JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **89**, 489–514 (1982)

On the Commutant Algebras Corresponding to the Permutation Representations of the Full Collineation Groups of $PG(k, s)$ and $EG(k, s)$, $s = p^r$, $k \geqslant 2$*

CATHERINE T. BURTON AND I. M. CHAKRAVARTI

Department of Statistics, University of North Carolina at Chapel Hill, Chapel Hill, North Carolina 27514

Submitted by G.-C. Rota

In this paper, the dimension t and a linear basis of the commutant algebra corresponding to the representation of the full collineation group as matrices permuting the flags (incident point-line or point-hyperplane pairs) have been determined for each one of the four geometries $PG(2, s)$, $EG(2, s)$, $PG(k, s)$, and $EG(k, s)$, $s = p^r$, $k \geqslant 3$. For the four geometries, $t = 6, 7, 7$, and 8, respectively, and the corresponding linear bases are (I, G, B, T, BT, TB), $(I, G, B, T, BT, TB, BTB)$, (I, G, B, T, BT, TB, S), and $(I, G, B, T, BT, TB, BTB, S)$. I, G, B, T are the relationship matrices of James (*Ann. Math. Statist.* **28** (1957), 993–1082) and the matrix S was introduced by Sysoev and Shaikin (*Avtomat. i Telemekh.* **5** (1976), 64–73).

1. INTRODUCTION

The analysis of variance of the observations $\mathbf{Y}$ from a block design, say, a balanced incomplete block (BIB) design, based on the commonly used linear model $(E(\mathbf{Y}) = \mu\mathbf{j} + A'\boldsymbol{\alpha} + L'\boldsymbol{\beta}$, $\operatorname{Var}(\mathbf{Y}) = \sigma^2 I$, is determined by the parameters (v, k, λ) of the BIB design alone and is insensitive (robust) to other structural properties of the design, such as its group of symmetries. Thus the two BIB designs having the same parameters $v = 31$, $k = 15$, $\lambda = 7$, one, the incidence matrix of the point-hyperplane pairs of the geometry $PG(4, 2)$ and the other generated by the difference set of the quadratic residues (mod 31), are not distinguishable by this model although they have different groups of symmetries.

In order to incorporate the group of symmetries of the design in the formulation of the linear model, we have followed the leads of McLaren [11], Hannan [8], and Sysoev and Shaikin [13] and assumed that the covariance matrix V of the observations $\mathbf{Y}$ is invariant under the action of the group of

* This research was supported in part by the National Science Foundation under Grant MCS78-01434.

0022-247X/82/100489–26$02.00/0

permutation matrices (permuting the incident point-block pairs (flags)) representing the group of symmetries (automorphisms) $\mathscr{G}$ of the design. This implies that $V = \sum_{i=1}^{t} c_i V_i$, where t is the dimension and V_i, $i = 1,..., t$, is a linear basis of the commutant algebra corresponding to the permutation representation of $\mathscr{G}$.

The dimension and two alternative linear bases of the commutant algebra corresponding to the full collineation group $PC(2, 2)$ (represented as a group of matrices permuting the 21 flags (incident point-line pairs)) of the projective plane $PG(2, 2)$ were determined by Chakravarti and Burton [4] using a theorem due to Schur (Wielandt [14, p. 80, Theorem 28.4]) and a theorem due to Burnside [1, p. 191, Theorem VII].

Using results from the theory of group representation and characters, Burton [1] and Chakravarti and Burton [5] have worked out the decompositions into a direct sum of irreducible representations of the permutation representations (permuting the flags of the plane) of the full collineation groups $PC(2, 2)$ and $EC(2, 3)$ of the finite planes $PG(2, 2)$ and $EG(2, 3)$, respectively, and the simple components of the corresponding commutant algebras. Further, the irreducible representations in the decompositions of the permutation representations of the groups $PC(2, 2)$ and $EC(2, 3)$ have been shown to be equivalent to real irreducible representations and hence to sets of real orthogonal matrices (Chakravarti and Burton [4] and Burton [2]).

In this paper, we have determined the dimension t and a linear basis of the commutant algebra corresponding to the representation of the full collineation group as matrices permuting the flags (incident point-line or point-hyperplane pairs) for each one of the four geometries $PG(2, s)$, $EG(2, s)$, $PG(k, s)$, and $EG(k, s)$, $s = p^r$ and $k \geqslant 3$. We have shown that $t = 6, 7, 7$, and 8, respectively, for the four geometries and the four bases are (I, G, B, T, BT, TB), $(I, G, B, T, BT, TB, BTB)$, (I, G, B, T, BT, TB, S), and $(I, G, B, T, BT, TB, BTB, S)$, respectively. I, G, B, and T are the relationship matrices of James [9] and the matrix S was defined by Sysoev and Shaikin [13].

2. Finite Projective and Affine Geometries and Their Collineations

A k-dimensional finite projective geometry $PG(k, s)$, $s = p^r$, where p is a prime and r an integer, has a concrete representation over the $GF(s)$. A *point* of this geometry is an ordered $(k+1)$-tuple $\mathbf{x} = (x_0, x_1,..., x_k)'$, where $x_0,..., x_k$ are elements of the field $GF(s)$, at least one of which is distinct from zero, and where it is understood that $(x_0,..., x_k)$ denotes the same point as $(\mu x_0,..., \mu x_k)$ for every nonzero μ in $GF(s)$. Similarly, a *hyperplane* ($(k-1)$)-space) is also defined by an ordered $(k+1)$-tuple $\mathbf{y} = [y_0, y_1,..., y_k]'$, where

$y_0, \ldots, y_k$ are elements of the $GF(s)$ not all zero, and $[y_0, y_1, \ldots, y_k]$ denotes the same hyperplane as $[\mu y_0, \ldots, \mu y_k]$ for every nonzero μ in $GF(s)$. A point $(x_0, \ldots, x_k)$ is incident with a hyperplane $[y_0, \ldots, y_k]$ if and only if $y_0 x_0 + \cdots + y_k x_k = 0$. $PG(2, s)$ will then denote the finite Desarguesian projective plane with $s + 1$ points on a line.

A k-dimensional finite affine geometry $EG(k, s)$ can be derived from $PG(k, s)$ by deleting a hyperplane and all the points incident with that hyperplane. A *point* in this geometry can be written as an ordered k-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_k)'$, where $x_1, \ldots, x_k$ are elements of $GF(s)$. A *hyperplane* of the geometry is an ordered $(k + 1)$-tuple $\mathbf{y} = [y_0, \ldots, y_k]'$, where $y_0, \ldots, y_k$ are elements in $GF(s)$ such that not all of $y_1, \ldots, y_k$ are zero. $[y_0, \ldots, y_k]$ and $[\mu y_0, \ldots, \mu y_k]$ denote the same hyperplane for all nonzero μ in $GF(s)$. A point $(x_1, \ldots, x_k)$ is incident with a hyperplane $[y_0, y_1, \ldots, y_k]$ if and only if $y_0 + y_1 x_1 + \cdots + y_k x_k = 0$. So $EG(2, s)$ will then denote the finite Desarguesian affine plane with s points on a line.

A *collineation* (automorphism) of a projective or affine geometry is a permutation of its points which maps lines into lines and hence, every subspace is then mapped into a subspace. A balanced incomplete block design (BIBD) constructed from a finite geometry by taking the points of the geometry as the points of the design and the subspaces of a given dimension of the geometry as *blocks* of the design will then have as its full group of automorphisms the full collineation group of the geometry.

The parameters of the BIB design constructed from the point-hyperplane incidences of $PG(k, s)$ are

$$v = b = (s^{k+1} - 1)/(s - 1), \qquad r = k^* = (s^k - 1)/(s - 1),$$
$$\lambda = (s^{k-1} - 1)/(s - 1).$$

The parameters of the BIB design derived from the point-hyperplane incidences of $EG(k, s)$ are

$$v = s^k, \qquad b = (s^{k+1} - s)/(s - 1), \qquad r = (s^k - 1)/(s - 1),$$
$$k^* = s^{k-1}, \qquad \lambda = (s^{k-1} - 1)/(s - 1).$$

The full collineation group $PC(k, s)$ of $PG(k, s)$ is represented analytically by the homogeneous transformations of the points (see, for instance, Carmichael [3, p. 362]):

$$\rho x_i^* = \sum_{j=0}^{k} a_{ij} x_j^{p^u} \qquad (i = 0, 1, \ldots, k;\ u = 0, 1, \ldots, r - 1), \tag{2.1}$$

where $\rho (\neq 0)$ and the a_{ij} are elements of $GF(s)$ such that the matrix $A = (a_{ij})$, $i, j = 0, 1, \ldots, k$, is nonsingular. (For a given value of u, two transformations are identical if the coefficients a_{ij} are identical despite distinct values of ρ.) The order of the group $PC(k, s)$ is

$$\frac{r}{(s-1)} \prod_{i=0}^{k} (s^{k+1} - s^i),$$

and when it is considered as a permutation group on the points, it is doubly transitive.

The following two lemmas are easy to prove and are stated without proof:

LEMMA 2.1. *For a finite projective geometry $PG(k, s)$, $s = p^r$, if the point vectors $\mathbf{x} = (x_0, x_1, \ldots, x_k)'$ are transformed by the collineation $A(x_0^m, x_1^m, \ldots, x_k^m)'$, $m = p^u$ for some $u = 0, 1, \ldots, r-1$, then the hyperplane vectors are transformed by the map*

$$(A')^{-1}(y_0^m, \ldots, y_k^m)'. \tag{2.2}$$

LEMMA 2.2. *The collineations in $PC(k, s)$ which fix the point* $(1, 0, \ldots, 0)$ *and the hyperplane* $[0, 0, \ldots, 0, 1]$ *form a subgroup $\mathscr{H}$ (stabilizer) of $PC(k, s)$ and these are of the form*

$$A\mathbf{x}^{(m)} = \begin{bmatrix} 1 & \mathbf{a}_1' & a_{0k} \\ \mathbf{0} & A_1 & \mathbf{a}_2 \\ \mathbf{0} & \mathbf{0}' & a_{kk} \end{bmatrix} \begin{pmatrix} x_0^m \\ \vdots \\ x_k^m \end{pmatrix}, \tag{2.3}$$

$m = p^u$, $u = 0, 1, \ldots, r-1$; $\mathbf{a}_1' = (a_{01}, \ldots, a_{0,k-1})$; $\mathbf{a}'_2 = (a_{1k}, \ldots, a_{k-1,k})$; $A_1 = (a_{ij})$, $i, j = 1, \ldots, k-1$; $|A_1| \neq 0$; $a_{kk} \neq 0$; a_{ii} in $GF(s)$. The order of $\mathscr{H}$ is $|\mathscr{H}| = rs^{2k-1}(s-1) \prod_{i=0}^{k-2} (s^{k-1} - s^i)$.

The full collineation group $EC(k, s)$ of the geometry $EG(k, s)$ is represented by the nonhomogeneous transformations

$$x_i^* = a_{i0} + \sum_{j=1}^{k} a_{ij} x^{p^u}; \qquad i = 1, \ldots, k; \quad u = 0, 1, \ldots, r-1; \tag{2.4}$$

a_{ij} are elements of $GF(s)$; and the matrix $A_1 = (a_{ij})$, $i, j = 1, \ldots, k$ is nonsingular. The order of $EC(k, s)$ is

$$rs^k \sum_{i=0}^{k-1} (s^k - s^i).$$

We also state the following two lemmas without proof, since they are easy to prove:

LEMMA 2.3. *For a finite affine geometry $EG(k, s)$, if the point vectors $\mathbf{x} = (x_1, x_2, \ldots, x_k)'$ are transformed by the collineation*

$$\mathbf{a} + A\mathbf{x}^{(m)} = \begin{pmatrix} a_{10} \\ \vdots \\ a_{k0} \end{pmatrix} + A \begin{pmatrix} x_1^m \\ \vdots \\ x_k^m \end{pmatrix}; \tag{2.5}$$

$m = p^u$, $u = 0, 1, \ldots, r-1$; $|A| \neq 0$; a_{ij} in $GF(s)$, then the hyperplane vectors $[y_0, y_1, \ldots, y_k]$ are transformed as follows:

$$y_0^m - (y_1^m, \ldots, y_k^m) A_1^{-1} \mathbf{a} = \rho y_0^*,$$

$$(A')^{-1} \begin{pmatrix} y_1^m \\ \vdots \\ y_k^m \end{pmatrix} = \rho \begin{pmatrix} y_1^* \\ \vdots \\ y_k^* \end{pmatrix}, \qquad \rho \neq 0 \quad \text{in} \quad GF(s). \tag{2.6}$$

LEMMA 2.4. *The collineations in $EC(k, s)$ which fix the point* $(0, \ldots, 0)$ *and the hyperplane* $[0, \ldots, 0, 1]$ *form a subgroup $\mathscr{H}$ of $EC(k, s)$ and are of the form*

$$A \begin{pmatrix} x_1^m \\ \vdots \\ x_k^m \end{pmatrix} = \begin{bmatrix} A_1 & \mathbf{a}_2 \\ \mathbf{0}' & a_{kk} \end{bmatrix} \begin{pmatrix} x_1^m \\ \vdots \\ x_k^m \end{pmatrix}; \tag{2.7}$$

$m = p^u$, $u = 0, 1, \ldots, r-1$; $\mathbf{a}_2' = (a_{1k}, \ldots, a_{k-1,k})$; $A_1 = (a_{ij})$, $i, j = 1, \ldots, k-1$; $|A_1| \neq 0$; $a_{kk} \neq 0$; a_{ij}'s *are elements of* $GF(s)$.

We also note that if the flags of a design are taken to be incident point-hyperplane pairs, the groups $PC(k, s)$ and $EC(k, s)$ are transitive on the flags in their respective sets of flags. This is seen to be true from a result in Dembowski [6, p. 80]: "Suppose that $(r, \lambda) = 1$. If Γ is doubly point transitive, then Γ is flag transitive." Γ here is the automorphism group of a design; (r, λ) indicates the greatest common divisor of r and λ. For both the BIB designs which are defined by the incident point-hyperplane pairs in $PG(k, s)$ and $EG(k, s)$,

$$r = (s^k - 1)/(s - 1) = s^{k-1} + \cdots + s + 1$$

$$\lambda = (s^{k-1} - 1)/(s - 1) = s^{k-2} + \cdots + s + 1.$$

Clearly, $(r, \lambda) = 1$, and since $PC(k, s)$ and $EC(k, s)$ are doubly point transitive, they are flag transitive. This flag transitivity also follows from the fact that the little projective group is transitive on *nests* (Dembowski [6, p. 37]).

For a block design, let $A = (a_{iu})$, $a_{iu} = 1$ if the uth flag (incident point-block pair, an experimental unit for a statistician) contains the ith point,

$a_{iu} = 0$ otherwise, and $L = (l_{ju})$, $l_{ju} = 1$ if the uth flag includes the jth block, $l_{ju} = 0$, otherwise. James [9] defined the relationship matrices B, T, I, G for a block design with n experimental units (flags) as $B = (b_{uu'})$, $b_{uu'} = 1$ if the flags u and u' share the same block, $b_{uu'} = 0$ otherwise; $T = (t_{uu'})$, $t_{uu'} = 1$ if the flags u and u' share the same point, $t_{uu'} = 0$, otherwise; I is the $n \times n$ identity matrix; and $G = (g_{uu'})$, $g_{uu'} = 1$ for all pairs (u, u'), $u, u' = 1, \ldots, n$. Note that $B = L'L$ and $T = A'A$. If the BIB design is asymmetric $(b > v)$, the symmetric nonnegative definite matrices I, G, B, and T generate a seven-dimensional noncommutative semi-simple algebra $\mathscr{R}_7$, of which I, G, B, T, BT, TB, and.BTB form a linear basis (James [9]). For a symmetric BIBD $(v = b)$, $BTB = \lambda G + (k - \lambda)B$ and (I, G, B, T, BT, TB) is a linear basis of the six-dimensional noncommutative semi-simple algebra $\mathscr{R}_6$ generated by I, G, B, and T (Mann [10]).

James [9, pp. 1001–1002] mentioned: "For certain designs, the relationship algebra is the *commutator* algebra (called *commutant* algebra or a centralizer ring these days) of the representtion of the experimental design. Such will be the subject of a further paper." (James told one of the authors in October, 1979, that he had obtained earlier some results in this area but never published them.)

For BIB designs, Sysoev and Shaikin [13] defined the matrix $S = (s_{uu'})$, where $s_{uu'} = 1$, if $(BT)_{uu'} = (TB)_{uu'} = 1$, $s_{uu'} = 0$, otherwise. For $\lambda = 1$, $S = B + T - I$.

The matrices I, G, B, T are invariant under the action of the permutation matrices $P(g)$ permuting the flags—incident point-block pairs—of the design, for every element g in the automorphism group $\mathscr{G}$ of the design. It follows that BT, TB, BTB, and S are similarly invariant and hence belong to the commutant algebra corresponding to the permutation representation of the automorphism group $\mathscr{G}$ of the design.

Sysoev and Shaikin [13[have shown that for the symmetric BIB designs defined by the point-hyperplane incidences of $PG(k, s)$, $k \geqslant 3$, $(I, G, B, T, BT, TB$, and $S)$ is a linear basis of a seven-dimensional noncommutative semi-simple algebra $\mathscr{P}_7$. Sysoev [12] has shown that for the asymmetric BIB designs defined by the point-hyperplane incidences of $EG(k, s)$, $k \geqslant 3$, $(I, G, B, T, BT, TB, BTB$ and $S)$ is a linear basis of an eight-dimensional noncommutative semi-simple algebra $\mathscr{P}_8$.

3. Dimensions and Linear Bases of Commutant Algebras Corresponding to the Permutation Representations of the Full Collineation Groups $PC(k, s)$ and $EC(k, s)$, $s = p^r$, $k \geqslant 2$

3.0 *Theorems Due to Schur and Burnside*

Two theorems, one due to Schur and the other to Burnside, are particularly useful in the determination of the dimensions and linear bases of the commutant algebras considered here.

We quote from Wielandt [14, p. 80] the following theorem due to Schur (1933):

> "Theorem 28.4. *If a transitive permutation group $\mathscr{G}$ is regarded as a matrix group $\mathscr{G}^*$, then the matrices of $\mathscr{G}^*$ form a ring $V = V(\mathscr{G})$. We call V the centralizer ring corresponding to $\mathscr{G}$. V is a vector space over the complex number field, which has the matrices $B(\Delta)$ corresponding to the orbits Δ of $\mathscr{G}_1$ (stabilizer—subgroup of $\mathscr{G}$ fixing the element* 1) *as a linear basis. In particular, the dimension of V coincides with the number k of orbits of $\mathscr{G}_1$.*"

The matrix $B(\Delta) = (v^{\Delta}_{\alpha,\beta})$, $\alpha, \beta = 1,..., n$, corresponding to the orbit Δ, is defined as

$$
\begin{aligned}
v^{\Delta}_{\alpha,\beta} &= 1, \qquad \text{if there exists } g \in \mathscr{G}, \delta \in \Delta \text{ with } 1^g = \beta, \delta^g = \alpha, \\
&= 0, \qquad \text{otherwise.}
\end{aligned}
$$

Here, $\Omega = \{1, 2,..., n\}$ is the set of n elements on which every permutation g of $\mathscr{G}$ acts.

The theorem due to Burnside [1, p. 191] states:

> "Theorem VII. *The sum of the numbers of symbols left unchanged by each of the permutations of a permutation group of order N is tN, where t is the number of transitive sets (orbits) in which the group permutes the symbols. The sum of the squares of the numbers of symbols left unchanged by each of the permutations of a transitive group of order N is sN, where s is the number of transitive sets in which a subgroup leaving one symbol unchanged (stabilizer) permutes the symbols.*"

3.1. *The Commutant Algebra Corresponding to the Permutation Representation of $PC(2, s)$, $s = p^r$*

The point-line incidences of the $PG(2, s)$ define the symmetric BIB designs with parameters $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$. There are $(s^2 + s + 1)(s + 1)$ flags (incident point-line pairs) in such a plane (or design). The full collineation group $PC(2, s)$ of the $PG(2, s)$, $s = p^r$, is defined by the $rs^3(s^2+s+1)(s+1)(s-1)^2$ point transformations

$$A\mathbf{x}^{(m)} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ x_2^m \end{pmatrix} = \alpha \begin{pmatrix} x_1^* \\ x_1^* \\ x_2^* \end{pmatrix}, \tag{3.1.1}$$

$\alpha \neq 0$ in $GF(s)$, $m = p^u$, $u = 0, 1, ..., r-1$, and A is nonsingular with coefficients in $GF(s)$. The collineations which fix the point (100) and the line [001] ($x_2 = 0$), forming the stabilizer $\mathscr{H}$, are from (2.3)

$$A\mathbf{x}^{(m)} = \begin{bmatrix} 1 & a_{01} & a_{02} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{bmatrix} \begin{pmatrix} x_0^m \\ x_1^m \\ x_2^m \end{pmatrix} = \alpha \begin{pmatrix} x_0^* \\ x_1^* \\ x_2^* \end{pmatrix}, \tag{3.1.2}$$

α, a_{11}, and a_{22} are nonzero elements of $GF(s)$. For any such point transformation, the lines are transformed as follows (Lemma 2.1):

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{01} & a_{11} & 0 \\ a_{02} & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} y_0^* \\ y_1^* \\ y_2^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ y_1^m \\ y_2^m \end{pmatrix}, \qquad \gamma (\neq 0) \text{ in } GF(s). \tag{3.1.3}$$

The stabilizer $\mathscr{H}$ has order $r(s-1)^2 s^3$. In order to determine the dimension of the commutant algebra corresponding to the full collineation group $PC(2, s)$, using Schur's theorem we need to find the number of orbits of $\mathscr{H}$, which by Burnside's theorem is equal to $(1/|\mathscr{H}|) \sum_{g \in PC(2,s)} \psi(g)$, where $\psi(g)$ is the number of fixed flags under the action of g in $PC(2, s)$ and $|\mathscr{H}|$

TABLE 3.1.1

Conditions on the Matrices of the Stabilizer which Fix Lines

Line type	Number of lines of this type	Conditions under which a line of this type is fixed (3.1.3)
[0 0 1]	1	Fixed for all A in $\mathscr{H}$
[0 1 a] $a \in GF(s)$	s	$a_{12} + a_{22}a = a_{11}a^m$
[1 a b] a, b in $GF(s)$	s^2	$a_{01} + a_{11}a = a^m$ $a_{02} + a_{12}a + a_{22}b = b^m$
	$s^2 + s + 1$ lines	

TABLE 3.1.2

Conditions on Matrices of the Stabilizer which Fix Points

Point type	Number of points of this type	Conditions under which a point of this type is fixed (3.1.2)
$(1\ 0\ 0)$	1	Fixed for all A in $\mathscr{H}$
$(c\ 1\ 0)$ c in $GF(s)$	s	$c^m + a_{01} = a_{11}c$
$(c\ d\ 1)$ c, d in $GF(s)$	s^2	$c^m + a_{01}d^m + a_{02} = a_{22}c$ $a_{11}d^m + a_{12} = a_{22}d$
	$s^2 + s + 1$ points	

denotes the order of $\mathscr{H}$. To find this number, we consider the conditions under which any given flag is fixed and then count the number of transformation in $\mathscr{H}$ satisfying these conditions. Types of lines and types of points are first considered.

For any fixed $m = p^u$, $u = 0, 1, \ldots, r-1$, we display the line types in Table 3.1.1. The third column gives the conditions on a matrix in the stabilizer, such that a line of the given type is fixed when transformed by such a matrix.

Likewise, for a given value of m, we show the point types in Table 3.1.2.

For each point type, we determine the lines with which it is incident. Then by examining the conditions under which a given flag (incident point-line

TABLE 3.1.3

Number of Matrices in the Stabilizer which Fix a Flag of Given Type

Point type	Number of points of this type	Type of lines incident with a point of given type	Number of lines of this type	Number of matrices which fix a flag of given type
$(1\,0\,0)$	1	(a) $[0\,0\,1]$	1	$(s-1)^2s^3$
		(b) $[0\,1\,a]$, a in $GF(s)$	s	$(s-1)^2s^2$
$(c\,1\,0)$, c in $GF(s)$	s	(c) $[0\,0\,1]$	1	$(s-1)^2s^2$
		(d) $[1\,a\,b]$, $a = -c$, b in $GF(s)$	s	$(s-1)^2s$
$(c\,d\,1)$, c, d in $GF(s)$	s^2	(e) $[0\,1\,a]$, $a = -d$	1	$(s-1)^2s$
		(f) $[1\,\mathrm{a}\,b]$, $b = -c - ad$, a in $GF(s)$	s	$(s-1)^2$

pair) is fixed, we determine the number of transformation matrices in the stabilizer $\mathscr{H}$ satisfying these conditions (see Table 3.1.3).

(a) The flag defined by the point (1 0 0) and the line $|0\ 0\ 1|$ is fixed for all matrices in the stabilizer. There are $(s-1)^2 s^3$ matrices fixing this flag for a fixed $m = p^u$, $u = 0, 1, \ldots, r-1$.

(b) The flag defined by the point (1 0 0) and the line $|0\ 1\ a|$, $a \in GF(s)$, $a_{12} + a_{22} a = a_{11} a^m$, is fixed by $(s-1)^2 s^2$ matrices ($a_{22} \neq 0$; $a_{11} \neq 0$; a_{12} is determined once a_{11}, a_{22}, and a are given; a_{01}, a_{02} are free). There are s such flags for s different a's.

(c) The flag defined by the point (c 1 0) and the line $|1\ 0\ 0|$, $c^m + a_{01} = a_{11} c$, $c \in GF(s)$, is fixed by $(s-1)^2 s^2$ collineations. There are s such flags.

(d) The flag defined by the point (c 1 0) and the line $|1, -c, b|$, c, b in $GF(s)$, is fixed by all collineations satisfying

$$c^m + a_{01} = a_{11} c, \qquad a_{02} - a_{12} c + a_{22} b = b^m. \tag{3.1.4}$$

There are $(s-1)^2 s$ matrices fixing this flag and s^2 such flags.

(e) The flag defined by the point $(c, d, 1)$ and the line $[0, 1, -d]$, c, d in $GF(s)$, is fixed by all collineations satisfying

$$c^m + a_{01} d^m + a_{02} = a_{22} c, \qquad a_{11} d^m + a_{12} = a_{22} d. \tag{3.1.5}$$

There are $(s-1)^2 s$ matrices fixing this flag and there are s^2 such flags.

(f) The flag consisting of the point $(c, d, 1)$ and the line $[1, a, -c - ad]$, c, d, and a in $GF(s)$, is fixed by all collineations satisfying

$$\begin{gathered} c^m + a_{01} d^m + a_{02} = a_{22} c, \\ a_{11} d^m + a_{12} = a_{22} d, \qquad a_{01} + a_{11} a = a^m. \end{gathered} \tag{3.1.6}$$

($a_{02} + a_{12} a + a_{22}(-c - ad) = (-c - ad)^m$ is a linear combination of the first three equations.) There are $(s-1)^2$ matrices fixing this flag and there are s^3 such flags.

The total number of flags fixed for a given $m = p^u$ is $6(s-1)^2 s^3$. For $u = 0, 1, \ldots, r-1$, there will thus be $6r(s-1)^2 s^3$ flags fixed by the $r(s-1)^2 s^3$ collineations of the stabilizer. Thus, the number of orbits of the stabilizer $= 6 =$ the dimension of the commutant algebra corresponding to the permutation representation of the collineations in $PC(2, s)$ permuting the incident point-line pairs. I, G, B, T, BT, and TB form a linear basis of this commutant algebra of dimension six, since the commutant algebra coincides with $\mathscr{R}_6$ (defined in Section 2).

3.2 *The Commutant Algebra Corresponding to the Permutation Representation of* $EC(2, s)$, $s = p^r$

The point-line incidences of the $EG(2, s)$ define the asymmetric BIB design with parameters $v = s^2$, $b = s^2 + s$, $r = s + 1$, $k = s$, $\lambda = 1$, and $(s^2 + s)s$ flags. The full collineation group $EC(2, s)$ of the $EG(2, s)$ is defined by the $rs^2(s^2 - 1)(s^2 - s)$ point transformations

$$A\mathbf{x}^{(m)} + \mathbf{c} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}; \qquad (3.2.1)$$

$m = p^u$, $u = 0, 1,\ldots, r-1$; a_{ij}'s and c_i's in $GF(s)$; and $A = (a_{ij})$, $i, j = 1, 2$ is nonsingular.

The collineations

$$A\mathbf{x}^{(m)} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}, \qquad (3.2.2)$$

defining the subgroup $\mathscr{H}$ of $EC(2, s)$, fix the point (0 0) and the line $[0\ 0\ 1]$ $(x_2 = 0)$. Both a_{11} and a_{22} are nonzero elements of $GF(s)$. The line $[y_0, y_1, y_2]$ is transformed as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} y_0^* \\ y_1^* \\ y_2^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ y_1^m \\ y_2^m \end{pmatrix},$$

$\gamma(\neq 0)$ in $GF(s)$. The stabilizer $\mathscr{H}$ has $r(s-1)^2 s$ elements.

As in the previous section, different line types, point types and flag types

TABLE 3.2.1

Conditions on Matrices of the Stabilizer which Fix Lines

Line type	Number of Lines of this type	Conditions under which a line of this type is fixed
$[0\ 0\ 1]$	1	Fixed for all A in $\mathscr{H}$
$[0\ 1\ a]$ a in $GF(s)$	s	$a_{12} + a_{22}a = a_{11}a^m$
$[1\ 0\ a]$ $a \neq 0)$ in $GF(s)$	$s - 1$	$a_{22}a = a^m$
$[1\ \mathrm{a}\ b)$ $a(\neq 0)$, b in $GF(s)$	$s(s-1)$	$a_{11}a = a^m$ $a_{12}a + a_{22}b = b^m$
	$s^2 + s$ lines	

are enumerated and conditions under which these are fixed by transformations of $\mathscr{H}$ are determined.

For any fixed $m = p^u$, we exhibit the line types in Table 3.2.1. Likewise, for a given value of m, we display in Table 3.2.2, conditions for a point type to be fixed.

Next, we examine the point-line incidences in order to determine the number of matrices which fix each flag type (see Table 3.2.3).

TABLE 3.2.2

Conditions on Matrices in the Stabilizer which Fix Points

Point type	Number of points of this type	Conditions under which a point of this type is fixed
$(0\ 0)$	1	Fixed for all A in $\mathscr{H}$
$(c\ 0)$ $c \neq 0$ in $GF(s)$	$s-1$	$a_{11}c^m = c$
(c, d) $d(\neq 0)$, c in $GF(s)$	$s(s-1)$	$a_{11}c^m + a_{12}d^m = c$ $a_{22}d^m = d$
	s^2 points	

TABLE 3.2.3

Number of Matrices in the Stabilizer which Fix a Flag of a Given Type for Fixed $m = p^u$

Point type	Number of points of this type	Type of lines incident with a point of given type	Number of lines of this type	Number of matrices which fix a flag of the given type
$(0\ 0)$	1	(a) $[0\ 0\ 1]$	1	$(s-1)^2 s$
		(b) $[0\ 1\ a]$ a in $GF(s)$	s	$(s-1)^2$
$(c\ 0)$ $c(\neq 0)$ in $GF(s)$	$s-1$	(c) $[0\ 0\ 1]$	1	$(s-1)s$
		(d) $[1\ a\ b]$ $a = -c^{-1}$, b in $GF(s)$	s	$s-1$
(c, d) $c, d(\neq 0)$ in $GF(s)$	$s(s-1)$	(e) $[0\ 1\ a]$ $a = -cd^{-1}$	1	$s-1$
		(f) $[1\ 0\ a]$ $a = -d^{-1}$	1	$s-1$
		(g) $[1\ a\ b]$ $a(\neq 0)$ in $GF(s)$ $b = -(1+ac)\,d^{-1}$	$s-1$	1

(a) The flag consisting of the point $(0, 0)$ and the line $[0, 0, 1]$ is fixed for all matrices A in the stabilizer. There are $(s-1)^2 s$ collineations which fix the flag.

(b) The flag consisting of the point $(0, 0)$ and the line $[0, 1, a]$, a in $GF(s)$, is fixed by all collineations satisfying

$$a_{12} + a_{22}a = a_{11}a^m.$$

There are $(s-1)^2$ such collineations and there are s such flags.

(c) The flag defined by the point $(c, 0)$ and the line $[0, 0, 1]$, $c \neq 0$ in $GF(s)$, is fixed by all collineations satisfying

$$a_{11}c^m = c.$$

There are $(s-1)s$ collineations which fix the flag and $(s-1)$ such flags.

(d) The flag defined by the point $(c, 0)$ and the line $[1, -c^{-1}, b]$, $c(\neq 0)$, b in $GF(s)$, is fixed by all collineations satisfying

$$a_{11}c^m = c, \qquad a_{22}b = b^m + a_{12}c^{-1}. \tag{3.2.4}$$

There are $(s-1)$ such collineations which fix the flag and $(s-1)s$ such flags.

(e) The flag consisting of the point (c, d) and the line $[0, 1, a]$, c, $d(\neq 0)$ in $GF(s)$, $a = -cd^{-1}$, is fixed by all collineations satisfying

$$a_{11}c^m + a_{12}d^m = c, \qquad a_{22}d^m = d. \tag{3.2.5}$$

(Equation $a_{12} + a_{22}(-cd^{-1}) = a_{11}c^m d^{-m}$ is a linear combination of the first two equations.) There are $(s-1)$ such collineations which fix the flag and $(s-1)s$ such flags.

(f) The flag consisting of the point (c, d) and the line $[1, 0, a]$, $a = -d^{-1}$, c, $d(\neq 0)$ in $GF(s)$, is fixed by all collineations satisfying

$$a_{11}c^m + a_{12}d^m = c, \qquad a_{22}d^m = d. \tag{3.2.6}$$

There are $(s-1)$ such collineations which fix the flag and there are $(s-1)s$ such flags.

(g) The flag consisting of the point (c, d) and the line $[1, a, b]$, c, $d(\neq 0)$ in $GF(s)$, and $b = -(1 + ac)d^{-1}$, is fixed by all collineations satisfying

$$a_{11}c^m + a_{12}d^m = c, \qquad a_{22}d^m = d, \qquad a_{11}a = a^m. \tag{3.2.7}$$

There is one collineation which fixes the flag and there are $(s-1)^2 s$ such flags.

Thus, there are $7(s-1)^2s$ flags fixed for a given u $(m = p^u)$ and hence, the collineations of the stabilizer fix $7r(s-1)^2s$ flags in all. The dimension of the commutant algebra is thus seven. I, G, B, T, BT, TB, and BTB form a linear basis for this seven-dimensional noncommutative semi-simple algebra since $\mathscr{R}_7$ coincides with the commutant algebra.

3.3 *The Commutant Algebra Corresponding to the Permutation Representation of $PC(k, s)$, $k \geqslant 3$, $s = p^r$*

The point-hyperplane incidences (flags) of the $PG(k, s)$ define a symmetric BIB design with parameters $v = b = (s^{k+1} - 1)/(s-1)$, $r = k^* = (s^k - 1)/(s-1)$, $\lambda = (s^{k-1} - 1)/(s-1)$. The full group of collineations $PC(k, s)$ of this geometry is given by transformations (2.1); hyperplanes are transformed as in Lemma 2.1.

As in Section 3.1, we proceed to determine the dimension of the commutant algebra corresponding to the full collineation group $PC(k, s)$, $k \geqslant 3$, by counting the number of flags fixed under the action of $PC(k, s)$ and then determining the number of orbits of $\mathscr{H}$ by Burnside's rule. For this purpose, we first enumerate point types, hyperplane types, and types of hyperplanes incident with given point types. For any $m = p^u$, $u = 0, 1, \ldots, r-1$, we have shown the hyperplane types in Table 3.3.1. Likewise, for any given $m = p^u$, we exhibit the point types in Table 3.3.2.

TABLE 3.3.1

Conditions on Matrices of the Stabilizer which Fix Hyperplanes

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed
$[0, \ldots, 0, 1]$	1	Fixed for all A in $\mathscr{H}$
$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$, $b_1, \ldots, b_{k-1}$ not all zero	$\dfrac{s(s^{k-1}-1)}{s-1}$	$A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$, $\gamma (\neq 0)$ in $GF(s)$; $\sum_{i=1}^{k} a_{ik} b_i = \gamma b_k^m$
$[1, b_1, \ldots, b_k]$ b_i in $GF(s)$	s^k	$\begin{pmatrix} a_{01} \\ \vdots \\ a_{0,k-1} \end{pmatrix} + A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$; $a_{0k} + a_{1k} b_1 + \cdots + a_{kk} b_k = b_k^m$
	$\dfrac{(s^{k+1}-1)}{(s-1)}$ hyperplanes	

TABLE 3.3.2

Conditions on the Matrices in the Stabilizer which Fix Points

Point type	Number of points of this type	Conditions under which a point of this type is fixed
$(1, 0, \ldots, 0)$	1	Fixed for all A in $\mathscr{H}$
$(c_1, \ldots, c_k, 0)$ c_i in $GF(s)$, $c_2, c_3, \ldots, c_k$ not all zero	$\dfrac{s(s^{k-1}-1)}{s-1}$	$c_1^m + a_{01}c_2^m + \cdots + a_{0,k-1}c_k^m = \alpha c_1$ $A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}$, $\alpha(\neq 0)$ in $GF(s)$
$(c_1, c_2, \ldots, c_k, 1)$ c_i in $GF(s)$	s^k	$c_1^m + a_{01}c_2^m + \cdots + a_{0,k-1}c_k^m + a_{0k} = a_{kk}c$ $A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}$
	$(s^{k+1}-1)/(s-1)$ points	

Next we determine for each point type the type of hyperplanes incident with that type. Then by examining the conditions under which a given flag (incident point-hyperplane pair) is fixed, we determine the number of transformation matrices in the stabilizer $\mathscr{H}$ satisfying these conditions.

(a) The flag consisting of the point $(1, 0, \ldots, 0)$ and the hyperplane $[0, \ldots, 0, 1]$ is fixed for all A in the stabilizer. For a fixed $m = p^u$, there are $(s-1)\, s^{2k-1}\phi(s, k-1)$ collineations fixing this flag, where

$$\phi(s, k-1) = \prod_{i=0}^{k-2} (s^{k-1} - s^i).$$

(b) The flag consisting of the point $(1, 0, \ldots, 0)$ and the hyperplane $[0, b_1, \ldots, b_k]$, b_i in $GF(s)$, $b_1, \ldots, b_{k-1}$ not all zero, is fixed by all collineations satisfying

$$A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \qquad \gamma(\neq 0) \quad \text{in} \quad \mathrm{GF}(s),$$

$$a_{1k}b_1 + \cdots + a_{kk}b_k = \gamma b_k^m. \tag{3.3.1}$$

The subgroup $PC_0(k-2, s)$ of $PC(k-2, s)$ (the full projective group, where $u = 0$ in (2.1)) is transitive on the hyperplanes of $PG(k-2, s)$. Hence, there exists a nonsingular matrix T such that $T\mathbf{b} = \mathbf{c}$ and $T^{(m)}\mathbf{b}^{(m)} = \mathbf{c}^{(m)}$, where

TABLE 3.3.3

Number of Matrices in the Stabilizer which Fix a Flag of a Given Type for Fixed $m = p^u$

Point type	Number of points of this type		Type of hyperplanes incident with a point of given type	Number of hyperplanes of this type	Number of Matrices which fix a flag of the given type
$(1, 0, \ldots, 0)$	1	(a)	$[0, \ldots, 0, 1]$	1	$(s-1)\, s^{2k-1} \phi(s, k-1)$
		(b)	$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero	$\frac{(s^k - s)}{(s-1)}$	$(s-1)^2 s^{3k-4} \phi(s, k-2)$
$(c_1, \ldots, c_k, 0)$ c_i in $GF(s)$, $c_2, \ldots, c_k$ not all zero	$\frac{(s^k - s)}{(s-1)}$	(c)	$[0, 0, \ldots, 0, 1]$	1	$(s-1)^2 s^{3k-4} \phi(s, k-2)$
		(d)	$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $c_2 b_1 + \cdots + c_k b_{k-1} = 0$	$\frac{s^{k-1} - s}{s-1}$	$(s-1)^3 s^{4k-8} \phi(s, k-3)$, $k \geqslant 4$; $(s-1)^2 s^4$, $k = 3$
		(e)	$[1, b_1, \ldots, b_k]$ b_i in $GF(s)$; $c_1 + c_2 b_1 + \cdots + c_k b_{k-1} = 0$	s^{k-1}	$(s-1)^2 s^{2k-3} \phi(s, k-2)$
$(c_1, \ldots, c_k, 1)$ c_i in $GF(s)$	s^k	(f)	$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $c_2 b_1 + \cdots + c_k b_{k-1} + b_k = 0$	$\frac{s^{k-1} - 1}{s-1}$	$(s-1)^2 s^{2k-3} \phi(s, k-2)$
		(g)	$[1, b_1, \ldots, b_k]$; b_i in $GF(s)$; $c_1 + c_2 b_1 + \cdots + c_k b_{k-1} + b_k = 0$	s^{k-1}	$(s-1)\, \phi(s, k-1)$

Note. $\phi(s, k) = \prod_{i=0}^{k-1} (s^k - s^i)$.

$\mathbf{b} \neq \mathbf{0}$, $\mathbf{c} \neq \mathbf{0}$, and $T^{(m)} = (t_{ij}^m)$ $(m = p^u,\ u = 0, \ldots, r-1)$ is nonsingular. The matrix equation of (3.3.1) can therefore be rewritten as $T^{(m)} A_1' T^{-1} \mathbf{c} = \gamma \mathbf{c}^{(m)}$. Thus, the number of matrices A_1 satisfying this matrix condition is the same for every choice of $b_1, \ldots, b_{k-1}$. For $b_1 = 1$, $b_2 = \cdots = b_{k-1} = 0$, the two conditions (3.3.1) reduce to

$$a_{11} = \gamma, \qquad a_{12} = \cdots = a_{1k-1} = 0, \qquad a_{1k} b_1 + a_{kk} b_k = \gamma b_k^m. \quad (3.3.2)$$

There are thus $(s-1)\, s^{3k-4} \phi(s, k-2)$ collineations fixing this flag. There are $s(s^{k-1} - 1)/(s-1)$ such flags.

(c) The flag defined by the point $(c_1, c_2, \ldots, c_k, 0)$ and the hyperplane $[0, \ldots, 0, 1]$, c_i in $GF(s)$, $c_2, c_3, \ldots, c_k$ not all zero, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}, \qquad \alpha (\neq 0) \quad \text{in} \quad GF(s),$$

$$c_1^m + a_{01} c_2^m + \cdots + a_{0,k-1} c_k^m = \alpha c_1. \quad (3.3.3)$$

Since the subgroup $PC_0(k-2, s)$, the full projective group of $PC(k-2, s)$, is transitive on the points of $PG(k-2, s)$, the number of matrices satisfying the first condition will be the same for any choice $c_2, c_3, \ldots, c_k$ not all zero. Thus, for $c_2 = 1$, $c_3 = c_4 = \cdots = c_k = 0$, the two conditions reduce to

$$\begin{aligned} &a_{11} = \alpha, \qquad a_{21} = a_{31} = \cdots = a_{k-1,1} = 0, \\ &c_1^m + a_{01} c_2^m = \alpha c_1, \qquad (\alpha \neq 0) \quad \text{in} \quad GF(s). \end{aligned} \quad (3.3.4)$$

There are $(s-1)^2 s^{3k-4} \phi(s, k-2)$ collineations which fix the flag. There are $s(s^{k-1} - 1)/(s-1)$ such flags.

(d) The flag consisting of the point $(c_1, \ldots, c_k, 0)$ and the hyperplane $[0, b_1, b_2, \ldots, b_k]$, c_i, b_i in $GF(s)$, $c_2, \ldots, c_k$ not all zero, $b_1, \ldots, b_k$ not all zero, $c_2 b_1 + \cdots + c_k b_{k-1} = 0$, is fixed by all collineations which satisfy the conditions

$$A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} = \alpha \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}, \qquad A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \qquad \begin{matrix} \alpha(\neq 0), \quad \gamma(\neq 0) \\ \text{in} \quad GF(s), \end{matrix}$$

$$c_1^m + a_{01} c_2^m + \cdots + a_{0,k-1} c_k^m = \alpha c_1, \qquad a_{1k} b_1 + \cdots + a_{kk} b_k = \gamma b_k^m. \quad (3.3.5)$$

(i) Let $k \geqslant 4$. Since the subgroup $PC_0(k-2, s)$ of $PC(k-2, s)$, the full projective group, is transitive on the flags (incident point-hyperplane pairs) of $PG(k-2, s)$, the number of matrices A_1 which satisfy the two matrix equations will be the same for any choice of $c_2,\ldots,c_k$ (not all zero), $b_1,\ldots,b_{k-1}$ (not all zero) consistent with the restriction $c_2b_1+\cdots+c_kb_{k-1}=0$. For the choice $c_2=b_{k-1}=1$, $c_3=\cdots=c_k=0=b_1=\cdots=b_{k-2}$, the conditions reduce to

$$\begin{gathered} a_{11}=\alpha, \qquad a_{k-1,k-1}=\gamma, \\ a_{21}=\cdots=a_{k-1,1}=a_{k-1,2}=\cdots=a_{k-1,k-2}=0, \\ c_1^m+a_{01}=\alpha c_1, \\ a_{k-1,k}+a_{kk}b_k=\gamma b_k^m, \qquad \alpha(\neq 0), \quad \gamma(\neq 0) \quad \text{in} \quad GF(s). \end{gathered} \tag{3.3.6}$$

There are $(s-1)^3s^{4k-8}\phi(s,k-3)$ collineations which fix the flag and $s^2(s^{k-1}-1)(s^{k-2}-1)/(s-1)^2$ such flags.

(ii) For $k=3$, the conditions are

$$A_1\begin{pmatrix}c_2^m\\c_3^m\end{pmatrix}=\alpha\begin{pmatrix}c_2\\c_3\end{pmatrix}, \qquad A_1'\begin{pmatrix}b_1\\b_2\end{pmatrix}=\gamma\begin{pmatrix}b_1^m\\b_2^m\end{pmatrix}, \qquad \alpha(\neq 0), \quad \gamma(\neq 0) \quad \text{in} \quad GF(s),$$

$$c_1^m+a_{01}c_2^m+a_{02}c_3^m=\alpha c_1, \qquad a_{13}b_1+a_{23}b_2+a_{33}b_3=\gamma b_3^m. \tag{3.3.7}$$

There are $(s-1)^2s^4$ collineations which fix the flag and $s^2(s+1)$ such flags.

(e) The flag consisting of the point $(c_1,\ldots,c_k,0)$ and the hyperplane $[1,b_1,\ldots,b_k]$, b_i, c_i in $GF(s)$, $c_2,c_3,\ldots,c_k$ not all zero, $c_1+c_2b_1+\cdots+c_kb_{k-1}=0$, is fixed by all collineations satisfying

$$A_1\begin{pmatrix}c_2^m\\ \vdots\\ c_k^m\end{pmatrix}=\alpha\begin{pmatrix}c_2\\ \vdots\\ c_k\end{pmatrix}, \qquad \begin{pmatrix}a_{01}\\ \vdots\\ a_{0,k-1}\end{pmatrix}+A_1'\begin{pmatrix}b_1\\ \vdots\\ b_{k-1}\end{pmatrix}=\begin{pmatrix}b_1^m\\ \vdots\\ b_{k-1}^m\end{pmatrix},$$
$$a_{0k}+a_{1k}b_1+\cdots+a_{kk}b_k=b_k^m, \qquad \alpha(\neq 0) \quad \text{in} \quad GF(s). \tag{3.3.8}$$

(Equation $c_1^m+a_{01}c_2^m+\cdots+a_{0,k-1}c_k^m=\alpha c_1$ can be easily seen to be a linear combination of these conditions and $c_1+c_2b_1+\cdots+c_kb_{k-1}=0$.) There are $(s-1)\,s^{k-2}\phi(s,k-2)$ choices of A_1 satisfying the first matrix equation. When A_1 is thus determined, $a_{01},\ldots,a_{0,k-1}$ are determined from the second set of equations. There are thus $(s-1)^2s^{2k-3}\phi(s,k-2)$ collineations fixing this flag and $s^k(s^{k-1}-1)/(s-1)$ such flags.

(f) The flag consisting of the point $(c_1,c_2,\ldots,c_k,1)$ and the hyperplane $[0,b_1,\ldots,b_k]$, c_i, b_i in $GF(s)$, $b_1,\ldots,b_{k-1}$ not all zero, $c_2b_1+\cdots+c_kb_{k-1}+b_k=0$, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}, \qquad A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix},$$

$$c_1^m + a_{01} c_2^m + \cdots + a_{0,k-1} c_k^m + a_{0k} = a_{kk} c_1, \qquad \gamma (\neq 0) \quad \text{in} \quad GF(s). \tag{3.39}$$

(Equation $a_{1k} b_1 + a_{2k} b_2 + \cdots + a_{kk} b_k = \gamma b_k^m$ is easily shown to be a linear combination of the other conditions.) There are $(s-1) s^{k-2} \phi(s, k-2)$ possible matrices A_1' satisfying the second matrix equation. Once A_1' and a_{kk} are determined, $a_{1k}, a_{2k}, \ldots, a_{k-1,k}$ are completely determined. There are thus $(s-1)^2 s^{2k-3} \phi(s, k-2)$ collineations fixing this flag and $s^k(s^{k-1}-1)/(s-1)$ such flags.

(g) The flag consisting of the point $(c_1, \ldots, c_k, 1)$ and the hyperplane $[1, b_1, b_2, \ldots, b_k]$, c_i, b_i in $GF(s)$, $c_1 + c_2 b_1 + \cdots + c_k b_{k-1} + b_k = 0$, is fixed by all collineations in $\mathscr{H}$ satisfying

$$A_1 \begin{pmatrix} c_2^m \\ \vdots \\ c_k^m \end{pmatrix} + \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = a_{kk} \begin{pmatrix} c_2 \\ \vdots \\ c_k \end{pmatrix}, \quad \begin{pmatrix} a_{01} \\ \vdots \\ a_{0,k-1} \end{pmatrix} + A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix},$$

$$a_{0k} + a_{1k} b_1 + \cdots + a_{kk} b_k = b_k^m. \tag{3.3.10}$$

There are $\phi(s, k-1)$ matrices A_1 satisfying the first matrix equation. Once A_1 and a_{kk} are determined, all the remaining a_{ij}'s are completely determined. Therefore, there are $(s-1)\,\phi(s, k-1)$ collineations fixing the flag and s^{2k-1} such flags.

For a fixed u, there are thus $7(s-1)\, s^{2k-1} \phi(s, k-1)$ flags fixed by the collineations of the stabilizer. Hence, there are $7r(s-1)\, s^{2k-1} \phi(s, k-1)$ flags fixed by all the collineations of the stabilizer. Thus, the dimension of the stabilizer is seven. The commutant algebra coincides with $\mathscr{R}_7$ (Section 2) and I, G, B, T, BT, TB, and S form a basis of this algebra.

3.4. *The Commutant Algebra Corresponding to the Permutation Representation of $EC(k, s)$, $s = p^r$, $k \geqslant 3$*

The point-hyperplane incidences of the $EG(k, s)$, $k \geqslant 3$, define an asymmetric BIB design with parameters $v = s^k$, $b = s(s^k - 1)/(s-1)$, $r = (s^k - 1)/(s-1)$, $k^* = s^{k-1}$, and $\lambda = (s^{k-1} - 1)/(s-1)$. There are $s^k(s^k - 1)/(s-1)$ flags. The full collineation group $EC(k, s)$ is defined by the linear nonhomogeneous transformations (2.4). The subgroup $\mathscr{H}$ of collineations which fix the point $(0, 0, \ldots, 0)$ and the hyperplane $[0, 0, \ldots, 0, 1]$ consists of the transformations found in Lemma 2.4. For each of the point transformations (2.7), we see from (2.6) that the hyperplanes are transformed as follows:

$$\begin{pmatrix} 1 & \mathbf{0}' & 0 \\ \mathbf{0} & A_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{a}' & a_{kk} \end{pmatrix} \begin{pmatrix} y_0^* \\ \vdots \\ y_k^* \end{pmatrix} = \gamma \begin{pmatrix} y_0^m \\ \vdots \\ y_k^m \end{pmatrix}, \qquad \gamma(\neq 0) \text{ in } GF(s). \tag{3.4.1}$$

For a fixed $m = p^u$, $u = 0, 1, \ldots, r-1$, we show in Table 3.4.1, the conditions for a line type to be fixed by the collineations of $\mathscr{H}$. Likewise, for any given value of m, we exhibit in Table 3.4.2, conditions for a point type to be fixed by the collineations of $\mathscr{H}$.

TABLE 3.4.1

Conditions on the Matrices of the Stabilizer which Fix Lines

Hyperplane type	Number of hyperplanes of this type	Conditions under which a hyperplane of this type is fixed
$[0, 0, \ldots, 1]$	1	Fixed for all A in $\mathscr{H}$
$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero	$\dfrac{s(s^{k-1}-1)}{s-1}$	$A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$, $\gamma(\neq 0)$ in $GF(s)$ $a_{1k}b_1 + \cdots + a_{k-1,k}b_{k-1} + a_{kk}b_k = \gamma b_k^m$
$[1, 0, \ldots, 0, b]$ $b(\neq 0)$ in $GF(s)$	$s-1$	$\gamma = 1$ $a_{kk} = b^{m-1}$
$[1, b_1, \ldots, b_k]$ $b_i$in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero	$s(s^{k-1}-1)$	$\gamma = 1$ $A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}$ $1 + a_{1k}b_1 + \cdots + a_{kk}b_k = b_k^m$
	$\dfrac{s(s^k-1)}{s-1}$ hyperplanes	

Next, we determine the flag types (incident point-hyperplane pairs) and the number of matrices in the stabilizer $\mathscr{H}$ fixing each flag type.

(a) The flag consisting of the point $(0, 0, \ldots, 0)$ and the hyperplane $[0, 0, \ldots, 0, 1]$ is fixed for all A in the stabilizer $\mathscr{H}$. There are $(s-1)\, s^{k-1} \phi(s, k-1)$ such collineations for a fixed u $(m = p^u$. $u = 0, 1, \ldots, r-1)$.

(b) The flag consisting of the point $(0, 0, \ldots, 0)$ and the hyperplane

TABLE 3.4.2

Conditions on the Matrices of the Stabilizer which Fix Points

Point type	Number of points of this type	Conditions under which a point of this type is fixed
$(0, 0, 0, \ldots, 0)$	1	Fixed for all A in $\mathscr{H}$
$(c_1, \ldots, c_{k-1}, 0)$ c_i in $GF(s)$; $c_1, \ldots, c_{k-1}$ not all zero	$s^{k-1} - 1$	$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}$
$(c_1, \ldots, c_{k-1}, c_k)$ $c_k(\neq 0)$; c_i in $GF(s)$	$(s-1)\, s^{k-1}$	$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} + c_{kk}^m \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}$
	s^k points	

$[0, b_1, \ldots, b_k]$, b_i in $GF(s)$, $b_1, \ldots, b_{k-1}$ not all zero, is fixed by all collineations satisfying

$$A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \qquad \gamma(\neq 0) \quad \text{in} \quad GF(s),$$

$$a_{1k} b_1 + \cdots + a_{kk} b_k = \gamma b_k^m. \tag{3.4.2}$$

The number of matrices A_1' which satisfy the matrix equation for a fixed m ($m = p^u$, $u = 0, 1, \ldots, r-1$) is the same for all $b_1, b_2, \ldots, b_{k-1}$ not all zero, since the subgroup $PC_0(k-2, s)$ of $PC(k-2, s)$, the full projective group, is transitive on the hyperplanes of $PG(k-2, s)$. For the choice $b_1 = 1$, $b_2 = b_3 = \cdots = b_{k-1} = 0$, the above conditions become

$$a_{11} = \gamma, \qquad a_{12} = \cdots = a_{1,k-1} = 0, \qquad a_{1k} + a_{kk} b_k = \gamma b_k^m. \tag{3.4.3}$$

There are $(s-1)^2 s^{2k-4} \phi(s, k-2)$ such collineations which fix the flag and there are $s(s^{k-1} - 1)/(s-1)$ such flags.

(c) The flag defined by the point $(c_1, \ldots, c_{k-1}, 0)$ and the hyperplane $[0, 0, \ldots, 0, 1]$, c_i in $GF(s)$, $c_1, c_2, \ldots, c_{k-1}$ not all zero, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}. \tag{3.4.4}$$

TABLE 3.4.3

Number of Matrices in the Stabilizer which Fix a Flag of Given Type for Fixed $m = p^u$, $u = 0, 1, \ldots, r-1$

Point type	Number of points of this type		Type of hyperplanes incident with a point of this type	Number of hyperplanes of this type	Number of matrices which fix a flag of the given type
$(0, 0, \ldots, 0)$	1	(a)	$[0, 0, \ldots, 0, 1]$	1	$(s-1)\, s^{k-1} \phi(s, k-1)$
		(b)	$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero	$\dfrac{s(s^{k-1}-1)}{(s-1)}$	$(s-1)^2 s^{2k-4} \phi(s, k-2)$
$(c_1, \ldots, c_{k-1}, 0)$ c_i in $GF(s)$; $c_1, \ldots, c_{k-1}$ not all zero	$s^{k-1}-1$	(c)	$[0, 0, \ldots, 0, 1]$	1	$(s-1)\, s^{2k-3} \phi(s, k-2)$
		(d)	$[0, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$	$\dfrac{s(s^{k-2}-1)}{(s-1)}$	$(s-1)^2 s^{3k-7} \phi(s, k-3)$ for $k \geqslant 4$; $(s-1)^2 s^2$ for $k = 3$
		(e)	$[1, b_1, \ldots, b_k]$ b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $1 + c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$	s^{k-1}	$(s-1)\, s^{k-2} \phi(s, k-2)$
$(c_1, \ldots, c_k)$ c_i in $GF(s)$; $c_k \neq 0$	$(s-1)\, s^{k-1}$	(f)	$[0, b_1, \ldots, b_k]$, b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $c_1 b_1 + \cdots + c_k b_k = 0$	$\dfrac{(s^{k-1}-1)}{(s-1)}$	$(s-1)\, s^{k-2} \phi(s, k-2)$
		(g)	$[1, 0, \ldots, 0, b]$; $b = -c_k^{-1}$	1	$\phi(s, k-1)$
		(h)	$[1, b_1, \ldots, b_k]$; b_i in $GF(s)$; $b_1, \ldots, b_{k-1}$ not all zero; $1 + c_1 b_1 + \cdots + c_k b_k = 0$	$s^{k-1}-1$	$s^{k-2} \phi(s, k-2)$

The number of A_1 which satisfy this equation is the same for all choices of $c_1, c_2, \ldots, c_{k-1}$ not all zero, since the subgroup $PC_0(k-1, s)$ is transitive on the points of $PG(k-1, s)$. For the choice $c_1 = 1$, $c_2 = \cdots = c_{k-1} = 0$, the above equations become $a_{11} = 1$, $a_{21} = a_{31} = \cdots = a_{k-1,1} = 0$. There are $(s-1)\, s^{2k-3}\phi(k, s-2)$ collineations which fix the flag and there are $(s^{k-1} - 1)$ such flags.

(d) The flag defined by the point $(c_1, \ldots, c_{k-1}, 0)$ and the hyperplane $[0, b_1, \ldots, b_k]$, c_i, b_i in $GF(s)$, $c_1, \ldots, c_{k-1}$ not all zero, $b_1, \ldots, b_{k-1}$ not all zero, $c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}, \qquad A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \qquad \gamma (\neq 0) \quad \text{in} \quad GF(s),$$
$$a_{1k} b_1 + \cdots + a_{kk} b_k = \gamma b_k^m. \tag{3.4.5}$$

The number of matrices A_1 satisfying the two matrix equations is the same for all choices of $c_1, \ldots, c_{k-1}$ (not all zero) and $b_1, \ldots, b_{k-1}$ (not all zero) subject to $c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$, since the subgroup $PC_0(k-2, s)$ of $PC(k-2, s)$ is transitive on the flags (incident point-hyperplane pairs) of $PG(k-2, s)$. Thus for the choice $c_1 = b_{k-1} = 1$, $c_2 = c_3 = \cdots = c_{k-1} = b_1 = \cdots = b_{k-2} = 0$, the above conditions become

$$a_{11} = 1, \quad a_{k-1,k-1} = \gamma, \quad a_{21} = \cdots = a_{k-1,1} = a_{k-1,2} = \cdots = a_{k-1,k-2} = 0,$$
$$a_{k-1,k} + a_{kk} b_k = \gamma b_k^m, \qquad \gamma \neq 0 \quad \text{in} \quad GF(s). \tag{3.4.6}$$

For $k \geqslant 4$, there are thus $(s-1)^2 s^{3k-7} \phi(s, k-3)$ such collineations which fix the flag and there are $s(s^{k-1} - 1)(s^{k-2} - 1)/(s-1)$ such flags. For $k = 3$, there are $(s-1)^2 s^2$ collineations which fix the flag and there are $s(s^2 - 1)$ such flags.

(e) The flag defined by the point $(c_1, \ldots, c_{k-1}, 0)$ and the hyperplane $[1, b_1, \ldots, b_k]$, c_i, b_i in $GF(s)$, $c_1, \ldots, c_{k-1}$ not all zero, $b_1, \ldots, b_{k-1}$ not all zero, $1 + c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}, \qquad A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix},$$
$$a_{1k} b_1 + \cdots + a_{kk} b_k = b_k^m. \tag{3.4.7}$$

The number of matrices A_1 satisfying the two matrix equations is the same for all $c_1, \ldots, c_{k-1}$ not all zero, $b_1, \ldots, b_{k-1}$ not all zero, subject to $1 + c_1 b_1 + \cdots + c_{k-1} b_{k-1} = 0$, since the subgroup $EC_0(k-1, s)$ of

$EC(k-1, s)$ (the full affine group, where $u=0$ in (2.4)) is transitive on the flags (incident point-hyperplanes) of $EG(k-1, s)$. For the choice $c_1 = 1$, $b_1 = -1$, $c_2 = c_3 = \cdots = c_{k-1} = b_2 = \cdots = b_{k-1} = 0$, the conditions reduce to

$$\begin{gathered} a_{11} = 1, \qquad a_{21} = a_{31} = \cdots = a_{k-1,1} = a_{12} = \cdots = a_{1,k-1} = 0, \\ -a_{1k} + a_{kk} b_k = b_k^m. \end{gathered} \tag{3.4.8}$$

There are $(s-1)\, s^{k-2} \phi(s, k-2)$ collineations which fix the flag and $s^{k-1}(s^{k-1}-1)$ such flags.

(f) The flag defined by the point $(c_1, c_2, ..., c_k)$ and the hyperplane $[0, b_1, b_2, ..., b_k]$, c_i, b_i in $GF(s)$, $c_k \neq 0$, $b_1, ..., b_{k-1}$ not all zero, $c_1 b_1 + \cdots + c_k b_k = 0$, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}, \qquad A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \gamma \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix},$$
$$\gamma \neq 0 \ \text{ in } \ GF(s), \qquad a_{kk} = c_k^{1-m}. \tag{3.4.9}$$

(The condition $a_{1k} b_1 + \cdots + a_{kk} b_k = \gamma b_k^m$ is a linear combination of the other conditions.) There are $(s-1)\, s^{k-2} \phi(s, k-2)$ matrices A_1' satisfying the second matrix equation. Once A_1' is determined, all the other coefficients in A are determined by the other equations. Thus, there are $(s-1)\, s^{k-2} \phi(s, k-2)$ collineations which fix the flag and $s^{k-1}(s^{k-1}-1)$ such flags.

(g) The flag defined by the point $(c_1, ..., c_k)$ and the hyperplane $[1, 0, ..., 0, b]$, c_i in $GF(s)$, $c_k \neq 0$, $b = -c_k^{-1}$, is fixed by all collineations satisfying

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}, \qquad a_{kk} = c_k^{1-m}. \tag{3.4.10}$$

There are $\phi(s, k-1)$ nonsingular matrices A_1. Given A_1, all the other coefficients of A are determined by the equations given above. Hence, there are $\phi(s, k-1)$ such collineations fixing the flag and there are $(s-1)\, s^{k-1}$ such flags.

(h) The flag consisting of the point $(c_1, ..., c_k)$ and the hyperplane $[1, b_1, ..., b_k]$, c_i, b_i in $GF(s)$, $c_k \neq 0$, $b_1, ..., b_{k-1}$ not all zero, $1 + c_1 b_1 + \cdots + c_k b_k = 0$, is fixed by all collineations satisfying

$$A_1' \begin{pmatrix} b_1 \\ \vdots \\ b_{k-1} \end{pmatrix} = \begin{pmatrix} b_1^m \\ \vdots \\ b_{k-1}^m \end{pmatrix}, \qquad a_{kk} = c_m^{1-m},$$

$$A_1 \begin{pmatrix} c_1^m \\ \vdots \\ c_{k-1}^m \end{pmatrix} + c_k^m \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}. \tag{3.4.11}$$

There are $s^{k-2}\phi(s, k-2)$ matrices A_1' satisfying the first matrix equation. Given A_1', all the remaining coefficients of A are determined from the other conditions. Thus, there are $s^{k-2}\phi(s, k-2)$ such matrices and $(s-1)\, s^{k-1}(s^{k-1}-1)$ such flags.

For a given u ($m = p^u$), there are thus $8(s-1)\, s^{k-1}\phi(s, k-1)$ flags fixed by the collineations of the stabilizer. Therefore, $8r(s-1)\, s^{k-1}\phi(s, k-1)$ flags are fixed by all the collineations of the stabilizer. Hence, the number of orbits of the stabilizer is eight, which is thus also the dimension of the commutant algebra corresponding to the permutation representation of $EC(k, s)$, $s = p^r$, representing the permutations of the point-hyperplane pairs of $EG(k, s)$. The commutant algebra here coincides with $\mathscr{P}_8$ (defined in Section 2) and thus I, G, B, T, BT, TB, BTB, and S form a linear basis of this algebra.

REFERENCES

1. W. BURNSIDE, "Theory of Groups of Finite Order," 2nd ed., Cambridge Univ. Press, London/New York, 1911 (Dover, 1955).
2. C. T. BURTON, Automorphism Groups of Balanced Incomplete Block Designs and Their Use in Statistical Model Construction and Analysis," Ph.D. Dissertation, Department of Statistics, University of North Carolina at Chapel Hill, 1980.
3. R. D. CARMICHAEL, "Introduction to the Theory of Groups of Finite Order," Dover, New York, 1956.
4. I. M. CHAKRAVARTI AND C. T. BURTON, "Symmetries (Groups of Automorphisms) of Desarguesian Finite Projective and Affine Planes and Their Role in Statistical Model Construction, *in* "Statistics and Probability: Essays in Honor of C. R. Rao," (G. Kallianpur, P. R. Krishnaiah, J. K. Ghosh, Eds.), pp. 169–178, North Holland-Publishing Company, Amsterdam, 1982.
5. I. M. CHAKRAVARTI AND C. T. BURTON, "On the Algebras of Symmetries (Groups of Collineations of Designs) from Finite Desarguesian Planes with Applications in Statistics," *J. Math. Anal. Appl.* **90** (1982), 515–529.
6. P. DEMBOWSKI, "Finite Geometries, Springer-Verlag, New York, 1968.
7. M. HALL, JR., "The Theory of Groups," MacMillan, New York, 1959.
8. E. J. HANNAN, Group representations and applied probability, *in* "Methuen Monographs on Applied Probability and Statistics," Supplementary Review Series in Applied Probability (J. Gani Ed.), Vol. 3, Methuen, London, 1965.
9. A. T. JAMES, The relationship algebra of an experimental design, *Ann. Math. Statist.* **28** (1957), 993–1002.
10. H. B. MANN, The algebra of a linear hypothesis, *Ann. Math. Statist.* **31** (1960), 1–15.
11.. A. D. MACLAREN, On group representations and invariant stochastic processes, *Proc. Cambridge Philos. Soc.* **59** (1963), 431–450.
12. L. P. SYSOEV, Properties of certain classes of experimental block designs and correlation

characteristic of observations, *Avtomat. i Telemekh.* **8** (1977), 41–53. (Translated as *Automat. Remote Control* **37** (1977), 1145–1156.

13. L. P. SYSOEV AND M. E . SHAIKIN, Algebraic methods of investigating the correlation connections in completely balanced block schemes for experiment design. I. Characterizations of covariance matrices in the case of asymmetric block schemes, *Avtomat. i Telemekh.* (1976), 64–83. (Translated as *Automat. Remote Control* **37** (6) (1976), 696–704); II. Relationship algebras and characterization of covariance matrices in the case of symmetric block schemes. **7** (1976), 57–67, translation, **37** (7) (1976), 1022–1031.
14. H. WIELANDT, "Finite Permutation Groups," Academic Press, New York, 1964.