \le 1 \text{ for each } b \in B \Big\}.$$

Theorem 3.1.6. *[22, 66, 78] Let $A \subset \mathbb{Z}^n$ be a set of lattice vectors of rank d and $Z(A)$ be the associated zonotope; that is, the Minkowski sum of the vectors in A.*

1. *The zonotope $Z(A)$ can be tiled using one translate of the parallelepiped $\Box B$ for each basis B of A. Therefore, the volume of the d-dimensional zonotope $Z(A)$ is*
$$\mathrm{Vol}\,(Z(A)) = \sum_{\substack{B\subseteq A\\ B \text{ basis}}} \mathrm{Vol}\,(\Box B).$$

2. *For each $B \subset \mathbb{Z}^n$ of rank d,* $\mathrm{Vol}(\Box B)$ *equals the index of $\mathbb{Z}B$ as a sublattice of* $(\mathrm{Span}\ B) \cap \mathbb{Z}^n$*. Using the vectors in B as the columns of an $n \times d$ matrix,* $\mathrm{Vol}(B)$ *is the greatest common divisor of the minors of rank d.*

By Theorem 3.1.5, the fixed polytope Π_n^σ is a translate of the zonotope generated by the set

$$F_\sigma = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,;\, 1 \le j < k \le m \right\}.$$

This set of vectors has a nice combinatorial structure, which will allow us to describe the bases B and the volumes $\mathrm{Vol}\,(\Box B)$ combinatorially. We do this in the next two lemmas. For a tree T whose vertex set is $[m]$, let

$$\begin{aligned} F_T &= \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, j<k \text{ and } jk \text{ is an edge of } T \right\}, \\ E_T &= \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \,:\, j<k \text{ and } jk \text{ is an edge of } T \right\}. \end{aligned}$$

Lemma 3.1.7. *The vector configuration*

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, 1 \le j < k \le m \right\}$$

has exactly m^{m-2} bases: they are the sets F_T as T ranges over the spanning trees on $[m]$.

Proof. The vectors in F_σ are positive scalar multiples of the vectors in

$$E_\sigma = \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \,:\, 1 \le j < k \le m \right\},$$

which are the images of the vector configuration $A_{m-1}^+ = \{f_j - f_k \,:\, 1 \le j < k \le m\}$ under the bijective linear map $\phi : \mathbb{R}^m \to (\mathbb{R}^n)^\sigma$ where

$$\phi(f_i) = \frac{1}{l_i} e_{\sigma_i} \text{ for } 1 \le i \le m. \tag{3.3}$$

The set A_{m-1}^+ is a set of positive roots for the Lie algebra $\mathfrak{gl}_m$; its bases are known [12] to correspond to the spanning trees T on $[m]$, and there are m^{m-2} of them by Cayley's formula [18]. It follows that the bases of F_σ are precisely the sets F_T as T ranges over those m^{m-2} trees. $\square$

Lemma 3.1.8. *For any tree T on $[m]$ we have*

$$\begin{aligned} 1.\quad \mathrm{Vol}(\Box F_T) &= \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(E_T), \\ 2.\quad \mathrm{Vol}(\Box E_T) &= \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}, \end{aligned}$$

where $\deg_T(i)$ is the number of edges containing vertex i in T.

Proof. 1. Since $l_k e_{\sigma_j} - l_j e_{\sigma_k} = l_j l_k \big(\frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \big)$ for each edge jk of T, and volumes scale linearly with respect to each edge length of a parallelepiped, we have

$$\begin{aligned} \mathrm{Vol}(\Box F_T) &= \Big(\prod_{jk \text{ edge of T}} l_j l_k \Big) \mathrm{Vol}(\Box E_T) \\ &= \prod_{i=1}^{m} l_i^{\deg_T(i)} \mathrm{Vol}(\Box E_T) \end{aligned}$$

as desired.

2. The parallelepipeds $\Box E_T$ are the images of the parallelepipeds $\Box A_T$ under the bijective linear map ϕ of (3.3), where

$$A_T := \{f_j - f_k \,:\, j < k,\, jk \text{ is an edge of } T\}.$$

Since the vector configuration $\{f_j - f_k \, : \, 1 \leq j < k \leq m\}$ is unimodular [61], all parallelepipeds $\Box A_T$ have unit volume. Therefore, the parallelepipeds $\Box E_T = \phi(\Box A_T)$ have the same normalized volume, so $\mathrm{Vol}(E_T)$ is independent of T.

It follows that we can use any tree T to compute $\mathrm{Vol}(E_T)$ or, equivalently, $\mathrm{Vol}(F_T)$. We choose the tree $T = \mathrm{Claw}_m$ with edges $1m, 2m, \ldots, (m-1)m$. Writing the $m-1$ vectors of

$$F_{\mathrm{Claw}_m} = \{l_m e_{\sigma_i} - l_i e_{\sigma_m} \, : \, 1 \leq i \leq m-1\}$$

as the columns of an $n \times (m-1)$ matrix, then $\mathrm{Vol}(F_{\mathrm{Claw}_m})$ is the greatest common divisor of the non-zero maximal minors of this matrix. This quantity does not change when we remove duplicate rows; the result is the $m \times (m-1)$ matrix

$$\begin{bmatrix} l_m & 0 & 0 & \cdots & 0 \\ 0 & l_m & 0 & \cdots & 0 \\ 0 & 0 & l_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & l_m \\ -l_1 & -l_2 & -l_3 & \cdots & -l_{m-1} \end{bmatrix}.$$

This matrix has m maximal minors, whose absolute values equal

$$l_m^{m-2} l_1, l_m^{m-2} l_2, \ldots l_m^{m-2} l_{m-1}, l_m^{m-1}.$$

Therefore,

$$\mathrm{Vol}(\Box F_{\mathrm{Claw}_m}) = l_m^{m-2} \gcd(l_1, \ldots, l_{m-1}, l_m)$$

and part 1 then implies that

$$\mathrm{Vol}(\Box E_{\mathrm{Claw}_m}) = \frac{\mathrm{Vol}(\Box F_{\mathrm{Claw}_m})}{l_1 \cdots l_{m-1} l_m^{m-1}} = \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m}$$

as desired. $\Box$

Lemma 3.1.9. *For any positive integer $m \geq 2$ and unknowns $x_1, \ldots, x_m$, we have*

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}.$$

Proof. We derive this from the analogous result for rooted trees [70, Theorem 5.3.4], which states that

$$\sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^{m} x_i^{\mathrm{children}_{(T,r)}(i)} = (x_1 + \cdots + x_m)^{m-1}$$

where $\mathrm{children}_{(T,r)}(v)$ counts the children of v; that is, the neighbors of v which are not on the unique path from v to the root r.

Notice that

$$\mathrm{children}_{(T,r)}(i) = \begin{cases} \deg_T(i) - 1 & \text{if } i \neq r, \\ \deg_T(i) & \text{if } i = r. \end{cases}$$

Therefore,

$$\begin{aligned} \sum_{\substack{(T,r) \text{ rooted} \\ \text{tree on } [m]}} \prod_{i=1}^{m} x_i^{\mathrm{children}_{(T,r)}(i)} &= \sum_{r=1}^{m} \left(\sum_{\substack{(T,r) \text{ tree on } [m] \\ \text{rooted at } r}} x_r \prod_{i=1}^{m} x_i^{\deg_T(i)-1} \right) \\ &= \left(\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} \right) (x_1 + \cdots + x_m) \end{aligned}$$

from which the desired result follows. □

Theorem 3.1.10. *If σ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the slice of Π_n fixed by σ is*

$$\operatorname{Vol} \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

Proof. Since Π_n^σ is a translate of the zonotope for the lattice vector configuration

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} \,:\, 1 \leq j < k \leq m \right\},$$

we invoke Theorem 3.1.6. Using Lemmas 3.1.7, 3.1.8, and 3.1.9, it follows that

$$\begin{aligned} \operatorname{Vol} \Pi_n^\sigma &= \sum_{T \text{ tree on } [m]} \operatorname{Vol}(\Box F_T) \\ &= \sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} l_i^{\deg_T(i)-1} \gcd(l_1, \ldots, l_m) \\ &= (l_1 + \cdots + l_m)^{m-2} \gcd(l_1, \ldots, l_m), \end{aligned}$$

as desired. □

When $\sigma = \mathrm{id}$ is the identity permutation, the fixed polytope is $\Pi_n^{\mathrm{id}} = \Pi_n$, and we recover Stanley's result that $\operatorname{Vol} \Pi_n = n^{n-2}$ [66].

Equivariant triangulations of the prism

Gel'fand, Kapranov, and Zelevinsky [29] introduced the *secondary polytope*, an $(n - d - 1)$-dimensional polytope $\Sigma(P)$ associated to a point configuration P of n points in dimension d. The vertices of $\Sigma(P)$ correspond to the *regular triangulations* of P, and more generally, the faces of $\Sigma(P)$ correspond to the *regular subdivisions* of P. Furthermore, face inclusion in $\Sigma(P)$ corresponds to refinement of subdivisions.

The permutahedron Π_n is the secondary polytope of the prism $\Delta_{n-1} \times I$ over the n-simplex. In fact, all subdivisions of the prism $\Delta_{n-1} \times I$ are regular, so the faces of the permutahedron are in order-preserving bijection with the ways of subdividing the prism $\Delta_{n-1} \times I$.

When the polytope P is invariant under the action of a group G, Reiner [58] introduced the *equivariant secondary polytope* $\Sigma^G(P)$, whose faces correspond to the G-invariant subdivisions of P. We call such a subdivision *fine* if it cannot be further refined into a G-invariant subdivision.

This equivariant framework applies to our setting, since a permutation $\sigma \in S_n$ acts naturally on the prism $\Delta_{n-1} \times I$ and on the permutahedron Π_n. The following is a direct consequence of [58, Theorem 2.10].

Proposition 3.1.11. *The fixed polytope Π_n^σ is the equivariant secondary polytope for the triangular prism $\Delta_{n-1} \times I$ under the action of σ.*

Thus, bearing in mind that the faces of the m-permutahedron are in order-preserving bijection with the ordered set partitions of $[m]$, our Theorem 3.1.5 has the following consequence.

Corollary 3.1.12. *The poset of σ-invariant subdivisions of the prism $\Delta_{n-1} \times I$ is isomorphic to the poset of ordered set partitions of $[m]$, where m is the number of cycles of σ. In particular, the number of finest σ-invariant subdivisions is $m!$.*

It is possible to describe the equivariant subdivisions of the prism combinatorially; we hope this will be a fun exercise for the interested reader.

Slices of Π_n fixed by subgroups of S_n

One might ask, more generally, for the subset of Π_n fixed by a subgroup of H in S_n; that is,

$$\Pi_n^H = \{x \in \Pi_n \,:\, \sigma \cdot x = x \text{ for all } \sigma \in H\}.$$

It turns out that this more general definition leads to the same family of fixed polytopes of Π_n.

Lemma 3.1.13. *For every subgroup H of S_n there is a permutation σ of S_n such that $\Pi_n^H = \Pi_n^\sigma$.*

Proof. Let $\{\sigma_1, \ldots, \sigma_r\}$ be a set of generators for H. Notice that a point $p \in \mathbb{R}^n$ is fixed by H if and only if it is fixed by each one of these generators. For each generator σ_t, the cycles of σ_t form a set partition π_t of $[n]$. Furthermore, a point $x \in \mathbb{R}^n$ is fixed by σ_t if and only if $x_j = x_k$ whenever j and k are in the same part of π_t.

Let $\pi = \pi_1 \vee \cdots \vee \pi_r$ in the lattice of partitions of $[n]$; the partition π is the finest common coarsening of $\pi_1, \ldots, \pi_r$. Then $x \in \mathbb{R}^n$ is fixed by each one of the generators of H if and only if $x_j = x_k$ whenever j and k are in the same part of π. Therefore, we may choose any permutation σ of $[n]$ whose cycles are supported on the parts of π, and we will have $\Pi_n^H = \Pi_n^\sigma$, as desired. □

Example 3.1.14. Consider the subset of Π_9 fixed by the subgroup

$$H = \langle (173)(46)(89), (27)(68) \rangle$$

of S_9. To be fixed by the two generators of H, a point $x \in \mathbb{R}^9$ must satisfy

$$\begin{aligned} \sigma_1 = (173)(46)(89): &\qquad x_1 = x_7 = x_3, \quad x_4 = x_6, \quad x_8 = x_9, \\ \sigma_2 = (27)(68): &\qquad x_2 = x_7, \quad x_6 = x_8, \end{aligned}$$

corresponding to the partitions $\pi_1 = 137|2|46|5|89$ and $\pi_2 = 1|27|3|4|5|68|9$. Combining these conditions gives

$$x_1 = x_2 = x_3 = x_7, \quad x_4 = x_6 = x_8 = x_9,$$

which corresponds to the join $\pi_1 \vee \pi_2 = 1237|4689|5$. For any permutation σ whose cycles are supported on the parts of $\pi_1 \vee \pi_2$, such as $\sigma = (1237)(4689)$, we have $\Pi_9^H = \Pi_9^\sigma$.

3.2 Equivariant Ehrhart Theory of the Permutahedron

Equivariant Ehrhart theory

This work fits into the framework of equivariant Ehrhart theory, as we now explain.

Let G be a finite group acting on $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a d-dimensional lattice polytope that is invariant under the action of G. Let M be the sublattice of $\mathbb{Z}^n$ obtained by translating the affine span of P to the origin, and consider the induced representation $\rho : G \to GL(M)$. We then obtain a family of permutation representations by looking at how ρ permutes the lattice points inside the dilations of P. Let $\chi_{tP} : G \to \mathbb{C}$ denote the permutation character associated to the action of G on the lattice points in the t^{th} dilate of P. We have

$$\chi_{tP}(g) = L_{P^g}(t)$$

where P^g is the polytope of points in P fixed by g and $L_{P^g}(t)$ is its lattice point enumerator.

The permutation characters χ_{tP} live in the ring $R(G)$ of virtual characters of G, which are the integer combinations of the irreducible characters of G. The positive integer combinations are called *effective*; they are the characters of representations of G.

Stapledon encoded the characters χ_{tP} in a power series $H^*[z] \in R(G)[[z]]$ given by

$$\sum_{t \geq 0} \chi_{tP}(g) z^t = \frac{H^*[z](g)}{(1-z)\det(I - g \cdot z)}. \tag{3.4}$$

We say that $H^*[z] =: \sum_{i \geq 0} H_i^* z^i$ is *effective* if each virtual character H_i^* is a character. Stapledon denoted this series $\varphi[t]$, but we denote it $H^*[z]$ and call it the *equivariant H^*-series* because for the identity element, $H^*[z](e) = h^*[z]$ is the well-studied h^*-polynomial of P.

The main open problem in equivariant Ehrhart theory is to characterize when $H^*[z]$ is effective, and Stapledon offered the following conjecture.

Conjecture 3.2.1 ([74, Effectiveness Conjecture 12.1])**.** *Let P be a lattice polytope fixed by the action of a group G. The following conditions are equivalent.*

1. *The toric variety of P admits a G-invariant non-degenerate hypersurface.*
2. *The equivariant H^*-series of P is effective.*
3. *The equivariant H^*-series of P is a polynomial.*

Examples

Example 3.2.2. Let us illustrate these results for the permutahedron Π_4 and the permutation $\sigma = (12)(3)(4)$ which has cycle type $\lambda = (2,1,1)$, illustrated in Figure 3.2. The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon, and one may verify manually that

$$L_{\Pi_4^{(12)}}(t) = \begin{cases} 4t^2+3t+1 & \text{if } t \text{ is even} \\ 4t^2+2t & \text{if } t \text{ is odd,} \end{cases} \qquad H^*[z](12) = 1+4z+11z^2-2z^3+\frac{4z^4}{1+z}.$$

Since the H^*-series of Π_4 is not polynomial when evaluated at (12), Stapledon's Conjecture 3.2.1 predicts that it is also not effective, and that the permutahedral variety X_{Π_4} does not admit an S_4-invariant non-degenerate hypersurface. We verify this in Section 3.2.

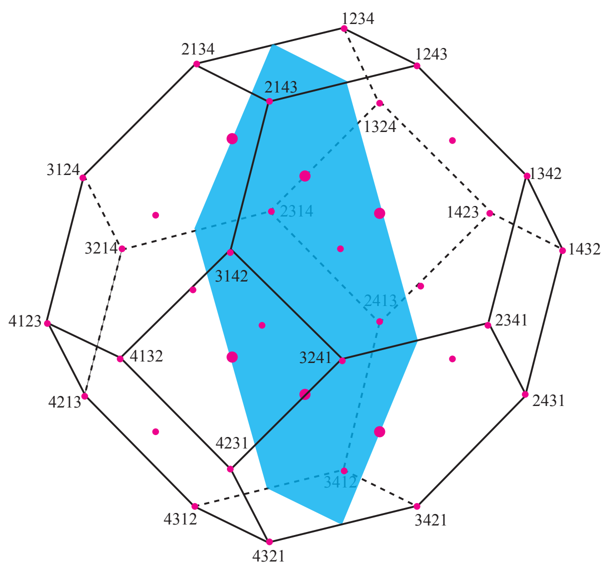

Figure 3.2: The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon containing 6 lattice points.

The equivariant Ehrhart quasipolynomials and H^*-series of Π_3 and Π_4 are shown in Tables 3.2 and 3.3.

Example 3.2.3. Further subtleties already arise in the simple case when Π_n^σ is a segment; this happens when σ has only two cycles of lengths ℓ_1 and ℓ_2. For even t, we simply have

$$L_{\Pi_n^\sigma}(t) = \gcd(\ell_1, \ell_2)t + 1.$$

However, for odd t we have

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \gcd(\ell_1, \ell_2)t + 1 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both odd,} \\ \gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ have different parity,} \\ \gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have the same 2-valuation,} \\ 0 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have different 2-valuations,} \end{cases}$$

Revisiting the fixed polytopes of the permutahedron

Recall from Theorem 3.1.5 that the fixed polytopes of the permutahedron Π_n^σ have the following zonotope description:

$$\Pi_n^\sigma = \sum_{1\le i<j\le m} [\ell_i \mathbf{e}_{\sigma_j}, \ell_j \mathbf{e}_{\sigma_i}] + \sum_{k=1}^{m} \frac{\ell_k + 1}{2} \mathbf{e}_{\sigma_k}. \tag{3.5}$$

Corollary 3.2.4. *The fixed polytope Π_n^σ is integral or half-integral. It is a lattice polytope if and only if all cycles of σ have odd length.*

Proof. From (3.5) and from the fact that all of the $\mathbf{e}_{\sigma_i}$ in (3.5) are linearly independent, we can see that all the vertices of Π_n^σ will be in the integer lattice if and only if $\ell_i + 1$ is even for all i. $\square$

Equation (3.5) also shows that Π_n^σ is a rational translation of the zonotope $Z(V)$ where

$$V = \{\ell_i \mathbf{e}_{\sigma_j} - \ell_j \mathbf{e}_{\sigma_i} : 1 \le i < j \le m\}.$$

Note that Lemma 3.1.7 characterizes the linearly linearly independent subsets of V, i.e., the linearly independent subsets of V are in bijection with forests with vertex set $[m]$, where the vector $\ell_i \mathbf{e}_{\sigma_j} - \ell_j \mathbf{e}_{\sigma_i}$ corresponds to the edge connecting vertices i and j.

In light of Lemma 3.1.7, the fixed polytope Π_n^σ gets subdivided into half-open parallelepipeds $\square_F$ of the form

$$\square_F = \sum_{\{i,j\}\in E(F)} [\ell_i \mathbf{e}_{\sigma_j}, \ell_j \mathbf{e}_{\sigma_i}) + \sum_{k=1}^{m} \frac{\ell_k + 1}{2} \mathbf{e}_{\sigma_k} + \mathbf{v}_F, \qquad \mathbf{v}_F \in \mathbb{Z}^n \tag{3.6}$$

for each forest F with vertex set $[m]$. When F is a tree T we have that

$$\mathrm{Vol}(\square_T) = \left(\prod_{i=1}^{m} \ell_i^{\deg_T(i)-1}\right) \gcd(\ell_1, \dots, \ell_m).$$

by Lemma 3.1.8. For a general forest F, the parallelepipeds $\square_T$ corresponding to each connected component T of F live in orthogonal subspaces, so

$$\mathrm{Vol}(\square_F) = \Big(\prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\Big)\Big(\prod_{\substack{\text{conn. comp.}\\ T \text{ of } F}} \gcd(\ell_j : j \in \mathrm{vert}(T))\Big). \tag{3.7}$$

The Ehrhart polynomial of the fixed polytope Π_n^σ: the lattice case

Suppose that $\lambda = (\ell_1, \ldots, \ell_m)$ is a partition of n into odd parts and that $\sigma \in S_n$ has cycle type λ. Then Corollary 3.2.4 says that Π_n^σ is a lattice zonotope, and hence we can use (2.1) to write a combinatorial expression for its Ehrhart polynomial.

Theorem 3.2.5. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, where ℓ_i is odd for all i. Then*

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|}$$

summing over all partitions $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ and

$$v_\pi := \prod_{i=1}^{k} \left(\gcd(\ell_j : j \in B_i) \cdot \Big(\sum_{j \in B_i} \ell_j\Big)^{|B_i|-2} \right). \tag{3.8}$$

Proof. Combining Theorem 2.2.16 with (3.7) gives us the following formula for the Ehrhart polynomial of Π_n^σ:

$$L_{\Pi_n^\sigma}(t) = \sum_{\substack{\text{Forests } F\\ \text{on } [m]}} \Big(\prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\Big) \cdot \Big(\prod_{\substack{\text{conn. comp.}\\ T \text{ of } F}} \gcd(\ell_j : j \in \mathrm{vert}(T))\Big) t^{|E(F)|}. \tag{3.9}$$

We can construct a forest with vertex set $[m]$ by first partitioning $[m]$ into nonempty sets $\{B_1, \ldots, B_k\}$ and then choosing a tree with vertex set B_j for each j. The number of edges in such a forest is $m - k$. Using these observations, we can rewrite (3.9) as

$$L_{\Pi_n^\sigma}(t) = \sum_{\{B_1,\ldots,B_k\} \vDash [m]} \Big(\prod_{i=1}^{k} \gcd(\ell_j : j \in B_i)\Big) \cdot \Big(\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\Big) t^{m-k}.$$

To complete the proof, it remains to show that for a given partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$, the following identity holds:

$$\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1} = \prod_{i=1}^{k} \Big(\sum_{j \in B_i} \ell_j\Big)^{|B_i|-2}. \tag{3.10}$$

This follows from the following identity, found in Lemma 3.1.9:

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_j^{\deg_T(j)-1} = (x_1 + \cdots + x_m)^{m-2}. \tag{3.11}$$

Using (3.11) we obtain

$$\begin{aligned}
\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1} &= \sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{i=1}^{k} \prod_{j\in B_i} \ell_j^{\deg_F(j)-1} \\
&= \prod_{i=1}^{k} \Bigg(\sum_{\substack{\text{trees } T\\ \text{on } B_i}} \prod_{j\in B_i} \ell_j^{\deg_T(j)-1} \Bigg) \\
&= \prod_{i=1}^{k} \Big(\sum_{j\in B_i} \ell_j \Big)^{|B_i|-2}
\end{aligned}$$

as desired. □

The Ehrhart quasipolynomial of the fixed polytope: the general case

In general, Π_n^σ is a half-integral polytope. This means that instead of an Ehrhart polynomial, it has an Ehrhart quasipolynomial with period at most 2. As in the lattice case from the previous subsection, we can decompose Π_n^σ into half-open parallelepipeds. However, there is a new feature that does not arise in the lattice case: some of the parallelepipeds in this decomposition may not contain any lattice points.

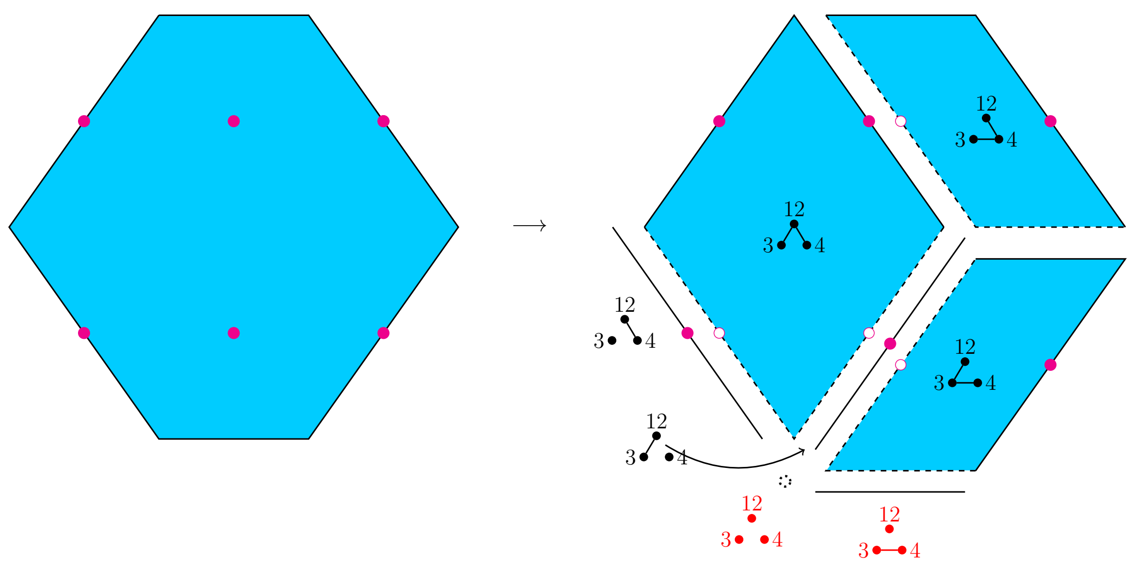

Figure 3.3: Decomposition of the fixed polytope $\Pi_4^{(12)}$ into half-open parallelepipeds.

Example 3.2.6. The fixed polytope $\Pi_4^{(12)}$ of Figure 3.2, which corresponds to the cycle type $\lambda = (2,1,1)$, is

$$\Pi_4^{(12)} = [2\mathbf{e}_3, \mathbf{e}_{12}] + [2\mathbf{e}_4, \mathbf{e}_{12}] + [\mathbf{e}_4, \mathbf{e}_3] + \frac{3}{2}\mathbf{e}_{12} + \mathbf{e}_3 + \mathbf{e}_4.$$

Figure 3.3 shows its decomposition into parallelograms indexed by the forests on vertex set $\{12, 3, 4\}$. The three trees give parallelograms with volumes $2, 1, 1$ that contain $2, 1, 1$ lattice points, respectively. The three forests with one edge give segments of volumes $1, 1, 1$ and $1, 1, 0$ lattice points, respectively. The empty forest gives a point of volume 1 and 0 lattice points. Hence, the Ehrhart quasipolynomial of $\Pi_4^{(12)}$ is

$$L_{\Pi_4^{(12)}}(t) = \begin{cases} (2+1+1)t^2 + (1+1+1)t + 1 & \text{if } t \text{ is even} \\ (2+1+1)t^2 + (1+1+0)t + 0 & \text{if } t \text{ is odd} \end{cases}.$$

Following the reasoning of Example 3.2.6, we will find the Ehrhart quasipolynomial of Π_n^σ by examining its decomposition into half-open parallelepipeds. In order to find the number of lattice points in each parallelepiped $\square_F$, the following observation is crucial.

Lemma 3.2.7. *[1, 49] If $\square$ is a lattice parallelepiped in $\mathbb{Z}^n$ and $\mathbf{v} \in \mathbb{Q}^n$, the number of lattice points in $\square + \mathbf{v}$ is*

$$|(\square + \mathbf{v}) \cap \mathbb{Z}^n| = \begin{cases} \mathrm{Vol}(\square) & \textit{if the affine span of } \square + \mathbf{v} \textit{ intersects the lattice } \mathbb{Z}^n \\ 0 & \textit{otherwise} \end{cases}.$$

We now apply Lemma 3.2.7 to the parallelepipeds $\square_F$. Surprisingly, whether $\mathrm{aff}(\square_F)$ contains lattice points does not depend on the forest F, but only on the set partition π of the vertex set $[m]$ induced by the connected components of F. To make this precise we need a definition. Recall that the 2-*valuation* of a positive integer is the largest power of 2 dividing that integer; for example, $\mathrm{val}(24) = 3$.

Definition 3.2.8. Let $\lambda = (\ell_1, \ldots, \ell_m)$ be a partition of the integer n. A set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is called λ-*compatible* if for each block $B_i \in \pi$, at least one of the following conditions holds:

1. ℓ_j is odd for some $j \in B_i$, or
2. the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ occurs an even number of times.

Example 3.2.9. Let $\lambda = (\ell_1, \ell_2, \ell_3)$ and $\mathrm{val}(\ell_i) = v_i$ for $i = 1, 2, 3$, and assume that $v_1 \geq v_2 \geq v_3$. Table 3.1 shows which partitions of [3] are λ-compatible depending on $\mathrm{val}(\lambda)$.

Lemma 3.2.10. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Let F be a forest on $[m]$ whose connected components induce the partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$. Then $\mathrm{aff}(\square_F)$ intersects the lattice $\mathbb{Z}^n$ if and only if π is λ-compatible.*

Proof. First we claim that

$$\mathrm{aff}(\square_F) = \left\{ \sum_{j=1}^m x_j \mathbf{e}_{\sigma_j} \; : \; \sum_{j \in B_i} \ell_j x_j = \sum_{j \in B_i} \frac{\ell_j(\ell_j + 1)}{2} \text{ for } 1 \leq i \leq k \right\}. \tag{3.12}$$

Table 3.1: λ-compatibility for $m = 3$.

	123	12|3	13|2	23|1	1|2|3
$v_1 = v_2 = v_3 = 0$	•	•	•	•	•
$v_1 = v_2 = v_3 > 0$					
$v_1 = v_2 > v_3 = 0$	•	•			
$v_1 = v_2 > v_3 > 0$					
$v_1 > v_2 = v_3 = 0$	•	•	•		
$v_1 > v_2 = v_3 > 0$	•				
$v_1 > v_2 > v_3 = 0$	•				
$v_1 > v_2 > v_3 > 0$					

Let $E(F)$ be the edge set of F. We have $\mathrm{aff}(\square_F) = \mathrm{Span}\{\ell_b \mathbf{e}_{\sigma_a} - \ell_a \mathbf{e}_{\sigma_b} : \{a,b\} \in E(F)\} + \sum_{a=1}^{m} \frac{1}{2}(\ell_a + 1)\mathbf{e}_{\sigma_a}$. A point $\mathbf{y} \in \mathrm{Span}\{\ell_b \mathbf{e}_{\sigma_a} - \ell_a \mathbf{e}_{\sigma_b} : \{a,b\} \in E(F)\}$ will satisfy $\sum_{j \in B_i} \ell_j y_j = 0$ for each block B_i. Furthermore, the translating vector $\mathbf{v} := \sum_{a=1}^{m} \frac{1}{2}(\ell_a + 1)\mathbf{e}_{\sigma_a}$ satisfies $\sum_{j \in B_i} \ell_j v_j = \sum_{j \in B_i} \frac{1}{2}\ell_j(\ell_j + 1)$ for each block B_i. Thus, every point $\mathbf{x}$ in the affine span of $\square_F$ satisfies the given equations. These are all the relations among the x_js because each block B_i contributes $|E(B_i)| = |B_i| - 1$ to the dimension of the affine span of $\square_F$.

This affine subspace intersects the lattice $\mathbb{Z}^n$ if and only if all equations in (3.12) have integer solutions. Elementary number theory tells us that this is the case if and only if each block B_i satisfies

$$\gcd(\ell_j : j \in B_i) \,\Big|\, \sum_{j \in B_i} \frac{\ell_j(\ell_j + 1)}{2}. \tag{3.13}$$

It is always true that $\gcd(\ell_j : j \in B_i)$ divides $\displaystyle\sum_{j \in B_i} \ell_j(\ell_j + 1)$, so (3.13) holds if and only if

$$\mathrm{val}\Big(\gcd(\ell_j : j \in B_i)\Big) < \mathrm{val}\Big(\sum_{j \in B_i} \ell_j(\ell_j + 1)\Big). \tag{3.14}$$

We consider two cases.

(i) Suppose ℓ_j is odd for some $j \in B_i$. Then $\gcd(\ell_j : j \in B_i)$ is odd, whereas $\sum_{j \in B_i} \ell_j(\ell_j + 1)$ is always even. Hence, (3.14) always holds in this case.

(ii) Suppose that ℓ_j is even for all $j \in B_i$. For each ℓ_j, write $\ell_j = 2^{p_j} q_j$ for some integer $p_j \geq 1$ and odd integer q_j. Then $\mathrm{val}(\gcd(\ell_j : j \in B_i)) = \min_{j \in B_i} p_j$; we will call this integer p. We have

$$\begin{aligned} \mathrm{val}\Big(\sum_{j \in B_i} \ell_j(\ell_j + 1)\Big) &= \mathrm{val}\Big(\sum_{j \in B_i} 2^{p_j} q_j(\ell_j + 1)\Big) \\ &= p + \mathrm{val}\Big(\sum_{j \in B_i} 2^{p_j - p} q_j(\ell_j + 1)\Big). \end{aligned}$$

Note that $q_j(\ell_j+1)$ is odd for each j. If the minimum 2-valuation p of $\{\ell_j : j \in B_i\}$ occurs an odd number of times, then $\sum_{j\in B_i} 2^{p_j-p} q_j(\ell_j+1)$ will be odd and we will have $\mathrm{val}(\sum_{j\in B_i} \ell_j(\ell_j+1)) = p$. Otherwise, this sum will be even and we will have $\mathrm{val}(\sum_{j\in B_i} \ell_j(\ell_j+1)) > p$. Therefore, (3.14) holds if and only if the minimum 2-valuation among the ℓ_j for $j \in B_i$ occurs an even number of times. This is precisely the condition of λ-compatibility. $\square$

We now have all of the necessary tools to compute the Ehrhart quasipolynomial of the fixed polytope Π_n^σ. Recall the definition of λ-compatibility in Definition 3.2.8 and the definition of v_π in (3.8).

Theorem 3.2.11. *Let σ be a permutation of $[n]$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Then the Ehrhart quasipolynomial of the fixed polytope of the permutahedron Π_n fixed by σ is*

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \displaystyle\sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\ \displaystyle\sum_{\substack{\pi \vDash [m] \\ \lambda-compatible}} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd} \end{cases}.$$

Proof. We calculate the number of lattice points in each integer dilate $t\Pi_n^\sigma$ by decomposing it into half-open parallelepipeds and adding up the number of lattice points inside of each parallelepiped.

First, suppose that t is even. Then $t\Pi_n^\sigma$ is a lattice polytope, all parallelepipeds in the decomposition of $t\Pi_n^\sigma$ have vertices on the integer lattice, and each i-dimensional parallelepiped $\square$ contains $\mathrm{Vol}(\square)t^i$ lattice points [9, Lemma 9.2]. The parallelepipeds correspond to linearly independent subsets of the vector configuration $\{\ell_i \mathbf{e}_{\sigma_j} - \ell_j \mathbf{e}_{\sigma_i} : 1 \le i < j \le m\}$, which is in bijection with forests on $[m]$. Following the reasoning used to prove Theorem 3.2.5, we conclude that when t is even,

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|}.$$

Next, suppose t is odd. Then $t\Pi_n^\sigma$ is half-integral, but it may not be a lattice polytope. As before, we may decompose $t\Pi_n^\sigma$ into half-open parallelepipeds that are in bijection with forests on $[m]$. Lemma 3.2.7, Lemma 3.2.10, and [9, Lemma 9.2] tell us that $\square_F$ contains $\mathrm{Vol}(\square_F)t^{m-|\pi|}$ lattice points if the set partition π induced by F is λ-compatible, and 0 otherwise. Therefore if t is odd

$$L_{\Pi_n^\sigma}(t) = \sum_{\substack{\pi \vDash [m] \\ \lambda-\text{compatible}}} v_\pi \cdot t^{m-|\pi|}$$

as desired. $\square$

The equivariant H^*-series of the permutahedron

We now compute the equivariant H^*-series of the permutahedron and characterize when it is polynomial and when it is effective, proving Stapledon's Effectiveness Conjecture 3.2.1 in this special case.

The *Ehrhart series* of a rational polytope P is

$$\mathrm{Ehr}_P(z) = 1 + \sum_{t=1}^{\infty} L_P(t) \cdot z^t.$$

In computing the Ehrhart series of Π_n^σ, Eulerian polynomials naturally arise. The *Eulerian polynomial* $A_k(z)$ is defined by the identity

$$\sum_{t \geq 0} t^k z^t = \frac{A_k(z)}{(1-z)^{k+1}}.$$

Proposition 3.2.12. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The Ehrhart series of Π_n^σ is*

$$\mathrm{Ehr}_{\Pi_n^\sigma}(z) = \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} \frac{v_\pi \cdot A_{m-|\pi|}(z)}{(1-z)^{m-|\pi|+1}} + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} \frac{v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2)}{(1-z^2)^{m-|\pi|+1}}$$

and the H^*-series of the permutahedron equals

$$H^*[z](\sigma) = \left(\prod_{i=1}^{m} (1 - z^{\ell_i}) \right) \cdot \mathrm{Ehr}_{\Pi_n^\sigma}(z)$$

Proof. The first statement follows readily from Theorem 3.2.11:

$$\begin{aligned}
\mathrm{Ehr}_{\Pi_n^\sigma}(z) &= \sum_{t \text{ even}} \left(\sum_{\pi \vDash [m]} v_\pi t^{m-|\pi|} \right) z^t + \sum_{t \text{ odd}} \left(\sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi t^{m-|\pi|} \right) z^t \\
&= \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi \left(\sum_{t=0}^{\infty} t^{m-|\pi|} z^t \right) + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} v_\pi \left(\sum_{t \text{ even}} t^{m-|\pi|} z^t \right) \\
&= \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi \frac{A_{m-|\pi|}(z)}{(1-z)^{m-|\pi|+1}} + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} v_\pi \cdot 2^{m-|\pi|} \frac{A_{m-|\pi|}(z^2)}{(1-z^2)^{m-|\pi|+1}}
\end{aligned}$$

For the second statement, recall that $H^*[z]$ is defined as in (3.4), where ρ is the standard representation of S_n in this case. The left hand side is the Ehrhart series. The denominator on the right side is $(1-z)\det(I - \rho(\sigma) \cdot z)$; it equals the characteristic polynomial of the permutation matrix of σ, which is $\prod_{i=1}^{m}(1 - z^{\ell_i})$. □

Table 3.2: The equivariant H^*-series of Π_3

Cycle type of $\sigma \in S_3$	$\chi_{t\Pi_3}(\sigma)$	$\sum_{t\geq 0} \chi_{t\Pi_3}(\sigma)z^t$	$\phi[z](\sigma)$
$(1,1,1)$	$3t^2+3t+1$	$\frac{1+4z+z^2}{(1-z)^3}$	$1+4z+z^2$
$(2,1)$	$\begin{cases} t+1 & \text{if } t \text{ is even} \\ t & \text{if } t \text{ is odd} \end{cases}$	$\frac{1+z^2}{(1-z)(1-z^2)}$	$1+z^2$
(3)	1	$\frac{1}{1-z} = \frac{1+z+z^2}{1-z^3}$	$1+z+z^2$

Table 3.3: The equivariant H^*-series of Π_4

Cycle type of $\sigma \in S_4$	$\chi_{t\Pi_4}(\sigma)$	$\sum_{t\geq 0} \chi_{t\Pi_4}(\sigma)z^t$	$\phi[z](\sigma)$
$(1,1,1,1)$	$16t^3+15t^2+6t+1$	$\frac{1+34z+55z^2+6z^3}{(1-z)^4}$	$1+34z+55z^2+6z^3$
$(2,1,1)$	$\begin{cases} 4t^2+3t+1 & \text{if } t \text{ is even} \\ 4t^2+2t & \text{if } t \text{ is odd} \end{cases}$	$\frac{1+6z+20z^2+24z^3+11z^4+2z^5}{(1-z)^2(1-z^2)(1+z)^2}$	$1+4z+11z^2-2z^3+\sum_{i=4}^{\infty} 4(-1)^i z^i$
$(3,1)$	$t+1$	$\frac{1}{(1-z)^2} = \frac{1+z+z^2}{(1-z)(1-z^3)}$	$1+z+z^2$
(4)	$\begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$	$\frac{1}{1-z^2} = \frac{1+z^2}{1-z^4}$	$1+z^2$
$(2,2)$	$\begin{cases} 2t+1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd} \end{cases}$	$\frac{1+2z+3z^2+2z^3}{(1-z^2)^2}$	$1+2z+3z^2+2z^3$

Tables 3.2 and 3.3 show the equivariant H^*-series of the permutahedra Π_3 and Π_4.

Stapledon writes, *"The main open problem is to characterize when $H^*[z]$ is effective"*, and he conjectures the following characterization:

Conjecture 3.2.13 (Effectiveness Conjecture, [74])**.** *Let P be a lattice polytope invariant under the action of a group G. The following conditions are equivalent.*

1. *The toric variety of P admits a G-invariant non-degenerate hypersurface.*
2. *The equivariant H^*-series of P is effective.*
3. *The equivariant H^*-series of P is a polynomial.*

He shows that (1) $\implies$ (2) $\implies$ (3), so only the reverse implications are conjectured. Our next goal is to verify Stapledon's conjecture for the action of S_n on the permutahedron Π_n. We do so by showing that the conditions of Conjecture 3.2.1 hold if and only if $n \leq 3$.

Polynomiality of $H^*[z]$

Lemma 3.2.14. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The equivariant H^*-series evaluated at σ, $H^*[z](\sigma)$, is a polynomial if and only if the number of even parts in λ is 0, $m-1$, or m.*

Proof. By Proposition 3.2.12, the Ehrhart series $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ may only have poles at $z = \pm 1$. The pole at $z = 1$ has order at most m. Since the polynomial $\prod_{i=1}^m (1 - z^{\ell_i})$ has a zero at $z = 1$ of order m, the series $H^*[z](\sigma)$ will not have a pole at $z = 1$. Hence we only need to check whether $H^*[z](\sigma)$ has a pole at $z = -1$.

(i) First, suppose no ℓ_i is even. Then all partitions of $[m]$ are λ-compatible, so $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ does not have a pole at $z = -1$. Thus $H^*[z](\sigma)$ is a polynomial in this case.

(ii) Next, suppose that some ℓ_i is even. Then the partition $\{\{\ell_i\}, [m] - \{\ell_i\}\}$ is λ-incompatible, so $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ does have a pole at $z = -1$. It is well known that $A_k(1) = k!$ so every numerator $v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2)$ is positive at $z = -1$. It follows that the order of the pole $z = -1$ of $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ is $m - d + 1$ where $d = \min\{|\pi| : \pi \text{ is } \lambda\text{-incompatible}\}$. This equals $m - 1$ if the partition $\{[m]\}$ is λ-compatible and m if it is λ-incompatible.

On the other hand, $\prod_{i=1}^m (1 - z^{\ell_i})$ has a zero at $z = -1$ of order equal to the number of even ℓ_i. Now consider three cases:

a) If the number of even ℓ_i is between 1 and $m - 2$, it is less than the the order of the pole of $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$, so $H^*[z](\sigma)$ is not polynomial.

b) If all ℓ_i are even, the zero at $z = -1$ in $\prod_{i=1}^m (1 - z^{\ell_i})$ has order m and cancels the pole in $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$. Thus $H^*[z](\sigma)$ is polynomial.

c) If $m-1$ of the ℓ_i are even, the partition $\{[m]\}$ is λ-compatible. Therefore the order of the pole in $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ and the order of the zero in $\prod_{i=1}^m (1 - z^{\ell_i})$ both equal $m - 1$, and $H^*[z](\sigma)$ is polynomial. $\square$

Proposition 3.2.15. *The equivariant H^*-series of the permutahedron Π_n is a polynomial if and only if $n \leq 3$.*

Proof. When $n \leq 3$, all partitions of n have 0, 1, or all odd parts. Hence $H^*[z](\sigma)$ is a polynomial for all $\sigma \in S_n$, so $H^*[z]$ is a polynomial.

Suppose $n \geq 4$. Then there always exists some partition of n with more than 1 but fewer than all odd parts: if n is even we can take the partition $(n-2, 1, 1)$, and if n is odd we can take the partition $(n-3, 1, 1, 1)$. Therefore $H^*[z]$ is not polynomial. $\square$

Effectiveness of $H^*[z]$

Proposition 3.2.16. *The equivariant H^*-series of the permutahedron Π_n is effective if and only if $n \leq 3$.*

Proof. Stapledon [74] observed that if H^* is effective then it is polynomial. Thus by Proposition 3.2.15 we only need to check effectiveness for $n = 1, 2, 3$.

Let us check it for $n = 3$. Table 3.2 shows that $H^*[z] = H_0^* + H_1^* z + H_2^* z^2$ for $H_0^*, H_1^*, H_2^* \in R(S_3)$. Comparing these with the character table of S_3 (see for example [28, pg.14]) gives

$$H_0^* = \chi_{triv}, \qquad H_1^* = \chi_{triv} + \chi_{alt} + \chi_{std}, \qquad H_2^* = \chi_{triv}.$$

Since all coefficients are nonnegative,

$$H_{\Pi_3}^*[z] = \chi_{triv} + (\chi_{triv} + \chi_{alt} + \chi_{std})z + \chi_{triv} z^2$$

is indeed effective.

Similarly, $H_{\Pi_2}^*[z] = \chi_{triv}$ and $H_{\Pi_1}^*[z] = \chi_{triv}$ are effective as well. $\square$

In contrast, a similar computation based on Table 3.3 gives

$$\begin{aligned} H_{\Pi_4}^* &= \chi_{triv} + (3\chi_{triv} + \chi_{alt} + 5\chi_{std} + 3\chi_{\ydiagram{2,1,1}} + 3\chi_{\ydiagram{2,2}})z \\ &+ (6\chi_{triv} + 9\chi_{std} + 4\chi_{\ydiagram{2,1,1}} + 5\chi_{\ydiagram{2,2}})z^2 + (\chi_{alt} + \chi_{\ydiagram{2,1,1}} + \chi_{\ydiagram{2,2}})z^3 \\ &+ (\chi_{triv} - \chi_{alt} + \chi_{std} - \chi_{\ydiagram{2,1,1}})(z^4 - z^5 + z^6 - z^7 + \cdots) \end{aligned}$$

which is not effective.

S_n-invariant non-degenerate hypersurfaces in the permutahedral variety

We begin by explaining condition (1) of Conjecture 3.2.1, which arises from Khovanskii's notion of non-degeneracy [44]. We refer the reader to [74, Section 7] for more details.

Let $P \subset \mathbb{R}^n$ be a lattice polytope with an action of a finite group G. For $\mathbf{v} \in \mathbb{Z}^n$ we write $x^{\mathbf{v}} := x_1^{v_1} \cdot \ldots \cdot x_n^{v_n}$. The coordinate ring of the projective toric variety X_P of P has the form $\mathbb{C}[x^{\mathbf{v}} : \mathbf{v} \in P \cap \mathbb{Z}^n]$, so a hypersurface in X_P is given by a linear equation $\sum_{\mathbf{v} \in P \cap \mathbb{Z}^n} a_{\mathbf{v}} x^{\mathbf{v}} = 0$ for some complex coefficients $a_{\mathbf{v}}$. The group G acts on the monomials $x^{\mathbf{v}}$ by its action on the lattice points $\mathbf{v} \in P \cap \mathbb{Z}^n$, so the equation of a G-invariant hypersurface should have $a_{\mathbf{v}} = a_{\mathbf{u}}$ whenever $\mathbf{u}$ and $\mathbf{v}$ are in the same G-orbit. A projective hypersurface in X_P with equation $f(x_1, \ldots, x_n) = 0$ is *smooth* if the gradient $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is never zero when $(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$. There is a unique polynomial in the $a_{\mathbf{v}}$'s, called the *discriminant*, such that the hypersurface is smooth when the discriminant does not vanish at the coefficients $a_{\mathbf{v}}$. A hypersurface in the toric variety of P is *non-degenerate* if it is smooth and for each face F of P, the hypersurface $\sum_{\mathbf{v} \in F \cap \mathbb{Z}^n} a_{\mathbf{v}} x^{\mathbf{v}} = 0$ is also smooth.

The *permutahedral variety* X_{Π_n} is the projective toric variety associated to the permutahedron Π_n.

Proposition 3.2.17. *The permutahedral variety X_{Π_n} admits an S_n-invariant non-degenerate hypersurface if and only if $n \leq 3$.*

Proof. Stapledon proved [75, Theorem 7.7] that if X_{Π_n} admits such a hypersurface, then $H^*[z]$ is effective. By Proposition 3.2.16, this can only occur for $n = 1, 2, 3$.

Case 1: $n = 1$.

A hypersurface in the toric variety of $\Pi_1 = \{1\} \subset \mathbb{R}$ has the form $ax = 0$, and since we are working over projective space, we can assume $a = 1$. The derivative of this never vanishes, so this is a smooth S_1-invariant hypersurface.

Case 2: $n = 2$.

The permutahedron Π_2 is the line segment with vertices $(1, 2), (2, 1) \in \mathbb{R}^2$ and no other lattice points. The vertices are in the same S_2-orbit, so we need to check that hypersurface with equation $xy^2 + x^2y = 0$ is non-degenerate. The gradient is $(y(y + 2x), x(2y + x))$, which never vanishes on $(\mathbb{C}^*)^2$. The vertex $(1, 2)$ corresponds to the hypersurface $xy^2 = 0$. The gradient of this is $(y^2, 2xy)$ which also never vanishes on $(\mathbb{C}^*)^2$. The computation for the other vertex is similar. Hence this is an S_2-invariant non-degenerate hypersurface.

Case 3: $n = 3$.

The permutahedron Π_3 is a hexagon with one interior point. Choosing the vertices to be all permutations of the point $(0, 1, 2) \in \mathbb{R}^3$ (instead of $(1, 2, 3)$) will simplify calculations. The six vertices of the hexagon are one S_3-orbit and the interior point is its own orbit. Hence (up to scaling) an S_3-invariant hypersurface must have the equation

$$a \cdot xyz + yz^2 + y^2z + xy^2 + x^2y + xz^2 + x^2z = 0 \tag{3.15}$$

which has one parameter a. We want to check whether there exists some choice of a for which this hypersurface is non-degenerate. We need to check this on each face.

The vertex $(0, 1, 2)$ gives the hypersurface $yz^2 = 0$ with gradient $(0, z^2, 2yz)$. This never vanishes on $(\mathbb{C}^*)^3$, so it is smooth. The computations for the other five vertices are similar.

For the edge connecting $(0, 1, 2)$ and $(0, 2, 1)$, the corresponding hypersurface is $yz^2 + y^2z = 0$. This is the same hypersurface as the line segment Π_2, so it is smooth; so are the hypersurfaces of the other five edges.

Finally, we need to show there exists a such that the entire hypersurface is smooth. This is the same as showing that the discriminant of (3.15) is not identically zero. Since (3.15) is a symmetric polynomial, we can write in terms of the power-sum symmetric polynomials, $p_k = x^k + y^k + z^k$; we obtain

$$\frac{a}{6}p_1^3 + \left(1 - \frac{a}{2}\right) p_1p_2 + \left(\frac{a}{3} - 1\right) p_3 = 0. \tag{3.16}$$

The discriminant of a degree 3 symmetric polynomial is given in [**?**, Equation 64]; substituting the coefficients $a/6$, $1 - a/2$, and $a/3 - 1$ gives a non-zero polynomial of degree 12:

$$\begin{gathered}
\frac{-512000}{16677181699666569}a^{12} + \frac{492800}{617673396283947}a^{10} - \frac{985600}{617673396283947}a^9 + \frac{6320}{7625597484987}a^8 \\
- \frac{25280}{7625597484987}a^7 + \frac{27431}{7625597484987}a^6 - \frac{478}{282429536481}a^5 + \frac{965}{282429536481}a^4 \\
- \frac{2128}{847288609443}a^3 + \frac{8}{10460353203}a^2 - \frac{32}{31381059609}a + \frac{16}{31381059609}
\end{gathered}$$

Any value of a that is not a root of this discriminant gives us an S_3-invariant non-degenerate hypersurface. $\square$

By contrast, we should not be able to find an S_n-invariant non-degenerate hypersurface in X_{Π_n} for $n \geq 4$. This can be seen from the fact that all permutahedra Π_n when $n \geq 4$ have a square face, and the hypersurface of this square face is not smooth. For example, consider the square face of Π_4 with vertices $(0,1,2,3)$, $(0,1,3,2)$, $(1,0,3,2)$, and $(1,0,2,3)$. The corresponding hypersurface is $yz^2w^3 + yz^3w^2 + xz^3w^2 + xz^2w^3 = 0$, and its gradient vanishes whenever $x = -y$ and $z = -w$.

Stapledon's Conjectures

Our second main result now follows as a corollary.

Theorem 3.2.18. *Stapledon's Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.*

Proof. This follows immediately from Propositions 3.2.15, 3.2.16, and 3.2.17. □

In closing, we verify the remaining three conjectures of Stapledon for the special case of the S_n–action on the permutahedron Π_n.

Conjecture 3.2.19. *[74, Conjecture 12.2] If $H^*[z]$ is effective, then $H^*[1]$ is a permutation representation.*

Conjecture 3.2.20. *[74, Conjecture 12.3] For a polytope $P \subset \mathbb{R}^n$, let $\operatorname{ind}(P)$ be the smallest positive integer k such that the affine span of kP contains a lattice point. For any $g \in G$, let M^g be the sublattice of M fixed by g, and define $\det(I - \rho(g))_{(M^g)^\perp}$ to be the determinant of $I - \rho(g)$ when the action of $\rho(g)$ is restricted to $(M^g)^\perp$. The quantity*

$$H^*[1](g) = \frac{\dim(P^g)! \cdot \operatorname{vol}(P^g) \cdot \det(I - \rho(g))_{(M^g)^\perp}}{\operatorname{ind}(P^g)}$$

is a non-negative integer.

Conjecture 3.2.21. *[74, Conjecture 12.4] If $H^*[z]$ is a polynomial and the i^{th} coefficient of the h^*-polynomial of P is positive, then the trivial representation occurs with non-zero multiplicity in the virtual character H^*_i.*

Proposition 3.2.22. *Conjectures 3.2.19, 3.2.20, and 3.2.21 hold for permutahedra under the action of the symmetric group.*

Proof. 3.2.19: This statement only applies to Π_1, Π_2, and Π_3. From the proof of Proposition 3.2.16 we obtain that $H^*[1]$ is the trivial character for Π_1 and Π_2 and the statement holds. For Π_3 we have

$$H^*[1] = 3\chi_{triv} + \chi_{alt} + \chi_{std} = \chi_{triv} + (\chi_{triv} + \chi_{alt}) + (\chi_{triv} + \chi_{std}). \tag{3.17}$$

Now $\chi_{triv} + \chi_{alt}$ is the permutation character of the sign action of S_3 on the set $[2]$, and $\chi_{triv} + \chi_{std}$ is the character of the permutation representation of S_3. Hence all summands on the right side of (3.17) are permutation characters, so their sum is as well.

3.2.20: For $\sigma \in S_n$ of cycle type $\lambda = (\ell_1, \dots, \ell_m)$, the dimension of Π_n^σ is $m-1$ and the volume is $n^{m-2}\gcd(\ell_1, \dots, \ell_m)$. Now, the fixed lattice $M^g = \mathbb{Z}\{\mathbf{e}_{\sigma_1}, \cdots, \mathbf{e}_{\sigma_m}\}$ has rank m, so

$$\det(I - \rho(\sigma) \cdot z)_{(M^\sigma)^\perp} = \frac{(1-z)\det(I - \rho(\sigma)\cdot z)}{(1-z)^m} = \prod_{i=1}^{m}(1 + z + \cdots + z^{\ell_i - 1}).$$

Therefore the numerator is $(m-1)! \cdot n^{m-2} \cdot \gcd(\ell_1, \dots, \ell_m) \cdot \ell_1 \cdots \ell_m$. The denominator is

$$\operatorname{ind}(\Pi_n^\sigma) = \begin{cases} 2 & \text{if all } \pi \vDash [m] \text{ are } \lambda\text{-incompatible,} \\ 1 & \text{otherwise.} \end{cases}$$

When the denominator is 2, all the ℓ_i must be even, so the numerator is even. The desired result follows.

3.2.21: We need to check this for Π_1, Π_2, and Π_3. For Π_1 and Π_2 the h^*-polynomial is 1 and $H_0^* = \chi_{triv}$. For Π_3, the h^*-polynomial is $1 + 4z + z^2$, and $\phi_0 = \chi_{triv}$, $\phi_1 = \chi_{triv} + \chi_{alt} + \chi_{std}$, and $\phi_2 = \chi_{triv}$ all contain a copy of the trivial character. $\square$

Conclusion and Further Directions

Unfortunately, Stapledon's conjecture does not hold in general. Francisco Stantos and Alan Stapledon communicated to us a counterexample. Considering P to be the unit 3-cube $[0,1]^3$ with a group G isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ that contains the identity, the two rotations of order two with horizontal diagonal axes, and the rotation of order two with vertical axis suggests that the implication from (2) $\implies$ (1) in the conjecture fail (F. Santos and A. Stapledon, 2019, personal communication). This does not mean all is lost, it is a matter of reformulating the question and related problems. This conjecture is significant since it would allow for the study of symmetries of a polytope from three different perspectives: the algebro-geometric through toric varieties and hypersurfaces, the representation theoretic through the effectiveness of irreducible characters, and the discrete geometrical through generating polynomials. We conclude this chapter with some future directions.

Problem 3.2.23. Characterize P and G such that the implications (3) $\implies$ (2) $\implies$ (1) hold or fail.

To solve Problem 3.2.23, we can first investigate the following questions.

Question 3.2.24. Does the Effectiveness Conjecture hold when $G = S_n$, but P is allowed to vary?

Question 3.2.25. Are there examples (or counterexamples) that arise from subgroups of S_n acting on the permutahedron that show that Stapledon's Effectiveness Conjecture holds (or is false)?

It would be interesting to see if a counterexample exists where the the equivariant H^*-series of P is a polynomial, but implications (3) $\implies$ (2) and (2) $\implies$ (1) in the Effectiveness Conjecture fail.

Chapter 4 Decompositions of the Ehrhart h^*-polynomial for Rational Polytopes

Introduction

In this chapter we are going to study decompositions of the h^*-polynomial, which is the numerator of the rational generating function introduced before as the Ehrhart series. Recall that the *Ehrhart series* is the rational generating function

$$\mathrm{Ehr}(P;z) := \sum_{t\geq 0} L(P;t)\,z^t = \frac{h^*(P;z)}{(1-z^q)^{d+1}}\,,$$

where $h^*(P;z)$ is a polynomial of degree less than $q(d+1)$, the h^**-polynomial* of P. Note that the h^*-polynomial depends not only on q (though that is implicitly determined by P), but also on our choice of representing the rational function $\mathrm{Ehr}(P;z)$, which in our form will not be in lowest terms.

Our first main contributions are generalizations of two well-known decomposition formulas of the h^*-polynomial for lattice polytopes due to Betke–McMullen [11] and Stapledon [73]. (All undefined terms are specified in the sections below.)

Theorem 4.0.1. *For a triangulation T with denominator q of a rational d-polytope P,*

$$\mathrm{Ehr}(P;z) = \frac{\sum_{\Omega\in T} B(\Omega;z)\,h(\Omega;z^q)}{(1-z^q)^{d+1}}\,.$$

Theorem 4.0.2. *Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a}\in\mathbb{Z}^d$ and $\ell\in\mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q. Then*

$$h^*(P;z) = \frac{1-z^q}{1-z^\ell}\sum_{\Omega\in T}\left(B(\Omega;z)+B(\Omega';z)\right)h(\Omega;z^q)\,.$$

Our second main result is a generalization of inequalities provided by Hibi [34] and Stanley [65] that are satisfied by the coefficients of the h^*-polynomial for lattice polytopes.

Theorem 4.0.3. *Let P be a rational d-polytope with denominator q and let $s := \deg h^*(P;z)$. The h^*-vector $(h_0^*,\dots,h_{q(d+1)-1}^*)$ of P satisfies the following inequalities:*

$$h_0^*+\cdots+h_{i+1}^* \geq h_{q(d+1)-1}^*+\cdots+h_{q(d+1)-1-i}^*\,, \qquad i=0,\dots,\left\lfloor\frac{q(d+1)-1}{2}\right\rfloor-1\,, \tag{4.1}$$

$$h_s^*+\cdots+h_{s-i}^* \geq h_0^*+\cdots+h_i^*\,, \qquad i=0,\dots,q(d+1)-1\,. \tag{4.2}$$

Inequality (4.1) is a generalization of a theorem by Hibi [34] for lattice polytopes, and (4.2) generalizes an inequality given by Stanley [65] for lattice polytopes, namely the case when $q = 1$. Both inequalities follow from the *a/b-decomposition* of the $\overline{h^*}$-polynomial for rational polytopes given in Theorem 4.2.7 in Section 4.2, which in turn generalizes results (and uses rational analogues of techniques) by Stapledon [73]. Stapledon's a/b-decomposition has been used by different authors to study connections to unimodality, dilated polytopes, open polytopes, order polytopes, and connections to chromatic polynomials [8, 40, 41, 45].

Set-Up and Notation

A *pointed simplicial cone* is a set of the form

$$K(\mathbf{W}) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : \lambda_i \geq 0 \right\},$$

where $\mathbf{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a set of n linearly independent vectors in $\mathbb{R}^d$. If we can choose $\mathbf{w}_i \in \mathbb{Z}^d$ then $K(\mathbf{W})$ is a *rational* cone and we assume this throughout this paper. Define the *open parallelepiped* associated with $K(\mathbf{W})$ as

$$\mathrm{Box}\,(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 < \lambda_i < 1 \right\}. \tag{4.3}$$

Observe that we have the natural involution $\iota : \mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d \to \mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d$ given by

$$\iota\left(\sum_i \lambda_i \mathbf{w}_i \right) := \sum_i (1 - \lambda_i) \mathbf{w}_i \,. \tag{4.4}$$

We set $\mathrm{Box}\,(\{0\}) := \{0\}$.

Let $u : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the last coordinate. We then define the *box polynomial* as

$$B(\mathbf{W}; z) := \sum_{\mathbf{v} \in \mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^d} z^{u(\mathbf{v})}. \tag{4.5}$$

If $\mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^d = \emptyset$, then we set $B(\mathbf{W}; z) = 0$. We also define $B(\emptyset; z) = 1$.

Example 4.0.4. Let $\mathbf{W} = \{(1,3), (2,3)\}$. Then

$$\mathrm{Box}\,(\mathbf{W}) = \{\lambda_1(1,3) + \lambda_2(2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$

Thus $\mathrm{Box}\,(\mathbf{W}) \cap \mathbb{Z}^2 = \{(1,2), (2,4)\}$ and its associated box polynomial is

$$B(\mathbf{W}; z) = z^2 + z^4.$$

Lemma 4.0.5. $B(\mathbf{W}; z) = z^{\sum_i u(\mathbf{w}_i)} B\left(\mathbf{W}; \frac{1}{z}\right)$.

Proof. Using the involution ι,

$$z^{\sum_i u(\mathbf{w}_i)} B\left(\mathbf{W}; \frac{1}{z}\right) = \sum_{\mathbf{v} \in \mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^d} z^{\sum_i u(\mathbf{w}_i) - u(\mathbf{v})} = \sum_{\mathbf{v} \in \mathrm{Box}(\mathbf{W}) \cap \mathbb{Z}^d} z^{u(\iota(\mathbf{v}))} = B(\mathbf{W}; z)\,. \square$$

Next, we define the *fundamental parallelepiped* $\mathcal{P}(\mathbf{W})$ to be a half-open variant of $\mathrm{Box}\,(\mathbf{W})$, namely,

$$\mathcal{P}(\mathbf{W}) := \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{w}_i : 0 \leq \lambda_i < 1 \right\}.$$

We also want to cone over a polytope P. If $P \subset \mathbb{R}^d$ is a rational polytope with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a 1 as the last coordinate. Then

$$\mathrm{cone}\,(P) = \left\{ \sum_{i=1}^{n} \lambda_i (\mathbf{v}_i, 1) : \lambda_i \geq 0 \right\} \subset \mathbb{R}^{d+1}. \tag{4.6}$$

We say a point is at *height* k in the cone if the point lies on $\mathrm{cone}\,(P) \cap \{\mathbf{x} : x_{d+1} = k\}$. Note that qP is embedded in $\mathrm{cone}\,(P)$ as $\mathrm{cone}\,(P) \cap \{\mathbf{x} : x_{d+1} = q\}$.

A *triangulation* T of a d-polytope P is a subdivision of P into simplices (of all dimensions). If all the vertices of T are rational points, define the *denominator* of T to be the least common multiple of all the vertex coordinate denominators of the faces of T. For each $\Delta \in T$, we define the *h-polynomial* of Δ with respect to T as

$$h_T(\Delta; z) := (1-z)^{d-\dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left(\frac{z}{1-z} \right)^{\dim(\Phi)-\dim(\Delta)}, \tag{4.7}$$

where the sum is over all simplices $\Phi \in T$ containing Δ. When T is clear from context, we omit the subscript. Note that when T is a boundary triangulation of P, the definition of the h-vector will be adjusted according to dimension, that is, d should be replaced by $d-1$ in (4.7).

For a d-simplex Δ with denominator p, let $\mathbf{W}$ be the set of ray generators of $\mathrm{cone}\,(\Delta)$ at height p, which are all integral. We then define the *h^*-polynomial of* Δ as the generating function of the last coordinate of integer points in $\mathcal{P}(\mathbf{W}) := \mathcal{P}(\Delta)$, that is,

$$h^*(\Delta; z) = \sum_{\mathbf{v} \in \mathcal{P}(\Delta) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})}.$$

With this consideration, the Ehrhart series of Δ can be expressed as

$$\mathrm{Ehr}(\Delta; z) = \frac{h^*(\Delta; z)}{(1-z^p)^{d+1}}.$$

We use a modified convention when Δ is a rational m-simplex of a triangulation T, where T has denominator q. In this case, it is possible that the denominator of Δ as an individual simplex might be different from q, but for coherence among all simplices in T we use q to select the height of the ray generators in Δ. Namely, we let $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$, where the $(\mathbf{r}_i, q)$ are integral ray generators of $\mathrm{cone}\,(\Delta)$ at height q. The corresponding h^*-polynomial of Δ is a function of q and the Ehrhart series of Δ can be expressed as

$$\mathrm{Ehr}(\Delta; z) = \frac{h^*(\Delta; z)}{(1-z^q)^{m+1}}.$$

We may think of $h^*(\Delta; z)$ as computed via $\sum_{\mathbf{v} \in \mathcal{P}(\mathbf{W}) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})}$.

4.1 Rational Betke–McMullen Decomposition

Decomposition à la Betke–McMullen

Let P be a rational d-polytope and T a triangulation of P with denominator q. For an m-simplex $\Delta \in T$, let $\mathbf{W} = \{(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)\}$, where the $(\mathbf{r}_i, q)$ are the integral ray generators of $\operatorname{cone}(\Delta)$ at height q as above. Further, set $B(\mathbf{W}; z) =: B(\Delta; z)$ and similarly $\operatorname{Box}(\mathbf{W}) =: \operatorname{Box}(\Delta)$. We emphasize that the h^*-polynomial, fundamental parallelepiped, and box polynomial of Δ depend on the denominator q of T.

A point $\mathbf{v} \in \operatorname{cone}(\Delta)$ can be uniquely expressed as $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q)$ for $\lambda_i \geq 0$. Define

$$I(\mathbf{v}) := \{i \in [m+1] : \lambda_i \in \mathbb{Z}\} \qquad \text{and} \qquad \overline{I(\mathbf{v})} := [m+1] \setminus I, \tag{4.8}$$

where $[m+1] := \{1, \ldots, m+1\}$.

Lemma 4.1.1. *Fix a triangulation T with denominator q of a rational d-polytope P and let $\Delta \in T$. Then $h^*(\Delta; z) = \sum_{\Omega \subseteq \Delta} B(\Omega; z)$.*

Proof. First we show that $\mathcal{P}(\Delta) = \biguplus_{\Omega \subseteq \Delta} \operatorname{Box}(\Omega)$. The reverse containment follows from the fact that any element in $\operatorname{Box}(\Omega)$ is a linear combination of the ray generators of $\operatorname{cone}(\Omega)$.

For the forward containment, if $\mathbf{v} \in \mathcal{P}(\Delta)$, then

$$\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) = \sum_{i \in \overline{I(\mathbf{v})}} \lambda_i(\mathbf{r}_i, q) \in \operatorname{Box}(\Omega)\,,$$

for $\Omega := \operatorname{conv}\left(\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v})}\right) \subseteq \Delta$. Note that $\mathbf{v}$ will always lie in a unique $\operatorname{Box}(\Omega)$ because every Ω corresponds to a different subset of $[m+1]$, which also tells us that the union we desire is disjoint.

Thus $\mathcal{P}(\Delta) = \biguplus_{\Omega \subseteq \Delta} \operatorname{Box}(\Omega)$, and so

$$h^*(\Delta; z) = \sum_{\mathbf{v} \in \mathcal{P}(\Delta) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} = \sum_{\Omega \subseteq \Delta} \sum_{\mathbf{v} \in \operatorname{Box}(\Omega) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} = \sum_{\Omega \subseteq \Delta} B(\Omega; z)\,. \qquad \square$$

Theorem 4.1.2. *For a triangulation T with denominator q of a rational d-polytope P,*

$$\operatorname{Ehr}(P; z) = \frac{\sum_{\Omega \in T} B(\Omega; z)\, h(\Omega; z^q)}{(1 - z^q)^{d+1}}\,.$$

Proof. We write P as the disjoint union of all open nonempty simplices in T and use

Ehrhart–Macdonald reciprocity [25, 47]:

$$\begin{aligned}
\mathrm{Ehr}(P;z) &= 1 + \sum_{\Delta\in T\setminus\{\emptyset\}} \mathrm{Ehr}(\Delta^\circ;z) = 1 + \sum_{\Delta\in T\setminus\{\emptyset\}} (-1)^{\dim(\Delta)+1}\mathrm{Ehr}\left(\Delta;\frac{1}{z}\right) \\
&= 1 + \sum_{\Delta\in T\setminus\{\emptyset\}} (-1)^{\dim(\Delta)+1} \frac{h^*\left(\Delta;\frac{1}{z}\right)}{\left(1-\frac{1}{z^q}\right)^{\dim(\Delta)+1}} \\
&= 1 + \sum_{\Delta\in T\setminus\{\emptyset\}} \frac{(z^q)^{\dim(\Delta)+1}(1-z^q)^{d-\dim(\Delta)}\, h^*\left(\Delta;\frac{1}{z}\right)}{(1-z^q)^{d+1}}.
\end{aligned}$$

Note that the Ehrhart series of each Δ is being written as a rational function with denominator $(1-z^q)^{d+1}$. Using Lemma 4.1.1,

$$\begin{aligned}
\mathrm{Ehr}(P;z) &= 1 + \sum_{\Delta\in T\setminus\emptyset} \frac{(z^q)^{\dim(\Delta)+1}(1-z^q)^{d-\dim(\Delta)} \sum_{\Omega\subseteq\Delta} B\left(\Omega;\frac{1}{z}\right)}{(1-z^q)^{d+1}} \\
&= \frac{\sum_{\Delta\in T}\left[(z^q)^{\dim(\Delta)+1}(1-z^q)^{d-\dim(\Delta)} \sum_{\Omega\subseteq\Delta} B\left(\Omega;\frac{1}{z}\right)\right]}{(1-z^q)^{d+1}}.
\end{aligned}$$

By Lemma 4.0.5,

$$\begin{aligned}
h^*(P;z) &= \sum_{\Delta\in T}\left[(z^q)^{\dim(\Delta)+1}(1-z^q)^{d-\dim(\Delta)} \sum_{\Omega\subseteq\Delta} B\left(\Omega;\frac{1}{z}\right)\right] \\
&= \sum_{\Delta\in T}\left[(z^q)^{\dim(\Delta)+1}(1-z^q)^{d-\dim(\Delta)} \sum_{\Omega\subseteq\Delta} (z^q)^{-\dim(\Omega)-1} B(\Omega;z)\right] \\
&= \sum_{\Omega\in T}\sum_{\Omega\subseteq\Delta} (z^q)^{\dim(\Delta)-\dim(\Omega)}(1-z^q)^{d-\dim(\Delta)} B(\Omega;z) \\
&= \sum_{\Omega\in T}\left[B(\Omega;z)(1-z^q)^{d-\dim(\Omega)} \sum_{\Omega\subseteq\Delta}\left(\frac{z^q}{1-z^q}\right)^{\dim(\Delta)-\dim(\Omega)}\right].
\end{aligned}$$

Using the definition of the h-polynomial, the theorem follows. □

Rational h^*-Monotonicity

We now show how the following theorem follows from our rational Betke–McMullen formula.

Theorem 4.1.3 (Stanley Monotonicity [68])**.** *Suppose that $P \subseteq Q$ are rational polytopes with qP and qQ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the h^*-polynomials via*

$$\mathrm{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{\dim(P)+1}} \qquad \text{and} \qquad \mathrm{Ehr}(Q;z) = \frac{h^*(Q;z)}{(1-z^q)^{\dim(Q)+1}}.$$

Then $h_i^*(P;z) \le h_i^*(Q;z)$ coefficient-wise.

In addition to Stanley's original proof, Beck and Sottile [10] provide a proof of Theorem 4.1.3 using irrational decompositions of rational polyhedra. In the case of lattice polytopes, Jochemko and Sanyal [42] prove Theorem 4.1.3 using combinatorial positivity of translation-invariant valuations and Stapledon [72] gives a geometric interpretation of Theorem 4.1.3 by considering the h^*-polynomials of lattice polytopes in terms of orbifold Chow rings. The following lemma assumes familiarity with Cohen–Macaulay complexes and related theory; see [69] for definitions and further reading.

Lemma 4.1.4. *Suppose P is a polytope and T a triangulation of P. Let $P \subseteq Q$ be a polytope and T' a triangulation of Q such that T' restricted to P is T. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that (1) the join $T * \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a subcomplex of T' and (2) $\dim(P * \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)) = \dim(Q)$. For every face $\Omega \in T$, the coefficient-wise inequality $h_T(\Omega; z) \leq h_{T'}(\Omega, z)$ holds.*

Proof. Suppose first that $\dim(P) = \dim(Q)$. Let T be a triangulation of P and T' a triangulation of Q such that T' restricted to P is T. Note that T and T' are geometric simplicial complexes covering P and Q, respectively. Let $\Omega \in T$. Then $\operatorname{link}_T(\Omega)$ and $\operatorname{link}_{T'}(\Omega)$ are either balls or spheres, hence Cohen–Macaulay. Now, consider $\mathcal{R} := \operatorname{link}_{T'}(\Omega) - \operatorname{link}_T(\Omega)$, which is a relative simplicial complex. By [69, Corollary 7.3(iv)] $\mathcal{R}$ is also Cohen–Macaulay. From [69, Proposition 7.1] it follows that

$$h_{\mathcal{R}}(\emptyset; z) = h_{T'}(\Omega; z) - h_T(\Omega; z) \qquad \text{and} \qquad h_{\mathcal{R}}(\emptyset; z),\ h_T(\Omega; z),\ h_{T'}(\Omega; z) \geq 0\,.$$

Rearranging, we obtain that $h_{T'}(\Omega; z) = h_{\mathcal{R}}(\emptyset; z) + h_T(\Omega; z)$, which implies that $h_T(\Omega; z) \leq h_{T'}(\Omega; z)$ Hence, for each face in T, the result follows.

Now, consider the case when $\dim(P) < \dim(Q)$. Again, let T be a triangulation of P and T' a triangulation of Q such that T' restricted to P is T, where we further assume that there exists a set of affinely independent vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of Q outside the affine span of P such that (1) the join $T * \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is a subcomplex of T' and (2) $\dim(P * \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)) = \dim(Q)$. Note that the affine independence of the $\mathbf{v}_i$'s implies that

$$\dim(\operatorname{conv}(P \cup \mathbf{v}_1 \cup \cdots \cup \mathbf{v}_k)) = \dim(\operatorname{conv}(P \cup \mathbf{v}_1 \cup \cdots \cup \mathbf{v}_{k-1})) + 1\,.$$

Let T_k denote the join of T with the simplex $\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Let $\Omega \in T_k$. Since $\Omega \subseteq \partial T_{k+1}$ and $\operatorname{link}_{T_k}(\Omega)$ and $\operatorname{link}_{T_{k+1}}(\Omega)$ are both balls, $\mathcal{R} := \operatorname{link}_{T_{k+1}}(\Omega) - \operatorname{link}_{T_k}(\Omega)$ is Cohen–Macaulay by [69, Proposition 7.3(iii)]. Thus, by a similar argument as given in the paragraph above,

$$h_{T_k}(\Omega; z) \leq h_{T_{k+1}}(\Omega; z)\,.$$

Combining this with the fact that $\dim(P * \operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n)) = \dim(Q)$, it follows by induction (for the first inequality) and our previous case (for the second inequality) that for $\Omega \in T$

$$h_T(\Omega; z) \leq h_{T_n}(\Omega; z) \leq h_{T'}(\Omega; z)\,. \qquad \square$$

Proof of Theorem 4.1.3. Let P be a polytope contained in Q. Let T be a triangulation of P and let T' be a triangulation of Q such that T' restricted to P is T, where if $\dim(P) < \dim(Q)$ the triangulation T' satisfies the conditions given in Lemma 4.1.4. (Note that such a triangulation T' can always be obtained from T, e.g., by extending T using a placing triangulation.) By Theorem 4.1.2, $h^*(P;z) = \sum_{\Omega\in T} B(\Omega;z)\, h_T(\Omega;z^q)$. Since P is contained in Q,

$$h^*(Q;z) = \sum_{\Omega\in T} B(\Omega;z)\, h_{T'|_P}(\Omega;z^q) + \sum_{\Omega\in T'\setminus T} B(\Omega;z)\, h_{T'}(\Omega;z^q).$$

By Lemma 4.1.4, the coefficients of $\sum_{\Omega\in T} B(\Omega;z)h_{T'}(\Omega;z^q)$ dominate the coefficients of $\sum_{\Omega\in T} B(\Omega;z)h_T(\Omega;z^q)$. This further implies that the coefficients of $h^*(Q;z)$ dominate the coefficients of $h^*(P;z)$ since

$$\begin{aligned}\sum_{\Omega\in T} B(\Omega;z)\, h_T(\Omega;z^q) &\le \sum_{\Omega\in T} B(\Omega;z)\, h_{T'}(\Omega;z^q)\\ &\le \sum_{\Omega\in T} B(\Omega;z)\, h_{T'|_P}(\Omega;z^q) + \sum_{\Omega\in T'\setminus T} B(\Omega;z)\, h_{T'}(\Omega;z^q)\,.\square\end{aligned}$$

4.2 h^*-Decompositions from Boundary Triangulations

Set-up

Throughout this section we will use the following set-up. Fix a boundary triangulation T with denominator q of a rational d-polytope P. Take $\ell \in \mathbb{Z}_{>0}$, such that ℓP contains a lattice point $\mathbf{a}$ in its interior. Thus $(\mathbf{a},\ell) \in \operatorname{cone}(P)^\circ \cap \mathbb{Z}^{d+1}$ is a lattice point in the interior of the cone of P at height ℓ, and $\operatorname{cone}((\mathbf{a},\ell))$ is the ray through the point $(\mathbf{a},\ell)$. We cone over each $\Delta \in T$ and define $\mathbf{W} = \{(\mathbf{r}_1,q),\dots,(\mathbf{r}_{m+1},q)\}$ where the $(\mathbf{r}_i,q)$ are integral ray generators of $\operatorname{cone}(\Delta)$ at height q. As before, we have the associated box polynomial $B(\mathbf{W};z) =: B(\Delta;z)$. Now, let $\mathbf{W}' = \mathbf{W}\cup\{(\mathbf{a},\ell)\}$ be the set of generators from $\mathbf{W}$ together with $(\mathbf{a},\ell)$ and we set $\operatorname{cone}(\Delta')$ to be the cone generated by $\mathbf{W}'$, with associated box polynomial $B(\mathbf{W}';z) =: B(\Delta';z)$.

Corollary 4.2.1. *For each face Δ of T,*

$$B(\Delta;z) = z^{q(\dim(\Delta)+1)} B\left(\Delta;\tfrac{1}{z}\right) \qquad \text{and} \qquad B(\Delta';z) = z^{q(\dim(\Delta)+1)+\ell} B\left(\Delta';\tfrac{1}{z}\right).$$

Proof. The height of $\sum_i(\mathbf{r}_i,q)$ is q times the number of summands, which gives us $q(\dim(\Delta)+1)$. The first equations now follow from the involution ι and Lemma 4.0.5; note that we will have to use $\mathbf{W}$ in the first case and $\mathbf{W}'$ in the second. $\square$

Observe that when $\Delta = \emptyset$ is the empty face, $B(\emptyset;z) = 1$, but $B(\emptyset';z) = B((\mathbf{a},\ell);z)$. This differs from the scenario in [73] where Stapledon's set-up determined that $B(\emptyset',z) = 0$.

For a real number x, define $\lfloor x\rfloor$ to be the greatest integer less than or equal to x. Additionally, define the *fractional part* of x to be $\{x\} = x - \lfloor x\rfloor$.

Boundary Triangulations

For each $\mathbf{v} \in \operatorname{cone}(P)$ we associate two faces $\Delta(\mathbf{v})$ and $\Omega(\mathbf{v})$ of T, as follows. The face $\Delta(\mathbf{v})$ is chosen to be the minimal face of T such that $\mathbf{v} \in \operatorname{cone}(\Delta'(\mathbf{v}))$, and we define

$$\Omega(\mathbf{v}) := \operatorname{conv}\left(\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v})}\right) \subseteq \Delta(\mathbf{v}),$$

where $\overline{I(\mathbf{v})}$ is defined as in (4.8) and the $(\mathbf{r}_i, q)$ are ray generators of $\operatorname{cone}(\Delta(\mathbf{v}))$. In an effort to make our statements and proofs less notation heavy, for the rest of this section we write $\Delta(\mathbf{v}) = \Delta$ and $\Omega(\mathbf{v}) = \Omega$ with the understanding that both depend on $\mathbf{v}$. Furthermore, for $\mathbf{v} = \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) + \lambda(\mathbf{a}, \ell)$ where $\lambda, \lambda_i \geq 0$, define

$$\{\mathbf{v}\} := \sum_{i \in \overline{I(\mathbf{v})}} \{\lambda_i\}(\mathbf{r}_i, q) + \{\lambda\}(\mathbf{a}, \ell).$$

Lemma 4.2.2. *Given* $\mathbf{v} \in \operatorname{cone}(P)$*, construct* $\Delta = \Delta(\mathbf{v})$ *as described above, with* $\operatorname{cone}(\Delta)$ *generated by* $(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)$*. Then* $\mathbf{v}$ *can be written uniquely as*

$$\{\mathbf{v}\} + \sum_{i \in I(\mathbf{v})} (\mathbf{r}_i, q) + \sum_{i=1}^{m+1} \mu_i(\mathbf{r}_i, q) + \mu(\mathbf{a}, \ell), \tag{4.9}$$

where $\mu, \mu_i \in \mathbb{Z}_{\geq 0}$*.*

Below we will note the dependence of the unique coefficients μ_i and μ on $\mathbf{v}$ by writing them as $\mu_i(\mathbf{v})$ and $\mu(\mathbf{v})$.

Proof. Since $\mathbf{v}$ is in $\operatorname{cone}(\Delta')$, it can be written as a linear combination of the generators of $\operatorname{cone}(\Delta)$ and $(\mathbf{a}, \ell)$. We further express $\mathbf{v}$ as a sum of its integer and fractional parts.

$$\begin{aligned}
\mathbf{v} &= \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) + \lambda(\mathbf{a}, \ell), \text{ for } \lambda_i > 0 \text{ and } \lambda \geq 0 \\
&= \sum_{i \in \overline{I(\mathbf{v})}} \{\lambda_i\}(\mathbf{r}_i, q) + \{\lambda\}(\mathbf{a}, \ell) + \sum_{i=1}^{m+1} \lfloor \lambda_i \rfloor (\mathbf{r}_i, q) + \lfloor \lambda \rfloor (\mathbf{a}, \ell) \\
&= \{\mathbf{v}\} + \sum_{i=1}^{m+1} \lfloor \lambda_i \rfloor (\mathbf{r}_i, q) + \lfloor \lambda \rfloor (\mathbf{a}, \ell).
\end{aligned}$$

Note that each $\lambda_i > 0$ because of the minimality of Δ. Recall that $\Omega = \operatorname{conv}\left(\frac{\mathbf{r}_i}{q} : i \in \overline{I(\mathbf{v})}\right) \subseteq \Delta$. Thus

- if $\lambda \notin \mathbb{Z}$, then $\{\mathbf{v}\} \in \operatorname{Box}(\Omega')$,
- if $\lambda \in \mathbb{Z}$, then $\{\mathbf{v}\} \in \operatorname{Box}(\Omega)$.

Further observe that when λ is an integer, $\{\mathbf{v}\}$ is an element on the boundary of cone(P).

If $i \in I(\mathbf{v})$, then $\lambda_i \in \mathbb{Z}$ and $\lfloor \lambda_i \rfloor = \lambda_i \geq 1$ for $i \in I(\mathbf{v})$. This allows us to represent $\mathbf{v}$ in the form

$$\mathbf{v} = \{\mathbf{v}\} + \sum_{i \in I(\mathbf{v})} (\mathbf{r}_i, q) + \sum_{i=1}^{m+1} \mu_i(\mathbf{r}_i, q) + \mu(\mathbf{a}, \ell),$$

where $\mu, \mu_i \in \mathbb{Z}_{\geq 0}$. $\square$

Corollary 4.2.3. *Continuing the notation above,*

$$u(\mathbf{v}) = u(\{\mathbf{v}\}) + q(\dim \Delta(\mathbf{v}) - \dim \Omega(\mathbf{v})) + \sum_{i=1}^{m+1} q\, \mu_i(\mathbf{v}) + \mu(\mathbf{v})\, \ell\,. \tag{4.10}$$

Proof. This follows from considering the height contribution of each part in (4.9). $\square$

The following theorem provides a decomposition of the h^*-polynomial of a rational polytope in terms of box and h-polynomials. It is important to note again that the h^*-polynomial depends on the denominator of the boundary triangulation.

Theorem 4.2.4. *Consider a rational d-polytope P that contains an interior point $\frac{\mathbf{a}}{\ell}$, where $\mathbf{a} \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation T of P with denominator q. Then*

$$h^*(P; z) = \frac{1 - z^q}{1 - z^\ell} \sum_{\Omega \in T} (B(\Omega; z) + B(\Omega'; z))\, h(\Omega; z^q).$$

Proof. By Corollary 4.2.3,

$$\begin{aligned}
\frac{h^*(P; z)}{(1 - z^q)^{d+1}} &= \sum_{\mathbf{v} \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} \\
&= \sum_{\mathbf{v} \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{u(\{\mathbf{v}\}) + q(\dim \Delta(\mathbf{v}) - \dim \Omega(\mathbf{v})) + \sum_{i=1}^{\dim(\Delta)+1} q\mu_i(\mathbf{v}) + \mu(\mathbf{v})\ell} \\
&= \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} z^{q(\dim \Delta - \dim \Omega)} \sum_{\mathbf{v} \in (\mathrm{Box}(\Omega) \cup \mathrm{Box}(\Omega')) \cap \mathbb{Z}^{d+1}} z^{u(\mathbf{v})} \sum_{\mu_i, \mu \geq 0} z^{\sum_{i=1}^{\dim(\Delta)+1} q\mu_i + \mu\ell} \\
&= \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} \frac{(B(\Omega; z) + B(\Omega'; z))\, z^{q(\dim \Delta - \dim \Omega)}}{(1 - z^q)^{\dim(\Delta)+1}(1 - z^\ell)} \\
&= \frac{1}{1 - z^\ell} \sum_{\Omega \in T} (B(\Omega; z) + B(\Omega'; z)) \sum_{\Omega \subseteq \Delta} \frac{(z^q)^{\dim(\Delta) - \dim(\Omega)}}{(1 - z^q)^{\dim(\Delta)+1}} \\
&= \frac{1}{(1 - z^\ell)(1 - z^q)^d} \sum_{\Omega \in T} (B(\Omega; z) + B(\Omega'; z))\, h(\Omega; z^q)\,. \square
\end{aligned}$$

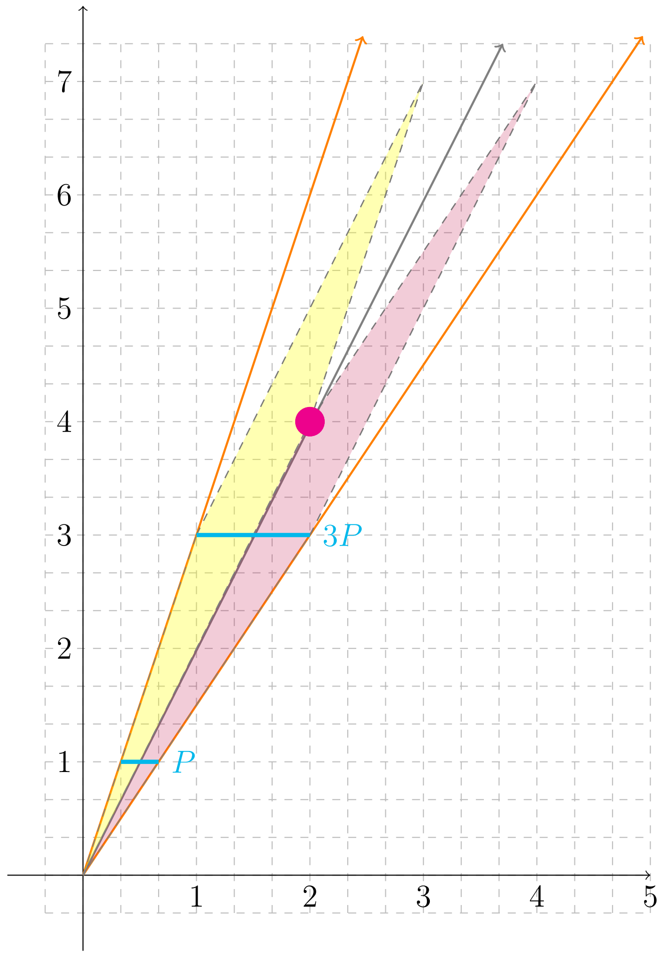

Figure 4.1: This figure shows cone (P) (in orange), P, $3P$, $(\mathbf{a}, \ell) = (2,4)$, Box (Δ_1') (in yellow), Box (Δ_2') (in pink).

Example 4.2.5. Following the setup in Section 4.2, consider the line segment $P = \left[\frac{1}{3}, \frac{2}{3}\right]$ and so our boundary triangulation T has denominator 3. In the cone over P, set $(\mathbf{a}, \ell) = (2,4)$. The simplices in T are the empty face $\emptyset$ and the two vertices $\Delta_1 = \frac{1}{3}$ and $\Delta_2 = \frac{2}{3}$. The cones over the vertices have integral ray generators $\mathbf{W}_1 = \{(1,3)\}$ and $\mathbf{W}_2 = \{(2,3)\}$. We see that if $\mathbf{v} \in$ cone (P) then the only options for $\Delta(\mathbf{v})$ to be chosen as a minimal face of T such that $\mathbf{v} \in$ cone $(\Delta'(\mathbf{v}))$ are again to consider $\emptyset$, Δ_1, and Δ_2. In this example, $\Omega(\mathbf{v}) = \Delta(\mathbf{v})$. Recall that since T is a boundary triangulation of P, the definition of the h-vector (4.7) is adjusted according to dimension, that is, d is be replaced by $d-1$.

From Figure 4.1 we determine the following:

$\Omega \in T$	$\dim(\Omega)$	$B(\Omega; z)$	$B(\Omega'; z)$	$h(\Omega, z^3)$
Δ_1	0	0	0	1
Δ_2	0	0	0	1
$\emptyset$	-1	1	z^2	$1 + z^3$

Applying Theorem 4.2.4, we obtain

$$\begin{aligned} h^*(P;z) &= \frac{1-z^3}{1-z^4}\left(1+z^3+z^2+z^5\right) \\ &= 1+z^2+z^4, \end{aligned}$$

which agrees with the computation obtained using Normaliz [17].

Rational Stapledon Decomposition and Inequalities

Using Theorem 4.2.4, we can rewrite the h^*-polynomial of a rational polytope P as

$$h^*(P;z) = \frac{1+z+\cdots+z^{q-1}}{1+z+\cdots+z^{\ell-1}} \sum_{\Omega\in T} \left(B(\Omega;z)+B(\Omega';z)\right) h(\Omega;z^q)\,.$$

Next, we turn our attention to the polynomial

$$\overline{h^*}(P;z) := \left(1+z+\cdots+z^{\ell-1}\right) h^*(P;z)\,. \tag{4.11}$$

We know that $h^*(P;z)$ is a polynomial of degree at most $q(d+1)-1$, thus $\overline{h^*}(P;z)$ has degree at most $q(d+1)+\ell-2$. We set f to be the degree of $\overline{h^*}(P;z)$ and s to be the degree of $h^*(P;z)$. We can recover $h^*(P;z)$ from $\overline{h^*}(P;z)$ for a chosen value of ℓ; if we write

$$\overline{h^*}(P;z) = \overline{h^*_0} + \overline{h^*_1}z + \cdots + \overline{h^*_f}z^f,$$

then

$$\overline{h^*_i} = h^*_i + h^*_{i-1} + \cdots + h^*_{i-l+1} \qquad i=0,\ldots,f, \tag{4.12}$$

and we set $h^*_i = 0$ when $i > s$ or $i < 0$.

Proposition 4.2.6. *Let P be a rational d-polytope with denominator q and Ehrhart series*

$$\operatorname{Ehr}(P;z) = \frac{h^*(P;z)}{(1-z^q)^{d+1}}\,.$$

Then $\deg h^*(P;z) = s$ if and only if $(q(d+1)-s)P$ is the smallest integer dilate of P that contains an interior lattice point.

Proof. Let $L(P;t)$ and $L(P^\circ;t)$ be the Ehrhart quasipolynomials of P and the interior of P, respectively. Using Ehrhart–Macdonald reciprocity [25, 47] we obtain

$$\begin{aligned} \operatorname{Ehr}(P^\circ;z) &= \sum_{t\geq 1} L(P^\circ;t)z^t = (-1)^{d+1}\frac{\sum_{j=0}^{s} h^*_j\left(\frac{1}{z}\right)^j}{\left(1-\frac{1}{z^q}\right)^{d+1}} = z^{q(d+1)}\frac{\sum_{j=0}^{s} h^*_j z^{-j}}{(1-z^q)^{d+1}} \\ &= \left(\sum_{j=0}^{s} h^*_j z^{q(d+1)-j}\right)\left(1+z^q+z^{2q}+\cdots\right)^{d+1}. \end{aligned}$$

Now, note that the minimum degree term of

$$\left(\sum_{j=0}^{s} h^*_j z^{q(d+1)-j}\right)\left(1+z^q+z^{2q}+\cdots\right)^{d+1}$$

is $h^*_s z^{q(d+1)-s}$, which implies that the term of $\sum_{t\geq 1} L(P^\circ; t) z^t$ with minimum degree is $q(d+1)-s$. Hence, the degree of $h^*(P;z)$ is s precisely if $(q(d+1)-s)P$ is the smallest integer dilate of P that contains an interior lattice point. $\square$

The following result provides a decomposition of the $\overline{h^*}$-polynomial which we refer to as an *a/b-decomposition*. It generalizes [73, Theorem 2.14] to the rational case.

Theorem 4.2.7. *Let P be a rational d-polytope with denominator q, and let $s := \deg h^*(P;z)$. Then $\overline{h^*}(P;z)$ has a unique decomposition*

$$\overline{h^*}(P;z) = a(z) + z^\ell b(z)\,,$$

where $\ell = q(d+1)-s$ and $a(z)$ and $b(z)$ are polynomials with integer coefficients satisfying $a(z) = z^{q(d+1)-1} a\left(\frac{1}{z}\right)$ and $b(z) = z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right)$. Moreover, the coefficients of $a(z)$ and $b(z)$ are nonnegative.

Proof. Let a_i and b_i denote the coefficients of z^i in $a(z)$ and $b(z)$, respectively. Set

$$a_{i+1} = h^*_0 + \cdots + h^*_{i+1} - h^*_{q(d+1)-1} - \cdots - h^*_{q(d+1)-1-i}, \tag{4.13}$$

and

$$b_i = -h^*_0 - \cdots - h^*_i + h^*_s + \cdots + h^*_{s-i}. \tag{4.14}$$

Using (4.12) and the fact that $\ell = q(d+1)-s$, we compute that

$$\begin{aligned}
a_i + b_{i-\ell} &= h^*_0 + \cdots + h^*_i - h^*_{q(d+1)-1} - \cdots - h^*_{q(d+1)-i} - h^*_0 - \cdots - h^*_{i-\ell} \\
&\quad + h^*_s + \cdots + h^*_{s-i+\ell} \\
&= h^*_{i-\ell+1} + \cdots + h^*_i = \overline{h^*_i}\,, \\
a_i - a_{q(d+1)-1-i} &= h^*_0 + \cdots + h^*_i - h^*_{q(d+1)-1} - \cdots - h^*_{q(d+1)-i} - h^*_0 - \cdots \\
&\quad - h^*_{q(d+1)-1-i} + h^*_{q(d+1)-1} + \cdots + h^*_{i+1} \\
&= 0\,, \\
b_i - b_{q(d+1)-1-\ell-i} &= -h^*_0 - \cdots - h^*_i + h^*_s + \cdots + h^*_{s-i} + h^*_0 + \cdots + h^*_i \\
&\quad - h^*_s - \cdots - h^*_{s-i-1} - h^*_s - \cdots - h^*_{i+1} \\
&= 0\,,
\end{aligned}$$

for $i = 0, \ldots, q(d+1)-1$. Thus, we obtain the decomposition desired. The uniqueness property follows from (4.13) and (4.14).

Let T be a regular boundary triangulation of P. By Theorem 4.2.4 and (4.11), we can set

$$a(z) = (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega; z)\, h(\Omega; z^q)\,, \tag{4.15}$$

and

$$b(z) = z^{-\ell}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z)\, h(\Omega; z^q)\,, \tag{4.16}$$

so that $\overline{h^*}(P;z) = a(z) + z^\ell b(z)$. By Proposition 4.2.6, the dilate kP contains no interior lattice points for $k = 1, \ldots, \ell - 1$, so if $\mathbf{v} \in \mathrm{Box}\,(\Omega') \cap \mathbb{Z}^{d+1}$ for $\Omega \in T$, then $u(\mathbf{v}) \geq \ell$. Hence, $b(z)$ is a polynomial. We now need to verify that

$$a(z) = z^{q(d+1)-1} a\left(\tfrac{1}{z}\right) \qquad \text{and} \qquad b(z) = z^{q(d+1)-1-\ell} b\left(\tfrac{1}{z}\right).$$

It is a well-known property of the h-vector in (4.7) that

$$h(\Omega, z^q) = z^{q(d-1-\dim(\Omega))} h(\Omega; z^{-q})$$

[27, 48, 64].

Using the aforementioned and Corollary 4.2.1, we determine that

$$\begin{aligned}
&z^{q(d+1)-1} a\left(\frac{1}{z}\right) \\
&= z^{q(d+1)-1}\left(1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}}\right) \sum_{\Omega \in T} B\left(\Omega; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\
&= z^{q(d+1)-1} z^{1-q} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B\left(\Omega; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\
&= z^{qd}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(\dim(\Omega)+1)} B(\Omega, z)\, z^{-q(d-1-\dim\Omega)} h(\Omega; z^q) \\
&= (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega, z)\, h(\Omega; z^q) = a(z)
\end{aligned}$$

and

$$\begin{aligned}
&z^{q(d+1)-1-\ell} b\left(\frac{1}{z}\right) \\
&= z^{q(d+1)-1-\ell} z^\ell \left(1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}}\right) \sum_{\Omega \in T} B\left(\Omega'; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\
&= z^{q(d+1)-1} z^{1-q} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B\left(\Omega'; \frac{1}{z}\right) h\left(\Omega; \frac{1}{z^q}\right) \\
&= z^{qd}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(\dim\Omega+1)-\ell} B(\Omega'; z)\, z^{-q(d-1-\dim\Omega)} h(\Omega; z^q) \\
&= z^{-\ell}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z)\, h(\Omega; z^q) = b(z)\,.
\end{aligned}$$

Lastly, recall that the box polynomials and the h-polynomials have nonnegative coefficients [67], so a sum of products of box polynomials and h-polynomials will also have nonnegative coefficients. Thus, the result holds. □

The next theorem follows as a corollary to Theorem 4.2.7 and gives inequalities satisfied by the coefficients of the h^*-polynomial for rational polytopes.

Theorem 4.2.8. *Let P be a rational d-polytope with denominator q and let $s := \deg h^*(P;z)$. The h^*-vector $(h_0^*, \ldots, h_{q(d+1)-1}^*)$ of P satisfies the following inequalities:*

$$h_0^* + \cdots + h_{i+1}^* \geq h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-i}^*\,, \qquad i = 0, \ldots, \left\lfloor \frac{q(d+1)-1}{2} \right\rfloor - 1\,, \tag{4.17}$$

$$h_s^* + \cdots + h_{s-i}^* \geq h_0^* + \cdots + h_i^*\,, \qquad i = 0, \ldots, q(d+1)-1\,. \tag{4.18}$$

Proof. By (4.13) and (4.14) if follows that (4.17) and (4.18) hold if and only if $a(z)$ and $b(z)$ have nonnegative coefficients, respectively, which in turn follows from Theorem 4.2.7. □

4.3 Applications

Rational Reflexive Polytopes

A lattice polytope is *reflexive* if its dual is also a lattice polytope. Reflexive polytopes have enjoyed a wealth of recent research activity (see, e.g., [5, 13, 15, 32, 33, 36, 37, 38, 52]), and Hibi [35] proved that a lattice polytope P is the translate of a reflexive polytope if and only if $\operatorname{Ehr}\left(P; \frac{1}{z}\right) = (-1)^{d+1} z \operatorname{Ehr}(P;z)$ as rational functions, that is, $h^*(z)$ is palindromic. More generally, Fiset and Kaspryzk [26, Corollary 2.2] proved that a rational polytope P whose dual is a lattice polytope has a palindromic h^*-polynomial, complementing previous results by De Negri and Hibi [23]. The following proposition provides an alternate route to Fiset and Kaspryzk's result.

Theorem 4.3.1. *Let P be a rational polytope containing the origin. The dual of P is a lattice polytope if and only if $\overline{h^*}(P;z) = h^*(z) = a(z)$, that is, $b(z) = 0$ in the a/b-decomposition of $\overline{h^*}(P;z)$ from Theorem 4.2.4.*

Proof. Let P be a rational polytope containing the origin in its interior. Following Set-up 4.2, we let T be a boundary triangulation of P and we set $(\mathbf{a}, \ell) = (\mathbf{0}, 1)$. Recall that this implies

$$b(z) = z^{-1}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z) h(\Omega; z^q)\,.$$

Thus, $b(z) = 0$ if and only if $B(\Omega'; z) = 0$ for every $\Omega \in T$, which is true if and only if $\operatorname{Box}(\Omega')$ contains no integer points for every $\Omega \in T$.

To establish the forward direction, assume that the dual of P is a lattice polytope. We want to show that $b(z) = 0$ in the a/b-decomposition of $\overline{h^*}(P;z) = h^*(P;z)$. Each $\Omega \in T$ is contained in a facet F of P. Since the dual of P is a lattice polytope, the vector normal to $\operatorname{cone}(F)$ is of the form $(\mathbf{p}, 1)$, where $\mathbf{p}$ is the vertex of the dual of P corresponding to F. Let $(\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)$ be the ray generators of $\operatorname{Box}(\Omega)$. If $\sum_{i=1}^{m+1} \lambda_i (r_i, q) \in \operatorname{Box}(\Omega)$ for $0 < \lambda_i < 1$, then $(\mathbf{p}, 1) \cdot \left(\sum_{i=1}^{m+1} \lambda_i (r_i, q)\right) = 0$. Also, note that $(\mathbf{p}, 1) \cdot (\mathbf{0}, 1) = 1$, which tells us that $(\mathbf{0}, 1)$ is at lattice distance 1 away from

Box (Ω) with respect to $(\mathbf{p}, 1)$. So, if

$$\sum_{i=1}^{m+1} \lambda_i(r_i, q) + \lambda(\mathbf{0}, 1) \in \text{Box}\,(\Omega')$$

then $(\mathbf{p}, 1) \cdot \left[\sum_{i=1}^{m+1} \lambda_i(r_i, q) + \lambda(\mathbf{0}, 1)\right] = \lambda$, where $0 < \lambda < 1$. This implies that $\sum_{i=1}^{m+1} \lambda_i(r_i, q) + \lambda(\mathbf{0}, 1)$ is not an integer point, from which it follows that Box (Ω') contains no lattice points. Thus $B(\Omega', z) = 0$ and so $b(z) = 0$ in the a/b-decomposition of $\overline{h^*}(P; z)$. Hence, $\overline{h^*}(P; z) = h^*(P; z) = a(z)$ is palindromic.

For the backward direction, assume that $b(z) = 0$, and thus for every $\Omega \in T$, the set Box (Ω') contains no integer points. Our goal is to use this fact to show that for every facet F of P, the vertex of the dual of P corresponding to F is a lattice point, i.e., to show that the primitive facet normal to cone (F) is given by $(\mathbf{p}, 1)$ for some lattice point $\mathbf{p}$. Let F be a facet of P, and let $\Omega = \text{conv}\,((\mathbf{r}_1, q), \ldots, (\mathbf{r}_{m+1}, q)) \in T$ be a full-dimensional simplex contained in F. Since the origin lies in the interior of P, the dual of P is a rational polytope containing the origin. Further, the vector normal to cone (F) can be written in the form $(\mathbf{p}, b)$ with $b > 0$, where $\mathbf{p}$ is an integer vector that is primitive, i.e., the greatest common divisor of the entries in $(\mathbf{p}, b)$ equals 1. Observe that $(\mathbf{p}, b) \cdot (\mathbf{0}, 1) = b$. If $b = 1$, then the vertex of the dual of P corresponding to F is a lattice point, and our proof is complete.

Otherwise, suppose that $b > 1$. Since $(\mathbf{p}, b)$ is primitive, there exists an integer vector $\mathbf{v}$ such that $(\mathbf{p}, b) \cdot \mathbf{v} = 1$. Since $b > 1 > 0$, $\mathbf{v}$ is an element of the subset S strictly contained between the hyperplane H_0 spanned by cone (F) and the affine hyperplane $H_b = H_0 + (\mathbf{0}, 1)$; we can precisely describe this subset as

$$S := \left\{ \sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) + \lambda(\mathbf{0}, 1) : \lambda_i \in \mathbb{R} \text{ and } 0 < \lambda < 1 \right\}.$$

Since $b(z) = 0$, it follows that for each $\tau \subseteq \Omega$ the set Box $(\tau') = \text{Box}\,(\tau, (\mathbf{0}, 1))$ contains no integer points. The key observation is that translates of $\bigcup_{\tau \subseteq \Omega} \text{Box}\,(\tau, (\mathbf{0}, 1))$ by the integer ray generators of cone (F) cover S, though this union is not disjoint, i.e.,

$$S = \bigcup_{\mu_1 \ldots, \mu_{m+1} \in \mathbb{Z}} \left(\left(\sum_i \mu_i(\mathbf{r}_i, q) \right) + \bigcup_{\tau \subseteq \Omega} \text{Box}\,(\tau, (\mathbf{0}, 1)) \right).$$

This cover property follows from taking an arbitrary $\sum_{i=1}^{m+1} \lambda_i(\mathbf{r}_i, q) + \lambda(\mathbf{0}, 1) \in S$ and expressing each coefficient as a sum of an integer and fractional part. It follows that S contains no integer points, since $\bigcup_{\tau \subseteq \Omega} \text{Box}\,(\tau, (\mathbf{0}, 1))$ contains no integer points. Hence, no such integer vector $\mathbf{v}$ exists, implying that $b = 1$. Since F was arbitrary, it follows that the dual of P is a lattice polytope. $\square$

Reflexive Polytopes of Higher Index

Kasprzyk and Nill [43] introduced the following class of polytopes .

Definition 4.3.2. A lattice polytope P is a *reflexive polytope of higher index* $\mathcal{L}$ (also known as an $\mathcal{L}$*-reflexive polytope*), for some $\mathcal{L} \in \mathbb{Z}_{>0}$, if the following conditions hold:

- P contains the origin in its interior;
- The vertices of P are primitive, i.e., the line segment joining each vertex to $\mathbf{0}$ contains no other lattice points;
- For any facet F of P the local index $\mathcal{L}_F$ equals $\mathcal{L}$, i.e., the integral distance of $\mathbf{0}$ from the affine hyperplane spanned by F equals $\mathcal{L}$.

The 1-reflexive polytopes are the reflexive polytopes mentioned earlier in the section. Kaspryzk and Nill proved that if P is a lattice polytope with primitive vertices containing the origin in its interior then P is $\mathcal{L}$-reflexive if and only if $\mathcal{L}P^*$ is a lattice polytope having only primitive vertices. In this case, $\mathcal{L}P^*$ is also $\mathcal{L}$-reflexive.

Kaspryzk and Nill investigated $\mathcal{L}$-reflexive polygons. In particular, they show that there is no $\mathcal{L}$-reflexive polygon of even index. Furthermore, they provide a family of $\mathcal{L}$-reflexive polygons arising for each odd index:

$$P_{\mathcal{L}} = \operatorname{conv}\left(\pm(0,1), \pm(\mathcal{L},2), \pm(\mathcal{L},1)\right).$$

We are interested in the dual of $P_{\mathcal{L}}$:

$$P_{\mathcal{L}}^* = \operatorname{conv}\left(\pm\left(\frac{1}{\mathcal{L}},0\right), \pm(\frac{2}{\mathcal{L}},-1), \pm\left(\frac{1}{\mathcal{L}},-1\right)\right).$$

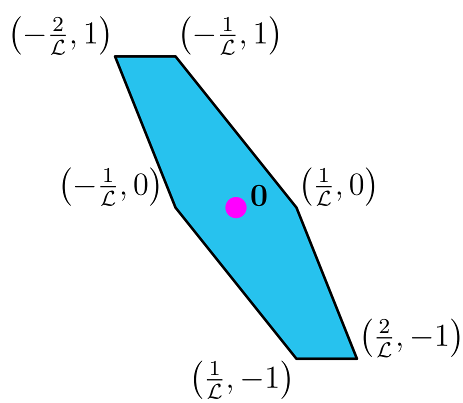

Figure 4.2: The rational hexagon $P_{\mathcal{L}}^*$.

Let $\mathcal{L}$ be odd. Our goal in the remainder of this subsection is to compute the h^*-polynomial of $P_{\mathcal{L}}^*$ using Theorem 4.2.4, to illustrate how this theorem can be applied. Consider the boundary as its own triangulation T (with denominator $\mathcal{L}$) of $P_{\mathcal{L}}^*$ and take the set of integral ray generators of $\operatorname{cone}(P_{\mathcal{L}}^*)$ to be

$$\{\pm(1,0,\mathcal{L}), \pm(2,-\mathcal{L},\mathcal{L}), \pm(1,-\mathcal{L},\mathcal{L})\}.$$

Observe that T contains six edges, six vertices, and the empty face $\emptyset$. It is not difficult to see that the box polynomials of the 0-simplices are 0. For example, in order for

$$\mathrm{Box}\left((2,-\mathcal{L},\mathcal{L})\right)=\{\lambda_1(2,-\mathcal{L},\mathcal{L}):0<\lambda_1<1\}\cap\mathbb{Z}^3$$

to contain any lattice points, $2\lambda_1$ must be an integer between 0 and 2, implying that $\lambda_1=\frac{1}{2}$. Also, $-\mathcal{L}\lambda_1$ and $\mathcal{L}\lambda_1$ must be integers, but since $\lambda_1=\frac{1}{2}$ and $\mathcal{L}$ is odd, $-\mathcal{L}\lambda_1$ and $\mathcal{L}\lambda_1$ are never integers. Therefore, $\mathrm{Box}\left((2,-\mathcal{L},\mathcal{L})\right)\cap\mathbb{Z}^3=\emptyset$.

Since $P^*_{\mathcal{L}}$ is a centrally symmetric hexagon, we can restrict our analysis to three of its facets: $F_1:=\mathrm{conv}\left(\pm\left(\frac{1}{\mathcal{L}},-1\right),\pm(\frac{2}{\mathcal{L}},-1)\right)$, $F_2:=\mathrm{conv}\left(\pm(\frac{2}{\mathcal{L}},-1),\pm\left(\frac{1}{\mathcal{L}},0\right)\right)$, and $F_3:=\mathrm{conv}\left(\pm\left(\frac{1}{\mathcal{L}},0\right),\pm\left(-\frac{1}{\mathcal{L}},1\right)\right)$. We consider each facet separately.

Case: F_1**.** Observe:

$$\begin{aligned}\mathrm{Box}\left((F_1,\mathcal{L})\right)&=\{\lambda_1(1,-\mathcal{L},\mathcal{L})+\lambda_2(2,-\mathcal{L},\mathcal{L}):0<\lambda_1,\lambda_2<1\}\\&=\{(\lambda_1+2\lambda_2,-\mathcal{L}\lambda_1-\mathcal{L}\lambda_2,\mathcal{L}\lambda_1+\mathcal{L}\lambda_2:0<\lambda_1,\lambda_2<1\}.\end{aligned}$$

Let $\mathcal{L}=2k+1$ for $k\in\mathbb{Z}_{\geq 0}$. We now want to determine when $(A,-B,B)\in\mathrm{Box}\left((F_1,\mathcal{L})\right)$ is a lattice point. This reduces to solving a system of linear equations between A and B. In order for A to be an integer it must be 1 or 2. When $A=\lambda_1+2\lambda_2=1$, $B=\mathcal{L}\lambda_1+\mathcal{L}\lambda_2$ equals $\mathcal{L}-k$, $\mathcal{L}-k+1$,..., $\mathcal{L}-2$, or $\mathcal{L}-1$ with the restriction that $0<\lambda_1,\lambda_2<1$. When $A=\lambda_1+2\lambda_2=2$, $B=\mathcal{L}\lambda_1+\mathcal{L}\lambda_2$ equals $\mathcal{L}+1$, $\mathcal{L}+2$,..., $\mathcal{L}+k-1$, or $\mathcal{L}+k$. Therefore, $\mathrm{Box}\left((F_1,\mathcal{L})\right)\cap\mathbb{Z}^3$ contains the elements $\{(1,k-\mathcal{L},\mathcal{L}-k),(1,k-\mathcal{L}-1,\mathcal{L}-k+1),\ldots(1,2-\mathcal{L},\mathcal{L}-2),(1,1-\mathcal{L},\mathcal{L}-1),(2,-\mathcal{L}-1,\mathcal{L}+1),(2,-\mathcal{L}-2,\mathcal{L}+2),\ldots,(2,1-\mathcal{L}-k,\mathcal{L}+k+1),(2,-\mathcal{L}-k,\mathcal{L}+k)\}$. Therefore, the box polynomial of F_1 is

$$B(F_1;z)=\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1}z^i+\sum_{i=\mathcal{L}+1}^{\mathcal{L}+k}z^i\,.$$

Case: F_2**.** Observe:

$$\begin{aligned}\mathrm{Box}\left(F_2,\mathcal{L}\right)&=\{\lambda_1(2,-\mathcal{L},\mathcal{L})+\lambda_2(1,0,\mathcal{L}):0<\lambda_1,\lambda_2<1\}\\&=\{(2\lambda_1+\lambda_2,-\mathcal{L}\lambda_1,\mathcal{L}\lambda_1+\mathcal{L}\lambda_2):0<\lambda_1,\lambda_2<1\}.\end{aligned}$$

Suppose (A,B,C) is an integer point in this set. Again, determining the integer points in the box reduces to solving a system of linear equations between A and C with the added condition coming from B that $\lambda_1=\frac{1}{\mathcal{L}},\ldots,\frac{\mathcal{L}-1}{\mathcal{L}}$. It is straightforward to verify that the resulting box polynomial of F_2 is the same as F_1.

Case: F_3**.** Observe:

$$\begin{aligned}\mathrm{Box}\left(F_3,\mathcal{L}\right)&=\{\lambda_1(-1,\mathcal{L},\mathcal{L})+\lambda_2(1,0,\mathcal{L}):0<\lambda_1,\lambda_2<1\}\\&=\{(-\lambda_1+\lambda_2,\mathcal{L}\lambda_1,\mathcal{L}\lambda_1+\mathcal{L}\lambda_2):0<\lambda_1,\lambda_2<1\}.\end{aligned}$$

Suppose (A,B,C) is an integer point in this set. For A to be an integer it must be equal to zero, so we obtain $\lambda_1=\lambda_2$. The expression for B implies that $\lambda_1=\frac{m}{\mathcal{L}}$

Table 4.1: Table for the $\Omega \in T, \dim(\Omega), B(\Omega; z)$, and $h(\Omega, z^{\mathcal{L}})$.

$\Omega \in T$	$\dim(\Omega)$	$B(\Omega; z)$	$h(\Omega, z^{\mathcal{L}})$
F_1	1	$\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i$	1
$-F_1$	1	$\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i$	1
F_2	1	$\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i$	1
$-F_2$	1	$\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i$	1
F_3	1	$\sum_{i=1}^{\mathcal{L}-1} z^{2i}$	1
$-F_3$	1	$\sum_{i=1}^{\mathcal{L}-1} z^{2i}$	1
$\left(\frac{1}{\mathcal{L}}, 0\right)$	0	0	$1 + z^{\mathcal{L}}$
$\left(-\frac{1}{\mathcal{L}}, 0\right)$	0	0	$1 + z^{\mathcal{L}}$
$\left(\frac{2}{\mathcal{L}}, -1\right)$	0	0	$1 + z^{\mathcal{L}}$
$\left(-\frac{2}{\mathcal{L}}, 1\right)$	0	0	$1 + z^{\mathcal{L}}$
$\left(\frac{1}{\mathcal{L}}, -1\right)$	0	0	$1 + z^{\mathcal{L}}$
$\left(-\frac{1}{\mathcal{L}}, 1\right)$	0	0	$1 + z^{\mathcal{L}}$
$\emptyset$	-1	1	$1 + 4z^{\mathcal{L}} + z^{2\mathcal{L}}$

for some integer $m \in [1, \mathcal{L} - 1]$. Lastly, C then reduces to $2\mathcal{L}\lambda_1 = 2m$. Therefore, we conclude $\operatorname{Box}((F_3, \mathcal{L}))$ contains $\mathcal{L} - 1$ lattice points of the form $(0, m, 2m)$, one for each integer $m \in [1, \mathcal{L} - 1]$. This implies the box polynomial of F_3 is given by

$$B(F_3; z) = \sum_{i=1}^{\mathcal{L}-1} z^{2i} \,.$$

Combining the above analysis with the values in Table 4.3, we apply Theorems 4.2.4 and 4.3.1 and conclude that for $\mathcal{L} = 2k + 1$,

$$h^*(P^*_{\mathcal{L}}; z) = (1 + z + \cdots + z^{\mathcal{L}}) \left(1 + 4z^{\mathcal{L}} + z^{2\mathcal{L}} + 4 \left(\sum_{i=\mathcal{L}-k}^{\mathcal{L}-1} z^i + \sum_{i=\mathcal{L}+1}^{\mathcal{L}+k} z^i \right) + 2 \sum_{i=1}^{\mathcal{L}-1} z^{2i} \right).$$

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Vita

Andrés Rodolfo Vindas Meléndez

Place of Birth:

- Bellflower, South East Los Ángeles, CA

Education:

- University of Kentucky, Lexington, KY
  Graduate Certificate in Latin American, Caribbean, and Latino/a Studies, December 2019
- San Francisco State University, San Francisco, CA
  M.A. in Mathematics, August 2017
- University of California, Berkeley, Berkeley, CA
  B.A. in Mathematics, May 2015

Professional Positions:

- National Science Foundation LSAMP Bridge to Doctorate Fellow, Kentucky-West Virginia Louis Stokes Alliance for Minority Participation (KY-WV LSAMP), University of Kentucky, Lexington, KY, Fall 2020 - Summer 2021
- National Science Foundation Graduate Research Fellow, Department of Mathematics, University of Kentucky, Lexington, KY, Fall 2017 - Summer 2020
- Teaching Associate, Department of Mathematics, San Francisco State University, San Francisco, CA, Fall 2015 - Summer 2017

Honors

- 2021-2024 Postdoctoral Research Fellowship, National Science Foundation
- 2020-2021 Bridge to Doctorate Fellowship, National Science Foundation
- 2017-2020 Graduate Research Fellowship, National Science Foundation
- 2020 Graduate Oral Presentation Award, SACNAS
- 2020 Latinxs & Hispanics in the Mathematical Sciences Honoree, Lathisms
- 2020 Endowed Doctoral Fellowship Award, University of Kentucky Association of Emeriti Faculty
- 2019-2020 Chancellor's Doctoral Incentive Program Award, California State University System

- 2019 Student Professional Development Grant, Center for Graduate and Professional Diversity Initiatives at the University of Kentucky
- 2018 Student Professional Development Grant, Center for Graduate and Professional Diversity Initiatives at the University of Kentucky
- 2017-2018 Hispanic Scholarship Fund Scholarship

Publications & Preprints:

- *Triangulations, order polytopes, and generalized snake posets* (with M. Bell, B. Braun, D. Hanely, K. Serhiyenko, J. Vega, and M. Yip), submitted: arXiv:2102.11306.
- *Counting k-Naples parking functions through permutations and the k-Naples area statistic* (with L. Colmenarejo, P.E. Harris, Z. Jones, C. Keller, A. Ramos Rodríguez, and E. Sukarto), Enumerative Combinatorics and Applications, 1:2 (2021) Article S2R11, arXiv:2009.01124.
- *Decompositions of Ehrhart h^*-polynomials for rational polytopes* (with M. Beck and B. Braun), to appear in Discrete & Computational Geometry, arXiv:2006.10076.
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- *A brief survey on lattice zonotopes* (with B. Braun), Algebraic and Geometric Combinatorics on Lattice Polytopes, Proceedings of the Summer Workshop on Lattice Polytopes, T. Hibi and A. Tsuchiya (eds), World Scientific, New Jersey, 2019, pp. 101-116.
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