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On the Local Stability of Nonautonomous Difference Equations in $\mathbb{R}^n$

MARC ARTZROUNI*

Department of Biostatistics, School of Public Health, University of North Carolina, Chapel Hill, North Carolina 27514

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We prove several theorems on the local stability of an iterative process associated with a nonautonomous difference equation in $\mathbb{R}^n$. These results provide general conditions under which the common fixed point X^* of a family of operators is uniformly stable, uniformly attractive, or uniformly exponentially stable. The stability conditions are obtained by majorizing products of Jacobian matrices in a neighborhood of X^*. When the Fréchet derivatives are equicontinuous at X^*, majorizations at X^* suffice to ensure stable behavior. Nonuniform stability conditions are discussed. Stability conditions are also investigated when the spectral radii of the Jacobian matrices at X^* are uniformly bounded below 1.

1. INTRODUCTION

We let G_k $(k = 0, 1, \ldots)$ be a sequence of operators in $\mathbb{R}^n$ defined by

$$X = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \longmapsto G_k(X) = \begin{matrix} g_1^k(x_1, x_2, \ldots, x_n) \\ g_2^k(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n^k(x_1, x_2, \ldots, x_n) \end{matrix} \tag{1.1}$$

We are concerned here with the dynamical behavior of the sequence X_k generated by the nonautonomous difference equation

$$X_{k+1} = G_k(X_k), \qquad k = 0, 1, \ldots. \tag{1.2}$$

The iterative process defined in Eq. (1.2) has applications in numerical analysis (e.g., the Newton–Gauss–Seidel method) and can also be a useful

* Research partially funded by NICHD Grant HD 07237. Current address: Mathematical Sciences, Clemson University, Clemson, SC 29634.

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tool when describing certain discrete dynamical processes in ecology, physics, population dynamics, etc. Whether from a theoretical point of view [3, 4, 5, 6, 10, 14, 18] or from an applied point of view [1, 9, 11, 12, 13], the iterative process of Eq. (1.2) has received a fair amount of attention.

In this paper we focus on the behavior of the sequence X_k in the neighborhood of a common fixed point X^* of all the operators G_k. That is, we assume that

$$X^* = G_k(X^*), \qquad k = 0, 1, \dots. \tag{1.3}$$

In the sequel we discuss various types of stabilities pertaining to non-autonomous difference equations of the form (1.2). We will review the classical concepts of uniform stability, uniform attractiveness, and uniform exponential stability of a common fixed point X^*. We will give general theorems, then more particular ones, under which the various types of stabilities occur. These theorems complement or generalize those by Ortega and Rheinbolt [15, pp. 349, 354], Cavanagh [3, p. 61], and Smith [18]. First, we give some notations which can be found in Ortega [14].

We let $X(k, p, X_p)$ $(k = p, p+1, \dots)$ be the sequence of iterates generated by Eq. (1.2) and starting at index p with the initial value X_p. We then have

$$X(k+1, p, X_p) = G_k(X(k, p, X_p)), \qquad k = p, p+1, \dots, \tag{1.4}$$

with

$$X(p, p, X_p) = X_p$$

and

$$X(r+i, r-1, X(r-1, p, X_p)) = X(r+i, p, X_p); \qquad r = p+1, \dots; \quad i = 0, \dots. \tag{1.5}$$

We next consider the following definitions.

DEFINITIONS 1.1. We assume that X^* is a common fixed point of the operators G_k. Then X^* is

(i) *uniformly stable* if given any $\varepsilon > 0$ there is $\delta > 0$ such that for any p

$$|X(k, p, X_p) - X^*| \leqslant \varepsilon \qquad \text{for} \quad k = p+1, p+2, \dots,$$

whenever $|X_p - X^*| \leqslant \delta$;

(ii) *uniformly attractive* if there is $\delta > 0$ such that given any $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that for any p,

$$|X(k+p, p, X_p) - X^*| \leqslant \varepsilon \qquad \text{for} \quad k = N(\varepsilon), N(\varepsilon)+1, \ldots,$$

whenever $|X_p - X^*| \leqslant \delta$;

(iii) *uniformly exponentially stable* if there is $\delta > 0$, $K > 0$, and $c > 0$ $(c < 1)$ such that for any p

$$|X(k, p, X_p) - X^*| \leqslant K|X_p - X^*|\, c^{k-p}; \qquad k = p+1, p+2, \ldots,$$

whenever $|X_p - X^*| \leqslant \delta$.

In what follows, we will assume, with no loss of generality, that the fixed point X^* is the origin 0. At times we will also assume that the operators G_k are differentiable in a neighborhood $S(0, r)$ of the origin ($S(0, r)$ in $\mathbb{R}^n$ is the ball of center 0 and radius r). The Fréchet derivative of G_k at any point X will then be the n-square Jacobian matrix $\mathbf{G}'_k(X)$ having in its ith row, jth column

$$\mathbf{G}'_k(X)_{i,j} = \frac{\partial g_i^k(X)}{\partial x_j}. \tag{1.6}$$

When the operators G_k are differentiable in $S(0, \varepsilon)$, we define $P(k, p, \varepsilon)$ for $k > p > 0$ by

$$P(k, p, \varepsilon) = \sup_{\substack{|W_i| \leqslant \varepsilon \\ i = 0, 1, \ldots, k-1-p}} \left\| \prod_{j=0}^{k-1-p} \mathbf{G}'_{k-j-1}(W_j) \right\|, \tag{1.7}$$

where $\| \ \|$ is any consistent norm (i.e., $\|\mathbf{AB}\| \leqslant \|\mathbf{A}\| \cdot \|\mathbf{B}\|$) on the metric space $M_n(\mathbb{C})$ of n-square complex matrices. The Euclidean norm of a vector W in $\mathbb{R}^n$ is denoted $|W|$. We are now in a position to prove the main theorem of this paper.

2. Main Theorem

We give general conditions under which a common fixed point of the operators G_k is uniformly stable, uniformly attractive, or uniformly exponentially stable.

THEOREM 2.1. *We assume that the origin* 0 *is a common fixed point of the operators* G_k *defined in* (1.1). *Now, we consider the following three sets of assumptions*:

(A1) *There is* $\varepsilon_1 > 0$ *and* $K_1 > 0$ *such that every* G_k *is differentiable in* $S(0, \varepsilon_1)$ *and*

$$P(k, p, \varepsilon_1) \leqslant K_1 \quad \forall p > 0, \qquad \forall k > p.$$

(A2) *The operators* G_k *are differentiable everywhere. There is* $K_1 > 0$ *and* $\delta_1 > 0$ *such that for any* $\varepsilon_1 > 0$ *there will be an integer* $N(\varepsilon_1)$ *with*

$$P(p + N(\varepsilon_1) - 1, p, \infty) \leqslant \varepsilon_1, \qquad \forall p > 0$$

and

$$n \geqslant N(\varepsilon_1) \Rightarrow P(p + n, p + N(\varepsilon_1) - 1, \delta_1) \leqslant K_1.$$

(A3) *There is* $K_1 > 0$, $\varepsilon_1 > 0$, *and* $c_1 > 0$ $(c_1 < 1)$ *such that the operators* G_k *are differentiable on* $S(0, \varepsilon_1)$ *and*

$$P(k, p, \varepsilon_1) \leqslant K_1 c_1^{k-p}, \qquad \forall p > 0, \quad \forall k < p.$$

Then under assumption (A1), (A2), or (A3), *respectively, the origin is uniformly stable, uniformly attractive, or uniformly exponentially stable.*

Proof. Throughout the paper we will repeatedly use

LEMMA 2.1. We assume that the operators are differentiable in a ball $S(0, \delta)$. For fixed values of k and p $(k > p > 0)$ we define

$$M(k, p) = \operatorname*{Max}_{p+1 \leqslant j \leqslant k} P(j, p, \delta), \tag{2.1}$$

and we choose ε smaller than δ and satisfying

$$M(k, p) \leqslant \delta/\varepsilon. \tag{2.2}$$

We then have

$$X_p \in S(0, \varepsilon) \Rightarrow |X(j, p, X_p)| \leqslant P(j, p, \delta)|X_p|, \qquad j = p+1, p+2, \ldots, k. \tag{2.3}$$

Proof. First, we assume that for some integer $r \geqslant 1$, we have

$$U_p \in S(0, \varepsilon) \Rightarrow X(p+i, p, U_p) \in S(0, \delta), \qquad i = 0, 1, \ldots, r-1. \tag{2.4}$$

We note that (2.4) holds for $r = 1$ because $\varepsilon \leqslant \delta$.

By use of a mean-value theorem [15, p. 69] and the chain rule [15, p. 62], we then have, if $X_p \in S(0, \varepsilon)$ and $k = p + r$,

$$|X(j, p, X_p)| \leqslant \operatorname*{Sup}_{0 \leqslant t \leqslant 1} \left\| \prod_{i=0}^{j-1-p} \mathbf{G}'_{j-i-1}(X(j-i-1, p, tX_p)) \right\| \cdot |X_p|,$$
$$j = p+1, p+2, \ldots, k. \tag{2.5}$$

Indeed, we recall that the Jacobian matrix (at tX_p) of the composition of m operators is the product of the m Jacobian matrices taken at the iterates of tX_p (provided the iterates belong to the domain of differentiability of the operators; this is ensured here by (2.4) with $U_p = tX_p$).

By definition of $P(j, p, \delta)$, the right-hand side of (2.5) is majorized by $P(j, p, \delta)|X_p|$ and therefore

$$|X(j, p, X_p)| \leqslant P(j, p, \delta)|X_p|. \tag{2.6}$$

As we noted above, under the conditions of the lemma, (2.4) holds for $r = 1$. Then (2.5) and (2.6) hold for $j = p + 1$ when $X_p \in S(0, \varepsilon)$. The conclusion (2.3) of the lemma is therefore proved for $j = p + 1$.

Recalling (2.1), (2.2), and the fact that $X_p \in S(0, \varepsilon)$, we now have

$$|X(p+1, p, X_p)| \leqslant (\delta/\varepsilon)\, \varepsilon = \delta. \tag{2.7}$$

Therefore if $X_p \in S(0, \varepsilon)$, then $X(p+1, p, X_p) \in S(0, \delta)$. Now (2.4) is true for $r = 2$ and therefore (2.5) and (2.6) are true for $j = p + 2$. This proves the lemma for $j = p + 1, p + 2$. Now $X(p+2, p, X_p)$ also belongs to $S(0, \delta)$ and the proof is then complete by finite induction. We now return to our main theorem.

Uniform Stability under Assumption (A1)

We let ε_2 be any positive number (we assume with no loss of generality that $\varepsilon_2 \leqslant \varepsilon_1$ and $\varepsilon_2 \leqslant K_1 \varepsilon_1$). We then apply Lemma 2.1 with

$$\delta = \varepsilon_1, \qquad \varepsilon = \varepsilon_2/K_1. \tag{2.8}$$

Then under assumption (A1) we have, for any $k > p > 0$,

$$M(k, p) \leqslant K_1 \leqslant K_1 \varepsilon_1/\varepsilon_2 = \delta/\varepsilon, \tag{2.9}$$

which proves that (2.2) of Lemma 2.1 holds. Therefore

$$X_p \in S(0, \varepsilon_2/K_1) \Rightarrow |X(j, p, X_p)| \leqslant K_1 \varepsilon_2/K_1 = \varepsilon_2, \tag{2.10}$$

which holds for any set of integers j, p satisfying $j > p > 0$. This completes the proof.

Uniform Attractiveness under Assumption (A2)

We let ε_2 be a positive number satisfying

$$\varepsilon_2 \leqslant K_1 \delta_1, \tag{2.11}$$

$$\varepsilon_2 \leqslant \delta_1. \tag{2.12}$$

We choose from (A2) a number ε_1 defined by

$$\varepsilon_1 = \varepsilon_2 / K_1 \delta_1. \tag{2.13}$$

We now apply Lemma 2.1 with

$$\begin{aligned} \delta &= \infty, \\ k &= p + N(\varepsilon_1) - 1, \\ \varepsilon &= \delta_1. \end{aligned} \tag{2.14}$$

Then (2.2) of Lemma 2.1 always holds, and for $j = k$, the conclusion (2.3) is

$$X_p \in S(0, \delta_1) \Rightarrow |X(p + N(\varepsilon_1) - 1, p, X_p)| \leqslant \varepsilon_1 \delta_1 = \varepsilon_2 / K_1. \tag{2.15}$$

Now, we recall Eq. (1.5) (with $i = s$ and $r = p + N(\varepsilon_1)$) and observe that

$$\begin{aligned} &X(p + N(\varepsilon_1) + s, p, X_p) \\ &\quad = X(p + N(\varepsilon_1) + s, p + N(\varepsilon_1) - 1, X(p + N(\varepsilon_1) - 1, p, X_p)), \\ &\qquad\qquad s = 0, 1, \ldots . \end{aligned} \tag{2.16}$$

Next, we apply Lemma 2.1, where

(i) the initial term X_p is $X(p + N(\varepsilon_1) - 1, p, X_p)$ which we denote $X_{p + N(\varepsilon_1) - 1}$.

(ii) p is $p + N(\varepsilon_1) - 1$ of (A2).

(iii) δ is δ_1 of (A2).

(iv) ε is ε_2 / K_1.

Recalling (A2), for any $k \geqslant p + N(\varepsilon)$ we then have

$$M(k, p + N(\varepsilon_1) - 1) \leqslant \operatorname*{Max}_{p + N(\varepsilon_1) \leqslant j} P(j, p + N(\varepsilon_1) - 1, \delta_1) \leqslant K_1. \tag{2.17}$$

Given (2.12), (iii), and (iv), we have

$$K_1 \leqslant K_1 \delta_1 / \varepsilon_2 = \delta / \varepsilon, \tag{2.18}$$

and therefore (2.2) holds. Bearing in mind (A2), (2.15), (i), and (iv), the conclusion (2.3) of the lemma is

$$|X(j, p + N(\varepsilon_1) - 1, X_{p + N(\varepsilon_1) - 1})| \leqslant K_1 \varepsilon_2 / K_1 = \varepsilon_2 \tag{2.19}$$

for $j = p + N(\varepsilon_1)$, $p + N(\varepsilon_1) + 1, \ldots$.

Recalling (2.16) we then have

$$|X(p+k, p, X_p)| \leqslant \varepsilon_2 \tag{2.20}$$

for $k = N(\varepsilon_1), N(\varepsilon_1)+1, \ldots$, which completes the proof.

Uniform Exponential Stability under Assumption (A3)

We apply Lemma 2.1, where

(i) δ is ε_1 of (A3).

(ii) ε is $\varepsilon_1/K_1 c_1$ of (A3).

Then under assumption (A3) we have

$$M(k, p) \leqslant K_1 c_1 = \delta/\varepsilon, \tag{2.21}$$

which proves that (2.2) holds for any $k > p > 0$. Therefore

$$X_p \in S(0, \varepsilon_1/K_1 c_1) \Rightarrow |X(k, p, X_p)| \leqslant P(k, p, \varepsilon_1)|X_p| \leqslant (\varepsilon_1/c_1)\, c_1^{k-p}, \qquad \forall p, \quad \forall k > p, \tag{2.22}$$

which proves that the origin is uniformly exponentially stable.

We should note an important fact concerning the conditions on the quantities $P(k, p, \varepsilon)$. Contrary to the classical results (of the form $\|\mathbf{G}'(X)\| < c < 1$ in a neighborhood of the origin) the various assumptions of Theorem 2.1 impose virtually no constraints on the individual Jacobian matrices (other than uniform boundedness in a neighborhood of the origin; for instance when $k = p+1$ in (A1)).

On the other hand, we observe that under assumption (A1), products of arbitrarily large numbers of matrices must be bounded. If we think of the autonomous case when all the operators are equal to some constant G, then such boundedness can occur only when the spectral radius $\rho(\mathbf{G}'(0))$ is less than 1 (we know of course that $\rho(\mathbf{G}'(0)) < 1$ is sufficient for stability [16]). We will see later that the boundedness conditions of Theorem 2.1 (specifically under assumption (A3)) may be obtained, in the non-autonomous case, by carefully generalizing conditions of the form $\rho(\mathbf{G}'(0)) < 1$.

The only simple way of ensuring that assumption (A1) holds is by imposing conditions of the form $\|\mathbf{G}_k'(X)\| < 1$ in a neighborhood of the origin. Indeed if there is ε such that

$$P(m+1, m, \varepsilon) \leqslant 1 \tag{2.23}$$

for all m, then Assumption A1 is satisfied and the origin is stable.

Theorem 2.1 also generalizes a known condition under which the origin is attractive. Indeed, it is known [15, p. 354, Ex. 11.1–11.4] that if

$$P(m+1, m, \varepsilon) \leqslant c < 1, \qquad \forall m > 0, \tag{2.24}$$

then the origin is attractive. However, (2.24) ensures that assumption (A3) is satisfied and we see that the origin is in fact uniformly exponentially stable.

We note that Theorem 2.1 immediately yields stability conditions in the nonuniform case. Indeed, if (A1), (A2), or (A3) hold only for $p=0$, then the corresponding iteration (starting at index 0) will be stable, attractive, or exponentially stable.

In the nonuniform context we will now discuss the possible behavior of an iteration that falls in between the cases (2.23) and (2.24). Indeed, it may be of interest to study an iteration in which the norms of the Jacobian matrices in a neighborhood of the origin are bounded by 1 but come arbitrarily close to 1. In other words, we assume there is an increasing sequence of integers $m(i)$ satisfying

$$m(i) \to \infty \qquad \text{as} \quad i \to \infty. \tag{2.25}$$

We assume that the boundedness condition (2.23) holds for $\varepsilon = \infty$ and

$$1 \geqslant P(m(i)+1, m(i), \infty) = 1 - s(i) \tag{2.26}$$

with

$$0 \leqslant s(i) \to 0 \qquad \text{as} \quad i \to \infty. \tag{2.27}$$

We know that stability and attractiveness are independent [14, p. 268]. Now, we will see that the origin will be either stable or attractive depending on the mode of convergence of $s(i)$.

We will say that $s(i)$ converges rapidly to 0 if $\sum s(i)$ converges. Similarly, $s(i)$ converges slowly to 0 if $\sum s(i)$ diverges. We then have the following result.

PROPOSITION 2.1. *With the notations given above, the origin is*

(i) *stable (and not necessarily attractive) if $s(i)$ converges rapidly to* 0,

(ii) *stable and attractive if $s(i)$ converges slowly to* 0.

Proof. The proof is based on the fact that for any k, we have

$$P(k, 0, \infty) \leqslant \prod_{j=1}^{k} P(j, j-1, \infty) \leqslant 1. \tag{2.28}$$

Because we assume that (2.23) holds, we already know that the origin is stable (whatever the mode of convergence of $s(i)$). It is then easy to see that the origin is not necessarily attractive if $s(i)$ converges rapidly to 0. The simplest example is the autonomous case where all the operators are the identity. Then $s(i)=0$ for all i and $s(i)$ trivially converges rapidly to 0. To prove that the origin is attractive when $s(i)$ converges slowly, we consider (2.28) for $k=m(i)+1$ and note that

$$P(m(i)+1, 0, \infty) \leqslant \prod_{j=0}^{i} (1-s(j)). \tag{2.29}$$

Because $s(i)$ converges slowly, given any ε, there is an index $m(i)$ such that

$$P(m(i)+1, 0, \infty) \leqslant \prod_{j=0}^{i} (1-s(j)) \leqslant \varepsilon \tag{2.30}$$

and clearly, because of (2.23),

$$n > m(i)+1 \Rightarrow P(n, m(i)+1, \infty) \leqslant 1, \tag{2.31}$$

which proves that assumption (A2) of Theorem 2.1 holds for $p=0$. Therefore the origin is attractive. In fact, because the origin is stable, the result would also hold if the domain of differentiability $S(0, \infty)$ considered in (2.26) were equal to some ball $S(0, \varepsilon)$.

The importance of the nonuniformity condition appears clearly in (2.30). Indeed, if we had wanted to prove uniform attractiveness, (2.30) would have been of the form

$$P(m(i)+1, p, \infty) \leqslant \prod_{j=j_1}^{i} (1-s(j)), \tag{2.32}$$

where $j_1 = \text{Min}\{k, m(k) \geqslant p\}$ and there would not be an index $m(i)$ such that the product of (2.32) could be made smaller than some ε for any value of p.

We have investigated conditions on the individual Jacobian matrices under which the various assumptions of Theorem 2.1 were satisfied. Next, we will examine more particular conditions under which (A3) is satisfied. Specifically, for some integer r, we will obtain uniform exponential stability by imposing boundedness conditions on lumped products of r Jacobian matrices at a time.

3. More on Uniform Exponential Stability

In this section, we generalize known conditions under which a common fixed point is attractive [15, pp. 349, 354; 61]. These generalizations lead to stronger results than simple attractiveness: we give below weak conditions under which a common fixed point of the operators G_k is uniformly exponentially stable. In the first corollary we make assumptions on the Jacobian matrices in a neighborhood of the origin.

Corollary 3.1. *We assume that the operators G_k all have the origin as a common fixed point. In addition, we make the following three assumptions:*

(A1) *Every G_k is differentiable in a ball $S(0, \delta_1)$.*

(A2) *There is $K_1 > 0$ such that*

$$P(m, m-1, \delta_1) \leqslant K_1, \qquad \forall m > 0,$$

i.e., the Jacobian matrices are uniformly bounded in $S(0, \delta_1)$.

(A3) *There is a positive integer r and a positive number c_1 ($c_1 < 1$) such that*

$$P(qr, (q-1)r, \delta_1) \leqslant c_1, \qquad \forall q > 0,$$

i.e., lumped products of r Jacobian matrices (in a neighborhood of the origin) are uniformly bounded below 1. Under these conditions the origin is uniformly exponentially stable.

Proof. It is a fairly simple and purely technical matter to prove (as we did in [2]) that if lumped products of Jacobian matrices are uniformly bounded below 1 (assumption (A3)) and if the matrices are uniformly bounded (assumption (A2)), then there are two positive numbers A_1 and c_2 ($c_2 < 1$) such that

$$P(k, p, \delta_1) \leqslant A_1 c_2^{k-p}, \qquad \forall p, \quad \forall k > p. \tag{3.1}$$

Assumption (A3) of Theorem 2.1 is now satisfied and the proof is complete.

In the next corollary we prove uniform exponential stability with conditions on the Jacobian matrices at the origin alone.

Corollary 3.2. *We again assume that the operators all have the orgin as a common fixed point. We also make the following four assumptions:*

(A1) *Each G_k is continuously differentiable in a ball $S(0, \delta_1)$.*

(A2) *There is* $K_1 > 0$ *such that*

$$P(m, m-1, 0) = \|\mathbf{G}'_{m-1}(0)\| \leqslant K_1, \qquad \forall m > 0,$$

i.e., the Jacobian matrices are uniformly bounded at the origin.

(A3) *There is a positive integer* r *and a positive number* c_1 $(c_1 < 1)$ *such that*

$$P(qr, (q-1)\,r, 0) \leqslant c_1, \qquad \forall q > 0,$$

i.e., products of lumped Jacobian matrices at the origin are uniformly bounded below 1.

(A4) *The Fréchet derivatives* $\mathbf{G}'_k(X)$ *are equicontinuous at the origin.*

Under these assumptions, the origin is uniformly exponentially stable.

Proof. First, we note the difference with Corollary 3.1. Assumptions (A2) and (A3) of Corollary 3.1 have been relaxed. Indeed, the given inequalities are now required only at the origin (and not uniformly in a neighborhood of the origin). The lost uniformity is recaptured by another type of uniformity imposed by the equicontinuity of the Fréchet derivatives at the origin.

We note that (A2) and (A4) imply that the Jacobian matrices are uniformly bounded in a neighborhood of the origin ((A2) of Corollary 3.1). In addition, equicontinuity of the Fréchet derivatives at the origin guarantees that the inequality of assumption (A3) will actually hold in a neighborhood of the origin. Hence there are two positive numbers ε_2 and c_2 $(c_2 < 1)$ such that

$$P(qr, (q-1)\,r, \varepsilon_2) \leqslant c_2, \qquad \forall q > 0. \tag{3.2}$$

Assumptions (A1), (A2), and (A3) of Corollary 3.1 are now satisfied and the proof is therefore complete.

If the equicontinuity of the Fréchet derivatives is replaced by uniform differentiability [15, p. 349], the fixed point is known to be attractive when assumption (A3) of Corollary 3.2 is satisfied with $r = 1$ [15, p. 354, Ex. 11.1–11.5]. We have therefore extended a known result by introducing equicontinuity which is slightly stronger than uniform differentiability. Indeed, the former clearly implies the latter. On the other hand, our assumption (A3) is more general than that of [15, p. 354, Ex. 11.1–11.5].

Equicontinuity is a weak condition which is easy to verify. As with uniform differentiability, we see that it is then sufficient to have information on the Jacobian matrices at the fixed point alone in order to have strong results (as opposed to conditions uniformly on a neighborhood of the fixed point, as in Theorem 2.1).

Also, if the Fréchet derivatives are equicontinuous, the majorization of the quantities $P(k, p, \varepsilon)$ of the general Theorem 2.1 becomes somewhat simpler. To see this we recall the definition of equicontinuity and note that for any $\varepsilon_1 > 0$ there is $\varepsilon_2 > 0$ such that

$$|X| \leqslant \varepsilon_2 \Rightarrow \|\mathbf{G}'_{k-i-1}(X)\| < \|\mathbf{G}'_{k-i-1}(0)\| + \varepsilon_1, \qquad \forall k-i > 0. \quad (3.3)$$

And therefore

$$P(k, p, \varepsilon_2) \leqslant \prod_{i=0}^{k-1-p} (\|\mathbf{G}'_{k-i-1}(0)\| + \varepsilon_1), \qquad \forall p > 0, \quad \forall k > p. \quad (3.4)$$

We may then be able to majorize the quantities $P(k, p, \varepsilon)$ of Theorem 2.1 with the knowledge of the Jacobian matrices at the origin alone.

We may wonder how critical the equicontinuity condition is in the above corollaries. To see this we consider an example taken from [14, p. 278]. By explicit calculation it can be shown that the iteration

$$X_{p+1} = \left(\frac{p+1}{2}\right) X_p^2 = G_p(X_p) \quad (3.5)$$

in $\mathbb{R}$ is attractive but not uniformly attractive. Therefore we know that at least one assumption of Corollary 3.1 must be violated. The derivative of each function G_p is

$$G'_p(X) = (p+1)\, X \quad (3.6)$$

and therefore every assumption of Corollary 3.2 is satisfied except the equicontinuity of the derivatives at the origin. This example shows that

(i) the equicontinuity condition is simple to verify and is violated only in fairly contrived situations (in our example when the parameter defining the functions becomes arbitrarily large);

(ii) equicontinuity is essential in the above corollaries.

We will now apply the above results to the cases where the only information we have on the Jacobian matrices at the origin relates to their spectral radius.

4. On the Spectral Radii of the Jacobian Matrices and Uniform Exponential Stability

As we indicated earlier, when all the operators are equal to some constant G, the origin will be attractive if the spectral radius $\rho(\mathbf{G}'(0))$ is less than 1. We also know [15, p. 354] that this result cannot be extended to the nonautonomous case. Indeed if

$$\rho(\mathbf{G}'_k(0)) \leqslant c < 1, \qquad \forall k > 0 \tag{4.1}$$

then the origin is not necessarily attractive.

In this section we give conditions under which Corollary 3.2 may be applied when the spectral radii are uniformly bounded below 1, as in (4.1) (we note that the assumptions of Corollary 3.2 are trivially satisfied in the autonomous case when the spectral radius is less than 1).

In a first corollary we prove that there will be a reordering of the operators under which the origin is uniformly exponentially stable.

COROLLARY 4.1. *We assume that the operators all have the origin as a fixed point. We make the following four assumptions*:

(A1) *Each G_k is continuously differentiable in a ball $S(0, \delta_1)$.*

(A2) *The Jacobian matrices are uniformly bounded at the origin.*

(A3) *The spectral radii $\rho(\mathbf{G}'_k(0))$ are uniformly bounded below* 1, *as in* (4.1).

(A4) *The Fréchet derivatives of the operators are equicontinuous at the origin.*

Under these conditions there is a bijection $w(i)$ of $\mathbb{N}$ onto $\mathbb{N}$ such that the origin is uniformly exponentially stable for the reordered nonautonomous difference equation

$$X_{k+1} = G_{w(k)}(X_k), \qquad k = 0, 1, \dots. \tag{4.2}$$

Proof. The proof hinges on a compactness argument based on the fact that $\mathbf{G}'_k(0)$ is uniformly bounded (assumption (A2)). In [2] we showed that under assumptions (A2) and (A3) there was a bijection $w(i)$ of $\mathbb{N}$ onto $\mathbb{N}$ such that there exist an integer r and a positive number $c_1 < 1$, with

$$P(w(qr), w((q-1)r), 0) \leqslant c_1, \qquad \forall q > 0. \tag{4.3}$$

Now, if we consider the reordered iteration $G_{w(i)}$ $(i = 0, 1, \dots)$, the result is then established by application of Corollary 3.2.

As we noted earlier, in general (4.1) will not suffice to guarantee that the origin is (uniformly) attractive or (uniformly) exponentially stable. In this context we observe that the critical assumption of Corollary 3.2 is (A3): in general products of lumped Jacobian matrices at the origin will not be majorized by some number c_1 smaller than 1.

Now, we will investigate conditions (in addition to (4.1)) under which assumption (A3) of Corollary 3.2 is satisfied. First, we note a well-known result based on the Jordan modified form of a matrix [15, p. 44].

PROPOSITION 4.1. *For any matrix $A \in M_n(\mathbb{C})$, and any number ε, there is an invertible matrix $Q(A, \varepsilon)$ such that*

$$\|\mathbf{A}\|_{Q(A,\varepsilon)} = \|\mathbf{Q}(A,\varepsilon)\,\mathbf{A}\mathbf{Q}(A,\varepsilon)^{-1}\| \leqslant \rho(\mathbf{A}) + \varepsilon, \tag{4.4}$$

where $\|\mathbf{A}\|_{Q(A,\varepsilon)}$ is the norm of $\mathbf{A}$ induced by $Q(\mathbf{A}, \varepsilon)$.

We also define $T(\mathbf{A}, \varepsilon) \subset M_n(\mathbb{C})$ as the set of matrices $Q(\mathbf{A}, \varepsilon)$ for which (4.4) holds. We then have the following result.

COROLLARY 4.2. *We assume that a sequence of operators G_k has the origin as a common fixed point and satisfies* (A1) *and* (A2) *of Corollary* 3.2. *In addition we make the following two assumptions*:

(A3) *The spectral radii of the Jacobian matrices at the origin are uniformly bounded by a positive number c_1 smaller than* 1 (*assumption of* (4.1)); *the Fréchet derivatives are equicontinuous at the origin.*

(A4) *With $a_k = (1 + c_1)/2 - \rho(\mathbf{G}_k'(0))$ for every k, the intersection*

$$I = \bigcap_{i=0}^{\infty} T(\mathbf{G}_i'(0), a_i)$$

is a nonempty set of $M_n(\mathbb{C})$.

Then (A3) *and* (A4) *guarantee that* (A3) *of Corollary* 3.2 *is satisfied and the origin is uniformly exponentially stable.*

Proof. Under assumptions (A3) and (A4) there exists a matrix $Q \in I$ that induces a norm $\| \ \|_Q$ for which the norm of every Jacobian matrix at the origin is bounded by $(1 + c_1)/2$. Then assumption (A3) of Corollary 3.2 is trivially satisfied if $r = 1$ and if the norm used to define $P(k, p, \varepsilon)$ is $\| \ \|_Q$. This completes the proof.

In this corollary we are essentially saying that there is a norm for which (2.24) holds with $\varepsilon = 0$. Equicontinuity then guarantees that (2.24) actually holds for some $\varepsilon > 0$. We know that the origin is then uniformly exponentially stable.

Finally, in the context of (4.1) we give a result which may be of practical interest when the Jacobian matrices $\mathbf{G}_k'(0)$ change sufficiently slowly for $k = 0, 1, \ldots$.

COROLLARY 4.3. *We assume that the sequence of operators has the origin as a common fixed point and satisfies* (A1), (A2) *of Corollary* 3.2, *and*

(A3) *of Corollary* 4.2 (*i.e., the spectral radii of the Jacobian matrices at the origin are uniformly bounded by some number* $c_1 < 1$; *the Fréchet derivatives are equicontinuous at the origin*). *Then there exists a number* ε_1 *such that* $\forall j \geqslant 1$, $\|\mathbf{G}'_j(0) - \mathbf{G}'_{j-1}(0)\| \leqslant \varepsilon_1 \Rightarrow$ *assumption* (A3) *of Corollary* 3.2 *is satisfied and the origin is uniformly exponentially stable.*

Proof. First, we recall that in $M_n(\mathbb{C})$ the Jacobian matrices belong to a compact ball $B(\mathbf{O}, K_1)$ of center $\mathbf{O}$ and radius K_1. Then, for any matrix $\mathbf{A} \in B(\mathbf{O}, K_1)$ and any number $\varepsilon > 0$ we know [17, p. 143] there are two numbers $K(\mathbf{A}) > 0$ and $s(\mathbf{A}) > 0$ such that

$$\mathbf{X}_i \in B(\mathbf{A}, s(\mathbf{A})); \quad i = 1, 2, ... \Rightarrow \forall p > 0,$$

$$\left\| \prod_{i=0}^{p-1} \mathbf{X}_{p-i} \right\| \leqslant K(\mathbf{A})(\rho(\mathbf{A}) + \varepsilon)^p. \tag{4.5}$$

Now, we recall that for any k we have $\rho(\mathbf{G}'_k(0)) \leqslant c_1 < 1$. We then fix ε of (4.5) at the value $\varepsilon = (1 - c_1)/2$. For ε so fixed, it can be seen, using a compactness argument, that $K(\mathbf{A})$ may be uniformly majorized by some number K, and $s(\mathbf{A})$ may be replaced by a (smallest) number s. If we now express (4.5) where $\mathbf{A}$ is each matrix $\mathbf{G}'_k(0)$, we will have, for any k and any p,

$$\mathbf{X}_i \in B(\mathbf{G}'_k(0), s), \qquad i = 1, 2, ... \Rightarrow \left\| \prod_{i=0}^{p-1} \mathbf{X}_{p-i} \right\| \leqslant K((1 + c_1)/2)^p. \tag{4.6}$$

Now, we choose p equal to some sufficiently large integer r, so that

$$K(1 + c_1)/2)^r = c_2 < 1. \tag{4.7}$$

$\mathbf{G}'_0(0)$ being fixed, we define $\varepsilon_1 = s/r$ and assume that

$$\forall j \geqslant 1, \qquad \|\mathbf{G}'_j(0) - \mathbf{G}'_{j-1}(0)\| \leqslant \varepsilon_1. \tag{4.8}$$

By the triangle inequality, it is then easy to see that

$$\forall q > 0, \qquad \mathbf{G}'_{qr-i-1}(0) \in B(\mathbf{G}'_{r(q-1)}(0), \mathrm{s}), \qquad i = 0, 1, ..., r - 1. \tag{4.9}$$

Bearing in mind (4.6) and (4.7), we then have, for any $q > 0$,

$$P(qr, (q-1)\, r, 0) = \left\| \prod_{i=0}^{r-1} \mathbf{G}'_{qr-i-1}(0) \right\| \leqslant c_2 \tag{4.10}$$

which proves (A3) of Corollary 3.2 and completes the proof.

When applying Corollary 4.3, the problem is to determine the quantity ε_1 of (4.8). In general, there are no conditions on the operators G_k that will guarantee that the successive Jacobian matrices will be close to each other. However, in some cases, the operators G_k will be defined (for some integer p) by a vector $A(k)$ of $\mathbb{R}^p$, i.e., Eq. (1.2) is

$$X_{k+1} = G(X_k, A(k)), \tag{4.11}$$

where G is a constant operator parameterized by $A(k)$. We assume that the vectors $A(k)$ belong to some set D of $\mathbb{R}^p$. Then the Jacobian matrix $\mathbf{G}'_k(0)$ may be considered a function $\mathbf{G}'(A(k))$ of the vector $A(k)$. Given ε_1 of (4.8), if $G'(A)$ is uniformly continuous on D, then there is δ_1 such that

$$A_1 \in D, A_2 \in D; \qquad |A_1 - A_2| \leqslant \delta_1 \Rightarrow \|\mathbf{G}'(A_1) - \mathbf{G}'(A_2)\| \leqslant \varepsilon_1 \tag{4.12}$$

and therefore, if

$$|A(j) - A(j-1)| \leqslant \delta_1, \qquad \forall j \geqslant 1, \tag{4.13}$$

then

$$\|\mathbf{G}'_j(0) - \mathbf{G}'_{j-1}(0)\| = \|\mathbf{G}'(A(j)) - \mathbf{G}'(A(j-1))\| \leqslant \varepsilon_1, \tag{4.14}$$

which shows that (4.8) will be satisfied. If D is compact, then continuity of $\mathbf{G}'(A)$ on D suffices to yield the desired result.

In sum, we have shown that if the operators G_k were defined by a sequence of vectors $A(k)$ such that $A(k)$ is close to $A(k-1)$ for every k, then, under the assumptions of Corollary 4.3, the origin will be uniformly exponentially stable.

As we already noted, the results of this section apply when the spectral radii of the Jacobian matrices at the origin are uniformly bounded below 1. These results are particularly relevant when the norms of these Jacobian matrices are larger than 1. Indeed in this case our earlier results may not be applicable. One important example of this occurrence is provided by the difference scheme (in $\mathbb{R}$) of the form

$$x_{k+1} = H_k(x_{k-n+1}, x_{k-n+2}, ..., x_k), \tag{4.15}$$

where each H_k is a real function of n variables.

Indeed, in this case, it is well known [8, p. 4] that the process of Eq.(4.15) may be reformulated in terms of the following nonautonomous difference equations in $\mathbb{R}^n$,

$$X = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \longmapsto G_k(X) = \begin{matrix} x_2 \\ x_3 \\ \vdots \\ H_k(x_1, x_2, ..., x_n). \end{matrix} \tag{4.16}$$

It is then easy to see that for every k, $\mathbf{G}'_k(0)$ is

$$\mathbf{G}'_k(0)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & & & \vdots \\ & & & & 1 \\ d_1(k) & d_2(k) & \cdots & & d_n(k) \end{bmatrix}, \tag{4.17}$$

where

$$d_i(k)=\frac{\partial H_k(0)}{\partial x_i}. \tag{4.18}$$

In other words $\mathbf{G}'_k(0)$ is a companion matrix having in its last row the partial derivatives of H_k at the origin.

We assume that for all k, 0 is a point of equilibrium of H_k (i.e., $H_k(0, 0,..., 0)=0$). This implies that the origin in $\mathbb{R}^n$ is a fixed point of G_k defined in (4.16). Clearly x_{k+1} in (4.15) is given by the last component of the vector $G_k(X)$ in (4.16).

We note that for the ordinary matrix norms (e.g., the Euclidean norm or the operator norm) we have

$$\|\mathbf{G}'_k(0)\| \geqslant 1, \qquad \forall k. \tag{4.19}$$

Therefore the results requiring a majorization by 1 are not applicable. However, if $\rho(\mathbf{G}'_k(0)) \leqslant c_1 < 1$ for all k, then the results of this section may apply (in particular when the operators G_k are equal to some constant G, then as we know, $\rho(\mathbf{G}'(0)) < 1$ guarantees that the origin is uniformly exponentially stable).

We note that the characteristic equation of the companion matrix $\mathbf{G}'_k(0)$ is

$$x^n - \sum_{j=1}^{n} d_{n+1-j}(k)\, x^{n-j} = 0. \tag{4.20}$$

There are simple circumstances under which the moduli of the solutions are bounded by 1. Indeed, if the partial derivatives are negative, then the moduli of the solutions are known [7] to be majorized by

$$\underset{j=1,2,\dots,n}{\mathrm{Max}} \frac{d_j(k)}{d_{j+1}(k)}, \tag{4.21}$$

where $d_{n+1}(k)$ is equal to 1 for every k.

Therefore if

$$-1 < d_n(k) < d_{n-1}(k) < \cdots < d_1(k) < 0 \tag{4.22}$$

then the spectral radius $\rho(\mathbf{G}'_k(0))$ will be less than 1. If this result holds uniformly as in (4.1), then the corollaries of this section become applicable. In particular, under the circumstances of Corollary 4.3, if the Jacobian matrices at the origin change sufficiently slowly, then the origin is uniformly exponentially stable. The quantity x_{k+1} of (4.15) then converges geometrically fast to 0.

5. Conclusions

In this paper we have extended classical stability conditions of the form $\|\mathbf{G}'_k(X)\| < 1$ in a neighborhood of a common fixed point X^*.

First, we generalized such results by introducing conditions of the form $P(k, p, \varepsilon) < K$, i.e., majorizations of products of arbitrarily large numbers of Jacobian matrices in a neighborhood of X^*. Second, by considering the equicontinuity of the Fréchet derivatives we have seen that majorizations of the form $P(k, p, 0) < K$ were sufficient to yield stable behavior of the iterative process.

Using these results, we discussed the critical case of Jacobian matrices bounded by 1 but coming arbitrarily close to 1. Depending on the nature of this proximity we saw that X^* was either stable (and not necessarily attractive) or stable and attractive.

Finally, we have investigated in some detail the case when the spectral radii of the Jacobian matrices were uniformly bounded below 1. In general, the fixed point is not stable in such a case. However, by adding certain conditions on the Fréchet derivatives we obtained uniform exponential stability. Such results are particularly relevant when the Jacobian matrices at the origin are of norm larger than 1. We illustrated this situation in the context of the iterative process formulation of (nonautonomous) difference schemes in $\mathbb{R}$. The results presented here then become applicable and may yield conditions under which such difference schemes converge.

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