Effect of Thomas Rotation on the Lorentz Transformation of Electromagnetic fields

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ABSTRACT

This is the supplementary information for the main article. It includes the mathematical equations for all the electromagnetic fields in different reference frames.

1 Notation

We begin by recalling that the electromagnetic field tensor $F^{\mu\nu}$ is an anti-symmetric matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{1}$$

Defining some dummy variables to make equations look cleaner:

$$\begin{aligned}
\lambda_1 &= \sqrt{\beta_x^2+\beta_y^2+\beta_z^2} \\
\lambda_2 &= \beta_x\delta\beta_x+\beta_y\delta\beta_y+\beta_z\delta\beta_z \\
\lambda_3 &= \beta_y^2\delta\beta_x-\beta_x\beta_y\delta\beta_y+\beta_z\left(\beta_z\delta\beta_x-\beta_x\delta\beta_z\right) \\
\lambda_4 &= \beta_x^2\delta\beta_y-\beta_x\beta_y\delta\beta_x+\beta_z\left(\beta_z\delta\beta_y-\beta_y\delta\beta_z\right) \\
\lambda_5 &= \beta_x^2\delta\beta_z-\beta_x\beta_z\delta\beta_x+\beta_y\left(\beta_y\delta\beta_z-\beta_z\delta\beta_y\right) \\
\lambda_6 &= \beta_x\delta\beta_y-\beta_y\delta\beta_x \\
\lambda_7 &= \beta_y\delta\beta_z-\beta_z\delta\beta_y \\
\lambda_8 &= \beta_z\delta\beta_x-\beta_x\delta\beta_z \\
\eta_1 &= \sqrt{\beta_x^2+\beta_y^2} \\
\eta_2 &= \sqrt{\beta_y^2+\beta_z^2} \\
\eta_3 &= \sqrt{\beta_x^2+\beta_z^2}
\end{aligned}$$

The detailed expressions of the electromagnetic fields (to the first order in $\delta\beta$) is given in the next two sections.

2 Longitudinal-Transverse ℓt-Frame

2.1 Direct Boosted Frame $(\vec{\beta}+\delta\vec{\beta})$

$$(F'')^{10}=\frac{1}{\lambda_1}\Bigg[\beta_x E_x+\beta_y E_y+\beta_z E_z-\lambda_3 E_x\gamma^2-\lambda_4 E_y\gamma^2-\lambda_5 E_z\gamma^2+B_z\lambda_6\gamma+B_y\lambda_8\gamma+B_x\lambda_7\gamma+\frac{\gamma(\gamma-1)\eta_2^2\delta\beta_x E_x}{\lambda_1^2} \\ +\frac{\gamma(\gamma-1)\eta_3^2\delta\beta_y E_y}{\lambda_1^2}+\frac{\gamma(\gamma-1)\eta_1^2\delta\beta_z E_z}{\lambda_1^2}-\frac{\gamma(\gamma-1)\beta_x\beta_y\delta\beta_y E_x}{\lambda_1^2}-\frac{\gamma(\gamma-1)\beta_x\beta_z\delta\beta_z E_x}{\lambda_1^2}-\frac{\gamma(\gamma-1)\beta_x\beta_y\delta\beta_x E_y}{\lambda_1^2} \\ -\frac{\gamma(\gamma-1)\beta_y\beta_z\delta\beta_z E_y}{\lambda_1^2}-\frac{\gamma(\gamma-1)\beta_x\beta_z\delta\beta_x E_z}{\lambda_1^2}-\frac{\gamma(\gamma-1)\beta_y\beta_z\delta\beta_y E_z}{\lambda_1^2}\Bigg]$$

$$(F'')^{20}=-\gamma B_z\eta_1+\frac{\gamma B_x\beta_x\beta_z}{\eta_1}+\frac{\gamma B_y\beta_y\beta_z}{\eta_1}-\frac{\gamma E_x\beta_y}{\eta_1}+\frac{\gamma\beta_x E_y}{\eta_1}-\frac{\gamma^3 B_z\beta_x\eta_1\delta\beta_x}{\lambda_1^2}+\frac{\gamma^2\delta\beta_x\beta_y\beta_z E_z}{\eta_1}-\frac{\gamma^3 B_z\beta_y\eta_1\delta\beta_y}{\lambda_1^2} \\ +\frac{\gamma\left(\gamma^2-1\right)B_x\beta_x^2\beta_z\delta\beta_x}{\eta_1\lambda_1^2}+\frac{\gamma\left(\gamma^2-1\right)B_y\beta_x\delta\beta_x\beta_y\beta_z}{\eta_1\lambda_1^2}-\frac{\gamma B_z\beta_x\delta\beta_x\beta_z^2}{\eta_1\lambda_1^2}-\frac{\gamma B_z\beta_y\delta\beta_y\beta_z^2}{\eta_1\lambda_1^2}+\frac{\gamma B_x\beta_x\eta_1\delta\beta_z}{\lambda_1^2} \\ +\frac{\gamma\left(\gamma^2-1\right)B_y\beta_y^2\delta\beta_y\beta_z}{\eta_1\lambda_1^2}+\frac{\gamma\left(\gamma^2-1\right)B_x\beta_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1^2}-\frac{\gamma\left(\gamma^2-1\right)B_z\beta_z\eta_1\delta\beta_z}{\lambda_1^2}-\frac{(\gamma-1)\gamma^2\beta_x\delta\beta_x E_x\beta_y}{\eta_1} \\ +\frac{\gamma^3 B_x\beta_x\beta_z^2\delta\beta_z}{\eta_1\lambda_1^2}+\frac{\gamma^3 B_y\beta_y\beta_z^2\delta\beta_z}{\eta_1\lambda_1^2}+\frac{\gamma B_y\beta_y\eta_1\delta\beta_z}{\lambda_1^2}-\frac{\gamma^3 E_x\beta_y^2\delta\beta_y}{\eta_1}-\frac{\gamma^2\beta_x^2 E_x\delta\beta_y}{\eta_1}-\frac{\gamma^3 E_x\beta_y\beta_z\delta\beta_z}{\eta_1}+\frac{\gamma^3\beta_x^2\delta\beta_x E_y}{\eta_1} \\ +\frac{(\gamma-1)\gamma\beta_x^2 E_x\delta\beta_y}{\eta_1\lambda_1^2}-\frac{(\gamma-1)\gamma\beta_x\delta\beta_x E_x\beta_y}{\eta_1\lambda_1^2}+\frac{(\gamma-1)\gamma^2\beta_x\beta_y\delta\beta_y E_y}{\eta_1}-\frac{(\gamma-1)\gamma\delta\beta_x\beta_y^2 E_y}{\eta_1\lambda_1^2}+\frac{\gamma^2\delta\beta_x\beta_y^2 E_y}{\eta_1} \\ +\frac{\gamma^3\beta_x E_y\beta_z\delta\beta_z}{\eta_1}-\frac{\gamma^2\beta_x\delta\beta_y\beta_z E_z}{\eta_1}+\frac{(\gamma-1)\gamma\beta_x\delta\beta_y\beta_z E_z}{\eta_1\lambda_1^2}-\frac{(\gamma-1)\gamma\delta\beta_x\beta_y\beta_z E_z}{\eta_1\lambda_1^2}+\frac{(\gamma-1)\gamma\beta_x\beta_y\delta\beta_y E_y}{\eta_1\lambda_1^2}$$

$$(F'')^{30}=\frac{\gamma B_y\beta_x\lambda_1}{\eta_1}-\frac{\gamma B_x\beta_y\lambda_1}{\eta_1}-\frac{\gamma\beta_x E_x\beta_z}{\eta_1\lambda_1}-\frac{\gamma\beta_y E_y\beta_z}{\eta_1\lambda_1}+\frac{\gamma E_z\eta_1}{\lambda_1}+\frac{\gamma^3 B_y\beta_x^2\delta\beta_x}{\eta_1\lambda_1}+\frac{\gamma B_y\delta\beta_x\beta_y^2}{\eta_1\lambda_1}-\frac{\gamma B_x\beta_x^2\delta\beta_y}{\eta_1\lambda_1} \\ -\frac{\gamma^3 B_x\beta_y^2\delta\beta_y}{\eta_1\lambda_1}-\frac{\gamma B_z\beta_z\lambda_6}{\eta_1\lambda_1}+\frac{\gamma\left(\gamma^2-1\right)B_y\beta_x\beta_y\delta\beta_y}{\eta_1\lambda_1}-\frac{\gamma\left(\gamma^2-1\right)B_x\beta_x\delta\beta_x\beta_y}{\eta_1\lambda_1}-\frac{(\gamma-1)\gamma\beta_x^2\delta\beta_x E_x\beta_z}{\eta_1\lambda_1^3} \\ +\frac{\gamma^3 B_y\beta_x\beta_z\delta\beta_z}{\eta_1\lambda_1}-\frac{\gamma^3 B_x\beta_y\beta_z\delta\beta_z}{\eta_1\lambda_1}-\frac{\gamma^2\beta_x E_x\delta\beta_z\eta_1}{\lambda_1}-\frac{(\gamma-1)\gamma^2\beta_x^2\delta\beta_x E_x\beta_z}{\eta_1\lambda_1}-\frac{(\gamma-1)\gamma^2\beta_x E_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1} \\ +\frac{(\gamma-1)\gamma\beta_x E_x\delta\beta_z\eta_1}{\lambda_1^3}-\frac{(\gamma-1)\gamma\beta_x E_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1^3}-\frac{(\gamma-1)\gamma^2\beta_x\delta\beta_x\beta_y E_y\beta_z}{\eta_1\lambda_1}-\frac{(\gamma-1)\gamma^2\beta_y^2\delta\beta_y E_y\beta_z}{\eta_1\lambda_1} \\ -\frac{\gamma^3\beta_x E_x\beta_z^2\delta\beta_z}{\eta_1\lambda_1}-\frac{\gamma^2\beta_y E_y\delta\beta_z\eta_1}{\lambda_1}+\frac{(\gamma-1)\gamma\beta_y E_y\lambda_5}{\eta_1\lambda_1^3}-\frac{\gamma^3\beta_y E_y\beta_z^2\delta\beta_z}{\eta_1\lambda_1}+\frac{(\gamma-1)\gamma\beta_z\delta\beta_z E_z\eta_1}{\lambda_1^3} \\ +\frac{\gamma^3\beta_y\delta\beta_y E_z\eta_1}{\lambda_1}+\frac{\gamma^3\beta_x\delta\beta_x E_z\eta_1}{\lambda_1}+\frac{\gamma^2\beta_x\delta\beta_x\beta_z^2 E_z}{\eta_1\lambda_1}-\frac{(\gamma-1)\gamma\beta_y\delta\beta_y\beta_z^2 E_z}{\eta_1\lambda_1^3}+\frac{(\gamma-1)\gamma^2\beta_z\delta\beta_z E_z\eta_1^3}{\lambda_1^3} \\ +\frac{\gamma^2\beta_y\delta\beta_y\beta_z^2 E_z}{\eta_1\lambda_1}+\frac{(\gamma-1)\gamma^2\beta_z^3\delta\beta_z E_z\eta_1}{\lambda_1^3}-\frac{(\gamma-1)\gamma\beta_x\delta\beta_x\beta_z^2 E_z}{\eta_1\lambda_1^3}$$

$$(F'')^{32}=\frac{1}{\lambda_1}\left[B_x\beta_x+B_y\beta_y+B_z\beta_z-\gamma E_x\lambda_7-\gamma E_y\lambda_8-\gamma E_z\lambda_6-\frac{(\gamma-1)B_z\lambda_5}{\lambda_1^2}-\frac{(\gamma-1)B_y\lambda_4}{\lambda_1^2}-\frac{(\gamma-1)B_x\lambda_3}{\lambda_1^2}\right]$$

$$
\begin{aligned}
(F'')^{13} = & -\frac{\gamma B_x\beta_y}{\eta_1}+\frac{\gamma B_y\beta_x}{\eta_1}-\frac{\gamma\beta_x E_x\beta_z}{\eta_1}-\frac{\gamma\beta_y E_y\beta_z}{\eta_1}+\gamma E_z\eta_1-\frac{\gamma^3 B_x\beta_x\delta\beta_x\beta_y}{\eta_1}-\frac{\gamma^3 B_x\beta_y^2\delta\beta_y}{\eta_1}-\frac{\gamma^3 B_x\beta_y\beta_z\delta\beta_z}{\eta_1}\\
&+\frac{\gamma^3 B_y\beta_x\beta_y\delta\beta_y}{\eta_1}+\frac{\gamma^3 B_y\beta_x^2\delta\beta_x}{\eta_1}+\frac{\gamma^3 B_y\beta_x\beta_z\delta\beta_z}{\eta_1}-\frac{\gamma^3\beta_x E_x\beta_z^2\delta\beta_z}{\eta_1\lambda_1^2}+\frac{(\gamma-1)B_x\beta_x\delta\beta_x\beta_y}{\eta_1\lambda_1^2}-\frac{(\gamma-1)B_x\beta_x^2\delta\beta_y}{\eta_1\lambda_1^2}\\
&+\frac{(\gamma-1)B_y\delta\beta_x\beta_y^2}{\eta_1\lambda_1^2}+\frac{(\gamma-1)B_z\delta\beta_x\beta_y\beta_z}{\eta_1\lambda_1^2}-\frac{(\gamma-1)B_y\beta_x\beta_y\delta\beta_y}{\eta_1\lambda_1^2}-\frac{(\gamma-1)B_z\beta_x\delta\beta_y\beta_z}{\eta_1\lambda_1^2}-\frac{\left(\gamma^2-1\right)\gamma\beta_x^2\delta\beta_x E_x\beta_z}{\eta_1\lambda_1^2}\\
&-\frac{\left(\gamma^2-1\right)\gamma\beta_x E_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1^2}-\frac{\gamma\beta_x E_x\delta\beta_z\eta_1}{\lambda_1^2}-\frac{\gamma\left(\gamma^2-1\right)\beta_y^2\delta\beta_y E_y\beta_z}{\eta_1\lambda_1^2}-\frac{\gamma\left(\gamma^2-1\right)\beta_x\delta\beta_x\beta_y E_y\beta_z}{\eta_1\lambda_1^2}-\frac{\gamma\beta_y E_y\delta\beta_z\eta_1}{\lambda_1^2}\\
&-\frac{\gamma^3\beta_y E_y\beta_z^2\delta\beta_z}{\eta_1\lambda_1^2}+\frac{\gamma^3\beta_y\delta\beta_y E_z\eta_1}{\lambda_1^2}+\frac{\gamma^3\beta_x\delta\beta_x E_z\eta_1}{\lambda_1^2}+\frac{\gamma\beta_x\delta\beta_x\beta_z^2 E_z}{\eta_1\lambda_1^2}+\frac{\gamma\left(\gamma^2-1\right)\beta_z\delta\beta_z E_z\eta_1}{\lambda_1^2}+\frac{\gamma\beta_y\delta\beta_y\beta_z^2 E_z}{\eta_1\lambda_1^2}
\end{aligned}
$$

$$
\begin{aligned}
(F'')^{12} = & \frac{\gamma B_z\eta_1}{\lambda_1}-\frac{\gamma B_x\beta_x\beta_z}{\eta_1\lambda_1}-\frac{\gamma B_y\beta_y\beta_z}{\eta_1\lambda_1}+\frac{\gamma E_x\beta_y\lambda_1}{\eta_1}-\frac{\gamma\beta_x E_y\lambda_1}{\eta_1}+\frac{(\gamma-1)B_x\beta_x^2\delta\beta_x\beta_z}{\eta_1\lambda_1^3}-\frac{\gamma^3 B_x\beta_x^2\delta\beta_x\beta_z}{\eta_1\lambda_1}-\frac{\gamma\delta\beta_x\beta_y^2 E_y}{\eta_1\lambda_1}\\
&-\frac{\gamma^3 B_y\beta_x\delta\beta_x\beta_y\beta_z}{\eta_1\lambda_1}+\frac{\gamma^3 B_z\beta_x\delta\beta_x\eta_1}{\lambda_1}+\frac{\gamma^3 B_z\beta_y\delta\beta_y\eta_1}{\lambda_1}-\frac{\gamma^3 B_x\beta_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1}-\frac{\gamma^3 B_y\beta_y^2\delta\beta_y\beta_z}{\eta_1\lambda_1}-\frac{\gamma^3 B_x\beta_x\beta_z^2\delta\beta_z}{\eta_1\lambda_1}\\
&+\frac{(\gamma-1)B_z\beta_x\delta\beta_x\beta_z^2}{\eta_1\lambda_1^3}+\frac{(\gamma-1)B_y\beta_x\delta\beta_x\beta_y\beta_z}{\eta_1\lambda_1^3}+\frac{(\gamma-1)B_x\beta_x\beta_y\delta\beta_y\beta_z}{\eta_1\lambda_1^3}+\frac{(\gamma-1)B_y\beta_y^2\delta\beta_y\beta_z}{\eta_1\lambda_1^3}+\frac{(\gamma-1)B_z\beta_y\delta\beta_y\beta_z^2}{\eta_1\lambda_1^3}\\
&-\frac{(\gamma-1)B_x\beta_x\delta\beta_z\eta_1}{\lambda_1^3}-\frac{(\gamma-1)B_y\beta_y\delta\beta_z\eta_1}{\lambda_1^3}-\frac{(\gamma-1)B_z\beta_z\delta\beta_z\eta_1}{\lambda_1^3}+\frac{\gamma\left(\gamma^2-1\right)\beta_x\delta\beta_x E_x\beta_y}{\eta_1\lambda_1}-\frac{\left(\gamma^2-1\right)\gamma\beta_x\beta_y\delta\beta_y E_y}{\eta_1\lambda_1}\\
&-\frac{\gamma^3 B_y\beta_y\beta_z^2\delta\beta_z}{\eta_1\lambda_1}+\frac{\gamma^3 B_z\beta_z\delta\beta_z\eta_1}{\lambda_1}+\frac{\gamma^3 E_x\beta_y^2\delta\beta_y}{\eta_1\lambda_1}+\frac{\gamma^3 E_x\beta_y\beta_z\delta\beta_z}{\eta_1\lambda_1}+\frac{\gamma\beta_x^2 E_x\delta\beta_y}{\eta_1\lambda_1}-\frac{\gamma^3\beta_x^2\delta\beta_x E_y}{\eta_1\lambda_1}-\frac{\gamma^3\beta_x E_y\beta_z\delta\beta_z}{\eta_1\lambda_1}\\
&-\frac{\gamma\delta\beta_x\beta_y\beta_z E_z}{\eta_1\lambda_1}+\frac{\gamma\beta_x\delta\beta_y\beta_z E_z}{\eta_1\lambda_1}
\end{aligned}
$$

2.2 Successively Boosted Frame $\vec{\beta}$ and $\Delta\vec{\beta}$

$$
\begin{aligned}
(F''')^{10} = & \frac{\beta_x E_x}{\lambda_1}+\frac{\beta_y E_y}{\lambda_1}+\frac{\beta_z E_z}{\lambda_1}+\frac{\gamma^2 B_z\lambda_6}{\lambda_1}+\frac{\gamma^2 B_y\lambda_8}{\lambda_1}+\frac{\gamma^2 B_x\lambda_7}{\lambda_1}+\frac{\gamma^2\beta_x E_x\beta_y\delta\beta_y}{\lambda_1}+\frac{\gamma^2\beta_x E_x\beta_z\delta\beta_z}{\lambda_1}-\frac{\gamma^2\delta\beta_x E_x\eta_2^2}{\lambda_1}\\
&+\frac{\gamma^2\beta_y E_y\beta_z\delta\beta_z}{\lambda_1}-\frac{\gamma^2\delta\beta_y E_y\eta_3^2}{\lambda_1}+\frac{\gamma^2\beta_x\delta\beta_x\beta_y E_y}{\lambda_1}+\frac{\gamma^2\beta_y\delta\beta_y\beta_z E_z}{\lambda_1}+\frac{\gamma^2\beta_x\delta\beta_x\beta_z E_z}{\lambda_1}-\frac{\gamma^2\delta\beta_z E_z\eta_1^2}{\lambda_1}
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{20} = & \frac{2\gamma B_y\beta_y\beta_z}{\eta_1}-\gamma B_z\eta_1-\frac{\gamma E_x\beta_y}{\eta_1}+\frac{\gamma\beta_x E_y}{\eta_1}-\frac{\gamma^3 B_z\eta_1\lambda_2}{\lambda_1^2}+\frac{\gamma B_z\beta_z\lambda_5}{\eta_1\lambda_1^2}-\frac{\gamma^3 E_x\beta_y\lambda_2}{\eta_1}+\frac{\gamma^3\beta_x E_y\lambda_2}{\eta_1}+\frac{2\gamma B_y\beta_y\delta\beta_z\eta_1}{\lambda_1^2}\\
&+\frac{2\gamma\left(\gamma^2-1\right)B_y\beta_y^2\delta\beta_y\beta_z}{\eta_1\lambda_1^2}+\frac{2\gamma\left(\gamma^2-1\right)B_y\beta_x\delta\beta_x\beta_y\beta_z}{\eta_1\lambda_1^2}+\frac{2\gamma^3 B_y\beta_y\beta_z^2\delta\beta_z}{\eta_1\lambda_1^2}
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{30} = & \frac{\gamma B_y\beta_x\lambda_1}{\eta_1}-\frac{\gamma B_x\beta_y\lambda_1}{\eta_1}-\frac{\gamma\beta_x E_x\beta_z}{\eta_1\lambda_1}-\frac{\gamma\beta_y E_y\beta_z}{\eta_1\lambda_1}+\frac{\gamma E_z\eta_1}{\lambda_1}-\frac{\gamma B_x\beta_x\lambda_6}{\eta_1\lambda_1}-\frac{\gamma B_y\beta_y\lambda_6}{\eta_1\lambda_1}-\frac{\gamma B_z\beta_z\lambda_6}{\eta_1\lambda_1}+\frac{\gamma^3 B_y\beta_x\lambda_2}{\eta_1\lambda_1}\\
&-\frac{\gamma^3 B_x\beta_y\lambda_2}{\eta_1\lambda_1}-\frac{\gamma^3\beta_x E_x\beta_z\lambda_2}{\eta_1\lambda_1}-\frac{\gamma^3\beta_y E_y\beta_z\lambda_2}{\eta_1\lambda_1}+\frac{\gamma^3 E_z\eta_1\lambda_2}{\lambda_1}
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{32} = & \frac{1}{\lambda_1}\left[B_x\beta_x+B_y\beta_y+B_z\beta_z-\gamma^2 E_x\lambda_7-\gamma^2 E_y\lambda_8-\gamma^2 E_z\lambda_6+\gamma^2 B_z\beta_x\delta\beta_x\beta_z+\gamma^2 B_x\beta_x\beta_y\delta\beta_y+\gamma^2 B_z\beta_y\delta\beta_y\beta_z\right.\\
&\left.-\gamma^2 B_x\delta\beta_x\eta_2^2-\gamma^2 B_y\delta\beta_y\eta_3^2-\gamma^2 B_z\delta\beta_z\eta_1^2+\gamma^2 B_y\beta_y\beta_z\delta\beta_z+\gamma^2 B_x\beta_x\beta_z\delta\beta_z+\gamma^2 B_y\beta_x\delta\beta_x\beta_y\right]
\end{aligned}
$$

$$(F''')^{13} = -\frac{\gamma B_x\beta_y}{\eta_1} + \frac{\gamma B_y\beta_x}{\eta_1} - \frac{\gamma\beta_x E_x\beta_z}{\eta_1} - \frac{\gamma\beta_y E_y\beta_z}{\eta_1} + \gamma E_z\eta_1 - \frac{\gamma^3 B_x\beta_y\lambda_2}{\eta_1} + \frac{\gamma^3 B_y\beta_x\lambda_2}{\eta_1} - \frac{\gamma^3\beta_x E_x\beta_z\lambda_2}{\eta_1\lambda_1^2} - \frac{\gamma^3\beta_y E_y\beta_z\lambda_2}{\eta_1\lambda_1^2}$$
$$- \frac{\gamma\beta_x E_x\lambda_5}{\eta_1\lambda_1^2} - \frac{\gamma\beta_y E_y\lambda_5}{\eta_1\lambda_1^2} + \frac{\gamma^3 E_z\eta_1\lambda_2}{\lambda_1^2} - \frac{\gamma\beta_z E_z\lambda_5}{\eta_1\lambda_1^2}$$

$$(F''')^{21} = \frac{\gamma B_z\eta_1}{\lambda_1} - \frac{\gamma B_x\beta_x\beta_z}{\eta_1\lambda_1} - \frac{\gamma B_y\beta_y\beta_z}{\eta_1\lambda_1} + \frac{\gamma E_x\beta_y\lambda_1}{\eta_1} - \frac{\gamma\beta_x E_y\lambda_1}{\eta_1} - \frac{\gamma^3 B_x\beta_x\beta_z\lambda_2}{\eta_1\lambda_1} - \frac{\gamma^3 B_y\beta_y\beta_z\lambda_2}{\eta_1\lambda_1} + \frac{\gamma^3 B_z\eta_1\lambda_2}{\lambda_1} + \frac{\gamma^3 E_x\beta_y\lambda_2}{\eta_1\lambda_1}$$
$$+ \frac{\gamma\beta_x E_x\lambda_6}{\eta_1\lambda_1} - \frac{\gamma^3\beta_x E_y\lambda_2}{\eta_1\lambda_1} + \frac{\gamma\beta_y E_y\lambda_6}{\eta_1\lambda_1} + \frac{\gamma\beta_z E_z\lambda_6}{\eta_1\lambda_1}$$

3 Laboratory xy-Frame

3.1 Direct Boosted Frame ($\vec{\beta} + \delta\vec{\beta}$)

$$(F'')^{10} = \gamma B_z\beta_y - \gamma B_y\beta_z - \gamma^2\beta_x^2 E_x + \frac{1}{\lambda_1^2}\Big[\gamma E_x\left(\gamma\beta_x^2 + \beta_y^2 + \beta_z^2\right) - (\gamma-1)\beta_x\beta_y E_y - (\gamma-1)\beta_x\beta_z E_z - \gamma^3 B_y\beta_z\lambda_2 + \gamma^3 B_z\beta_y\lambda_2$$
$$+\gamma B_z\lambda_4 + \gamma^3\beta_x\delta\beta_x E_x\eta_2^2 + \gamma^3 E_x\beta_y\delta\beta_y\eta_2^2 + \gamma^3 E_x\beta_z\delta\beta_z\eta_2^2 - \gamma^3\beta_x\beta_y\delta\beta_y\beta_z E_z - \gamma^3\beta_x\beta_z^2\delta\beta_z E_z - \gamma^3\beta_x^2\delta\beta_x\beta_y E_y$$
$$-\gamma B_y\lambda_5 - \gamma^3\beta_x\beta_y^2\delta\beta_y E_y - \gamma^3\beta_x\beta_y E_y\beta_z\delta\beta_z - \gamma^3\beta_x^2\delta\beta_x\beta_z E_z + \frac{2(\gamma-1)\beta_x^2 E_x\beta_y\delta\beta_y}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x^2 E_x\beta_z\delta\beta_z}{\lambda_1^2}$$
$$-\frac{(\gamma-1)\delta\beta_x\beta_y E_y\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\beta_y E_y\beta_z\delta\beta_z}{\lambda_1^2} - \frac{(\gamma-1)\beta_x\delta\beta_y E_y\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\beta_y\delta\beta_y\beta_z E_z}{\lambda_1^2}$$
$$-\frac{(\gamma-1)\delta\beta_x\beta_z E_z\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)\beta_x\delta\beta_z E_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2} - \frac{2(\gamma-1)\beta_x\delta\beta_x E_x\eta_2^2}{\lambda_1^2}\Big]$$

$$(F'')^{20} = \gamma B_x\beta_z - \gamma B_z\beta_x - \gamma^2\beta_x E_x\beta_y - \gamma^2\beta_y^2 E_y - \gamma^2\beta_y\beta_z E_z + \frac{1}{\lambda_1^2}\Big[\gamma^2\beta_y^2 E_y + \gamma E_y\eta_3^2 + \gamma^3\beta_x\delta\beta_x E_y\eta_3^2 + \gamma^3\beta_y\delta\beta_y E_y\eta_3^2$$
$$+(\gamma-1)\gamma\beta_x E_x\beta_y + (\gamma-1)\gamma\beta_y\beta_z E_z - \gamma^3 B_z\beta_x^2\delta\beta_x + \left(\gamma^2-1\right)\gamma B_x\beta_x\delta\beta_x\beta_z - \left(\gamma^2-1\right)\gamma B_z\beta_x\beta_y\delta\beta_y - \gamma B_z\delta\beta_x\eta_2^2$$
$$+\gamma^3 B_x\beta_z^2\delta\beta_z - \left(\gamma^2-1\right)\gamma B_z\beta_x\beta_z\delta\beta_z + \gamma\left(\gamma^2-1\right)B_x\beta_y\delta\beta_y\beta_z + \gamma B_x\delta\beta_z\eta_1^2 - \gamma^3\beta_x E_x\beta_y\beta_z\delta\beta_z + \gamma^3 E_y\beta_z\delta\beta_z\eta_3^2$$
$$-\gamma^3\beta_x^2\delta\beta_x E_x\beta_y - \gamma^3\beta_x E_x\beta_y^2\delta\beta_y - \gamma^3\beta_x\delta\beta_x\beta_y\beta_z E_z - \gamma^3\beta_y^2\delta\beta_y\beta_z E_z - \gamma^3\beta_y\beta_z^2\delta\beta_z E_z - \frac{(\gamma-1)\beta_y\delta\beta_z E_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2}$$
$$-\frac{(\gamma-1)\delta\beta_x E_x\beta_y\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)\beta_x E_x\delta\beta_y\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\delta\beta_x\beta_y^2 E_y}{\lambda_1^2} - \frac{2(\gamma-1)\beta_y\delta\beta_y E_y\eta_3^2}{\lambda_1^2}$$
$$+\frac{2(\gamma-1)\beta_y^2 E_y\beta_z\delta\beta_z}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\delta\beta_x\beta_y\beta_z E_z}{\lambda_1^2} - \frac{(\gamma-1)\delta\beta_y\beta_z E_z\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x E_x\beta_y\beta_z\delta\beta_z}{\lambda_1^2}\Big]$$

$$(F'')^{30} = \gamma B_y\beta_x - \gamma B_x\beta_y - \gamma^2\beta_z^2 E_z + \frac{1}{\lambda_1^2}\Big[\gamma^3 B_y\beta_x^2\delta\beta_x - (\gamma-1)\beta_x E_x\beta_z - (\gamma-1)\beta_y E_y\beta_z + \gamma E_z\left(\beta_x^2+\beta_y^2+\gamma\beta_z^2\right)$$
$$+\gamma^3 B_y\beta_x\beta_y\delta\beta_y - \gamma^3 B_x\beta_y^2\delta\beta_y - \left(\gamma^2-1\right)\gamma B_x\beta_x\delta\beta_x\beta_y + \gamma B_y\delta\beta_x\eta_2^2 - \gamma B_y\beta_x\beta_y\delta\beta_y + \gamma^3 B_y\beta_x\beta_z\delta\beta_z$$
$$-\gamma^3 B_x\beta_y\beta_z\delta\beta_z + \gamma B_x\beta_y\beta_z\delta\beta_z - \gamma B_x\delta\beta_y\eta_3^2 - \gamma B_y\beta_x\beta_z\delta\beta_z - \gamma^3\beta_x^2\delta\beta_x E_x\beta_z - \gamma^3\beta_x E_x\beta_y\delta\beta_y\beta_z - \gamma^3\beta_y E_y\beta_z^2\delta\beta_z$$
$$-\gamma^3\beta_x E_x\beta_z^2\delta\beta_z - \gamma^3\beta_x\delta\beta_x\beta_y E_y\beta_z - \gamma^3\beta_y^2\delta\beta_y E_y\beta_z + \gamma^3\beta_x\delta\beta_x E_z\eta_1^2 + \gamma^3\beta_y\delta\beta_y E_z\eta_1^2 + \gamma^3\beta_z\delta\beta_z E_z\eta_1^2$$
$$-\frac{(\gamma-1)\delta\beta_x E_x\beta_z\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x E_x\beta_y\delta\beta_y\beta_z}{\lambda_1^2} - \frac{(\gamma-1)\beta_x E_x\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\delta\beta_x\beta_y E_y\beta_z}{\lambda_1^2}$$
$$-\frac{(\gamma-1)\delta\beta_y E_y\beta_z\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)\beta_y E_y\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)\beta_x\delta\beta_x\beta_z^2 E_z}{\lambda_1^2} + \frac{2(\gamma-1)\beta_y\delta\beta_y\beta_z^2 E_z}{\lambda_1^2}$$
$$-\frac{2(\gamma-1)\beta_z\delta\beta_z E_z\eta_1^2}{\lambda_1^2}\Big]$$

$$(F'')^{32} = \gamma E_y\beta_z - \gamma\beta_y E_z + \frac{1}{\lambda_1^2}\Big[B_x\beta_x^2 + \gamma B_x\eta_2^2 - (\gamma-1)B_y\beta_x\beta_y - (\gamma-1)B_z\beta_x\beta_z - \gamma^3 B_y\beta_x^2\delta\beta_x\beta_y - \gamma^3 B_z\beta_x^2\delta\beta_x\beta_z$$
$$+\gamma^3 B_x\beta_x\delta\beta_x\eta_2^2 + \gamma^3 B_x\beta_y\delta\beta_y\eta_2^2 - \gamma^3 B_y\beta_x\beta_y^2\delta\beta_y - \gamma^3 B_z\beta_x\beta_y\delta\beta_y\beta_z + \gamma^3 B_x\beta_z\delta\beta_z\eta_2^2 - \gamma^3 B_z\beta_x\beta_z^2\delta\beta_z$$
$$-\gamma^3 B_y\beta_x\beta_y\beta_z\delta\beta_z - \gamma\delta\beta_y E_z\eta_3^2 + \gamma^3 E_y\beta_z^2\delta\beta_z + \left(\gamma^2-1\right)\gamma\beta_y\delta\beta_y E_y\beta_z + \gamma\left(\gamma^2-1\right)\beta_x\delta\beta_x E_y\beta_z + \gamma E_y\delta\beta_z\eta_1^2$$
$$-\gamma^3\beta_y^2\delta\beta_y E_z - \left(\gamma^2-1\right)\gamma\beta_x\delta\beta_x\beta_y E_z - \left(\gamma^2-1\right)\gamma\beta_y\beta_z\delta\beta_z E_z - \frac{2(\gamma-1)B_x\beta_x\delta\beta_x\eta_2^2}{\lambda_1^2} + \frac{2(\gamma-1)B_x\beta_x^2\beta_z\delta\beta_z}{\lambda_1^2}$$
$$+\frac{2(\gamma-1)B_y\beta_x\beta_y\beta_z\delta\beta_z}{\lambda_1^2} + \frac{2(\gamma-1)B_x\beta_x^2\beta_y\delta\beta_y}{\lambda_1^2} - \frac{(\gamma-1)B_z\delta\beta_x\beta_z\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)B_z\beta_x\beta_y\delta\beta_y\beta_z}{\lambda_1^2}$$
$$-\frac{(\gamma-1)B_y\beta_x\delta\beta_y\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)B_y\delta\beta_x\beta_y\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)B_z\beta_x\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2}\Big]$$

$$(F'')^{13} = -\gamma E_x\beta_z + \gamma\beta_x E_z + \frac{1}{\lambda_1^2}\Big[B_y\beta_y^2 + \gamma B_y\eta_3^2 - (\gamma-1)B_z\beta_y\beta_z - (\gamma-1)B_x\beta_x\beta_y - \gamma^3 B_x\beta_x^2\delta\beta_x\beta_y - \gamma^3 B_z\beta_y^2\delta\beta_y\beta_z$$
$$-\gamma^3 B_x\beta_x\beta_y^2\delta\beta_y - \gamma^3 B_x\beta_x\beta_y\beta_z\delta\beta_z + \gamma^3\beta_x\beta_z\delta\beta_z E_z + \gamma^3\beta_x^2\delta\beta_x E_z + \gamma\delta\beta_x E_z\eta_2^2 - \gamma\beta_x\beta_y\delta\beta_y E_z - \gamma\beta_x\beta_z\delta\beta_z E_z$$
$$-\gamma^3 E_x\beta_z^2\delta\beta_z - \left(\gamma^2-1\right)\gamma\beta_x\delta\beta_x E_x\beta_z - \left(\gamma^2-1\right)\gamma E_x\beta_y\delta\beta_y\beta_z - \gamma E_x\delta\beta_z\eta_1^2 + \gamma^3\beta_x\beta_y\delta\beta_y E_z - \gamma^3 B_z\beta_y\beta_z^2\delta\beta_z$$
$$+\gamma^3 B_y\beta_x\delta\beta_x\eta_3^2 + \gamma^3 B_y\beta_z\delta\beta_z\eta_3^2 + \gamma^3 B_y\beta_y\delta\beta_y\eta_3^2 - \gamma^3 B_z\beta_x\delta\beta_x\beta_y\beta_z + \frac{2(\gamma-1)B_y\beta_x\delta\beta_x\beta_y^2}{\lambda_1^2} - \frac{2(\gamma-1)B_y\beta_y\delta\beta_y\eta_3^2}{\lambda_1^2}$$
$$+\frac{2(\gamma-1)B_y\beta_y^2\beta_z\delta\beta_z}{\lambda_1^2} + \frac{2(\gamma-1)B_z\beta_x\delta\beta_x\beta_y\beta_z}{\lambda_1^2} - \frac{(\gamma-1)B_z\delta\beta_y\beta_z\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)B_z\beta_y\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2}$$
$$+\frac{2(\gamma-1)B_x\beta_x\beta_y\beta_z\delta\beta_z}{\lambda_1^2} - \frac{(\gamma-1)B_x\delta\beta_x\beta_y\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)B_x\beta_x\delta\beta_y\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2}\Big]$$

$$(F'')^{21} = \gamma E_x\beta_y - \gamma\beta_x E_y + \frac{1}{\lambda_1^2}\Big[B_z\beta_z^2 + \gamma B_z\eta_1^2 - (\gamma-1)B_x\beta_x\beta_z - (\gamma-1)B_y\beta_y\beta_z + \gamma^3 B_z\beta_x\delta\beta_x\eta_1^2 - \gamma^3 B_x\beta_x^2\delta\beta_x\beta_z$$
$$-\gamma^3\beta_x^2\delta\beta_x E_y - \gamma^3\beta_x\beta_y\delta\beta_y E_y - \gamma^3\beta_x E_y\beta_z\delta\beta_z + \gamma\beta_x\beta_y\delta\beta_y E_y + \gamma\beta_x E_y\beta_z\delta\beta_z - \gamma\delta\beta_x E_y\eta_2^2 + \gamma E_x\delta\beta_y\eta_3^2$$
$$+\left(\gamma^2-1\right)\gamma E_x\beta_y\beta_z\delta\beta_z + \gamma\left(\gamma^2-1\right)\beta_x\delta\beta_x E_x\beta_y - \gamma^3 B_y\beta_x\delta\beta_x\beta_y\beta_z + \gamma^3 B_z\beta_y\delta\beta_y\eta_1^2 - \gamma^3 B_y\beta_y^2\delta\beta_y\beta_z$$
$$-\gamma^3 B_x\beta_x\beta_y\delta\beta_y\beta_z + \gamma^3 B_z\beta_z\delta\beta_z\eta_1^2 - \gamma^3 B_x\beta_x\beta_z^2\delta\beta_z - \gamma^3 B_y\beta_y\beta_z^2\delta\beta_z + \gamma^3 E_x\beta_y^2\delta\beta_y + \frac{2(\gamma-1)B_z\beta_y\delta\beta_y\beta_z^2}{\lambda_1^2}$$
$$+\frac{2(\gamma-1)B_y\beta_x\delta\beta_x\beta_y\beta_z}{\lambda_1^2} + \frac{2(\gamma-1)B_z\beta_x\delta\beta_x\beta_z^2}{\lambda_1^2} - \frac{(\gamma-1)B_y\delta\beta_y\beta_z\left(\beta_x^2-\beta_y^2+\beta_z^2\right)}{\lambda_1^2} - \frac{2(\gamma-1)B_z\beta_z\delta\beta_z\eta_1^2}{\lambda_1^2}$$
$$-\frac{(\gamma-1)B_x\beta_x\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2} - \frac{(\gamma-1)B_x\delta\beta_x\beta_z\left(-\beta_x^2+\beta_y^2+\beta_z^2\right)}{\lambda_1^2} + \frac{2(\gamma-1)B_x\beta_x\beta_y\delta\beta_y\beta_z}{\lambda_1^2}$$
$$-\frac{(\gamma-1)B_y\beta_y\delta\beta_z\left(\beta_x^2+\beta_y^2-\beta_z^2\right)}{\lambda_1^2}\Big]$$

3.2 Successively Boosted Frame $\vec{\beta}$ and $\Delta\vec{\beta}$

$$(F''')^{10} = \gamma B_z\beta_y - \gamma B_y\beta_z - \gamma^2\beta_x^2 E_x - \gamma^2\beta_x\beta_z E_z - \gamma^2\beta_x\beta_y E_y + \frac{1}{\lambda_1^2}\big[\gamma^2\beta_x^2 E_x + \gamma E_x\eta_2^2 + \gamma^2\beta_x\beta_y E_y - \gamma\beta_x\beta_y E_y - \gamma^2\beta_x E_z\lambda_5$$
$$+\gamma^2\beta_x\beta_z E_z - \gamma\beta_x\beta_z E_z + \gamma^2 B_y\beta_x\delta\beta_x\beta_z - \gamma^2 B_z\beta_x\delta\beta_x\beta_y + \gamma B_y\beta_y\delta\beta_y\beta_z - \gamma B_z\beta_y\beta_z\delta\beta_z - (\gamma-1)\gamma B_x\beta_x\delta\beta_y\beta_z$$
$$-\gamma^3 B_y\beta_z\lambda_2 + \gamma B_z\delta\beta_y\left(\gamma\beta_x^2+\beta_z^2\right) - \gamma B_y\delta\beta_z\left(\gamma\beta_x^2+\beta_y^2\right) - \gamma^2\beta_x E_x\lambda_3 - \gamma^3\beta_x\beta_y E_y\lambda_2 + (\gamma-1)\gamma B_x\beta_x\beta_y\delta\beta_z$$
$$-\gamma^2\beta_x E_y\lambda_4 - \gamma^3\beta_x\beta_z E_z\lambda_2 + \gamma^3 E_x\eta_2^2\lambda_2 + \gamma^3 B_z\beta_y\lambda_2\big]$$

$$
\begin{aligned}
(F''')^{20} &= \gamma B_x\beta_z - \gamma B_z\beta_x - \gamma^2\beta_x E_x\beta_y - \gamma^2\beta_y^2 E_y - \gamma^2\beta_y\beta_z E_z + \frac{1}{\lambda_1^2}\left[\gamma^3 B_x\beta_y\delta\beta_y\beta_z(\gamma-1)\gamma\beta_x E_x\beta_y + (\gamma-1)\gamma\beta_y\beta_z E_z\right.\\
&+\gamma^3 B_x\beta_x\delta\beta_x\beta_z - \gamma^3 B_z\beta_x^2\delta\beta_x + \gamma^2 B_z\beta_x\beta_y\delta\beta_y + (\gamma-1)\gamma B_y\delta\beta_x\beta_y\beta_z - \gamma B_x\beta_x\delta\beta_x\beta_z + \gamma^2\beta_x\delta\beta_x\beta_y^2 E_y - \gamma^3\beta_y^2\delta\beta_y\beta_z E_z\\
&+\gamma E_y\left(\beta_x^2+\gamma\beta_y^2+\beta_z^2\right) - \gamma^3 B_z\beta_x\beta_y\delta\beta_y - \gamma^2 B_x\beta_y\delta\beta_y\beta_z + \gamma B_x\delta\beta_z\left(\beta_x^2+\gamma\beta_y^2\right) + \gamma B_z\beta_x\beta_z\delta\beta_z - (\gamma-1)\gamma B_y\beta_x\beta_y\delta\beta_z\\
&-\gamma^3 B_z\beta_x\beta_z\delta\beta_z + \gamma^3 B_x\beta_z^2\delta\beta_z - \gamma^3\beta_x^2\delta\beta_x E_x\beta_y - \gamma^3\beta_x E_x\beta_y^2\delta\beta_y + \gamma^2\beta_x E_x\beta_y^2\delta\beta_y - \gamma^2\delta\beta_x E_x\beta_y\eta_2^2 + \gamma^2\beta_x E_x\beta_y\beta_z\delta\beta_z\\
&-\gamma^3\beta_x E_x\beta_y\beta_z\delta\beta_z + \gamma^3\beta_x\delta\beta_x E_y\eta_3^2 - \gamma^2\beta_y\delta\beta_y E_y\eta_3^2 + \gamma^3\beta_y\delta\beta_y E_y\eta_3^2 + \gamma^2\beta_y^2 E_y\beta_z\delta\beta_z + \gamma^3 E_y\beta_z\delta\beta_z\eta_3^2 - \gamma^2\beta_y\delta\beta_z E_z\eta_1^2\\
&\left.-\gamma^3\beta_x\delta\beta_x\beta_y\beta_z E_z - \gamma^3\beta_y\beta_z^2\delta\beta_z E_z + \gamma^2\beta_y^2\delta\beta_y\beta_z E_z + \gamma^2\beta_x\delta\beta_x\beta_y\beta_z E_z - \gamma B_z\delta\beta_x\left(\gamma\beta_y^2+\beta_z^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{30} &= \gamma B_y\beta_x - \gamma B_x\beta_y - \gamma^2\beta_x E_x\beta_z - \gamma^2\beta_y E_y\beta_z - \gamma^2\beta_z^2 E_z + \frac{1}{\lambda_1^2}\left[\gamma^3 B_y\beta_x^2\delta\beta_x(\gamma-1)\gamma\beta_x E_x\beta_z + (\gamma-1)\gamma\beta_y E_y\beta_z\right.\\
&+\gamma E_z\left(\beta_x^2+\beta_y^2+\gamma\beta_z^2\right) - \gamma^3 B_x\beta_x\delta\beta_x\beta_y + \gamma B_y\delta\beta_x\left(\beta_y^2+\gamma\beta_z^2\right) + \gamma B_x\beta_x\delta\beta_x\beta_y - (\gamma-1)\gamma B_z\delta\beta_x\beta_y\beta_z - \gamma B_y\beta_x\beta_y\delta\beta_y\\
&+\gamma^3 B_y\beta_x\beta_y\delta\beta_y - \gamma^2 B_y\beta_x\beta_z\delta\beta_z + (\gamma-1)\gamma B_z\beta_x\delta\beta_y\beta_z - \gamma B_x\delta\beta_y\left(\beta_x^2+\gamma\beta_z^2\right) - \gamma^3 B_x\beta_y\beta_z\delta\beta_z + \gamma^3 B_y\beta_x\beta_z\delta\beta_z\\
&+\gamma^2 B_x\beta_y\beta_z\delta\beta_z - \gamma^3\beta_x^2\delta\beta_x E_x\beta_z - \gamma^3\beta_x E_x\beta_y\delta\beta_y\beta_z + \gamma^2\beta_x E_x\beta_y\delta\beta_y\beta_z - \gamma^2\delta\beta_x E_x\beta_z\eta_2^2 + \gamma^2\beta_x E_x\beta_z^2\delta\beta_z\\
&-\gamma^3\beta_x E_x\beta_z^2\delta\beta_z - \gamma^3\beta_x\delta\beta_x\beta_y E_y\beta_z - 2\gamma^3\beta_y^2\delta\beta_y E_y\beta_z - 2\gamma^2\delta\beta_y E_y\beta_z\eta_3^2 + \gamma^2\beta_x\delta\beta_x\beta_y E_y\beta_z + \gamma^2\beta_y E_y\beta_z^2\delta\beta_z\\
&-\gamma^3\beta_y E_y\beta_z^2\delta\beta_z + \gamma^3\beta_y\delta\beta_y E_z\eta_1^2 + \gamma^3\beta_z\delta\beta_z E_z\eta_1^2 + \gamma^3\beta_x\delta\beta_x E_z\eta_1^2 + \gamma^2\beta_x\delta\beta_x\beta_z^2 E_z - \gamma^2\beta_z\delta\beta_z E_z\eta_1^2 - \gamma^3 B_x\beta_y^2\delta\beta_y\\
&\left.+\gamma^2\beta_y\delta\beta_y\beta_z^2 E_z\right]
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{32} &= \gamma E_y\beta_z - \gamma\beta_y E_z + \gamma^3\beta_y\delta\beta_y E_y\beta_z - \gamma^3\beta_y^2 E_y\delta\beta_z + \gamma^3\delta\beta_y\beta_z^2 E_z - \gamma^3\beta_y\beta_z\delta\beta_z E_z + \frac{1}{\lambda_1^2}\left[-\gamma^2 B_y\beta_x\lambda_4 + \gamma^3 B_x\eta_2^2\lambda_2\right.\\
&-(\gamma-1)B_z\beta_x\beta_z + B_x\left(\beta_x^2+\gamma\left(\beta_y^2+\beta_z^2\right)\right) + (\gamma-1)\gamma\beta_x E_x\delta\beta_y\beta_z - (\gamma-1)\gamma\beta_x E_x\beta_y\delta\beta_z - \gamma^2 B_z\beta_x\lambda_5 - \gamma^3 B_z\beta_x\beta_z\lambda_2\\
&+\gamma^2 E_y\delta\beta_z\left(\beta_x^2+\gamma\left(\beta_y^2+\beta_z^2\right)\right) - \gamma^2\delta\beta_y E_z\left(\beta_x^2+\gamma\left(\beta_y^2+\beta_z^2\right)\right) + (\gamma-1)\gamma^2\beta_x\delta\beta_x E_y\beta_z - (\gamma-1)B_y\beta_x\beta_y - \gamma^3 B_y\beta_x\beta_y\lambda_2\\
&\left.-\gamma^2 B_x\beta_x\lambda_3 - (\gamma-1)\gamma^2\beta_x\delta\beta_x\beta_y E_z\right]
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{13} &= \gamma\beta_x E_z - \gamma E_x\beta_z - \gamma^3\beta_x\delta\beta_x E_x\beta_z + \gamma^3\beta_x^2 E_x\delta\beta_z + \gamma^3\beta_x\beta_z\delta\beta_z E_z + \frac{1}{\lambda_1^2}\left[\gamma^3 B_y\eta_3^2\lambda_2 - \gamma^2 B_x\beta_y\lambda_3 - \gamma^3 B_x\beta_x\beta_y\lambda_2\right.\\
&+\gamma\delta\beta_x\beta_z^2 E_z - (\gamma-1)B_x\beta_x\beta_y - (\gamma-1)B_z\beta_y\beta_z - \gamma^2 B_y\beta_y\lambda_4 - \gamma^2\delta\beta_x\beta_y E_y\beta_z - \gamma^2 B_z\beta_y\lambda_5 - \gamma\beta_x\beta_y E_y\delta\beta_z\\
&-\gamma^3 B_z\beta_y\beta_z\lambda_2 - \gamma^2 E_x\beta_y^2\delta\beta_z - (\gamma-1)\gamma^2 E_x\beta_y\delta\beta_y\beta_z - \gamma^3 E_x\delta\beta_z\eta_3^2 + \gamma^2\beta_x\beta_y E_y\delta\beta_z + \gamma^3\beta_x^2\delta\beta_x E_z + \gamma^2\delta\beta_x\beta_y^2 E_z\\
&\left.+\gamma\delta\beta_x\beta_y E_y\beta_z + (\gamma-1)\gamma^2\beta_x\beta_y\delta\beta_y E_z + B_y\left(\gamma\beta_x^2+\beta_y^2+\gamma\beta_z^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
(F''')^{21} &= \gamma^3\beta_x\delta\beta_x E_x\beta_y - \gamma^3\beta_x^2 E_x\delta\beta_y + \gamma E_x\beta_y + \gamma^3\delta\beta_x\beta_y^2 E_y - \gamma\beta_x E_y - \gamma^3\beta_x\beta_y\delta\beta_y E_y + \gamma^3\delta\beta_x\beta_y\beta_z E_z - \gamma^3\beta_x\delta\beta_y\beta_z E_z\\
&+\frac{1}{\lambda_1^2}\left[\gamma^3 B_z\eta_1^2\lambda_2 - (\gamma-1)B_x\beta_x\beta_z - (\gamma-1)B_y\beta_y\beta_z + B_z\left(\gamma\beta_x^2+\gamma\beta_y^2+\beta_z^2\right) - \gamma^3 B_x\beta_x\beta_z\lambda_2 - \gamma^3 B_y\beta_y\beta_z\lambda_2\right.\\
&+\gamma^3 E_x\delta\beta_y\eta_1^2 - \gamma^3\delta\beta_x E_y\eta_1^2 - \gamma^2 B_x\beta_z\lambda_3 - \gamma^2 B_y\beta_z\lambda_4 - \gamma^2 B_z\beta_z\lambda_5 + (\gamma-1)\gamma^2 E_x\beta_y\beta_z\delta\beta_z - (\gamma-1)\gamma^2\beta_x E_y\beta_z\delta\beta_z\\
&\left.+(\gamma-1)\gamma^2\beta_z E_z\lambda_6 + \gamma^2 E_x\delta\beta_y\beta_z^2 - \gamma^2\delta\beta_x E_y\beta_z^2\right]
\end{aligned}
$$

4 Comparison with the References

As mentioned in the Introduction, this work is inspired by Ungar et al.[9–11,13–16] so it is natural to compare our approach with Ungar to see if we share the same overall features. To check that we will work for a 2-dimensional case for simplicity by letting $\beta_y = \beta_z = \delta\beta_z = 0$ in the first part of Eq. (38) of the main article which will give[6]:

$$
A(\Delta\vec{\beta}) = \begin{pmatrix} 1 & -\gamma^2\delta\beta_x & -\gamma\delta\beta_y & 0 \\ -\gamma^2\delta\beta_x & 1 & 0 & 0 \\ -\gamma\delta\beta_y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}
$$

From $A(\Delta\vec{\beta})$, the relativistic velocity composition can be easily extracted:

$$\Delta\vec{\beta} = \gamma^2 \delta\beta_x \hat{x} + \gamma\delta\beta_y \hat{y} \tag{3}$$

Now for the three different inertial frames discussed in the Introduction Σ, Σ' and Σ'', we have

Relative velocity of Σ' with respect to $\Sigma : \vec{\beta}$
Relative velocity of Σ'' with respect to $\Sigma' : \Delta\vec{\beta}$

We can now use the relativistic velocity composition rule mentioned in Eq. (8) of the main article to calculate the relative velocity of Σ'' with respect to Σ[9–11,13–16] by defining:

$$\begin{aligned}
\vec{u} &= \vec{\beta} = \beta\hat{x} \\
\vec{v} &= \Delta\vec{\beta} = \gamma^2 \delta\beta_x \hat{x} + \gamma\delta\beta_y \hat{y} \\
\gamma_u &= \gamma \\
\gamma_v &\approx 1
\end{aligned}$$

After plugging in the values and simplifying, we get:

$$\begin{aligned}
(\vec{u} \oplus \vec{v})_x &= \frac{\beta + \gamma^2 \delta\beta_x}{1 + \gamma^2 \beta\delta\beta_x} \\
(\vec{u} \oplus \vec{v})_y &= \frac{\delta\beta_y}{1 + \gamma^2 \beta\delta\beta_x} \\
\gamma_{\vec{u}\oplus\vec{v}} &\approx \gamma(1 + \gamma^2 \beta\delta\beta_x)
\end{aligned} \tag{4}$$

Constructing the boost matrix from the above equation:

$$A(\vec{\beta} + \delta\vec{\beta}) = \begin{pmatrix} \gamma + \gamma^3 \beta\delta\beta_x & -(\gamma\beta + \gamma^3 \delta\beta_x) & -\gamma\delta\beta_y & 0 \\ -(\gamma\beta + \gamma^3 \delta\beta_x) & \gamma + \gamma^3 \beta\delta\beta_x & (\gamma - 1)\frac{\delta\beta_y}{\beta} & 0 \\ -\gamma\delta\beta_y & (\gamma - 1)\frac{\delta\beta_y}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

We can compare this boost matrix with Eq. (40) of the main article and notice that they are completely identical. Therefore our approach gives the identical final result as discussed in[9–11,13–16].

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