JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **208,** 425–445 (1997)
ARTICLE NO. AY975325

Oscillatory Radial Solutions of Semilinear Elliptic Equations

William R. Derrick and Shaohua Chen

Department of Mathematics, University of Montana, Missoula, Montana 59812

and

Joseph A. Cima

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

Submitted by Mark J. Balas

Received August 10, 1995

We study the oscillatory behavior of radial solutions of the nonlinear partial differential equation $\Delta u + f(u) + g(|x|, u) = 0$ in R^n, where f and g are continuous restoring functions, $uf(u) > 0$ and $ug(|x|, u) > 0$ for $u \neq 0$. We assume that for fixed q $\lim_{u\to 0}(|f(u)|/|u|^q) = B > 0$, for $1 < q < n/(n-2)$, and, additionally, that $2F(u) \geq (1 - 2/n)uf(u)$ when $n/(n-2) \leq q < (n+2)/(n-2)$, where $F(u) = \int_0^u f(s)ds$. We give conditions that guarantee that the solution oscillates infinitely and tends to zero as $r \to \infty$. Finally, we give bounds for the amplitude of the oscillations and show that the period of the oscillations increases as $r \to \infty$. © 1997 Academic Press

1. INTRODUCTION

Numerous authors (see [DN, DN2, L, LN1, LN2, LN3, N2, NY, P, Po]) have given substantial attention to the existence, nonexistence, uniqueness, and nonuniqueness of positive solutions of semilinear elliptic equations of the form

$$\Delta u + K(x)u^q = 0, \tag{1.1}$$

on the ball $|x| \leq R$ or in R^n. Equation (1.1) is said to involve critical Sobolev exponents if $q = n^* \equiv (n+2)/(n-2)$, and is then known to

0022-247X/97 $25.00

have a one parameter family of positive solutions ($u(x) > 0$) in R^n when $K \equiv 1$. Berestycki *et al*. [BLP] have shown that if $K \equiv 1$ and $q > n^*$ then (1.1) has a positive solution provided $u(0) > 0$. For radial $K(r) > Cr^p$, $C > 0$, $p \geq -2$, near infinity, and $n \geq 3$, Ni [N2, Theorem 3.35] has shown that if u is a positive solution to (1.1) on R^n, then

$$cr^{(p+2)/(1-q)} \geq u(r) \geq cr^{2-n}, \tag{1.2}$$

for large r. Hence for constant $K > 0$, so that $p = 0$, the existence of a positive solution to (1.1) in R^n requires that $q \geq n/(n-2)$. A comprehensive treatment of (1.1) for a wide range of functions $K(x)$ is given in [LN1].

It is well known (see [DN, DN2, G, L, LN1, LN2, LN3, N2]) that, under suitable conditions, the analysis of positive solutions of (1.1) extends to the more general equation

$$\Delta u + f(x,u) = 0, \qquad \text{in } R^n. \tag{1.3}$$

If $f(x,u)$ behaves like $K(|x|)u^q$ for large $|x|$ and small $u > 0$, then it is reasonable to expect that a solution to (1.3) may behave like a solution to (1.1). Gui [G] has established several comparison theorems for solutions of (1.1) and (1.3).

In this paper we shall study the radial solutions of the nonlinear partial differential equation

$$\Delta u + f(u) + g(|x|,u) = 0, \qquad \text{in } R^n,$$

where f and g are continuous restoring functions; that is, where $uf(u) > 0$ and $ug(|x|,u) > 0$ whenever $u \neq 0$. Such equations arise in many areas of applied mathematics (see [BN, JK, KNY, N2, NY, Po2]); positive solutions that exist in R^n and satisfy $u(x) \to 0$ as $|x| \to \infty$ are called ground states. Since we consider only radial solutions, our problem reduces to studying the singular initial-value problem

$$u'' + \frac{n-1}{r}u' + f(u) + g(r,u) = 0$$
$$\text{for } 0 < r < \infty,\ u(0) = u_0 \neq 0 = u'(0), \tag{1.4}$$

where f and g are continuous restoring functions. At various times we shall assume that $g(r,u) \equiv 0$, so our results will then apply to solutions of

$$u'' + \frac{n-1}{r}u' + f(u) = 0, \qquad \text{for } 0 < r < \infty,\ u(0) = u_0 \neq 0 = u'(0). \tag{1.5}$$

Suppose

$$\lim_{u \to 0} |f(u)|/|u|^q = B > 0, \tag{1.6}$$

for fixed q, $1 < q < n/(n-2)$. It is not hard to show for restoring functions f and g that (1.4) will have no positive or negative solutions in $a \le r < \infty$, $a \ge 0$. Consequently, under these conditions, solutions to (1.4) must oscillate about 0 infinitely often. But what is the nature of these oscillations? Do the solutions converge to 0 or oscillate but have no limit as $r \to \infty$? In Section 3 we prove.

THEOREM 1.7. *Let u be the solution to* (1.5) *with $f(u)$ a continuous restoring function. Suppose that* (1.6) *holds for fixed q, $1 < q < n^*$, and that*

$$2F(u) \ge \left(1 - \frac{2}{n}\right) u f(u) \qquad \text{for } u \ne 0, \tag{1.8}$$

where

$$F(u) = \int_0^u f(s)\, ds. \tag{1.9}$$

Then u will oscillate infinitely and tend to 0 *as $r \to \infty$.*

Having shown that the solution oscillates to zero, it is interesting to examine the decay of the solution as $r \to \infty$ and the period between consecutive zeros of the solution. In Section 4 we prove:

THEOREM 1.10. *Let t_j be the jth local extremum of the solution u of*

$$u'' + \frac{n-1}{r} u' + \left(b|u|^{q-1} + |u|^{p-1}\right) u = 0, \qquad u(0) \ne 0 = u'(0), \tag{1.11}$$

$1 < q < p \le n^*$. *Then there is a positive constant c such that*

$$\lim_{j \to \infty} t_j^{2(n-1)/(q+3)} |u(t_j)| = c.$$

Thus, the decay rate is of the order $|u(r)| \le c r^{-2(n-1)/(q+3)}$, as $r \to \infty$. Finally, we prove:

THEOREM 1.12. *Let r_j be the jth zero of the solution of* (1.11). *Then there is a positive constant c such that*

$$r_{j+1} - r_j \ge c r_j^{(n-1)(q-1)/(q+3)}.$$

Hence, the period between zeros increases as $r \to \infty$.

Although Section 4 is limited to solutions of (1.11), it is not hard to find conditions that extend these results to more general restoring functions.

2. ELEMENTARY RESULTS

In what follows, we shall need some elementary facts concerning solutions of the initial value problem (1.4). Suppose f and g are continuous. Then

$$(r^{n-1}u')' = -r^{n-1}[f(u) + g(r,u)].$$

Integration of this equality from 0 to r yields

$$u'(r) = -\frac{1}{r^{n-1}}\int_0^r [f(u(s)) + g(s,u(s))] s^{n-1}\, ds. \tag{2.1}$$

LEMMA 2.2. *If $f(u)$ and $g(r,u)$ are continuous restoring functions, and u is a positive solution of the equation* (1.4), *then u is strictly decreasing and tends to* 0 *as $r \to \infty$.*

Proof. Since the integrand in (2.1) is positive, $u' < 0$ so the solution is strictly decreasing. Hence, there is a number $c \geq 0$ such that $u(r)$ decreases to c as $r \to \infty$ and $u'(r) \to 0$ as $r \to \infty$. Suppose $c > 0$, then since f is continuous on the interval $[c, u_0]$ it has a minimum $f_{\min} > 0$ on this interval. Hence,

$$-r^{n-1}u'(r) = \int_0^r [f(u(s)) + g(s,u(s))] s^{n-1}\, ds \geq f_{\min}\frac{r^n}{n},$$

implying that $u'(r) \leq -f_{\min}(r/n) \to -\infty$. This is a contradiction and hence u tends to zero as $r \to \infty$.

Suppose the solution u of (1.4) oscillates about zero a finite number of times and has a local maximum at r_0 for which $u(r) > 0$ for all $r \geq r_0$. We call such solutions *eventually positive solutions*. Then, because $u'(r_0) = 0$,

$$u'(r) = -\frac{1}{r^{n-1}}\int_{r_0}^r [f(u(s)) + g(s,u(s))] s^{n-1}\, ds, \tag{2.3}$$

so that again u is strictly decreasing for $r \geq r_0$, and the same proof as above shows that u and u' converge to 0 as $r \to \infty$. A similar argument holds for *eventually negative solutions*.

LEMMA 2.4. *Let u be a positive solution of the initial value problem* (1.4), *and suppose* (1.6) *holds for $q > 1$. Then, for sufficiently large r,*

$$u(r) \le c r^{-2/(q-1)}. \tag{2.5}$$

Proof. Since u decreases to 0 by Lemma 2.2, some r_0 exists for which $|f(u)|/|u|^q > B/2$ for $r \ge r_0$. By (2.1) and Lemma 2.2

$$\begin{aligned}\frac{du}{dr} &\le -\frac{1}{r^{n-1}}\int_{r_0}^{r}\left[f(u(s)) + g(s,u(s))\right]s^{n-1}\,ds \\ &\le -\frac{1}{r^{n-1}}\int_{r_0}^{r} f(u(s))s^{n-1}\,ds \\ &\le -\frac{B}{2r^{n-1}}\int_{r_0}^{r} u^q(s)s^{n-1}\,ds \le -\frac{Bu^q(r)}{2r^{n-1}}\int_{r_0}^{r} s^{n-1}\,ds.\end{aligned}$$

Integrating the resulting inequality:

$$\int_{u(r_0)}^{u(r)} u^{-q}\,du \le \frac{-c}{n}\int_{r_0}^{r}\left[r - r_0^n r^{1-n}\right]dr$$

yields an inequality from which the result follows for $r \ge 2r_0$.

Inequality (2.5) also holds for eventually positive or eventually negative solutions of (1.4) by applying the argument above at the last value r_* at which $u'(r_*) = 0$. For positive solutions, (2.5) corresponds to the upper half of (1.2).

Unless otherwise stated, assume from now on in this section that $g(r,u) \equiv 0$. By the uniqueness theorem for solutions of initial value problems, a solution of (1.5) cannot satisfy both $u'(r) = 0$ and $f(u(r)) = 0$, unless u is constant. Thus, except for such cases, the critical points of any solution of (1.5) are isolated, and are minima whenever $f(u(r)) < 0$ and maxima whenever $f(u(r)) > 0$. Let $u(r)$ be a solution of (1.5) and define the "energy function" of [MTW]:

$$Q(u(r)) = \frac{(u'(r))^2}{2} + F(u(r)). \tag{2.6}$$

If (1.5) is multiplied by u', one obtains

$$\frac{dQ}{dr} = \left(\frac{(u')^2}{2} + F(u)\right)' = -\frac{n-1}{r}(u')^2 \le 0.$$

This implies that the "energy" function Q is strictly decreasing because the critical points of u are isolated.

LEMMA 2.7. *Let u have a critical point at r_0. If $u(r_0)$ is a local maximum, then $u(r) < u(r_0)$ for all $r > r_0$, and if $u(r_0)$ is a local minimum, then $u(r) > u(r_0)$ for all $r > r_0$.*

Proof. Suppose $u(r_1) = u(r_0)$ for $r_1 > r_0$. Then

$$Q(u(r_1)) = \frac{u'(r_1)^2}{2} + F(u(r_1)) \geq F(u(r_0)) = Q(u(r_0)),$$

contradicting the fact that Q is strictly decreasing.

We also need the following "energy" version of Pokhozhaev's second identity valid for continuous f and functions u that are $C^2(R^n)$ and radial (see [Po]):

$$\begin{aligned}
\int_0^r (\Delta u + f(u))(su' + \alpha u) s^k \, ds
&= r^{k+1} Q(u(r)) + \alpha r^k u(r) u'(r) \\
&\quad + \frac{\alpha}{2}(n - 1 - k) r^{k-1} u^2(r) \\
&\quad + (2n - 3 - k - 2\alpha) \int_0^r Q(u(s)) s^k \, ds \\
&\quad - \alpha \frac{(n - 1 - k)(k - 1)}{2} \int_0^r u^2(s) s^{k-2} \, ds \\
&\quad + \int_0^r [\alpha u f(u) - 2(n - 1 - \alpha) F(u)] s^k \, ds
\end{aligned}$$

for integers $k > 1$, and α real. (2.8)

Here $u = u(s)$ inside the integrals. If $\Delta u + f(u) = 0$, then the left side of (2.8) is zero. Verification of this identity is a routine task by using (1.5) instead of $\Delta u + f(u)$.

LEMMA 2.9. *Let u be a solution of* (1.5) *and let*

$$J(r, u) = r^n u'^2(r) + (n - 2) r^{n-1} u(r) u'(r) + 2 r^n F(u). \quad (2.10)$$

If $u'(r_0) = 0$ *for some* $r_0 \geq 0$, *then for all* $r \geq r_0$,

$$J(r,u) = \int_{r_0}^{r} \left[2nF(u(s)) - (n-2)u(s)f(u(s))\right] s^{n-1}\, ds + 2r_0^n F(u(r_0)). \tag{2.11}$$

Proof. Differentiate (2.10) with respect to r and substitute (1.5) into the resulting equation to obtain

$$\frac{dJ}{dr} = \left[2nF(u) - (n-2)uf(u)\right] r^{n-1}.$$

An integration yields the desired result.

LEMMA 2.12. *If u is a solution of* (1.5) *and* $\lim_{r\to\infty} Q(u(r)) = \beta$, *then, for any* r_0

$$\lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r} F(u(s))\, ds = \beta. \tag{2.13}$$

Proof. Equation (2.8) also holds when $k = \alpha = 0$, yielding

$$rQ(u(r)) + (2n-3)\int_0^r Q(u(s))\, ds = 2(n-1)\int_0^r F(u(s))\, ds. \tag{2.14}$$

Evaluating (2.14) at r_0 and subtracting from (2.14) yields

$$rQ(u(r)) - r_0 Q(u(r_0)) + (2n-3)\int_{r_0}^{r} Q(u(s))\, ds = 2(n-1)\int_{r_0}^{r} F(u(s))\, ds.$$

Dividing by r and taking the limit as $r \to \infty$ yields

$$\beta + (2n-3)\lim_{r\to\infty} \frac{1}{r}\int_{r_0}^{r} Q(u(s))\, ds = 2(n-1)\lim_{r\to\infty} \frac{1}{r}\int_{r_0}^{r} F(u(s))\, ds.$$

Using L'Hopital's rule, we obtain (2.13).

As mentioned in Section 1, problem (1.5) does not have positive radial solutions if the order of growth q of $f(u)$ to zero as $u \to 0+$ is in the interval $1 < q < n/(n-2)$. For q outside this interval, positive solutions may exist: for example, if $n = 3$ and $q \geq n/(n-2) = 3$, then

$$u'' + \frac{2}{r}u' + \frac{4qu_0}{(q-1)^2}\left(\frac{u}{u_0}\right)^{2q-1} + \frac{2(q-3)u_0}{(q-1)^2}\left(\frac{u}{u_0}\right)^{q} = 0,$$

$$u(0) = u_0 > 0,\ u'(0) = 0,$$

has the positive solution $u(r) = u_0(1 + r^2)^{1/(1-q)}$. The following result provides an elementary proof of the nonexistence of positive (or eventually positive) solutions for problem (1.4), and is included for completeness.

LEMMA 2.15. *If f and g are continuous restoring functions and* (1.6) *holds for* $1 < q < n/(n-2)$, *then problem* (1.4) *has no positive or negative solutions in* $a \leq r < \infty$, $a \geq 0$.

Proof. We prove the result for positive solutions; the case for negative solutions follows immediately by setting $v = -u$ and applying the positive case. Suppose a local maximum of the solution exists at r_0 such that $u(r) > 0$ for all $r \geq r_0$. By Lemma 2.2 and the remark following its proof, it follows that $u \to 0$ and $u' \to 0$ as $r \to \infty$.

By L'Hopital's rule and (2.3) we have

$$\lim_{r\to\infty} \frac{u(r)}{r^{2-n}} = \lim_{r\to\infty} \frac{u'(r)}{(2-n)r^{1-n}} = \lim_{r\to\infty} \frac{\int_{r_0}^{r} [f(u(s)) + g(s,u(s))] s^{n-1}\, ds}{n-2}. \tag{2.16}$$

Since the integral in (2.16) is positive, some constant $c_0 > 0$ exists so that

$$u(r) \geq c_0 r^{2-n} \qquad \text{for all } r \geq r_0. \tag{2.17}$$

This proves the bottom half of (1.2). Using (1.2), $2 \leq (n-2)(q-1)$ or $n \leq (n-2)q$, contradicting the hypothesis. Hence no positive solution exists in $r_0 \leq r < \infty$.

Remark 2.18. A similar result can be obtained for problem (1.5) when $n/(n-2) \leq q < n^*$, provided the restoring function f satisfies a Pokhozaev inequality (1.8). Suppose that the conclusion is not true. Then there exists a point $r_* > 0$ such that $u(r) > 0$ for all $r > r_* \geq 0$. Using (2.3) with $g(r,u) \equiv 0$ and Lemma 2.4, we get

$$\begin{aligned} |u'(r)| &= \left| \frac{1}{r^{n-1}} \int_0^r f(u(s)) s^{n-1}\, ds \right| \\ &\leq \frac{C}{r^{n-1}} + \frac{2Bc^q}{r^{n-1}} \int_{r_0}^{r} s^{n-1-2q/(q-1)}\, ds \leq \frac{C}{r^{n-1}} + C_1 r^{-(q+1)/(q-1)}, \end{aligned}$$

where we choose $r_0 \geq r_*$, such that $0 < f(u(r))/u^q(r) \leq 2B$ for $r \geq r_0$. Since $q \geq n/(n-2)$, it follows that $n - 1 \geq (q+1)/(q-1)$ and

$$|u'(r)| \leq c_2 r^{-(q+1)/(q-1)} \qquad \text{for large } r. \tag{2.19}$$

By L'Hopital's rule, we have from (1.6)

$$\lim_{r\to\infty} \frac{F(u(r))}{u^{q+1}(r)} = \lim_{r\to\infty} \frac{\int_0^{u(r)} f(u)\, du}{u^{q+1}(r)} = \lim_{r\to\infty} \frac{f(u(r))}{(q+1)u^q(r)} = \frac{B}{q+1}. \tag{2.20}$$

Now, using (2.5), (2.19), and (2.20) and $\alpha = 2(q+1)/(q-1) - n > 0$, we get

$$r^n F(u(r)) \le r^n \frac{F(u(r))}{u^{q+1}(r)} \left[cr^{-2/(q-1)}\right]^{q+1} \le cr^{-\alpha},$$

and both

$$r^{n-1} u(r)|u'(r)| \le cr^{-\alpha},$$

$$r^n |u'(r)|^2 \le cr^{-\alpha},$$

for large r. Let r_1 be the first maximum point such that $u(r) > 0$ for all $r \ge r_1$. Then $u'(r_1) = 0$. Using Lemma 2.9 and (1.8) we get

$$\begin{aligned} |J(r,u)| \le r^n |u'(r)|^2 + (n-2) r^{n-1} |u(r)|\,|u'(r)| \\ + 2r^n F(u) \le (n+1) cr^{-\alpha} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} J(r,u) &= \int_{r_1}^{r} \left[2nF(u(s)) - (n-2)u(s) f(u(s))\right] s^{n-1}\, ds + 2r_1^n F(u(r_1)) \\ &\ge 2r_1^n F(u(r_1)) > 0. \end{aligned} \tag{2.22}$$

Letting $r \to \infty$, we see that (2.21) contradicts (2.22), so the remark is proved. The result also holds for negative solutions by a similar argument.

EXAMPLE 2.23. The function $f(u) = u^5 + u^3\sqrt{1+u^2}$ satisfies the hypotheses of Remark 2.18, so (1.5) has no eventually positive or negative solutions in R^3 for this function. The function $f(u) = u^5 + u^4$ satisfies (2.18) except that f is not restoring since $f(u) > 0$ in $-1 < u < 0$. Thus, (1.5) has no eventually positive solution, but may (and does; see [CCD]) have an eventually negative solution.

3. OSCILLATORY BEHAVIOR

As was mentioned in Section 1, a solution to (1.4) or (1.5) satisfying the hypotheses of Lemma 2.15 or Remark 2.18, respectively, must oscillate infinitely as $r \to \infty$. In this section we will give conditions guaranteeing that the solution converges to 0 as $r \to \infty$. For simplicity, we will assume that $g(r,u) \equiv 0$.

First we show that with a stronger condition than Remark 2.18 we can prove that the solution converges to zero as $r \to \infty$.

LEMMA 3.1. *Let $f(u)$ be a continuous restoring function satisfying* (1.6) *for $1 < q < n^*$, and assume that*

$$(n-2)uf(u) \geq 2F(u) \geq \left(1 - \frac{2}{n}\right)uf(u) > 0 \qquad \text{for } u \neq 0. \quad (3.2)$$

Then, if u is a solution of the initial-value problem (1.5),

$$|u(r)| \leq cr^{-2/(q+1)} \qquad \text{for large } r. \quad (3.3)$$

Proof. Pokhozhaev's first identity [Po] with $\alpha = n-2$ can be rewritten as

$$r^2F(u(r)) + \tfrac{1}{2}[ru'(r) + (n-2)u(r)]^2 + \int_0^r [(n-2)u(s)f(u(s)) - 2F(u(s))]s\,ds = \frac{(n-2)^2u^2(0)}{2}.$$

Since the second and third terms on the left side of this equation are nonnegative, we have by (3.2)

$$0 < \frac{(n-2)u(r)f(u(r))}{n} \leq 2F(u(r)) \leq \frac{(n-2)^2u^2(0)}{r^2}.$$

Thus, $F(u(r)) \to 0$ as $r \to \infty$, and by the first inequality, F is only zero at $u = 0$. Hence, $u \to 0$ as $r \to \infty$. Then

$$\frac{|u|^{q+1}|f(u)|}{|u|^q} \leq \frac{n(n-2)u^2(0)}{r^2},$$

and using (1.6) the conclusion in (3.3) follows.

Observe that the function $f(u) = u^5 + u^3$ in R^3 satisfies the hypotheses of Lemma 3.1 with $q = 3$. Hence, the solution to (1.5) with this f will decay as $|u(r)| \leq cr^{-1/2}$ as $r \to \infty$. On the other hand, $f(u) = (u/(1+u^2))^3$ satisfies only the second inequality in (3.2) for $u \neq 0$, so the conclusion (3.3) cannot be assumed.

Lemma 3.4. *Let u be the solution of the initial value problem* (1.5). *Suppose that u oscillates infinitely about the value b and converges to b as $r \to \infty$. Then $f(b) = 0$.*

Proof. Assume $f(b) \neq 0$. By continuity it follows that $f(u(r))$ does not change sign for r sufficiently large. Without loss of generality assume $f(u(r)) < 0$ for $r > r_*$. Let $r_0 > r_*$ be the value of the first local maximum of $u(r)$ after r_*. Then by (2.3), for all $r > r_0$,

$$u'(r) = -\frac{1}{r^{n-1}} \int_{r_0}^{r} f(u(s)) s^{n-1}\, ds > 0,$$

which is impossible, since $u(r_0)$ is a local maximum. Thus $f(b) = 0$.

Next we show that for certain functions $f(u)$, we need not assume convergence of u to b, as $r \to \infty$, in order to get that $f(b) = 0$.

LEMMA 3.5. *Let $f(u)$ be continuous and suppose $F(b) = \inf\{F(u) : u \in R\}$. Suppose the solution of* (1.5) *oscillates about b as $r \to \infty$, and assumes no other root of $f(u) = 0$ for large r. Then* $\lim_{r\to\infty} u(r) = b$.

Proof. Since f is continuous, $f(b) = F'(b) = 0$, so b is a root of f. Suppose

$$\limsup_{r\to\infty} u(r) \neq \liminf_{r\to\infty} u(r).$$

Let the local minimums of $u(r)$ occur at r_{2j-1} and the local maximums at r_{2j}. Then, by Lemma 2.7,

$$u(r_1) < u(r_3) < u(r_5) < \cdots < u(r_6) < u(r_4) < u(r_2).$$

Thus, the maximums are bounded below, while the minimums are bounded above, so both sequences converge, say to u_+ and u_-, respectively, with $u_+ > u_-$. The energy function $Q(u(r))$ decreases and is bounded below since

$$\begin{aligned} F(u_+) &= \lim_{j\to\infty} F\big(u(r_{2j})\big) = \lim_{j\to\infty} Q\big(u(r_{2j})\big) = \lim_{r\to\infty} Q(u(r)) \\ &= \lim_{j\to\infty} Q\big(u(r_{2j-1})\big) = \lim_{j\to\infty} F\big(u(r_{2j-1})\big) = F(u_-). \end{aligned}$$

By Lemma 2.12, with $\beta = F(u_+)$,

$$\lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r} [F(u_+) - F(u(s))]\, ds = 0. \tag{3.6}$$

Let $g(s) = F(u_+) - F(u(s))$. Then $g(s)$ oscillates between the maximums $g(t_i) = \beta - F(b)$ at $u(t_i) = b$, and local minimums $g(r_i) = \beta - F(u(r_i)) \to 0$ as $i \to \infty$, with $t_i < r_i$. Select i_0 sufficiently large so that $g(r_i) > -\delta$ for $i \geq i_0$, where $\delta < [\beta - F(b)]/5$ and the only root of $f(u) = 0$ is $u = b$ for $r \geq r_{i_0}$. Then $|f(u)| \leq N$ and, by the mean value theorem for $j > i_0$,

$$\frac{1}{t_j}\int_{t_{i_0}}^{t_j} g(s)\,ds = \frac{1}{t_j}\sum_{i=i_0}^{j-1}\int_{t_i}^{t_{i+1}} g(s)\,ds$$
$$= \frac{1}{t_j}\sum_{i=i_0}^{j-1} g(s_i^*)(t_{i+1} - t_i) \geq \left[\inf_i g(s_i^*)\right]\left(1 - \frac{t_{i_0}}{t_j}\right).$$

If we can show that $\inf_i g(s_i^*) > 0$, then

$$\lim_{j\to\infty}\frac{1}{t_j}\int_{t_{i_0}}^{t_j} g(s)\,ds > 0,$$

which will be a contradiction of (3.6).

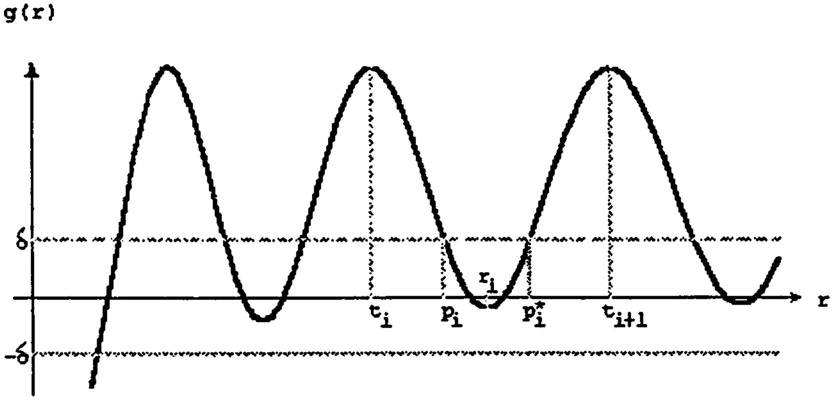
Set $A_i = \{s \in (t_i, t_{i+1}) : |g(s)| < \delta\}$ and $B_i = (t_i, t_{i+1}) \setminus A_i$. Denote the Lebesgue measure of a set A by meas(A). Then

$$g(s_i^*) = \frac{1}{t_{i+1} - t_i}\int_{t_i}^{t_{i+1}} g(s)\,ds$$
$$= \frac{1}{\operatorname{meas}(A_i) + \operatorname{meas}(B_i)}\left(\int_{A_i} g(s)\,ds + \int_{B_i} g(s)\,ds\right)$$
$$\geq \frac{\delta \operatorname{meas}(B_i)}{\operatorname{meas}(A_i) + \operatorname{meas}(B_i)} - \frac{\delta \operatorname{meas}(A_i)}{\operatorname{meas}(A_i) + \operatorname{meas}(B_i)}.$$

If we can prove that $\operatorname{meas}(A_i) \leq \frac{1}{4}(t_{i+1} - t_i)$, then it will follow that $g(s_i^*) \geq \frac{3}{4}\delta - \frac{1}{4}\delta = \frac{1}{2}\delta$. From (2.6), since Q and F are bounded, it follows that $|u'| \leq M < \infty$. By applying the mean value theorem,

$$0 < F(u_+) - F(b) \leq |g(r_i) - g(t_{i+1})|$$
$$= |f(u^*)|\,|u'(t^*)|\,|r_i - t_{i+1}| \leq NM(t_{i+1} - r_i).$$

Similarly, $F(u_+) - F(b) < NM(r_i - t_i)$, so it follows that $(t_{i+1} - t_i) \geq a > 0$ for all i (see Fig. 1).



Since $|u''| \le M_0$ for all $r > 0$, we have

$$|u'(r)| = \left| \int_{r_i}^{r} u''(s)\, ds \right| \le M_0 |r - r_i| \qquad \text{for all } i.$$

For every $\varepsilon > 0$, we can find an η independent of i such that $|u'(r)| \le \varepsilon$ when $|r - r_i| \le \eta$. Since $f^2(u(r_i)) \ge \varepsilon_0 > 0$, for all i, and

$$g''(r) = f^2(u(r)) + \left\{ \frac{(n-1)f(u(r))}{r} - f'(u(r))u'(r) \right\} u'(r), \quad (3.7)$$

there is an η independent of i such that $g''(r) \ge \frac{1}{2}\varepsilon_0$, for $|r - r_i| \le \eta$. Then $|g'(r)| \ge \frac{1}{2}\varepsilon_0 |r - r_i|$, so that

$$g(r_i \pm \eta) - g(r_i) \ge \tfrac{1}{4}\varepsilon_0 \eta^2, \qquad \text{or } g(r_i \pm \eta) \ge \tfrac{1}{8}\varepsilon_0 \eta^2, \quad (3.8)$$

for i sufficiently large. Choose $\delta < \frac{1}{8}\varepsilon_0 \eta^2$, and let $\delta = g(p_i) = g(p_i^*)$. Then $p_i^* - p_i \le 2\eta$ and η can be chosen less than $\frac{1}{8}a$, so that

$$\text{meas}(A_i) = p_i^* - p_i \le 2\eta \le \tfrac{1}{4}a \le \tfrac{1}{4}(t_{i+1} - t_i). \quad (3.9)$$

THEOREM 3.10. *Let $f(u)$ be a continuous restoring function satisfying* (1.8) *and assume that* (1.6) *holds for* $1 < q < n^*$. *Then the solution of* (1.5) *will oscillate infinitely and tend to* 0 *as* $r \to \infty$.

Proof. From (1.6) and (1.8) it follows that the hypotheses of Remark 2.18 are satisfied, so the solution u of (1.5) must oscillate infinitely about 0. By (1.8) and continuity, it follows that $u = 0$ is the only root of $f(u) = 0$ and $F(u) \ge F(0) = 0$. The conclusion follows from Lemma 3.5.

EXAMPLE 3.11. From Example 2.23 it follows that $f(u) = u^5 + u^3\sqrt{1+u^2}$ satisfies the hypotheses of Theorem 3.10. Hence, the solution to (1.5) with this function and $u_0 \neq 0$ oscillates infinitely and tends to 0 as $r \to \infty$. A similar conclusion holds in R^n for

$$f(u) = u\left(|u|^{n^*-1} + a|u|^{q-1}\right), \qquad 1 < q < n^*. \tag{3.12}$$

However, Theorem 3.10 does not hold in R^3 for solutions u of (1.5) when $f(u) = u^5 + u^4$, since inequality (1.8) fails to be satisfied because $f(u) > 0$ in $-1 < u < 0$. The behavior in this case, where $f(u)$ is not restoring, is substantially more complicated and will be explored in another paper [CCD].

It is worthwhile to comment on Theorem 2.B of [J], where he points out that nonlinearities of the form $f(u) = |u|^\sigma u + |u|^\rho u$, $0 < \rho < \sigma$, arise in the study of stationary or standing waves and in the Klein–Gordon or Schrödinger equations. Our Theorem 3.10 shows that for many choices of σ and ρ (e.g., $n = 3$, $\sigma = 4$, $\rho = 3$) a solution exists which oscillates to zero. Theorem 2.B of [J] states that for $\sigma = 4/(n-2)$, $n > 2$, and for $0 < \rho < \sigma - 1$, the problem

$$u'' + \frac{n-1}{r}u' + |u|^\sigma u + |u|^\rho u = 0, \qquad u(R) = 0,\ u'(0) = 0,$$

has no solutions with zeros in $0 < R \ll 1$, which means that $u(r)$ is positive in $0 \leq r < R$. The statements in [J] about positive solutions are relative to some bounded interval $[0, R)$ and not to $[0, \infty)$.

4. AMPLITUDES AND PERIODS OF OSCILLATION

In this section we will study the amplitude and period of the oscillations of the singular initial-value problem

$$u'' + \frac{n-1}{r}u' + \left(b|u|^{q-1} + |u|^{p-1}\right)u = 0, \qquad u(0) \neq 0 = u'(0), \tag{4.1}$$

with $b > 0$ and $1 < q < p \leq n^* = (n+2)/(n-2)$. While our proof will be limited to this binomial expression for $f(u)$, it is not difficult to extend our technique to more general $f(u)$. Following our results on the amplitude and period of the oscillations of (4.1), we will indicate some conditions that can be used in proving a more general result.

Theorem 3.10 guarantees that the solution u of (4.1) oscillates infinitely often and that u converges to 0 as $r \to \infty$. However, it does not provide a rate of convergence. Lemma 3.1 provides such a rate: $|u| \le cr^{-2/(q+1)}$, but requires additional hypotheses. However, this rate is far too coarse and can be improved on substantially.

Let r_j be the zeros of $u(r)$ and t_j be the local extremum points of $u(r)$ with $r_j < t_j < r_{j+1}$. Then the zeros and extrema of u satisfy the following result.

THEOREM 4.2. *There exists a positive constant c_1, such that*

$$\lim_{r\to\infty} r^{2(n-1)(q+1)/(q+3)} Q(u(r)) = c_1, \tag{4.3}$$

implying that

$$\lim_{j\to\infty} t_j^{2(n-1)/(q+3)} |u(t_j)| = \left[(q+1)c_1/b\right]^{1/(q+1)},$$

$$\lim_{j\to\infty} r_j^{(n-1)(q+1)/(q+3)} |u'(r_j)| = (2c_1)^{1/2}. \tag{4.4}$$

Proof. Applying Pokhozhaev's second identity with $\alpha = 2(n-1)/(q+3)$ and

$$1 < k = \frac{2(q+1)(n-2)-4}{q+3} + 1 = \frac{2(q+1)(n-1)}{q+3} - 1$$

$$= n - 1 - \frac{n+2-(n-2)q}{q+3} < n-1, \tag{4.5}$$

we have

$$r^{k+1}Q(u(r)) + \alpha r^k u(r)u'(r) + c_2 r^{k-1}u^2(r) + c_3\int_0^r |u(s)|^{p+1}s^k ds$$

$$= c_4\int_0^r u^2(s)s^{k-2}ds, \tag{4.6}$$

or

$$r^{k+1}\left(\frac{b|u(r)|^{q+1}}{q+1} + \frac{|u(r)|^{p+1}}{p+1}\right) + \frac{1}{4}r^{k+1}u'^2(r)$$

$$+ \frac{1}{4}r^{k-1}\left[ru'(r) + 2\alpha u(r)\right]^2 + c_3\int_0^r |u(s)|^{p+1}s^k\,ds$$

$$\le c_4\int_0^r u^2(s)s^{k-2}\,ds + \alpha^2 r^{k-1}u^2(r), \tag{4.7}$$

where

$$c_2 = \frac{(n-1)[n+2-q(n-2)]}{(q+3)^2}, \qquad c_3 = \frac{2(n-1)(p-q)}{(q+3)(p+1)},$$

$$c_4 = c_2(k-1) > 0.$$

By Young's inequality with $p' = (p+1)/2$ and $q' = (p+1)/(p-1)$, we have

$$c_4 u^2(s) s^{k-2} = \left(u^2(s) s^{2k/(p+1)}\right)\left(c_4 s^{k-2-2k/(p+1)}\right) \le c_3 |u(s)|^{p+1} s^k + c s^{\mu}, \tag{4.8}$$

where

$$\begin{aligned}
\mu &= \left(k - 2 - \frac{2k}{p+1}\right)\frac{p+1}{p-1} = k - 2 - \frac{4}{p-1} \\
&= \frac{2(q+1)(n-1)}{q+3} - 3 - \frac{4}{p-1} \\
&= -\frac{2(q+1)([(n+2)-p(n-2)] + 4(p-q)}{(p-1)(q+3)} - 1 < -1.
\end{aligned} \tag{4.9}$$

Also, by (4.8), $\alpha^2 r^{k-1} u^2(r) = r[\alpha^2 r^{k-2} u^2(r)] \le r[(1/2(p+1)) r^k |u(r)|^{p+1} + c r^{\mu}]$, so substituting this inequality and (4.8) into (4.7) yields

$$\begin{aligned}
&r^{k+1}\left(\frac{b|u(r)|^{q+1}}{q+1} + \frac{|u(r)|^{p+1}}{p+1}\right) + \frac{1}{4} r^{k+1} u'^2(r) + c_3 \int_0^r |u(s)|^{p+1} s^k \, ds \\
&\qquad \le c_4 \int_0^1 u^2(s) s^{k-2} \, ds + c_3 \int_1^r |u(s)|^{p+1} s^k \, ds + c \int_1^r s^{\mu} \, ds \\
&\qquad\qquad + \frac{|u(r)|^{p+1}}{2(p+1)} r^{k+1} + c r^{\mu+1}.
\end{aligned} \tag{4.10}$$

Thus we have proved simultaneously that

$$r^{k+1} Q(u(r)) \le c, \qquad |u(r)| \le c r^{-2(n-1)/(q+3)},$$
$$|u'(r)| \le c r^{-(q+1)(n-1)/(q+3)}. \tag{4.11}$$

Consequently we have, from (4.11),

$$\begin{aligned}
|\alpha r^k u(r) u'(r)| &\le c r^k r^{-2(n-1)/(q+3)} r^{-(q+1)(n-1)/(q+3)} \\
&\le c r^{-((n+2)-q(n-2))/(q+3)},
\end{aligned} \tag{4.12}$$

$$r^{k-1} u^2(r) \le c r^{-2[(n+2)-q(n-2)]/(q+3)}. \tag{4.13}$$

Letting $r \to \infty$ in (4.6) and using (4.12) and (4.13), we have

$$\lim_{r\to\infty} r^{k+1} Q(u(r)) = c_4 \int_0^\infty u^2(s) s^{k-2}\, ds - c_3 \int_0^\infty |u(s)|^{p+1} s^k\, ds. \quad (4.14)$$

We claim that

$$\lim_{r\to\infty} r^{k+1} Q(u(r)) > 0. \quad (4.15)$$

Suppose otherwise, then $\lim_{r\to\infty} r^{k+1} Q(u(r)) = 0$, since $Q(u(r)) > 0$ for all $r > 0$. Set

$$\lambda = \min\left\{ \frac{n+2-(n-2)q}{q+3}, \frac{2(n-1)(p-q)}{q+3} \right\} - \varepsilon_0 > 2(q+1)\varepsilon_0 > 0,$$

for small ε_0. By L'Hopital's rule, (4.11)–(4.14), we get

$$\begin{aligned} &\left| \lim_{r\to\infty} \frac{c_4 \int_0^r u^2(s) s^{k-2}\, ds - c_3 \int_0^r |u(s)|^{p+1} s^k\, ds}{r^{-\lambda}} \right| \\ &\quad \le \frac{c}{\lambda} \left[\lim_{r\to\infty} u^2(r) r^{\lambda+k-1} + \lim_{r\to\infty} |u(r)|^{p+1} r^{\lambda+1+k} \right] \\ &\quad \le \frac{c}{\lambda} \left[\lim_{r\to\infty} r^{\lambda - 2[n+2-(n-2)q]/(q+3)} + \lim_{r\to\infty} r^{\lambda - 2(p-q)(n-1)/(q+3)} \right] = 0. \end{aligned} \quad (4.16)$$

Hence, multiplying (4.6) by r^λ and using (4.12) and (4.13), we have

$$\lim_{r\to\infty} r^{k+1+\lambda} Q(u(r)) = 0, \qquad \text{or } |u(r)| \le c r^{-2(n-1)/(q+3) - \lambda(q+1)}. \quad (4.17)$$

Consequently, we have from (4.11) and (4.17),

$$|u(r)|^{p+1} r^{k+1} \le c r^{-2(n-1)(p-q)/(q+3) - \lambda(p+1)/(q+1)}, \quad (4.18)$$

and

$$\begin{aligned} &|\alpha r^k u(r) u'(r)| \\ &\quad \le c r^{-((n+2)-q(n-2))/(q+3) - \lambda/(q+1)} \le c r^{-((n+2)-q(n-2))/(q+3) - 2\varepsilon_0}. \end{aligned} \quad (4.19)$$

We can repeat the process in (4.16) using

$$\begin{aligned} \lambda_1 = \min&\left\{ \frac{n+2-q(n-2)}{q+3} + 2\varepsilon_0, \frac{2(n-1)(p-q)}{q+3} + \frac{\lambda(p+1)}{q+1} \right\} \\ &- \varepsilon_0 > 2(q+1)\varepsilon_0 > 0, \end{aligned}$$

to obtain

$$\lim_{r\to\infty} r^{k+1+\lambda_1} Q(u(r)) = 0, \qquad \text{or } |u(r)| \le c r^{-2(n-1)/(q+3)-\lambda_1/(q+1)}. \tag{4.20}$$

If

$$\frac{n+2-q(n-2)}{q+3} + 2\varepsilon_0 \le \frac{2(n-1)(p-q)}{q+3} + \frac{\lambda(p+1)}{q+1},$$

then, by Lemma 2.9,

$$\begin{aligned} 2r_j^n Q\big(u(r_j)\big) &= \int_0^{r_j}\left[\frac{n+2-q(n-2)}{q+1} b|u(s)|^{q+1}\right.\\ &\qquad \left. + \frac{n+2-p(n-2)}{p+1}|u(s)|^{p+1}\right] s^{n-1}\,ds \ge \varepsilon_1 > 0, \end{aligned} \tag{4.21}$$

contradicting (4.20), since $k+1+\lambda_1 = n+\varepsilon_0 > n$. If

$$\frac{n+2-q(n-2)}{(q+3)} + 2\varepsilon_0 > \frac{2(n-1)(p-q)}{(q+3)} + \frac{\lambda(p+1)}{q+1},$$

then, from (4.20),

$$\begin{aligned} &|u(r)|^{p+1} r^{k+1} \\ &\quad \le c r^{-2(n-1)(p-q)/(q+3)-2(n-1)(p-q)(p+1)/(q+3)(q+1)-\lambda(p+1)^2/(q+1)^2+\varepsilon_0(p+1)/(q+1)} \\ &\quad \le c r^{-2(n-1)(p-q)/(q+3)-\lambda(p+1)^2/(q+1)^2}. \end{aligned}$$

Repeating the process in (4.16) with

$$\begin{aligned} \lambda &= \lambda_2 \\ &= \min\left\{\frac{n+2-q(n-2)}{(q+3)} + 2\varepsilon_0, \frac{2(n-1)(p-q)}{(q+3)} + \frac{\lambda(p+1)^2}{(q+1)^2}\right\} - \varepsilon_0, \end{aligned}$$

yields

$$\lim_{r\to\infty} r^{k+1+\lambda_2} Q(u(r)) = 0.$$

Continue in this fashion until there is an i such that

$$\frac{2(n-1)(p-q)}{(q+3)} + \frac{\lambda(p+1)^i}{(q+1)^i} \geq \frac{n+2-q(n-2)}{(q+3)} + 2\varepsilon_0.$$

Then repeat the process in (4.21) to get a contradiction. Hence (4.15) is always true. The rest of the assertions follow immediately from the definition of the energy function.

We now show that the period between zeros of (4.1) grows longer as $r \to \infty$.

THEOREM 4.22. *There exits a positive constant c, such that*

$$\frac{r_{j+1} - r_j}{r_j^{(n-1)(q-1)/(q+3)}} \geq c. \tag{4.23}$$

Proof. Since

$$r_j^{k+1} Q\big(u(r_j)\big) = \frac{1}{2} r_j^{k+1} u'^2(r_j),$$

$$u'(r) = -\frac{1}{r^{n-1}} \int_{t_j}^{r} \Big[b|u(s)|^{q-1} + |u(s)|^{p-1} \Big] u(s) s^{n-1}\, ds$$

for $t_j < r$ and since $u'(t_j) = 0$, it follows from (4.4) and (4.11) that, for j large enough,

$$\begin{aligned} \sqrt{c_1} &\leq r_{j+1}^{(k+1)/2} |u'(r_{j+1})| \leq r_{j+1}^{-2(n-1)/(q+3)} \int_{t_j}^{r_{j+1}} \big[b|u(s)|^q + |u(s)|^p \big] s^{n-1}\, ds \\ &\leq c r_{j+1}^{-2(n-1)/(q+3)} \int_{t_j}^{r_{j+1}} s^{n-1-2q(n-1)/(q+3)}\, ds \\ &= c r_{j+1}^{-2(n-1)/(q+3)} \int_{t_j}^{r_{j+1}} s^{(3-q)(n-1)/(q+3)}\, ds. \end{aligned} \tag{4.24}$$

If $q \leq 3$ then the integrand is monotone increasing so that

$$\sqrt{c_1} \leq c r_{j+1}^{-(q-1)(n-1)/(q+3)} \int_{t_j}^{r_{j+1}} ds \leq c r_{j+1}^{-(q-1)(n-1)/(q+3)} (r_{j+1} - r_j).$$

If $q > 3$, then the integrand in (4.24) is monotone decreasing and

$$\begin{aligned}\sqrt{c_1} &\le c r_{j+1}^{-2(n-1)/(q+3)} t_j^{(3-q)(n-1)/(q+3)} \int_{t_j}^{r_{j+1}} ds \\ &\le c t_j^{-(q-1)(n-1)/(q+3)} \left(r_{j+1} - r_j \right).\end{aligned}$$

The rest of the proof is immediate.

Remark 4.25. Theorems 4.2 and 4.22 can be extended, with very minor modifications in their proofs, to restoring functions $f(u)$ satisfying (1.6), (1.8) for all u, and the condition:

$$uf(u) - (q+1)F(u) \ge c_1 \min\left(|u|^{p+1}, 1\right), \qquad F(u) \ge c_2 \min\left(|u|^{q+1}, 1\right) \tag{4.26}$$

for all u, for $1 < q < p \le n^*$. For large r, it follows, since $u \to 0$, that $c_2 r^{k+1} |u(r)|^{q+1} \le r^{k+1} F(u(r))$ and the integrand $uf(u) - (q+1)F(u) \ge c|u|^{p+1}$, so we can replace (4.7) by

$$\begin{aligned}&c_2 r^{k+1} |u(r)|^{q+1} + \frac{1}{4} r^{k+1} u'^2(r) + \frac{1}{4} r^{k-1} \left[r u'(r) + 2\alpha u(r) \right]^2 \\ &\qquad + \alpha c_1 \int_{r_0}^{r} |u(s)|^{p+1} s^k \, ds + \alpha \int_0^{r_0} \left[u(s) f(u(s)) - (q+1) F(u(s)) \right] s^k ds \\ &\qquad \le c_3 \int_0^r u^2(s) s^{k-2} \, ds + \alpha^2 r^{k-1} u^2(r).\end{aligned} \tag{4.27}$$

Using Young's inequality on the two terms on the right involving u^2, we bound the right side of (4.27) by

$$c_3 \int_0^{r_0} u^2(s) s^{k-2} \, ds + \varepsilon \int_{r_0}^{r} |u(s)|^{p+1} s^k ds + \varepsilon r^{k+1} |u(r)|^{q+1} + c,$$

with ε sufficiently small so that (4.11) applies. Minor adjustments to the proof also need to be made in (4.16) and (4.21). Note that Theorem 4.22 implies that the period between zeros is bounded by a constant when $q = 1$.

REFERENCES

[BLP] H. Berestycki, P. L. Lions, and L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in R^n, *Indiana University Math. J.* **30** (1981), 141–157.

[BN] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[CCD] S. Chen, J. A. Cima, and W. R. Derrick, Oscillations of eventually negative radial solutions of semilinear elliptic equations, in preparation.

[DN] W.-Y. Ding and W.-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, *Arch. Rational Mech. Anal.* **91** (1986), 283–308.

[DN2] W.-Y. Ding and W.-M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, and related topics, *Duke Math. J.* **52** (1985), 485–506.

[G] C. Gui, Positive entire solutions of the equation $\Delta u + f(x,u) = 0$, *J. Differential Equations* **99** (1992), 245–280.

[J] C. K. R. T. Jones, Radial solutions of a semilinear elliptic equation at a critical point, *Arch. Rational Mech. Anal.* **104** (1988), 251–270.

[JK] C. Jones and T. Küpper, On the infinitely many solutions of a semilinear elliptic equation, *SIAM J. Math. Anal.* **17** (1986), 803–835.

[KNY] N. Kawano, W.-M. Ni, and S. Yotsutani, A generalized Pohozaev identity and its applications, *J. Math. Soc. Japan* **42** (1990), 541–564.

[L] Y. Li, Asymptotic behavior of positive solutions of $\Delta u + K(x)u^p = 0$ in R^n, *J. Differential Equations* **95** (1992), 304–330.

[LN1] Y. Li and W.-M. Ni, On conformal scalar curvature equations in R^n, *Duke Math. J.* **57** (1988), 895–924.

[LN2] Y. Li and W.-M. Ni, On the existence and symmetry properties of finite total mass solutions of the Matakuma equation, the Eddington equation, and their generalizations, *Arch. Rational Mech. Anal.* **108** (1989), 175–194.

[LN3] Y. Li and W.-M. Ni, On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in R^n. I. Asymptotic behavior, *Arch. Rational Mech. Anal.* **118** (1992), 195–222.

[LN] C.-S. Lin and W.-M. Ni, A counterexample to the nodal domain conjecture and a related semilinear equation, *Proc. Amer. Math. Soc.* **102** (1988), 271–277.

[M] S. Maier, Radial solutions of semilinear elliptic equations with prescribed numbers of zeros, *J. Differential Equations* **107** (1994), 175–205.

[MS] K. McLeod and J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in R^n, *Arch. Rational Mech. Anal.* **99** (1987), 115–145.

[MTW] K. McLeod, W. C. Troy, and F. B. Weissler, Radial solutions of $\Delta u + f(u) = 0$ with prescribed numbers of zeros, *J. Differential Equations* **83** (1990), 368–378.

[N] W.-M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50** (1983), 289–304.

[N2] W.-M. Ni, On the elliptic equation $\Delta u + K(x)u^{(n+2)(n-2)} = 0$, its generalizations and applications in geometry, *Indiana Univ. Math. J.* **31** (1982), 493–529.

[NY] W.-M. Ni and S. Yotsutani, Semilinear elliptic equations of Mutukuma-type and related topics, *Japan J. Appl. Math.* **5** (1988), 1–32.

[P] X. Pan, Positive solutions of the elliptic equation $\Delta u + u^{(n+2)/(n-2)} + K(x)u^q = 0$ in R^n and in balls, *J. Math. Anal. Appl.* **172** (1993), 323–338.

[Po] S. I. Pokhozhaev, On the asymptotics of entire radial solutions of quasilinear elliptic equations, *Soviet Math. Dokl.* **44** (1992), 548–553.

[Po2] S. I. Pokhozhaev, On quasilinear elliptic problems in R^n in the supercritical case, *Proc. Steklov Inst. Math.* **2** (1994), 273–285.

[Y] E. Yamagida, Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ in R^n, *Arch. Rational Mech. Anal.* **115** (1991), 257–274.