Journal of Functional Analysis **191**, 132–209 (2002)
doi:10.1006/jfan.2001.3867, available online at http://www.idealibrary.com on IDEAL®

Singular Pseudodifferential Operators, Symmetrizers, and Oscillatory Multidimensional Shocks

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Communicated by Richard Melrose

Received July 31, 2001; revised August 26, 2001; accepted September 11, 2001

We introduce a calculus of singular pseudodifferential operators (SPOs) depending on wavelength ε and use them to solve three different types of singular quasilinear hyperbolic systems. Such systems arise in nonlinear geometric optics and also

variables and are singular not only because their symbols have finite regularity and depend on $\frac{1}{\varepsilon}$, but also because their derivatives fail to decay in the usual way in the dual variables. There is a necessarily crude calculus with large parameter (e.g., residual operators are just bounded on L^2), but the calculus admits the proof of Garding inequalities and enables us to symmetrize and sometimes even diagonalize the singular systems being considered by microlocalizing simultaneously in both slow and fast variables. The paper culminates in a proof of the existence of oscillatory multidimensional shocks on a fixed time interval independent of the wavelength ε as $\varepsilon \to 0$. The use of SPOs allows us to eliminate the small divisor assumptions made in earlier work and also to construct more general oscillatory solutions in which elliptic boundary layers are present on one or both sides of the shock.

Contents.

[1] This research was supported in part by NSF Grants DMS-9706489 and DMS-0070684.

0022-1236/02 $35.00

PART 1. SINGULAR PSEUDODIFFERENTIAL OPERATORS

1. *Introduction*

In this paper we introduce a class of singular pseudodifferential operators (SPOs) depending on wavelength (ε) and apply them to several types of singular quasilinear hyperbolic systems. These systems include initial-value and fixed-boundary problems as well as a free-boundary problem associated with multidimensional shocks. Our focus here is on singular systems arising from nonlinear geometric optics, but similar systems appear also, for example, in the study of incompressible limits and of nonlinear wave equations with small nonlinear terms or small data [M3, S].

The SPOs act in both "slow" and "fast" variables and are singular not only because their symbols have finite regularity and depend on $\frac{1}{\varepsilon}$, but also because their derivatives fail to decay in the usual way in the dual variables. The paper culminates in a proof of the existence of oscillatory multidimensional shocks on a fixed time interval independent of the wavelength ε as $\varepsilon \to 0$. Such a theorem was proved in our paper [W2] by a method that depended on a (generically valid) small divisor assumption to first construct high-order approximate solutions and then find exact solutions nearby. Here the use of SPOs as symmetrizers allows us to dispense with small divisor assumptions and construct the exact shock more directly by solving an appropriate *singular shock problem* (9.15). A formal construction of oscillatory shocks was given in [MA].

Earlier work on singular quasilinear hyperbolic systems in free space such as [JMR1, JMR2, S] used *symmetric hyperbolicity* assumptions to obtain L^2 estimates uniform in ε by a simple integration by parts in which the $\frac{1}{\varepsilon}$ terms cancel out. In symmetric hyperbolic *boundary* problems this argument fails except for rather special (e.g., maximally dissipative) boundary conditions. The use of SPO symmetrizers allows us to obtain L^2 estimates in a variety of situations where symmetry is either not assumed (e.g., Kreiss well-posed boundary problems, nonsymmetric initial value problems in free space) or not sufficient (e.g., symmetric hyperbolic systems with Kreiss boundary conditions, multidimensional shocks).

For quasilinear applications L^2 estimates are just a start: one needs estimates uniform in ε in norms that define algebras of bounded functions. In singular boundary problems it is typically easy to estimate tangential derivatives by differentiating the equations and applying L^2 estimates, but it is impossible to estimate uniformly even a single normal derivative D_{x_N} (one would be enough). We show that in some circumstances (absence of glancing boundary layers) microlocalization and diagonalization with SPOs can be used to circumvent this problem.

SPO cutoffs are used to split the solution into two pieces. For one piece it *is* possible to estimate a single normal derivative uniformly. The other piece (the hard one) satisfies a problem that can be diagonalized by SPOs. For this piece one can prove $C(x_N, H^k)$ estimates. Recombining the pieces we find a norm giving L^∞ control in which it is possible to estimate the solution without loss of derivatives ((1.36) and (1.37)). The norm is therefore suitable for Picard iteration.

A. *Quasilinear boundary problems*

On $\overline{\mathbb{R}}^{N+1}_+ = \{x = (x', x_N) = (x_0, x'', x_N) : x_N \geqslant 0\}$ consider the $m \times m$ quasilinear boundary problem

$$(1.1)\quad \begin{aligned} &L(v_\varepsilon, D_x)\, v_\varepsilon = \sum_{j=0}^{N} A_j(v_\varepsilon)\, D_{x_j} v_\varepsilon = F(v_\varepsilon) \\ &\phi(v_\varepsilon)|_{x_N=0} = g_0 + \varepsilon G\left(x', \frac{x' \cdot \beta'}{\varepsilon}\right) \\ &v_\varepsilon = u_0 \qquad \text{in} \quad x_0 < 0, \end{aligned}$$

where x_0 is time, $G(x', \theta) \in C^\infty(\mathbb{R}^N \times \mathbb{T}^1, \mathbb{R}^\mu)$ with $\operatorname{supp} G \subset \{x_0 \geqslant 0\}$, and the boundary frequency $\beta' \in \mathbb{R}^N \backslash 0$. The matrices $A_j \in C^\infty(\mathbb{R}^m, \mathbb{R}^{m^2})$, $iF \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, $\phi \in C^\infty(\mathbb{R}^m, \mathbb{R}^\mu)$, and $D_{x_j} = \frac{1}{i}\partial_{x_j}$. Looking for v_ε as a perturbation $v_\varepsilon = u_0 + \varepsilon u_\varepsilon$ of a constant state u_0 such that $F(u_0) = 0$, $\phi(u_0) = g_0$, we obtain for u_ε the system (with slightly different A_j)

$$(1.2)\quad \begin{aligned} &L(u_\varepsilon, D_x)\, u_\varepsilon = \sum_{j=0}^{N} A_j(\varepsilon u_\varepsilon)\, D_{x_j} u_\varepsilon = u_\varepsilon \mathscr{F}(\varepsilon u_\varepsilon) \equiv F_\varepsilon(u_\varepsilon) \\ &B(\varepsilon u_\varepsilon)\, u_\varepsilon|_{x_N=0} = G\left(x', \frac{x' \cdot \beta'}{\varepsilon}\right) \\ &u_\varepsilon = 0 \qquad \text{in} \quad x_0 < 0, \end{aligned}$$

where $B(v)$ is a C^∞ $\mu \times m$ real matrix defined by

$$\phi(u_0 + \varepsilon u_\varepsilon) = \phi(u_0) + B(\varepsilon u_\varepsilon)\, \varepsilon u_\varepsilon$$

and $F_\varepsilon(0) = 0$.

We seek exact solutions of the form $u_\varepsilon(x) = U_\varepsilon(x, x' \cdot \beta'/\varepsilon)$, where $U_\varepsilon(x, \theta)$ satisfies the singular system

$$(1.3)\quad \begin{aligned} &\sum_{j=0}^{N} A_j(\varepsilon U_\varepsilon)\, D_{x_j} U_\varepsilon + \sum_{j=0}^{N-1} A_j(\varepsilon U_\varepsilon)\, \beta_j \frac{D_\theta}{\varepsilon} U_\varepsilon = F_\varepsilon(U_\varepsilon), \\ &B(\varepsilon U_\varepsilon)(U_\varepsilon)|_{x_N=0} = G(x', \theta), \\ &U_\varepsilon = 0 \qquad \text{in} \quad x_0 < 0. \end{aligned}$$

We assume $A_N(0)$ is invertible, that is, $x_N = 0$ is a noncharacteristic boundary. After multiplication by $A_N^{-1}(\varepsilon U_\varepsilon)$ and relabeling, (1.3) becomes

$$(1.4)\quad \begin{aligned} &D_{x_N} U_\varepsilon + \left(\sum_{j=0}^{N-1} A_j(\varepsilon U_\varepsilon)\, D_{x_j} U_\varepsilon + \sum_{j=0}^{N-1} A_j(\varepsilon U_\varepsilon) \frac{\beta_j}{\varepsilon} D_\theta U_\varepsilon \right) \\ &\quad \equiv D_{x_N} U_\varepsilon - \mathscr{A}\left(\varepsilon U_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon} \right) U_\varepsilon = F_\varepsilon(U_\varepsilon), \\ &B(\varepsilon U_\varepsilon)(U_\varepsilon)|_{x_N=0} = G(x', \theta), \\ &U_\varepsilon = 0 \quad \text{in} \quad x_0 < 0. \end{aligned}$$

To obtain $U_\varepsilon(x, \theta)$ as a limit of iterates U_ε^n satisfying

$$(1.5)\quad \begin{aligned} &D_{x_N} U_\varepsilon^{n+1} - \mathscr{A}\left(\varepsilon U_\varepsilon^n, D_{x'} + \frac{\beta' D_\theta}{\varepsilon} \right) U_\varepsilon^{n+1} = F_\varepsilon(U_\varepsilon^n) \\ &B(\varepsilon U_\varepsilon^n)\, U_\varepsilon^{n+1}|_{x_N=0} = G(x', \theta) \\ &U_\varepsilon^{n+1} = 0 \quad \text{in} \quad x_0 < 0 \end{aligned}$$

we need energy estimates uniform in ε for the linear problem

$$(1.6)\quad \begin{aligned} &\mathscr{L}(\varepsilon V_\varepsilon, D_{x,\theta}^\varepsilon)\, U_\varepsilon \equiv D_{x_N} U_\varepsilon - \mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon} \right) U_\varepsilon = F_\varepsilon(x, \theta) \\ &B(\varepsilon V_\varepsilon)\, U_\varepsilon|_{x_N=0} = G(x', \theta) \\ &U_\varepsilon = 0 \quad \text{in} \quad x_0 < 0 \end{aligned}$$

in norms that define algebras of bounded functions.

The operator $\mathscr{L}(\varepsilon V_\varepsilon, D_{x,\theta}^\varepsilon)$ is singular not just because of the $\frac{1}{\varepsilon}$ dependence, but also because $\xi' + \beta' m/\varepsilon$ can vanish for $|\xi', m| \neq 0$. (Here $(\xi', \xi_N) \in \mathbb{R}^{N+1}$ are dual to (x', x_N) and $m \in \mathbb{Z}$ is dual to θ.) Thus, if $D_{x_N} - \mathscr{A}(0, D_{x'})$ is, for example, strictly hyperbolic with respect to x_0, the same is not true of $\mathscr{L}(\varepsilon V_\varepsilon, D_{x,\theta}^\varepsilon)$, even for fixed ε. Even if Kreiss symmetrizers can be constructed for the original system (1.2), one cannot construct symmetrizers for (1.6) by mimicking in $(x', x_N, \theta, \xi', m)$-space the usual construction in (x', x_N, ξ')-space. Instead, in situations where Kreiss symmetrizers $R(v, \xi', \gamma)$ can be constructed for the boundary problem $(D_{x_N} - \mathscr{A}(v, D_{x'}), B(v))$ when $v \in \mathbb{R}^m$ is near 0, we shall define an operator associated to the "symbol" $R(\varepsilon V(x, \theta), \xi' + m\beta'/\varepsilon, \gamma)$ and show it can be used as a symmetrizer for the singular problem (1.6).

B. *Singular Operators*

Most of the singular operators we use are defined by singular symbols built from standard symbols in the following way. Consider for $k \in \mathbb{R}$ the usual class of smooth symbols depending on a large parameter $\gamma \geqslant \gamma_0 > 0$,

$$S^k = \{p(v, X, \gamma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^N \times [1, \infty)) : \\ |\partial_v^\alpha \partial_X^\beta p(v, X, \gamma)| \leqslant C_{\alpha,\beta} \langle X, \gamma \rangle^{k-|\beta|} \text{ for all } (v, X, \gamma)\}, \tag{1.7}$$

where $\langle X, \gamma \rangle = (|X|^2 + \gamma^2)^{1/2}$. For $M \in \mathbb{N}$ let

$$V(x, \theta) \in C_c^{0,M}(\overline{\mathbb{R}}_+^{N+1} \times \mathbb{T}^1) \\ \equiv \{V(x', x_N, \theta) \in C(\overline{\mathbb{R}}_+, C^M(\mathbb{R}_{x'}^N \times \mathbb{T}^1, \mathbb{R}^m)) : \text{supp } V \text{ is compact}\}. \tag{1.8}$$

Given $p \in S^k$, $V \in C_c^{0,M}$, and a boundary frequency $\beta' \in \mathbb{R}^N$ we define the singular symbol

$$p_s(x, \theta, \xi', m, \varepsilon, \gamma) \equiv p\left(\varepsilon V(x, \theta), \xi' + \frac{m\beta'}{\varepsilon}, \gamma\right) \tag{1.9}$$

and let

$$\mathscr{S}_{\beta'}^{k,M} = \{p_s\colon p_s \text{ is given by (1.9) for some } p \in S^k, \\ V \in C_c^{0,M}, \text{ and } \beta' \in \mathbb{R}^N \backslash 0\}. \tag{1.10}$$

Note that since $|\xi' + m\beta'/\varepsilon|$ can be small when $|\xi', m|$ is large, elements of $\mathscr{S}_{\beta'}^{k,M}$ fail to satisfy the usual decay estimates in $|\xi', m, \gamma|$ even for ε fixed.

To each $p_s \in \mathscr{S}_{\beta'}^{k,M}$ we associate an operator whose action on $U(x, \theta) \in C_c^{0,\infty}$, for example, is defined by

$$p_s(D_{x',\theta})\, U \equiv p_s(x, \theta, D_{x'}, D_\theta, \varepsilon, \gamma)\, U = p\left(\varepsilon V(x, \theta), D_{x'} + \frac{\beta' D_\theta}{\varepsilon}, \gamma\right) U \\ = \int e^{ix'\xi' + i\theta m} p\left(\varepsilon V(x, \theta), \xi' + \frac{m\beta'}{\varepsilon}, \gamma\right) U\hat{\ }(\xi', x_N, m)\, d\xi'\, dm. \tag{1.11}$$

Here the dm integral is a sum over $m \in \mathbb{Z}$ and

$$U\hat{\ }(\xi', x_N, m) = \hat{U}_m(\xi', x_N) \qquad \text{where} \quad U(x, \theta) = \sum_{m \in \mathbb{Z}} U_m(x)\, e^{im\theta}. \tag{1.12}$$

It is clear that for $p_s \in \mathcal{S}^{k,M}_{\beta'}$

$$p_s(D_{x',\theta})\colon C^{0,\infty}_c \to C(\overline{\mathbb{R}}_+, C^\infty(\mathbb{R}^N \times \mathbb{T}^1)). \tag{1.13}$$

Notation 1.1. We will often write spaces like the one on the right in (1.13) as follows:

$$C(x_N, C^\infty(x', \theta)) \equiv C(\overline{\mathbb{R}}_+, C^\infty(\mathbb{R}^N \times \mathbb{T}^1)).$$

Letting $OP\mathcal{S}^{k,M}_{\beta'} = \{p_s(D_{x',\theta})\colon p_s \in \mathcal{S}^{k,M}_{\beta'}\}$, we have the following L^2 continuity result for $OP\mathcal{S}^{0,M}_{\beta'}$.

PROPOSITION 1.1. *If $M > N+1$ and $p_s \in \mathcal{S}^{0,M}_{\beta'}$, then*

$$p_s(D_{x',\theta})\colon L^2(\overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^1) \to L^2(\overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^1).$$

Proof. A simple argument in [T1] works here as well. When $p(v, X, \gamma) = p(X, \gamma)$, $p_s(D_{x',\theta})$ is just a bounded Fourier multiplier in the tangential (x', θ) variables. Reduce the general case to this case by writing

$$\begin{aligned} p(\varepsilon V(x,\theta), X, \gamma) &= p(0, X, \gamma) + \varepsilon V b(\varepsilon V, X, \gamma) \\ &= p(0, X, \gamma) + \varepsilon c(x, \theta, X, \gamma), \end{aligned} \tag{1.14}$$

$$c(x, \theta, D_{x',\theta}, \gamma) = \int e^{ix'\xi' + im\theta} \varepsilon c^\wedge(\xi', x_N, m, D_{x',\theta}, \gamma)\, d\xi'\, dm, \tag{1.15}$$

and using

$$|c^\wedge(\xi', x_N, m, X, \gamma)| \leqslant C\langle \xi', m\rangle^{-M}, \tag{1.16}$$

where $\langle \xi', m\rangle \equiv (1 + |\xi'|^2 + m^2)^{1/2}$ and C is independent of (ε, X, γ).

The argument shows that the L^2 operator norm of $p_s(D_{x',\theta})$ satisfies

$$|p_s(D_{x',\theta})| \leqslant C_1 + \varepsilon C_2(|V|_{C^{0,M}_c}), \tag{1.17}$$

where C_1 is independent of V. ∎

Notation 1.2. (a) In Proposition 1.1 and henceforth, when we write

$$T_{\varepsilon,\gamma}\colon \mathcal{X} \to \mathcal{Y}$$

for a family of linear operators mapping one function space into another, we mean that the operator norm is uniformly bounded with respect to ε, γ for $0 < \varepsilon \leqslant 1$ and $\gamma_0 \leqslant \gamma < \infty$.

(b) We also regard as elements of $OP\mathcal{S}^{k,M}_{\beta'}$ operators defined by symbols like $p_s = p(\varepsilon V(x,\theta), \varepsilon W(x,\theta), X, \gamma)$ depending on more than one member of $C^{0,M}_c$, as long as p satisfies the obvious analogue of (1.7).

(c) We ignore powers of 2π in all formulas involving Fourier transforms.

In Sections 2–4 we present an "L^2-calculus" for these operators which includes adjoints and products and allows the proof of Garding inequalities. In Section 5 we take the Kreiss symmetrizer $R(v, X, \gamma) \in S^0$ constructed in the classical way [CP] for a Kreiss well-posed boundary problem $(D_{x_N} - \mathscr{A}(v, D_{x'}), B(v))$ and apply the calculus to show that

$$R_s(D_{x',\theta}) = R\left(\varepsilon V_\varepsilon(x,\theta), D_{x'} + \frac{\beta' D_\theta}{\varepsilon}, \gamma\right) \tag{1.18}$$

symmetrizes the singular problem

$$\left(D_{x_N} - \mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right), B(\varepsilon V_\varepsilon)\right)$$

as in (1.6).

For use in the remainder of the Introduction and later, we collect some notation here.

Notation 1.3. (a) Let $\Omega = \overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^1$, $\Omega_T = \Omega \cap \{-\infty < x_0 < T\}$, $b\Omega = \mathbb{R}^N \times \mathbb{T}^1$, $b\Omega_T = \Omega_T \cap \{x_N = 0\}$, and set $\omega_T = \overline{\mathbb{R}}^{N+1}_+ \cap \{-\infty < x_0 < T\}$.

(b) For $k \in \mathbb{N} = \{0, 1, 2, ...\}$ $H^k \equiv H^k(b\Omega)$, the standard Sobolev space with norm $\langle V(x', \theta)\rangle_k$, $H^k_\gamma = e^{\gamma x_0} H^k$ and $|V|_{H^k_\gamma} = \langle V\rangle_{k,\gamma} \equiv \langle e^{-\gamma x_0} V\rangle_k$.

(c) $L^2 H^k \equiv L^2(x_N, H^k)$ with $|U(x,\theta)|_{L^2 H^k} \equiv |U|_{0,k}$.

(d) $CH^k \equiv C(x_N, H^k)$ with $|U(x,\theta)|_{CH^k} \equiv \sup_{x_N \geqslant 0} |U(., x_N, .)|_{H^k} \equiv |U|_{\infty,k}$.

(e) Similarly, $H^k_T \equiv H^k(b\Omega_T)$ with norm $\langle V\rangle_{k,T}$ and $L^2 H^k_T \equiv L^2(x_N, H^k_T)$, $CH^k_T \equiv C(x_N, H^k_T)$ have norms $|U|_{0,k,T}$, $|U|_{\infty,k,T}$, respectively.

(f) $L^2 H^k_\gamma \equiv L^2(x_N, H^k_\gamma)$ and $CH^k_\gamma \equiv C(x_N, H^k_\gamma)$ have norms $|U(x,\theta)|_{0,k,\gamma}$, $|U|_{\infty,k,\gamma}$ respectively.

(g) $\mathbb{H}^k_T = \{U \in CH^k_T \cap L^2 H^{k+1}_T : U|_{x_N=0} \in H^{k+1}_T\}$ with the norm

$$\|U\|_{k,T} = |U|_{\infty,k,T} + |U|_{0,k+1,T} + \sqrt{T}\,\langle U\rangle_{k+1,T}.$$

(h) $L^\infty W^{1,\infty} \equiv L^\infty(x_N, W^{1,\infty}(b\Omega))$ with norm $|U|_{L^\infty W^{1,\infty}} \equiv |U|^*$. We also write $|U|_{L^\infty(\Omega)} = |U|_*$, $\langle V\rangle_{L^\infty(b\Omega)} = \langle V\rangle_*$, $\langle V\rangle_{W^{1,\infty}(b\Omega)} \equiv \langle V\rangle^*$, $|U|_{L^\infty(\Omega_T)} = |U|_*$, etc.

(i) For $k, l \in \mathbb{N}$ denote by $h: (\mathbb{R}_+)^l \to \mathbb{R}_+$ or $h_k: (\mathbb{R}_+)^l \to \mathbb{R}_+$ an increasing function of each of its arguments independent of ε, γ. C always denotes a constant independent of ε, γ. h and constants C may change from line to line or even from term to term in the text. $C(K)$ denotes a constant that depends on K.

(j) For any function U, $U^\gamma \equiv e^{-\gamma x_0} U$.

(k) For $r \geqslant 0$ $[r]$ is the smallest integer $> r$.

(l) $M_0 = 2(N+2)+1$.

(m) $\nabla^\varepsilon = \partial_{x'} + \beta' \partial_\theta / \varepsilon$, $D^\varepsilon_{x',\theta} = \frac{1}{i} \nabla^\varepsilon$, $D^\varepsilon_{x,\theta} = (D^\varepsilon_{x',\theta}, D_{x_N})$.

Using $R_s(D_{x',\theta})$ as in (1.18) we obtain a uniform Kreiss-type estimate for the singular problem (1.6),

$$|U^\gamma_\varepsilon|_{0,0} + \frac{\langle U^\gamma_\varepsilon \rangle_0}{\sqrt{\gamma}} \leqslant C \left(\frac{|F^\gamma_\varepsilon|_{0,0}}{\gamma} + \frac{\langle G^\gamma \rangle_0}{\sqrt{\gamma}} \right), \tag{1.19}$$

where C is *independent* of ε and γ and depends on the C^{0, M_0}_c norm of $\mathscr{V}_\varepsilon$.

C. *Higher Derivatives and L^∞ Control*

Next one can differentiate (1.6) tangentially ($D_{x',\theta}$) and apply (1.19) to obtain

$$|U^\gamma_\varepsilon|_{0,k} + \frac{\langle U^\gamma_\varepsilon \rangle_k}{\sqrt{\gamma}} \leqslant C \left(\frac{|F^\gamma_\varepsilon|_{0,k}}{\gamma} + \frac{\langle G^\gamma \rangle_k}{\sqrt{\gamma}} \right). \tag{1.20}$$

At this point the major obstacle posed by singular multidimensional boundary problems presents itself: *it is impossible to estimate even a single normal (D_{x_N}) derivative uniformly in ε.* Uniform control of $|D_{x_N} U_\varepsilon|_{0,k,T}$ for $k > \frac{N+1}{2}$ would imply control of $|U_\varepsilon|_{L^\infty(\Omega_T)}$. The expression for $D_{x_N} U_\varepsilon$ in terms of tangential derivatives obtained from (1.6) contains a factor of $\frac{1}{\varepsilon}$. This difficulty does not arise for initial-value problems in free space, where the natural norms are $C(x_0, H^k(x'', x_N, \theta))$ norms and it is never necessary to estimate derivatives transverse to $x_0 = 0$, or for boundary problems in 1D, where L^∞ estimates follow by integrating along characteristics. One might try for uniform estimates of U_ε using $C(x_N, H^k_T(x', \theta))$ norms, but in fact this does not work. The examples of [W3] show that if the boundary frequency β' triggers glancing modes of order $\geqslant 3$ (Definition 1.1) then $|U_\varepsilon|_{L^\infty(\Omega_T)}$ does in general blow up for U_ε in (1.6) as $\varepsilon \to 0$, and this leads to examples where the maximal time of existence $T_\varepsilon \to 0$ as $\varepsilon \to 0$ in quasilinear problems (1.2) and (1.3).

Definition 1.1. Let $\mathscr{L}(\xi', \xi_N)$ be the $m \times m$ matrix symbol of the constant coefficient operator

$$\mathscr{L}(D_x) = D_{x_N} - \mathscr{A}(0, D_{x'}) \tag{1.21}$$

and set

$$p(\xi', \xi_N) = \det \mathscr{L}(\xi', \xi_N). \tag{1.22}$$

For $\beta' \in \mathbb{R}^N \setminus 0$ if $p(\beta', \xi_N(\beta')) = 0$, we call $\beta = (\beta', \xi_N(\beta'))$ a *characteristic mode*. A characteristic mode is *hyperbolic, glancing of order k* or *elliptic* if the root $\xi_N(\beta')$ is respectively simple real, real of order $k \geqslant 2$, or nonreal. A hyperbolic mode is *incoming* (resp., *outgoing*) if $\partial\xi_N/\partial\xi_0(\beta') > 0$ (resp., < 0). The *hyperbolic region* of $\mathscr{L}(D_x)$ is $\mathscr{H} = \{\beta' \in \mathbb{R}^N \setminus 0 : p(\beta', \xi_N) = 0$ has m simple real roots $\lambda_j(\beta')$, $j = 1, ..., m\}$. The *nonglancing region* is $\mathscr{G}^c = \{\beta' \in \mathbb{R}^N \setminus 0 : p(\beta', \xi_N) = 0$ has no real roots of order $\geqslant 2\}$. Finally, we set $\mathscr{G} = (\mathbb{R}^N \setminus 0) \setminus \mathscr{G}^c$, the *glancing region*.

In [W1] we studied the constant coefficient semilinear case (replace $A_j(\varepsilon U_\varepsilon)$ by $A_j(0)$ in (1.3)) under the assumption that any glancing modes associated to β' were of order $\leqslant 2$. There we controlled $|U_\varepsilon|_{L^\infty(\Omega_T)}$ by introducing $\mathscr{F}_{\infty, s, \gamma}$ spaces and directly estimating solutions constructed by Fourier–Laplace transforms in the associated norms

$$U \in \mathscr{F}_{\infty, s, \gamma} \Leftrightarrow ||U\hat{\ }(\xi_0' - i\gamma, \xi'', x_N, m)|_{L^\infty(x_N)} \langle \xi', m, \gamma \rangle^s|_{L^2(\xi', m)} < \infty. \tag{1.23}$$

That method does not work in the quasilinear case, even for $\beta' \in \mathscr{G}^c$, since (1.6) has variable coefficients.

Remark 1.1. The blow-up examples of [W3] show that the restriction to glancing modes of order $\leqslant 2$ cannot be relaxed without losing uniform $|U_\varepsilon|_{L^\infty(\Omega_T)}$ estimates.

The strategy of this paper for handling the quasilinear case begins with the observation that for noncharacteristic boundaries one can use Eq. (1.6) to estimate D_{x_N} derivatives in terms of F_ε and tangential $D_{x'}$ derivatives *alone*, microlocally in regions of $(\xi', \frac{m}{\varepsilon}, \gamma)$-space where

$$|\xi', \gamma| > \delta \left| \frac{m\beta'}{\varepsilon} \right|, \qquad \text{for some} \quad \delta > 0. \tag{1.24}$$

This leads us to split the problem into two pieces using pseudodifferential cutoffs $\chi_\varepsilon = \chi(D_{x'}, \beta' D_\theta/\varepsilon, \gamma)$ such that $\chi \equiv 1$ on the region

$$\begin{aligned} &|\xi', \gamma| \leqslant \delta \left| \frac{m\beta'}{\varepsilon} \right|, \qquad \text{and} \\ &\operatorname{supp} \chi\left(\xi', \frac{m\beta'}{\varepsilon}, \gamma\right) \subset \left\{ |\xi', \gamma| \leqslant \delta' \left| \frac{m\beta'}{\varepsilon} \right| \right\}, \end{aligned} \tag{1.25}$$

for $\delta' > \delta$ to be chosen (1.31). The action of the Fourier multiplier $\chi(D_{x'}, \beta' D_\theta/\varepsilon, \gamma)$ is defined by an integral just like (1.11). For the relatively easy piece $(1-\chi_\varepsilon) U_\varepsilon$ we use a 1-D Sobolev estimate and the above observation to obtain

(1.26)
$$\begin{aligned} |(1-\chi_\varepsilon) U_\varepsilon^\gamma|_{\infty,0} &\leqslant C\,|(1-\chi_\varepsilon) D_{x_N} U_\varepsilon^\gamma|_{0,0} \\ &\leqslant C\,|U_\varepsilon^\gamma|_{0,1} + |F_\varepsilon^\gamma|_{0,0}, \end{aligned}$$

for $\gamma > \gamma_0$, where C, γ_0 are independent of ε.

Remark 1.2. χ_ε is nonlocal, so it is more convenient to work on Ω than Ω_T here.

To deal with the hard piece $\chi_\varepsilon U_\varepsilon$ we assume

(1.27) β' lies in the nonglancing region, $\mathscr{G}^c$, of $\mathscr{L}(D_x)$.

Suppose for example that β' lies in the hyperbolic region $\mathscr{H}$ and that $V_\varepsilon(x, \theta)$ in (1.6) satisfies

(1.28)
$$|V_\varepsilon(x, \theta)|_{L^\infty(\Omega)} \leqslant C$$

uniformly in ε. Set

(1.29)
$$\xi'_\gamma = (\xi_0 - i\gamma, \xi''), \qquad D_{x'_\gamma} = (D_{x_0} - i\gamma, D_{x''}).$$

Then for ε small the symbol $\mathscr{A}(\varepsilon V_\varepsilon(x, \theta), \xi'_\gamma + m\beta'/\varepsilon)$ in (1.6) has m simple eigenvalues and associated eigenvectors

(1.30)
$$\lambda_j\left(\varepsilon V_\varepsilon(x, \theta), \xi'_\gamma + \frac{m\beta'}{\varepsilon}\right), \qquad r_j\left(\varepsilon V_\varepsilon(x, \theta), \xi'_\gamma + \frac{m\beta'}{\varepsilon}\right), \qquad j = 1, ..., m$$

microlocally in the region

(1.31)
$$|\xi', \gamma| \leqslant \delta' \left|\frac{m\beta'}{\varepsilon}\right|$$

for δ' small enough.

This suggests trying to diagonalize (1.6) by conjugation with S, $S^{-1}(D_{x_N} - \mathscr{A}) S$, where $S(\varepsilon V_\varepsilon(x, \theta), D_{x'_\gamma} + \beta' D_\theta/\varepsilon) \in OP\mathscr{S}^{0, M}_{\beta'}$ has an $m \times m$ matrix symbol whose columns are the $r_j(\varepsilon V_\varepsilon, \xi'_\gamma + m\beta'/\varepsilon)$. Use of the L^2-calculus shows that the conjugation introduces errors uniformly bounded with respect to ε in $C(x_N, L^2(x', \theta))$. We thereby reduce to considering m *scalar* pseudodifferential equations of the form

(1.32)
$$\left(D_{x_N} - \lambda_j\left(\varepsilon V_\varepsilon, D_{x'_\gamma} + \frac{\beta' D_\theta}{\varepsilon}\right)\right) W_{j,\gamma} = \mathscr{F}_{j,\gamma},$$

with appropriate boundary conditions, where $\mathscr{F}_{j,\gamma}$ depends on $(F_\varepsilon, U_\varepsilon)$. An integration by parts argument yields

$$(1.33)\qquad |W_{j,\gamma}|_{\infty,0} \leqslant C\left[\frac{|F^\gamma|^2_{\infty,0}}{\gamma} + |U^\gamma|^2_{0,0} + \langle G^\gamma\rangle_0^2 + \frac{\langle U^\gamma(0)\rangle_0^2}{\gamma^2}\right].$$

Combined with $(1.20)_1$ and $(1.26)_0$, this gives for γ large

$$(1.34)\qquad |U^\gamma_\varepsilon|_{\infty,0} + |U^\gamma_\varepsilon|_{0,1} + \frac{\langle U^\gamma_\varepsilon\rangle_1}{\sqrt{\gamma}} \leqslant C\left[\left(\frac{|F^\gamma_\varepsilon|_{\infty,0}}{\sqrt{\gamma}} + \langle G^\gamma\rangle_0\right) + \left(\frac{|F^\gamma_\varepsilon|_{0,1}}{\gamma} + \frac{\langle G^\gamma\rangle_1}{\sqrt{\gamma}}\right)\right].$$

Differentiating tangentially, applying Moser estimates to nonlinear functions and commutators, and restricting to finite time intervals, we obtain the *main linear estimate* (Corollary 7.2):

$$(1.35)\qquad |U_\varepsilon|_{\infty,k,T} + |U_\varepsilon|_{0,k+1,T} + \frac{\langle U_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}} \leqslant e^{\gamma T} C(K,E)\left[\left(\frac{|F_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}} + \frac{|F_\varepsilon|_{0,k+1,T}}{\gamma} + \frac{\langle G\rangle_{k+1,T}}{\sqrt{\gamma}}\right)\right].$$

where γ and C are independent of ε.

This shows that we can estimate the iterates U^n_ε in the $\mathbb{H}^k_T$ norm, $\|U^n_\varepsilon\|_{k,T}$, without loss of derivatives for $k \geqslant [M_0 + \frac{N+1}{2}]$:

$$(1.36)\qquad \|U^n_\varepsilon\|_{k,T} \equiv |U^n_\varepsilon|_{0,k+1,T} + |U^n_\varepsilon|_{\infty,k,T} + \sqrt{T}\,\langle U^n_\varepsilon\rangle_{k+1,T}.$$

After proving uniform boundedness of the iterates in the $\mathbb{H}^k_T$ norm, we contract in $\mathbb{H}^{k-1}_T$ for small enough T to obtain convergence to the exact solution U_ε.

Remark 1.3. It is an open question whether or not the maximal time of existence $T_\varepsilon \to 0$ as $\varepsilon \to 0$ when the boundary frequency β' in the quasilinear problem (1.4) lies in the glancing region $\mathscr{G}$, but is such that $p(\beta', \xi_N) = 0$ has real roots only of order $\leqslant 2$. There are competing effects that make the behavior of the solution as $\varepsilon \to 0$ hard to predict.

When ε is small, (1.4) is close to a constant coefficient semilinear problem for which double real roots do not cause blow-up [W1]. (Recall that real roots of order $\geqslant 3$ do cause blow-up in such problems [W3].) However, for $\varepsilon > 0$ fixed the operator $\mathscr{L}(\varepsilon U_\varepsilon, D^\varepsilon_{x,\theta})$ has variable coefficients and thus may, for example, have gliding rays [MS, H]. We have constructed Friedlander-type, variable coefficient, semilinear examples in which $T_\varepsilon \to 0$ as $\varepsilon \to 0$ when β' as above triggers gliding rays. Here the solution operator is a Fourier multiplier defined by a quotient of Airy functions, and blow-up occurs for β' corresponding to zeros of the denominator. Unlike (1.4) though, these examples do not approach constant coefficient problems as $\varepsilon \to 0$.

We hope to resolve the question of double real roots in the quasilinear case by constructing solutions to the linearized problem (1.6) using singular Fourier integral operators with complex phases.

D. *Initial-Value Problems*

In Section 8 we extend results of [JMR1, JMR2, S] for quasilinear symmetric hyperbolic systems in free space with oscillatory initial data to nonsymmetric hyperbolic systems for which one can construct pseudodifferential symmetrizers. This class includes nonsymmetric strictly hyperbolic systems, for example. Again, the problem is to demonstrate a time of existence independent of the wavelength ε as $\varepsilon \to 0$. Estimates in $C(x_0, L^2(x'', x_N, \theta))$ norms for the corresponding linearized singular initial-value problem (6.1) are obtained using SPOs, which now act in the variables (x'', x_N, θ) tangent to $x_0 = 0$ (actually, in Sections 6 and 8 we relabel coordinates so that x_N denotes *time*). $C(x_0, H^k(x'', x_N, \theta))$ estimates and L^∞ control follow by differentiating the equations. Thus, obtaining L^∞ control is much easier here than in the case of boundary problems, since $C(x_0, L^2(x'', x_N, \theta))$ norms are natural for the initial-value problem.

E. *Multidimensional Shocks*

In Section 9 we start with a planar shock solution $(U^{\pm}, x_N = \sigma x_0)$ of an $m \times m$ system of conservation laws (9.1) and then perturb it with high-frequency $(\frac{1}{\varepsilon})$ plane waves that reflect transversally off the shock. Assuming the planar solutions are *uniformly stable* in the sense of Majda [M1], our aim is to show that the perturbed solution $(U^{\pm} + \varepsilon v_\varepsilon^{\pm}, x_N = \sigma x_0 + \varepsilon \phi_\varepsilon(x'))$ exists on a fixed time interval independent of the wavelength ε as $\varepsilon \to 0$. Note that the amplitude of the perturbation is the critical amplitude of weakly nonlinear geometric optics.

The problem can be formulated as a hyperbolic mixed problem with the perturbed shock as a free boundary. After flattening the boundary by a change of variables depending on the unknown $\phi_\varepsilon(x')$, one reduces to a forward mixed problem similar to (1.2). The restriction to plane waves reflecting transversally off the shock corresponds to a choice of boundary frequency β' in the nonglancing region of the hyperbolic operator obtained by linearizing at the planar shock. In contrast to [W2] here our choice of β' also allows the formation of an elliptic boundary layer on one or both sides of the shock (for elliptic boundary layers see, e.g., [W3]). The uniform stability assumption implies Kreiss well-posedness, so we are able to solve the problem using SPO symmetrizers and the strategy for quasilinear boundary problems outlined above.

We remark here that the right norm for iteration is the shock analogue of the $\mathbb{H}^k_T$ norm, namely:

$$(1.37)\quad |\|U_n, \phi_n\||_{k,T} = |U_n|_{\infty,k,T} + |U_n|_{0,k+1,T} + \sqrt{T}\left\langle U_n, \frac{\phi_n}{T}, \nabla^\varepsilon \phi_n \right\rangle_{k+1,T} + \frac{\langle \nabla^\varepsilon \phi_n \rangle_{k,T}}{\sqrt{T}}.$$

2. *SPO Calculus: Adjoints, Products, Mollifiers*

In order to study adjoints and products of elements of $OP\mathscr{S}^{k,M}_{\beta'}$ (1.10), it is helpful to define classes of amplitudes

$$\text{(a)}\quad T^k = \{a(v, w, X, \gamma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^N \times [\gamma_0, \infty)) : |\partial^\alpha_{(v,w)} \partial^\beta_X a(v, w, X, \gamma)| \leqslant C_{\alpha,\beta} \langle X, \gamma\rangle^{k-|\beta|}\}$$

$$(2.1)\qquad \text{(b)}\quad \mathscr{T}^{k,M}_{\beta'} = \left\{ a_s = a\left(\varepsilon V(x,\theta), \varepsilon W(y,\omega), \xi' + \frac{m\beta'}{\varepsilon}, \gamma \right) : a \in T^k, V \in C^{0,M}_c, W \in C^{0,M}_c, \text{ and } y_N = x_N \right\}$$

and associated operators

$$\tilde{a}_s \in OP\mathscr{T}^{k,M}_{\beta'}$$

whose action on $U(x, \theta)$ is defined (formally, at first) by

$$(2.2)\qquad \tilde{a}_s(U)(x,\theta) \equiv \int e^{i(x'-y')\xi' + i(\theta-\omega)m} a\left(\varepsilon V(x,\theta), \varepsilon W(y,\omega), \xi' + \frac{m\beta'}{\varepsilon}, \gamma \right) \times U(y,\omega)\, dy'\, d\omega\, d\xi'\, dm.$$

For example, if W and $U \in C^{0,M}_c$ for M large enough, (2.2) makes sense if the $dy'\, d\omega$ integral is done first. As always the dm integral is a sum over $m \in \mathbb{Z}$.

Notation 2.1. We will often write $\xi' + m\beta/\varepsilon = X$ and $\eta' + m'\beta'/\varepsilon = Y$, where (η', m') are variables dual to (y, ω).

In order to clarify the mapping properties of SPOs and define residual operators we introduce the following scale of spaces.

DEFINITION 2.1. Let $\Lambda_{\varepsilon,\gamma}$ ($=\Lambda$, for short) $\equiv p_s(D_{x',\theta})$, where

$$p_s = \langle X, \gamma\rangle,$$

and $\varepsilon \leqslant 1$, $\gamma \geqslant 1$. For $r \in \mathbb{R}$ we set

$$\begin{aligned}
&\text{(a)} \quad \mathcal{H}^{0,r} = \Lambda^{-r}L^2(\overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^1)\\
&\qquad\quad = \{U_{\varepsilon,\gamma}(x,\theta) : |U_{\varepsilon,\gamma}|_{\mathcal{H}^{0,r}} \equiv |\Lambda^r U_{\varepsilon,\gamma}|_{L^2(x_N, L^2(x',\theta))} < \infty\}.\\
&\text{(b)} \quad \mathcal{H}^{1,r} = \{U_{\varepsilon,\gamma}(x,\theta) \in \mathcal{H}^{0,r} : D_{x_N} U_{\varepsilon,\gamma} \in \mathcal{H}^{0,r-1}\}.\\
&\text{(c)} \quad \mathcal{H}^{r} = \Lambda^{-r}L^2(\mathbb{R}^N \times \mathbb{T}^1)\\
&\qquad\quad = \{V_{\varepsilon,\gamma}(x',\theta) : |V_{\varepsilon,\gamma}|_{\mathcal{H}^{r}} \equiv |\Lambda^r V_{\varepsilon,\gamma}|_{L^2(x',\theta)} < \infty\}.
\end{aligned} \tag{2.3}$$

Remark 2.1. (a) For all $t \in \mathbb{R}$, $r \in \mathbb{R}$ Λ^t: $\mathcal{H}^{0,r} \to \mathcal{H}^{0,r-t}$, and in fact

$$|\Lambda^t U_{\varepsilon,\gamma}|_{\mathcal{H}^{0,r-t}} = |U_{\varepsilon,\gamma}|_{\mathcal{H}^{0,r}}.$$

(b) $\mathcal{H}^{0,0} = L^2(x_N, L^2(x',\theta))$, but for $r \neq 0$ $\mathcal{H}^{0,r}$ has only a limited resemblance to the standard Sobolev space $L^2(x_N, H^r(x',\theta))$. Suppose $U_{\varepsilon,\gamma}(x', x_N, \theta) \in \mathcal{H}^{0,r}$ is continuous in x_N. Then for x_N fixed, $U_{\varepsilon,\gamma}$ is H^r microlocally away from $\xi' + m\beta'/\varepsilon = 0$ in (ξ', m)-space, but the H^r regularity is not uniform in ε. Moreover, even if $r \gg 0$, $U_{\varepsilon,\gamma}$ is only microlocally L^2 near $\xi' + m\beta'/\varepsilon = 0$.

PROPOSITION 2.1. *If $M > N+1$ and $p_s \in \mathcal{S}^{k,M}_{\beta'}$ for some $k \in \mathbb{R}$, then*

$$p_s(D_{x',\theta})\colon \mathcal{H}^{0,r} \to \mathcal{H}^{0,r-k} \qquad \textit{for all } r.$$

Proof. The result is clear for Fourier multipliers of order k. Reduce to that case just as in Proposition 1.1. ▮

Since $\partial^\beta_X a(\varepsilon V, \varepsilon W, \xi' + \frac{m\beta}{\varepsilon}, \gamma)$ fails to decay in $|\xi', m|$, we can not hope for residual operators that are smoothing in the usual sense. Instead we have

DEFINITION 2.2. Let $k \geqslant 0$. A linear operator $r_{\varepsilon,\gamma}$ is called *residual of order k* if

$$r_{\varepsilon,\gamma}\colon \mathcal{H}^{0,0} \to \mathcal{H}^{0,0}$$

with operator norm $\leqslant C\gamma^{-k}$, for C independent of ε, γ.

Parallel to Proposition 1.1 we have

PROPOSITION 2.2. *If* $M > N+1$ *and*

$$a_s = a\left(\varepsilon V(x,\theta), \varepsilon W(y,\omega), \xi' + \frac{m\beta'}{\varepsilon}, \gamma\right) \in \mathcal{T}^{0,M}_{\beta'},$$

then

$$\tilde{a}_s \colon \mathcal{H}^{0,0} \to \mathcal{H}^{0,0}.$$

Proof. **1.** First assume $a(v, w, X, \gamma) = a(w, X, \gamma)$, write

$$\begin{aligned}(2.4)\qquad a(\varepsilon V, \varepsilon W, X, \gamma) &= a(\varepsilon W, X, \gamma) = a(0, X, \gamma) + \varepsilon W b(\varepsilon W, X, \gamma)\\ &= a(0, X, \gamma) + \varepsilon c(y, \omega, X, \gamma),\end{aligned}$$

and so reduce to considering

$$(2.5)\qquad \tilde{c}_s(U)(x,\theta) \equiv \int e^{i(x'-y')\xi' + i(\theta-\omega)m} c(y,\omega,X,\gamma)\, U(y,\omega)\, dy'\, d\omega\, d\xi'\, dm,$$

where $X = \xi' + m\beta'/\varepsilon$. c has compact support in y' and $\omega \in \mathbb{T}^1$, so after integrating $dy' d\omega$ and using the Plancherel theorem, we find

(2.6)
$$|\tilde{c}_s(U)|_{L^2(x,\theta)} = \left| \int c\hat{\ }(\xi'-\eta', x_N, m-m', X, \gamma)\, U\hat{\ }(\eta', x_N, m')\, d\eta'\, dm' \right|_{L^2(\xi', x_N, m)}.$$

Now

$$(2.7)\qquad |c\hat{\ }(\xi'-\eta', x_N, m-m', X, \gamma)| \leqslant C\langle \xi'-\eta', m-m'\rangle^{-M},$$

so Young's inequality gives (2.6) $\leqslant C\, |U|_{L^2}$ since $M > N+1$.

2. Reduce the general case to the case just treated by writing

$$(2.8)\qquad a(\varepsilon V, \varepsilon W, X, \gamma) = a(0, \varepsilon W, X, \gamma) + \varepsilon c(x, \theta, \varepsilon W, X, \gamma)$$

and using

$$|c\hat{\ }(\xi', x_N, m, \varepsilon W, X, \gamma)| \leqslant C\langle \xi', m\rangle^{-M}$$

as in the proof of Proposition 1.1. ∎

PROPOSITION 2.3. *Suppose $M > N+2$ and*

$$a_s = a(\varepsilon V(x,\theta), \varepsilon W(y,\omega), X, \gamma) \in \mathscr{T}^{k,M}_{\beta'}$$

for some $k \in [0,1]$. Then

(2.9) $$\tilde{a}_s = p_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $p_s = a(\varepsilon V(x,\theta), \varepsilon W(x,\theta), X, \gamma) \in \mathscr{S}^{k,M}_{\beta'}$ and $r_{\varepsilon,\gamma}$ is residual of order $1-k$ (Definition 2.2).

Proof. **1.** Write

(2.10) $$a(\varepsilon V(x,\theta), \varepsilon W(y,\omega), X, \gamma) = a(\varepsilon V(x,\theta), 0, X, \gamma) + \varepsilon W(y,\omega)\, a_1(\varepsilon V, \varepsilon W, X, \gamma),$$

and thereby reduce to considering operators defined by symbols

$$c_s = \varepsilon c(x,\theta,y,\omega,X,\gamma)$$

(convenient abuse of notation here) for which

(2.11) there exists a compact set $K \subset \overline{\mathbb{R}}^{N+1}_+$ such that $c_s \equiv 0$ for $y \in K^c$.

2. Choose $\phi \in C^\infty$ with proper support in $\overline{\mathbb{R}}^{N+1}_+ \times \overline{\mathbb{R}}^{N+1}_+$ and $\phi \equiv 1$ on $|x-y| \leqslant 1$. Consider first

(2.12) $$\varepsilon c_{1s} = \varepsilon\phi(x,y)\, c(x,\theta,y,\omega,X,\gamma),$$

which has compact support in (x, y). With $\tilde{c}_{1s}$ the SPO defined by c_{1s}, as usual set

(2.13) $$p_{1s}(x,\theta,\xi',m,\varepsilon,\gamma) = e^{-ix'\xi' - i\theta m}\varepsilon\tilde{c}_{1s}(e^{ix'\xi' + i\theta m}),$$

let

$$b(x,\theta,y,\omega,X,\gamma) = \varepsilon c_{1s}(x,\theta,x+y,\theta+\omega,X,\gamma)$$

and note that

(2.14) $$p_{1s} = \int b\hat{\ }(x,\theta,\eta',m',X+Y,\gamma)\, d\eta'\, dm'$$

(recall Notation 2.1). Since $W \in C^{0,M}_c$ we have for $|\alpha| = 1$

(2.15) $$|\partial^\alpha_X b\hat{\ }(x,\theta,\eta',m',X,\gamma)| \leqslant C\varepsilon\langle\eta',m'\rangle^{-M}\langle X,\gamma\rangle^{k-1}.$$

Taking a first-order Taylor expansion of $b^\wedge$ in (2.14) gives

(2.16)

$$b^\wedge(x,\theta,\eta',m',X+Y,\gamma)$$
$$=b^\wedge(x,\theta,\eta',m',X,\gamma)+\sum_{|\alpha|=1}\int_0^1\frac{\partial^\alpha b^\wedge}{\partial X^\alpha}(x,\theta,\eta',m',X+tY,\gamma)\,Y^\alpha\,dt$$
$$=b^\wedge+\int_0^1 b_1^\wedge\,dt.$$

Observe that

$$\int b^\wedge(x,\theta,\eta',m',X,\gamma)\,d\eta'\,dm'=\varepsilon c_{1s}(x,\theta,x,\theta,X,\gamma).$$

3. We claim that $\int b_1^\wedge\,dt$ in (2.16) corresponds to a residual operator of order $1-k$. Indeed,

$$|Y^\alpha|\leqslant C\frac{\langle\eta',m'\rangle}{\varepsilon},$$

so (2.15) implies

$$\begin{aligned}|b_1^\wedge|&\leqslant C\langle\eta',m'\rangle^{-(M-1)}\langle X+tY,\gamma\rangle^{k-1}\\&\leqslant C\langle\eta',m'\rangle^{-(M-1)}\gamma^{k-1}.\end{aligned}\tag{2.17}$$

If $b_1^\wedge$ had no (x,θ) dependence, the Fourier multiplier given by

$$\int b_1^\wedge\,d\eta'\,dm'$$

would be residual of order $1-k$, since $M-1>N+1$ in (2.17). Reduce to that case, using the fact that b_1 has compact support and is C^M in (x,θ), by arguing just as in Proposition 1.1.

4. It remains to consider

$$\varepsilon c_{2s}=\varepsilon(1-\phi(x,y))\,c(x,\theta,y,\omega,X,\gamma).\tag{2.18}$$

We claim

$$\tilde{c}_{2s}\text{ is residual of order } j,\text{ for all } j>0.\tag{2.19}$$

For $j\in\mathbb{N}$, $j>>k$, write

$$\tilde{c}_{2s}(U)=\int e^{i(x'-y')\xi'+i(\theta-\omega)m}\frac{1-\phi(x,y)}{|x'-y'|^{2j}}(\triangle_{\xi'})^j\,c(x,\theta,y,\omega,X,\gamma)\times U(y,\omega)\,dy'\,d\omega\,d\xi'\,dm.\tag{2.20}$$

Note that

(2.21) $$|(\triangle_{\xi'})^j c| \leqslant C < X, \gamma >^{k-2j}.$$

Suppose first that

(2.22) $$\frac{1-\phi(x, y)}{|x'-y'|^{2j}} (\triangle_{\xi'})^j c(x, \theta, y, \omega, X, \gamma)$$

in (2.20) is replaced by a function $d(y, \omega, X, \gamma)$ satisfying

(2.23) $$|d\hat{}(\eta', y_N, m', X, \gamma| \leqslant C\langle \eta', m' \rangle^{-M} \gamma^{k-2j}.$$

Then part **1** in the proof of Proposition 2.2 gives (2.19). Reduce to this case by taking the Fourier transform in (x', θ) of (2.22), using (2.11), and superposing operators via the inverse Fourier transform as in the proof of Proposition 1.1. ∎

Proposition 2.3 yields results on adjoints and products as corollaries.

Notation 2.2. $p_s(D_{x',\theta})^*$ denotes the adjoint of the operator $p_s(D_{x',\theta})$, while $p_s^*(D_{x',\theta})$ denotes the operator associated to the matrix symbol $p^*(\varepsilon V, X, \gamma)$.

PROPOSITION 2.4 (Adjoints). *Suppose $M > N+2$ and*

$$p_s = p(\varepsilon V(x, \theta), X, \gamma) \in \mathscr{S}^{k, M}_{\beta'}$$

for some $k \in [0, 1]$. Then

(2.24) $$p_s(D_{x',\theta})^* = p_s^*(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $r_{\varepsilon,\gamma}$ is residual of order $1-k$.

Proof. $p_s(D_{x',\theta})^* = \tilde{a}_s$, where $a_s = p^*(\varepsilon V(y, \omega), X, \gamma) \in \mathscr{T}^{k, M}_{\beta'}$, so the conclusion follows immediately from Proposition 2.3. ∎

Observe that if $\tilde{a}_s$ is defined by the amplitude $a(\varepsilon V(x, \theta), \varepsilon W(y, \omega), X, \gamma)$, then $\tilde{a}_s^*$ is defined by $a^*(\varepsilon V(y, \omega), \varepsilon W(x, \theta), X, \gamma)$.

COROLLARY 2.1. *Let $r_{\varepsilon,\gamma}$ denote the residual operator of order $1-k$ appearing in (2.9) or (2.13). Then $r^*_{\varepsilon,\gamma}$ is also residual of order $1-k$.*

Proof. This follows immediately from Propositions 2.3 and 2.4 and the preceding observation. ∎

Proposition 2.5 (Products, I). *Suppose $M > N+2$ and*

$$p_s = p(\varepsilon V(x,\theta), X, \gamma) \in \mathscr{S}^{k_1, M}_{\beta'}, \qquad q_s = q(\varepsilon W(x,\theta), X, \gamma) \in \mathscr{S}^{k_2, M}_{\beta'}$$

for $k_1, k_2 \in \mathbb{R}$ such that $0 \leqslant k_1 + k_2 \leqslant 1$. Set

$$t_s = p(\varepsilon V(x,\theta), X, \gamma)\, q^*(\varepsilon W(x,\theta), X, \gamma).$$

Then

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta})^* = t_s(D_{x',\theta}) + r_{\varepsilon,\gamma}, \tag{2.25}$$

where $r_{\varepsilon,\gamma}$ is residual of order $1-(k_1+k_2)$.

Proof. $q_s(D_{x',\theta})^* = \tilde{a}_s$, where $a_s = q^*(\varepsilon W(y,\omega), X, \gamma)$. Thus

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta})^* = \tilde{b}_s,$$

where

$$b_s = p(\varepsilon V(x,\theta), X, \gamma)\, q^*(\varepsilon W(y,\omega), X, \gamma) \in \mathscr{T}^{k_1+k_2, M}_{\beta'},$$

and Proposition 2.3 gives (2.25). ∎

Proposition 2.6 (Products, II). *Suppose $M > N+2$ and*

$$p_s = p(\varepsilon V(x,\theta), X, \gamma) \in \mathscr{S}^{k_1, M}_{\beta'}, \qquad q_s = q(\varepsilon W(x,\theta), X, \gamma) \in \mathscr{S}^{k_2, M}_{\beta'}$$

for $k_1 \leqslant 0$, $0 \leqslant k_1 + k_2 \leqslant 1$. Set

$$t_s = p(\varepsilon V(x,\theta), X, \gamma)\, q(\varepsilon W(x,\theta), X, \gamma).$$

Then

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta}) = t_s(D_{x',\theta}) + r_{\varepsilon,\gamma}, \tag{2.26}$$

where $r_{\varepsilon,\gamma}$ is residual of order $1-(k_1+k_2)$.

Proof. Using Proposition 2.4 and its proof and Corollary 2.1, write

$$\begin{aligned} q_s(D_{x',\theta}) &= q_s(D_{x',\theta})^{**} = q_s^*(D_{x',\theta})^* + r_{\varepsilon,\gamma} \\ &= \tilde{Q}_s + r_{\varepsilon,\gamma}, \end{aligned} \tag{2.27}$$

where $Q_s = q(\varepsilon W(y,\omega), X, \gamma)$ and $r_{\varepsilon,\gamma}$ is residual of order $1-k_2$. Set $T_s = p(\varepsilon V(x,\theta), X, \gamma)\, q(\varepsilon W(y,\omega), X, \gamma) \in \mathscr{T}^{k_2, M}_{\beta'}$, and note that

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta}) = \tilde{T}_s + p_s(D_{x',\theta})\, r_{\varepsilon,\gamma}, \tag{2.28}$$

where $p_s(D_{x',\theta})\, r_{\varepsilon,\gamma}$ is residual of order $1-k_1-k_2$ since

$$p_s(D_{x',\theta}) \colon \mathscr{H}^{0,k_1} \to \mathscr{H}^{0,0}.$$

Finally, apply Proposition 2.3 to get (2.26). ∎

Remark 2.2. If $k_1>0$ and $0 \leqslant k_1+k_2 \leqslant 1$, the error term $p_s(D_{x',\theta})\, r_{\varepsilon,\gamma}$ in (2.28) is not obviously residual or even bounded on L^2. We use a different argument in the following proposition.

PROPOSITION 2.7 (Products, III). *Suppose* $M_1 \geqslant N+2$, $M_2 \geqslant 2(N+2)+1$, $p_s = p(\varepsilon V(x,\theta), X, \gamma) \in \mathscr{S}^{k_1, M_1}_{\beta'}$, *and* $q_s = q(\varepsilon W(x,\theta), X, \gamma) \in \mathscr{S}^{k_2, M_2}_{\beta'}$ *for* $0 \leqslant k_1 \leqslant 1$, $k_2 = 0$. *Set*

$$t_s = t(\varepsilon V(x,\theta), \varepsilon W(x,\theta), X, \gamma) = p(\varepsilon V, X, \gamma)\, q(\varepsilon W, X, \gamma).$$

Then

$$A \equiv p_s(D_{x',\theta})\, q_s(D_{x',\theta}) = t_s(D_{x',\theta}) + r_{\varepsilon,\gamma}, \tag{2.29}$$

where $r_{\varepsilon,\gamma}$ *is residual of order* $1-k_1$.

Proof. **1.** As usual there is nothing to prove if $q(\varepsilon W, X, \gamma) = q(X,\gamma)$, so reduce to considering

$$p_s(D_{x',\theta})\, \varepsilon c_s(D_{x',\theta}),$$

where $c_s = c(x,\theta,X,\gamma)$ has compact support in x and is given by

$$\begin{aligned} q(\varepsilon W, X, \gamma) &= q(0,X,\gamma) + \varepsilon W b(\varepsilon W, X, \gamma) \\ &= q(0,X,\gamma) + \varepsilon c(x,\theta,X,\gamma). \end{aligned} \tag{2.30}$$

2. Write

$$AU(x,\theta) = \int e^{ix'\xi' + i\theta m} d_\varepsilon(x,\theta,X,\gamma)\, U^\wedge(\xi', x_N, m)\, d\xi'\, dm, \tag{2.31}$$

where, with $X = \xi' + \frac{m\beta'}{\varepsilon}$, $Y = \eta' + \frac{m'\beta'}{\varepsilon}$

(2.32)

$$\begin{aligned} &d_\varepsilon(x,\theta,X,\gamma) \\ &\quad = \int e^{i(x'-y')\eta' + i(\theta-\omega)m'} p(\varepsilon V(x,\theta), X+Y, \gamma)\, \varepsilon c(y,\omega,X,\gamma)\, dy'\, d\omega\, d\eta'\, dm'. \end{aligned}$$

Expand $p(\varepsilon V, X+Y, \gamma)$ about X to obtain

$$d_\varepsilon = p(\varepsilon V(x,\theta)\, X, \gamma)\, \varepsilon c(x,\theta,X,\gamma) + R_\varepsilon,$$

where

$$R_\varepsilon(x,\theta,X,\gamma)=\sum_{|\alpha|=1}\iint_0^1 e^{ix'\eta'+i\theta m'}\frac{\partial^\alpha p}{\partial X^\alpha}(\varepsilon V(x,\theta),X+tY,\gamma)\times Y^\alpha \varepsilon c\hat{\ }(\eta',x_N,m',X,\gamma)\,dt\,d\eta'\,dm'. \tag{2.33}$$

3. It remains to show that R_ε defines a residual operator of order $1-k_1$. The modulus of the integrand in (2.33) is

$$\leqslant C\langle X+tY,\gamma\rangle^{k_1-1}\frac{\langle\eta',m'\rangle}{\varepsilon}\cdot\varepsilon\cdot\langle\eta',m'\rangle^{-M_2} \leqslant C\langle\eta',m'\rangle^{-(M_2-1)}\,\gamma^{k_1-1}. \tag{2.34}$$

Write

$$\frac{\partial^\alpha p}{\partial X^\alpha}(\varepsilon V,X+tY,\gamma)=\frac{\partial^\alpha p}{\partial X^\alpha}(0,X+tY,\gamma)+\varepsilon h(x,\theta,X+tY,\gamma), \tag{2.35}$$

where h has compact support and regularity of order M_1 in (x,θ). Corresponding to (2.35)

$$R_\varepsilon=R_{\varepsilon,1}(x,\theta,X,\gamma)+R_{\varepsilon,2}(x,\theta,X,\gamma). \tag{2.36}$$

As in (2.34) we have

$$|R\hat{\ }_{\varepsilon,1}(\eta',x_N,m',X,\gamma)|\leqslant C\langle\eta',m'\rangle^{-(M_2-1)}\,\gamma^{k_1-1}, \tag{2.37}$$

so $R_{\varepsilon,1,s}(D_{x',\theta})$ is residual of order $1-k_1$. We claim that $R_{\varepsilon,2}(x,\theta,X,\gamma)$ has compact support and regularity of order $N+2$ in (x,θ). This follows by differentiating under the integral sign since

$$|h(x,\theta,X+tY,\gamma)\,Y^\alpha\varepsilon c\hat{\ }(\eta',x_N,m',X,\gamma)|\leqslant C\langle\eta',m'\rangle^{-(M_2-1)}\,\gamma^{k_1-1},$$

with $M_1\geqslant N+2$ and $(M_2-1)-(N+2)\geqslant N+2$. This implies

$$|R\hat{\ }_{\varepsilon,2}(\eta',x_N,m',X,\gamma)|\leqslant C\langle\eta',m'\rangle^{-(N+2)}\,\gamma^{k_1-1}, \tag{2.38}$$

so $R_{\varepsilon,2,s}(D_{x',\theta})$ is also residual of order $1-k_1$. ∎

Remark 2.3. This argument fails when $k_2>0$. Suppose, for example, that $0<k_2\leqslant 1-k_1$. When $|Y|<\rho\,|X|$ for some $\rho<1$, the contribution $\langle X+tY,\gamma\rangle^{k_1-1}$ from $\partial^\alpha p/\partial X^\alpha$ in (2.33) controls the growth $\langle X,\gamma\rangle^{k_2}$ of $c\hat{\ }$. In spite of the good decay in $\langle\eta',m'\rangle$, we do not see how to control $c\hat{\ }$ uniformly in ε when $|Y|\geqslant\rho\,|X|$.

Mollifiers.

PROPOSITION 2.8. *For $\delta > 0$ let*

$$\Lambda_\delta = 1 + \delta\Lambda,$$

where $\Lambda = t_s(D_{x',\theta})$ for $t_s = \langle X, \gamma\rangle$. Suppose $U(x,\theta) \in \mathscr{H}^{0,0}$, $M > N+2$. Then

(a) $\Lambda_\delta^{-1} U \in \mathscr{H}^{0,1}$ *and* $\Lambda_\delta^{-1} U \to U$ *in* $\mathscr{H}^{0,0}$ *as* $\delta \to 0$.

(b) *For* $p_s(D_{x',\theta}) \in OP\mathscr{S}^{1,M}_{\beta'}$, $[\Lambda_\delta^{-1}, p_s(D_{x',\theta})]$ *is bounded on* $\mathscr{H}^{0,0}$ *uniformly with respect to* ε, γ, δ.

Proof. **1.** (a) is immediate, since

$$(\Lambda_\delta^{-1} U)^\wedge\, (\xi', x_N, m) = (1 + \delta\langle X, \gamma\rangle)^{-1}\, U^\wedge(\xi', x_N, m).$$

2. We have $p_s(D_{x',\theta})\, \Lambda_\delta^{-1} = q_s(D_{x',\theta})$, where

$$q_s = p(\varepsilon V, X, \gamma)(1 + \delta\langle X, \gamma\rangle)^{-1}.$$

Proposition 2.6 implies

$$\Lambda_\delta^{-1} p_s(D_{x',\theta}) = q_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $r_{\varepsilon,\gamma}$ is residual of order 0 uniformly with respect to δ since

$$\{(1 + \delta\langle X, \gamma, \rangle)^{-1}\}_{\delta > 0}$$

is a bounded subset of S^0. This gives (b). ∎

Extended calculus. It is now easy to extend the SPO calculus to a larger class of symbols that includes pseudodifferential cutoffs $\chi_\varepsilon = \chi(D_{x'}, \beta' D_\theta/\varepsilon, \gamma)$ like those described in the Introduction (1.25). Define extended classes of symbols and amplitudes

(2.39)

(a) $$eS^k = \{p(v, X, Z, \gamma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^{2N} \times [1,\infty)) : |\partial_v^\alpha \partial_{(X,Z)}^\beta p(v, X, Z, \gamma)| \leqslant C_{\alpha,\beta}\langle X, \gamma\rangle^{k-|\beta|} \text{ whenever } |Z| \geqslant |X|\},$$

(b) $$e\mathscr{S}^{k,M}_{\beta'} = \left\{p_s = p\left(\varepsilon V(x,\theta), \xi' + \frac{m\beta'}{\varepsilon}, \xi', \frac{m\beta'}{\varepsilon}, \gamma\right)\right. \left. \text{for some } p \in eS^k, V \in C_c^{0,M}, \text{ and } \beta' \in \mathbb{R}^N\setminus 0\right\},$$

(2.40)

$$\text{(a)}\quad eT^k = \{a(v,w,X,Z,\gamma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^{2N} \times [1,\infty)) : |\partial^\alpha_{(v,w)} \partial^\beta_{(X,Z)} a(v,w,X,Z,\gamma)| \leqslant C_{\alpha,\beta} \langle X,\gamma\rangle^{k-|\beta|} \text{ whenever } |Z| \geqslant |X|\},$$

$$\text{(b)}\quad e\mathscr{T}^{k,M}_{\beta'} = \left\{ a_s = a\left(\varepsilon V(x,\theta), \varepsilon W(y,\omega), \xi' + \frac{m\beta'}{\varepsilon}, \xi', \frac{m\beta'}{\varepsilon}, \gamma \right) : a \in eT^k, V \in C^{0,M}_c, W \in C^{0,M}_c, \text{ and } y_N = x_N \right\}.$$

Given $p_s \in e\mathscr{S}^{k,M}_{\beta'}$, $a_s \in e\mathscr{T}^{k,M}_{\beta'}$ we define corresponding operators $p_s(D_{x',\theta}) \in OPe\mathscr{S}^{k,M}_{\beta'}$, $\tilde{a}_s \in OPe\mathscr{T}^{k,M}_{\beta'}$ by the same formulas as before, (1.11) and (2.2), respectively.

Remark 2.4. (a) The analogues of Propositions (1.1), and (2.2)–(2.7) all remain true for the extended operators. The proofs involve only minor changes. For example, with $X = \xi' + m\beta'/\varepsilon$, $Y = \eta' + m'\beta'/\varepsilon$, $Z = (\xi', m\beta'/\varepsilon)$ and $Z' = (\eta', m'\beta'/\varepsilon)$, (2.15) and (2.17) become respectively

$$|\partial^\alpha_{(X,Z)} b^\wedge(x,\theta,\eta',m',X,Z,\gamma)| \leqslant C\varepsilon \langle \eta',m'\rangle^{-M} \langle X,\gamma\rangle^{k-1}, \tag{2.41}$$

$$|b^\wedge_1| \leqslant C\langle \eta',m'\rangle^{-(M-1)} \langle X+tY,\gamma\rangle^{k-1} \leqslant C\langle \eta',m'\rangle^{-(M-1)} \gamma^{k-1}, \tag{2.42}$$

where

$$b^\wedge_1 = \sum_{|\alpha|=1} \partial^\alpha_{(X,Z)} b^\wedge(x,\theta,\eta',m',X+tY,Z+tZ',\gamma)(Y,Z')^\alpha.$$

(b) Suppose $\chi(Z,\gamma) \in C^\infty(\mathbb{R}^{2N} \times [1,\infty))$ and

$$|\partial^\beta_Z \chi(Z,\gamma)| \leqslant C_\beta \langle Z,\gamma\rangle^{-|\beta|}.$$

Then

$$\chi(Z,\gamma) \in eS^0. \tag{2.43}$$

If $p(v,X,Z,\gamma) \in eS^k$, then

$$p(v,X,Z,\gamma)\, \chi(Z,\gamma) \in eS^k. \tag{2.44}$$

(c) Let $Z = (Z_1, Z_2) \in \mathbb{R}^{2N}$. To arrange (1.25) we will choose $\chi \in eS^0$ such that

$$\operatorname{supp} \chi(Z,\gamma) \subset \{|Z_1,\gamma| \leqslant \delta' |Z_2|\}. \tag{2.45}$$

3. *Garding Inequalities*

Notation 3.1. In Sections 3–5 we let $(,)$ (resp. $\langle , \rangle$) and $|\ |_0$ (resp. $\langle\ \rangle_0$) denote L^2 inner products and L^2 norms on $\overline{\mathbb{R}}^{N+1}_+$ (resp. $\mathbb{R}^N$). We shall use the same notation for L^2 inner products and norms on $\Omega \equiv \overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^1$ and $b\Omega = \mathbb{R}^N \times \mathbb{T}^1$. In Section 7 we will revert to Notation 1.3.

PROPOSITION 3.1. *Suppose* $M > N+2$, $p_s = p(\varepsilon V, X, \gamma) \in \mathcal{S}^{0,M}_{\beta'}$, *and* $Re\ p(\varepsilon V, X, \gamma) \geqslant C > 0$. *Fix* $\delta > 0$. *Then there exists* $\gamma_0 > 0$ *such that for* $\gamma \geqslant \gamma_0$, $U \in \mathcal{H}^{0,0}$

$$Re\ (p_s(D_{x',\theta})\, U, U) \geqslant (C-\delta)\ |U|_0^2. \tag{3.1}$$

Proof. Let

$$q_s = q(\varepsilon V, X, \gamma) = \mathrm{Re}\ p(\varepsilon V, X, \gamma) - (C-\delta)$$

and

$$b_s(\varepsilon V, X, \gamma) = (q(\varepsilon V, X, \gamma))^{\frac{1}{2}} \in \mathcal{S}^{0,M}_{\beta'},$$

a positive self-adjoint matrix. By Proposition 2.4

$$b_s(D_{x',\theta})^* = b_s(D_{x',\theta}) + r_{\varepsilon,\gamma}, \tag{3.2}$$

where $r_{\varepsilon,\gamma}$ is residual of order 1. Proposition 2.6 and (3.2) imply

$$b_s(D_{x',\theta})^*\, b_s(D_{x',\theta}) = q_s(D_{x',\theta}) + r_{\varepsilon,\gamma}, \tag{3.3}$$

where again $r_{\varepsilon,\gamma}$ is residual of order 1. Equation (3.3) yields

$$(q_s(D_{x',\theta})\, U, U) = (b_s(D_{x',\theta})\, U, b_s(D_{x',\theta})\, U) + (r_{\varepsilon,\gamma} U, U). \tag{3.4}$$

This implies (3.1) for γ_0 large enough, since

$$\begin{aligned}\mathrm{Re}\ p_s(D_{x',\theta}) - (C-\delta) &= \tfrac{1}{2}\,[p_s(D_{x',\theta}) + p_s(D_{x',\theta})^*] - (C-\delta)\\ &= q_s(D_{x'\theta}) + r_{\varepsilon,\gamma},\end{aligned}$$

where $r_{\varepsilon,\gamma}$ is residual of order 1. ∎

COROLLARY 3.1. *Suppose* $M > N+2$, $p_s = p(\varepsilon V, X, \gamma) \in \mathcal{S}^{k,M}_{\beta'}$, $k \geqslant 0$, *and*

$$Re\ p(\varepsilon V, X, \gamma) \geqslant C\langle X, \gamma\rangle^k, \qquad C > 0.$$

Fix $\delta > 0$. Then there exists $\gamma_0 > 0$ such that for $\gamma \geqslant \gamma_0$, $U \in \mathscr{H}^{0,k}$

$$\text{(3.5)} \qquad Re\,(p_s(D_{x',\theta})\,U, U) \geqslant (C-\delta)\,|U|^2_{\mathscr{H}^{0,k/2}}.$$

Proof. Let

$$\text{(3.6)} \qquad q_s = q(\varepsilon V, X, \gamma) = \langle X, \gamma\rangle^{-\frac{k}{2}}\, p(\varepsilon V, X, \gamma)\langle X, \gamma\rangle^{-\frac{k}{2}} \in \mathscr{S}^{0,M}_{\beta'}.$$

Proposition 2.6 with $k_1 = -\frac{k}{2}$, $k_2 = \frac{k}{2}$ implies

$$\text{(3.7)} \qquad \Lambda^{-\frac{k}{2}} p_s(D_{x',\theta})\,\Lambda^{-\frac{k}{2}} = q_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $r_{\varepsilon,\gamma}$ is residual of order 1. Setting $V = \Lambda^{k/2}U$ and applying Proposition 3.1 gives

$$\text{(3.8)} \qquad \mathrm{Re}(q_s(D_{x',\theta})\,V, V) \geqslant \left(C - \frac{\delta}{2}\right)|V|^2_{\mathscr{H}^{0,0}} = \left(C - \frac{\delta}{2}\right)|U|^2_{\mathscr{H}^{0,k/2}}$$

for γ_0 large enough. This implies (3.5) for γ_0 possibly larger since

$$\text{(3.9)} \qquad (r_{\varepsilon,\gamma}V, V) \leqslant \frac{C}{\gamma}\,|V|^2_{\mathscr{H}^{0,0}} = \frac{C}{\gamma}\,|U|^2_{\mathscr{H}^{0,k/2}}. \qquad \blacksquare$$

4. *Modifications*

A. ($\gamma = 1$) In contrast to Section 3, it was never necessary to take γ large in the proofs of Section 2. Consequently, we obtain a singular calculus without large parameter by setting $\gamma = 1$ in the proofs of that section. Thus, $\langle X\rangle = (|X|^2+1)^{1/2}$, and all residual operators r_ε are residual of order 0, i.e., bounded on L^2 uniformly with respect to ε. Garding inequalities can be proved by letting ε instead of $\frac{1}{\gamma}$ play the role of a small parameter (see Proposition 4.6).

B. (Homogeneous symbols) In the application to initial value problems we will need to work with operators defined by symbols homogeneous in X and depending on $\theta \in \mathbb{T}^L$.

For $j = 1, \ldots, L$ let $\beta'_j \in \mathbb{R}^N$ and set $\beta' = (\beta'_1, \ldots, \beta'_L)$. For $m \in \mathbb{Z}^L$ let $m\beta' = \sum_{j=1}^L m_j\beta'_j$.

Definition 4.1. For $k \geqslant 0$ let

(a) $S^k_h = \{p(v, X) \in C^\infty(\mathbb{R}^m \times (\mathbb{R}^N \backslash 0)) : p$ is homogeneous of degree k in $X\}$.

(b) $\mathscr{S}^{k,M}_{\beta',h} = \{p_s = p(\varepsilon V(x,\theta), X) : p \in S^k_h,\ X = \xi' + \frac{m\beta'}{\varepsilon},\ V \in C^{0,M}_c(\overline{\mathbb{R}}^{N+1}_+ \times \mathbb{T}^L)\}$.

(c) $T^k_h = \{a(v, w, X) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m \times (\mathbb{R}^N \setminus 0))$: a is homogeneous of degree k in $X\}$.

(d) $\mathscr{T}^{k,M}_{\beta',h} = \{a_s = a(\varepsilon V(x,\theta), \varepsilon W(y,\omega), X) : a \in T^k_h, \quad X = \xi' + \frac{m\beta'}{\varepsilon}, V \in C^{0,M}_c$, and $y_N = x_N\}$.

To symbols $p_s \in \mathscr{S}^{k,M}_{\beta',h}$ and amplitudes $a_s \in \mathscr{T}^{k,M}_{\beta',h}$ we associate operators $p_s(D_{x',\theta})$ and $\tilde{a}_s$ defined by the obvious analogues of (1.11) and (2.2). In these formulas the symbols are homogeneous in all of $|X| \neq 0$; i.e., we do not modify the symbols to be $\equiv 0$ for $|X|$ small. Such a modification introduces an error that is generally no better than bounded on L^2, which is too large if, for example, $k = 0$.

Versions of Propositions 1.1 and 2.2 for $OP\mathscr{S}^{0,M}_{\beta',h}$ and $OP\mathscr{T}^{0,M}_{\beta',h}$ follow by the same proofs if one assumes the regularity index $M > N+L$. In Section 6 we will need the following calculus results for $OP\mathscr{S}^{k,M}_{\beta',h}$ and $OP\mathscr{T}^{k,M}_{\beta',h}$. When using the homogeneous calculus, we define

$$r_\varepsilon \text{ is } \textit{residual} \Leftrightarrow r_\varepsilon : \mathscr{H}^{0,0} \to \mathscr{H}^{0,0}. \tag{4.1}$$

PROPOSITION 4.1. *Suppose $M > N+L+1$ and $a_s = a(\varepsilon V(x,\theta), \varepsilon W(y,\omega), X) \in \mathscr{T}^{k,M}_{\beta',h}$ for some $k \in [0,1]$. Then*

$$\tilde{a}_s = p_s(D_{x',\theta}) + r_\varepsilon,$$

where $p_s = a(\varepsilon V(x,\theta), \varepsilon W(x,\theta), X) \in \mathscr{S}^{k,M}_{\beta',h}$ and r_ε is residual. (The statement is trivial when $k = 0$.)

PROPOSITION 4.2. *Suppose $M > N+L+1$ and $p_s = p(\varepsilon V(x,\theta), X) \in \mathscr{S}^{k,M}_{\beta',h}$ for some $k \in [0,1]$. Then*

$$p_s(D_{x',\theta})^* = p_s^*(D_{x',\theta}) + r_\varepsilon,$$

where r_ε is residual.

PROPOSITION 4.3. *Suppose $M > N+L+1$ and $p_s = p(\varepsilon V(x,\theta), X) \in \mathscr{S}^{k_1,M}_{\beta',h}$, $q_s = q(\varepsilon W(x,\theta), X) \in \mathscr{S}^{k_2,M}_{\beta',h}$ where $k_1 \geqslant 0$, $k_2 \geqslant 0$, $k_1 + k_2 \leqslant 1$. Set*

$$t_s = p(\varepsilon V(x,\theta), X)\, q^*(\varepsilon W(x,\theta), X).$$

Then

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta})^* = t_s(D_{x',\theta}) + r_\varepsilon,$$

where r_ε is residual.

PROPOSITION 4.4. *Suppose $M > N+L+1$ and*

$$p_s = p(\varepsilon V(x,\theta), X) \in \mathscr{S}^{0,M}_{\beta',h}, \qquad q_s = q(\varepsilon W(x,\theta), X) \in \mathscr{S}^{k_2,M}_{\beta',h},$$

where $0 \leqslant k_2 \leqslant 1$. Set

$$t_s = p(\varepsilon V(x,\theta), X)\, q(\varepsilon W(x,\theta), X).$$

Then

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta}) = t_s(D_{x',\theta}) + r_\varepsilon,$$

where r_ε is residual.

PROPOSITION 4.5. *Suppose $M_1 \geqslant N+L+1$, $M_2 \geqslant 2(N+L+1)+1$, $p_s = p(\varepsilon V(x,\theta), X) \in \mathscr{S}^{1,M_1}_{\beta',h}$, $q_s = q(\varepsilon W(x,\theta), X) \in \mathscr{S}^{0,M_2}_{\beta',h}$. Set*

$$t_s = p(\varepsilon V(x,\theta), X)\, q(\varepsilon W(x,\theta), X).$$

Then

$$p_s(D_{x',\theta})\, q_s(D_{x',\theta}) = t_s(D_{x',\theta}) + r_\varepsilon,$$

where r_ε is residual.

Proof. Proof of Propositions 4.1–4.5. We will point out just the changes needed in the corresponding proofs of Section 2.

1. (Proposition 4.1) Write $a_s = a_{1s} + a_{2s}$, where

$$a_{1s} \equiv 0 \quad \text{for} \quad |X| \leqslant \tfrac{1}{2}, \qquad a_{2s} = a_s \quad \text{for} \quad |X| \geqslant 1.$$

$\tilde{a}_{1s}$ is residual, so the result follows from Proposition 2.3 in the $\gamma = 1$ calculus (modification A with $\theta \in \mathbb{T}^L$) applied to a_{2s}.

2. (Propositions 4.2–4.4) No changes in the proofs.

3. (Proposition 4.5) In (2.34) $\langle X + tY, \gamma\rangle^{k_1 - 1}$ is now replaced by $|X+tY|^{k_1-1}$, but this is not a problem since $k_1 = 1$. ∎

C. ($\theta \in \mathbb{T}^L$, $\gamma \geqslant 1$) The propositions of sections 2 and 3 remain true for the original large parameter calculus when $\theta \in \mathbb{T}^L$, provided the regularity index M is increased as in Propositions 4.1–4.5.

D. (Tangential calculus) In the definition of the operators $p_s(D_{x',\theta})$ and $\tilde{a}_s$, the x_N variable is just a continuous parameter. All the results of Sections 2, 3, and above hold for the operators on $\mathbb{R}^N \times \mathbb{T}^L$ obtained by fixing x_N.

The following simple Garding inequality for tangential, homogeneous SPOs is needed in Section 6.

PROPOSITION 4.6. *Suppose* $M > N + L$, $R_s = R(\varepsilon V(x, \theta), X) \in \mathscr{S}^{0, M}_{\beta', h}$, *and* $Re\ R(\varepsilon V, X) \geqslant C > 0$. *Let* $R_{s, x_N}(D_{x', \theta})$ *be the tangential operator obtained by fixing a particular* x_N. *For some* $K > 0$ *assume* $|V|_{C^{0, M}_c} \leqslant K$. *Then there exists* $\varepsilon_1(K)$ *such that for* $0 < \varepsilon \leqslant \varepsilon_1(K)$, $U \in \mathscr{H}^0$,

$$Re\langle R_{s, x_N}(D_{x', \theta})\, U, U\rangle \geqslant \frac{C}{2} \langle U\rangle_0^2.$$

Proof. Write

$$R(\varepsilon V, X) = R(0, X) + \varepsilon V b(\varepsilon V, X) = r_{1s} + \varepsilon r_{2s}.$$

Clearly, Re $\langle r_{1s}(D_{x', \theta})\, V, V\rangle \geqslant C\langle V\rangle_0^2$, so the L^2 boundedness of r_{2s} implies the result. ∎

E. (Paradifferential calculus) There may be a paradifferential version of the SPO calculus that would allow a large reduction in the amount of (x, θ) regularity assumed for the symbols $p(\varepsilon V(x, \theta), X, \gamma)$. We have decided not to pursue that possibility here.

PART 2. SYMMETRIZERS FOR SINGULAR SYSTEMS

5. L^2 *Estimates for Boundary Problems*

In this section we obtain L^2 estimates uniform in ε for singular boundary problems

(5.1)
$$\mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x, \theta})\, U_\varepsilon$$
$$= D_{x_N} U_\varepsilon - \mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) U_\varepsilon = F_\varepsilon(x, \theta) \qquad \text{in} \quad x_N > 0$$
$$B(\varepsilon V_\varepsilon)\, U_\varepsilon = G_\varepsilon(x', \theta) \qquad \text{on} \quad x_N = 0,$$

where

$$\mathscr{A}(v, \xi') = \sum_{j=0}^{N-1} A_j(v)\, \xi_j \qquad \text{for} \quad A_j \in C^\infty(\mathbb{R}^m, \mathbb{R}^{m^2})$$

and $B(v)$ is a real $\mu \times m$ matrix C^∞ in v.

Consider first the nonsingular analogue

$$\begin{aligned} \mathscr{L}(v, D_x)\, u &= D_{x_N} u - \mathscr{A}(v, D_{x'})\, u = f \\ B(v)\, u &= g \qquad \text{on} \quad x_N = 0, \end{aligned} \tag{5.2}$$

where $v(x) \in C_c^\infty$ (for now) with values in $B_R = \{v \in \mathbb{R}^m : |v| \leqslant R\}$ for some fixed R. Set $w = u^\gamma \equiv e^{-\gamma x_0} u$,

$$\mathscr{A}(v, D_{x'_\gamma}) = \mathscr{A}(v, D_{x_0} - i\gamma, D_{x''}), \qquad \mathscr{A}(v, \xi'_\gamma) = \mathscr{A}(v, \xi_0 - i\gamma, \xi'')$$

and rewrite (5.2)

$$\begin{aligned} D_{x_N} w - \mathscr{A}(v, D_{x'_\gamma})\, w &= f^\gamma \\ B(v)\, w &= g^\gamma \text{ on } x_N = 0. \end{aligned} \tag{5.3}$$

Kreiss Symmetrizers

We now briefly recall the method of Kreiss symmetrizers [K] for proving energy estimates. More detailed expositions may be found in [CP, Met2].

DEFINITION 5.1. A *symmetrizer* for (5.3) is a family $R_\gamma(x_N)$, $x_N \geqslant 0$, $\gamma \geqslant 1$ of bounded self-adjoint operators on $L^2(\mathbb{R}^N)$ which define bounded self-adjoint operators on $L^2(\overline{\mathbb{R}}^{N+1}_+)$ via

$$(R_\gamma w)(., x_N) = R_\gamma(x_N)\, w(., x_N).$$

In addition there exist constants C, c, γ_0 such that for all $w \in H^1(\overline{\mathbb{R}}^{N+1}_+)$ and $\gamma \geqslant \gamma_0$

$$\begin{aligned} &\text{(a)} \quad |R_\gamma w|_0 \leqslant C\, |w|_0 \\ &\text{(b)} \quad |[\partial_{x_N}, R_\gamma]\, w|_0 \leqslant C\, |w|_0 \\ &\text{(c)} \quad \operatorname{Im}(R_\gamma \mathscr{A}(v, D_{x'_\gamma})\, w, w) \geqslant c\gamma\, |w|_0^2 \\ &\text{(d)} \quad \langle R_\gamma w, w\rangle + C\langle B(v)\, w\rangle_0^2 \geqslant c\langle w\rangle_0^2. \end{aligned} \tag{5.4}$$

PROPOSITION 5.1. *Suppose R_γ is a symmetrizer for* (5.3). *Then there exist C_1, γ_1 depending only on C, c, γ_0 in* (5.4) *such that for all $w \in H^1(\overline{\mathbb{R}}^{N+1}_+)$ and $\gamma \geqslant \gamma_1$*

$$|w|_0 + \frac{1}{\sqrt{\gamma}} \langle w\rangle_0 \leqslant C_1 \left(\frac{1}{\gamma} |(D_{x_N} - \mathscr{A}(D_{x'_\gamma}))\, w|_0 + \frac{1}{\sqrt{\gamma}} \langle B(v)\, w\rangle_0 \right). \tag{5.5}$$

Proof. Integrate $\partial_{x_N}(R_\gamma w, w)$ in x_N from 0 to ∞ to obtain

$$\langle R_\gamma w, w\rangle + \operatorname{Im}(R_\gamma \mathscr{A}(v, D_{x'_\gamma})\, w, w) = 2\operatorname{Im}((D_{x_N} - \mathscr{A}(v, D_{x'_\gamma}))\, w, R_\gamma w) - ([\partial_{x_N}, R_\gamma]\, w, w). \tag{5.6}$$

The estimate (5.5) is now a simple consequence of (5.6) and the properties (5.4). ▮

A symmetrizer can be constructed as a pseudodifferential operator when the boundary problem (5.3) is *Kreiss well-posed.* Set

$$B_R = \{v \in \mathbb{R}^m : |v| \leqslant R\} \qquad \text{and} \qquad z = (v, \xi', \gamma) \in B_R \times \mathbb{R}^N \times (0, \infty).$$

DEFINITION 5.2 [K, Met2]. The problem (5.2) is *Kreiss well-posed* for $v \in B_R$ if there exists an $m \times m$ matrix-valued function

$$R(z) \in C^\infty(B_R \times \mathbb{R}^N \times (0, \infty)),$$

homogeneous of degree zero in (ξ', γ), and satisfying:

(a) $R(z) = R^*(z)$;

(b) there exist $C > 0$, $c > 0$ such that for all z

$$R(z) + C\, B^*(v)\, B(v) \geqslant c; \tag{5.7}$$

(c) there exist finite sets of C^∞ matrices on $B_R \times \mathbb{R}^N \times (0, \infty)$, $T_l(z)$, $H_l(z)$, and $E_l(z)$ such that

$$\text{(i)}\quad \operatorname{Im} R(z)\, \mathscr{A}(v, \xi'_\gamma) = \sum_l T_l(z) \begin{bmatrix} \gamma H_l(z) & 0 \\ 0 & E_l(z) \end{bmatrix} T_l^*(z); \tag{5.8}$$

(ii) T_l, H_l are homogeneous of degree zero in (ξ', γ), E_l is homogeneous of degree one;

(iii) $H_l(z) = H_l^*(z)$, $E_l(z) = E_l^*(z)$;

(iv) there exist $C, c > 0$ such that

$$\sum_l T_l(z)\, T_l^*(z) \geqslant c, \qquad H_l(z) \geqslant c, \qquad E_l(z) \geqslant c(|\xi'| + \gamma). \tag{5.9}$$

The dimension of H_l and E_l can vary with l.

Remark 5.1. (a) Kreiss [K] proved that strictly hyperbolic systems with boundary conditions satisfying the uniform Lopatinski condition are well-posed in the sense of Definition 5.2. A *Kreiss symmetrizer* for (5.3) is given by

$$R_\gamma = \tfrac{1}{2}\,[R(v, D_{x'}, \gamma) + R(v, D_{x'}, \gamma)^*],$$

where $R(v, D_{x'}, \gamma)$ is the (standard) pseudodifferential operator associated to $R(v, \xi', \gamma)$. Using paradifferential operators Metivier [Met1, Met2] and Mokrane [Mo] extended the construction to the case where $v(x)$ is only Lipschitz.

(b) Many physical examples of hyperbolic systems are not strictly hyperbolic, for example, the linearized Euler equations of gas dynamics. In the Kreiss construction strict hyperbolicity is only used to show that $\mathscr{A}(v, \xi'_\gamma)$ has a suitable block structure (the main difficulty being near glancing boundary frequencies $\xi' \in \mathscr{G}$). Majda [M1] showed that this *block structure condition* is satisfied by several nonstrictly hyperbolic systems, in particular the linearized shock front equations of gas dynamics. Recently, Metivier [Met3] has shown that, more generally, all symmetrizable hyperbolic systems with constant multiplicity satisfy the block structure assumption. Boundary problems for such systems which satisfy the uniform Lopatinski condition are therefore Kreiss well-posed.

(c) Sometimes one has to allow the boundary operator in (5.3) to be a pseudodifferential operator $B(v, D_{x'}, \gamma)$ associated to a C^∞ symbol $B(v, \xi', \gamma)$ homogeneous of degree zero in (ξ', γ). Such boundary conditions arise, for example, in the study of shocks (9.51) and also when one reduces an mth order scalar problem with differential boundary conditions to an $m \times m$ first-order system. The formulation of Kreiss well-posedness for such problems is unchanged except for the replacement of $B(v)$ by $B(z)$ in (5.7). Again, block structure and the uniform Lopatinski condition together imply Kreiss well-posedness and the existence of Kreiss symmetrizers.

Singular Symmetrizers

Notation 5.1. For $0 < \varepsilon \leqslant 1$, $\gamma \geqslant 1$, $V_\varepsilon \in C_c^{0,M}$, $\Omega = \overline{\mathbb{R}}_+^{N+1} \times \mathbb{T}^1$, let

(a) $$\mathscr{A}_s = \mathscr{A}\left(\varepsilon V_\varepsilon, (\xi_0 - i\gamma, \xi'') + \frac{m\beta'}{\varepsilon}\right) \in \mathscr{S}^{1,M}_{\beta'} \qquad \text{and}$$

$$\mathscr{A}_s(D_{x',\theta}) = \mathscr{A}\left(\varepsilon V_\varepsilon, (D_{x_0} - i\gamma, D_{x''}) + \frac{\beta' D_\theta}{\varepsilon}\right).$$

(b) For $k \in \mathbb{R}$

$$\mathscr{H}^{0,k}_\gamma = e^{\gamma x_0}\mathscr{H}^{0,k} = \{U(x, \theta) \colon |\Lambda^k(e^{-\gamma x_0}U)|_{L^2(\Omega)} < \infty\}$$
$$(\Lambda, \mathscr{H}^{0,k} \text{ as in Definition 2.1}).$$

(c) $\mathscr{H}^{1,k}_\gamma = e^{\gamma x_0}\mathscr{H}^{1,k}$.

(d) $\mathscr{H}^k_\gamma(b\Omega) = \{V(x', \theta) : |\Lambda^k(e^{-\gamma x_0}V)|_{L^2(b\Omega)} < \infty\}$.

(e) $|U|_{\mathscr{H}^{0,0}_\gamma} = |U|_{0,\gamma}$ and $\langle V\rangle_{\mathscr{H}^0_\gamma} = \langle V\rangle_{0,\gamma}$.

The main result of this section is that if $R(v, D_{x'}, \gamma)$ symmetrizes (5.2), then $R_s(D_{x',\theta}) = R(\varepsilon V_\varepsilon, D_{x'} + \beta' D_\theta/\varepsilon, \gamma)$ symmetrizes the singular problem (5.1).

DEFINITION 5.3. A symmetrizer for the singular problem (5.1) is a family $\mathscr{R}_{\varepsilon,\gamma} \in OP\mathscr{S}^{0,M}_{\beta'}$, $\varepsilon \in (0, \varepsilon_0]$, $\gamma \geqslant 1$, such that

$$\mathscr{R}_{\varepsilon,\gamma} - \mathscr{R}^*_{\varepsilon,\gamma}$$

is residual of order 1. In addition there exist constants C, c, γ_0 independent of ε such that for all $W(x, \theta) \in \mathscr{H}^{0,1}$ and $\gamma \geqslant \gamma_0$

$$\text{(5.10)}\qquad \begin{aligned} &\text{(a)} \quad |\mathscr{R}_{\varepsilon,\gamma} W|_0 \leqslant C\,|W|_0 \\ &\text{(b)} \quad |[\partial_{x_N}, \mathscr{R}_{\varepsilon,\gamma}]\, W|_0 \leqslant C\,|W|_0 \\ &\text{(c)} \quad \operatorname{Im}(\mathscr{R}_{\varepsilon,\gamma}\mathscr{A}_s(D_{x',\theta})\, W, W) \geqslant c\gamma\,|W|_0^2 \\ &\text{(d)} \quad \operatorname{Re}\langle \mathscr{R}_{\varepsilon,\gamma} W, W\rangle + C\langle B(\varepsilon V_\varepsilon)\, W\rangle_0^2 \geqslant c\langle W\rangle_0^2. \end{aligned}$$

THEOREM 5.1. *Suppose* (5.2) *is Kreiss well-posed for* $v \in B_R$ *and let* $R(z) = R(v, \xi', \gamma)$ *be as in Definition* 5.2. *Fix* $K > 0$ *and* $\varepsilon_0 > 0$. *Suppose* $V_\varepsilon(x, \theta) \in C^{0,M_0}_c$ *for* $M_0 = 2(N+2)+1$ *and satisfies for* $0 < \varepsilon \leqslant \varepsilon_0$

$$\text{(5.11)}\qquad \begin{aligned} &\text{(a)} \quad |\varepsilon V_\varepsilon(x, \theta)|_{L^\infty(\Omega)} \leqslant R \\ &\text{(b)} \quad |V_\varepsilon|_{C^{0,M_0}_c} \leqslant K \\ &\text{(c)} \quad |\varepsilon \partial_{x_N} V_\varepsilon|_{L^\infty(\Omega)} \leqslant h(|V_\varepsilon|_{C^{0,1}_c}), \end{aligned}$$

for some increasing function h: $\mathbb{R}_+ \to \mathbb{R}_+$. *Set*

$$R_s = R(\varepsilon V_\varepsilon(x, \theta), X, \gamma) \in \mathscr{S}^{0,M}_{\beta'}.$$

Then

$$\text{(5.12)}\qquad \mathscr{R}_{\varepsilon,\gamma} = R_s(D_{x',\theta}), \qquad \varepsilon \in (0, \varepsilon_0], \quad \gamma \geqslant 1$$

is a symmetrizer for the singular problem (5.1).

Proof. **1.** We use the SPO calculus to check properties (5.10.a)–(5.10d). The argument parallels the classical one, but extra care is needed because our residual operators are not smoothing. See Remark 5.2.

Property (5.10a) follows directly from Proposition 1.1.

2. Note that

$$[\partial_{x_N}, R_s(D_{x',\theta})] = \varepsilon \frac{\partial V_\varepsilon}{\partial x_N}(x, \theta)\, t_s(D_{x',\theta}), \qquad \text{where}$$

$$t_s = \frac{\partial R}{\partial v}(\varepsilon V_\varepsilon(x, \theta), X, \gamma)$$

so (5.11c) and Proposition 1.1 give (5.10b).

3. Let $T_l(\varepsilon V_\varepsilon, X, \gamma)$, $H_l(\varepsilon V_\varepsilon, X, \gamma) \in \mathscr{S}^{0,M}_{\beta'}$ and $E_l(\varepsilon V_\varepsilon, X, \gamma) \in \mathscr{S}^{1,M}_{\beta'}$ be given by the functions T_l, H_l, E_l in (5.8). For $W(x, \theta) \in \mathscr{H}^1$ set

$$Y_l = T_l(D_{x',\theta})^* W = (Y_{l_1}, Y_{l_2}), \tag{5.13}$$

where the components of Y_l correspond to the blocks in (5.8).

With

$$F_l(\varepsilon V_\varepsilon, X, \gamma) = T_l(\varepsilon V_\varepsilon, X, \gamma) \begin{bmatrix} \gamma H_l(\varepsilon V_\varepsilon, X, \gamma) & 0 \\ 0 & E_l(\varepsilon V_\varepsilon, X, \gamma) \end{bmatrix}$$

Proposition 2.6 gives

$$F_l(D_{x',\theta}) = T_l(D_{x',\theta}) \begin{bmatrix} \gamma H_l(D_{x',\theta}) & 0 \\ 0 & E_l(D_{x',\theta}) \end{bmatrix} + r_{\varepsilon,\gamma}, \tag{5.14}$$

where $r_{\varepsilon,\gamma}$ is residual of order 0. Setting

$$G(\varepsilon V, X, \gamma) = \operatorname{Im} R(\varepsilon V, X, \gamma)\, \mathscr{A}_s(\varepsilon V, X, \gamma) \tag{5.15}$$

and using Proposition 2.5 to compute $F_l(D_{x',\theta})\, T_l(D_{x',\theta})^*$, we obtain

$$G(D_{x',\theta}) = \sum_l T_l(D_{x',\theta}) \begin{bmatrix} \gamma H_l(D_{x',\theta}) & 0 \\ 0 & E_l(D_{x',\theta}) \end{bmatrix} T_l(D_{x',\theta})^* + r_{\varepsilon,\gamma}, \tag{5.16}$$

where $r_{\varepsilon,\gamma}$ is residual of order 0.

Equation (5.9) and Corollary 3.1 (Garding inequality) imply for γ large enough

$$\begin{aligned} &\operatorname{Re}(\gamma H_l(D_{x',\theta})\, Y_{l_1}, Y_{l_1}) \geqslant c\gamma\, |Y_{l_1}|_0^2 \\ &\operatorname{Re}(E_l(D_{x',\theta})\, Y_{l_2}, Y_{l_2}) \geqslant c\, |Y_{l_2}|^2_{\mathscr{H}^{0,1/2}} \geqslant c\gamma\, |Y_{l_2}|_0^2, \end{aligned} \tag{5.17}$$

and hence, summing over l,

$$\operatorname{Re} G(D_{x',\theta})\, W, W) \geqslant c\gamma \sum_l |Y_l|_0^2 \tag{5.18}$$

for γ large.

Proposition 2.5 shows that for $r_{\varepsilon,\gamma}$ residual of order 1

(5.19) $$\sum_l T_l(D_{x',\theta})\, T_l(D_{x',\theta})^* = q_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $q_s = \sum_l T_l(\varepsilon V_\varepsilon, X, \gamma)\, T_l^*(\varepsilon V_\varepsilon, X, \gamma)$. Thus, by (5.9) and Corollary 3.1

(5.20) $$|W|_0^2 \leqslant C \sum_l |Y_l|_0^2 \qquad \text{for } \gamma \text{ large.}$$

Now Proposition 2.6 (for $R_s(D_{x',\theta})\, \mathscr{A}_s(D_{x',\theta})$) and Proposition 2.5 (for $\mathscr{A}_s(D_{x',\theta})^*\, R_s(D_{x',\theta})^*$) imply

(5.21) $$\operatorname{Im} R_s(D_{x',\theta})\, \mathscr{A}_s(D_{x',\theta}) = \operatorname{Re} G_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $r_{\varepsilon,\gamma}$ is residual of order 0. Property (5.10c) now follows from (5.18), (5.20), and (5.21).

4. (5.7) and Corollary 3.1 imply for γ large

(5.22) $$\operatorname{Re}\langle R_s(D_{x',\theta})\, W, W\rangle + C\langle B(\varepsilon V_\varepsilon)\, W, B(\varepsilon V_\varepsilon)\, W\rangle \geqslant c\langle W\rangle_0^2,$$

which is (5.10d) in Definition 5.3. This concludes the proof of Theorem 5.1. Note that the constants C, c, γ_0 in (5.10) depend only on K in (5.11) and the constants in Definition 5.3. ∎

Remark 5.2. If $p_s(D_{x',\theta}) \in OP\mathscr{S}^{1,M}_{\beta'}$ and $r_{\varepsilon,\gamma}$ is residual of order 1, the product $p_s(D_{x',\theta})\, r_{\varepsilon,\gamma}$ is not necessarily bounded on L^2. We avoid this problem in the above proof by defining

$$\mathscr{R}_{\varepsilon,\gamma} = R_s(D_{x',\theta}) \text{ instead of } \tfrac{1}{2}\,[R_s(D_{x',\theta}) + R_s(D_{x',\theta})^*]$$

and also by using Proposition 2.5 instead of a combination of Propositions 2.4 and 2.7.

COROLLARY 5.1. *Under the hypotheses of Theorem* 5.1 *there exist constants* $C_1(K)$, $\gamma_1(K)$, *such that for all* $W(x,\theta) \in \mathscr{H}^{1,1}$ *and* $\gamma \geqslant \gamma_1$

(5.23)
$$|W|_0 + \frac{1}{\sqrt{\gamma}}\langle W\rangle_0 \leqslant C_1\left(\frac{1}{\gamma}\,|(D_{x_N} - \mathscr{A}_s(D_{x',\theta}))\, W|_0 + \frac{1}{\sqrt{\gamma}}\langle B(\varepsilon V_\varepsilon)\, W\rangle_0\right).$$

Proof. Propositions 2.7 (for $\mathscr{A}_s(D_{x',\theta})^*\, R_s(D_{x',\theta})$) and 2.5 (for $\mathscr{A}_s(D_{x',\theta})^*\, R_s(D_{x',\theta})^*$) show that

(5.24)
$$\operatorname{Im} R_s(D_{x',\theta})\, \mathscr{A}_s(D_{x',\theta}) = R_s(D_{x',\theta})\, \mathscr{A}_s(D_{x',\theta}) - \mathscr{A}_s(D_{x',\theta})^*\, R_s(D_{x',\theta}) + r_{\varepsilon,\gamma},$$

where $r_{\varepsilon,\gamma}$ is residual of order 0. Thus, integrating $\partial_{x_N}(R_s(D_{x',\theta})W, W)$ in x_N from 0 to ∞ yields

(5.25)

$$\begin{aligned}\langle R_s(D_{x',\theta})W, W\rangle + \operatorname{Im}(R_s(D_{x',\theta})\mathscr{A}_s(D_{x',\theta})W, W) \\ = 2\operatorname{Im}((D_{x_N} - \mathscr{A}_s(D_{x',\theta}))W, R_s(D_{x',\theta})W) - ([\partial_{x_N}, R_s(D_{x',\theta})]W, W) \\ + O(|W|_0^2).\end{aligned}$$

With (5.10) this easily implies (5.23). ∎

Applying Corollary 5.1 to $W = e^{-\gamma x_0}U_\varepsilon$, we now rephrase (5.23) in terms of the original singular system (5.1).

Corollary 5.2. *Under the hypotheses of Theorem* 5.1 *there exist constants* $C_1(K)$, $\gamma_1(K)$ *independent of* ε *such that for all* $U_\varepsilon(x,\theta) \in \mathscr{H}_\gamma^{1,1}(\Omega)$ *and* $\gamma \geqslant \gamma_1$, $0 < \varepsilon \leqslant \varepsilon_0$

(5.26)

$$|U_\varepsilon|_{0,\gamma} + \frac{1}{\sqrt{\gamma}}\langle U_\varepsilon\rangle_{0,\gamma} \leqslant C_1\left(\frac{1}{\gamma}|\mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})U_\varepsilon|_{0,\gamma} + \frac{1}{\sqrt{\gamma}}\langle B(\varepsilon V_\varepsilon)U_\varepsilon\rangle_{0,\gamma}\right).$$

Existence and Uniqueness

Corresponding to the (nonsingular) boundary problem (5.2) there is a *dual boundary problem* and a notion of *backward* uniform Lopatinski condition for the dual problem obtained by changing x_0 to $-x_0$ in the original condition [CP, Chap. 7]. It is known that when the original problem satisfies the uniform Lopatinski condition and is either strictly hyperbolic [CP] or symmetrizable hyperbolic with constant multiplicity [Met2, Met3], then the dual problem satisfies the backward uniform Lopatinski condition and is thus Kreiss well-posed.

When the dual problem is Kreiss well-posed for $v \in B_R$, Corollary 5.1 implies that the corresponding singular dual problem satisfies an estimate like (5.23) for V_ε as in (5.11). Standard arguments [CP] using functional analysis (for existence), the mollifiers of Proposition 2.8 (to show L^2 solutions satisfy the estimate (5.23) and are thus unique), and the fact that

$$\sup_j |V|_{0,\gamma+j} < \infty \Rightarrow V = 0 \quad \text{in} \quad x_0 < 0$$

(for the support property) yield the following theorem for the system (5.1).

THEOREM 5.2. *Suppose* (5.2) *and the corresponding dual problem are both Kreiss well-posed for* $v \in B_R$. *Fix* $K > 0$ *and* $\varepsilon_0 > 0$. *Assume* $V_\varepsilon(x, \theta) \in C_c^{0, M}$ *for* $M \geqslant 2(N+1)+1$ *and satisfies* (5.11) *for* $0 < \varepsilon \leqslant \varepsilon_0$. *There is a constant* $\gamma_0(K)$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$, $\gamma \geqslant \gamma_0$, $F_\varepsilon(x, \theta) \in \mathscr{H}_\gamma^{0,0}(\Omega)$, $G_\varepsilon(x', \theta) \in \mathscr{H}_\gamma^{0}(b\Omega)$, *there is a unique solution* $U_\varepsilon(x, \theta) \in \mathscr{H}_\gamma^{0,0}(\Omega)$ *to the singular problem*

$$\text{(5.27)} \qquad \begin{aligned} \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon &= F_\varepsilon(x, \theta) \qquad \textit{in} \quad x_N > 0 \\ B(\varepsilon V_\varepsilon)\, U_\varepsilon &= G(x', \theta) \qquad \textit{on} \quad x_N = 0, \end{aligned}$$

and U_ε *satisfies the estimate* (5.26). *Moreover, if* $F_\varepsilon, G_\varepsilon$ *vanish in* $x_0 < 0$, *the same is true of* U_ε.

6. L^2 Estimates for Initial Value Problems

We continue to work on $\overline{\mathbb{R}}_+^{N+1} = \{(x', x_N) : x_N \geqslant 0\}$, but in this section x_N *denotes time*. For $j = 1, ..., L$ let $\beta'_j \in \mathbb{R}^N$, and set $\beta' = (\beta'_1, ..., \beta'_L)$. For $\theta \in \mathbb{T}^L$ we set $D_\theta = (D_{\theta_1}, ..., D_{\theta_L})$ and consider the singular initial value problem

$$\text{(6.1)} \qquad \begin{aligned} \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon &\equiv D_{x_N} U_\varepsilon - \mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) U_\varepsilon = F_\varepsilon(x, \theta) \\ U_\varepsilon &= G(x', \theta) \qquad \text{on} \quad x_N = 0 \end{aligned}$$

with $\mathscr{A}$, F_ε as in (1.4). Quasilinear initial value problems

$$\text{(6.2)} \qquad \begin{aligned} \sum_{j=0}^{N} A_j(v_\varepsilon)\, D_{x_j} v_\varepsilon &= F(v_\varepsilon) \qquad \text{in} \quad x_N > 0 \\ v_\varepsilon &= u_0 + \varepsilon G\left(x', \frac{x'\beta'_1}{\varepsilon}, ..., \frac{x'\beta'_L}{\varepsilon}\right) \qquad \text{on} \quad x_N = 0 \end{aligned}$$

lead to problems like (6.1) in the same way that (1.1) led to (1.6); namely, one looks for

$$\text{(6.3)} \qquad v_\varepsilon = u_0 + \varepsilon U_\varepsilon(x, \theta)|_{\theta = \frac{x'\beta'}{\varepsilon}}.$$

DEFINITION 6.1 [CP]. $D_{x_N} - \mathscr{A}(v, D_{x'})$ is *symmetrizable* for $v \in B_R$ if there exist $R(v, \xi') \in C^\infty(B_R \times (\mathbb{R}^N \backslash 0))$ homogeneous of degree 0 in ξ' and $c > 0$ such that for all (v, ξ')

(a) $R(v, \xi') = R^*(v, \xi')$,

(b) $R(v, \xi')\, \mathscr{A}(v, \xi')$ is hermitian,

(c) $R(v, \xi') \geqslant c$.

Remark 6.1. (a) First-order systems strictly hyperbolic with respect to x_N are symmetrizable [CP].

(b) If $D_{x_N} - \mathscr{A}(v, D_{x'})$ is symmetrizable by $R(v, \xi')$, then $D_{x_N} - \mathscr{A}(v, D_{x'})^*$ is symmetrizable by $R^{-1}(v, \xi')$.

Note that $R(v, X) \in S^0_h$ (Definition 4.1). Our goal is to show that for $R(v, \xi')$ as in Definition 6.1, if $R_s = R(\varepsilon V_\varepsilon(x, \theta), X)$ then $R_s(D_{x',\theta}) \in OP\mathscr{S}^{0,M}_{\beta',h}$ symmetrizes the singular problem (6.1).

Notation 6.1. (a) Let $\mathcal{O}_T = \{(x, \theta) \in \mathbb{R}^{N+1} \times \mathbb{T}^L : x_N \in [0, T]\}$ and set $C^{0,M}_{c,T} = \{V(x, \theta) \in C([0, T], C^M(\mathbb{R}^N \times \mathbb{T}^L, \mathbb{R}^m)) :$ supp V is compact$\}$.

(b) $\langle , \rangle$ (resp. $\langle\ \rangle_0$) denotes the L^2 inner product (resp. norm) on $\mathbb{R}^N \times \mathbb{T}^L$.

(c) For $t \in [0, T]$, $\mathscr{A}_s(t) \equiv \mathscr{A}(\varepsilon V_\varepsilon(x', t, \theta), D_{x'} + \frac{\beta' D_\theta}{\varepsilon})$.

(d) Set $\mathscr{F}(t) = (\mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon)(t)$.

THEOREM 6.1. *Let $D_{x_N} - \mathscr{A}(v, D_{x'})$ be symmetrizable for $v \in B_R$. Fix $K > 0$ and $\varepsilon_0 > 0$. Suppose $V_\varepsilon(x, \theta) \in C^{0,M_0}_{c,T}$ for $M_0 = 2(N+L+1)+1$ and satisfies for $0 < \varepsilon \leqslant \varepsilon_0$*

$$\text{(6.4)}\qquad \begin{aligned} &\text{(a)}\quad |\varepsilon V_\varepsilon|_{L^\infty(\mathcal{O}_T)} \leqslant R\\ &\text{(b)}\quad |V_\varepsilon|_{C^{0,M_0}_{c,T}} \leqslant K\\ &\text{(c)}\quad |\varepsilon \partial_{x_N} V_\varepsilon|_{L^\infty(\mathcal{O}_T)} \leqslant h(|V_\varepsilon|_{C^{0,1}_{c,T}}),\end{aligned}$$

for some increasing function h: $\mathbb{R}_+ \to \mathbb{R}_+$. Then there exist $C(K)$, $C_1(\varepsilon, K)$, $\varepsilon_1(K) \leqslant \varepsilon_0$ such that for $x_N \in [0, T]$ and $U_\varepsilon \in C^0([0, T], \mathscr{H}^1) \cap C^1([0, T], \mathscr{H}^0)$ ($\mathscr{H}^k$ as in Definition 2.1), we have for $0 < \varepsilon \leqslant \varepsilon_1(K)$ the estimate

(6.5)
$$\langle U_\varepsilon(x_N)\rangle_0 \leqslant C_1(\varepsilon, K)\, e^{C(K)\, x_N} \langle U_\varepsilon(0)\rangle_0 + C(K) \int_0^{x_N} e^{C(K)(x_N - t)} \langle \mathscr{F}(t)\rangle_0\, dt.$$

Here

$$\text{(6.6)}\qquad C_1(\varepsilon, K) = C_2 + \varepsilon C_3(K),$$

where C_2 is independent of K.

Proof. **1.** Letting $\mathscr{R}_\varepsilon(t) = R_{s,t}(D_{x',\theta})$, where $R_{s,t} = R(\varepsilon V_\varepsilon(x', t, \theta), X)$ for $R(v, \xi')$ as in Definition 6.1, as usual we have

(6.7)

$$\frac{\partial}{\partial t}\langle \mathcal{R}_\varepsilon(t)\, U_\varepsilon(t), U_\varepsilon(t)\rangle$$

$$= i\langle(\mathcal{R}_\varepsilon(t)\, \mathcal{A}_s(t) - \mathcal{A}_s(t)^*\, \mathcal{R}_\varepsilon(t))\, U_\varepsilon(t), U_\varepsilon(t)\rangle + i\langle \mathcal{R}_\varepsilon(t)\, \mathcal{F}(t), U_\varepsilon(t)\rangle$$
$$+ \langle \mathcal{R}_\varepsilon(t)\, U_\varepsilon(t), i\mathcal{F}(t)\rangle + \langle[\partial_t, \mathcal{R}_\varepsilon(t)]\, U_\varepsilon(t), U_\varepsilon(t)\rangle$$

(6.8) $$= i\langle(\mathcal{R}_\varepsilon(t)\, \mathcal{A}_s(t) - \mathcal{A}_s(t)^*\, \mathcal{R}_\varepsilon(t))\, U_\varepsilon(t), U_\varepsilon(t)\rangle + H(t).$$

Here

$$|H(t)| \leqslant C(K)(\langle U_\varepsilon(t)\rangle_0^2 + \langle \mathcal{F}(t)\rangle_0^2),$$

since $\mathcal{R}_\varepsilon(t)$ and $[\partial_t, \mathcal{R}_\varepsilon(t)]$ are bounded on $\mathcal{H}^0$ (see part 2 of the proof of Theorem 5.1).

In view of Definition 6.1(a), (b), use of Proposition 4.4 in the homogeneous calculus (for $\mathcal{R}_\varepsilon(t)\, \mathcal{A}_s(t)$), Proposition 4.5 (for $\mathcal{A}_s(t)^*\, \mathcal{R}_\varepsilon(t)$), and Proposition 4.3 (for $\mathcal{A}_s(t)^*\, \mathcal{R}_\varepsilon(t)^*$) implies that the first term in (6.8) is dominated by $C(K)\langle U_\varepsilon(t)\rangle_0^2$. Thus, integrating (6.7) and taking real parts gives

(6.9) $$\mathrm{Re}\langle \mathcal{R}_\varepsilon(x_N)\, U_\varepsilon(x_N), U_\varepsilon(x_N)\rangle$$
$$\leqslant C_1(\varepsilon, K)\langle U_\varepsilon(0)\rangle_0^2 + C(K)\int_0^{x_N} (\langle U_\varepsilon(t)\rangle_0^2 + \langle \mathcal{F}(t)\rangle_0^2)\, dt,$$

for $C_1(\varepsilon, K)$ as in (6.6) (recall (1.17)). The Garding inequality for homogeneous SPOs of Proposition 4.6 and Gronwall's inequality now provide a choice of $\varepsilon_1(K)$ such that (6.5) holds. ∎

Standard arguments [CP] using functional analysis (for existence of L^2 solutions) and the mollifiers of Proposition 2.4 (to show that L^2 solutions satisfy the estimate (6.5) and are thus unique) yield:

THEOREM 6.2. *Let $F_\varepsilon(x, \theta) \in L^2(\mathcal{O}_T)$, $G_\varepsilon(x', \theta) \in L^2(\mathbb{R}^N \times \mathbb{T}^L)$. Under the hypotheses of Theorem 6.1, there exist $C(K)$, $\varepsilon_1(K) \leqslant \varepsilon_0$ such that for $0 < \varepsilon \leqslant \varepsilon_1(K)$ the singular problem*

$$\mathcal{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon = F_\varepsilon \qquad \text{in } \mathcal{O}_T$$
$$U_\varepsilon = G_\varepsilon \qquad \text{on } \quad x_N = 0$$

has a unique solution $U_\varepsilon \in C^0([0, T], \mathcal{H}^0)$, and U_ε satisfies the estimate (6.5).

PART 3. QUASILINEAR PROBLEMS WITH OSCILLATORY DATA

7. *Fixed Boundaries*

In this section we revert to the use of Notation 1.3.

7.1. *Main Results*

THEOREM 7.1. (a) *Consider the quasilinear boundary problem* (1.1), *where* $G(x', \theta) \in H^{k+1}(b\Omega)$, $k \geqslant [M_0 + \frac{N+1}{2}]$ *satisfies*

$$\text{supp}\, G \subset \{x_0 \geqslant 0\} \cap \{|x'| \leqslant D\} \quad \text{for some} \quad D > 0 \tag{7.1}$$

and $\beta' \in \mathcal{G}^c$ *(Definition* 1.1*). Fix* $R > 0$ *and, with* $\mathcal{A}(v, \xi')$, $B(v)$, F_ε *as in* (1.4), *suppose*

$$(D_{x_N} - \mathcal{A}(v, D_{x'}), B(v)) \tag{7.2}$$

and the corresponding dual problem are Kreiss well-posed for $v \in B_R$ *(Definition* 4.10*). Suppose* $\alpha < \infty$ *is an upper bound for the propagation speed of* (7.2) *when* $v \in B_R$. *There exist an* $\varepsilon_1(k) > 0$, *a* $\mathbb{T}_k > 0$ *independent of* $\varepsilon \in (0, \varepsilon_1(k)]$, *and a unique* $U_\varepsilon(x, \theta) \in CH^k_{T_k}$ *satisfying the singular problem*

$$\begin{aligned} &D_{x_N} U_\varepsilon - \mathcal{A}\left(\varepsilon U_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) U_\varepsilon = F_\varepsilon(U_\varepsilon), \\ &B(\varepsilon U_\varepsilon)(U_\varepsilon)|_{x_N = 0} = G(x', \theta), \\ &U_\varepsilon = 0 \qquad \text{in} \quad x_0 < 0, \end{aligned} \tag{7.3}$$

and such that

$$v_\varepsilon = u_0 + \varepsilon U_\varepsilon\left(x, \frac{x'\beta'}{\varepsilon}\right)$$

is the unique C^1 *solution of* (1.1) *on* $\omega_{\mathbb{T}_k}$.

(b) *(Finite propagation speed)* α *is an upper bound for the propagation speed of* (7.3). *Thus, if* $y = (y_0, \bar{y}) \in \overline{\mathbb{R}}^{N+1}_+$ *and* G *vanishes inside the backward cone*

$$\Gamma_y = \{(x_0, \bar{x}, \theta) \in \Omega : |\bar{x} - \bar{y}| \leqslant \alpha\, |x_0 - y_0|, x_0 \leqslant y_0\},$$

then U_ε *vanishes in* Γ_y *(recall* $F_\varepsilon(0) = 0$*). For the systems described in Remark* 5.1 *there exists an upper bound* $\alpha < \infty$ *for the propagation speed.*

7.2. $L^2H^1_\gamma$ *and* CH^0_γ *Estimates*

When the data in Theorem 5.2 have higher tangential regularity, so does U_ε.

Proposition 7.1. (a) *Fix $R>0$ and $\varepsilon_0>0$. Suppose*

$$(D_{x_N}-\mathscr{A}(v,\xi'),B(v))$$

and the corresponding dual problem are Kreiss well-posed for $v\in B_R$. Suppose $V_\varepsilon(x,\theta)\in C_c^{0,M_0}$ and satisfies for $0<\varepsilon\leqslant\varepsilon_0$

$$\text{(7.4)}\qquad \begin{aligned}&\text{(a)}\quad |\varepsilon V_\varepsilon(x,\theta)|_*\leqslant R\\ &\text{(b)}\quad |\varepsilon\partial_{x_N}V_\varepsilon|_*\leqslant h(|V_\varepsilon|_{C_c^{0,1}})\end{aligned}$$

(with h as in Notation 1.3). Then there is a constant $\gamma_0(|V_\varepsilon|_{C_c^{0,M_0}})$ such that for $0<\varepsilon\leqslant\varepsilon_0$, $\gamma\geqslant\gamma_0$, $F_\varepsilon(x,\theta)\in L^2H^1_\gamma$, $G_\varepsilon(x',\theta)\in H^1_\gamma$ there is a unique solution $U_\varepsilon(x,\theta)\in L^2H^1_\gamma$ to the singular problem

$$\text{(7.5)}\qquad \begin{aligned}&\text{(a)}\quad \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\,U_\varepsilon\\ &\qquad = D_{x_N}U_\varepsilon-\mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'}+\frac{\beta' D_\theta}{\varepsilon}\right)U_\varepsilon=F_\varepsilon \qquad \text{in } \Omega\\ &\text{(b)}\quad B(\varepsilon V_\varepsilon)\,U_\varepsilon=G(x',\theta)\qquad \text{on } b\Omega,\end{aligned}$$

and U_ε satisfies

$$\text{(7.6)}\qquad \begin{aligned}&|U_\varepsilon|_{0,1,\gamma}+\frac{1}{\sqrt{\gamma}}\langle U_\varepsilon\rangle_{1,\gamma}\\ &\qquad\leqslant h(|V_\varepsilon|_{C_c^{0,M_0}})\left[\frac{1}{\gamma}|F_\varepsilon|_{0,1,\gamma}+\frac{1}{\sqrt{\gamma}}\langle G\rangle_{1,\gamma}\right].\end{aligned}$$

(b) *If F_ε, G_ε vanish in $x_0<0$, the same is true of U_ε.*

(c) *(Finite propagation speed) Suppose α is an upper bound for the propagation speed of $(D_{x_N}-\mathscr{A}(v,D_{x'}),B(v))$ when $v\in B_R$. Then α is an upper bound for the propagation speed of (7.5). Thus, if $y=(y_0,\bar{y})\in\overline{\mathbb{R}}^{N+1}_+$ and F_ε, G vanish inside the backward cone*

$$\Gamma_y=\{(x_0,\bar{x},\theta)\in\Omega:|\bar{x}-\bar{y}|\leqslant\alpha\,|x_0-y_0|,\ x_0\leqslant y_0\},$$

then U_ε vanishes in Γ_y.

Proof. (a) Write

$$\mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'}+\frac{\beta' D_\theta}{\varepsilon}\right) = \mathscr{A}\left(0, D_{x'}+\frac{\beta' D_\theta}{\varepsilon}\right)+\varepsilon V_\varepsilon \mathscr{A}_1\left(\varepsilon V_\varepsilon, D_{x'}+\frac{\beta' D_\theta}{\varepsilon}\right).$$

A standard mollifier argument [CP, Chap. 7] based on the L^2 estimate (5.26) shows $U_\varepsilon \in L^2 H^1_\gamma$, the key point here being that the mollifiers commute exactly with the singular part $\mathscr{A}(0, D_{x'}+\frac{\beta' D_\theta}{\varepsilon})$.

To establish (7.6) differentiate (7.5) with $\partial^\alpha_{(x', \theta)}$ where $|\alpha| \leqslant 1$, apply the L^2 estimate (5.26), and observe that

$$\left|\left[\mathscr{A}\left(\varepsilon V_\varepsilon, D_{x'}+\frac{\beta' D_\theta}{\varepsilon}\right), \partial^\alpha_{(x', \theta)}\right] U_\varepsilon\right|_{0, 0, \gamma} \leqslant h(|V_\varepsilon|_{C^{0,1}_c})\, |U_\varepsilon|_{0, 1, \gamma}, \tag{7.7}$$

$$\begin{aligned}\langle [B(\varepsilon V_\varepsilon), \partial^\alpha_{(x', \theta)}]\, U_\varepsilon \rangle_{0, \gamma} &\leqslant h(|V_\varepsilon|_{C^{0,1}_c}) \langle U_\varepsilon \rangle_{0, \gamma} \\ &\leqslant h(|V_\varepsilon|_{C^{0,1}_c})\, \frac{\langle U_\varepsilon \rangle_{1, \gamma}}{\gamma}.\end{aligned} \tag{7.8}$$

For (7.7) we have used $[\mathscr{A}(0, D_{x'}+\beta' D_\theta/\varepsilon), \partial^\alpha_{(x', \theta)}]=0$.

(b) This follows from Theorem 5.2.

(c) Let $v_\varepsilon(x)=V_\varepsilon(x, \beta' x'/\varepsilon)$ and $u_\varepsilon(x)=U_\varepsilon(x, \beta' x'/\varepsilon)$, and note that

$$\begin{aligned} D_{x_N} u_\varepsilon - \mathscr{A}(\varepsilon v_\varepsilon, D_{x'})\, u_\varepsilon &= F_\varepsilon\left(x, \frac{\beta' x'}{\varepsilon}\right) \\ B(\varepsilon v_\varepsilon)\, u_\varepsilon &= G\left(x', \frac{\beta' x'}{\varepsilon}\right). \end{aligned} \tag{7.9}$$

Thus, the propagation speed of U_ε is bounded above by α as well. ∎

PROPOSITION 7.2. *Fix $0<\delta<\delta'<1$. Choose $\chi(Z, \gamma)=\chi(Z_1, Z_2, \gamma) \in eS^0$* (2.39) *such that*

$$\begin{aligned} &0 \leqslant \chi \leqslant 1 \\ &\chi \equiv 1 \quad \text{on} \quad |Z_1, \gamma| \leqslant \delta\, |Z_2| \\ &\operatorname{supp} \chi \subset \{|Z_1, \gamma| \leqslant \delta'\, |Z_2|\}, \end{aligned} \tag{7.10}$$

and let $\chi_s(D_{x',\theta}) \in OPe\mathscr{S}^{0,\infty}_{\beta'}$ *be the associated operator. Let* V_ε, U_ε, F_ε *be as in Proposition* 7.1. *Then for* ε, γ *as in Proposition* 7.1

$$\text{(7.11)}\quad \begin{aligned} &\text{(a)}\quad |(1-\chi_s(D_{x',\theta}))\, U^\gamma_\varepsilon|_{\infty,0} \leqslant C|D_{x_N}(1-\chi_s(D_{x',\theta}))\, U^\gamma_\varepsilon|_{0,0} \\ &\text{(b)}\quad |D_{x_N}(1-\chi_s(D_{x',\theta}))\, U^\gamma_\varepsilon|_{0,0} \leqslant h(|V_\varepsilon|_*)\, |U_\varepsilon|_{0,1,\gamma} + |F_\varepsilon|_{0,0,\gamma}. \end{aligned}$$

Proof. (a) is a $1-D$ Sobolev estimate.

(b) With $\mathscr{A}_s(D_{x',\theta}) = \mathscr{A}(\varepsilon V_\varepsilon, (D_{x_0}-i\gamma, D_{x''}) + \frac{\beta' D_\theta}{\varepsilon}) \in OP\mathscr{S}^{1,M_0}_{\beta'}$ we have

(7.12)

$$\begin{aligned} &|D_{x_N}(1-\chi_s(D_{x',\theta}))\, U^\gamma_\varepsilon|_{0,0} \\ &\qquad \leqslant |\mathscr{A}_s(D_{x',\theta})(1-\chi_s)\, U^\gamma_\varepsilon|_{0,0} + |(1-\chi_s)\, F^\gamma_\varepsilon|_{0,0} + |[1-\chi_s, \mathscr{A}_s]\, U^\gamma_\varepsilon|_{0,0}. \end{aligned}$$

$[1-\chi_s, \mathscr{A}(0, D_{x'_\gamma} + \frac{\beta' D_\theta}{\varepsilon})] = 0$, so the commutator term is dominated by

$$\text{(7.13)}\qquad h(|V_\varepsilon|_*)\, |U^\gamma_\varepsilon|_{0,1}.$$

Note the calculus is not needed here since $\mathscr{A}_s$ is differential and χ_s is a bounded Fourier multiplier.

The first term on the right in (7.12) is $\leqslant$ (7.13) as well since

$$\text{(7.14)}\quad \left\langle (\xi_0 - i\gamma, \xi'') + \frac{m\beta'}{\varepsilon} \right\rangle \left| 1 - \chi\left(\xi', \frac{m\beta'}{\varepsilon}, \gamma \right) \right| \leqslant C\langle \xi_0 - i\gamma, \xi'' \rangle$$

by (7.10). ∎

We estimate $|\chi_s(D_{x',\theta})\, U^\gamma_\varepsilon|_{\infty,0}$ in the following key proposition.

PROPOSITION 7.3 (CH^0_γ estimate). *Fix* $R > 0$, $\varepsilon_0 > 0$, *and* $R' \leqslant R$ *as in* (7.23). *Suppose*

$$(D_{x_N} - \mathscr{A}(v, D_{x'}), B(v))$$

and the corresponding dual problem are Kreiss well-posed for $v \in B_R$. *Assume* $\beta' \in \mathscr{G}^c$ *and suppose* $V_\varepsilon \in C^{0,M_0}_c$ *satisfies for* $0 < \varepsilon \leqslant \varepsilon_0$

$$\text{(7.15)}\quad \begin{aligned} &\text{(a)}\quad |\varepsilon V_\varepsilon(x,\theta)|_* \leqslant R' \\ &\text{(b)}\quad |V_\varepsilon|_{C^{0,M_0}_c} \leqslant K \\ &\text{(c)}\quad |\varepsilon \partial_{x_N} V_\varepsilon|_* \leqslant h(|V_\varepsilon|_{C^{0,1}_c}), \end{aligned}$$

for $M_0 = 2(N+2)+1$. *Suppose* U_ε *satisfies*

$$(7.16)\qquad \begin{aligned} \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon &= F_\varepsilon(x,\theta) && \text{in } \Omega \\ B(\varepsilon V_\varepsilon)\, U_\varepsilon &= G(x',\theta) && \text{on } b\Omega, \end{aligned}$$

where $F_\varepsilon \in L^2H^1_\gamma \cap CH^0_\gamma$, $G \in H^1_\gamma$, *and supp* $F_\varepsilon \subset \{0 \leqslant x_N \leqslant E\}$.

If $\delta' > 0$ *in* (7.10) *is small enough, there exists a constant* $\gamma_0(K)$ *such that for* $\gamma \geqslant \gamma_0$, $0 < \varepsilon \leqslant \varepsilon_0$

$$(7.17)\qquad |\chi_s U^\gamma_\varepsilon|_{\infty,0} \leqslant C(K,E)\left[\frac{|F^\gamma_\varepsilon|_{\infty,0}}{\sqrt{\gamma}} + \langle G^\gamma\rangle_0 + |U^\gamma_\varepsilon|_{0,0} + \frac{\langle U^\gamma_\varepsilon\rangle_0}{\gamma}\right],$$

$$(7.18)\qquad |U^\gamma_\varepsilon|_{\infty,0} \leqslant C(K,E)\left[\left(\frac{|F^\gamma_\varepsilon|_{\infty,0}}{\sqrt{\gamma}} + \langle G^\gamma\rangle_0\right) + \left(\frac{|F^\gamma_\varepsilon|_{0,1}}{\gamma} + \frac{\langle G^\gamma\rangle_1}{\sqrt{\gamma}}\right)\right].$$

The first step in the proof of Proposition 7.3 is a block structure lemma. Let $\lambda_i(\pm\beta')$, $i = 1, \ldots, \mathscr{M}(\pm\beta')$ be the *distinct* roots of

$$p(\pm\beta', \xi_N) = \det(\xi_N - \mathscr{A}(0, \pm\beta')) = 0,$$

indexed so that $\lambda_i(-\beta') = -\lambda_i(\beta')$. Denoting the (algebraic) multiplicity of $\lambda_i(\pm\beta')$ by $m_i(\pm\beta')$, we have $m_i(\beta') = m_i(-\beta')$.

Since $\beta' \in \mathscr{G}^c$ we can write the index set $\{1, 2, \ldots, \mathscr{M}(\pm\beta')\}$ as a disjoint union of subsets $\mathscr{O}(\pm\beta')$, $\mathscr{P}(\pm\beta')$, $\mathscr{I}(\pm\beta')$, $\mathscr{N}(\pm\beta')$ corresponding to the roots $\lambda_i(\pm\beta')$ for which the associated modes $(\pm\beta', \lambda_i(\pm\beta'))$ are respectively outgoing, such that Im $\lambda_i(\pm\beta')$ is positive, incoming or such that Im $\lambda_i(\pm\beta')$ is negative (Definition 1.1). When $(\beta', \lambda_i(\beta'))$ is outgoing (resp. incoming), $(-\beta', -\lambda_i(\beta'))$ is also outgoing (resp. incoming). Moreover, $p(\beta', \xi_N)$ has real coefficients, so we may take $\mathscr{O}(\beta') = \mathscr{O}(-\beta'), \ldots, \mathscr{N}(\beta') = \mathscr{N}(-\beta')$.

Set

$$m_+ = \sum_{i \in \mathscr{P}(\beta')} m_i(\beta'), \qquad m_- = \sum_{i \in \mathscr{N}(\beta')} m_i(\beta')$$

and note $m_+ = m_-$. For $i \in \mathscr{O}(\beta') \cup \mathscr{I}(\beta')$ we have, of course, $m_i(\beta') = 1$. Set

$$m_{\mathscr{O}} = \operatorname{card} \mathscr{O}(\beta') \qquad \text{and} \qquad m_{\mathscr{I}} = \operatorname{card} \mathscr{I}(\beta').$$

LEMMA 7.1 [K, CP]. *Let*

$$\Sigma = \{z = (v, X, \gamma) \in B_R \times \mathbb{R}^N \times [0, \infty) : (X, \gamma) \neq 0\}$$

and $\mathscr{A}(z) = \mathscr{A}(v, \xi_0 - i\gamma, \xi'')$. *There exists an invertible* $m \times m$ *matrix* $S(z)$, *homogeneous of degree zero in* (X, γ) *and* C^∞ *for* z *in a conic neighborhood* Γ *of* $\{(0, \beta', 0), (0, -\beta', 0)\}$ *in* Σ, *such that*

$$S^{-1}(z)\, \mathscr{A}(z)\, S(z) = \Lambda(z), \tag{7.19}$$

where $\Lambda(z)$ *has the block diagonal form with blocks of dimension* $m_{\mathcal{O}}$, m_+, $m_{\mathscr{I}}$, m_-:

$$\Lambda(z) = \begin{bmatrix} \Lambda_{\mathcal{O}}(z) & & & \\ & \Lambda_+(z) & & \\ & & \Lambda_{\mathscr{I}}(z) & \\ & & & \Lambda_-(z) \end{bmatrix}. \tag{7.20}$$

$\Lambda_{\mathcal{O}}(z)$ *(resp.* $\Lambda_{\mathscr{I}}(z)$*) is a diagonal matrix whose entries are simple eigenvalues* $\lambda_i(z)$ *(abuse of notation here) of* $\mathscr{A}(z)$ *satisfying for some* $C > 0$ *and* $z \in \Gamma$:

$$\begin{gathered} Im\, \lambda_i(z) = \gamma H_i(z) \geqslant C\gamma, \qquad i \in \mathcal{O}(\beta') \\ (resp.,\ Im\, \lambda_i(z) = -\gamma H_i(z) \leqslant -C\gamma,\ i \in \mathscr{I}(\beta')), \end{gathered} \tag{7.21}$$

where $H_i(z)$ *is homogeneous of degree zero. Moreover,*

$$\begin{gathered} Im\, \Lambda_+(z) \geqslant C\langle X, \gamma\rangle \\ Im\, \Lambda_-(z) \leqslant -C\langle X, \gamma\rangle \end{gathered} \tag{7.22}$$

for $z \in \Gamma$.

For $\underline{z} = (0, \pm\beta', 0)$ *the entries of* $\Lambda_{\mathcal{O}}(\underline{z})$ *(resp.* $\Lambda_{\mathscr{I}}(\underline{z})$*) are* $\lambda_i(\pm\beta')$, $i \in \mathcal{O}(\beta')$ *(resp.* $\mathscr{I}(\beta')$*). The eigenvalues of* $\Lambda_+(\underline{z})$ *(resp.* $\Lambda_-(\underline{z})$*) are* $\lambda_i(\pm\beta')$, $i \in \mathscr{P}(\beta')$ *(resp.* $\mathscr{N}(\beta')$*).*

We may take

$$\Gamma = B_{R'} \times \Gamma'_{\beta'} \tag{7.23}$$

for some $R' \leqslant R$, $\Gamma'_{\beta'} \subset \mathbb{R}^N \times [0, \infty)$.

Remark 7.1. (a) (Lopatinski determinant) Denote the columns of $S(z)$, ordered from left to right, by $r_i(z)$, where i belongs consecutively to $\mathcal{O}(\beta')$, $\mathscr{P}(\beta')$, $\mathscr{I}(\beta')$, and $\mathscr{N}(\beta')$. Write

$$S(z) = [S^+(z) \quad S^-(z)], \tag{7.24}$$

where $S^+(z)$ (resp. $S^-(z)$) is the matrix whose columns are $r_i(z)$, $i \in \mathcal{O}(\beta') \cup \mathscr{P}(\beta')$ (resp. $i \in \mathscr{I}(\beta') \cup \mathscr{N}(\beta')$), and set

$$B^{\pm}(z) = B(v)\, S^{\pm}(z) \tag{7.25}$$

for $z \in \Gamma$. A necessary condition for the Kreiss well-posedness assumed in Proposition 7.3 is that $B^+(z)$ is a $\mu \times \mu$ matrix such that

$$\det B^+(z) > C > 0 \tag{7.26}$$

for $z \in \Gamma$ ([K]).

(b) (Extension to Σ) We need to extend $\mathscr{A}(z)|_\Gamma$, $S(z)$, and $\Lambda(z)$ so that (7.19)–(7.22), (7.26) hold for all $z \in \Sigma$. For this, first extend $\Lambda(z)$ to $\tilde{\Lambda}(z) \in S^1$ (1.7) of the form (7.20) satisfying (7.21), (7.22), and extend the $r_i(z)$ to m independent columns defining

$$\tilde{S}(z) = [\tilde{S}^+(z) \quad \tilde{S}^-(z)] \in S^0$$

such that $\tilde{B}^+(z) = B(v)\, \tilde{S}^+(z)$ satisfies (7.26). Then set

$$\tilde{\mathscr{A}}(z) = \tilde{S}(z)\, \tilde{\Lambda}(z)\, \tilde{S}^{-1}(z) \in S^1. \tag{7.27}$$

Henceforth, we will drop the tildes on $\tilde{\Lambda}$, $\tilde{S}$, $\tilde{B}^+$, but not on $\tilde{\mathscr{A}}(z)$ (in order to avoid confusion with $\mathscr{A}(z)$ which is already defined on Σ).

(c) With $\Gamma = B_{R'} \times \Gamma'_{\beta'}$ as in (7.23) observe that if $\delta' > 0$ in (7.10) is small enough, then

$$(X, \gamma) \in \Gamma'_{\beta'} \text{ for } X = Z_1 + Z_2 \text{ when } (Z_1, Z_2, \gamma) = \left(\xi', \frac{m\beta'}{\varepsilon}, \gamma\right) \in \operatorname{supp} \chi. \tag{7.28}$$

Notation 7.1. (a) In the next proof we will sometimes suppress the subscript ε and also write simply χ, S, $\mathscr{A}$, etc., instead of $\chi_s(D_{x',\theta})$, $S_s(D_{x',\theta})$, $\mathscr{A}_s(D_{x',\theta})$ for SPOs in $OPe\mathscr{S}^{k,M}_{\beta'}$ (2.39).

(b) Residual operators $r_{\varepsilon,\gamma}$ of order 0 (resp. 1) will be denoted r_0 (resp. r_1) and may change from line to line.

Proof of Proposition 7.3. **1.** (Microlocalize) Rewrite (7.16) as

$$\begin{aligned} D_{x_N} U^\gamma_\varepsilon - \mathscr{A}_s(D_{x',\theta})\, U^\gamma_\varepsilon &= F^\gamma_\varepsilon \\ B(\varepsilon V_\varepsilon)\, U^\gamma_\varepsilon &= G^\gamma. \end{aligned} \tag{7.29}$$

Choose δ' in (7.10) such that (7.28) holds and apply $\chi_s(D_{x',\theta}) \in OPe\mathscr{S}^{0,\infty}_{\beta'}$ to (7.29) to get

$$\begin{aligned} &\text{(a)} \quad D_{x_N} \chi U^\gamma - \mathscr{A} \chi U^\gamma = \chi F^\gamma + [\chi, \mathscr{A}]\, U^\gamma \\ &\text{(b)} \quad B(\varepsilon V)\, \chi U^\gamma = \chi G^\gamma + [B(\varepsilon V), \chi]\, U^\gamma. \end{aligned} \tag{7.30}$$

Product theorems in the extended SPO calculus give

$$[\chi, \mathscr{A}]\, U^{\gamma} = r_0 U^{\gamma}$$
(7.31)
$$[B(\varepsilon V), \chi]\, U^{\gamma} = r_1 U^{\gamma}.$$

2. (Solve $SW = \chi U^{\gamma}$ exactly) To find $W \in L^2(\Omega)$ apply $S_s^{-1}(D_{x', \theta}) \in OP\mathscr{S}^{0, M_0}_{\beta'}$ to obtain

(7.32) $$(1 + r_1)\, W = \chi U^{\gamma},$$

where r_1 is an operator like that defined by R_ε in (2.33) with L^2 operator norm

$$|r_1| \leqslant \frac{C(K)}{\gamma}.$$

The dependence of C on K is clear from the proof of Proposition 2.7. Choose $\gamma \geqslant 4C(K)$ and invert $1 + r_1$ with a Neumann series to obtain $W \in L^2(\Omega)$.

This step allows us to avoid unacceptable error terms like $D_{x_N} r_1 U^{\gamma}$ in Step 3.

3. (Diagonalize) Apply S^{-1} to (7.30) to get

$$S^{-1} D_{x_N} SW - S^{-1} \mathscr{A} SW = r_0 F^{\gamma} + r_0 U^{\gamma}$$
(7.33)
$$B(\varepsilon V)\, SW = \chi G^{\gamma} + r_1 U^{\gamma}.$$

To avoid terms like $D_{x_N} r_1 W$ we set $\mathscr{W} = S^{-1} SW$ and use $[S^{-1}, D_{x_N}] = r_0$ to rewrite (7.33):

$$D_{x_N} \mathscr{W} - S^{-1} \mathscr{A} S \mathscr{W} = r_0 F^{\gamma} + r_0 U^{\gamma}$$
(7.34)
$$B(\varepsilon V)\, S\mathscr{W} = \chi G^{\gamma} + r_1 U^{\gamma}.$$

Here we have used the calculus to conclude

$$S^{-1} \mathscr{A} SW - S^{-1} \mathscr{A} S (S^{-1} SW) = r_0 U^{\gamma}$$
(7.35)
$$B(\varepsilon V)\, SW - B(\varepsilon V)(S^{-1} SW) = r_1 U^{\gamma}.$$

Recall $\Gamma = B_{R'} \times \Gamma'_{\beta'}$ and for $0 < \varepsilon \leqslant \varepsilon_0$

(7.36) $$|\varepsilon V_\varepsilon|_{L^\infty(\Omega)} \leqslant R'.$$

$\tilde{\mathscr{A}}(z) = \mathscr{A}(z)$ for $z \in \Gamma$, so we deduce from (7.36), (7.28), and product theorems that

(7.37) $$S^{-1} \tilde{\mathscr{A}} S \mathscr{W} - S^{-1} \mathscr{A} S \mathscr{W} = r_0 U^{\gamma}.$$

Finally, (7.27), (7.37), and the calculus allow us to rewrite (7.34) in block diagonal form

$$(7.38)\qquad \begin{aligned} &\text{(a)}\quad D_{x_N}\mathscr{W}-\Lambda\mathscr{W}=r_0F^\gamma+r_0U^\gamma\\ &\text{(b)}\quad B(\varepsilon V)\,S\mathscr{W}=r_0G^\gamma+r_1U^\gamma, \end{aligned}$$

where $\Lambda=\Lambda_s(D_{x',\theta})$ is the element of $OP\mathscr{S}_c^{1,M_0}$ associated to $\Lambda(z)$.

4. (Invert boundary condition) Write $\mathscr{W}=(\mathscr{W}^+,\mathscr{W}^-)$, corresponding to the decomposition

$$S(z)=[S^+(z)\quad S^-(z)],$$

and let $S^\pm$, $B^\pm$ denote the elements of $OP\mathscr{S}_{\beta'}^{0,M_0}$ defined by the symbols $S^\pm(z)$, $B^\pm(z)$ ((7.24), (7.25)). Thus, (7.38b) becomes

$$(7.39)\qquad B^+\mathscr{W}^++B^-\mathscr{W}^-=r_0G^\gamma+r_1U^\gamma,$$

and using (7.26) and the calculus gives

$$(7.40)\qquad \mathscr{W}^+=r_0\mathscr{W}^-+r_0G^\gamma+r_1U^\gamma\qquad \text{on } b\Omega.$$

5. ($|U_\varepsilon^\gamma|_{\infty,0}\leqslant\frac{C}{\varepsilon}$) This 1-D Sobolev estimate follows from (7.6) with $k=1$ by rewriting $D_{x_N}U_\varepsilon^\gamma$ using Eq. (7.29). In fact, the usual approximation argument gives

$$(7.41)\qquad \langle U_\varepsilon^\gamma(x_N)\rangle_0\to 0 \text{ as } x_N\to+\infty,\qquad \text{for each fixed } \varepsilon.$$

We need to improve this to the uniform estimate (7.18).

6. (Estimate $|\mathscr{W}|_{\infty,0}$) Write $\mathscr{W}^+=(\mathscr{W}_{\mathscr{O}},\mathscr{W}_+)$, $\mathscr{W}^-=(\mathscr{W}_{\mathscr{I}},\mathscr{W}_-)$, corresponding to the blocks of (7.20), and set

$$(7.42)\qquad \Lambda^-(z)=\begin{bmatrix}\Lambda_{\mathscr{I}}(z) & \\ & \Lambda_-(z)\end{bmatrix}.$$

First obtain energy estimates for the system with boundary conditions "at $+\infty$"

$$(7.43)\qquad (D_{x_N}-\Lambda^-)\,\mathscr{W}^-=r_0F^\gamma+r_0U^\gamma=\mathscr{F}^-,$$

where r_0 is an $(m_{\mathscr{I}}+m_-)\times m$ matrix operator. For each x_N take the L^2 inner product $\langle,\rangle$ of both sides of (7.43) with $\mathscr{W}^-(x_N)$, and integrate from x_N to $+\infty$. Take the imaginary part of both sides and use (7.41), (7.21), (7.22), and SPO Garding inequalities (Corollary 3.1) to obtain

$$(7.44)\qquad \langle \mathscr{W}^-(x_N)\rangle_0^2+\gamma\,|\mathscr{W}^-|_{0,0}^2$$
$$\leqslant |\mathrm{Im}(\mathscr{F}^-,\mathscr{W}^-)|$$
$$\leqslant C(K)\,|F^\gamma|_{\infty,0}\int_0^E \langle \mathscr{W}^-(x_N')\rangle_0\,dx_N'+C(K)\,|U^\gamma|_{0,0}^2$$
$$\leqslant C(K,E)\,\frac{|F^\gamma|_{\infty,0}^2}{\gamma}+\frac{\gamma\,|\mathscr{W}^-|_{0,0}^2}{2}+C(K)\,|U^\gamma|_{0,0}^2.$$

Next consider

$$(7.45)\qquad \begin{aligned}&\text{(a)}\quad (D_{x_N}-\Lambda^+)\,\mathscr{W}^+=r_0F^\gamma+r_0U^\gamma=\mathscr{F}^+ &&\text{on } \Omega\\ &\text{(b)}\quad \mathscr{W}^+=r_0\mathscr{W}^-+r_0G^\gamma+r_1U^\gamma &&\text{on } b\Omega.\end{aligned}$$

Proceed as above, but now integrate from 0 to x_N to obtain

$$(7.46)\qquad \langle \mathscr{W}^+(x_N)\rangle_0^2+\gamma\,|\mathscr{W}^+|_{0,0}^2$$
$$\leqslant C(K,E)\,\frac{|F^\gamma|_{\infty,0}^2}{\gamma}+\frac{\gamma\,|\mathscr{W}^+|_{0,0}^2}{2}+C(K)\,|U^\gamma|_{0,0}^2+\langle \mathscr{W}^+(0)\rangle_0^2.$$

Add (7.44) and (7.46), rewrite $\mathscr{W}^+(0)$ using (7.45b), and use (7.44) to estimate $\langle \mathscr{W}^-(0)\rangle_0$ to conclude for γ large

$$(7.47)\qquad \langle \mathscr{W}(x_N)\rangle_0^2+\gamma\,|\mathscr{W}|_{0,0}^2$$
$$\leqslant C(K,E)\left[\frac{|F^\gamma|_{\infty,0}^2}{\gamma}+|U^\gamma|_{0,0}^2+\langle G^\gamma\rangle_0^2+\frac{\langle U^\gamma(0)\rangle_0^2}{\gamma^2}\right].$$

7. (Estimate $|\chi_s U_\varepsilon^\gamma|_{\infty,0}$) The estimate (7.17) now follows from (7.47) and (7.32).

8. (Estimate $|U_\varepsilon^\gamma|_{\infty,0}$) The estimate (7.18) follows directly from (7.17), the estimates for $(1-\chi_s)\,U_\varepsilon^\gamma$ in (7.11), and (7.6). ▌

As an immediate consequence of Propositions 7.1 and 7.3 we have

COROLLARY 7.1. *Under the hypotheses of Proposition* 7.3 *(in particular, recall* $|V_\varepsilon|_{C_c^{0,M_0}}\leqslant K$ *for* $0<\varepsilon\leqslant\varepsilon_0$*), we have for* $0<\varepsilon\leqslant\varepsilon_0$, $\gamma\geqslant\gamma_0(K)$

$$(7.48)\quad |U_\varepsilon|_{\infty,0,\gamma}+|U_\varepsilon|_{0,1,\gamma}+\frac{\langle U\rangle_{1,\gamma}}{\sqrt{\gamma}}$$
$$\leqslant C(K,E)\left[\left(\frac{|F_\varepsilon|_{\infty,0,\gamma}}{\sqrt{\gamma}}+\langle G\rangle_{0,\gamma}\right)+\left(\frac{|F_\varepsilon|_{0,1,\gamma}}{\gamma}+\frac{\langle G\rangle_{1,\gamma}}{\sqrt{\gamma}}\right)\right].$$

In preparation for the main linear estimate we now recall some standard tools with slight modifications.

7.3. *Seeley Extensions and Nonlinear Estimates*

LEMMA 7.2 (Seeley extensions, [CP]). *For any $T>0$, $k\in\mathbb{N}$, given $U(x,\theta)\in CH^k_T$ (resp. $L^2H^k_T$), there is an extension $\tilde{U}\in CH^k$ (resp. L^2H^k) such that*

$$\text{(a)}\quad \tilde{U}=U \quad \text{on } \Omega_T$$

$$\text{(b)}\quad |\tilde{U}|_{\infty,k}\leqslant h(k)\,|U|_{\infty,k,T} \qquad (\text{resp. } |\tilde{U}|_{0,k}\leqslant h(k)\,|U|_{0,k,T}).$$

$h(k)$ depends just on the C^k norm of the cutoff function used to define the extension. It is independent of T even though $\tilde{U}$ depends on T.

Notation 7.2. (a) For $k\in\mathbb{N}$ let ∂^k denote the collection of tangential operators $\partial^\alpha_{(x',\theta)}$ with $|\alpha|=k$ (α is a multi-index). Sometimes ∂^k is used to denote a particular member of this collection. Set $\partial^0\phi=\phi$.

(b) For $k\in\{1,2,3,...\}$ denote by $\partial^{\langle k\rangle}\phi$ the set of products of the form $(\partial^{\alpha_1}\phi_{i_1})\cdots(\partial^{\alpha_r}\phi_{i_r})$ where $1\leqslant r\leqslant k$, $\alpha_1+\cdots\alpha_r=k$, $\alpha_i\geqslant 1$. Set $\partial^{\langle 0\rangle}\phi=1$.

LEMMA 7.3 (Moser estimates). *For $k\in\mathbb{N}$ let $\alpha_1+\cdots+\alpha_r\leqslant k$, $\alpha_i\in\mathbb{N}$. Suppose $v_i\in H^k_T\cap L^\infty(b\Omega_T)$, $u_i, u\in CH^k_T\cap L^\infty(\Omega_T)$, $w_i, w\in L^2H^k_T\cap L^\infty(\Omega_T)$, and $F(\cdot)$ is C^∞ with $F(0)=0$. There exists C independent of T such that*

$$\text{(a)}\quad |(\partial^{\alpha_1}v_1)\cdots(\partial^{\alpha_r}v_r|_{0,T}\leqslant C\sum_{i=1}^r |v_i|_{k,T}\left(\prod_{j\neq i}|v_i|_*\right)$$

$$\text{(b)}\quad |(\partial^{\alpha_1}w_1)\cdots(\partial^{\alpha_r}w_r|_{0,0,T}\leqslant C\sum_{i=1}^r |w_i|_{0,k,T}\left(\prod_{j\neq i}|w_i|_*\right)$$

$$\text{(c)}\quad |(\partial^{\alpha_1}u_1)\cdots(\partial^{\alpha_r}u_r|_{\infty,0,T}\leqslant C\sum_{i=1}^r |u_i|_{\infty,k,T}\left(\prod_{j\neq i}|u_i|_*\right)$$

$$\text{(d)}\quad |F(w)|_{0,k,T}\leqslant h(|w|_*)\,|w|_{0,k,T}$$

$$\text{(e)}\quad |F(u)|_{\infty,k,T}\leqslant h(|u|_*)\,|u|_{\infty,k,T}$$

LEMMA 7.4 (Commutator estimates). *Assume $A(\cdot)$, $B(\cdot)\in C^\infty$, u_1, $u_2\in L^\infty W^{1,\infty}\cap CH^k_T$, $w_1, w_2\in L^\infty W^{1,\infty}\cap L^2H^{k+1}_T$, and $v_1\in W^{1,\infty}\cap H^{k+1}_T$, $v_2\in L^\infty(b\Omega_T)\cap H^k_T$. Then*

$$\text{(a)}\quad |[A(u_1)\,\partial^1,\partial^k]\,u_2|_{\infty,0,T}$$
$$\leqslant h(|u_1|^*)\,|u_2|_{\infty,k,T}+h(|u_1|^*)\,|u_2|^*\,|u_1|_{\infty,k,T}$$

$$\text{(b)}\quad |[A(w_1)\,\partial^1,\partial^k]\,w_2|_{0,1,T}$$
$$\leqslant h(|w_1|^*)\,|w_2|_{0,k+1,T}+h(|w_1|^*)\,|w_2|^*\,|w_1|_{0,k+1,T}$$

$$\text{(c)}\quad \langle[B(v_1),\partial^k]\,v_2\rangle_{1,T}\leqslant h(\langle v_1\rangle^*)\langle v_2\rangle_{k,T}+h(\langle v_1\rangle^*)\langle v_2\rangle_*\,\langle v_1\rangle_{k+1,T}.$$

Proof of Lemmas 7.3 *and* 7.4 **1.** (Lemma 7.3) (a) is a classical consequence of the Gagliardo–Nirenberg and Hölder inequalities. (b)–(e) are trivial consequences of (a).

2. (Lemma 7.4) (a) The commutator is a sum of terms of the form

$$\phi(u_1)\,\partial^{\langle j\rangle}u_1\partial^l u_2 \tag{7.49}$$

where $\phi(\cdot)\in C^\infty$ and $j+l=k+1$, $l\geqslant 1$, $j\geqslant 1$. Rewrite (7.49) with obvious notation as

$$\phi(u_1)\,\partial^{\langle j'\rangle}(\partial u_1)\,\partial^{l'}(\partial u_2), \tag{7.50}$$

where $j'+l'=k-1$, and apply Lemma 7.3(c).

Lemma 7.3(b) (resp. (a)) and the same kind of argument yield part (b) (resp. (c)) of Lemma 7.4. ∎

7.4. *The Main Linear Estimate*

Notation 7.3. $C^{0,M}_{c,T}=\{V(x,\theta)\in C(x_N\colon C^M(b\Omega_T,\mathbb{R}^m))\colon \operatorname{supp} V$ is compact$\}$.

THEOREM 7.2. *Fix* $T>0$, $R>0$, $\varepsilon_0>0$, $k\in\mathbb{N}$, *and* $R'\leqslant R$ *as in* (7.23). *Suppose* $(D_{x_N}-\mathscr{A}(v,D_{x'}),B(v))$ *and the corresponding dual problem are Kreiss well-posed for* $v\in B_R$. *Assume* $\beta'\in\mathscr{G}^c$ *and suppose* $V_\varepsilon\in C^{0,M_0}_{c,T}\cap\mathbb{H}^k_T$ *satisfies for* $0<\varepsilon\leqslant\varepsilon_0$

$$\text{(a)}\quad |\varepsilon V_\varepsilon(x,\theta)|_*\leqslant R',\qquad |\varepsilon\partial_{x_N}V_\varepsilon|_*\leqslant h(|V_\varepsilon|_{C^{0,1}_{c,T}})$$

$$\text{(b)}\quad |V_\varepsilon|_{C^{0,M_0}_{c,T}}\leqslant K.$$

Suppose U_ε *satisfies*

$$\begin{aligned} \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta})\,U_\varepsilon &= F_\varepsilon(x,\theta) && \text{in } \Omega_T\\ B(\varepsilon V_\varepsilon)\,U_\varepsilon &= G(x',\theta) && \text{on } b\Omega_T,\\ U_\varepsilon &= 0 && \text{in } x_0<0, \end{aligned} \tag{7.51}$$

where F_ε *and* G_ε *vanish in* $x_0<0$, $F_\varepsilon\in L^2H^{k+1}_T\cap CH^k_T$, $G\in H^{k+1}_T$, *and* supp $F_\varepsilon\subset\{0\leqslant x_N\leqslant E\}$. *Then there exists a constant* $\gamma_0(K,|V_\varepsilon|_{\infty,k,T},|V_\varepsilon|_{0,k+1,T})$ *such that for* $\gamma\geqslant\gamma_0$, $0<\varepsilon\leqslant\varepsilon_0$

$$\begin{aligned} &|U_\varepsilon|_{\infty,k,T}+|U_\varepsilon|_{0,k+1,T}+\frac{\langle U_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}\\ &\quad\leqslant e^{\gamma T}C(K,E)\Bigg[\left(\frac{|F_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}}+\frac{|F_\varepsilon|_{0,k+1,T}}{\gamma}+\frac{\langle G\rangle_{k+1,T}}{\sqrt{\gamma}}\right)\\ &\qquad+\Bigg(h(|V_\varepsilon|^*)\,|U_\varepsilon|^*\left(\frac{|V_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}}+\frac{|V_\varepsilon|_{0,k+1,T}}{\gamma}\right)\\ &\qquad+h(\langle V_\varepsilon\rangle^*)\langle U_\varepsilon\rangle_*\frac{\langle V_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}\Bigg)\Bigg]. \end{aligned} \tag{7.52}$$

Proof. **1.** ($\mathbb{H}_T^0$ estimate) Use Seeley extensions of (F_ε, G) (resp. V_ε) to extend the data (resp. coefficients) of (7.51) from Ω_T to Ω. Observe that for $k \in \mathbb{N}, T > 0$

$$\begin{aligned} |W|_{\infty, k, T} &\leqslant h(k)\, e^{\gamma T}\, |W^\gamma|_{\infty, k} \\ |W|_{0, k, T} &\leqslant h(k)\, e^{\gamma T}\, |W^\gamma|_{0, k}. \end{aligned} \tag{7.53}$$

Modifying F_ε, G, and V_ε in $x_0 > T$ does not change U_ε in Ω_T, so from the estimate (7.48), (7.53), and Lemma 7.2(b) we obtain for $0 < \varepsilon \leqslant \varepsilon_0$, $\gamma \geqslant \gamma_0(K)$,

$$\begin{aligned} &|U_\varepsilon|_{\infty, 0, T} + |U_\varepsilon|_{0, 1, T} + \frac{\langle U_\varepsilon \rangle_{1, T}}{\sqrt{\gamma}} \\ &\quad \leqslant e^{\gamma T} C(K, E) \left[\left(\frac{|F_\varepsilon|_{\infty, 0, T}}{\sqrt{\gamma}} + \frac{|F_\varepsilon|_{0, 1, T}}{\gamma} + \frac{\langle G \rangle_{1, T}}{\sqrt{\gamma}} \right) \right]. \end{aligned} \tag{7.54}$$

Here we have used $\langle G \rangle_{0, \gamma} \leqslant \langle G \rangle_{1, \gamma} / \gamma$.

2. To obtain (7.52) we apply (7.54) to the system

$$\begin{aligned} \mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x, \theta})\, \partial^k U_\varepsilon &= \partial^k F_\varepsilon + [\mathscr{L}, \partial^k]\, U_\varepsilon \\ B(\varepsilon V_\varepsilon)\, \partial U_\varepsilon &= \partial^k G + [B, \partial^k]\, U_\varepsilon, \end{aligned} \tag{7.55}$$

noting that ∂^k commutes exactly with the singular part of $\mathscr{L}$:

$$\left[\mathscr{A}\left(0, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right), \partial^k \right] = 0. \tag{7.56}$$

Thus, Lemma 7.4(a),(c) imply

$$\begin{aligned} &\text{(a)} \quad |[\mathscr{L}, \partial^k]\, U_\varepsilon|_{\infty, 0, T} \leqslant h(|V_\varepsilon|^*)\, |U_\varepsilon|_{\infty, k, T} + h(|V_\varepsilon|^*)\, |U_\varepsilon|^*\, |V_\varepsilon|_{\infty, k, T} \\ &\text{(b)} \quad \langle [B, \partial^k]\, U_\varepsilon \rangle_{1, T} \leqslant h(\langle V_\varepsilon \rangle^*) \langle U_\varepsilon \rangle_{k, T} + h(\langle V_\varepsilon \rangle^*) \langle U_\varepsilon \rangle_*\, \langle V_\varepsilon \rangle_{k+1, T}. \end{aligned} \tag{7.57}$$

The first term on the right in each of (a) and (b) can be absorbed by the left side of (7.52) by taking γ large.

$|[\mathscr{L}, \partial^k]\, U_\varepsilon|_{0, 1, T}$ is handled the same way using Lemma 7.4(b). ▮

COROLLARY 7.2. *Make the same hypotheses as in Theorem 7.2, but take* $k \geqslant [1 + \frac{N+1}{2}]$. *There exists a constant* $\mathscr{T}_0(K)$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0$, $T < \mathscr{T}_0$, *and* $\gamma = \frac{1}{T}$

(7.58)
$$|U_\varepsilon|_{\infty,k,T}+|U_\varepsilon|_{0,k+1,T}+\frac{\langle U_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}} \leqslant C(K,E)\left[\left(\frac{|F_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}}+\frac{|F_\varepsilon|_{0,k+1,T}}{\gamma}+\frac{\langle G\rangle_{k+1,T}}{\sqrt{\gamma}}\right)\right].$$

Proof. $|U_\varepsilon|^* \leqslant C\,|U_\varepsilon|_{\infty,k,T}$ so the term involving $|U_\varepsilon|^*$ on the right in (7.52) can be absorbed by taking $\mathcal{T}_0$ small.

Since $U_\varepsilon = 0$ on $x_0 < 0$ we have

(7.59)
$$\langle U_\varepsilon\rangle_* \leqslant C\langle U_\varepsilon\rangle_{\frac{N+1}{2},T} \leqslant CT\langle U_\varepsilon\rangle_{k,T}.$$

Writing

$$\frac{\langle V_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}=\frac{\sqrt{T}\,\langle V_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}\sqrt{T}}$$

and using (7.59), we can absorb the term involving $\langle U_\varepsilon\rangle_*$ by taking $\mathcal{T}_0$ small enough. ∎

7.5. *Iteration: $\mathbb{H}^k_T$ Boundedness*

The proof of Theorem 7.1 is based on the iteration scheme (1.5) and will be contained in the next two propositions. $\mathbb{H}^k_T$ spaces were defined in Notation 1.3.

PROPOSITION 7.4 (Uniform $\mathbb{H}^k_T$ estimates). *Fix $\delta>0$, $k\geqslant[M_0+\frac{N+1}{2}]$, and consider the iteration scheme* (1.5) *under the hypotheses of Theorem* 7.1. *There is an $\varepsilon_0>0$ and a $T_k=T_k(\delta)>0$ independent of $\varepsilon\in(0,\varepsilon_0]$ such that*

(7.60)
$$\|U^n_\varepsilon\|_{k,T_k}\leqslant\delta \qquad \textit{for all } n \textit{ and all } \varepsilon\in(0,\varepsilon_0].$$

Proof. **1.** (Preliminaries) We will suppress ε and write the iterates as U_n (instead of U^n_ε). Take $U_0=0$.

Fix $T_0\in(0,1]$. We will choose $T_k\leqslant T_0$. Proposition 7.1 (b),(c) imply that for all n

(7.61)
$$\operatorname{supp} U_n|_{\Omega_{T_0}}\subset\{x_0\geqslant 0\}\cap\{|x|\leqslant h(D,T_0)\},$$

where h depends on the propagation speed α.

For any $l\in\mathbb{N}$, $T>0$ Sobolev estimates imply

(7.62)
$$|U_n|_{C^{0,l}_{c,T}}\leqslant h_1(l)\,|U_n|_{\infty,[l+\frac{N+1}{2}],T}.$$

2. (Choice of ε_0) Recall $\Gamma=B_{R'}\times\Gamma'_{\beta'}$ (7.23). Choose

(7.63)
$$\varepsilon_0\leqslant\frac{R'}{h_1(0)\,\delta}.$$

3. (Induction step) Let $\mathscr{T}_0(K)$ be as in Corollary 7.2. Fix n and assume for $T(k) \leqslant \min\{T_0, \mathscr{T}_0(h_1(M_0)\,\delta)\}$, $n' \leqslant n$

$$\|U_{n'}\|_{k,T(k)} \leqslant \delta \qquad \text{for} \quad 0<\varepsilon\leqslant\varepsilon_0. \tag{7.64}$$

The choice of ε allows us to use Corollary 7.2 to estimate $\|U_{n+1}\|_{k,T(k)}$. Note that the equation and (7.64) imply

$$|\varepsilon\partial_{x_N} U_n|_* \leqslant h(|U_n|_{C^{0,1}_{c,T(k)}}).$$

Set

$$\|V\|'_{k,T} \equiv |V|_{\infty,k,T} + |V|_{0,k+1,T} \tag{7.65}$$

and take $T = T(k)$, $\gamma = \frac{1}{T(k)}$ in (7.58) to obtain

$$\|U_{n+1}\|_{k,T(k)} \leqslant eC(h_1(M_0)\,\delta)\,\sqrt{T(k)}\,(\|F_\varepsilon(U_n)\|'_{k,T(k)} + \langle G\rangle_{k+1,T(k)}). \tag{7.66}$$

Lemma 7.3 (d),(e) give for some h_2:

$$\|F_\varepsilon(U_n)\|'_{k,T(k)} \leqslant h_2(h_1(0)\,\delta)\,\|U_n\|'_{k,T(k)}. \tag{7.67}$$

Thus,

$$\|U_{n+1}\|_{k,T(k)} \leqslant \sqrt{T(k)}\,eC(h_1(M_0)\,\delta)[\delta h_2(h_1(0)\,\delta) + \langle G\rangle_{k+1,T_0}]. \tag{7.68}$$

Reduce $T(k)$ if necessary so the right side of (7.68) is $\leqslant \delta$. This gives T_k. ∎

7.6. *Contraction*

PROPOSITION 7.5. *Fix δ, k, T_k as in Proposition 7.4 and ε_0 as in (7.63). There exist $\mathbb{T}_k \in (0, T_k]$, $\varepsilon_1(\delta) \leqslant \varepsilon_0$, and $U_\varepsilon \in \mathbb{H}^k_{\mathbb{T}_k}$ such that*

$$\|U^n_\varepsilon - U_\varepsilon\|_{k-1,\mathbb{T}_k} \to 0 \qquad \textit{as} \quad n\to\infty$$

uniformly for $\varepsilon \in (0, \varepsilon_1(\delta)]$.

Proof. As before write $U^n_\varepsilon = U_n$, and set $W_n = U_{n+1} - U_n$. The problem satisfied by W_n has the form

$$\begin{aligned} D_{x_N} W_n - \mathscr{A}\left(\varepsilon U_n, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) W_n &= \mathscr{F}(\varepsilon, U_{n-1}, U_n, \varepsilon D_{x'} U_n, D_\theta U_n)\, W_{n-1} \\ B(\varepsilon U_n)\, W_n &= \mathscr{G}(\varepsilon, U_{n-1}, U_n)\,\varepsilon W_{n-1} \\ W_n &= 0 \qquad \text{in} \quad x_0 < 0, \end{aligned} \tag{7.69}$$

where $\mathscr{F}$, $\mathscr{G}$ are C^∞ functions of $\varepsilon \in [0, \varepsilon_0]$ and their other arguments.

From Lemma 7.3 we obtain

$$\text{(7.70)}\quad \begin{aligned} &\text{(a)}\quad |\mathscr{F}W_{n-1}|_{\infty,k-1,T} \leqslant h(|U_{n-1},U_n,\partial^1 U_n|_*)\,|W_{n-1}|_{\infty,k-1,T}\\ &\qquad +h(|U_{n-1},U_n,\partial^1 U_n|_*)\,|W_{n-1}|_*\,|U_{n-1},U_n,\partial^1 U_n|_{\infty,k-1,T}\\ &\text{(b)}\quad |\mathscr{F}W_{n-1}|_{0,k,T} \leqslant h(|U_{n-1},U_n,\partial^1 U_n|_*)\,|W_{n-1}|_{0,k,T}\\ &\qquad +h(|U_{n-1},U_n,\partial^1 U_n|_*)\,|W_{n-1}|_*\,|U_{n-1},U_n,\partial^1 U_n|_{0,k,T}\\ &\text{(c)}\quad \langle \mathscr{G}\varepsilon W_{n-1}\rangle_{k,T} \leqslant \varepsilon[h(\langle U_{n-1},U_n\rangle_*)\langle W_{n-1}\rangle_{k,T}\\ &\qquad +h(\langle U_{n-1},U_n\rangle_*)\langle W_{n-1}\rangle_*\,\langle U_{n-1},U_n\rangle_{k,T}]. \end{aligned}$$

Take $T \leqslant T_k$ in (7.70) and use (7.60), (7.62) to obtain for some h_3:

$$\text{(7.71)}\qquad h(|U_{n-1},U_n,\partial^1 U_n|_*) \leqslant h_3(\delta).$$

Apply Corollary 7.2 to (7.69), letting $\gamma=\frac{1}{T}$ in (7.58) and using (7.62) (for $|W_{n-1}|_*$ and $C(K)$) and (7.70), to find for some $h_4(\delta)$:

$$\text{(7.72)}\qquad \|W_n\|_{k-1,T} \leqslant \sqrt{T}\,h_4(\delta)\,\|W_{n-1}\|'_{k-1,T} + \varepsilon h_4(\delta)\sqrt{T}\,\langle W_{n-1}\rangle_{k,T}.$$

Finally, choose $T=\mathbb{T}_k(\delta)$ and $\varepsilon(\delta) \leqslant \varepsilon_0$ small enough so that

$$\text{(7.73)}\qquad \sqrt{T}\,h_4(\delta) \leqslant \tfrac{1}{4}, \qquad \varepsilon h_4(\delta) \leqslant \tfrac{1}{4}.$$

This gives

$$\text{(7.74)}\qquad \|W_n\|_{k-1,\mathbb{T}_k} \leqslant \tfrac{1}{2}\,\|W_{n-1}\|_{k-1,\mathbb{T}_k} \quad \text{for} \quad 0<\varepsilon\leqslant\varepsilon_1. \quad \blacksquare$$

This also completes the proof of Theorem 7.1.

8. *Initial Value Problems*

We revert to the notation of Section 6. In particular, x_N now denotes time. Recall

(8.1)

(a) $\mathcal{O}_T=\{(x,\theta)\in\mathbb{R}^{N+1}\times\mathbb{T}^L : x_N\in[0,T]\}$

(b) $C^{0,M}_{c,T}=\{V(x,\theta)\in C([0,T],C^M(\mathbb{R}^N\times\mathbb{T}^L,\mathbb{R}^m)) : \operatorname{supp} V \text{ is compact}\}$

(c) $M_0=2(N+L+1)+1$

Notation 8.1. (a) For $k \in \mathbb{N}$ $H^k = H^k(\mathbb{R}^N \times \mathbb{T}^L)$, the standard Sobolev space with norm $\langle U(x', \theta)\rangle_k$.

(b) $C_T H^k = \{V(\cdot, x_N, \cdot) \in C([0, T] : H^k)\}$ with

$$|V|_{C_T H^k} \equiv \sup_{x_N \in [0,T]} |V(\cdot, x_N, \cdot)|_{H^k} \equiv |V|_{\infty, k, T}.$$

(c) $\omega_T = \{x \in \mathbb{R}^{N+1} : x_N \in [0, T]\}$.

The goal of this section is to extend a result of [JMR1, JMR2, S] for symmetric hyperbolic quasilinear initial value problems of the form (6.2) to the case of systems that are just *symmetrizable* in the weaker sense of Definition 6.1.

Consider the singular problem

$$\text{(8.2)} \quad \begin{aligned} &\mathscr{L}(\varepsilon U_\varepsilon, D^\varepsilon_{x',\theta})\, U_\varepsilon \\ &= D_{x_N} U_\varepsilon - \mathscr{A}\left(\varepsilon U_\varepsilon, D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) U_\varepsilon = F_\varepsilon(U_\varepsilon) \qquad \text{on } \mathcal{O}_T \\ &U_\varepsilon = G(x', \theta) \qquad \text{on} \quad x_N = 0 \end{aligned}$$

with $\mathscr{A}(v, \xi')$, F_ε as in (1.4).

THEOREM 8.1. *Fix* $k \geqslant [M_0 + \frac{N+L}{2}]$ *and assume*

$$\text{(8.3)} \qquad G(x', \theta) \in H^k, \qquad \textit{supp}\ G \subset \{|x'| \leqslant D\}.$$

Fix $R > 0$, $K = \langle G\rangle_k$, *and suppose*

$$\text{(8.4)} \qquad D_{x_N} - \mathscr{A}(v, D_{x'})$$

is symmetrizable for $v \in B_R$ (*Definition* 6.1). *Suppose* $\alpha < \infty$ *is an upper bound for the propagation speed of* (8.4) *when* $v \in B_R$. *There exist* $\varepsilon_2(K)$, $T(K)$, *and a unique* $U_\varepsilon(x, \theta) \in C_{T(K)} H^k$ *satisfying* (8.2) *for* $0 < \varepsilon \leqslant \varepsilon_2(K)$ *on* $\mathcal{O}_{T(K)}$ *and such that*

$$\text{(8.5)} \qquad v_\varepsilon = u_0 + \varepsilon U_\varepsilon\left(x, \frac{x'\beta'}{\varepsilon}\right)$$

is the unique C^1 *solution of* (6.2) *on* $\omega_{T(K)}$.

The main step in the proof of Theorem 8.1 is a $C_T H^k$ estimate for the linearized problem (6.1).

Proposition 8.1. *Fix* $R>0$, $k\geqslant[M_0+\frac{N+L}{2}]$, $T>0$, *and* $\varepsilon_0>0$. *Let* $D_{x_N}-\mathscr{A}(v, D_{x'})$ *be symmetrizable for* $v\in B_R$. *Suppose* $V_\varepsilon(x,\theta)\in C_TH^k$ *has compact support and satisfies for* $0<\varepsilon\leqslant\varepsilon_0$

$$\text{(a)}\quad |\varepsilon V_\varepsilon|_{L^\infty(\mathscr{O}_T)}\leqslant R$$

$$\text{(b)}\quad |V_\varepsilon|_{C^{0,M_0}_{c,T}}\leqslant K_1, \qquad |V_\varepsilon|_{\infty,k,T}\leqslant K_2$$

$$\text{(c)}\quad |\varepsilon\partial_{x_N}V_\varepsilon|_{L^\infty(\mathscr{O}_T)}\leqslant h(|V_\varepsilon|_{C^{0,1}_{c,T}}),$$

and let $\varepsilon_1(K_1)\leqslant\varepsilon_0$ *be as in Theorem* 6.1. *Assume* $F_\varepsilon\in C_TH^k$ *and* $G\in H^k$. *Then the solution* U_ε *of the singular problem* 6.1 *belongs to* C_TH^k *and there exist constants* $C_1(\varepsilon, K_1)$, $C(K_2)$ *such that for* $x_N\in[0,T]$, $0<\varepsilon\leqslant\varepsilon_1(K_1)$

(8.6)
$$\langle U_\varepsilon(x_N)\rangle_k\leqslant\left[C_1(\varepsilon,K_1)\,e^{C(K_2)\,x_N}\langle G\rangle_k+C(K_2)\int_0^{x_N}e^{C(K_2)(x_N-t)}\langle F_\varepsilon(x_N)\rangle_k\,dt\right].$$

Here

$$C_1(\varepsilon,K_1)=C_2+\varepsilon C_3(K_1), \tag{8.7}$$

where C_2 *is independent of* V_ε.

Proof of Proposition 8.1. Apply the estimate (6.9) to $\partial^\alpha_{(x',\theta)}U_\varepsilon$, $|\alpha|\leqslant k$, and note (as in [JMR1]) that since $\partial^\alpha_{(x',\theta)}$ commutes with the singular part $\mathscr{A}(0, D_{x'}+\beta' D_\theta/\varepsilon)$, we have

$$|[\mathscr{L}(\varepsilon V_\varepsilon, D^\varepsilon_{x,\theta}), \partial^\alpha_{(x',\theta)}]\,U_\varepsilon|_{\infty,0,T}\leqslant C(K_2)\,|U_\varepsilon|_{\infty,k,T}. \tag{8.8}$$

An application of Gronwall's inequality yields (8.6). ∎

Proof of Theorem 8.1. Fix $T>0$. The proof is a standard iteration based on the scheme

$$\begin{aligned}\mathscr{L}(\varepsilon U^n_\varepsilon, D^\varepsilon_{x,\theta})\,U^{n+1}_\varepsilon&=F_\varepsilon(U^n_\varepsilon) &&\text{on } \mathscr{O}_T\\ U^{n+1}_\varepsilon&=G(x',\theta) &&\text{on}\quad x_N=0,\end{aligned} \tag{8.9}$$

where $U^0_\varepsilon\equiv G$. Finite propagation speed for the linearized problem (which follows as in Proposition 7.1) implies

$$\operatorname{supp} U^n_\varepsilon|_{\mathscr{O}_T}\subset\{(x,\theta)\in\mathscr{O}_T : |x'|\leqslant h(D)\}\qquad\text{for all } n. \tag{8.10}$$

To prove uniform boundedness, fix n, let C_2 be as in (8.7), and assume, for $\varepsilon_2(K)$, $T(K)\leqslant T$ (determined below)

$$|U^n_\varepsilon|_{\infty,k,T(K)}\leqslant(C_2+1)(K+1)\qquad\text{for}\quad 0<\varepsilon\leqslant\varepsilon_2(K). \tag{8.11}$$

Since $k \geqslant [M_0 + \frac{N+L}{2}]$, (8.11) implies $|U_\varepsilon^n|_{C^{0,M_0}_{c,T(K)}} \leqslant h(K) \equiv K_3$. Let $\varepsilon_1(K_3)$ be as in Proposition 8.1. Choose $\varepsilon_2(K) \leqslant \varepsilon_1(K_3)$ such that for $0 < \varepsilon \leqslant \varepsilon_2(K)$

$$\text{(8.12)} \qquad \begin{aligned} &\text{(a)} \quad |\varepsilon U_\varepsilon^n|_{L^\infty(\mathcal{O}_{T(K)})} \leqslant R \qquad \text{and} \\ &\text{(b)} \quad |\varepsilon C_3(|U_\varepsilon^n|_{C^{0,M_0}_{c,T(K)}})| \leqslant 1 \qquad (C_3 \text{ as in (8.7)}). \end{aligned}$$

Thus, estimate (8.6) applies with $U_\varepsilon = U_\varepsilon^{n+1}$, $V_\varepsilon = U_\varepsilon^n$ and yields

$$\text{(8.13)} \qquad \begin{aligned} \langle U_\varepsilon^{n+1}(x_N)\rangle_k &\leqslant (C_2+1)\, e^{C(K)\,x_N} K + C(K)\, x_N e^{C(K)\,x_N} \\ &\leqslant (C_2+1)(K+1), \end{aligned}$$

where $T(K)$ is chosen so the last inequality holds for $x_N \in [0, T(K)]$. This completes the induction step.

The existence of $U_\varepsilon \in C_{T(K)} H^k$ such that $U_\varepsilon^n \to U_\varepsilon$ in $C_{T(K)} H^{k-1}$ as $n \to \infty$, uniformly for $0 < \varepsilon \leqslant \varepsilon_2(K)$, now follows in the usual way. This implies U_ε is the unique solution of (8.2) on $\mathcal{O}_{T(K)}$. ∎

9. *Multidimensional Shocks*

In this section we return to the use of Notations 1.3 and 5.1.

9.1. *The Initial Boundary Value Problem for Oscillatory Shocks*

Consider the system of conservation laws

$$\text{(9.1)} \qquad \sum_{j=0}^{N} \partial_{x_j} f_j(u) = 0$$

on $\mathbb{R}^{N+1}$, where the $f_j\colon \mathbb{R}^m \to \mathbb{R}^m$ are C^∞ functions. Set $x = (x', x_N)$ and let S be a noncharacteristic surface for (9.1) defined by $x_N = \psi(x')$, where ψ is C^1. Suppose u is a C^1 function up to S on each side of S whose restriction u^+ (respectively, u^-) to $x_N > \psi(x')$ (respectively, $x_N < \psi(x')$) satisfies (9.1). Then u is a *multidimensional shock* if, in addition, the $u^\pm$ satisfy the jump condition

$$\text{(9.2)} \qquad \sum_{j=0}^{N-1} \psi_{x_j} [f_j(u)] - [f_N(u)] = 0 \qquad \text{on } S.$$

Functions u as above satisfying (9.1) on each side of S are weak solutions of (9.1) in $\mathbb{R}^{N+1}$ if and only if (9.2) holds. Equations (9.1) and (9.2) constitute a hyperbolic free boundary problem for the unknowns $(u^\pm, \psi)$. This problem was solved by Majda [M1, M2] with several improvements by Metivier [Met1, Met2] under an appropriate stability hypothesis.

Following the classical idea we reduce to a problem with fixed boundary by making the change of variables $\tilde{x}' = x'$, $\tilde{x}_N = \pm(x_N - \psi(x'))$, and putting $\tilde{u}^{\pm}(\tilde{x}) = u^{\pm}(x)$. Setting

$$A_j(v) = f'_j(v),\ j = 0, ..., N$$

$$A_N(v, d\psi) = A_N(v) - \sum_{j=0}^{N-1} \psi_{x_j} A_j(v)$$

and dropping the tildes, we obtain the equations

$$\sum_{j=0}^{N-1} A_j(u^{\pm})\, \partial_{x_j} u^{\pm} \pm A_N(u^{\pm}, d\psi)\, \partial_{x_N} u^{\pm} = 0 \qquad \text{on} \quad x_N > 0 \tag{9.3}$$

$$G(u^{\pm}, d\psi) = \sum_{j=0}^{N-1} \psi_{xj} [f_j(u)] - [f_N(u)] = 0 \qquad \text{on} \quad x_N = 0. \tag{9.4}$$

Consider a planar shock solution $(U^{\pm}, \sigma x_0)$ of (9.3) and (9.4). This means that $(U^{\pm}, \sigma)$ are constants satisfying

$$\sigma[U] - [f_N(U)] = 0. \tag{9.5}$$

Our main assumptions are:

(M.A.) (a) $(U^{\pm}, \sigma x_0)$ is uniformly stable [M1].

(b) The system (9.3) is symmetric hyperbolic and satisfies Majda's *block structure condition* for $(u^{\pm}, d\psi)$ in some neighborhood $\mathcal{O} \subset \mathbb{R}^{2m+N}$ of $(U^{\pm}, d(\sigma x_0))$ (see Remark 5.1).

Symmetric hyperbolic means there are smooth matrix-valued functions $\mathcal{S}^{\pm}(u^{\pm})$ such that $\mathcal{S}^{\pm}(u^{\pm})\, A_j(u^{\pm})$ is symmetric for all j and $\mathcal{S}^{\pm}(u^{\pm})\, A_0(u^{\pm})$ is positive definite.

We study oscillatory perturbations of $(U^{\pm}, \sigma x_0)$. To describe these, observe first that Eqs. (9.3) and (9.4) admit the following linearization

$$\begin{aligned} L^{\pm}(u^{\pm}, d\psi, \partial_x)\, v^{\pm} &= f^{\pm} \qquad \text{in} \quad x_N > 0 \\ B^{\pm}(u^{\pm}, d\psi)(v^{\pm}, d\phi) &= g \qquad \text{on} \quad x_N = 0, \end{aligned} \tag{9.6}$$

where

$$L^{\pm}(u^{\pm}, d\psi, \partial_x)\, v^{\pm} = \sum_{j=0}^{N-1} A_j(u^{\pm})\, \partial_{x_j} v^{\pm} \pm A_N(u^{\pm}, d\psi)\, \partial_{x_N} v^{\pm}, \tag{9.7}$$

$$B(u^{\pm}, d\psi)(v^{\pm}, d\phi) = \sum_{j=0}^{N-1} \phi_{x_j} [f_j(u)] - [A_N(u, d\psi)\, v]. \tag{9.8}$$

Note that if we let $\omega = (u^+, u^-, d\psi)$, then $B(u^{\pm}, d\psi) = d_\omega G(u^{\pm}, d\psi)$.

The linearized interior operators at $(U^{\pm}, \sigma x_0)$ take the form

$$\mathscr{L}^{\pm}(\partial_x) = \sum_{j=0}^{N-1} A_j(U^{\pm})\,\partial_{x_j} \pm (A_N(U^{\pm}) - \sigma)\,\partial_{x_N} \tag{9.9}$$

with corresponding symbols

$$p^{\pm}(\xi', \xi_N) = \det \mathscr{L}^{\pm}(\xi', \xi_N). \tag{9.10}$$

For each choice of sign define the hyperbolic $\mathscr{H}_{\pm}$ and nonglancing regions $\mathscr{G}^c_{\pm} \subset \mathbb{R}^N \backslash 0$ as in Definition 1.1 ($\mathscr{H}_{\pm} \subset \mathscr{G}^c_{\pm}$), and choose a boundary frequency

$$\beta' \in \mathscr{G}^c_{+} \cap \mathscr{G}^c_{-}. \tag{9.11}$$

Thus, for each choice of sign the characteristic modes $(\beta', \lambda^{\pm}_j(\beta'))$, $j = 1, \ldots, m$, associated to β' are some combination of hyperbolic and elliptic modes. Let

$$\phi^{\pm}_j(x) = \beta' x' + \lambda^{\pm}_j(\beta')\, x_N, \qquad j = 1, \ldots, m \tag{9.12}$$

be the associated real or complex characteristic phases.

The perturbed shock $(u^{\pm}_{\varepsilon}, d\psi_{\varepsilon})$ is the solution to an initial boundary value problem on $\overline{\mathbb{R}}^{N+1}_{+} = \{(x_0, x'', x_N) : x_N > 0\}$

$$\begin{aligned}
&\text{(a)} \quad L^{\pm}(u^{\pm}_{\varepsilon}, d\psi_{\varepsilon}, \partial_x)\, u^{\pm}_{\varepsilon} = 0 \qquad \text{in} \quad x_N > 0\\
&\text{(b)} \quad G(u^{\pm}_{\varepsilon}, d\psi_{\varepsilon}) = 0 \qquad \text{on} \quad x_N = 0\\
&\text{(c)} \quad u^{\pm}_{\varepsilon} = U^{\pm} + \varepsilon w^{\pm}_{\varepsilon}(x'', x_N, \theta)|_{\theta = \frac{\beta' x'}{\varepsilon}} \qquad \text{on} \quad x_0 = 0\\
&\text{(d)} \quad \psi_{\varepsilon} = 0 \qquad \text{on} \quad x_0 = 0,
\end{aligned} \tag{9.13}$$

where $w^{\pm}_{\varepsilon}(x'', x_N, \theta)$ satisfies appropriate corner and phase compatibility conditions.

In [W2] we chose $\beta' \in \mathscr{H}_{+} \cap \mathscr{H}_{-}$ and perturbed the planar shock with oscillatory plane waves whose associated characteristics reflected strictly transversally off the shock. Here our choice of β' (9.11) allows the formation of an elliptic boundary layer on one or both sides of the shock as well.

9.2. *The Singular Shock Problem and Choice of Initial Data*

One can look for solutions to (9.13) of the form

$$\begin{aligned}
u^{\pm}_{\varepsilon}(x) &= \mathscr{U}^{\pm}_{\varepsilon}(x, \theta)|_{\theta = \frac{\beta' x'}{\varepsilon}}\\
\psi_{\varepsilon}(x') &= \Psi_{\varepsilon}(x', \theta)|_{\theta = \frac{\beta' x'}{\varepsilon}},
\end{aligned} \tag{9.14}$$

where $(\mathscr{U}_\varepsilon^\pm, \Psi_\varepsilon)$ satisfies a *singular shock problem* obtained by manipulations such as (1.1)–(1.4) (here $\nabla^\varepsilon = \partial_{x'} + \beta' \partial_\theta / \varepsilon$):

$$\begin{aligned} &\mathscr{L}^\pm(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon, D_{x,\theta}^\varepsilon)\, \mathscr{U}_\varepsilon^\pm \\ &\qquad \equiv D_{x_N} \mathscr{U}_\varepsilon^\pm - \mathscr{A}^\pm(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon, D_{x,\theta}^\varepsilon)\, \mathscr{U}_\varepsilon^\pm = 0 \\ &G(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon) = 0 \qquad \text{on} \quad x_N = 0 \\ &\mathscr{U}_\varepsilon^\pm = U^\pm + \varepsilon w_\varepsilon^\pm(x'', x_N, \theta) \qquad \text{on} \quad x_0 = 0 \\ &\Psi_\varepsilon = 0 \qquad \text{on} \quad x_0 = 0. \end{aligned} \tag{9.15}$$

Here

$$-\mathscr{A}^\pm = [\pm A_N(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon)]^{-1} \left[\sum_{j=0}^{N-1} A_j(\mathscr{U}_\varepsilon^\pm) \left(D_{x_j} + \frac{\beta_j D_\theta}{\varepsilon} \right) \right]. \tag{9.16}$$

Fix $k_0 \geqslant [M_0 + \frac{N+1}{2}]$ and $D > 0$. Choose initial data $w_\varepsilon^\pm(x'', x_N, \theta) \in C^\infty(\mathbb{R}^N \times \mathbb{T}^1)$ supported in $|x'', x_N| \leqslant D$, vanishing to high order (for convenience) at $x_N = 0$, extended to be zero in $x_N < 0$, and such that the pair $(\sigma x_0, U^\pm + \varepsilon w_\varepsilon^\pm)$ satisfies both corner compatibility conditions to order k_0 and phase compatibility conditions (see [W2, Sect. 5 and Appendix A] for details on satisfying corner and phase compatibility, including the construction of initial data that do not vanish at $x_N = 0$).

Let $W_\varepsilon^\pm(x, \theta) \in C^\infty$ satisfy for some $T_0 > 0$ and $\varepsilon \in (0, 1]$ the singular initial value problems on $\mathscr{O}_{T_0} \equiv \{(x, \theta) \in \mathbb{R}^{N+1} \times \mathbb{T}^1 : x_0 \in [-T_0, T_0]\}$:

$$\begin{aligned} &\mathscr{L}^\pm(U^\pm + \varepsilon W_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0), D_{x,\theta}^\varepsilon)(U^\pm + \varepsilon W_\varepsilon^\pm) = 0 \qquad \text{on } \mathscr{O}_{T_0} \\ &W_\varepsilon^\pm = w_\varepsilon^\pm(x'', x_N, \theta) \qquad \text{on} \quad x_0 = 0, \end{aligned} \tag{9.17}$$

where, with $\mathscr{O}_{T_0, x_N} = \{(x', \theta) : x_0 \in [-T_0, T_0]\}$,

$$\begin{aligned} &\{(W_\varepsilon^\pm, \varepsilon \partial_{x_N} W_\varepsilon^\pm) : \varepsilon \in (0, 1]\} \text{ is bounded in} \\ &\qquad C(\mathbb{R}_{x_N} : H^{k_0+1}(\mathscr{O}_{T_0, x_N})) \times C(\mathbb{R}_{x_N} : H^{k_0}(\mathscr{O}_{T_0, x_N})). \end{aligned} \tag{9.18}$$

By finite propagation speed, which holds for (9.17) by the argument of Proposition 7.1,

$$\operatorname{supp} W_\varepsilon^\pm|_{\mathscr{O}_{T_0}} \subset \{|x| \leqslant h(D)\}. \tag{9.19}$$

Remark 9.1. (a) Corner $(x_0 = 0, x_N = 0)$ compatibility to order k_0 is designed to ensure that for $W_\varepsilon^\pm$ satisfying (9.17) we have

$$\partial_{x_0}^j \mathscr{G}_\varepsilon(x', \theta) = 0 \qquad \text{at } x_0 = 0 \text{ for } 0 \leqslant j \leqslant k_0, \qquad \text{where} \tag{9.20}$$

$$G(U^\pm + \varepsilon W_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0)) = \varepsilon \mathscr{G}_\varepsilon(x', \theta) \in C^\infty. \tag{9.21}$$

(b) The initial value problems (9.17) are *not* of the type treated in Section 8, since θ is a placeholder for a phase $\beta' x'/\varepsilon$ that involves time (x_0); hence, phase compatibility conditions are needed here.

One way to construct $w^+_\varepsilon(x'', x_N, \theta)$ and $W^+_\varepsilon(x, \theta)$, for example, is to choose an incoming phase ϕ^+_j in (9.12) and solve the symmetric hyperbolic initial value problem for $v^+_\varepsilon(x)$

$$\text{(9.22)}\qquad \begin{aligned} &L^+(U^+ + \varepsilon v^+_\varepsilon, d(\sigma x_0), \partial_x)\, v^+_\varepsilon = 0 \qquad \text{in } \mathcal{O}_{T_0}\\ &v^+_\varepsilon = v^+\left(x'', x_N, \frac{\phi^+_j}{\varepsilon}\right) \qquad \text{on} \quad x_0 = 0, \end{aligned}$$

where $v^+(x'', x_N, \zeta) \in C^\infty$ is periodic in ζ and satisfies corner and phase compatibility conditions to high enough order. Geometric optics (e.g., [JR, G]) yields for some $T_0 > 0$ and $\varepsilon \in (0, 1]$ an exact solution

$$v^+_\varepsilon(x) = V^+_\varepsilon(x, \zeta)|_{\zeta = \phi^+_j/\varepsilon}$$

on $[-T_0, T_0] \times \mathbb{R}^N$ such that

$$\text{(9.23)}\qquad \begin{aligned} &\{(V^+_\varepsilon, \varepsilon\partial_{x_N} V^+_\varepsilon) : \varepsilon \in (0, 1]\} \text{ is bounded in}\\ &C(\mathbb{R}_{x_N} : H^{k_0+1}(\mathcal{O}_{T_0, x_N})) \times C(\mathbb{R}_{x_N} : H^{k_0}(\mathcal{O}_{T_0, x_N})). \end{aligned}$$

Now set

$$\text{(9.24)}\qquad \begin{aligned} &w^+_\varepsilon(x'', x_N, \theta) = v^+\left(x'', x_N, \theta + \frac{\lambda_j(\beta')\, x_N}{\varepsilon}\right) \qquad \text{and}\\ &W^+_\varepsilon(x, \theta) = V^+_\varepsilon\left(x, \theta + \frac{\lambda_j(\beta')\, x_N}{\varepsilon}\right). \end{aligned}$$

(c) Without phase compatibility conditions the solution to (9.22) would generally depend on additional phases not in the set (9.12), which would therefore correspond to boundary frequencies $\tilde{\beta} \neq \beta'$.

9.3. L^2 *Estimates for the Forward Singular Shock Problem*

Define $\mathscr{B}(\vartheta^\pm, \nabla^\varepsilon\chi)$ by

$$\text{(9.25)}\qquad \begin{aligned} &G(\vartheta^\pm, \nabla^\varepsilon\chi) - G(\mu^\pm_\varepsilon, \nabla^\varepsilon(\sigma x_0))\\ &= \mathscr{B}(\vartheta^\pm, \nabla^\varepsilon\chi)(\vartheta^\pm - \mu^\pm_\varepsilon, \nabla^\varepsilon(\chi - \sigma x_0))\\ &= \sum_{j=0}^{N-1}\left(\int_0^1 [f_j(\mu_\varepsilon + s(\vartheta - \mu_\varepsilon))]\, ds\right)\left(\partial_{x_j} + \frac{\beta_j\partial_\theta}{\varepsilon}\right)(\chi - \sigma x_0)\\ &\quad - \int_0^1 [A_N(\mu_\varepsilon + s(\vartheta - \mu_\varepsilon), \nabla^\varepsilon(\sigma x_0 + s(\chi - \sigma x_0)))(\vartheta - \mu_\varepsilon)]\, ds \end{aligned}$$

(recall (9.8)). Observe the notation suppresses the dependence of $\mathscr{B}$ on $(\mu_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0))$.

In order to reduce (9.15) to a forward problem on Ω, we choose a C^∞ cutoff $\chi(x_0)$ such that $\chi = 1$ on $[-T_0/2, T_0/2]$, supp $\chi \subset [-T_0, T_0]$, and set

$$\tilde{W}_\varepsilon^\pm = \chi W_\varepsilon^\pm, \qquad \tilde{\mu}_\varepsilon^\pm = U^\pm + \varepsilon \tilde{W}_\varepsilon^\pm \tag{9.26}$$

$$\varepsilon \tilde{\mathscr{F}}_\varepsilon^\pm = \begin{cases} -\mathscr{L}^\pm(\tilde{\mu}_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0), D_{x,\theta}^\varepsilon)(\tilde{\mu}_\varepsilon^\pm), & x_0 \geqslant 0 \\ 0, & x_0 < 0 \end{cases} \tag{9.27}$$

$$\varepsilon \tilde{\mathscr{G}}_\varepsilon = \begin{cases} G(\tilde{\mu}_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0)), & x_0 \geqslant 0 \\ 0, & x_0 < 0. \end{cases} \tag{9.28}$$

Equation (9.17) and the compatibility conditions imply

$$\begin{aligned} &\text{(a)} \quad \{(\tilde{W}_\varepsilon^\pm, \varepsilon \partial_{x_N} \tilde{W}_\varepsilon^\pm) : \varepsilon \in (0, 1]\} \text{ is bounded in} \\ &\qquad CH^{k_0+1} \times CH^{k_0} \text{ with supp } \tilde{W}_\varepsilon^\pm \subset \{|x| \leqslant h(D)\}. \\ &\text{(b)} \quad \{(\tilde{\mathscr{F}}_\varepsilon^\pm, \tilde{\mathscr{G}}_\varepsilon) : \varepsilon \in (0, 1]\} \text{ is bounded in } CH^{k_0+1} \times H^{k_0+1} \\ &\qquad \text{with } \operatorname{supp}_{x,x'}(\tilde{\mathscr{F}}_\varepsilon^\pm, \tilde{\mathscr{G}}_\varepsilon) \subset \{x_0 \geqslant 0\} \cap \{|x, x'| \leqslant h(D)\}. \end{aligned} \tag{9.29}$$

Dropping the tildes we set $\mu_\varepsilon^\pm = U^\pm + \varepsilon W_\varepsilon^\pm$ and look for a solution $(\mathscr{U}_\varepsilon^\pm, \Psi_\varepsilon)$ to (9.15) of the form

$$\begin{aligned} \mathscr{U}_\varepsilon^\pm &= \mu_\varepsilon^\pm + \varepsilon U_\varepsilon^\pm \\ \Psi_\varepsilon &= \sigma x_0 + \varepsilon \phi_\varepsilon, \end{aligned} \tag{9.30}$$

where $(U_\varepsilon^\pm, \phi_\varepsilon)$ satisfies the forward singular shock problem

$$\begin{aligned} &\text{(a)} \quad \mathscr{L}^\pm(\mu_\varepsilon^\pm + \varepsilon U_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0 + \varepsilon \phi_\varepsilon), D_{x,\theta}^\varepsilon)\, \varepsilon U_\varepsilon^\pm \\ &\qquad = [\mathscr{L}^\pm(\mu_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0), D_{x,\theta}^\varepsilon) \\ &\qquad\quad - \mathscr{L}^\pm(\mu_\varepsilon^\pm + \varepsilon U_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0 + \varepsilon \phi_\varepsilon), D_{x,\theta}^\varepsilon)]\, \mu_\varepsilon^\pm + \varepsilon \mathscr{F}_\varepsilon^\pm, \\ &\text{(b)} \quad \mathscr{B}(\mu_\varepsilon^\pm + \varepsilon U_\varepsilon^\pm, \nabla^\varepsilon(\sigma x_0 + \varepsilon \phi_\varepsilon))(\varepsilon U_\varepsilon^\pm, \nabla^\varepsilon(\varepsilon \phi_\varepsilon)) = -\varepsilon \mathscr{G}_\varepsilon \qquad \text{on} \quad x_N = 0, \\ &\text{(c)} \quad U_\varepsilon^\pm = 0, \ \phi_\varepsilon = 0 \qquad \text{in} \quad x_0 < 0. \end{aligned} \tag{9.31}$$

We solve (9.31) using the following iteration scheme:

$$\begin{aligned} &\mathscr{L}^{\pm}(\mu_\varepsilon^{\pm}+\varepsilon U_\varepsilon^{n,\pm}, \nabla^\varepsilon(\sigma x_0+\varepsilon\phi_\varepsilon^n), D_{x,\theta}^\varepsilon)\, U_\varepsilon^{n+1,\pm} \\ &\qquad =\frac{1}{\varepsilon}\,[\mathscr{L}^{\pm}(\mu_\varepsilon^{\pm}, \nabla^\varepsilon(\sigma x_0), D_{x,\theta}^\varepsilon) \\ &\qquad\quad -\mathscr{L}^{\pm}(\mu_\varepsilon^{\pm}+\varepsilon U_\varepsilon^{n,\pm}, \nabla^\varepsilon(\sigma x_0+\varepsilon\phi_\varepsilon^n), D_{x,\theta}^\varepsilon)]\,\mu_\varepsilon^{\pm}+\mathscr{F}_\varepsilon^{\pm}, \\ &\mathscr{B}(\mu_\varepsilon^{\pm}+\varepsilon U_\varepsilon^{n,\pm}, \nabla^\varepsilon(\sigma x_0+\varepsilon\phi_\varepsilon^n))(U_\varepsilon^{n+1,\pm}, \nabla^\varepsilon\phi_\varepsilon^{n+1})=-\mathscr{G}_\varepsilon \quad \text{on} \quad x_N=0, \\ &U_\varepsilon^{n+1,\pm}=0,\ \phi_\varepsilon^{n+1}=0 \quad \text{in} \quad x_0<0. \end{aligned} \tag{9.32}$$

Notation 9.1. Let $\mathscr{C}_c^{0,M}\equiv C_c^{0,M}\times C_c^{0,M}\times C_c^M$, where $C_c^M=\{\beta(x',\theta)\in C^M(\mathbb{R}^N\times\mathbb{T}^1:\mathbb{R}^N):\operatorname{supp}\beta \text{ is compact}\}$. $\mathscr{C}_{c,T}^{0,M}$ is defined similarly for $T>0$.

THEOREM 9.1 (L^2 estimate). *Assume (M.A.) and set*

$$\begin{aligned} &\text{(a)}\quad \mathscr{L}^{\pm}(\mathscr{U}_\varepsilon^{\pm}, \nabla^\varepsilon\Psi_\varepsilon, D_{x,\theta}^\varepsilon)\, U_\varepsilon^{\pm}=F_\varepsilon^{\pm}(x,\theta) \qquad \textit{in} \quad x_N>0 \\ &\text{(b)}\quad \mathscr{B}(\mathscr{U}_\varepsilon^{\pm}, \nabla^\varepsilon\Psi_\varepsilon)(U_\varepsilon^{\pm}, \nabla^\varepsilon\phi^\varepsilon)=G_\varepsilon(x',\theta) \qquad \textit{on} \quad x_N=0. \end{aligned} \tag{9.33}$$

Fix $K>0$, $\varepsilon_0>0$, *and let*

$$\mathscr{V}_\varepsilon(x,\theta)=(W_\varepsilon, \varepsilon^{-1}[(\mathscr{U}_\varepsilon, \nabla^\varepsilon\Psi_\varepsilon)-(U,\nabla^\varepsilon(\sigma x_0))]), \tag{9.34}$$

where $W_\varepsilon=\binom{W_\varepsilon^+}{W_\varepsilon^-}$, $U=\binom{U^+}{U^-}$, *etc. There exist positive constants R, C, C(K), γ(K) such that if* $\mathscr{V}_\varepsilon\in\mathscr{C}_c^{0,M_0}$ *and satisfies for* $0<\varepsilon\leqslant\varepsilon_0$

$$\begin{aligned} &\text{(a)}\quad |\varepsilon\mathscr{V}_\varepsilon|_*\leqslant R \\ &\text{(b)}\quad |\mathscr{V}_\varepsilon|_{\mathscr{C}_c^{0,M_0}}\leqslant K \\ &\text{(c)}\quad |\varepsilon\partial_{x_N}\mathscr{V}_\varepsilon|_*\leqslant h(|\mathscr{V}_\varepsilon|_{\mathscr{C}_c^{0,1}}), \end{aligned} \tag{9.35}$$

then $|A_N^{-1}(\mathscr{U}_\varepsilon^{\pm}, \nabla^\varepsilon\Psi_\varepsilon)|_{L^\infty(\Omega)}\leqslant C$ *and for all* $(U_\varepsilon^{\pm}, \phi_\varepsilon)\in\mathscr{H}_\gamma^{1,1}(\Omega)\times\mathscr{H}_\gamma^1(b\Omega)$ *(Notation 5.1),* $0<\varepsilon\leqslant\varepsilon_0$, $\gamma\geqslant\gamma(K)$ *we have*

$$\begin{aligned} &|U_\varepsilon|_{0,0,\gamma}+\frac{1}{\sqrt{\gamma}}\langle U_\varepsilon\rangle_{0,\gamma}+\frac{1}{\sqrt{\gamma}}\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{0,\gamma} \\ &\qquad\leqslant C(K)\left(\frac{|F_\varepsilon|_{0,0,\gamma}}{\gamma}+\frac{\langle G_\varepsilon\rangle_{0,\gamma}}{\sqrt{\gamma}}\right). \end{aligned} \tag{9.36}$$

Here we have set $|U_\varepsilon|_{0,0,\gamma}=|U_\varepsilon^+|_{0,0,\gamma}+|U_\varepsilon^-|_{0,0,\gamma}$, *etc.*

Remark 9.2. $\gamma\langle\phi_\varepsilon\rangle_{0,\gamma}\leqslant\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{0,\gamma}$, so one may also include $\sqrt{\gamma}\,\langle\phi_\varepsilon\rangle_{0,\gamma}$ on the left side of (9.36).

Proof of Theorem 9.1. The proof has much in common with that of Theorem 5.1 and Corollary 5.2, so we will concentrate mainly on the differences.

1. (Choose R,C) Let $\mathcal{O} \ni (U, d(\sigma x_0)) = (U^+, U^-, d(\sigma x_0))$ be as in (M.A.). The first requirement on R is that for $(u, \eta) \in \mathbb{R}^{2m+N}$

$$|(u, \eta) - (U, d(\sigma x_0))| \leqslant R \Rightarrow (u, \eta) \in \mathcal{O}.$$

Write

$$\mathcal{B}(\mathcal{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi)(U_\varepsilon^\pm, \nabla^\varepsilon \phi_\varepsilon) = i \sum_{j=0}^{N-1} b_j(\mu_\varepsilon, \nabla^\varepsilon(\sigma x_0), \mathcal{U}_\varepsilon, \nabla^\varepsilon \Psi_\varepsilon) \left(D_{x_j} + \frac{\beta_j D_\theta}{\varepsilon} \right) \phi_\varepsilon + M(\mu_\varepsilon, \nabla^\varepsilon(\sigma x_0), \mathcal{U}_\varepsilon, \nabla^\varepsilon \Psi_\varepsilon)\, U_\varepsilon, \tag{9.37}$$

where the $b_j \in \mathbb{R}^m$ and $m \times 2m$ matrix M (acting on $U_\varepsilon = \binom{U_\varepsilon^+}{U_\varepsilon^-}$) are defined in (9.25).

Let $L^\pm$ be as in (9.7) and set

$$\mathbb{B}(a_1, a_2)(v, \phi) = i \sum_{j=0}^{N-1} b_j(a_1, a_2)\, D_{x_j} \phi + M(a_1, a_2)\, v. \tag{9.38}$$

Since $(U^\pm, \sigma x_0)$ is uniformly stable, provided R is chosen small enough, the constant coefficient system

$$(L^\pm(a_2), \mathbb{B}(a_1, a_2)) \tag{9.39}$$

obtained by freezing coefficients at $a_1 = (\mu_\varepsilon, \nabla^\varepsilon(\sigma x_0))$, $a_2 = (\mathcal{U}_\varepsilon, \nabla^\varepsilon \Psi_\varepsilon)$ is uniformly stable. This implies uniform invertibility of $A_N(\mathcal{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon)$ and the existence of C.

2. (Eliminate ϕ_ε) With $\mathcal{V}_\varepsilon(x, \theta)$ as in (9.34) let $\xi'_\gamma = (\xi_0 - i\gamma, \xi'')$ and define singular symbols

$$b_s = b(\varepsilon \mathcal{V}_\varepsilon(x, \theta), X, \gamma) = i \sum_{j=0}^{N-1} b_j(\mu_\varepsilon, \nabla^\varepsilon(\sigma x_0), \mathcal{U}_\varepsilon, \nabla^\varepsilon \Psi_\varepsilon) \left(\xi'_{\gamma, j} + \frac{m\beta_j}{\varepsilon} \right) \in \mathcal{S}^{1, M_0}_{\beta'}, \tag{9.40}$$

$$m_s = m(\varepsilon \mathcal{V}_\varepsilon(x, \theta)) = M(\mu_\varepsilon, \nabla^\varepsilon(\sigma x_0), \mathcal{U}_\varepsilon, \nabla^\varepsilon \Psi_\varepsilon) \in \mathcal{S}^{0, M_0}_{\beta'},$$

and the $m \times m$ orthogonal projector on $b_s^\perp$ given by

$$\pi_s = \pi(\varepsilon \mathcal{V}_\varepsilon(x, \theta), X, \gamma) \in \mathcal{S}^{0, M_0}_{\beta'}:$$

$$\pi(\varepsilon \mathcal{V}_\varepsilon(x, \theta), X, \gamma)\, h = h - \frac{(h, b(\varepsilon \mathcal{V}_\varepsilon, X, \gamma))}{|b(\varepsilon \mathcal{V}_\varepsilon, X, \gamma)|^2}\, b(\varepsilon \mathcal{V}_\varepsilon, X, \gamma). \tag{9.41}$$

In (9.41) we use the fact that for ε_0, R as above, the b_j are independent and there exists a $C>0$ such that

$$C\langle X,\gamma\rangle \leqslant |b(\varepsilon\mathscr{V}_\varepsilon, X,\gamma)| \leqslant \frac{1}{C}\langle X,\gamma\rangle. \tag{9.42}$$

We can now use the SPO calculus to eliminate ϕ_ε by arguments parallel to [M1, Met2].

Conjugate (9.33) by $e^{-\gamma x_0}$ and rewrite (9.33b):

$$ib_s(D_{x',\theta})\,\phi_\varepsilon^\gamma + m_s(D_{x',\theta})\,U_\varepsilon^\gamma = G_\varepsilon^\gamma \qquad \text{on} \quad x_N = 0. \tag{9.43}$$

$\pi_s b_s = 0$ so the calculus gives (with $\langle\ \rangle_0 = |\ |_{\mathscr{H}^0(b\Omega)}$)

$$\langle \pi_s(D_{x',\theta})\,b_s(D_{x',\theta})\,\phi_\varepsilon^\gamma\rangle_0 \leqslant C(K)\langle\phi_\varepsilon^\gamma\rangle_0 \leqslant C(K)\,\frac{|\phi_\varepsilon^\gamma|_{\mathscr{H}^1}}{\gamma}. \tag{9.44}$$

Set $B_{M,s} = B_M(\varepsilon\mathscr{V}_\varepsilon, X,\gamma) \equiv \pi_s m_s \in \mathscr{S}_{\beta'}^{0,M_0}$ and deduce

$$\langle B_{M,s}(D_{x',\theta})\,U_\varepsilon^\gamma - \pi_s(D_{x',\theta})\,m_s(D_{x',\theta})\,U_\varepsilon^\gamma\rangle_0 \leqslant \frac{C(K)}{\gamma}\langle U_\varepsilon^\gamma\rangle_0. \tag{9.45}$$

Thus, (9.43)–(9.45) imply

$$\langle B_{M,s}(D_{x',\theta})\,U_\varepsilon^\gamma\rangle_0 \leqslant C(K)\langle G_\varepsilon^\gamma\rangle_0 + \frac{C(K)}{\gamma}\left(|\phi_\varepsilon^\gamma|_{\mathscr{H}^1} + \langle U_\varepsilon^\gamma\rangle_0\right). \tag{9.46}$$

Next, introduce the row vector b_s^* and note the scalar $p_s = b_s^* b_s \in \mathscr{S}_{\beta'}^{2,M_0}$ is *elliptic* by (9.42). The SPO Garding inequality (Corollary 3.1) and the calculus imply for $\gamma \geqslant \gamma(K)$

$$\begin{aligned} |\phi_\varepsilon^\gamma|_{\mathscr{H}^1}^2 &\leqslant C(K)\,\mathrm{Re}(p_s(D_{x',\theta})\,\phi_\varepsilon^\gamma, \phi_\varepsilon^\gamma) \leqslant C(K)\langle b_s(D_{x',\theta})\,\phi_\varepsilon^\gamma\rangle_0^2 \\ &\leqslant C(K)(\langle G_\varepsilon^\gamma\rangle_0^2 + \langle U_\varepsilon^\gamma\rangle_0^2). \end{aligned} \tag{9.47}$$

Although the calculus does not apply directly to $b_s(D_{x',\theta})^*\,b_s(D_{x',\theta})$, we can write

$$\langle b_s(D_{x',\theta})\,\phi_\varepsilon^\gamma, b_s(D_{x',\theta})\,\phi_\varepsilon^\gamma\rangle = \langle \Lambda^{-1}b_s(D_{x',\theta})^*\,b_s(D_{x',\theta})\,\Lambda^{-1}(\Lambda\phi_\varepsilon^\gamma), \Lambda\phi_\varepsilon^\gamma\rangle$$

and use Proposition 2.4 and Theorem 2.6 to see that

$$\Lambda^{-1}p_s(D_{x',\theta})\,\Lambda^{-1} - \Lambda^{-1}b_s(D_{x',\theta})^*\,b_s(D_{x',\theta})\,\Lambda^{-1} \text{ is residual of order 1.} \tag{9.48}$$

Now

$$|\phi_\varepsilon^\gamma|_{\mathscr{H}^1} \sim \langle (\nabla^\varepsilon \phi_\varepsilon)^\gamma \rangle_0$$

so in view of (9.46) and (9.47) to prove the estimate (9.36) it will suffice to show for $\gamma \geqslant \gamma(K)$,

$$|U_\varepsilon^\gamma|_{0,0} + \frac{1}{\sqrt{\gamma}} \langle U_\varepsilon^\gamma \rangle_0 \leqslant C(K) \left(\frac{|F_\varepsilon^\gamma|_{0,0}}{\gamma} + \frac{\langle B_{M,s}(D_{x',\theta})\, U_\varepsilon^\gamma \rangle_0}{\sqrt{\gamma}} \right). \tag{9.49}$$

3. (Estimate U_ε^γ) With $\mathscr{V}_\varepsilon(x, \theta)$ as in (9.34) let $v = (v_1, v_2, v_3) \in \mathbb{R}^{4m+N}$ be a placeholder for $\varepsilon \mathscr{V}_\varepsilon$: v_1 is a placeholder for $\varepsilon W_\varepsilon$, v_2 for $\mathscr{U}_\varepsilon - U$, v_3 for $\nabla^\varepsilon(\Psi_\varepsilon - \sigma x_0)$. As before set $\xi'_\gamma = (\xi_0 - i\gamma, \xi'')$ and let

(9.50)

$$\tilde{\mathscr{A}}(v, \xi'_\gamma) = \begin{pmatrix} \mathscr{A}^+(U^+ + v_2^+, \nabla^\varepsilon(\sigma x_0) + v_3, \xi'_\gamma) & 0 \\ 0 & \mathscr{A}^-(U^- + v_2^-, \nabla^\varepsilon(\sigma x_0) + v_3, \xi'_\gamma) \end{pmatrix}$$

$$B_M(v, \xi', \gamma) = \pi(v, \xi', \gamma)\, m_s(v).$$

($\tilde{\mathscr{A}}$ is $2m \times 2m$, B_M is $m \times 2m$.)

The uniform stability of $(U^\pm, \sigma x_0)$ and our choice of R imply the system

$$(D_{x_N} - \tilde{\mathscr{A}}(v, D_{x'_\gamma}),\ B_M(v, D_{x'}, \gamma)) \tag{9.51}$$

is Kreiss well-posed (see Definition 5.2 and Remark 5.1(c)) for v in $B_R = \{v: |v| \leqslant R\}$.

We are now in precisely the situation of Section 5. System (9.51) has a Kreiss symmetrizer $R(v, \xi', \gamma)$. Set $R_s = R(\varepsilon \mathscr{V}_\varepsilon(x, \theta), \xi' + m\beta'/\varepsilon, \gamma)$. Then the SPO $R_s(D_{x',\theta})$ is a symmetrizer for the singular problem corresponding to (9.51). The estimate (9.36) now follows from Corollary 5.1. ∎

THEOREM 9.2 (Linear existence and uniqueness). *Assume (M.A.), fix $K > 0$, $\varepsilon_0 > 0$, and suppose $\mathscr{V}_\varepsilon$ (9.34) satisfies (9.35) for R as in Theorem 9.1. There is a constant $\gamma_0(K)$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, $\gamma \geqslant \gamma_0(K)$, $F_\varepsilon^\pm(x, \theta) \in \mathscr{H}_\gamma^{0,0}(\Omega)$, $G_\varepsilon(x', \theta) \in \mathscr{H}_\gamma^0(b\Omega)$, there is a unique solution $(U_\varepsilon^\pm, \phi_\varepsilon) \in \mathscr{H}_\gamma^{0,0}(\Omega) \times \mathscr{H}_\gamma^1(b\Omega)$ to the singular problem*

$$\begin{aligned} &\mathscr{L}^\pm(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon, D_{x,\theta}^\varepsilon)\, U_\varepsilon^\pm = F_\varepsilon^\pm \qquad \textit{in} \quad x_N > 0 \\ &\mathscr{B}(\mathscr{U}_\varepsilon^\pm, \nabla^\varepsilon \Psi_\varepsilon)(U_\varepsilon^\pm, \nabla^\varepsilon \phi_\varepsilon) = G_\varepsilon \qquad \textit{on} \quad x_N = 0, \end{aligned} \tag{9.52}$$

and $(U_\varepsilon^\pm, \phi_\varepsilon)$ satisfies the estimate (9.36). If $F_\varepsilon^\pm$, G_ε vanish in $x_0 < 0$, the same is true of $(U_\varepsilon^\pm, \phi_\varepsilon)$.

Proof. Let $L^{\pm}$ be as in (9.7), $\mathbb{B}(a_1, a_2)$ as in (9.38), and set $\tilde{\mu}_\varepsilon(x) = \mu_\varepsilon(x, \beta' x'/\varepsilon)$. Corresponding to the (nonsingular) shock problem

$$(9.53)\qquad \begin{aligned} &L^{\pm}(u^{\pm}, d\psi, \partial)\, v^{\pm} = f^{\pm} && \text{in} \quad x_N > 0 \\ &\mathbb{B}(\tilde{\mu}_\varepsilon, d(\sigma x_0), u, d\psi)(v, d\phi) = g && \text{on} \quad x_N = 0 \end{aligned}$$

there is a dual shock problem (see [Met2]) which satisfies the backward uniform stability condition. The associated singular problem dual to (9.33) thus satisfies an estimate like (9.36), so the same arguments that gave Theorem 5.2 apply here. ∎

We proceed directly to the shock analogue of Corollary 7.1. Let

$$(9.54)\qquad \mathscr{V}_\varepsilon(x, \theta) = (W_\varepsilon, \varepsilon^{-1}[\mathscr{U}_\varepsilon - U, \nabla^\varepsilon(\Psi_\varepsilon - \sigma x_0)])$$

as before, let $\tilde{\mathscr{A}}$ be as in (9.50), and set

$$\tilde{\mathscr{B}}(\varepsilon\mathscr{V})(U_\varepsilon, \nabla^\varepsilon\phi_\varepsilon) \equiv \mathscr{B}(\mathscr{U}_\varepsilon^{\pm}, \nabla^\varepsilon\Psi_\varepsilon)(U_\varepsilon^{\pm}, \nabla^\varepsilon\phi_\varepsilon).$$

To emphasize the parallels with Section 7, we note $\tilde{\mathscr{B}}$ is a zeroth order operator acting on $(U_\varepsilon, \nabla^\varepsilon\phi_\varepsilon)$, and rewrite the system (9.52) as

$$(9.55)\qquad \begin{aligned} &\tilde{\mathscr{L}}(\varepsilon\mathscr{V}_\varepsilon, D^\varepsilon_{x,\theta})\, U_\varepsilon \equiv D_{x_N} U_\varepsilon - \tilde{\mathscr{A}}(\varepsilon\mathscr{V}_\varepsilon, D^\varepsilon_{x',\theta})\, U_\varepsilon = F_\varepsilon \\ &\tilde{\mathscr{B}}(\varepsilon\mathscr{V}_\varepsilon)(U_\varepsilon, \nabla^\varepsilon\phi_\varepsilon) = G_\varepsilon. \end{aligned}$$

PROPOSITION 9.1 ($L^2H^1_\gamma - CH^0_\gamma$ estimate). *Assume (M.A.) and $\beta' \in \mathscr{G}^c_+ \cap \mathscr{G}^c_-$. Fix $k \in \mathbb{N}$, $\varepsilon_0 > 0$, and suppose for R as in Theorem* 9.1 *and $R' \leqslant R$ as in* (9.64) *that $\mathscr{V}_\varepsilon \in \mathscr{C}^{0, M_0}_c$, F_ε, G_ε satisfy for $0 < \varepsilon \leqslant \varepsilon_0$*

$$(9.56)\qquad \begin{aligned} &\text{(a)}\quad |\varepsilon\mathscr{V}_\varepsilon|_* \leqslant R' \\ &\text{(b)}\quad |\mathscr{V}_\varepsilon|_{\mathscr{C}^{0,M_0}_c} \leqslant K,\ |\varepsilon\partial_{x_N}\mathscr{V}_\varepsilon|_* \leqslant h(|\mathscr{V}_\varepsilon|_{\mathscr{C}^{0,1}_c}) \\ &\text{(c)}\quad F_\varepsilon \in L^2H^1_\gamma \cap CH^0_\gamma,\ G_\varepsilon \in H^1_\gamma,\ \mathit{supp}\ F_\varepsilon \subset \{0 \leqslant x_N \leqslant E\}. \end{aligned}$$

Then there exists a constant $\gamma_0(K)$ such that for $\gamma \geqslant \gamma_0$, $0 < \varepsilon \leqslant \varepsilon_0$, the solution $(U_\varepsilon, \phi_\varepsilon)$ to the singular problem (9.55) *satisfies*

$$(9.57)\qquad \begin{aligned} &|U_\varepsilon|_{\infty,0,\gamma} + |U_\varepsilon|_{0,1,\gamma} + \frac{\langle U_\varepsilon, \nabla^\varepsilon\phi_\varepsilon\rangle_{1,\gamma}}{\sqrt{\gamma}} \\ &\quad \leqslant C(K, E)\left[\left(\frac{|F_\varepsilon|_{\infty,0,\gamma}}{\sqrt{\gamma}} + \langle G_\varepsilon\rangle_{0,\gamma}\right) + \left(\frac{|F_\varepsilon|_{0,1,\gamma}}{\gamma} + \frac{\langle G_\varepsilon\rangle_{1,\gamma}}{\sqrt{\gamma}}\right)\right] \end{aligned}$$

and exhibits finite propagation speed.

Proof. **1.** ($L^2H^1_\gamma$ estimate) Conjugate (9.55) with $e^{-\gamma x_0}$ to get

$$\text{(9.58)}\qquad \begin{aligned} &D_{x_N}U^\gamma_\varepsilon - \tilde{\mathscr{A}}\left(\varepsilon\mathscr{V}_\varepsilon(x,\theta), D_{x'_\gamma}+\frac{\beta' D_\theta}{\varepsilon}\right)U^\gamma_\varepsilon = F^\gamma_\varepsilon \quad \text{in} \quad x_N>0\\ &\tilde{\mathscr{B}}(\varepsilon\mathscr{V}_\varepsilon)(U^\gamma_\varepsilon, (\nabla^\varepsilon\phi_\varepsilon)^\gamma) = G^\gamma_\varepsilon \quad \text{on} \quad x_N=0.\end{aligned}$$

The proof of Proposition 7.1 shows there is a constant $\gamma_0(K)$ such that for $0<\varepsilon\leqslant\varepsilon_0$, $\gamma\geqslant\gamma_0$,

$$\text{(9.59)}\qquad |U^\gamma_\varepsilon|_{0,1} + \frac{\langle U^\gamma_\varepsilon, (\nabla^\varepsilon\phi_\varepsilon)^\gamma\rangle_1}{\sqrt{\gamma}} \leqslant h(K)\left[\frac{|F^\gamma_\varepsilon|_{0,1}}{\gamma}+\frac{\langle G^\gamma_\varepsilon\rangle_1}{\sqrt{\gamma}}\right].$$

For example, in place of (7.8) we have for $|\alpha|\leqslant 1$

$$\text{(9.60)}\qquad \langle[\tilde{\mathscr{B}}(\varepsilon\mathscr{V}_\varepsilon), \partial^\alpha_{(x',\theta)}](U^\gamma_\varepsilon, (\nabla^\varepsilon\phi_\varepsilon)^\gamma)\rangle_{0,\gamma} \leqslant h(|\mathscr{V}_\varepsilon|_{\mathscr{C}^{0,1}_c})\,\frac{\langle U^\gamma_\varepsilon, (\nabla^\varepsilon\phi_\varepsilon)^\gamma\rangle_1}{\gamma}.$$

2. (Propagation speed) The argument in Proposition 7.1 shows the propagation speed for (9.58) is bounded above by the (finite) propagation speed of the corresponding nonsingular problem.

3. (CH^0_γ estimate, I) Apply $\pi_s(D_{x',\theta})$ to the boundary equation in the form (9.43) and use the SPO calculus to obtain the reduced system for U^γ_ε

$$\text{(9.61)}\qquad \begin{aligned} &(D_{x_N}-\tilde{\mathscr{A}})\,U^\gamma_\varepsilon = F^\gamma_\varepsilon\\ &B_{M,s}(D_{x',\theta})\,U^\gamma_\varepsilon = H_{\varepsilon,\gamma} \quad \text{on} \quad x_N=0,\end{aligned}$$

where $H_{\varepsilon,\gamma} = r_0G^\gamma_\varepsilon + r_1U^\gamma_\varepsilon + r_0\phi^\gamma_\varepsilon$ (as before r_j is residual of order j). We microlocalize and diagonalize (9.61) just as we did for (7.29).

As in Lemma 7.1 let $v=(v_1, v_2, v_3)\in\mathbb{R}^{4m+N}$ be a placeholder for $\varepsilon\mathscr{V}_\varepsilon(x,\theta)$, and define

$$\text{(9.62)}\qquad \Sigma = \{z=(v,X,\gamma)\in B_R\times\mathbb{R}^N\times[0,\infty) : (X,\gamma)\neq 0\}.$$

Since we will use + (resp. −) to label outgoing (resp. incoming) hyperbolic and elliptic modes, it is desirable now to rewrite the $2m\times 2m$ matrix $\tilde{\mathscr{A}}$ as

$$\text{(9.63)}\qquad \tilde{\mathscr{A}}(z) = \begin{bmatrix}\tilde{\mathscr{A}}^R & 0\\ 0 & \tilde{\mathscr{A}}^L\end{bmatrix}.$$

For each choice of R, L define $\mathscr{O}^{R,L}(\beta')$, $\mathscr{P}^{R,L}(\beta')$, $\mathscr{I}^{R,L}(\beta')$, $\mathscr{N}^{R,L}(\beta')$, $\lambda^{R,L}_i(z)$, $r^{R,L}_i(z)$ as before. For z in a conic neighborhood Γ of $\{(0,\beta',0), (0,-\beta',0)\}$,

$$\text{(9.64)}\qquad \Gamma = B_{R'}\times\Gamma'_{\beta'}\subset\Sigma$$

we have

$$S^{-1}(z)\,\tilde{\mathscr{A}}(z)\,S(z)=\Lambda(z), \tag{9.65}$$

where $S(z)$, $\Lambda(z)$ are $2m\times 2m$ block diagonal matrices with $m\times m$ blocks $S^{R,L}$, $\Lambda^{R,L}$, respectively. For each choice of R, L we have the analogues of (7.20), (7.21), and (7.22).

As in Remark 7.1(a) write

$$S^R(z)=[S^{R,+}(z)\quad S^{R,-}(z)], \tag{9.66}$$

where $S^{R,+}$ (resp. $S^{R,-}$) is the matrix whose columns are $r_i^R(z)$, $i\in\mathcal{O}^R(\beta')\cup\mathscr{P}^R(\beta')$ (resp. $i\in\mathscr{I}^R(\beta')\cup\mathscr{N}^R(\beta')$), and similarly write

$$S^L(z)=[S^{L,+}(z)\quad S^{L,-}(z)].$$

Let $m^{R,+}$ (resp. $m^{L,+}$) be the number of columns of $S^{R,+}$ (resp. $S^{L,+}$). Uniform stability implies

$$m^{R,+}+m^{L,+}=m-1. \tag{9.67}$$

$B_M(z)=\pi(z)\,m(z)$ and $m(z)$ (9.40) (9.41) are $m\times 2m$ matrices we can write as

$$B_M=[B_M^R\quad B_M^L],\qquad m=[m^R\quad m^L], \tag{9.68}$$

where $B_M^{R,L}=\pi m^{R,L}$. Uniform stability implies the $m\times(m-1)$ matrix

$$B_M^+(z)=[B_M^R S^{R,+}\quad B_M^L S^{L,+}] \tag{9.69}$$

has rank $m-1$ for $z\in\Gamma$. More precisely, there is a smooth, bounded $(m-1)\times m$ left inverse $(B_M^+)^{-1}$ such that

$$(B_M^+)^{-1}(z)\,B_M(z)=I_{m-1}\qquad\text{for}\quad z\in\Gamma. \tag{9.70}$$

Now extend $\tilde{\mathscr{A}}(z)|_\Gamma$, $B_M(z)|_\Gamma$, $S(z)$, $\Lambda(z)$ to elements of S^1, S^0, S^0, S^1, respectively (Notation 1.7), so that (9.65), (9.70), and the analogues of (7.20)–(7.22) hold for all $z\in\Sigma$.

4. (CH_γ^0 estimate, II) With $\Gamma=B_{R'}\times\Gamma'_{\beta'}$ as in (9.64) choose $\delta'>0$ in (7.10) small enough so that

(9.71)
$$(Z_1,Z_2,\gamma)=\left(\xi',\frac{m\beta'}{\varepsilon},\gamma\right)\in\operatorname{supp}\chi\Rightarrow(X,\gamma)\in\Gamma'_{\beta'}\qquad\text{for}\quad X=Z_1+Z_2.$$

Now proceed as in the proof of Proposition 7.3 using Notation 7.1. Solve $SW=\chi U^\gamma$, set $\mathscr{W}=S^{-1}SW$, and use the calculus to rewrite (9.61)

$$(9.72)\quad \begin{aligned}&\text{(a)}\quad D_{x_N}\mathscr{W}-\Lambda\mathscr{W}=r_0F^\gamma+r_0U^\gamma\\ &\text{(b)}\quad B_{M,s}(D_{x',\theta})\,S\mathscr{W}=r_0H_{\varepsilon,\gamma}+r_1U^\gamma=r_0G^\gamma+r_1U^\gamma+r_0\phi^\gamma.\end{aligned}$$

Write $\mathscr{W}=(\mathscr{W}^{R+},\mathscr{W}^{R-},\mathscr{W}^{L+},\mathscr{W}^{L-})$ corresponding to the decomposition of $S^{R,L}$ (9.66). For $B_M^+(z)$ as in (9.69), (9.72b) becomes

$$(9.73)\quad B_M^+\begin{pmatrix}\mathscr{W}^{R+}\\ \mathscr{W}^{L+}\end{pmatrix}=r_0\begin{pmatrix}\mathscr{W}^{R-}\\ \mathscr{W}^{L-}\end{pmatrix}+r_0G^\gamma+r_1U^\gamma+r_0\phi^\gamma \text{ on } x_N=0.$$

(9.70) and the calculus now give

$$(9.74)\quad \begin{pmatrix}\mathscr{W}^{R+}\\ \mathscr{W}^{L+}\end{pmatrix}=r_0\begin{pmatrix}\mathscr{W}^{R-}\\ \mathscr{W}^{L-}\end{pmatrix}+r_0G^\gamma+r_1U^\gamma+r_0\phi^\gamma.$$

Instead of (7.47) the arguments of Section 7 yield

$$(9.75)\quad \begin{aligned}&\langle\mathscr{W}(x_N)\rangle_0^2+\gamma\,|\mathscr{W}|_{0,0}^2\\ &\leqslant C(K,E)\left[\frac{|F^\gamma|_{\infty,0}^2}{\gamma}+|U^\gamma|_{0,0}^2+\langle G^\gamma\rangle_0^2+\frac{\langle U^\gamma(0)\rangle_0^2}{\gamma}+\langle\phi^\gamma\rangle_0^2\right],\end{aligned}$$

which implies the shock analogue of (7.17):

$$(9.76)\quad |\chi_sU_\varepsilon^\gamma|_{\infty,0}\leqslant C(K,E)\left[\frac{|F_\varepsilon^\gamma|_{\infty,0}}{\sqrt{\gamma}}+\langle G_\varepsilon^\gamma\rangle_0+|U_\varepsilon^\gamma|_{0,0}+\frac{\langle U_\varepsilon^\gamma\rangle_0}{\gamma}+\langle\phi_\varepsilon^\gamma\rangle_0\right].$$

The proof of (7.11) gives

$$(9.77)\quad |(1-\chi_s)\,U_\varepsilon^\gamma|_{\infty,0}\leqslant h(|\mathscr{V}_\varepsilon|_{\mathscr{C}_c^{0,0}})\,|U_\varepsilon^\gamma|_{0,1}+|F_\varepsilon^\gamma|_{0,0}.$$

Together with (9.76) and (9.59), this implies the estimate of (9.57) for $|U_\varepsilon^\gamma|_{\infty,0}$. ▮

Notation 9.2. (a) For $k\in\mathbb{N}$ let $\mathbb{D}_T^k=\{(Z(x,\theta),\eta(x',\theta))\in\mathbb{H}_T^k\times H_T^k$: (Z,η) is valued in $\mathbb{R}^{4m}\times\mathbb{R}^N\}$, and set

$$|(Z,\eta)|_{\mathbb{D}_T^k}\equiv|\langle(Z,\eta)\rangle|_{k,T}=\|Z\|_{k,T}+\sqrt{T}\,\langle\eta\rangle_{k+1,T}.$$

(b) For $\mathscr{V}=(Z,\eta)\in\mathbb{D}_T^k$ we will sometimes write $|\mathscr{V}|_{0,k+1,T}$ or $|\mathscr{V}|_{\infty,k,T}$. The meanings of $|\mathscr{V}|_{\infty,k,T}$ and $|Z|_{0,k+1,T}$ are clear, but $|\eta(x',\theta)|_{0,k+1,T}$ is not. For $T\in(0,1]$, the solution $(U_\varepsilon,\phi_\varepsilon)$ to (9.55) satisfies

$$(9.78)\quad \operatorname{supp} U_\varepsilon|_{\Omega_T}\subset\{0\leqslant x_N\leqslant h(E)\}$$

by finite propagation speed. Choose $\rho(x_N)\in C_c^\infty(\overline{\mathbb{R}}_+)$, $\rho\equiv 1$ on $[0,h(E)]$ and define

$$|\eta|_{0,k+1,T}\equiv|\rho(x_N)\,\eta(x',\theta)|_{0,k+1,T}\leqslant C(E)\langle\eta\rangle_{k+1,T}. \tag{9.79}$$

THEOREM 9.3 (Main linear estimate). *Assume (M.A.) and* $\beta'\in\mathscr{G}_+^c\cap\mathscr{G}_-^c$. *Fix* $k\in\mathbb{N}$, $\varepsilon>0$, $T\in(0,1]$, *and suppose for R as in Theorem 9.1 and* $R'\leqslant R$ *as in (9.64)* $(\Gamma=B_{R'}\times\Gamma'_{\beta'})$ *that* $\mathscr{V}_\varepsilon$, F_ε, G_ε *satisfy for* $0<\varepsilon\leqslant\varepsilon_0$

$$\begin{aligned}&\text{(a)}\quad |\varepsilon\mathscr{V}_\varepsilon|_*\leqslant R',\ |\varepsilon\partial_{x_N}\mathscr{V}_\varepsilon|_*\leqslant h(|\mathscr{V}_\varepsilon|_{\mathscr{C}^{0,1}_{c,T}})\\ &\text{(b)}\quad |\mathscr{V}_\varepsilon|_{\mathscr{C}^{0,M_0}_{c,T}}\leqslant K,\ \mathscr{V}_\varepsilon\in\mathbb{D}^k_T\\ &\text{(c)}\quad F_\varepsilon\in L^2H^{k+1}_T\cap CH^k_T,\ G_\varepsilon\in H^{k+1}_T,\ \textit{supp}\ F_\varepsilon\subset\{0\leqslant x_N\leqslant E\}.\\ &\text{(d)}\quad F_\varepsilon,\ G_\varepsilon\ \textit{vanish in}\ x_0<0.\end{aligned}\tag{9.80}$$

Then there exists a constant $\gamma_0(K)$ *such that for* $\gamma\geqslant\gamma_0$, $0<\varepsilon\leqslant\varepsilon_0$, *the unique solution* $(U_\varepsilon,\phi_\varepsilon)$ *to (9.55) vanishing in* $x_0<0$ *satisfies*

$$\begin{aligned}&|U_\varepsilon|_{\infty,k,T}+|U_\varepsilon|_{0,k+1,T}+\frac{\langle U_\varepsilon,\gamma\phi_\varepsilon,\nabla^\varepsilon\phi_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}+\sqrt{\gamma}\,\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{k,T}\\ &\leqslant e^{\gamma T}C(K,E)\Bigg[\left(\frac{|F_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}}+\frac{|F_\varepsilon|_{0,k+1,T}}{\gamma}+\frac{\langle G_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}\right)\\ &+\Bigg(h(|\mathscr{V}_\varepsilon|^*)\,|U_\varepsilon|^*\left(\frac{|\mathscr{V}_\varepsilon|_{\infty,k,T}}{\sqrt{\gamma}}+\frac{|\mathscr{V}_\varepsilon|_{0,k+1,T}}{\gamma}\right)\\ &+h(\langle\mathscr{V}_\varepsilon\rangle^*)\langle U_\varepsilon,\nabla^\varepsilon\phi_\varepsilon\rangle_*\frac{\langle\mathscr{V}_\varepsilon\rangle_{k+1,T}}{\sqrt{\gamma}}\Bigg)\Bigg].\end{aligned}\tag{9.81}$$

Proof. Write $\mathscr{V}_\varepsilon=(Z(x,\theta),\eta(x',\theta))$ and note that $\eta(x',\theta)$ may be replaced by $\rho(x_N)\,\eta$ in the first equation of (9.55), where $\rho(x_N)$ was chosen in Notation 9.2(b). An argument identical to the proof of Theorem 7.2 now gives (9.81), where

$$\begin{aligned}&|\mathscr{V}_\varepsilon|_{\infty,k,T}=|Z|_{\infty,k,T}+\langle\eta\rangle_{k,T}\\ &|\mathscr{V}_\varepsilon|_{0,k+1,T}=|(Z,\rho(x_N)\,\eta)|_{0,k+1,T}\leqslant|Z|_{0,k+1,T}+C(E)\langle\eta\rangle_{k+1,T}.\end{aligned}\tag{9.82}$$

The term $\sqrt{\gamma}\,\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{k,T}$ may be included in (9.81) since in (9.57)

$$\frac{\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{1,\gamma}}{\sqrt{\gamma}}\geqslant\sqrt{\gamma}\,\langle\nabla^\varepsilon\phi_\varepsilon\rangle_{0,\gamma}.\quad\blacksquare$$

COROLLARY 9.1. *Make the same hypotheses as in Theorem 9.3, but take* $k\geqslant[2+\frac{N+1}{2}]$. *There exists a constant* $\mathscr{T}_0(K)$ *such that for* $0<\varepsilon\leqslant\varepsilon_0$, $T\leqslant\mathscr{T}_0$, *and* $\gamma=\frac{1}{T}$

$$|U_\varepsilon|_{\infty, k, T} + |U_\varepsilon|_{0, k+1, T} + \frac{\langle U_\varepsilon, \gamma\phi_\varepsilon, \nabla^\varepsilon \phi_\varepsilon \rangle_{k+1, T}}{\sqrt{\gamma}} + \sqrt{\gamma}\, \langle \nabla^\varepsilon \phi_\varepsilon \rangle_{k, T} \leqslant C(K, E) \left(\frac{|F_\varepsilon|_{\infty, k, T}}{\sqrt{\gamma}} + \frac{|F_\varepsilon|_{0, k+1, T}}{\gamma} + \frac{\langle G_\varepsilon \rangle_{k+1, T}}{\sqrt{\gamma}} \right). \tag{9.83}$$

Proof. We have

$$|U_\varepsilon|^* \leqslant C\, |U_\varepsilon|_{\infty, [1+\frac{N+1}{2}], T} \leqslant CT\, |U_\varepsilon|_{\infty, k, T} \tag{9.84}$$

since $k \geqslant [2 + \frac{N+1}{2}]$. Using (9.82) we obtain

$$\frac{|\mathscr{V}_\varepsilon|_{\infty, k, T}}{\sqrt{\gamma}} + \frac{|\mathscr{V}_\varepsilon|_{0, k+1, T}}{\gamma} \leqslant \frac{|Z|_{\infty, k, T}}{\sqrt{\gamma}} + \frac{\sqrt{T}\, \langle \eta \rangle_{K, T}}{\sqrt{T}\, \sqrt{\gamma}} + \frac{|Z|_{0, k+1, T}}{\gamma} + \frac{C(E)\, \sqrt{T}\, \langle \eta \rangle_{k+1, T}}{\sqrt{T}\, \gamma}. \tag{9.85}$$

Since

$$|\langle \mathscr{V}_\varepsilon \rangle|_{k, T} = |Z|_{\infty, k, T} + |Z|_{0, k+1, T} + \sqrt{T}\, (\langle Z \rangle_{k+1, T} + \langle \eta \rangle_{k+1, T}),$$

(9.84) and (9.85) imply we can absorb the term involving $|U^\varepsilon|^*$ in (9.81) by taking $\mathscr{T}_0$ small enough.

Since

$$\langle U_\varepsilon, \nabla^\varepsilon \phi_\varepsilon \rangle_* \leqslant CT \langle U_\varepsilon, \nabla^\varepsilon \phi_\varepsilon \rangle_{k, T} \qquad \text{and}$$
$$\frac{\langle \mathscr{V}_\varepsilon \rangle_{k+1, T}}{\sqrt{\gamma}} = \frac{\sqrt{T}\, (\langle Z \rangle_{k+1, T} + \langle \eta \rangle_{k+1, T})}{\sqrt{T}\, \sqrt{\gamma}}, \tag{9.86}$$

we can also absorb the term involving $\langle U_\varepsilon, \nabla^\varepsilon \phi_\varepsilon \rangle_*$ by taking $\mathscr{T}_0$ small enough. ∎

9.4. *Main Results*

THEOREM 9.4 (Oscillatory multidimensional shocks). *Assume (M.A.). Fix $k \geqslant [M_0 + \frac{N+1}{2}]$, set $k_0 = k+1$, and let $W_\varepsilon^\pm$ be as chosen in Section 9.2 (see (9.29)). Consider the singular initial boundary value problem (9.15) for $(\mathscr{U}_\varepsilon^\pm, \Psi_\varepsilon)$, where $w_\varepsilon^\pm(x'', x_N, \theta) = W_\varepsilon^\pm(0, x'', x_N, \theta)$ and $\beta' \in \mathscr{G}_+^c \cap \mathscr{G}_-^c$, and the associated (nonsingular) problem (9.13) for $(u_\varepsilon^\pm(x), \psi_\varepsilon(x'))$. There exist an $\varepsilon_1(k) > 0$, a $\mathbb{T}_k > 0$ independent of $\varepsilon \in (0, \varepsilon_1(k)]$, and a unique $(\mathscr{U}_\varepsilon^\pm, \Psi_\varepsilon) \in CH^k_{\mathbb{T}_k} \times H^{k+1}_{\mathbb{T}_k}$ satisfying (9.15) and such that*

$$u_\varepsilon^\pm(x) = \mathscr{U}_\varepsilon^\pm\left(x, \frac{\beta' x'}{\varepsilon}\right), \qquad \psi_\varepsilon(x') = \Psi_\varepsilon\left(x', \frac{\beta' x'}{\varepsilon}\right)$$

is the unique C^1 solution of (9.13) *on* $\omega_{\mathbb{T}_k}$. *More detailed information on the regularity of* $(\mathscr{U}_\varepsilon^\pm, \Psi_\varepsilon)$ *is given in Proposition* 9.3.

The proof of Theorem 9.4 is contained in the next two propositions. Suppressing some epsilons and writing iterates $(U_\varepsilon^{n,\pm}, \phi_\varepsilon^n)$ as (U_n, ϕ_n), we set

$$\mathscr{V}_n = (W, W+U_n, \nabla^\varepsilon \phi_n), \qquad \mathscr{V}_W = (W, W, 0), \tag{9.87}$$

and rewrite the forward problem for (U_{n+1}, ϕ_{n+1}) (9.32) in the notation (9.55):

$$\begin{aligned}
\tilde{L}(\varepsilon\mathscr{V}_n, D_{x,\theta}^\varepsilon)\, U_{n+1} &= D_{x_N} U_{n+1} - \tilde{\mathscr{A}}(\varepsilon\mathscr{V}_n, D_{x',\theta}^\varepsilon)\, U_{n+1} \\
&= [\tilde{\mathscr{A}}(\varepsilon\mathscr{V}_n, D_{x',\theta}^\varepsilon) - \tilde{\mathscr{A}}(\varepsilon\mathscr{V}_W, D_{x',\theta}^\varepsilon)]\, W_\varepsilon + \mathscr{F}_\varepsilon \\
&\equiv \mathscr{H}_\varepsilon + \mathscr{F}_\varepsilon \qquad \text{in} \quad x_N > 0 \\
\tilde{\mathscr{B}}(\varepsilon\mathscr{V}_n)(U_{n+1}, \nabla^\varepsilon \phi_{n+1}) &= -\mathscr{G}_\varepsilon \qquad \text{on} \quad x_N = 0 \\
U_{n+1} = 0, \qquad \phi_{n+1} &= 0 \qquad \text{in} \quad x_0 < 0 \\
U_0 \equiv 0, \qquad \phi_0 &\equiv 0.
\end{aligned} \tag{9.88}$$

Here $\{(W_\varepsilon, \varepsilon\partial_{x_N} W_\varepsilon) : \varepsilon \in (0,1]\}$ is bounded in $CH^{k_0+1} \times CH^{k_0}$ with supp $W_\varepsilon \subset \{|x| \leqslant h(D)\}$, and $\{(\mathscr{F}_\varepsilon, \mathscr{G}_\varepsilon) : \varepsilon \in (0,1]\}$ is bounded in $CH^{k_0+1} \times H^{k_0+1}$ with $\text{supp}_{x,x'}(\mathscr{F}_\varepsilon, \mathscr{G}_\varepsilon) \subset \{x_0 \geqslant 0\} \cap \{|x, x'| \leqslant h(D)\}$.

DEFINITION 9.1. For $T > 0$ let

$$\mathbb{A}_T^k = \Big\{ (V(x,\theta), \zeta(x',\theta)) : (V,\zeta) \text{ is valued in } \mathbb{R}^{2m} \times \mathbb{R} \text{ and}$$

$$|\|V, \zeta\||_{k,T} \equiv \|V\|_{k,T} + \sqrt{T} \left\langle \frac{\zeta}{T}, \nabla^\varepsilon \zeta \right\rangle_{k+1,T} + \frac{\langle \nabla^\varepsilon \zeta \rangle_{k,T}}{\sqrt{T}} < \infty \Big\}.$$

We have

$$\begin{aligned} &|\|U_n, \phi_n\||_{k,T} \\ &\quad = |U_n|_{\infty,k,T} + |U_n|_{0,k+1,T} + \sqrt{T} \left\langle U_n, \frac{\phi_n}{T}, \nabla^\varepsilon \phi_n \right\rangle_{k+1,T} + \frac{\langle \nabla^\varepsilon \phi_n \rangle_{k,T}}{\sqrt{T}}. \end{aligned} \tag{9.89}$$

PROPOSITION 9.2 (Iteration: Uniform $\mathbb{A}_T^k$ estimates). *Fix* $k \geqslant [M_0 + \frac{N+1}{2}]$, $k_0 = k+1$, $\delta > 0$ *and consider the iteration scheme* (9.88) *under the hypotheses of Theorem* 9.4. *There is an* $\varepsilon_0 > 0$ *and a* $T_k = T_k(\delta) > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0]$ *such that*

$$|\|U_n, \phi_n\||_{k,T_k} \leqslant \delta \qquad \textit{for all } n \textit{ and all } \varepsilon \in (0, \varepsilon_0]. \tag{9.90}$$

Proof. **1.** (Preliminaries) We will choose $T_k \leqslant 1$. By finite propagation speed

$$\operatorname{supp}_{x,x'}(U_n, \phi_n) \subset \{x_0 \geqslant 0\} \cap \{|x, x'| \leqslant h(D)\}. \tag{9.91}$$

For any $l \in \mathbb{N}$, $T > 0$ Sobolev estimates imply

$$|\mathscr{V}_n|_{\mathscr{C}^{0,l}_{c,T}} \leqslant h_1(l)(|W, U_n|_{\infty, [l+\frac{N+1}{2}], T} + \langle \nabla^\varepsilon \phi_n \rangle_{[l+\frac{N+1}{2}], T}). \tag{9.92}$$

We also have for $T \in (0, 1]$, $\varepsilon \in (0, 1]$, and some $C_W > 0$

$$\begin{aligned} |\langle \mathscr{V}_n \rangle|_{k,T} &= \|(W, W+U_n)\|_{k,T} + \sqrt{T}\,\langle \nabla^\varepsilon \phi_n \rangle_{k+1,T} \\ &\leqslant C_W + |U_n|_{\infty,k,T} + |U_n|_{0,k+1,T} + \sqrt{T}\,\langle U_n, \nabla^\varepsilon \phi_n \rangle_{k+1,T} \\ &\leqslant C_W + |\|U_n, \phi_n\||_{k,T}, \end{aligned} \tag{9.93}$$

$$|\mathscr{V}_n|_{\infty,k,T} \leqslant C_W + |U_n|_{\infty,k,T} + \langle \nabla^\varepsilon \phi_n \rangle_{k,T}, \tag{9.94}$$

$$|\mathscr{V}_n|_{0,k+1,T} \leqslant C_W + |U_n|_{0,k+1,T} + h(D) \langle \nabla^\varepsilon \phi_n \rangle_{k+1,T} \tag{9.95}$$

(see (9.82)).

2. (Choice of ε_0) Assuming (9.90) we have on Ω_{T_k}

$$|\varepsilon \mathscr{V}_n|_* \leqslant \varepsilon h_1(0)(C_W + \delta).$$

With $R' \leqslant R$ as in Theorem 9.3 choose

$$\varepsilon_0 \leqslant \min\left\{1, \frac{R'}{h_1(0)(C_W+\delta)}\right\}. \tag{9.96}$$

3. (Induction step) Let $\mathscr{T}_0(K)$ be as in Corollary 9.1. Fix n and assume for C_W as in (9.93)–(9.94), $n' \leqslant n$, and

$$T(k) \leqslant \min\{1, \mathscr{T}_0(h_1(M_0)(C_W+\delta))\} \tag{9.97}$$

that

$$|\|U_{n'}, \phi_{n'}\||_{k,T(k)} \leqslant \delta \qquad \text{for} \quad 0 < \varepsilon \leqslant \varepsilon_0. \tag{9.98}$$

Equations (9.96)–(9.98) allow us to use Corollary 9.1 to estimate $|\|U_{n+1}, \phi_{n+1}\||_{k,T(k)}$.

For $\mathscr{F}_\varepsilon$, $\mathscr{G}_\varepsilon$ as in (9.88) there exist constants $C_{\mathscr{F}}(k)$, $C_{\mathscr{G}}(k)$ such that for $T \leqslant 1$, $\varepsilon \in (0, 1]$

$$\begin{aligned} |\mathscr{F}_\varepsilon|_{\infty,k,T} + |\mathscr{F}_\varepsilon|_{0,k+1,T} &\leqslant C_{\mathscr{F}} \\ \langle \mathscr{G}_\varepsilon \rangle_{k+1,T} &\leqslant C_{\mathscr{G}}. \end{aligned} \tag{9.99}$$

$\mathscr{H}_\varepsilon$ may be written with obvious notation

$$(9.100)\qquad \mathscr{H}_\varepsilon = \sum_{j=0}^{N-1} (\tilde{\mathscr{A}}_j(\varepsilon\mathscr{V}_n) - \tilde{\mathscr{A}}_j(\varepsilon\mathscr{V}_W))\left(D_{x'} + \frac{\beta' D_\theta}{\varepsilon}\right) W_\varepsilon$$
$$= \sum_{j=0}^{N-1} C_j(\varepsilon\mathscr{V}_n, \varepsilon\mathscr{V}_W)(0, U_n, \nabla^\varepsilon\phi_n)(\varepsilon D_{x'} W_\varepsilon + \beta' D_\theta W_\varepsilon).$$

To estimate $|\mathscr{H}_\varepsilon|_{0,k+1,T}$, let $k' \leqslant k+1$ and observe that $\partial^{k'}\mathscr{H}_\varepsilon$ is a sum of terms of the form (in Notation 7.2)

$$(9.101)\qquad \phi(\varepsilon, \varepsilon\mathscr{V}_n, \varepsilon\mathscr{V}_W)\, \partial^{\langle j\rangle}(\mathscr{V}_n, \mathscr{V}_W)\, \partial^l(0, U_n, \nabla^\varepsilon\phi_n)\, \partial^m(D_{x'}W, D_\theta W),$$

where $\phi \in C^\infty$ and $j+l+m = k'$. Now for $T \leqslant T(k)$ (9.92), (9.95), and the induction assumption imply

(9.102)
$$|(\mathscr{V}_n, \mathscr{V}_W)|_* \leqslant C_W + h_1(0)\,\delta, \qquad |(\mathscr{V}_n, \mathscr{V}_W)|_{0,k+1,T} \leqslant C_W + h(D)\frac{\delta}{\sqrt{T}}$$
$$|U_n, \nabla^\varepsilon\phi_n|_* \leqslant C_W, \qquad |U_n, \nabla^\varepsilon\phi_n|_{0,k+1,T} \leqslant h(D)\frac{\delta}{\sqrt{T}} \text{ (recall (9.79))}$$
$$|D_{x'}W, D_\theta W|_* \leqslant C_W, \qquad |D_{x'}W, D_\theta W|_{0,k+1,T} \leqslant |W|_{0,k_0+1,T} \leqslant h(D)\, C_W.$$

Applying the Moser estimate of Lemma 7.3(b) gives for $T \leqslant T(k)$ as in (9.97)

$$(9.103)\qquad |\mathscr{H}_\varepsilon|_{0,k+1,T} \leqslant \frac{h_2(C_W, \delta, D)}{\sqrt{T}}.$$

In just the same way one finds

$$(9.104)\qquad |\mathscr{H}_\varepsilon|_{\infty,k,T} \leqslant h_3(C_W, \delta).$$

Finally, take $T = T(k)$, $\gamma = \frac{1}{T(k)}$ in (9.83) and use (9.99), (9.103), (9.104) to obtain

(9.105)
$$|\|U_{n+1}, \phi_{n+1}\||_{k,T(k)}$$
$$\leqslant eh(C_W, \delta)\Big[(h_3(C_W, \delta) + C_{\mathscr{F}})\sqrt{T(k)} + \left(\frac{h_2(C_W, \delta, D)}{\sqrt{T(k)}} + C_{\mathscr{F}}\right) T(k)$$
$$+ C_{\mathscr{G}}\sqrt{T(k)}\Big].$$

Reduce $T(k)$ if necessary to make the right side of (9.105) $\leqslant \delta$. This gives T_k. ∎

PROPOSITION 9.3 (Contraction). *Fix* δ, $k \geqslant [M_0 + \frac{N+1}{2}]$, $T_k \leqslant 1$, ε_0 *as in Proposition* 9.2. *There exist* $\mathbb{T}_k \in (0, T_k]$, $\varepsilon_1(k, \delta) \leqslant \varepsilon_0$, *and* $(U_\varepsilon, \phi_\varepsilon) \in \mathbb{A}^k_T$ *such that*

$$|\|(U^n_\varepsilon, \phi^n_\varepsilon) - (U_\varepsilon, \phi_\varepsilon)\||_{k-1, \mathbb{T}_k} \to 0 \qquad \textit{as} \quad n \to \infty$$

uniformly for $\varepsilon \in (0, \varepsilon_1]$.

Proof. Let $(\Delta_n, \delta_n) = (U_{n+1} - U_n, \phi_{n+1} - \phi_n)$ and consider the problem satisfied by (Δ_n, δ_n):

$$\begin{aligned} \tilde{\mathscr{L}}(\varepsilon \mathscr{V}_n, D^\varepsilon_{x,\theta})\, \Delta_n &= (\tilde{\mathscr{A}}(\varepsilon \mathscr{V}_n, D^\varepsilon_{x',\theta}) - \tilde{\mathscr{A}}(\varepsilon \mathscr{V}_{n-1}, D^\varepsilon_{x',\theta}))(U_n + W_\varepsilon) \\ &\equiv \mathscr{F}_n \qquad \text{in} \quad x_N > 0 \\ \tilde{\mathscr{B}}(\varepsilon \mathscr{V}_n)(\Delta_n, \nabla^\varepsilon \delta_n) &= (\tilde{\mathscr{B}}(\varepsilon \mathscr{V}_{n-1}) - \tilde{\mathscr{B}}(\varepsilon \mathscr{V}_n))(U_n, \nabla^\varepsilon \phi_n) \\ &\equiv \mathscr{G}_n \qquad \text{on} \quad x_N = 0 \\ (\Delta_n, \delta_n) &= 0 \qquad \text{in} \quad x_0 < 0. \end{aligned} \tag{9.106}$$

To apply Corollary 9.1 we need to estimate

$$|\mathscr{F}_n|_{\infty, k-1, T}, \qquad |\mathscr{F}_n|_{0, k, T}, \qquad \text{and} \qquad \langle \mathscr{G}_n \rangle_{k, T}.$$

For $k' \leqslant k$, $\partial^{k'} \mathscr{F}_n$ and $\partial^{k'} \mathscr{G}_n$ are sums of terms of the form

$$\begin{aligned} &\text{(a)} \quad \phi_1(\varepsilon, \varepsilon \mathscr{V}_n, \varepsilon \mathscr{V}_{n-1})\, \partial^{\langle j \rangle}(\mathscr{V}_n, \mathscr{V}_{n-1}) \\ &\qquad \times \partial^l(0, \Delta_{n-1}, \nabla^\varepsilon \delta_{n-1})\, \partial^m(D_{x'}(U_n + W_\varepsilon), D_\theta(U_n + W_\varepsilon)) \\ &\text{(b)} \quad \phi_2(\varepsilon, \varepsilon \mathscr{V}_n, \varepsilon \mathscr{V}_{n-1})\, \partial^{\langle j \rangle}(\mathscr{V}_n, \mathscr{V}_{n-1}) \\ &\qquad \times \partial^l(0, \Delta_{n-1}, \nabla^\varepsilon \delta_{n-1})\, \partial^m(U_n, \nabla^\varepsilon \phi_n), \end{aligned} \tag{9.107}$$

respectively, where $\phi_1, \phi_2 \in C^\infty$ and $j + l + m = k'$.

As in (9.102) note that for $T \leqslant T_k$

$$\begin{aligned} |\Delta_{n-1}, \nabla^\varepsilon \delta_{n-1}|_* &\leqslant h_1(0)\, |\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T} \\ |\Delta_{n-1}, \nabla^\varepsilon \delta_{n-1}|_{0, k, T} &\leqslant h(D)\, \frac{|\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}}{\sqrt{T}} \\ |\Delta_{n-1}, \nabla^\varepsilon \delta_{n-1}|_{\infty, k-1, T} &\leqslant |\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T} \\ \langle \Delta_{n-1}, \nabla^\varepsilon \delta_{n-1} \rangle_{k, T} &\leqslant h(D)\, \frac{|\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}}{\sqrt{T}}. \end{aligned} \tag{9.108}$$

From (9.102), (9.108), and Lemma 7.3 we deduce

$$
(9.109)\qquad
\begin{aligned}
|\mathscr{F}_n|_{\infty, k-1, T} &\leqslant h_4(C_W, \delta)\, |\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}\\
\mathscr{F}_n|_{0, k, T} &\leqslant h_5(C_W, \delta, D)\, \frac{|\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}}{\sqrt{T}}\\
\langle \mathscr{G}_n \rangle_{k, T} &\leqslant \varepsilon h_6(C_W, \delta)\, \frac{|\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}}{\sqrt{T}}.
\end{aligned}
$$

Apply Corollary 9.1 to (9.106), letting $\gamma = \frac{1}{T}$ in (9.83), to obtain for h_4, h_5, h_6 as in (9.109)

(9.110)

$$
|\|\Delta_n, \delta_n\||_{k-1, T} \leqslant eh(C_W, \delta)(h_4\sqrt{T} + h_5\sqrt{T} + h_6\varepsilon)\, |\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, T}.
$$

For $\mathbb{T}_k \leqslant T_k$ and $\varepsilon_1(\delta) \leqslant \varepsilon_0$ small enough, (9.110) implies

$$
|\|\Delta_n, \delta_n\||_{k-1, \mathbb{T}_k} \leqslant \tfrac{1}{2}\, |\|\Delta_{n-1}, \delta_{n-1}\||_{k-1, \mathbb{T}_k}. \quad \blacksquare
$$

This also completes the proof of Theorem 9.4.

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