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Potential Theory on Lipschitz Domains in Riemannian Manifolds: Sobolev–Besov Space Results and the Poisson Problem

Marius Mitrea[1]

Department of Mathematics, University of Missouri, Columbia, Missouri 65211

and

Michael Taylor[2]

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

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We continue a program to develop layer potential techniques for PDE on Lipschitz domains in Riemannian manifolds. Building on L^p and Hardy space estimates established in previous papers, here we establish Sobolev and Besov space estimates on solutions to the Dirichlet and Neumann problems for the Laplace operator plus a potential, on a Lipschitz domain in a Riemannian manifold with a metric tensor smooth of class $C^{1+\gamma}$, for some $\gamma > 0$. We treat the inhomogeneous problem and extend it to the setting of manifolds results obtained for the constant-coefficient Laplace operator on a Lipschitz domain in Euclidean space, with the Dirichlet boundary condition, by D. Jerison and C. Kenig.

1. INTRODUCTION

There has been a substantial amount of work in the area of elliptic, constant coefficient PDEs in Lipschitz domains via layer potential methods. In particular, the classical Dirichlet and Neumann boundary value problems for the flat-space Laplacian $\partial_1^2 + \cdots + \partial_n^2$ with boundary data in L^p spaces for optimal ranges of p have been solved. There are also sharp results for the inhomogeneous Dirichlet and Neumann problems. See [Ve, DK, JK2, FMM, Za] and the references therein for more on this subject. Similar

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issues are studied in [AP] for the flat-space bi-Laplacian in Euclidean Lipschitz domains.

In [MT, MMT] the authors initiated a program to extend the applicability of the layer potential theory from the setting of the flat-space Laplacian on Lipschitz domains in Euclidean space to the setting of variable coefficients, and more generally to the context of Lipschitz domains in Riemannian manifolds. The paper [MT] dealt with the scalar Laplace–Beltrami operator, and [MMT] treated natural boundary problems for the Hodge Laplacian acting on differential forms.

Quite recently, more progress in this direction has been made in [MT2], where a sharp L^p theory for scalar layer potentials associated with the Laplace–Beltrami operator in Lipschitz domains has been developed. Let us now recall the general setting and describe some of the main results from [MT] and [MT2].

Consider a smooth, connected, compact Riemannian manifold M, of real dimension $\dim M = n \geqslant 3$, equipped with a Riemannian metric tensor $g = \sum_{j,k} g_{jk}\, dx_j \otimes dx_k$ whose coefficients are Lipschitz continuous. The Laplace–Beltrami operator on M is then given in local coordinates by

$$\Delta u := \operatorname{div}(\operatorname{grad} u) = g^{-1/2}\, \partial_j (g^{jk} g^{1/2}\, \partial_k u), \tag{1.1}$$

where we use the summation convention, take (g^{jk}) to be the matrix inverse to (g_{jk}), and set $g := \det(g_{jk})$. For $V \in L^\infty(M)$ we introduce the second order, elliptic differential operator

$$L := \Delta - V. \tag{1.2}$$

Assume $V \geqslant 0$ and, also, $V > 0$ on a set of positive measure in M. This implies that

$$L\colon L^p_1(M) \to L^p_{-1}(M) \tag{1.3}$$

is an isomorphism for each $p \in (1, \infty)$, where $L^p_s(M)$ denotes the class of L^p-Sobolev spaces on M. See [MT].

Throughout this paper, we let $\Omega \subset M$ be a connected open set that is a Lipschitz domain; i.e., $\partial\Omega$ is locally representable as the graph of a Lipschitz function. We will always assume that $V > 0$ on a set of positive measure in each connected component of $M \backslash \bar{\Omega}$. Consider the Dirichlet boundary problem

$$Lu = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f, \tag{1.4}$$

and the Neumann boundary problem

$$Lu = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = g, \tag{1.5}$$

where ν denotes the outward unit normal to $\partial\Omega$, and $\partial_\nu = \partial/\partial\nu$ is the normal derivative on $\partial\Omega$. When the boundary data are from $L^p(\partial\Omega)$ (for appropriate p) then, in (1.4)–(1.5), the boundary traces are taken in the nontangential limit sense and natural estimates (involving the nontangential maximal function) are sought. More specifically, if $\{\gamma(x)\}_{x\in\partial\Omega}$ is a family of nontangential approach regions (cf. [MT] for more details) and u is defined in Ω then u^*, the nontangential maximal function of u, is defined at boundary points by

$$u^*(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{1.6}$$

Then the natural accompanying estimates for (1.4) and (1.5) are, respectively,

$$\|u^*\|_{L^p(\partial\Omega)} \leqslant C\, \|f\|_{L^p(\partial\Omega)} \tag{1.7}$$

and

$$\|(\nabla u)^*\|_{L^p(\partial\Omega)} \leqslant C\, \|g\|_{L^p(\partial\Omega)}. \tag{1.8}$$

In the case when Ω is a Lipschitz domain in the Euclidean space and $L = \partial_1^2 + \cdots + \partial_n^2$ is the flat-space Laplacian, these problems were first treated in [Da, JK] by means of harmonic measure estimates. Shortly thereafter, building on [FJR] where the case of C^1 domains was treated, a new approach using layer potential techniques was developed in [Ve, DK]. In these latter papers, a key ingredient was the L^2 boundedness of Cauchy type integrals on Lipschitz surfaces due to [CMM] (following the pioneering work of [C]).

In [MT] we have extended such operator norm estimates on Cauchy integrals to a variable coefficient setting, allowing for an analysis of potential type operators in the manifold setting described above. To be more specific, denote by $E(x, y)$ the integral kernel of L^{-1}, so

$$L^{-1}u(x) = \int_M E(x, y)\, u(y)\, d\,\mathrm{Vol}(y), \qquad x \in M, \tag{1.9}$$

where $d\,\mathrm{Vol}$ is the volume element on M determined by its Riemannian metric. Then, for functions $f\colon \partial\Omega \to \mathbb{R}$, define the single layer potential

$$\mathscr{S}f(x) := \int_{\partial\Omega} E(x, y)\, f(y)\, d\sigma(y), \qquad x \notin \partial\Omega, \tag{1.10}$$

and define the double layer potential by

$$\mathscr{D}f(x) := \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x, y)\, f(y)\, d\sigma(y), \qquad x \notin \partial\Omega. \tag{1.11}$$

Here $d\sigma$ denotes the canonical surface measure on $\partial\Omega$ induced by the Riemannian metric on M. The following results on the behavior of these potentials, which extend previously known results for the flat Euclidean case, were demonstrated in [MT].

Define $\Omega_+ := \Omega$ and $\Omega_- := M \backslash \bar{\Omega}$; note that $\Omega_\pm$ are Lipschitz domains. Given $f \in L^p(\partial\Omega)$, $1 < p < \infty$, we have, for a.e. $x \in \partial\Omega$,

$$\mathscr{S}f|_{\partial\Omega_\pm}(x) = Sf(x) := \int_{\partial\Omega} E(x, y) f(y)\, d\sigma(y), \tag{1.12}$$

$$\mathscr{D}f|_{\partial\Omega_\pm}(x) = (\pm\tfrac{1}{2}I + K) f(x), \tag{1.13}$$

where, for a.e. $x \in \partial\Omega$,

$$Kf(x) := \text{P.V.} \int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x, y)\, f(y)\, d\sigma(y). \tag{1.14}$$

Here P.V. $\int_{\partial\Omega}$ indicates that the integral is taken in the principal value sense. Specifically, we fix a smooth background metric which, in turn, induces a distance function on M. In particular, we can talk about balls and P.V. $\int_{\partial\Omega}$ is defined in the usual sense, of removing such small geodesic balls and passing to the limit. Moreover, for a.e. $x \in \partial\Omega$,

$$\partial_\nu \mathscr{S}f|_{\partial\Omega_\pm}(x) = (\mp\tfrac{1}{2}I + K^*)\, f(x), \tag{1.15}$$

where K^* is the formal transpose of K.

The operators

$$K, K^*: L^p(\partial\Omega) \to L^p(\partial\Omega), \qquad 1 < p < \infty, \tag{1.16}$$

and

$$S: L^p(\partial\Omega) \to L^p_1(\partial\Omega), \qquad 1 < p < \infty, \tag{1.17}$$

are bounded.

Turning to the issue of invertibility, extending results produced in the Euclidean, constant coefficient case by [Ve, DK] we showed in [MT, MT2] that there exists $\varepsilon = \varepsilon(\Omega) > 0$ so that the operators

$$\begin{aligned} \pm\tfrac{1}{2}I + K&: L^p(\partial\Omega) \to L^p(\partial\Omega), \qquad 2 - \varepsilon < p < \infty, \\ \pm\tfrac{1}{2}I + K^*&: L^p(\partial\Omega) \to L^p(\partial\Omega), \qquad 1 < p < 2 + \varepsilon, \end{aligned} \tag{1.18}$$

are Fredholm, of index zero. In particular, (recall that we are assuming $V>0$ on a set of positive measure in each connected component of $M\setminus\bar{\Omega}$) the operators

$$\begin{aligned} &\tfrac{1}{2}I+K\colon L^p(\partial\Omega)\to L^p(\partial\Omega), && 2-\varepsilon<p<\infty,\\ &\tfrac{1}{2}I+K^*\colon L^p(\partial\Omega)\to L^p(\partial\Omega), && 1<p<2+\varepsilon, \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} &S\colon L^p(\partial\Omega)\to L^p_1(\partial\Omega), && 1<p<2+\varepsilon,\\ &\tfrac{1}{2}I+K\colon L^p_1(\partial\Omega)\to L^p_1(\partial\Omega), && 1<p<2+\varepsilon, \end{aligned} \tag{1.20}$$

are invertible. Also, if $V>0$ on a set of positive measure in Ω, then

$$\begin{aligned} &-\tfrac{1}{2}I+K\colon L^p(\partial\Omega)\to L^p(\partial\Omega), && 2-\varepsilon<p<\infty,\\ &-\tfrac{1}{2}I+K^*\colon L^p(\partial\Omega)\to L^p(\partial\Omega), && 1<p<2+\varepsilon,\\ &-\tfrac{1}{2}I+K\colon L^p_1(\partial\Omega)\to L^p_1(\partial\Omega), && 1<p<2+\varepsilon, \end{aligned} \tag{1.21}$$

are isomorphisms, while, if $V=0$ on $\bar{\Omega}$, then

$$\begin{aligned} &-\tfrac{1}{2}I+K\colon L^p(\partial\Omega)/\mathscr{C}\to L^p(\partial\Omega)/\mathscr{C}, && 2-\varepsilon<p<\infty,\\ &-\tfrac{1}{2}I+K^*\colon L^p_0(\partial\Omega)\to L^p_0(\partial\Omega), && 1<p<2+\varepsilon,\\ &-\tfrac{1}{2}I+K\colon L^p_1(\partial\Omega)/\mathscr{C}\to L^p_1(\partial\Omega)/\mathscr{C}, && 1<p<2+\varepsilon, \end{aligned} \tag{1.22}$$

are isomorphisms, where $\mathscr{C}$ is the set of constant functions on $\partial\Omega$ and $L^p_0(\partial\Omega)$ consists of elements of $L^p(\partial\Omega)$ integrating to zero.

Given these results, solutions to the Dirichlet, Regularity and Neumann problems for the operator L in Lipschitz subdomains of M can be produced for L^p boundary data, for appropriate (sharp ranges of) p and optimal nontangential function estimates; cf. [MT2]. The paper [MT2] also contains invertibility results in the local Hardy space $\mathfrak{h}^1(\partial\Omega)$, its dual $\mathrm{bmo}(\partial\Omega)$, and spaces of Hölder continuous functions $C^\alpha(\partial\Omega)$, for small $\alpha>0$. Parenthetically, let us point out that real interpolation and (1.21)–(1.22) also allow one to solve the Dirichlet and the Neumann problems with boundary data in Lorentz spaces $L_{p,q}(\partial\Omega)$ for appropriate ranges of indices.

One of our primary goals in this paper is to continue this line of work and to prove invertibility results for the boundary layer potentials in the right sides of (1.12)–(1.14) in the class of Besov spaces $B^p_s(\partial\Omega)$ for the optimal range of the parameters p and s. We mention a sample result. Let

$\Omega \subset M$ be an arbitrary Lipschitz domain. There exists $\varepsilon \in (0, 1]$ so that if $p \in [1, \infty]$ and $s \in (0, 1)$ satisfy one of the three conditions

$$\begin{aligned} \frac{2}{1+\varepsilon} < p < \frac{2}{1-\varepsilon} \qquad &\text{and} \qquad 0 < s < 1; \\ 1 \leqslant p < \frac{2}{1+\varepsilon} \qquad &\text{and} \qquad \frac{2}{p} - 1 - \varepsilon < s < 1; \\ \frac{2}{1-\varepsilon} < p \leqslant \infty \qquad &\text{and} \qquad 0 < s < \frac{2}{p} + \varepsilon, \end{aligned} \tag{1.23}$$

then, with $q \in [1, \infty]$ denoting the conjugate exponent of p, the operators

$$\tfrac{1}{2}I + K \colon B^p_s(\partial\Omega) \to B^p_s(\partial\Omega), \qquad S \colon B^q_{-s}(\partial\Omega) \to B^q_{1-s}(\partial\Omega), \tag{1.24}$$

are invertible. For a complete statement see Section 11. This extends work in [FMM] where the same issue has been studied in the flat-space setting. Our approach parallels that in [FMM] but the setting of Lipschitz domains in Riemannian manifolds introduces significant additional difficulties, which require new techniques and ideas to overcome. On the analytical side, one major concern is understanding the behavior of the fundamental solution $E(x, y)$ near the diagonal in $M \times M$ in the context of a metric tensor whose coefficients have a rather limited amount of smoothness; here we build on results in [MT, MMT, MT2].

Even though our main results are for scalar functions, we are occasionally lead to consider differential forms of higher degrees. Given this, we develop mapping properties of layer potentials in this context. See Section 6–8. To obtain such invertibility results, in the present paper we shall assume more regularity on the metric tensor g than the Lipschitz condition mentioned above. We say more about this below.

Having these invertibility results, we then proceed to solve the Poisson problem for the Laplace–Beltrami operator with Dirichlet and Neumann boundary conditions for data in Sobolev–Besov spaces. Concretely, consider

$$\begin{cases} Lu = f \in L^p_{s+1/p-2}(\Omega), \\ \mathrm{Tr}\, u = g \in B^p_s(\partial\Omega), \\ u \in L^p_{s+1/p}(\Omega), \end{cases} \tag{1.25}$$

and

$$\begin{cases} Lu = f \in L^q_{1/q-s-1,\,0}(\Omega), \\ \partial^f_\nu u = g \in B^q_{-s}(\partial\Omega), \\ u \in L^q_{1-s+1/q}(\Omega). \end{cases} \tag{1.26}$$

The technical definition of $\partial_\nu^f u$ is given in Section 12. Here we would like to point out that $\partial_\nu^f u$ can be thought of as a "renormalization" of the normal gradient of u on $\partial\Omega$, in a fashion that depends decisively on f. Assume that $p \in (1, \infty)$ and $s \in (0, 1)$ satisfy one of the three conditions listed in (1.23) and that q is the conjugate exponent of p. Then (1.25) and (1.26) are solvable with naturally accompanying estimates. Certain limiting cases of (1.25)–(1.26) are also considered; see Sections 9–10.

Our main results regarding the solvability of (1.25)–(1.26) extend previous work in [JK2, FMM, Za] where similar problems for the flat-space Laplacian have been considered. As in [FMM] we employ the method of layer potentials. In addition to establishing invertibility results, like (1.24), we also need to understand the mapping properties of the operators (1.10)–(1.11), sending functions from Sobolev–Besov spaces on $\partial\Omega$ into functions in appropriate Sobolev–Besov spaces in Ω. We do this in Sections 6–8.

We outline the structure of the rest of this paper. In Section 2, following [MT, MMT], we include a discussion of the nature of the main singularity of the fundamental solution $E(x, y)$. In Section 3 we establish some useful interior estimates for null-solutions of the operator L. In Sections 4–5 we collect basic definitions and several important properties of Sobolev and Besov spaces, as well as atomic Hardy spaces. Mapping properties for Newtonian potentials are presented in Section 6. In Sections 7–8 we then establish a number of useful results for single and double layer potential operators on Sobolev–Besov spaces.

Two endpoint results for the Poisson problem for L (with Neumann and Dirichlet boundary conditions, respectively) are studied in Sections 9–10. Then, in Section 11, invertibility results for boundary layer potentials, including (1.24), are presented. Subsequently, these are used in Sections 12–13 to solve the general Poisson problem for L, respectively with Neumann and Dirichlet boundary conditions, for data in Sobolev–Besov spaces. Applications to complex powers of the Laplace–Beltrami operator (with either homogeneous Dirichlet or homogeneous Neumann boundary conditions) in an arbitrary Lipschitz domain are presented in Section 14. These extend results on the flat, Euclidean case given in [JK2, JK3, MM].

As mentioned above, some of our arguments require more regularity of the metric tensor than the Lipschitz condition. For example, most of our scalar results are obtained under the assumption that

$$g_{jk} \in C^{1+\gamma}, \qquad \text{some} \quad \gamma > 0. \tag{1.27}$$

Some of our arguments, as in [MMT], need

$$g_{jk} \in L_2^r, \qquad \text{some} \quad r > n, \tag{1.28}$$

and we shall occasionally require that

$$g_{jk} \in L^r_2, \qquad \text{some} \quad r \geqslant 2n. \tag{1.29}$$

We would like to stress that, for all our main results, the assumption (1.27) suffices. Nonetheless, we will explicitly state which of these hypotheses we need in each segment of the analysis.

In closing, let us point out that, for the results that we have in mind, such as the well posedness of (1.25)–(1.26), *some smoothness* of the metric tensor is required. More specifically, the condition $g_{jk} \in L^\infty$ $(\forall\, j, k)$ is, generally speaking, not enough in order to guarantee, e.g., the well-posedness of the L^p-Poisson problem (1.25) (with $s := 1 - 1/p$) for p in some *a priori* given open interval containing 2 (even when the underlying domain is smooth). This can be seen from an adaptation of an example given in Section 5 of [Me]; we omit the details here. The interested reader is also referred to Propositions 1.7–1.10 in [Ta2, Chap. 3] for some related results.

2. ESTIMATES ON THE FUNDAMENTAL SOLUTION

Retain the hypotheses made in Section 1 on M. As for the metric tensor

$$g = \sum_{j,k} g_{jk}\, dx_j \otimes dx_k, \tag{2.1}$$

unless otherwise stated, in this section we will assume that the coefficients (g_{jk}) satisfy (1.28), i.e.,

$$g \in L^r_2(M, \operatorname{Hom}(TM \otimes TM, \mathbb{R})), \qquad \text{for some} \quad r > n = \dim M, \tag{2.2}$$

where TM is the tangent bundle of M.

As mentioned in the introduction, we will primarily study the Laplace operator on scalar functions, but some constructions involving the Hodge Laplacian on differential forms will arise, and in this section we work in the context of forms.

Let $\Lambda^\ell TM$, $\ell \in \{0, 1, ..., n\}$, stand for the ℓ-th exterior power of the cotangent bundle T^*M. Its sections consist of ℓ-differential forms. If $(x_1, ..., x_n)$ are local coordinates in an arbitrary coordinate patch U on M and $u \in \Lambda^\ell TM$ then $u|_U = \sum_{|I|=\ell} u_I\, dx^I$, where the sum is performed over ordered ℓ-tuples $I = (i_1, ..., i_\ell)$, $1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n$ and, for each such I, $dx^I := dx_{i_1} \wedge \cdots \wedge dx_{i_\ell}$. Here, of course, the wedge stands for the usual exterior product of forms, while $|I|$ denotes the cardinality of I.

The Hermitian structure in the fibers on TM extends naturally to T^*M by setting $\langle dx_j, dx_k \rangle_x := g^{jk}(x)$. The latter further induces a Hermitian structure on $\Lambda^\ell TM$ by selecting $\{\omega^I\}_{|I|=\ell}$ to be an orthonormal frame in $\Lambda^\ell TM$ provided $\{\omega_j\}_{1\leqslant j\leqslant n}$ is an orthonormal frame in T^*M (locally). Note that

$$\langle dx^I, dx^J\rangle_x = \det((g^{ij}(x))_{i\in I,\, j\in J}). \tag{2.3}$$

Recall that the exterior derivative operator d is given locally by

$$d = \sum_{j=1}^n \frac{\partial}{\partial x_j}\, dx_j \wedge \tag{2.4}$$

and denote by δ its formal adjoint with respect to the metric induced by (2.3). The latter acts on a differential form $u \in \Lambda^\ell TM$, locally written as $u|_U = \sum_{|I|=\ell} u_I\, dx^I$, by $\delta u = \sum_{|I|=\ell-1} (\delta u)_I\, dx^I$ where, for each I,

$$(\delta u)_I := -\sum_{|J|=\ell}\ \sum_{j,k} g^{jk} \varepsilon^J_{jI} \frac{\partial u_J}{\partial x_k} + \sum_{|K|=\ell-1}\ \sum_{j,k,r,s} \varepsilon^{rK}_{jI}\, g^{jk} \Gamma^s_{rk} u_{sK} \qquad \text{in } U. \tag{2.5}$$

Here, for $1 \leqslant j, k, s \leqslant n$,

$$\Gamma^s_{jk} := \tfrac{1}{2} \sum_t g^{st} \left(\frac{\partial g_{tj}}{\partial x_k} + \frac{\partial g_{tk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_t} \right) \tag{2.6}$$

are the Christoffel symbols. Also, for any two ordered arrays J, K, the generalized Kronecker symbol ε^J_K is given by

$$\varepsilon^J_K := \begin{cases} \det((\delta_{j,k})_{j\in J,\, k\in K}), & \text{if } \ |J| = |K|, \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

where, as usual, $\delta_{j,k} := 1$ if $j = k$, and zero if $j \neq k$.

Denote by $\Delta_\ell := -d\delta - \delta d$ the Hodge–Laplacian on ℓ-forms, $\ell \in \{0, 1, ..., n\}$ and fix some positive, not identically zero function $V \in C^\infty(M)$. Recall from [MMT] that under these conditions, the operator $L := \Delta_\ell - V$, acting on suitable spaces of ℓ-forms has an inverse, $(\Delta_\ell - V)^{-1}$, whose Schwartz kernel, $E_\ell(x, y)$, is a symmetric double form of bidegree (ℓ, ℓ).

In local coordinates, in which the metric tensor is given by (2.1), we can set

$$e^\ell_0(x-y, y) := C_n \left(\sum_{j,k} g_{jk}(y)(x_j - y_j)(x_k - y_k) \right)^{-(n-2)/2} \Gamma_\ell(x, y) \tag{2.8}$$

for appropriate $C = C_n$, where Γ_ℓ is the double form of bidegree (ℓ, ℓ) given by

$$\Gamma_\ell(x, y) := \begin{cases} \sum\limits_{|I| = \ell} \sum\limits_{|J| = \ell} \det((g_{ij}(y))_{i \in I,\, j \in J})\, dx^I \otimes dy^J, & \text{if} \quad \ell \geqslant 1, \\ 1, & \text{if} \quad \ell = 0. \end{cases} \tag{2.9}$$

Note that $e_0^\ell(z, y)$ is smooth and homogeneous of degree $-(n-2)$ in $z \in \mathbb{R}^n \setminus 0$ and $C^{2+\gamma}$ in y, for some $\gamma > 0$. Then define the remainder $e_1^\ell(x, y)$ so that

$$E_\ell(x, y) \sqrt{g(y)} = e_0^\ell(x - y, y) + e_1^\ell(x, y). \tag{2.10}$$

In the theorem below we collect useful estimates for $e_1^\ell(x, y)$ and its derivatives. These estimates follow from Propositions 2.5 and 2.8 of [MMT].

THEOREM 2.1. *For each $\ell \in \{0, 1, ..., n\}$ and each $\varepsilon \in (0, 1)$, the remainder $e_1^\ell(x, y)$ satisfies*

$$|\nabla_x^j \nabla_y^k e_1^\ell(x, y)| \leqslant C_\varepsilon\, |x - y|^{-(n-3+j+k+\varepsilon)}, \tag{2.11}$$

for each $j, k \in \{0, 1\}$.

Furthermore, for $\ell = 0$ the same conclusion is valid under the (weaker) hypothesis (1.27) on the metric tensor.

In the sequel, we shall also need information about the "commutators" between d, δ on the one hand and the forms $E_\ell(x, y)$ on the other hand. This is made precise in the proposition below, which is taken from Section 6 of [MMT].

PROPOSITION 2.2. *There exists a double form $R_\ell(x, y)$ of bidegree $(\ell, \ell+1)$ so that*

$$\begin{gathered} R_\ell \in C^{1+\gamma}_{\mathrm{loc}}((M \times M \setminus \mathrm{diag}) \cup \{(x, y) \colon x \notin \operatorname{supp} dV\}), \qquad \text{some} \quad \gamma > 0, \\ |\nabla_x^j \nabla_y^k R_\ell(x, y)| \leqslant C\, |x - y|^{-(n-4+j+k)}, \qquad 0 \leqslant j, k \leqslant 1, \end{gathered} \tag{2.12}$$

and so that

$$\delta_x(E_{\ell+1}(x, y)) = d_y(E_\ell(x, y)) + R_\ell(x, y). \tag{2.13}$$

Furthermore,

$$d_x(E_\ell(x, y)) = \delta_y(E_{\ell+1}(x, y)) - R_\ell(y, x). \tag{2.14}$$

We shall occasionally also use R_ℓ to denote the boundary integral operator with kernel $R_\ell(x, y)$, that is,

$$R_\ell f(x) := \int_{\partial\Omega} \langle R_\ell(x, y), f(y)\rangle \, d\sigma(y), \qquad f \in L^p(\partial\Omega, \Lambda^{\ell+1}TM). \quad (2.15)$$

Also, set R_ℓ^t for the analogous operator with kernel $R_\ell(y, x)$.

Next, let Ω be a Lipschitz domain in M and denote by $\mathcal{S}_\ell$ the single layer potential operator on $\partial\Omega$ with kernel $E_\ell(x, y)$, i.e.,

$$\mathcal{S}_\ell f(x) := \int_{\partial\Omega} \langle E_\ell(x, y), f(y)\rangle \, d\sigma(y), \qquad x \in M \backslash \partial\Omega, \quad (2.16)$$

where $f \in L^p(\partial\Omega, \Lambda^\ell TM)$. Note that at the level of scalar functions, i.e., when $\ell = 0$, this agrees with (1.12). Also, $(\Delta_\ell - V)\,\mathcal{S}_\ell f = 0$ in $M \backslash \partial\Omega$. Finally, set $S_\ell f := \mathcal{S}_\ell f|_{\partial\Omega}$ for $\ell = 0, 1, ..., n$.

3. INTERIOR ESTIMATES

In this section we derive interior estimates (L^p-style) for elliptic systems of PDEs. General standard references are [ADN, DN]. Here we obtain quite sharp estimates for operators whose coefficients have the limited regularity described by (3.2) below.

To proceed, let L be a second order, elliptic differential operator in a compact Riemannian manifold M, locally given by

$$Lu = \sum_{j,k} \partial_j A^{jk}(x)\, \partial_k u + \sum_j B^j(x)\, \partial_j u - V(x)\, u. \quad (3.1)$$

Above, $A^{jk} := (a_{jk}^{\alpha\beta})_{\alpha,\beta}$, $B^j := (b_j^{\alpha\beta})_{\alpha,\beta}$ and $V := (v^{\alpha\beta})_{\alpha,\beta}$ are matrix-valued functions with entries satisfying

$$a_{jk}^{\alpha\beta} \in \mathrm{Lip}, \qquad b_j^{\alpha\beta} \in L_1^r, \quad v^{\alpha\beta} \in L^r, \qquad \text{for some} \quad r > n, \quad (3.2)$$

We note that the Laplace–Beltrami operator on scalar functions satisfies (3.2) when the metric tensor is Lipschitz. Also, the Hodge Laplacian Δ_ℓ on ℓ-forms, for $1 \leqslant \ell \leqslant n-1$, satisfies (3.2) if the metric tensor satisfies (1.28).

Our first proposition is a local regularity result that sharpens part of Proposition 3.3 in [MT].

PROPOSITION 3.1. *Let $\mathcal{O}$ be an open subset of M, and assume $r/(r-1) < p \leqslant q < r$. Then*

$$u \in L^p_{\mathrm{loc}}(\mathcal{O}), \qquad Lu \in L^q_{\mathrm{loc}}(\mathcal{O}) \Rightarrow u \in L^q_{2,\mathrm{loc}}(\mathcal{O}), \quad (3.3)$$

and natural estimates hold. If one strengthens the hypotheses in (3.2) *to*

$$a_{jk}^{\alpha\beta} \in C^{1+\gamma}, \qquad b_j^{\alpha\beta} \in C^{1+\gamma}, \qquad \gamma > 0, \qquad v^{\alpha\beta} \in L^\infty, \tag{3.4}$$

then (3.3) *holds for any* $q \in (1, \infty)$ *with the hypothesis* $u \in L^p_{\mathrm{loc}}(\mathcal{O})$ *weakened to* $u \in L^1_{\mathrm{loc}}(\mathcal{O})$.

Proof. We first treat (3.3) when the hypothesis on u is strengthened to $u \in L^p_{\tau, \mathrm{loc}}(\mathcal{O})$ for all $\tau < 1$. For notational simplicity, we use L^p_s in place of $L^p_{s, \mathrm{loc}}(\mathcal{O})$, etc. Recall that $S^m_{\rho, \delta}$ stands for Hörmander's classes of symbols. Since we find it necessary to work with symbols which only exhibit a limited amount of regularity in the spatial variable (while still C^∞ in the Fourier variable) we define

$$p(x, \xi) \in \mathrm{Lip}\, S^m_{\rho, \delta} \Leftrightarrow |D^\beta_\xi p(x, \xi)| \leqslant C_\beta \langle \xi \rangle^{m - \rho|\beta|}, \qquad \text{and}$$
$$\|D^\beta_\xi p(\cdot, \xi)\|_{\mathrm{Lip}(\mathbb{R}^n)} \leqslant C_\beta \langle \xi \rangle^{m - \rho|\beta| + \delta}. \tag{3.5}$$

We also denote by $\mathrm{OPLip}\, S^m_{\rho, \delta}$ the corresponding class of pseudodifferential operators $p(x, D)$ with symbols $p(x, \xi) \in \mathrm{Lip}\, S^m_{\rho, \delta}$, and by $\varnothing\mathrm{PLip}\, S^m_{\rho, \delta}$ the class of operators $q(D, x)$ with symbols $q(\xi, y) = p(y, \xi)$ when $p(x, \xi) \in \mathrm{Lip}\, S^m_{\rho, \delta}$.

Turning to the problem at hand, we use a symbol decomposition on the first-order differential operator $A_j := \sum_k A^{jk}(x)\, \partial_k$, of a sort introduced by [KN, Bo] (cf. also the exposition in [Ta]). Picking $\delta \in (0, 1)$, write

$$A_j = A_j^{\#} + A_j^b, \qquad \text{where} \quad A_j^{\#} \in \mathrm{OP}\, S^1_{1, \delta} \qquad \text{and} \qquad A_j^b \in \mathrm{OPLip}\, S^{1-\delta}_{1, \delta}. \tag{3.6}$$

Then set $L^{\#} := \sum \partial_j A_j^{\#} \in \mathrm{OP}\, S^2_{1, \delta}$, elliptic, and denote by $E^{\#} \in \mathrm{OP}\, S^{-2}_{1, \delta}$ a parametrix of $L^{\#}$. Make all pseudodifferential operators properly supported. We have, on $\mathcal{O}$,

$$u = E^{\#} f - \sum_j E^{\#}\, \partial_j A_j^b u - \sum_j E^{\#} B^j\, \partial_j u + E^{\#} V u, \qquad \mathrm{mod} \quad C^\infty, \tag{3.7}$$

where $f := Lu$. Note that the hypotheses imply $E^{\#} f \in L^q_2$.

Next we shall need mapping properties for pseudodifferential operators whose symbols have a limited amount of smoothness in the spatial variables. Concretely, the result that is relevant for us here reads as follows. If $1 < p < \infty$ and $\delta \in (0, 1)$ then

$$p(x, \xi) \in \mathrm{Lip}\, S^m_{1, \delta} \Rightarrow p(x, D) \colon L^p_{s+m} \to L^p_s, \qquad -(1 - \delta) < s \leqslant 1. \tag{3.8}$$

When $s<1$, this is a special case of results in [Bou] (cf. also Proposition 2.1.E in [Ta]). The case when $s=1$ requires a separate analysis, which we now give.

To prove (3.8) with $s=1$ it suffices to take $m=-1$. We need to show that

$$p(x,\xi)\in \mathrm{Lip}\, S^{-1}_{1,\delta},\qquad u\in L^p\Rightarrow D_j(p(x,D)\,u)\in L^p. \tag{3.9}$$

If D_j falls on u, the estimate is clear from standard results. If D_j falls on the coefficients of $p(x,D)$, we need only note that

$$p(x,\xi)\in L^\infty S^{-a}_{1,\delta},\qquad a>0\Rightarrow p(x,D)\colon L^p\to L^p,\qquad 1\leqslant p\leqslant\infty, \tag{3.10}$$

which follows from elementary integral kernel estimates. This finishes the justification of (3.8).

Going further, based on the discussion above we have

$$\begin{aligned}\forall v\in L^p_\tau,\qquad 0<\tau\leqslant 2-\delta &\Rightarrow A^b_j v\in L^p_{\tau-1+\delta}\\ &\Rightarrow E^{\#}\partial_j A^b_j v\in L^p_{\tau+\delta}.\end{aligned} \tag{3.11}$$

In order to continue, we need a multiplication result to the effect that

$$L^r_1\cdot L^p_{\tau-1}\hookrightarrow L^{p^*}_{\tau-1} \tag{3.12}$$

whenever

$$\max\{0,n(1/p+1/r-1)\}<\tau\leqslant 1,\qquad n<r<\infty,\quad\text{and}\quad 1<p^*<p<\infty. \tag{3.13}$$

This is a special case of [RS, Theorem 1, p. 171]. Hence, from (3.2) and (3.12), it follows that

$$u\in L^p_\tau,\qquad \forall\tau<1\Rightarrow E^{\#}B^j\,\partial_j u\in L^p_{\tau+1},\qquad \forall\tau<1. \tag{3.14}$$

Indeed, we only need to check that $\tau\in(0,1)$ can be chosen so that $\tau>n(1/p+1/r-1)$, i.e., that $1/n>1/p+1/r-1$. Nonetheless, the largest right side in this last inequality occurs when $p\searrow 1$ and $r\searrow n$, and that is $1/n$. A direct inspection shows that the limiting case of (3.14) corresponding to $\tau=1$ is also true.

Going further, $u\in L^p_\tau$ and $V\in L^r$ imply that $Vu\in L^q$ with $1/q:=1/p+1/r-\tau/n$ and, hence, $E^{\#}Vu\in L^q_2\hookrightarrow L^p_{\tau+\delta}$, as long as $\delta\in(0,1)$ and $r>\max\{p,n\}$. Thus, at this stage we see that the right side of (3.7) belongs to $L^p_{\tau+\delta}$ and so does u. This is an improvement of regularity for u and the argument can be iterated a finite number of times, to produce the conclusion in (3.3) in the case under discussion.

To treat the case $\tau = 0$, i.e., when $u \in L^p$, we apply symbol smoothing to the first order differential operator $\sum_j \partial_j A^{jk}$, which this time we denote $A_k = A_k(D, x) \in \emptyset\mathrm{PLip}\, S^1_{\mathrm{cl}}$. Symbol smoothing produces

$$A_k = A_k^{\#} + A_k^b, \qquad A^{\#} \in \emptyset\mathrm{P}\, S^1_{1,\delta} = \mathrm{OP}\, S^1_{1,\delta}, \quad A_k^b \in \emptyset\mathrm{PLip}\, S^{1-\delta}_{1,\delta}. \tag{3.15}$$

Note that $\emptyset\mathrm{PLip}\, S^m_{1,\delta}$ consists of adjoints of elements of $\mathrm{OPLip}\, S^m_{1,\delta}$, so instead of (3.8) we have

$$p(\xi, y) \in \mathrm{Lip}\, S^m_{1,\delta} \Rightarrow p(D, x)\colon L^p_s \to L^p_{s-m}, \qquad -1 \leqslant s < 1 - \delta. \tag{3.16}$$

This time, let $E^{\#} \in \mathrm{OP}\, S^{-2}_{1,\delta}$ denote a parametrix for $\sum A_k^{\#} \partial_k$. Then, if $u \in L^p$ and $Lu = f \in L^q$ we have

$$u = E^{\#} f - \sum_k E^{\#} A_k^b \partial_k u - \sum_j E^{\#} B^j \partial_j u + E^{\#} V u, \qquad \mathrm{mod}\ C^\infty. \tag{3.17}$$

Now

$$u \in L^p \Rightarrow A_k^b \partial_k u \in L^p_{-2+\delta} \Rightarrow E^{\#} A_k^b \partial_k u \in L^p_\delta. \tag{3.18}$$

To treat terms like $E^{\#} B^j \partial_j u$, note that $\partial_j u \in L^p_{-1}$ and $B^j \in L^r_1$ for each j. Now, the inclusion

$$L^p_{-1} \cdot L^r_1 \hookrightarrow L^p_{-1} \tag{3.19}$$

is true for $r/(r-1) < p < \infty$, as long as $r > n$. Thus, $E^{\#} B^j \partial_j u \in L^p_1$. Also, $Vu \in L^{p^*}$ where $1/p^* := 1/r + 1/p$, by Hölder's inequality so that $E^{\#} V u \in L^{p^*}_2 \hookrightarrow L^p_1$, by Sobolev's embedding theorem.

We see that the right side of (3.17) then belongs to L^p_δ, if $\delta \in (0, 1)$, so $u \in L^p_\delta$ for $\delta < 1$. This reduces our problem to the case already treated, so (3.3) is established.

The rest of the proposition, with stronger hypotheses on the coefficients of L and weaker hypotheses on u, has a similar proof. In fact, to get started, it suffices to assume $u \in L^{1+\varepsilon}_{-\delta, \mathrm{loc}}$, for some (small) $\varepsilon, \delta > 0$. ∎

We now produce some weighted estimates, which will be very useful for subsequent analysis.

PROPOSITION 3.2. *Let Ω be a Lipschitz domain in M and take $s \in [0, 1]$, $r/(r-1) < p < r$. For $u \in L^p_{2,\mathrm{loc}}(\Omega)$ satisfying $Lu = 0$ in Ω we have, under the hypothesis* (3.2),

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)^{(1-s)p} |\nabla^2 u(x)|^p \, d\operatorname{Vol}(x)$$

$$\leqslant C \int_\Omega \operatorname{dist}(x, \partial\Omega)^{-sp} |\nabla u(x)|^p \, d\operatorname{Vol}(x) + C \int_\Omega |u|^p \, d\operatorname{Vol}. \tag{3.20}$$

Proof. Denote by $Q_r(x)$ the cube in $\mathbb{R}^n$ centered at x and whose side length is r. We claim that for each arbitrary, fixed $x_0 \in \Omega$ and $0 < \rho < \frac{16}{9} n^{-1/2} \operatorname{dist}(x_0, \partial\Omega)$ there holds

$$\int_{Q_\rho(x_0)} |\nabla^2 u|^p \, d\operatorname{Vol} \leqslant C\rho^{-p} \int_{Q_{9\rho/8}(x_0)} |\nabla u|^p \, d\operatorname{Vol} + C \int_{Q_{9\rho/8}(x_0)} |Vu|^p \, d\operatorname{Vol}. \tag{3.21}$$

Passing from (3.21) to (3.20) is done by multiplying both sides with an appropriate power of ρ and summing over cubes in a Whitney decomposition of Ω. Specifically, let $\{Q_i\}_{i \in I}$ be an open covering of Ω with cubes $Q_i \subseteq \Omega$ so that $\frac{9}{8} Q_i \subseteq \Omega$, $\operatorname{dist}(Q_i, \partial\Omega) \approx \rho_i := \operatorname{diam} Q_i$ and the collection $\{\frac{9}{8} Q_i\}_{i \in I}$ has the finite intersection property; cf. [St]. Then, applying (3.21) to each Q_i, multiplying both sides by $\rho_i^{(1-s)p}$ and summing up the resulting inequalities yields a version of (3.20) with the only difference that, in the current scenario, the last integral is $\| Vu \|^p_{L^p(\Omega)}$.

To further transform this we use Hölder's inequality and a Sobolev-type estimate to write

$$\left(\int_\Omega |Vu|^p \, d\operatorname{Vol} \right)^{1/p} \leqslant C \left(\int_\Omega |u|^{p^*} \, d\operatorname{Vol} \right)^{1/p^*}$$

$$\leqslant C \left(\left(\int_\Omega |\nabla u|^p \, d\operatorname{Vol} \right)^{1/p} + \left(\int_\Omega |u|^p \, d\operatorname{Vol} \right)^{1/p} \right), \tag{3.22}$$

where $1/p^* := 1/p - 1/r$. Note that for the second estimate to hold we need $1 \leqslant p \leqslant p^* \leqslant \infty$ and $1/p - 1/p^* < 1/n$ (cf., e.g., [GT, Lemmas 7.12 and 7.16]). These are true since we are assuming $p \in (1, r)$ and $r > n$. Now, based on the above discussion, (3.22) can be used to conclude (3.20).

To prove (3.21), use x_0 as the center of a coordinate system and, as in [MT, MMT], introduce the dilation operators $v_\rho(x) := v(\rho x)$, $x \in Q_{9/8} := Q_{9/8}(x_0)$. Thus, if $Lu = 0$, then u_ρ satisfies the equation

$$\partial_j A^{jk}_\rho \, \partial_k u_\rho + \rho B^j_\rho \, \partial_j u_\rho = f_\rho, \tag{3.23}$$

where

$$
\begin{aligned}
A_\rho^{jk}(x) &:= (a_{jk}^{\alpha\beta}(\rho x))_{\alpha,\beta}, \\
B_\rho^{j}(x) &:= (b_{j}^{\alpha\beta}(\rho x))_{\alpha,\beta}, \\
V_\rho(x) &:= (v^{\alpha\beta}(\rho x))_{\alpha,\beta},
\end{aligned} \tag{3.24}
$$

and

$$
f_\rho := \rho^2 V_\rho u_\rho. \tag{3.25}
$$

Furthermore, Eq. (3.23) holds on $Q_{9/8}$, for ρ in some interval $(0, \rho_0]$, on which the collection A_ρ^{jk} is uniformly Lipschitz and ρB_ρ^j is bounded in L_1^r, since $r > n$.

The regularity result in Proposition 3.1 then gives, as long as $p \in (r/(r-1), r)$,

$$
\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \leqslant C_p \int_{Q_{9/8}} [\,|f_\rho|^p + |u_\rho|^p]\, d\,\mathrm{Vol}, \tag{3.26}
$$

for any solution to (3.23) on $Q_{9/8}$. Now if u_ρ solves (3.23), so does $u_\rho - a$ for any constant (vector) a, so (3.26) holds with u_ρ replaced by $u_\rho - a$, for any a. Take a to be the average of u_ρ on $Q_{9/8}$ and use Poincaré's inequality

$$
\int_{Q_{9/8}} |u_\rho - a|^p \, d\,\mathrm{Vol} \leqslant C_p \int_{Q_{9/8}} |\nabla u_\rho|^p \, d\,\mathrm{Vol}, \tag{3.27}
$$

to get

$$
\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \leqslant C_p \int_{Q_{9/8}} |f_\rho|^p \, d\,\mathrm{Vol} + C_p \int_{Q_{9/8}} |\nabla u_\rho|^p \, d\,\mathrm{Vol}, \tag{3.28}
$$

for any solution u_ρ to (3.23) on $Q_{9/8}$. Now, if $Lu = 0$, then u_ρ satisfies (3.23) with f_ρ given by (3.25), so we have

$$
\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \leqslant C_p \int_{Q_{9/8}} |\nabla u_\rho|^p \, d\,\mathrm{Vol} + C_p \rho^{2p} \int_{Q_{9/8}} |V_\rho u_\rho|^p \, d\,\mathrm{Vol}. \tag{3.29}
$$

In turn, this estimate is equivalent to (3.21) and the proof of (3.20) is finished. ∎

PROPOSITION 3.3. *If we strengthen* (3.2) *to*

$$a_{jk}^{\alpha\beta} \in \mathrm{Lip}, \qquad b_j^{\alpha\beta} \in L_1^r, \qquad v^{\alpha\beta} \in L^r, \qquad \textit{for some} \quad r \geqslant 2n, \tag{3.30}$$

then the conclusion in Proposition 3.2 *holds for* $1 \leqslant p < r$.

In passing, let us point out that (3.30) is automatically fulfilled by the Hodge–Laplacian Δ_ℓ, $1 \leqslant \ell \leqslant n$, if the metric tensor satisfies (1.29), and by $\Delta - V$ on scalars given (1.27).

Proof. Let $1 \leqslant p \leqslant r/(r-1)$ and let $q \in (r/(r-1), r)$ be arbitrary. In this case, we know by Proposition 3.1 that $u \in L^q_{\mathrm{loc}}(\Omega)$, $Lu \in L^q_{\mathrm{loc}}(\Omega) \Rightarrow u \in L^q_{2,\,\mathrm{loc}}(\Omega)$. Hölder's inequality plus a quantitative form of this implication give

$$\left(\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \right)^{1/p} \leqslant C \left(\int_{Q_{9/8}} [\,|f_\rho|^q + |u_\rho|^q\,] \, d\,\mathrm{Vol} \right)^{1/q}, \tag{3.31}$$

for any solution to (3.23). Again we can replace u_ρ by $u_\rho - a$. This time use

$$\left(\int_{Q_{9/8}} |u_\rho - a|^q \, d\,\mathrm{Vol} \right)^{1/q} \leqslant C \left(\int_{Q_{9/8}} |\nabla u_\rho|^{q^*} \, d\,\mathrm{Vol} \right)^{1/q^*}, \tag{3.32}$$

provided

$$1 \leqslant q^* \leqslant q \leqslant \infty \qquad \text{and} \qquad \frac{1}{q^*} - \frac{1}{q} < \frac{1}{n} \tag{3.33}$$

(see, e.g., [GT, Lemmas 7.12 and 7.16]), to get that, for any solution to (3.23),

$$\left(\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \right)^{1/p} \leqslant C \left(\int_{Q_{9/8}} |\nabla u_\rho|^{q^*} \, d\,\mathrm{Vol} \right)^{1/q^*} + C \left(\int_{Q_{9/8}} |f_\rho|^q \, d\,\mathrm{Vol} \right)^{1/q}. \tag{3.34}$$

Thus, if (3.25) holds, we have

$$\left(\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \right)^{1/p} \leqslant C \left(\int_{Q_{9/8}} |\nabla u_\rho|^{q^*} \, d\,\mathrm{Vol} \right)^{1/q^*} + C\rho^2 \left(\int_{Q_{9/8}} |V_\rho u_\rho|^q \, d\,\mathrm{Vol} \right)^{1/q}. \tag{3.35}$$

Now use

$$\left(\int_{Q_{9/8}} |V_\rho u_\rho|^q \, d\,\mathrm{Vol}\right)^{1/q} \leqslant C\rho^{-n/r} \left(\int_{Q_{9/8}} |u_\rho|^{\tilde{q}} \, d\,\mathrm{Vol}\right)^{1/\tilde{q}}$$
$$\leqslant C\rho^{-n/r} \left(\int_{Q_{9/8}} |\nabla u_\rho|^p + |u_\rho|^p \, d\,\mathrm{Vol}\right)^{1/p}, \tag{3.36}$$

where $1/\tilde{q} := 1/q - 1/r$ and the last inequality is Sobolev's. For this to hold, we need

$$1 \leqslant p \leqslant \tilde{q} \leqslant \infty \qquad \text{and} \qquad \frac{1}{p} - \frac{1}{\tilde{q}} < \frac{1}{n}. \tag{3.37}$$

Granted (3.37) and the possibility of choosing q so that

$$p = q^* \tag{3.38}$$

we get

$$\int_{Q_1} |\nabla^2 u_\rho|^p \, d\,\mathrm{Vol} \leqslant C \int_{Q_{9/8}} |\nabla u_\rho|^p \, d\,\mathrm{Vol} + C\rho^{(2-n/r)\,p} \int_{Q_{9/8}} |u_\rho|^p \, d\,\mathrm{Vol}, \tag{3.39}$$

and, further,

$$\int_{Q_\rho} |\nabla^2 u|^p \, d\,\mathrm{Vol} \leqslant C\rho^{-p} \int_{Q_{9\rho/8}} |\nabla u|^p \, d\,\mathrm{Vol} + C\rho^{-np/r} \int_{Q_{9\rho/8}} |u|^p \, d\,\mathrm{Vol}. \tag{3.40}$$

As before, this latter inequality leads to

$$\begin{aligned}\int_\Omega \mathrm{dist}(\cdot, \partial\Omega)^{(1-s)\,p} \, |\nabla^2 u|^p \, d\,\mathrm{Vol} \\ \leqslant C \int_\Omega \mathrm{dist}(\cdot, \partial\Omega)^{-sp} \, |\nabla u|^p \, d\,\mathrm{Vol} \\ + C \int_\Omega \mathrm{dist}(\cdot, \partial\Omega)^{(1-s-n/r)\,p} \, |u|^p \, d\,\mathrm{Vol}\end{aligned} \tag{3.41}$$

and, further, to (3.20), via the elementary estimate

$$\int_\Omega \mathrm{dist}\,(\cdot, \partial\Omega)^{-\alpha p} \, |F|^p \, d\,\mathrm{Vol} \leqslant C \int_\Omega |F|^p \, d\,\mathrm{Vol} + C \int_\Omega |\nabla F|^p \, d\,\mathrm{Vol}, \tag{3.42}$$

which is valid for $1 \leqslant p < \infty$ and $\alpha < 1/p$. For this program to work it suffices to have

$$\frac{n}{r} < \frac{1}{p}. \tag{3.43}$$

The conclusion is that (3.20) holds for $1 \leqslant p \leqslant r/(r-1)$ if, given such a p, one can choose $q \in (r/(r-1), r)$ so that (3.33), (3.37), (3.38), and (3.43) are satisfied. Some elementary algebra shows that this is indeed the case provided $r \geqslant 2n$. ▌

For later reference we record below two more related estimates.

PROPOSITION 3.4. *Assume* $p \in (r/(r-1), r)$ *and let* $u \in L^p_{\mathrm{loc}}(\Omega)$ *satisfy* $Lu = 0$ *in* Ω. *Here* L *is as in* (3.1) *and its coefficients are as in* (3.2). *Then, for any ball* $B_\rho(x)$ *with* $x \in \Omega$, $0 < \rho < \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$, *there holds*

$$\rho \, \|\nabla u\|_{L^\infty(B_\rho(x))} \leqslant C \left(\frac{1}{\rho^n} \int_{B_{2\rho}(x)} |u(z) - u(x)|^p \, d\,\mathrm{Vol}(z) \right)^{1/p} + C\rho^{2-n/r} \, |u(x)|. \tag{3.44}$$

In particular, for p as above, for any $\alpha \in \mathbb{R}$ *we have*

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)^{\alpha+p} \, |\nabla u(x)|^p \, d\,\mathrm{Vol}(x) \leqslant C \int_\Omega \operatorname{dist}(x, \partial\Omega)^\alpha \, |u(x)|^p \, d\,\mathrm{Vol}(x). \tag{3.45}$$

These results also hold for $1 \leqslant p < \infty$ *if the coefficients of* L *satisfy* (3.4).

Proof. Note that under the current smoothness assumptions for the coefficients of L, in each coordinate patch $\mathcal{O}$, Proposition 3.1 gives

$$u \in L^p_{\mathrm{loc}}(\mathcal{O}), \qquad Lu \in L^q_{\mathrm{loc}}(\mathcal{O}) \Rightarrow u \in C^{1+\varepsilon}_{\mathrm{loc}}(\mathcal{O}) \qquad \text{some} \quad \varepsilon > 0, \quad \text{if} \quad q > \frac{n}{2}. \tag{3.46}$$

A natural estimate accompanies (3.46) also. As before, we apply (3.46) to the equation

$$\partial_j A^{jk}_\rho \, \partial_k (u_\rho - a) + \rho B^j_\rho \, \partial_j (u_\rho - a) + \rho^2 V_\rho (u_\rho - a) = f_\rho, \tag{3.47}$$

where, this time, $f_\rho := a\rho^2 V_\rho$ and $a \in \mathbb{R}$. Note that $\{\rho^2 V_\rho; \rho \in (0, \rho_0]\}$ is bounded in L^r and that $\|f_\rho\|_{L^q(\mathcal{O})} \leqslant Ca\rho^{2-n/r}$. Choosing $a := u(x)$ yields (3.44) after a rescaling.

Finally, (3.45) is a simple consequence of (3.44), and the last part of the statement follows similarly with the help of (the last part in) Proposition 3.1. ∎

4. SOBOLEV AND BESOV SPACES ON LIPSCHITZ DOMAINS

The theory of Besov spaces and Sobolev spaces on Lipschitz domains in Euclidean space has been thoroughly treated in [JK2]. Other references for various aspects of the theory include [BL, Pe, BS, Tr, Gr]. We recall some of these results here, to fix notation, and we indicate extensions to the manifold setting.

We recall that, if Ω is a Lipschitz domain in $\mathbb{R}^n$, then, for $1 \leqslant p, q < \infty$ and $0 < s < 1$, one definition of the Besov space $B_s^{p,q}(\partial\Omega)$ is the collection of all measurable functions f on $\partial\Omega$ such that

$$\|f\|_{B_s^{p,q}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\left(\int_{\partial\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{(n-1+sq)p/q}}\,d\sigma(y)\right)^{q/p} d\sigma(x)\right)^{1/q} < \infty. \tag{4.1}$$

In the special situation when $p = q$, it is customary to simplify the notation a bit and write $B_s^p(\partial\Omega)$ in place of $B_s^{p,p}(\partial\Omega)$.

The case when $p = \infty$ for the "diagonal" scale corresponds to the non-homogeneous version of the space of Hölder continuous functions on $\partial\Omega$. More precisely, $B_s^\infty(\partial\Omega)$, $0 < s < 1$, is defined as the Banach space of measurable functions on f so that

$$\|f\|_{B_s^\infty(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sup_{\substack{x,\, y \in \partial\Omega \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty. \tag{4.2}$$

Also, recall that $B^p_{-s}(\partial\Omega) := (B_s^q(\partial\Omega))^*$ for each $0 < s < 1$, $1 < p \leqslant \infty$ and $q = (1 - \frac{1}{p})^{-1}$. Let Lip $(\partial\Omega)$ denote the collection of all Lipschitz functions on $\partial\Omega$.

Next, we consider the case when Ω is a Lipschitz domain in the compact Riemannian manifold M. It is then natural to say that f belongs to $B_s^{p,q}(\partial\Omega)$ for some $1 \leqslant p,\ q \leqslant \infty$ and $0 < s < 1$ if (and only if) for any smooth chart $(\mathcal{O}, \Phi)$ and any smooth cut-off function $\theta \in C^\infty_{\text{comp}}(\mathcal{O})$, $(f\theta) \circ \Phi \in B_s^{p,q}(\Phi(\mathcal{O} \cap \partial\Omega))$. The following elementary lemma allows for transferring many results about Besov spaces originally proved in the classical Euclidean setting to the case of boundaries of Lipschitz domains in Riemannian manifolds.

LEMMA 4.1. *Let* $(\mathcal{O}_i, \Phi_i)_{i\in I}$ *be a finite set of local bi-Lipschitz charts so that* $\partial\Omega \subseteq \bigcup_{i\in I} \mathcal{O}_i$ *and* $\Phi_i(\partial\Omega \cap \mathcal{O}_i) = \mathbb{R}^{n-1}$. *Also, consider* $(\chi_i)_{i\in I}$ *a partition of unity subordinate to* $(\mathcal{O}_i)_{i\in I}$. *Finally, for a measurable function* f *on* $\partial\Omega$ *denote by* F_i *the extension by zero outside its support of* $(f\chi_i) \circ \Phi_i$, *for each* $i \in I$.

Then, for $1 \leqslant p, q \leqslant \infty$ *and* $0 < s < 1$, *the following two statements are equivalent*:

(1) $f \in B_s^{p,q}(\partial\Omega)$;

(2) $F_i \in B_s^{p,q}(\mathbb{R}^{n-1})$ *for each* $i \in I$.

Moreover, if these conditions hold, then

$$\|f\|_{B_s^{p,q}(\partial\Omega)} \approx \sum_{i\in I} \|F_i\|_{B_s^{p,q}(\mathbb{R}^{n-1})}. \tag{4.3}$$

To illustrate the point made before the statement of this result we note that from well known interpolation results in $\mathbb{R}^n$ the following can be proved. First, recall that $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,p}$ denote, respectively, the brackets for the complex and real interpolation method (cf., e.g., [Ca, BL, BS]).

PROPOSITION 4.2. *For* $0 < \theta < 1$, $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ *and* $0 < s_0, s_1 < 1$, *there holds*

$$[B_{s_0}^{p_0,q_0}(\partial\Omega), B_{s_1}^{p_1,q_1}(\partial\Omega)]_\theta = B_s^{p,q}(\partial\Omega), \tag{4.4}$$

where $1/p := (1-\theta)/p_0 + \theta/p_1$, $1/q := (1-\theta)/q_0 + \theta/q_1$ *and* $s := (1-\theta)\, s_0 + \theta s_1$.

A similar result is valid for $-1 < s_0, s_1 < 0$, *and for the real method of interpolation.*

Another result of interest for us is an atomic characterization of the Besov space $B_s^1(\partial\Omega)$. First, we shall need a definition. For $0 < s < 1$, a $B_s^1(\partial\Omega)$-atom is a function $a \in \mathrm{Lip}(\partial\Omega)$ with support contained in $B_r(x_0) \cap \partial\Omega$ for some $x_0 \in \partial\Omega$, $r \in (0, \operatorname{diam} \Omega]$, and satisfying the normalization conditions

$$\|a\|_{L^\infty(\partial\Omega)} \leqslant r^{s-n+1}, \qquad \|\nabla_{\tan} a\|_{L^\infty(\partial\Omega)} \leqslant r^{s-n}. \tag{4.5}$$

Here $\nabla_{\tan}$ stands for the tangential gradient operator on $\partial\Omega$ and, throughout the paper, distances on M are considered with respect to some fixed, smooth background Riemannian metric g_0. The atomic theory of [FJ] lifted to $\partial\Omega$ gives the following.

PROPOSITION 4.3. *Let $0<s<1$ and $f\in B^1_s(\partial\Omega)$. Then there exist a sequence of $B^1_s(\partial\Omega)$-atoms $\{a_j\}_j$ and a sequence of scalars $\{\lambda_j\}_j\in\ell^1$ such that*

$$f=\sum_{j\geqslant 0}\lambda_j a_j, \tag{4.6}$$

with convergence in $B^1_s(\partial\Omega)$, and

$$\|f\|_{B^1_s(\partial\Omega)}\approx\inf\left\{\sum_{j\geqslant 0}|\lambda_j| : f=\sum_{j\geqslant 0}\lambda_j a_j,\ a_j\ B^1_s(\partial\Omega)\text{-atom},\ (\lambda_j)_j\in\ell^1\right\}. \tag{4.7}$$

In the sequel, we shall also need to work with the Besov spaces $B^1_{-s}(\partial\Omega)$, $s\in(0,1)$. Inspired by the corresponding atomic characterization from [FJ], set

$$B^1_{-s}(\partial\Omega):=\mathscr{C}+\left\{f=\sum_{j\geqslant 0}\lambda_j a_j : a_j\, B^1_{-s}(\partial\Omega)\text{-atom},\ (\lambda_j)_j\in\ell^1\right\}, \tag{4.8}$$

where the series converges in the sense of distributions, and $\mathscr{C}$ is the space of constant functions on $\partial\Omega$. In this context, a $B^1_{-s}(\partial\Omega)$-atom, $0<s<1$, is a function $a\in L^\infty(\partial\Omega)$ with support contained in $B_r(x_0)\cap\partial\Omega$ for some $x_0\in\partial\Omega$, $0<r<\operatorname{diam}\Omega$, and satisfying

$$\int_{\partial\Omega}a\,d\sigma=0, \qquad \|a\|_{L^\infty(\partial\Omega)}\leqslant r^{-s-n+1}. \tag{4.9}$$

Furthermore, for $f\in B^1_{-s}(\partial\Omega)$, $0<s<1$,

$$\|f\|_{B^1_{-s}(\partial\Omega)}:=\inf\left\{\|g\|_{L^\infty}+\sum_{j\geqslant 0}|\lambda_j| : f=g+\sum_{j\geqslant 0}\lambda_j a_j\right\}, \tag{4.10}$$

where $g\in\mathscr{C}$, a_j's and $(\lambda_j)_j$ are as in (4.8). Parenthetically, observe that $\nabla_{\tan}: B^1_s(\partial\Omega)\to B^1_{s-1}(\partial\Omega)$ is a bounded operator; this is trivially checked using the above atomic decompositions.

Next, we include a brief discussion of the Besov and Sobolev classes in the interior of a Lipschitz domain $\Omega\subset M$. First, for $1\leqslant p,\ q\leqslant\infty$, $s>0$, the Besov space $B^{p,q}_s(M)$ is defined by localizing and transporting via local charts its Euclidean counterpart, i.e., $B^{p,q}_s(\mathbb{R}^n)$ (for the latter see, e.g., [Pe], [BL, BS, Tr, JW]). Going further, $B^{p,q}_s(\Omega)$ consists of restrictions to Ω of functions from $B^{p,q}_s(M)$. This is equipped with the natural norm, i.e., defined by taking the infimum of the $\|\cdot\|_{B^{p,q}_s(M)}$-norms of all possible extensions to M. The spaces $B^{p,p}_s(\Omega)$ will be abbreviated as $B^p_s(\Omega)$.

Using Stein's extension operator and then invoking well known real interpolation results (cf., e.g., [BL]), it follows that for any Lipschitz domain $\Omega \subset M$,

$$(B_{s_0}^{p_0, q_0}(\Omega), B_{s_1}^{p_1, q_1}(\Omega))_{\theta, p} = B_s^{p, q}(\Omega) \tag{4.11}$$

if $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, $s = (1-\theta)\, s_0 + \theta s_1$, $0 < \theta < 1$, $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, $s_0 \neq s_1$, $s_0, s_1 > 0$.

A similar discussion applies to the Sobolev (or potential) spaces $L_s^p(\Omega)$, this time starting with the potential spaces $L_s^p(M)$ (lifted to M from $\mathbb{R}^n$ via an analogue of Lemma 4.1; for the latter context see, e.g., [St, BL]). For $s \in \mathbb{R}$ we define the space $L_{s,0}^p(\Omega)$ to consist of distributions in $L_s^p(M)$ supported in $\bar{\Omega}$ (with the norm inherited from $L_s^p(M)$). It is known that $C_{\text{comp}}^\infty(\Omega)$ is dense in $L_{s,0}^p(\Omega)$ for all values of s and p.

Recall (cf. [JW]) that the trace operator

$$\text{Tr}\colon L_s^p(\Omega) \to B_{s-1/p}^p(\partial\Omega) \tag{4.12}$$

is well defined, bounded and onto if $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. This also has a bounded right inverse whose operator norm is controlled exclusively in terms of p, s and the Lipschitz character of Ω. Similar results are valid for $\text{Tr}\colon B_s^{p,q}(\Omega) \to B_{s-1/p}^p(\partial\Omega)$. In this latter case we may allow $1 \leqslant p \leqslant \infty$; cf. [BL].

Next, if $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$, the space $L_{s,0}^p(\Omega)$ is the kernel of the trace operator Tr acting on $L_s^p(\Omega)$. This follows from the Euclidean result [JK2, Proposition 3.3]. In fact, for the same range of indices, $L_{s,0}^p(\Omega)$ is the closure of $C_{\text{comp}}^\infty(\Omega)$ in the $L_s^p(\Omega)$ norm.

For positive s, $L_{-s}^p(\Omega)$ is defined as the space of linear functionals on test functions in Ω equipped with the norm

$$\|f\|_{L_{-s}^p(\Omega)} := \sup\{|\langle f, g\rangle| : g \in C_{\text{comp}}^\infty(\Omega),\ \|\tilde{g}\|_{L_s^q(M)} \leqslant 1\}, \tag{4.13}$$

where tilde denotes the extension by zero outside Ω and $\frac{1}{p} + \frac{1}{q} = 1$. For all values of p and s, $C^\infty(\Omega)$ is dense in $L_s^p(\Omega)$. Also, for any $s \in \mathbb{R}$,

$$L_{-s,0}^q(\Omega) = (L_s^p(\Omega))^* \qquad \text{and} \qquad L_{-s}^p(\Omega) = (L_{s,0}^q(\Omega))^*. \tag{4.14}$$

For later reference, let us point out that

$$L_{s-1+1/p}^p(\Omega) = (L_{-s+1/q}^q(\Omega))^*, \qquad L_{-s+1/q}^q(\Omega) = (L_{s-1+1/p}^p(\Omega))^*, \tag{4.15}$$

for $0 < s < 1$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. These follow from (4.14) and the fact that $L_{\alpha,0}^p(\Omega) = L_\alpha^p(\Omega)$ for $0 \leqslant \alpha < \frac{1}{p}$, $1 < p < \infty$ (the latter can be easily deduced from [JK2, Proposition 3.5]).

We shall also need the fact that the exterior derivative operator

$$d\colon L^p_s(\Omega, \Lambda^\ell TM) \to L^p_{s-1}(\Omega, \Lambda^{\ell+1} TM) \tag{4.16}$$

is well defined and bounded for $0 \leqslant \ell \leqslant n$, $1 < p < \infty$ and $s \geqslant 0$. See, e.g., [FMM] for a discussion. A similar result holds for δ, the formal adjoint of d, provided the metric tensor on M is sufficiently smooth. Here, $L^p_s(\Omega, \Lambda^\ell TM)$ stands for the space of ℓ-forms on Ω with coefficients in $L^p_s(\Omega)$.

As for $L^p_s(\partial\Omega)$, $1 < p < \infty$, $-1 \leqslant s \leqslant 1$, define this to be $L^p(\partial\Omega)$ when $s = 0$ and $\{f \in L^p(\partial\Omega); \nabla_{\tan} f \in L^p(\partial\Omega)\}$ when $s = 1$. The case when $0 < s < 1$ can be handled by defining $L^p_s(\partial\Omega)$ by complex interpolation:

$$L^p_s(\partial\Omega) := [L^p(\partial\Omega), L^p_1(\partial\Omega)]_s. \tag{4.17}$$

Finally, when $-1 \leqslant s < 0$, set

$$L^p_s(\partial\Omega) := (L^q_{-s}(\partial\Omega))^*, \qquad \frac{1}{q} + \frac{1}{p} = 1.$$

Going further, it is well known that $L^p_s(\Omega)$, $L^p_{s,0}(\Omega)$, $L^p_{-s}(\Omega)$, $L^p_{-s,0}(\Omega)$, $L^p_s(\partial\Omega)$, and $L^p_{-s}(\partial\Omega)$ are complex interpolation scales for $1 < p < \infty$ and nonnegative s (in the case of the last two scales we also require that $s \leqslant 1$). Also, the Besov and Sobolev spaces on the domain are related via real interpolation. For instance, we have the formula

$$(L^p(\Omega), L^p_k(\Omega))_{s,q} = B^{p,q}_{sk}(\Omega) \tag{4.18}$$

when $0 < s < 1$, $1 < p < \infty$, and k is a nonnegative integer. It is also known that for $s > 0$,

$$\begin{aligned} 1 < p \leqslant 2 &\Rightarrow B^{p,p}_s(\Omega) \hookrightarrow L^p_s(\Omega) \hookrightarrow B^{p,2}_s(\Omega), \\ 2 \leqslant p < \infty &\Rightarrow B^{p,2}_s(\Omega) \hookrightarrow L^p_s(\Omega) \hookrightarrow B^{p,p}_s(\Omega). \end{aligned} \tag{4.19}$$

Also, the same inclusions hold with Ω replaced by $\partial\Omega$.

A more detailed discussion and further properties of these spaces, as well as proofs for some of the statements in this paragraph for Euclidean domains can be found in [BL, BS, JK2]. We consider next yet other characterizations of membership in the classes of Sobolev and Besov spaces, together with some of their consequences.

PROPOSITION 4.4. *Let Ω be a Lipschitz domain in M and fix $1 \leqslant p \leqslant \infty$, $k \in \{0, 1\}$, and a function u in Ω so that $\nabla^j u \in L^p_{\mathrm{loc}}(\Omega)$ for $0 \leqslant j \leqslant k+1$. Then, for $0 < s < 1$,*

$$\operatorname{dist}(\cdot, \partial\Omega)^{1-s} |\nabla^{k+1} u| + |\nabla^k u| + |u| \in L^p(\Omega) \Rightarrow u \in B^p_{k+s}(\Omega). \tag{4.20}$$

Suppose next that $1 < p < \infty$ and $0 \leqslant s \leqslant 1$; in the case $p > 2$ assume further that $s \neq 1/p$. Then

$$\operatorname{dist}(\cdot, \partial\Omega)^{1-s} |\nabla^{k+1} u| + |\nabla^k u| + |u| \in L^p(\Omega) \Rightarrow u \in L^p_{k+s}(\Omega). \tag{4.21}$$

Finally, if $0 < s < \frac{1}{p} < 1$, then

$$u \in B^p_{k+s}(\Omega) \Rightarrow \operatorname{dist}(\cdot, \partial\Omega)^{-s} |\nabla^k u| + |u| \in L^p(\Omega). \tag{4.22}$$

Moreover, naturally accompanying estimates are valid in each case.

Proof. The implications (4.20)–(4.21) are proved in [JK2]. This was done in the Euclidean setting but it can easily be adapted to manifolds. Also, the implication (4.22) follows by observing that $u \in B^p_{k+s}(\Omega)$ entails $\nabla^k u \in B^p_s(\Omega)$ and then invoking [Gr, Theorems 1.4.2.4 and 1.4.4.4]. ∎

We conclude this section with a discussion of membership in the classes of Sobolev and Besov spaces for null-solutions of elliptic PDEs. First, we need a few lemmas.

LEMMA 4.5. *Let L be as in (3.1) and assume that its coefficients satisfy (3.2). Assume $u \in B^p_s(\Omega)$, for some $p \in (r/(r-1), r)$, $s \in (0, 1)$, and $Lu = 0$ in Ω. Then*

$$\operatorname{dist}(\cdot, \partial\Omega)^{1-s} |\nabla u| \in L^p(\Omega). \tag{4.23}$$

The same conclusion holds if $p = \infty$. Also, if the coefficients of L satisfy (3.4) then (4.23) is actually true for $1 \leqslant p \leqslant \infty$.

Proof. Note that it suffices to estimate $\operatorname{dist}(x, \partial\Omega)^{1-s} |\nabla u(x)|$ for x near $\partial\Omega$. If $0 < \rho < \operatorname{dist}(x, \partial\Omega)$ then, working in local coordinates, Proposition 3.4 gives

$$\operatorname{dist}(x, \partial\Omega) |\nabla u(x)| \leqslant C \left(\rho^{sp} \int_{B_\rho(x)} \frac{|u(z) - u(x)|^p}{|z - x|^{n+sp}} dz \right)^{1/p} + C \operatorname{dist}(x, \partial\Omega)^{2-n/r} |u(x)| \tag{4.24}$$

so that

$$(\operatorname{dist}(x, \partial\Omega)^{1-s}\, |\nabla u(x)|)^p \leqslant C \int_{B_\rho(x)} \frac{|u(z)-u(x)|^p}{|z-x|^{n+sp}}\, dz + C\, |u(x)|^p. \qquad (4.25)$$

Since $u \in B^p_s(\Omega)$, this readily yields a local version of (4.23), near $\partial\Omega$. Away from $\partial\Omega$, the desired conclusion follows from (3.45).

Next, if $p=\infty$, then (4.23) is a simple consequence of (3.44). The last part of the statement of the lemma can be proved in a similar fashion. ∎

LEMMA 4.6. *Let L be as in* (1.2), *with Lipschitz metric tensor, and assume that $u \in L^p_s(\Omega)$, with $1<p<\infty$ and $0 \leqslant s \leqslant 1$, satisfies $Lu=0$ in Ω. Then*

$$\operatorname{dist}(\cdot, \partial\Omega)^{1-s}\, |\nabla u| \in L^p(\Omega). \qquad (4.26)$$

Proof. First note that (4.26) is obvious when $s=1$ and that (the last part in) Proposition 3.4 furnishes (4.26) when $s=0$. In particular, the assignment $u \mapsto \nabla u$ is continuous from $\mathscr{H} \cap L^p(\Omega)$ and $\mathscr{H} \cap L^p_1(\Omega)$ into the weighted Lebesgue space $L^p(\Omega, \operatorname{dist}(\cdot, \partial\Omega)^p\, d\operatorname{Vol})$ and $L^p(\Omega)$, respectively, where we set

$$\mathscr{H} = \{u \in L^p_{\mathrm{loc}}(\Omega) : Lu=0\}, \qquad (4.27)$$

a space that is independent of $p \in (1, \infty)$, by (the last part in) Proposition 3.1. Interpolating with change of measure (cf. [SW]) gives that the map

$$\nabla\colon [\mathscr{H} \cap L^p(\Omega), \mathscr{H} \cap L^p_1(\Omega)]_s \to L^p(\Omega, \operatorname{dist}\,(\cdot, \partial\Omega)^{(1-s)p}\, d\operatorname{Vol}) \qquad (4.28)$$

is continuous for $0 \leqslant s \leqslant 1$. At this stage, the desired result is a consequence of the fact that

$$[\mathscr{H} \cap L^p(\Omega), \mathscr{H} \cap L^p_1(\Omega)]_s = \mathscr{H} \cap L^p_s(\Omega), \qquad 0 \leqslant s \leqslant 1. \qquad (4.29)$$

This concludes the proof of the lemma, modulo that of (4.29) which is treated separately below. ∎

The result (4.29) is proven by an argument similar to one used in the proof of Theorem 4.2 in [JK2], for $L=\Delta_0$, the flat-space Laplacian. We present the argument here, in order to make clear what properties we need on the metric tensor.

LEMMA 4.7. *Whenever the metric tensor is Lipschitz, the result* (4.29) *is true for all $p \in (1, \infty)$.*

Proof. To begin with, note that the claim about (1.3) extends, via (the last part in) Proposition 3.1, to include the fact that

$$L: L^p_2(M) \to L^p(M) \qquad \text{is an isomorphism} \quad \forall p \in (1, \infty). \tag{4.30}$$

Next, for $j = 0, 1$, let $\mathscr{E}: L^p_j(\Omega) \to L^p_j(M)$ denote Stein's extension operator, and let $\mathscr{R}: L^p_j(M) \to L^p_j(\Omega)$ denote restriction. Then set

$$\mathscr{D} = L\mathscr{E}: L^p_j(\Omega) \to L^p_{j-2}(M), \qquad \mathscr{G} = \mathscr{R}L^{-1}: L^p_{j-2}(M) \to L^p_j(\Omega). \tag{4.31}$$

To justify the mapping property stated for $\mathscr{G}$, note that (4.30) implies $L^{-1}: L^p(M) \to L^p_2(M)$, so by duality $L^{-1}: L^p_{-2}(M) \to L^p(M)$, $1 < p < \infty$.

One sees that $\mathscr{G}\mathscr{D} = I$ on $L^p_j(\Omega)$, while $Q := \mathscr{D}\mathscr{G}$ satisfies $Q^2 = Q$ and $f \in L^p_{j-2}(M) \Rightarrow Qf = f$ on Ω, i.e.,

$$I - Q: L^p_{j-2}(M) \to L^p_{j-2,0}(M \backslash \bar{\Omega}), \qquad j = 1, 2. \tag{4.32}$$

Also, given $u \in L^p_j(\Omega)$ we have $u \in \mathscr{H}$ if and only if $\mathscr{D}u \in L^p_{j-2,0}(M \backslash \bar{\Omega})$ for $j = 1, 2$. With this set-up, [LM, Theorem 14.3] applies to give (4.29). ∎

We now present a result which for the flat-space Laplacian was established in [JK2, Sect. 4].

PROPOSITION 4.8. *Let L be as in* (1.2), *with Lipschitz metric tensor, and consider a function u so that $Lu = 0$ in Ω. Then, if $1 < p < \infty$ and $0 < s < 1$,*

$$u \in L^p_s(\Omega) \Rightarrow u \in B^p_s(\Omega). \tag{4.33}$$

Conversely, retain the same hypotheses and, in the case $p > 2$, assume further that $s \neq 1/p$. Then

$$u \in B^p_s(\Omega) \Rightarrow u \in L^p_s(\Omega). \tag{4.34}$$

Proof. This is a direct consequence of Lemmas 4.5–4.6 and Proposition 4.4. ∎

In order to state our last result in this section, recall that the nontangential maximal operator acts on a section u defined in the Lipschitz domain Ω, by

$$u^*(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega. \tag{4.35}$$

Here, $\gamma(x) \subset \Omega$ is a suitable nontangential approach region with "vertex" at $x \in \partial\Omega$.

Our aim is to relate L^2-estimates on the nontangential maximal function to membership in the Sobolev space $L^2_{1/2}$ for null-solutions of L. We have

PROPOSITION 4.9. *Consider L as in* (3.1), *a second order strongly elliptic, formally self-adjoint operator so that*

$$a_{jk}^{\alpha\beta} \in C^{1+\gamma}, \qquad \gamma > 0, \qquad b_j^{\alpha\beta} \in L_1^r, \qquad v^{\alpha\beta} \in L^r, \qquad r > n. \tag{4.36}$$

Then, for each u so that $Lu = 0$ in Ω, there holds

$$u \in L_{1/2}^2(\Omega) \Leftrightarrow u^* \in L^2(\partial\Omega). \tag{4.37}$$

Also, naturally accompanying estimates are valid.

Proof. We start with the right-to-left implication in (4.37). As a consequence of the well-posedness of the Dirichlet problem for $L - \lambda$ (where the real parameter λ is assumed to be large) and the mapping properties of the layer potential operators involved in the integral representation of the solution, the estimate

$$\int_{\Omega} \operatorname{dist}(x, \partial\Omega)\, |\nabla u(x)|^2 \, d\,\mathrm{Vol}(x) \leqslant C \int_{\partial\Omega} |u^*|^2 \, d\sigma, \tag{4.38}$$

uniformly in u satisfying $Lu = 0$ in Ω, has been established in [MMT]. See, e.g., (3.55) and Corollary 3.2 in that paper. In concert with (4.19), this readily yields the right-to-left implication in (4.37).

As for the opposite implication, since the problem has local character, there is no loss of generality in assuming that Ω is small, e.g., contained in some coordinate patch. Moreover, so we claim, it suffices to consider the case when u, satisfying $Lu = 0$ in Ω, also belongs to $C^0(\bar{\Omega}) \cap L_1^2(\Omega)$ and, for each $\varepsilon > 0$, prove the estimate

$$\int_{\partial\Omega} |u|^2 \, d\sigma \leqslant C_\varepsilon \int_{\Omega} (\operatorname{dist}(x, \partial\Omega)\, |\nabla u(x)|^2 + |u(x)|^2) \, d\,\mathrm{Vol}(x) + \varepsilon \int_{\partial\Omega} |u^*|^2 \, d\sigma. \tag{4.39}$$

Hereafter we shall work in local coordinates and the constant C_ε is supposed to depend exclusively on the Lipschitz nature of Ω, ε and L.

To justify the claim, let us show how (4.39) can be used to finish the proof of the left-to-right implication in (4.37). Indeed, granted (4.36), Lemma 4.5 and a simple limiting argument which utilizes an approximating sequence $\Omega_j \nearrow \Omega$ with bounded Lipschitz character lead to

$$\sup_j \|u\|_{L^2(\partial\Omega_j)} \leqslant C_\varepsilon \, \|u\|_{L_{1/2}^2(\Omega)} + \varepsilon \, \|u^*\|_{L^2(\partial\Omega)}. \tag{4.40}$$

In order to continue, we invoke an estimate from [MMT], to the effect that

$$\|u\|_{L^2(\partial\Omega)} \approx \|u^*\|_{L^2(\partial\Omega)}, \tag{4.41}$$

uniformly for null-solutions $u \in C^0(\bar{\Omega})$ of L, with constants depending only on the Lipschitz nature of Ω and L. This is a consequence of [MMT, Theorem 3.1] (cf. especially the estimate (3.5) there) and has been established under the requirement that the operator L satisfies the non-singularity hypothesis:

$$\forall D(\subseteq M) \quad \text{Lipschitz domain,} \quad u \in L^2_{1,0}(D), \quad Lu = 0 \Rightarrow u = 0 \quad \text{in } D. \tag{4.42}$$

Let us assume for now that this is the case and explain how the proof of the proposition can be finished.

With Ω_j in place of Ω, (4.41) readily gives that, in our current setting, $\|u^*\|_{L^2(\partial\Omega)} \approx \sup_j \|u\|_{L^2(\partial\Omega_j)}$. Utilizing this back in (4.40) and choosing ε small enough, establishes the estimate $\|u^*\|_{L^2(\partial\Omega)} \leqslant C(\partial\Omega) \|u\|_{L^2_{1/2}(\Omega)}$, as desired.

Finally, (4.39) can be established as [DKPV, Sect. 2]. The approach devised in the aforementioned paper is sufficiently flexible, as it rests on the following ingredients: (i) the assumptions of symmetry and strong ellipticity for L, (ii) interior estimates for L, (iii) the existence of an adapted distance function (see [DKPV, p. 1432–1433] for details), and (iv) integrations by parts, Schwarz's inequality and the standard Carleson estimate. All these are available in our case (cf. Section 3 for (ii)) and the argument in [DKPV] can be adapted to our situation to yield (4.39). We leave the details to the interested reader.

At this point we return to (4.42) and indicate how one can dispose of this extra assumption. The first step is to observe that if L is negative-definite on M then (4.42) is automatically verified. Indeed, if $D \subset M$ is a Lipschitz domain and $u \in L^2_{1,0}(D)$ is such that $Lu = 0$ in D then, with tilde denoting extension by zero outside D, we see that $\tilde{u} \in L^2_1(M)$ satisfies $\operatorname{supp} L\tilde{u} \subseteq \partial D$. In particular, $\langle L\tilde{u}, \tilde{u} \rangle = 0$ due to support considerations. Since we are assuming $L < 0$ on M, this forces $\tilde{u} = 0$, as desired.

In the second step we shall show how L can be altered off Ω so that it becomes negative-definite on M. In the process, we can (and will) assume that Ω is very small (relative to L, in a sense made more precise below). To this end, fix some constant $A \in (0, \infty)$ so that, in the current setting, $L - A$ is negative-definite on $L^2(M)$. Also, pick $B > A$ and set $B_j = B_j(x) = B(1 - \chi_{\Omega_j})$ where, for some fixed $p \in \Omega$, $\Omega_j = \{x \in M : \operatorname{dist}(x, p) < 1/j\}$.

Here is the claim which finishes the proof of the proposition.

$$L - B_j \qquad \text{is negative-definite on } M \text{ for large } j. \tag{4.43}$$

Since this is a consequence of $0 < (-L + B_j + A)^{-1} < A^{-1}$ (all operators considered on $L^2(M)$) and standard functional analysis, it suffices to prove that

$$(L - B_j - A)^{-1} \to (L - B - A)^{-1}, \qquad \text{in } L^2\text{-operator norm, as } j \to \infty. \tag{4.44}$$

To see this, given $f \in L^2(M)$ set $u_j = (L - B_j - A)^{-1} f$ and $u = (L - B - A)^{-1} f$. We have $\|u_j\|_{L^2} \leqslant \|f\|_{L^2}$. Also, the fact that

$$(L - A)\, u_j = B_j u_j + f \tag{4.45}$$

is bounded in $L^2(M)$ entails $\|u_j\|_{L^2_1} \leqslant C_0 \, \|f\|_{L^2}$. Now $(L - A)(u - u_j) = Bu - B_j u_j = B(u - u_j) + (B - B_j)\, u_j$ which implies $(L - A - B)(u - u_j) = (B - B_j)\, u_j$ and, further,

$$\|u - u_j\|_{L^2} \leqslant B^{-1} \, \|(B - B_j)\, u_j\|_{L^2}. \tag{4.46}$$

Note that, generally speaking,

$$\|(B - B_j)\, v\|_{L^2} \leqslant \delta_j B \, \|v\|_{L^2_1}, \tag{4.47}$$

with $\delta_j \to 0$ as $j \to \infty$. Hence

$$\|u - u_j\|_{L^2} \leqslant B^{-1} \, \|(B - B_j)\, u_j\|_{L^2} \leqslant C_0 \delta_j \, \|f\|_{L^2}, \tag{4.48}$$

and (4.44) is proven. ∎

5. THE BANACH ENVELOPE OF ATOMIC HARDY SPACES

Fix an arbitrary Lipschitz domain Ω in M and for each $p \in ((n-1)/n, 1)$ set $s := (n-1)(\frac{1}{p} - 1)$. Note that $0 \leqslant s < 1$. A function $a \in L^\infty(\partial\Omega)$ is called an $\mathfrak{H}^p(\partial\Omega)$-atom if $\int_{\partial\Omega} a \, d\sigma = 0$ and, for some boundary point $x_0 \in \partial\Omega$ and some $0 < r < \operatorname{diam} \Omega$,

$$\operatorname{supp} a \subseteq \partial\Omega \cap B_r(x_0), \qquad \|a\|_{L^\infty(\partial\Omega)} \leqslant r^{-(n-1)/p}. \tag{5.1}$$

Recall that $\mathscr{C}$ is the space of constant functions on $\partial\Omega$. For $(n-1)/n < p < 1$ (and $p = 1$, respectively) we recall that the atomic Hardy space $\mathfrak{H}^p(\partial\Omega)$ is defined as the vector subspace of $(B^\infty_s(\partial\Omega)/\mathscr{C})^*$ (and

$L^1(\partial\Omega)$, respectively) consisting of all linear functionals f that can be represented as

$$f = \sum_{j \geqslant 0} \lambda_j a_j, \qquad (\lambda_j)_j \in \ell^p, \qquad a_j\text{'s are } \mathfrak{H}^p(\partial\Omega)\text{-atoms,} \tag{5.2}$$

in the sense of convergence in $(B_s^\infty(\partial\Omega))^*$ (and $L^1(\partial\Omega)$, respectively). We equip $\mathfrak{H}^p(\partial\Omega)$ with the (quasi-) norm

$$\|f\|_{\mathfrak{H}^p(\partial\Omega)} := \inf\left\{\left(\sum_{j\geqslant 0} |\lambda_j|^p\right)^{1/p} : f = \sum_{j\geqslant 0} \lambda_j a_j \text{ as in (5.2)}\right\}. \tag{5.3}$$

We shall also need the localized version of $\mathfrak{H}^p(\partial\Omega)$, i.e., $\mathfrak{h}^p(\partial\Omega) \subseteq (B_s^\infty(\partial\Omega))^*$ defined by

$$\mathfrak{h}^p(\partial\Omega) := \mathfrak{H}^p(\partial\Omega) + \mathscr{C} = \mathfrak{H}^p(\partial\Omega) + L^q(\partial\Omega), \qquad \forall q > 1. \tag{5.4}$$

As is well known,

$$(\mathfrak{h}^p(\partial\Omega))^* = \begin{cases} B_s^\infty(\partial\Omega), & \text{if } \ p < 1, \\ \mathrm{bmo}(\partial\Omega), & \text{if } \ p = 1. \end{cases} \tag{5.5}$$

See, e.g., [CW] for a more detailed account. It is also well known that

$$\{\mathfrak{h}^p(\partial\Omega)\}_{(n-1)/n < p \leqslant 1} \cup \{L^p(\partial\Omega)\}_{1 < p < \infty} \qquad \text{is an interpolation scale for the complex method.} \tag{5.6}$$

See the discussion in [KM].

Next we discuss the "minimal enlargement" of $\mathfrak{h}^p(\partial\Omega)$ to a Banach space, its so-called Banach envelope. To define this properly, we digress momentarily for the purpose of explaining a somewhat more general functional analytic setting. A good reference is [KPR].

Let $\mathscr{V}$ be a locally bounded topological vector space, whose dual separates points, and fix U a bounded neighborhood of the origin. Then, with co A standing for the convex hull of a set $A \subseteq \mathscr{V}$, the functional given by

$$\|x\|_{\hat{\mathscr{V}}} := \inf\{\gamma > 0;\, \gamma^{-1} x \in \mathrm{co}(U)\} \tag{5.7}$$

defines a (continuous) norm on $\mathscr{V}$ (any other such norm corresponding to a different choice of U is in fact equivalent to (5.7)). Then, $\hat{\mathscr{V}}$, the Banach envelope of $\mathscr{V}$, is the completion of $\mathscr{V}$ in the norm (5.7). Thus, $\hat{\mathscr{V}}$ is a well defined Banach space, uniquely defined up to an isomorphism. Also, the inclusion $\mathscr{V} \hookrightarrow \hat{\mathscr{V}}$ is continuous and has a dense image. Furthermore, $\mathscr{V}$

and $\hat{\mathscr{V}}$ have the same dual; see [KPR]. Another useful observation is contained in the next lemma (whose proof is an exercise).

LEMMA 5.1. *Let $\mathscr{V}_1$, $\mathscr{V}_2$ be two topological vector spaces as above and consider a bounded linear operator $T: \mathscr{V}_1 \to \mathscr{V}_2$. Then T extends to a bounded linear operator $\hat{T}: \hat{\mathscr{V}}_1 \to \hat{\mathscr{V}}_2$ with*

$$\sup_x \|\hat{T}(x)\|_{\hat{\mathscr{V}}_2} \leqslant \|x\|_{\hat{\mathscr{V}}_1} \inf\{\varepsilon > 0 : T(U_1) \subset \varepsilon U_2\}, \tag{5.8}$$

where $U_j \subset \mathscr{V}_j$ is the bounded neighborhood with respect to which the norm $\|\cdot\|_{\hat{\mathscr{V}}_j}$ has been defined, $j = 1, 2$.

If, moreover, T is an isomorphism then so is $\hat{T}$.

Our main result in this section relates $\hat{\mathfrak{h}}^p(\partial\Omega)$ to the atomic Besov spaces $B^1_{-s}(\partial\Omega)$, defined by (4.8)–(4.10). It is an improvement of [FMM, Theorem 5.4].

PROPOSITION 5.2. *Let Ω be a bounded Lipschitz domain in M and fix some index $\frac{n-1}{n} < p < 1$. Then $\hat{\mathfrak{h}}^p(\partial\Omega)$, the Banach envelope of $\mathfrak{h}^p(\partial\Omega)$, coincides with $B^1_{-s}(\partial\Omega)$, where $s = (n-1)(\frac{1}{p} - 1)$.*

Proof. The crucial observation is that $\hat{\mathfrak{h}}^p(\partial\Omega)$, as a subspace of $(B^\infty_s(\partial\Omega))^*$, is given by

$$\mathscr{C} + \left\{ f = \sum_{j \geqslant 1} \lambda_j a_j \text{ in } (B^\infty_s(\partial\Omega))^* : (\lambda_j)_j \in \ell^1, \, a_j\text{'s are } \mathfrak{H}^p(\partial\Omega)\text{-atoms} \right\} \tag{5.9}$$

endowed with the norm

$$f \mapsto \inf \left\{ \|g\|_{L^\infty(\partial\Omega)} + \sum |\lambda_j| : f = g + \sum \lambda_j a_j, \, g \in \mathscr{C}, \, \lambda_j\text{'s}, \, a_j\text{'s as in (5.9)} \right\}. \tag{5.10}$$

For Euclidean domains this has been proved in [FMM] and the proof given there readily adapts to our current setting. Now the desired conclusion follows by noting that each $\mathfrak{H}^p(\partial\Omega)$-atom is in fact a $B^1_{-s}(\partial\Omega)$-atom, granted that $s = (n-1)(\frac{1}{p} - 1)$. ∎

We mention that Proposition 5.2 plus previously noted duality results yield $(B^1_{-s}(\partial\Omega))^* = B^\infty_s(\partial\Omega)$, a special case of duality results that are contained in [Tr, Theorem 2.11.2].

6. NEWTONIAN POTENTIALS ON SOBOLEV AND BESOV SPACES

Let the manifold M, the Lipschitz domain Ω and the potential V be as in Section 1. The goal is to study the mapping properties on scales of Sobolev–Besov spaces for the associated Newtonian potential operator

$$\Pi_\ell f(x) := \int_\Omega \langle E_\ell(x, y), f(y) \rangle \, d\,\mathrm{Vol}(y), \qquad x \in \Omega. \tag{6.1}$$

In (6.1), f is a differential form of degree $\ell \in \{0, 1, ..., n\}$ in Ω, and $E_\ell(x, y)$ is the Schwartz kernel of

$$(\Delta_\ell - V)^{-1} : L^2_{-1}(M, \Lambda^\ell TM) \to L^2_1(M, \Lambda^\ell TM), \tag{6.2}$$

as in Section 2. In other words, $\Pi_\ell = R(\Delta_\ell - V)^{-1} E$, where E denotes extension by 0 off Ω and R denotes restriction to Ω.

Proposition 6.1. *Assume the metric tensor satisfies* (1.28). *For each* $\ell \in \{1, ..., n\}$,

$$\begin{gathered}\Pi_\ell : (L^q_{s+1}(\Omega, \Lambda^\ell TM))^* \to L^p_{1-s}(\Omega, \Lambda^\ell TM), \qquad \forall s \in [-1, 1], \\ r/(r-1) < p, \quad q < r, \quad 1/p + 1/q = 1\end{gathered} \tag{6.3}$$

and

$$\begin{gathered}\Pi_\ell : (B^q_{s+1}(\Omega, \Lambda^\ell TM))^* \to B^p_{1-s}(\Omega, \Lambda^\ell TM), \qquad \forall s \in (-1, 1), \\ r/(r-1) < p, \quad q < r, \quad 1/p + 1/q = 1,\end{gathered} \tag{6.4}$$

are well defined, bounded operators.

For $\ell = 0$ *and under the* (*weaker*) *assumption that the metric tensor is Lipschitz*, (6.3)–(6.4) *are valid for the full range* $1 < p, q < \infty$.

Proof. Given our assumption on the metric, if $s = -1$ then (6.3) holds for $1 < p < r$; see [MMT, Theorem 2.9]. The full range $-1 \leqslant s \leqslant 1$ is then easily seen from this, duality and interpolation (note that Π_ℓ is formally self-adjoint).

The assertion on Besov spaces is a corollary of the preceding result and repeated applications of the method of real interpolation (together with the corresponding duality and reiteration theorems).

Finally, the last part in the proposition follows much as before; this time, however, we invoke (4.30). ∎

7. SINGLE LAYER POTENTIALS ON SOBOLEV AND BESOV SPACES

In this section we continue to retain the same hypotheses made in Section 1 for M, Ω and V. Recall the fundamental solution $E_\ell(x, y)$ for the operator $\Delta_\ell - V$ and the single layer potential operator

$$\mathscr{S}_\ell\colon (\mathrm{Lip}(\partial\Omega, \Lambda^\ell TM))^* \to C^{1+\gamma}_{\mathrm{loc}}(\Omega, \Lambda^\ell TM), \qquad \text{some} \quad \gamma > 0, \tag{7.1}$$

defined by

$$\mathscr{S}_\ell f(x) := \langle E_\ell(x, \cdot)|_{\partial\Omega}, f\rangle, \qquad x \in \Omega, \tag{7.2}$$

for each $f \in (\mathrm{Lip}(\partial\Omega, \Lambda^\ell TM))^*$. Hereafter, $\langle \cdot, \cdot \rangle$ stands for the natural duality pairing between a topological vector space and its dual (in the case at hand, between $\mathrm{Lip}(\partial\Omega, \Lambda^\ell TM)$ and $(\mathrm{Lip}(\partial\Omega, \Lambda^\ell TM))^*$).

Our main result in this section summarizes the mapping properties of the operator (7.1) on scales of Sobolev–Besov spaces.

THEOREM 7.1. *Assume the metric tensor satisfies* (1.29) *Then, for* $1 \leqslant p \leqslant \infty$, $0 < s < 1$ *and* $1 \leqslant \ell \leqslant n$, *the operator*

$$\mathscr{S}_\ell\colon B^p_{-s}(\partial\Omega, \Lambda^\ell TM) \to B^p_{1+1/p-s}(\Omega, \Lambda^\ell TM) \tag{7.3}$$

is well-defined and bounded. In fact, if $1 < p < \infty$, *then*

$$\mathscr{S}_\ell\colon B^p_{-s}(\partial\Omega, \Lambda^\ell TM) \to L^p_{1+1/p-s}(\Omega, \Lambda^\ell TM) \tag{7.4}$$

is also a bounded linear mapping, so that $\forall s \in (0, 1)$, $\forall p \in (1, \infty)$,

$$\begin{aligned} \|\mathscr{S}_\ell f\|_{B^p_{1+1p-s}(\Omega, \Lambda^\ell TM)} &+ \|\mathscr{S}_\ell f\|_{L^p_{1+1p-s}(\Omega, \Lambda^\ell TM)} \\ &\leqslant C(\Omega, s, p)\, \|f\|_{B^p_{-s}(\partial\Omega, \Lambda^\ell TM)}, \end{aligned} \tag{7.5}$$

uniformly for $f \in B^p_{-s}(\partial\Omega, \Lambda^\ell TM)$.

In particular, if Tr *stands for the trace map on* $\partial\Omega$, *the operator* $S_\ell := \mathrm{Tr}\, \mathscr{S}_\ell$ *has the property*

$$S_\ell\colon B^p_{-s}(\partial\Omega, \Lambda^\ell TM) \to B^p_{1-s}(\partial\Omega, \Lambda^\ell TM),$$

for $1 \leqslant p \leqslant \infty$, $0 < s < 1$, *and* $1 \leqslant \ell \leqslant n$.

Finally, if $\ell = 0$, *the same results are valid for* $\mathscr{S}_\ell$ *under the assumption that the metric tensor satisfies* (1.27).

Handling $\mathscr{S}_\ell$ on Besov spaces utilizes size estimates for the integral kernel (and its derivatives). A general statement to this effect is formalized and proved in the Lemmas 7.2–7.3 below. Since these lemmas have some interest of their own, we state (and prove) them in a slightly more general version than what is needed here.

LEMMA 7.2. *Let* Ω *be a Lipschitz domain in* M *and consider a positive integer* N. *For* $k(x, y)$ *defined on* $M \times M\backslash$diagonal *and satisfying*

$$|\nabla_x^i \nabla_y^j k(x, y)| \leqslant \kappa \operatorname{dist}(x, y)^{-(n-2+i+j)}, \qquad \forall i = 0, 1, ..., N, \quad \forall j = 0, 1, \tag{7.6}$$

for some $\kappa > 0$ *independent of* x, y, *introduce*

$$\mathscr{K} f(x) := \langle k(x, \cdot)|_{\partial\Omega}, f\rangle, \qquad x \in \Omega. \tag{7.7}$$

Then, for $0 < s < 1$, *this operator satisfies the estimates*

$$\begin{aligned} &\|\operatorname{dist}(\cdot, \partial\Omega)^{s-1+i}\, |\nabla^{1+i} \mathscr{K} f|\|_{L^1(\Omega)} + \|\nabla^i \mathscr{K} f\|_{L^1(\Omega)} + \|\mathscr{K} f\|_{L^1(\Omega)} \\ &\quad \leqslant C(\Omega, \kappa, s)\, \|f\|_{(B_s^\infty(\partial\Omega))^*}, \qquad \forall i = 0, 1, ..., N-1, \end{aligned} \tag{7.8}$$

uniformly in $f \in (B_s^\infty(\partial\Omega))^*$.

Proof. Let us estimate the leading term in the left side of (7.8); the argument for the remaining terms is simpler. The point is to establish the estimate

$$\begin{aligned} &\left\| \int_\Omega \operatorname{dist}(x, \partial\Omega)^{s-1+i}\, \nabla_x^{1+i} k(x, \cdot)\, g(x)\, d\operatorname{Vol}(x) \right\|_{B_s^\infty(\partial\Omega)} \\ &\quad \leqslant C(\Omega, \kappa, s)\, \|g\|_{L^\infty(\Omega)}, \end{aligned} \tag{7.9}$$

uniformly for $g \in L^\infty_{\text{comp}}(\Omega)$. To this end, for a fixed, arbitrary $g \in L^\infty_{\text{comp}}(\Omega)$, we shall focus on establishing

$$\begin{aligned} &\left| \int_\Omega g(x) \operatorname{dist}(x, \partial\Omega)^{s-1+i} (\nabla_x^{1+i} k(x, p) - \nabla_x^{1+i} k(x, q))\, d\operatorname{Vol}(x) \right| \\ &\quad \leqslant C \operatorname{dist}(p, q)^s, \end{aligned} \tag{7.10}$$

uniformly for $p, q \in \partial\Omega$. Now, fix two arbitrary boundary points $p, q \in \partial\Omega$ and, for a large constant C, bound the integral above by

$$\int_{\mathrm{dist}(x,\,p)<C\,\mathrm{dist}(p,\,q)} |g(x)|\,\mathrm{dist}(x,\partial\Omega)^{s-1+i}\,|\nabla_x^{1+i}k(x,p)|\,d\,\mathrm{Vol}(x)$$

$$+\int_{\mathrm{dist}(x,\,p)<C\,\mathrm{dist}(p,\,q)} |g(x)|\,\mathrm{dist}(x,\partial\Omega)^{s-1+i}\,|\nabla_x^{1+i}k(x,q)|\,d\,\mathrm{Vol}(x)$$

$$+\int_{\mathrm{dist}(x,\,p)>C\,\mathrm{dist}(p,\,q)} |g(x)|\,\mathrm{dist}(x,\partial\Omega)^{s-1+i}$$

$$\times\,|\nabla_x^{1+i}k(x,p)-\nabla_x^{1+i}\,k(x,q)|\,d\,\mathrm{Vol}(x)$$

$$=: I+II+III. \tag{7.11}$$

To deal with I, in the light of (7.6), by localizing and pulling back to $\mathbb{R}^n_+$, it suffices to consider

$$\int_{|x'-p'|+|t+\varphi(x')-\varphi(p')|<C|p'-q'|} \frac{t^{s-1+i}}{(|x'-p'|+|t+\varphi(x')-\varphi(p')|)^{n-1+i}}\,dt\,dx', \tag{7.12}$$

where $\varphi\colon\mathbb{R}^{n-1}\to\mathbb{R}$ is a Lipschitz function and $x=(x',t+\varphi(x'))$, $p=(p',\varphi(p'))$, $q=(q',\varphi(q'))$. Accordingly, we seek a bound of order $|p'-q'|^s$. To this effect, we note that the integral (7.12) is majorized by

$$C\int_{|x'-p'|+t<C\,|p'-q'|} \frac{t^{s-1+i}}{(|x'-p'|+t)^{n-1+i}}\,dt\,dx'$$

$$\leqslant C\left(\int_0^\infty \frac{t^{s-1+i}}{(1+t)^{n-1+i}}\,dt\right)\left(\int_{|x'|<C|p'-q'|} \frac{1}{|x'|^{n-1-s}}\,dx'\right)$$

$$\leqslant C_{n,\,s}\,|p'-q'|^s, \tag{7.13}$$

and the last bound has the right order. Similar arguments also apply to II in (7.11) since $\mathrm{dist}(x,q)<\mathrm{dist}(x,p)+\mathrm{dist}(p,q)<C\,\mathrm{dist}(p,q)$. Thus, we are left with estimating III. For this, an application of the mean-value theorem together with (7.6) and a change of variables allow us to write

$$III\leqslant C\int_{|x'-p'|+|t+\varphi(x')-\varphi(p')|>C\,|p'-q'|}$$

$$\times\frac{t^{s-1+i}\,|p'-q'|}{(|x'-p'|+|t+\varphi(x')-\varphi(p')|)^{n+i}}\,dx'\,dt$$

$$\leqslant C\int_{|x'-p'|+t>C\,|p'-q'|} \frac{t^{s-1+i}\,|p'-q'|}{(|x'-p'|+t)^{n+i}}\,dx'\,dt. \tag{7.14}$$

Making $x' \mapsto x'' := (x' - p')/|p' - q'|$ and $t \mapsto t' := t/|p' - q'|$ in the last integral above readily leads to a bound of order $|p' - q'|^s$, i.e., the proper size. This finishes the proof of (7.8). ∎

LEMMA 7.3. *Retain the same hypotheses as in Lemma 7.2 and recall the operator* $\mathcal{K}$ *introduced in* (7.7). *Then, for* $0 < s < 1$, *this operator satisfies the estimates*

$$\|\mathrm{dist}(\cdot, \partial\Omega)^{s+i} |\nabla^{1+i}\mathcal{K}f|\|_{L^\infty(\Omega)} + \|\nabla^i \mathcal{K}f\|_{L^\infty(\Omega)} + \|\mathcal{K}f\|_{L^\infty(\Omega)}$$
$$\leqslant C(\Omega, \kappa, s)\, \|f\|_{B^\infty_{-s}(\partial\Omega)}, \qquad \forall i = 0, 1, ..., N-1, \tag{7.15}$$

uniformly in $f \in B^\infty_{-s}(\partial\Omega) := (B^1_s(\partial\Omega))^*$.

Proof. Consider the leading term in the left side of (7.15); all the others can be handled similarly. Here the idea is to prove that

$$\|\nabla_x^{1+i} k(x, \cdot)\|_{B^1_s(\partial\Omega)} \leqslant C(\Omega, s)\, \mathrm{dist}(x, \partial\Omega)^{-s-i}, \qquad \forall\, i = 0, 1, ..., N-1, \tag{7.16}$$

uniformly in $x \in \Omega$. Clearly, this suffices in order to conclude (7.15). The remainder of the proof, modeled upon [FMM], consists of a verification of (7.16).

The problem localizes and, hence, it suffices to prove the estimate

$$\int_{\partial\Omega} \int_{\partial\Omega} \frac{|\nabla_x^{1+i} k(x, p) - \nabla_x^{1+i} k(x, q)|}{|p - q|^{n-1+s}}\, d\sigma(p)\, d\sigma(q) \leqslant C\, \mathrm{dist}(x, \partial\Omega)^{-s-i}, \tag{7.17}$$

uniformly for $x \in \Omega$, in the case when Ω is the domain above the graph of a Lipschitz function $\varphi: \mathbb{R}^{n-1} \to \mathbb{R}$.

Now, for a fix, sufficiently large $C > 0$, split the inner integral according to whether $|x - p| < C\,|p - q|$ or $|x - p| > C\,|p - q|$. Thus, it suffices to treat $\int_{\partial\Omega} |I|\, d\sigma$, $\int_{\partial\Omega} |II|\, d\sigma$, and $\int_{\partial\Omega} |III|\, d\sigma$, where

$$\begin{aligned}
I &:= \int_{|x-p| < C\,|p-q|} \frac{|\nabla_x^{1+i} k(x, p)|}{|p - q|^{n-1+s}}\, d\sigma(p), \\
II &:= \int_{|x-p| < C\,|p-q|} \frac{|\nabla_x^{1+i} k(x, q)|}{|p - q|^{n-1+s}}\, d\sigma(p), \\
III &:= \int_{|x-p| > C\,|p-q|} \frac{|\nabla_x^{1+i} k(x, p) - \nabla_x^{1+i} k(x, q)|}{|p - q|^{n-1+s}}\, d\sigma(p).
\end{aligned} \tag{7.18}$$

To this end, note first that a change of variables based on the representations $x=(x', \varphi(x')+t)$, $p=(p', \varphi(p'))$, and $q=(q', \varphi(q'))$ gives

$$
\begin{aligned}
|I| &\leqslant C \int_{|x'-p'|+|t+\varphi(x')-\varphi(p')|<C\,|p'-q'|} \\
&\qquad \times \frac{|p'-q'|^{-n+1-s}}{(|t+\varphi(x')-\varphi(p')|+|x'-p'|)^{n-1+i}}\, dp' \\
&\leqslant C \int_{|x'-p'|+t<C\,|p'-q'|} \frac{dp'}{|p'-q'|^{n-1+s}\,(t+|x'-p'|)^{n-1+i}}.
\end{aligned} \tag{7.19}
$$

Substituting $x'-p'=th$ in the last integral above and then integrating against $\int_{\mathbb{R}^{n-1}} dq'$ yields

$$
\begin{aligned}
\int_{\partial\Omega} |I|\, d\sigma \leqslant C\frac{1}{t^i} \int_{\mathbb{R}^{n-1}} \Bigg(\int_{|h|+1 \leqslant C\,|x'-th-q'|/t} \\
\times \frac{dh}{(|h|+1)^{n-1+i}\,|x'-th-q'|^{n-1+s}} \Bigg)\, dq'.
\end{aligned} \tag{7.20}
$$

Substituting again, this time first $x'-q'=tw$ and then $w-h=r\omega$, $r>0$, $\omega\in S^{n-2}$, we may further bound the last integral in (7.20) by

$$
\begin{aligned}
&C\frac{1}{t^{s+i}} \int_{\mathbb{R}^{n-1}} \int_{|h|+1\leqslant C\,|w-h|} \frac{dh\, dw}{(|h|+1)^{n-1+i}\,|w-h|^{n-1+s}} \\
&\quad = \frac{C}{t^{s+i}} \int \frac{1}{(|h|+1)^{n-1}} \left(\int_{|h|+1}^{\infty} \frac{dr}{r^{s+1}} \right) dh \\
&\quad = C_{n,s,i}\, t^{-s-i},
\end{aligned} \tag{7.21}
$$

which is a bound of the right order for $\int_{\partial\Omega} |I|\, d\sigma$. The same arguments work to bound $\int_{\partial\Omega} |II|\, d\sigma$, by observing that $|x-p|<C|p-q| \Rightarrow |x-q|<C'|p-q|$ and using Fubini's theorem.

As for $\int_{\partial\Omega} |III|\, d\sigma$, we note first that since $|x-z|\geqslant C\,|x-p|$ uniformly for $z\in[p,q]$,

$$
\int_{\partial\Omega} |III|\, d\sigma \leqslant C \int_{\partial\Omega} \int_{|x-p|>C\,|p-q|} \frac{1}{|p-q|^{n-2+s}\,|x-p|^{n+i}}\, d\sigma(p)\, d\sigma(q). \tag{7.22}
$$

As before, pulling back everything to $\mathbb{R}^{n-1}$, it is enough to bound

$$\int_{\mathbb{R}^{n-1}} \int_{|x'-p'|+t>C\,|p'-q'|} \frac{1}{|p'-q'|^{n-2+s}\,(|x'-p'|+t)^{n+i}}\,dp'\,dq'. \qquad (7.23)$$

Substituting $x'-p'=th$ and then $x'-q'=tw$ gives

$$\begin{aligned} &\frac{1}{t^{i+1}} \int_{\mathbb{R}^{n-1}} \left(\int_{|h|+1>C\,|p'-q'|/t} \frac{1}{(|h|+1)^{n+i}\,|q'-x'+ht|^{n-2+s}}\,dh \right) dq' \\ &= \frac{1}{t^{s+i}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|h|+1)^{n+i}} \left(\int_{|h|+1>C\,|w-h|} \frac{dw}{|w-h|^{n-2+s}} \right) dh \\ &= C_{n,s,i}\, t^{-s-i}, \end{aligned} \qquad (7.24)$$

as desired. This finishes the proof of (7.16) and, with it, the proof of the lemma. ∎

We are now ready to present the

Proof of Theorm 7.1. Consider first the single layer potential (7.1) on the scale of Besov spaces and recall the decomposition (2.10). This and (2.11) show that the kernel of $\mathscr{S}_\ell$ satisfies the estimates (7.6) and, hence, Lemmas 7.2–7.3 apply. From Lemma 7.2 (with $N=1$) we get, dropping the dependence of the various norms on the exterior power bundle,

$$\|\mathrm{dist}(\cdot,\partial\Omega)^{s-1}\,|\nabla\mathscr{S}_\ell f|\,\|_{L^1(\Omega)} + \|\mathscr{S}_\ell f\|_{L^1(\Omega)} \leqslant C(\Omega,s)\,\|f\|_{(B_s^\infty(\partial\Omega))^*}, \qquad (7.25)$$

uniformly in $f\in(B_s^\infty(\partial\Omega))^*$. Since $(\Delta_\ell - V)\,\mathscr{S}_\ell f=0$ in Ω, it follows from (7.25) and Proposition 3.3 that

$$\begin{aligned} &\|\mathrm{dist}(\cdot,\partial\Omega)^{s}\,|\nabla^2\mathscr{S}_\ell f|\,\|_{L^1(\Omega)} + \|\nabla\mathscr{S}_\ell f\|_{L^1(\Omega)} + \|\mathscr{S}_\ell f\|_{L^1(\Omega)} \\ &\quad \leqslant C(\Omega,s)\,\|f\|_{(B_s^\infty(\partial\Omega))^*}, \end{aligned} \qquad (7.26)$$

uniformly in $f\in(B_s^\infty(\partial\Omega))^*$. Up to this point, the hypothesis (1.28) on the metric sufficed. However, it is here that the hypothesis (1.29) on the metric tensor is needed for the first time. To be more precise, (1.29) is needed when $\ell>0$; for the case $\ell=0$, (1.27) suffices.

Let us digress momentarily and point out that we could have arrived at (7.26) solely based on Lemma 7.2 in which we take $N=2$. However, this approach requires the pointwise control of (mixed) derivatives of order three for the kernel $E_\ell(x,y)$; cf. (7.6). Under the present approach, this would require a version of (2.11) with (mixed) derivatives of order three placed on the residual part $e_1^\ell(x,y)$ which, in turn, would require a metric tensor of class L_3^r, $r>n$, (for the techniques of [MT, MMT] to apply unaltered). This is why we choose to establish (7.25) by using Lemma 7.2 with $N=1$ and then, further, invoke the interior estimates of Section 3 in order to obtain (7.26). In this latter scenario, the hypothesis (1.29) suffices.

Returning to the mainstream discussion, note that (7.26) in concert with Proposition 4.4 give that the operator

$$\mathscr{S}_\ell : (B_s^\infty(\partial\Omega))^* \to B^1_{2-s}(\Omega), \qquad s \in (0,1), \tag{7.27}$$

is well-defined and bounded. Observe that, by Proposition 5.2, this is a stronger result than the one corresponding to $p=1$, $0<s<1$ in Theorem 7.1.

Going further, Lemma 7.3 (with $N=1$) applied to the single layer operator gives

$$\|\mathrm{dist}(\cdot,\partial\Omega)^s\,|\nabla\mathscr{S}_\ell f|\|_{L^\infty(\Omega)} + \|\mathscr{S}_\ell f\|_{L^\infty(\Omega)} \leqslant C(\Omega,s)\,\|f\|_{B^\infty_{-s}(\partial\Omega)}, \tag{7.28}$$

uniformly in $f\in B^\infty_{-s}(\partial\Omega)$. Now the conclusion in the first part of Theorem 7.1 follows from (7.26) and (7.28) by virtue of Proposition 4.4 and interpolation.

Turning our attention to the second part, i.e., when the range of $\mathscr{S}_\ell$ is taken on the scale of Sobolev spaces, note that (7.25), (7.28) and Stein's interpolation theorem for analytic families of operators give that for $1\leqslant p\leqslant\infty$, $0<s<1$,

$$\|\mathrm{dist}(\cdot,\partial\Omega)^{s-1/p}\,|\nabla\mathscr{S}_\ell f|\|_{L^p(\Omega)} + \|\mathscr{S}_\ell f\|_{L^p(\Omega)} \leqslant C(\Omega,p,s)\|f\|_{B^p_{-s}(\partial\Omega)}, \tag{7.29}$$

uniformly in $f\in B^p_{-s}(\partial\Omega)$. Now, this already leads to the desired conclusion when $s-1/p\geqslant 0$, thanks to (4.21) in Proposition 4.4. In the case when $s-1/p<0$, we first invoke Proposition 3.3 (here the hypothesis (1.29) is used again when $\ell>0$; when $\ell=0$, (1.27) suffices) and, proceeding as before, we arrive at the same conclusion if $1<p<r$, $s\in(0,1/p)$.

Finally, interpolation between this range and the one treated earlier, covers the full unit square, i.e., $s\in(0,1)$, $1/p\in(0,1)$. This finishes the proof of the Theorem 7.1. ∎

Parenthetically, let us point out that (7.3)–(7.4) and real interpolation also give that for $1<p,q<\infty$,

$$\begin{aligned} \|\mathscr{S}_\ell f\|_{B^{p,q}_{1+1/p-s}(\Omega,\Lambda^\ell TM)} &\leqslant C(\Omega,s,p,q)\,\|f\|_{B^{p,q}_{-s}(\partial\Omega,\Lambda^\ell TM)},\\ \|S_\ell f\|_{B^{p,q}_{1-s}(\partial\Omega,\Lambda^\ell TM)} &\leqslant C(\Omega,s,p,q)\,\|f\|_{B^{p,q}_{-s}(\partial\Omega,\Lambda^\ell TM)}, \end{aligned} \tag{7.30}$$

uniformly for $f\in B^{p,q}_{-s}(\partial\Omega,\Lambda^\ell TM)$, $\forall s\in(0,1)$.

Our next result deals with the case when the domain of $\mathscr{S}_\ell$ is in the scale of Sobolev spaces. To state it, recall that for any two real numbers a, b we set $a \vee b := \max\{a, b\}$.

THEOREM 7.4. *Assume that the metric tensor satisfies* (1.28). *Then for* $1<p<\infty$, $0 \leqslant s \leqslant 1$, *and* $1 \leqslant \ell \leqslant n$, *the operator*

$$\mathscr{S}_\ell : L^p_{-s}(\partial\Omega, \Lambda^\ell TM) \to B^{p,\, p \vee 2}_{1-s+1/p}(\Omega, \Lambda^\ell TM) \tag{7.31}$$

is well defined and bounded. When $\ell = 0$, *this holds under hypothesis* (1.27).

Proof. Note that it suffices to treat only the cases when $s=0$ and $s=1$ since the rest follows by complex interpolation (cf. [BL]). In fact, we shall only consider the situation when $s=0$, the rest being similar.

Recall the residual kernel $e^\ell_1(x, y)$ from Section 2 and denote by $\mathscr{B}$, $\tilde{\mathscr{B}}$ the integral operators

$$\begin{aligned} \mathscr{B}f(x) &:= \int_{\partial\Omega} \langle \nabla_x e^\ell_1(x, y), f(y) \rangle \, d\sigma(y), \qquad x \in \Omega, \\ \tilde{\mathscr{B}}f(x) &:= \int_{\partial\Omega} \langle \nabla_y e^\ell_1(x, y), f(y) \rangle \, d\sigma(y), \qquad x \in \Omega. \end{aligned} \tag{7.32}$$

Then, if the metric tensor satisfies (1.28), or (1.27) in case $\ell = 0$,

$$\mathscr{B}: L^p(\partial\Omega) \to B^{p,\,p}_s(\Omega), \qquad 0<s<1, \qquad 1<p<r, \tag{7.33}$$

and

$$\tilde{\mathscr{B}}: L^p(\partial\Omega) \to L^p_1(\Omega), \qquad 1<p<\infty, \tag{7.34}$$

are bounded operators. This follows from Lemmas 2.11–2.12 in [MMT]. In particular,

$$\mathscr{B}, \tilde{\mathscr{B}}: L^p(\partial\Omega) \to B^{p,\,p}_{1/p}(\Omega), \qquad \forall p \in (1, \infty), \tag{7.35}$$

are also bounded. For $\mathscr{B}$, this is contained in (7.33) if $1<p<r$ whereas, for $n<p<\infty$, it is a consequence of the fact that $\mathscr{B}$ in (7.35) factors as

$$L^p(\partial\Omega) \hookrightarrow L^n(\partial\Omega) \xrightarrow{\mathscr{B}} B^{n,\,n}_{1-\varepsilon}(\Omega) \hookrightarrow B^{p,\,p}_{1/p}(\Omega), \qquad \forall \varepsilon > 0 \text{ small}. \tag{7.36}$$

The corresponding statement for $\tilde{\mathscr{B}}$ (in (7.35)) is clear from (7.34).

To continue, denote by $C^\mu S^m_{\mathrm{cl}}$ the class of classical symbols $q(\xi, x)$ of order m which are C^μ in x, for some $\mu \in [0, \infty]$, while still smooth in $\xi \in \mathbb{R}^n \setminus 0$. The problem at hand localizes and, granted the fact that the operators (7.35) are bounded, when Ω is an Euclidean Lipschitz domain it suffices to show the following. If $q(\xi, x) \in C^\mu S^{-2}_{\mathrm{cl}}$, $\mu \geqslant 1$, has a principal symbol that is even in ξ, then the Schwartz kernels of $\partial_j q(D, x)$, $q(D, x)\,\partial_j \in \mathcal{O}\mathrm{P}\, C^0 S^{-1}_{\mathrm{cl}}$ are all kernels of operators mapping $L^p(\partial\Omega)$ boundedly into $B^{p,\, p \vee 2}_{1/p}(\Omega)$ for each $1 < p < \infty$.

Take for instance the case of the Schwartz kernel $k(x-y, y)$ of $\nabla_x q(D, x) \in \mathcal{O}\mathrm{P} C^0 S^{-1}_{\mathrm{cl}}$, for some $q(\xi, x) \in C^\mu S^{-2}_{\mathrm{cl}}$, $\mu \geqslant 1$, whose principal symbol is even in ξ and denote by $\mathscr{K}$ the corresponding integral operator, i.e.,

$$\mathscr{K} f(x) := \int_{\partial\Omega} k(x-y, y)\, f(y)\, d\sigma(y), \qquad x \in \Omega. \tag{7.37}$$

By performing a decomposition in spherical harmonics (cf. [MT] for details in similar circumstances), there is no loss of generality in assuming that $\mu = \infty$.

Next, fix $f \in L^p(\partial\Omega)$ and set $u := \mathscr{K} f$ in Ω. Analogously to [JK2], [Ve2], we use the fact that $\|u\|_{B^{p,q}_{1/p}(\Omega)}$ is controlled by a finite sum of expressions of the type

$$\begin{aligned}
&\left(\int_0^r t^{q-q/p} \left(\int_{S_r} \int_t^r |\nabla u(x', \varphi(x') + s)|^p \, dx' \, ds\right)^{q/p} \frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\int_0^r t^{q-q/p} \left(\int_{S_r} \int_t^r |u(x', \varphi(x') + s)|^p \, dx' \, ds\right)^{q/p} \frac{dt}{t}\right)^{1/q} \\
&=: I + II.
\end{aligned} \tag{7.38}$$

Here $\varphi \colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function used to describe $\partial\Omega$ locally and $S_r \subseteq \partial\Omega$ is a surface ball of fixed radius $r > 0$.

Our aim is to bound I and II by $\|f\|_{L^p(\partial\Omega)}$ in the case when $1 < p < \infty$ and $q := p \vee 2$. The first observation is that II in (7.38) can easily be controlled using

$$\|u^*\|_{L^p(\partial\Omega)} = \|(\mathscr{K} f)^*\|_{L^p(\partial\Omega)} \leqslant C\, \|f\|_{L^p(\partial\Omega)}, \tag{7.39}$$

where the last estimate is proved in [MT] (recall that $(\cdot)^*$ has been introduced in Section 1). As for I, following [JK2] we invoke Hardy's inequality (cf., e.g., [St, Appendix A]) in the case $1 < p \leqslant 2$ plus Minkowski's inequality in order to write

$$\begin{aligned}
|I| &\leqslant C \left(\int_0^r \left(\int_{S_r} |s\, \nabla u(x', \varphi(x') + s)|^p \, dx'\right)^{q/p} \frac{ds}{s}\right)^{1/q} \\
&\leqslant C \left(\int_{S_r} \left(\int_0^r |s\, \nabla u(x', \varphi(x') + s)|^q \frac{ds}{s}\right)^{p/q} dx'\right)^{1/p}.
\end{aligned} \tag{7.40}$$

Note that we can arrive at same majorand for I as above in the case $2 \leqslant p < \infty$ simply by using Fubini's theorem since, in this case, $p = q$.

At this point observe that matters are reduced to proving the L^p-boundedness, $1 < p < \infty$, of the $L^q((0, r), ds/s)$-valued operator T given by the assignment:

$$L^p(S_r) \ni f \mapsto s\, \nabla \mathscr{K} f(x', \varphi(x') + s) \in L^p(S_r, L^q((0, r), ds/s)). \tag{7.41}$$

It is preferable to deal first with the Hilbert space setting, i.e., when $q = 2$, since in this case the vector-valued Calderón–Zygmund theory works. Concretely, setting

$$\tilde{k}_s(x', y') := s\, \nabla_1 k((x' - y', \varphi(x') - \varphi(y') + s), (y', \varphi(y'))), \qquad x', y' \in \mathbb{R}^{n-1}, \tag{7.42}$$

the estimates

$$|\nabla_1^{i+1} \nabla_2^j k(x, y)| \leqslant C\, |x - y|^{-(n-i-j)}, \qquad i, j \geqslant 0, \tag{7.43}$$

readily imply that

$$\left(\int_0^r |\nabla_{x'}^i \nabla_{y'}^j \tilde{k}_s(x', y')|^2 \frac{ds}{s} \right)^{1/2} \leqslant C\, |x' - y'|^{-(n-1+i+j)}, \qquad 0 \leqslant i + j \leqslant 1. \tag{7.44}$$

In turn, these express the fact that the kernel of T in (7.41) is standard. The boundedness of the operator T when $p = 2$ follows from

$$\begin{aligned} \|Tf\|_{L^2(S_r, L^2((0, r), ds/s))} &\leqslant C \left(\int_{S_r} \int_0^r s\, |\nabla u(x', \varphi(x') + s)|^2\, ds\, dx' \right)^{1/2} \\ &\leqslant C \left(\int_\Omega \operatorname{dist}(x, \partial\Omega)\, |\nabla u(x)|^2\, dx \right)^{1/2} \\ &\leqslant C\, \|f\|_{L^2(\partial\Omega)}. \end{aligned} \tag{7.45}$$

The crucial step in (7.45) is the last inequality and this has been proved in [MMT, Theorem 1.1]. This finishes the proof of the L^p-boundedness of T when $q = 2$ and takes care of the $1 < p \leqslant 2$ part in the statement of the theorem.

Next, consider the case when $p \geqslant 2$. When f has small support, contained in an open subset of $\partial\Omega$ where $\partial\Omega$ is given as the graph of $\varphi : \{x' \in \mathbb{R}^{n-1} : |x'| < r\} \to \mathbb{R}$ and $x_0 = (x_0', \varphi(x_0')) \in \partial\Omega$ is an arbitrary boundary point, then

$$\begin{aligned} &s\,|\nabla u(x_0', \varphi(x_0') + s)| \\ &\quad \leqslant \int_{|y'|<r} s\,|\nabla k((x_0', \varphi(x_0') + s), (y', \varphi(y')))| \cdot |f(y', \varphi(y'))|\, dy' \\ &\quad \leqslant C\mathscr{M}f(x_0), \end{aligned} \tag{7.46}$$

uniformly in s, where $\mathscr{M}$ denotes the Hardy–Littlewood maximal operator on $\partial\Omega$. The last inequality in (7.46) is a consequence of the fact that the expression

$$s\,|\nabla k((x', \varphi(x') + s), (y', \varphi(y')))| \tag{7.47}$$

behaves like the Poisson kernel for the upper-half space; see, e.g., [St, Theorem 2, pp. 62–63]. Thus,

$$\sup_{s \in (0,r)} s\,|\nabla u(x_0', \varphi(x_0') + s)| \leqslant C\mathscr{M}f(x_0). \tag{7.48}$$

Using this and the fact that $\mathscr{M}$ is bounded on $L^p(\partial\Omega)$, $1 < p < \infty$, it follows that

$$\left(\int_{S_r} \big(\sup_{s \in (0,r)} |s\,\nabla u(x', \varphi(x') + s)| \big)^p\, dx' \right)^{1/p} \leqslant C\, \|f\|_{L^p(\partial\Omega)}, \tag{7.49}$$

i.e., that T in (7.41) is bounded when $q = \infty$.

The case when $p = q > 2$ now follows by interpolating the end-point results corresponding to $q = 2$ and $q = \infty$. (Note that here we use the fact that $B_s^{p,q_1} \hookrightarrow B_s^{p,q_2}$ for $1 \leqslant q_1 < q_2 \leqslant \infty$.) This completes the proof of Theorem 7.4. ∎

The last result of this section deals with the mapping properties of layer potential operators associated with *general* elliptic, second order differential operators on Lipschitz domains. As such, it further augments the results in [MMT, Sect. 2] where some partial results in this direction where first proved.

THEOREM 7.5. *Let $\mathscr{E}, \mathscr{F} \to M$ be two (smooth) vector bundles over the (smooth) compact, boundaryless manifold M, of real dimension n. It is assumed that the metric structures on $\mathscr{E}$, $\mathscr{F}$, and M have coefficients in L_r^2 for some $r > n$.*

Let

$$Lu = \sum_{j,k} \partial_j A^{jk}(x)\, \partial_k u + \sum_j B^j(x)\, \partial_j u - V(x)\, u, \tag{7.50}$$

where $A^{jk} = (a^{\alpha\beta}_{jk})$, $B^j = (b^{\alpha\beta}_j)$, *and* $V = (d^{\alpha\beta})$ *are matrix-valued functions, be an elliptic, second-order differential operator mapping* C^2 *sections of* $\mathscr{E}$ *into measurable sections of* $\mathscr{F}$. *It is assumed that, when written in local coordinates, the coefficients of L and* L^* *satisfy*

$$a^{\alpha\beta}_{jk} \in L^2_r, \qquad b^{\alpha\beta}_j \in L^1_r, \qquad d^{\alpha\beta} \in L^r, \qquad \textit{for some} \quad r > n. \tag{7.51}$$

Moreover, suppose that L is invertible as a map from $H^{1,2}(M, \mathscr{E})$ *onto* $H^{-1,2}(M, \mathscr{F})$ *and denote by E the Schwartz kernel of* L^{-1}.

Let Ω *be an arbitrary Lipschitz domain in M. For a first order differential operator* $P \in \mathrm{Diff}_1(\mathscr{F}, \mathscr{F})$ *with continuous coefficients, consider the integral operator* $\mathscr{D}$ *with kernel* $(\mathrm{Id}_x \otimes P_y)\, E(x, y)$, *i.e.,*

$$\mathscr{D}f(x) := \int_{\partial\Omega} \langle (\mathrm{Id}_x \otimes P_y)\, E(x, y), f(y) \rangle \, d\sigma(y), \qquad x \in \Omega. \tag{7.52}$$

Then, for each $1 < p < \infty$,

$$\mathscr{D}\colon L^p(\partial\Omega, \mathscr{F}) \to B^{p, p^*}_{1/p}(\Omega, \mathscr{E}) \tag{7.53}$$

is a bounded operator.

Consider next the integral operator $\mathscr{S}$ *which is constructed as before but, this time, in connection with the kernel* $E(x, y)$. *Then, for each* $1 < p < \infty$ *and* $0 \leqslant s \leqslant 1$,

$$\mathscr{S}\colon L^p_{-s}(\partial\Omega, \mathscr{F}) \to B^{p, p^*}_{-s+1+1/p}(\Omega, \mathscr{E}) \tag{7.54}$$

is a bounded operator. Moreover, if $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$, *then*

$$\mathscr{S}\colon B^p_{-s}(\partial\Omega, \mathscr{F}) \to B^p_{-s+1+1/p}(\Omega, \mathscr{E}) \tag{7.55}$$

is a bounded operator also. Finally, the same conclusion applies to

$$\mathscr{S}\colon B^p_{-s}(\partial\Omega, \mathscr{F}) \to L^p_{-s+1+1/p}(\Omega, \mathscr{E}) \tag{7.56}$$

provided $1 < p < \infty$ *and* $0 < s < 1$.

Proof. This follows much as Theorem 7.4 and Theorem 7.1, via a decomposition in spherical harmonics; cf. [MT] for details in similar circumstances. ∎

8. DOUBLE LAYER POTENTIALS ON SOBOLEV AND BESOV SPACES

We shall work with our usual set of hypotheses on M, Ω and V made in Section 1. The aim is to describes the action of the operators (1.11) and (1.15) on scales of Sobolev–Besov spaces. We begin with the case when the domain of the operators is the scale of Besov spaces.

THEOREM 8.1. *Assume that the metric tensor satisfies* (1.27). *Then, for* $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$, *the operator*

$$\mathscr{D}\colon B_s^p(\partial\Omega) \to B_{s+1/p}^p(\Omega) \tag{8.1}$$

is well defined and bounded. In fact, if $1 < p < \infty$ *and* $0 < s < 1$, *then*

$$\mathscr{D}\colon B_s^p(\partial\Omega) \to L_{s+1/p}^p(\Omega) \tag{8.2}$$

is also bounded. Furthermore,

$$K\colon B_s^p(\partial\Omega) \to B_s^p(\partial\Omega) \tag{8.3}$$

is well defined and bounded for $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$.

This is based on a series of lemmas, which we now formulate and prove. We debut with the following Hölder result which, on the Besov scale, corresponds to $p = \infty$.

LEMMA 8.2. *Assume that the metric tensor on* M *satisfies* (1.27). *Then, for* $0 < s < 1$, $\mathscr{D}$ *is a bounded linear map from* $B_s^\infty(\partial\Omega)$ *into* $B_s^\infty(\Omega)$.

Proof. This has been proved in [MT2, Sect. 7] as a consequence of the estimate

$$\sup_{x \in \Omega} (\operatorname{dist}(x, \partial\Omega)^{1-s} \, |\nabla \mathscr{D} f(x)|) \leqslant C(\Omega, s) \, \|f\|_{B_s^\infty(\partial\Omega)}, \tag{8.4}$$

uniformly for $f \in B_s^\infty(\partial\Omega)$, if $0 < s < 1$. Cf. [MT2, (7.24)]. ∎

We next turn attention to the case $p = 1$ on the Besov scale. The main result in this regard is the lemma below.

LEMMA 8.3. *Assume that the metric tensor satisfies* (1.27). *Then the operator* $\mathscr{D}$ *maps* $B_s^1(\partial\Omega)$ *linearly and boundedly into* $B_{s+1}^1(\Omega)$ *for each* $0 < s < 1$.

Proof. Since $\mathscr{D}f$ is a null-solution for L in Ω, by Proposition 3.3 and Proposition 4.4, it suffices to show that

$$\int_{\Omega} \operatorname{dist}(x, \partial\Omega)^{-s} |\nabla \mathscr{D} f(x)| \, d\operatorname{Vol}(x) \leqslant C \, \|f\|_{B_s^1(\partial\Omega)}, \tag{8.5}$$

uniformly for $f \in B_s^1(\partial\Omega)$. Fix an arbitrary $f \in B_s^1(\partial\Omega)$, $0 < s < 1$. Note that, in the left side of (8.5), the contribution away from the boundary is easily estimated, e.g., by the interior estimates in Section 3. Thus, we may replace Ω by $\mathscr{C} \cap \Omega$, where $\mathscr{C}$ is a small collar neighborhood of $\partial\Omega$ in M. Furthermore, via a partition of unity, there is no loss of generality in assuming that f has small support, contained in a coordinate patch.

To continue, let us assume for a moment that $V = 0$ on $\bar{\Omega}$. In this case, since $\nabla \mathscr{D}$ annihilates constants, we can replace f in (8.5) by $f - f(\pi(x))$ where $\pi: \mathscr{C} \cap \Omega \to \partial\Omega$ is some Lipschitz continuous map so that $\operatorname{dist}(x, \partial\Omega) \approx \operatorname{dist}(x, \pi(x))$, uniformly for $x \in \mathscr{C} \cap \Omega$. Next, since the metric satisfies (1.27), the decomposition (2.10) in concert with Theorem 2.1 give that the kernel $k(x, y)$ of $\nabla \mathscr{D}$ satisfies

$$|k(x, y)| \leqslant C \operatorname{dist}(x, y)^{-n}, \qquad \forall x \in \bar{\Omega}, \qquad \forall y \in \partial\Omega. \tag{8.6}$$

Thus, we need to estimate

$$\int_{\mathscr{C} \cap \Omega} \int_{\partial\Omega} \operatorname{dist}(x, \partial\Omega)^{-s} \operatorname{dist}(x, y)^{-n} \, |f(y) - f(\pi(x))| \, d\sigma(y) \, d\operatorname{Vol}(x). \tag{8.7}$$

Note that $\max\{\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \pi(x))\} \leqslant C \operatorname{dist}(x, y)$. Also, set $t := \operatorname{dist}(x, \partial\Omega)$ and $\tilde{x} := \pi(x)$. Then, the integral in (8.7) is bounded by $I + II$ where

$$I := \int_0^\infty \int_{\partial\Omega} \int_{\substack{\operatorname{dist}(y, \tilde{x}) \geqslant Ct \\ y \in \partial\Omega}} t^{-s} \operatorname{dist}(y, \tilde{x})^{-n} \, |f(y) - f(\tilde{x})| \, d\sigma(y) \, d\sigma(\tilde{x}) \, dt \tag{8.8}$$

and

$$II := \int_0^\infty \int_{\partial\Omega} \int_{\substack{\operatorname{dist}(y, \tilde{x}) \leqslant Ct \\ y \in \partial\Omega}} t^{-s-1} \operatorname{dist}(y, \tilde{x})^{-n+1} \, |f(y) - f(\tilde{x})| \, d\sigma(y) \, d\sigma(\tilde{x}) \, dt, \tag{8.9}$$

where $C > 0$ is a fixed, sufficiently large constant. To continue, pull-back everything to $\mathbb{R}^{n-1}$ and denote by $(\omega g)(x) := \|g(x + \cdot) - g(\cdot)\|_{L^1(\mathbb{R}^{n-1})}$ the L^1-modulus of continuity of an arbitrary function $g: \mathbb{R}^{n-1} \to \mathbb{R}$. Then, changing the order of integration and integrating first with respect to t, we obtain

$$|I| \leqslant C \int_0^\infty \frac{1}{t^s} \int_{|z|>Ct} \frac{(\omega f)(z)}{|z|^n} \, dz \, dt$$

$$\leqslant C \int_{\mathbb{R}^{n-1}} \frac{(\omega f)(z)}{|z|^{n-1+s}} \, dz \leqslant C \, \|f\|_{B_s^1(\mathbb{R}^{n-1})}, \tag{8.10}$$

which has the right order. In a similar manner,

$$|II| \leqslant C \int_0^\infty \frac{1}{t^{1+s}} \int_{|z|<Ct} \frac{(\omega f)(z)}{|z|^{n-1}} \, dz \, dt$$

$$\leqslant C \int_{\mathbb{R}^{n-1}} \frac{(\omega f)(z)}{|z|^{n-1+s}} \, dz \leqslant C \, \|f\|_{B_s^1(\mathbb{R}^{n-1})}. \tag{8.11}$$

This concludes the proof of the lemma in the case when $V=0$ on $\bar{\Omega}$.

To treat the general case, let $\mathscr{D}_0$ be the double layer potential corresponding to the choice of a potential V_0 which vanishes in $\bar{\Omega}$. By the previous discussion, it suffices to prove that

$$\|\mathrm{dist}(\cdot, \partial\Omega)^{-s} \, |\nabla(\mathscr{D} - \mathscr{D}_0) f| \|_{L^1(\Omega)} \leqslant C_s \, \|f\|_{L^1(\partial\Omega)}, \tag{8.12}$$

for each $s \in (0, 1)$, uniformly for $f \in L^1(\partial\Omega)$. Now, by (2.10) and Theorem 2.1, the integral operator in the left side of (8.12) has a kernel $k(x, y)$ satisfying, for each $\varepsilon > 0$,

$$|k(x, y)| \leqslant C_\varepsilon \, \mathrm{dist}(x, \partial\Omega)^{-s} \, \mathrm{dist}(x, y)^{-(n-1+\varepsilon)}, \qquad \forall x \in \bar{\Omega}, \qquad \forall y \in \partial\Omega. \tag{8.13}$$

This and elementary estimates then readily yield (8.12). ∎

Let us pause for a moment and discuss an alternative approach to the Lemma 8.3. While this works for differential forms of higher degree, the metric tensor would have to satisfy (1.29) in place of (1.27). Nonetheless, this result (or rather its proof) will be useful for us in the sequel.

To get started, let us consider a generalization of (1.11) at the level of differential forms of arbitrary degrees. Specifically, for each $\ell \in \{0, 1, ..., n\}$, we introduce the double layer potential of a ℓ-form f in, say, $L^2(\partial\Omega, \Lambda^\ell TM)$ by setting

$$\mathscr{D}_\ell f(x) := \int_{\partial\Omega} \langle (\nu(y) \vee d_y - \nu(y) \wedge \delta_y) \, E_\ell(x, y), f(y) \rangle \, d\sigma(y),$$

$$x \in \Omega, \tag{8.14}$$

where $\wedge$, $\vee$ are the usual exterior and interior, respectively, products of forms. Its (nontangential) boundary trace is

$$\mathscr{D}_\ell f|_{\partial\Omega} = (\tfrac{1}{2}I + K_\ell) f, \qquad \text{a.e. on } \partial\Omega, \tag{8.15}$$

where

$$K_\ell f(x) := \text{P.V.} \int_{\partial\Omega} \langle (\nu(y) \vee d_y - \nu(y) \wedge \delta_y)\, E_\ell(x, y), f(y) \rangle \, d\sigma(y), \qquad x \in \partial\Omega. \tag{8.16}$$

See [MMT, Sect. 6] for more on these. Note that when $\ell = 0$, the operators (8.14), (8.16) reduce precisely to (1.11) and (1.15), respectively.

Lemma 8.4. *Assume that the metric tensor satisfies* (1.29). *Then, for* $0 < s < 1$ *and* $\ell \in \{0, 1, ..., n\}$, *the operators*

$$d\mathscr{D}_\ell \colon B^1_s(\partial\Omega, \Lambda^\ell TM) \to B^1_s(\Omega, \Lambda^{\ell+1} TM) \tag{8.17}$$

and

$$\delta\mathscr{D}_\ell \colon B^1_s(\partial\Omega, \Lambda^\ell TM) \to B^1_s(\Omega, \Lambda^{\ell-1} TM) \tag{8.18}$$

are well defined and bounded.

Proof. We deal only with (8.17) since the case of (8.18) follows from this via an application of the Hodge star isomorphism.

A basic ingredient in the proof is an identity to the effect that

$$\begin{aligned} d\mathscr{D}_\ell f(x) &= \int_{\partial\Omega} \langle d_x d_y E_\ell(x, y), \nu(y) \wedge f(y) \rangle \, d\sigma(y) \\ &\quad + \int_{\partial\Omega} \langle d_x R_{\ell-1}(y, x), \nu(y) \vee f(y) \rangle \, d\sigma(y) \\ &= d\delta\mathscr{S}_{\ell+1}(\nu \wedge f)(x) - R_\ell(\nu \wedge f)(x) + dR^t_{\ell-1}(\nu \vee f)(x) \\ &= -\delta d\mathscr{S}_{\ell+1}(\nu \wedge f)(x) - R_\ell(\nu \wedge f)(x) - V\mathscr{S}_{\ell+1}(\nu \wedge f)(x) \\ &\quad + dR^t_{\ell-1}(\nu \vee f)(x) \\ &= \delta\mathscr{S}_{\ell+2}(\nu \wedge df)(x) - \delta R^t_{\ell+1}(\nu \wedge f)(x) - R_\ell(\nu \wedge f)(x) \\ &\quad - V\mathscr{S}_{\ell+1}(\nu \wedge f)(x) + dR^t_{\ell-1}(\nu \vee f)(x). \end{aligned} \tag{8.19}$$

Recall that $\mathscr{S}_\ell$ is the single layer potential on ℓ-forms, i.e., the boundary integral operator with kernel $E_\ell(x, y)$; cf. Section 2. Also, for the last equality see [MMT, Sect. 6].

Now, if f is an ℓ-form with components $B^1_s(\partial\Omega)$-atoms, then (componentwise) $\nu \wedge df \in \mathfrak{h}^p(\partial\Omega)$ with $p \in ((n-1)/n, 1)$, $s \in (0,1)$ related by $1-s=(n-1)(\frac{1}{p}-1)$, and $\|\nu \wedge df\|_{\mathfrak{h}^p(\partial\Omega)} \leqslant C$ with C independent of f. This follows from (4.5). In particular,

$$\|\nu \wedge df\|_{(B^{\infty}_{1-s}(\partial\Omega))^*} \leqslant C. \tag{8.20}$$

Now the desired conclusion about $\delta\mathscr{S}_{\ell+2}(\nu \wedge df)$ follows from Proposition 4.3, (8.20), and Theorem 7.1 (it is at this point that the hypothesis (1.29) is used).

The remaining terms after the last equality sign in (8.19) are residual and can be handled more directly. In fact, it suffices to assume (1.28), which we shall do. To this end, it helps to note that $B^1_s(\partial\Omega) \hookrightarrow L^{q(s)}(\partial\Omega)$ for $1/q(s) := 1 - s/(n-1)$. Take, for instance, the case of $\delta R^t_{\ell+1}(\nu \wedge f)$; it suffices to show that

$$R^t_{\ell+1}\colon L^{q(s)}(\partial\Omega, \Lambda^{\ell+1}TM) \to B^1_{1+s}(\Omega, \Lambda^{\ell}TM), \qquad 0<s<1, \tag{8.21}$$

is bounded. Indeed, as we shall see momentarily, we even have

$$R^t_{\ell+1}\colon L^{p}(\partial\Omega, \Lambda^{\ell}TM) \to L^p_2(\Omega, \Lambda^{\ell+1}TM), \qquad \forall p \in (1, r). \tag{8.22}$$

To justify (8.22), let us introduce

$$\tilde{R}_{\ell+1}g(x) := \int_{\Omega} \langle R_{\ell+1}(x,y), g(y)\rangle \, d\,\mathrm{Vol}(y), \qquad x \in \Omega, \tag{8.23}$$

and observe that when the metric tensor satisfies (1.28),

$$\begin{aligned} \tilde{R}_{\ell+1} : L^{q'}_{-2,0}(\Omega) = (L^q_2(\Omega))^* \to L^{q'}_2(\Omega), \\ r/(r-1) < q, \qquad q' < r, \qquad 1/q + 1/q' = 1, \end{aligned} \tag{8.24}$$

is bounded. This follows by invoking an observation made in [MMT, Sect. 6] to the effect that $\tilde{R}_{\ell+1} = (\Delta_\ell - V)^{-1}(dV \vee)(\Delta_{\ell+1} - V)^{-1}$ and by appealing to the mapping properties of the Newtonian potentials established in Section 6 of the present paper. The point is that (8.22) is a consequence of (8.24) and duality, upon noticing that

$$\int_{\Omega} \langle R^t_{\ell+1} f, g\rangle \, d\,\mathrm{Vol} = \int_{\partial\Omega} \langle \mathrm{Tr}\,(\tilde{R}_{\ell+1} g), f\rangle \, d\sigma \tag{8.25}$$

for any reasonable forms f, on $\partial\Omega$, and g, on Ω. Specifically, since

$$\mathrm{Tr}\colon L^{q'}_2(\Omega) \to L^p(\partial\Omega), \qquad \forall p \in (1, \infty), \qquad \forall q' > \frac{n}{2}, \tag{8.26}$$

is bounded, (8.25) entails

$$\left|\int_{\Omega} \langle R^t_{\ell+1} f, g\rangle \, d\,\mathrm{Vol}\right| \leqslant C\, \|g\|_{(L^q_2(\Omega,\, \Lambda^{\ell+1}TM))^*}\, \|f\|_{L^p(\partial\Omega,\, \Lambda^{\ell}TM)}, \tag{8.27}$$

for $1<p<\infty$ as long as $q'>n/2$. This last condition can be arranged given the validity range for (8.24) and the fact that we are assuming $r>n$. For such a q', (8.27) proves the membership of $R^t_{\ell+1}f$ to $L^q_2(\Omega, \Lambda^{\ell+1}TM)$ whenever $f\in L^p(\partial\Omega, \Lambda^{\ell}TM)$ with $1<p<\infty$, plus natural estimates. Thus, (8.22) holds for p close to 1.

In fact, a similar reasoning as above but with (8.26) replaced by

$$\mathrm{Tr}\colon L^{q'}_2(\Omega)\to L^{q'}(\partial\Omega), \qquad \forall q', \tag{8.28}$$

shows that (8.22) holds for any $p\in(r/(r-1), r)$. Then the full range $1<p<r$ in (8.22) follows by interpolation.

Hence, the proof of (8.22) is finished and this, in turn, completes the proof of (8.21). As the remaining terms in the right side of the last equality in (8.19) are treated similarly, the proof of the lemma is finished. ∎

We are now ready for the

Proof of Theorem 8.1. Let us first deal with the operator (8.1). The case $p=\infty$, $0<s<1$ is contained in Lemma 8.2 whereas Lemma 8.3 covers the case $p=1$, $0<s<1$. The full range then follows by interpolation.

Turning now attention to the operator (8.2), recall from the proof of Lemma 8.3 that

$$\|\mathrm{dist}(\cdot, \partial\Omega)^{-s}\, |\nabla \mathscr{D}f|\|_{L^1(\Omega)} + \|\mathscr{D}f\|_{L^1(\Omega)} \leqslant C(\Omega, s)\, \|f\|_{B^1_s(\partial\Omega)} \tag{8.29}$$

holds uniformly for $f\in B^1_s(\partial\Omega)$. Then, (8.29) in concert with the Hölder estimate (8.4) and Stein's interpolation theorem for analytic families of operators yield the estimate

$$\|\mathrm{dist}(\cdot, \partial\Omega)^{1-s-1/p}\, |\nabla \mathscr{D}f|\|_{L^p(\Omega)} + \|\mathscr{D}f\|_{L^p(\Omega)} \leqslant C(\Omega, p, s)\, \|f\|_{B^p_s(\partial\Omega)}, \tag{8.30}$$

uniformly for $f\in B^p_s(\partial\Omega)$ for each $1\leqslant p\leqslant\infty$, $0<s<1$.

To continue, we distinguish two cases. First, if $1-s-1/p\geqslant 0$ then (8.30) and Proposition 4.4 yield that $\mathscr{D}f\in L^p_{s+1/p}(\Omega)$ plus natural estimates. Second, if $1-s-1/p<0$ we proceed similarly and arrive at the same conclusion as before. The only difference is that we involve Proposition 3.3 (cf. the comment following its statement) before invoking Proposition 4.4. This completes the proof of the claim regarding the operator (8.2).

Finally, the last point in the statement of the theorem is a consequence of (8.1) and the trace formula (1.13). ∎

In the remainder of this section we analyze the action of the double layer potential on the scale of Sobolev spaces.

THEOREM 8.5. *Assume that the metric tensor satisfies* (1.28). *Then, for* $1 < p < \infty$ *and* $0 \leqslant s \leqslant 1$, *the operator*

$$\mathscr{D}\colon L^p_s(\partial\Omega) \to B^{p,\, p \vee 2}_{s+1/p}(\Omega) \tag{8.31}$$

is bounded.

Proof. For (8.31) with $s = 0$, arguments similar to the ones used in the proof of Theorem 7.4 apply. Matters can again be reduced to the same pattern in the case $s = 1$, thanks to the identity (8.19). Note that, in this later situation, we need estimates like

$$R^t_{\ell+1}: L^p(\partial\Omega, \Lambda^\ell TM) \to B^{p,\, p}_{1+1/p}(\Omega, \Lambda^{\ell+1} TM), \qquad \forall p \in (1, \infty). \tag{8.32}$$

In turn, the estimate (8.32) follows easily from (8.22) and embedding results (since we are assuming $r > n$). Now the claim about (8.31) is obtained by interpolation. ∎

For our next theorem, the following observation is useful. To state it, recall the nontangential maximal operator $(\cdot)^*$ from Section 1.

PROPOSITION 8.6. *If the metric tensor satisfies* (1.28) *and* $1 < p < \infty$, *then*

$$f \in L^p_1(\partial\Omega) \Rightarrow (\nabla \mathscr{D} f)^* \in L^p(\partial\Omega), \tag{8.33}$$

plus a natural estimate.

Proof. This follows from (8.19), with $\ell = 0$, plus estimates on the single layer potential $\mathscr{S}_\ell$ established in [MMT, Sect. 2]. ∎

Our next result contains an improvement of (8.31) and (7.31) with $\ell = 0$ in the range $2 \leqslant p < \infty$.

THEOREM 8.7. *Assume that the metric tensor satisfies* (1.28). *Then, for* $2 \leqslant p < \infty$ *and* $0 \leqslant s \leqslant 1$, *the operators*

$$\mathscr{D}\colon L^p_s(\partial\Omega) \to L^p_{s+1/p}(\Omega) \tag{8.34}$$

and

$$\mathscr{S}\colon L^p_{-s}(\partial\Omega) \to L^p_{1-s+1/p}(\Omega) \tag{8.35}$$

are well defined and bounded.

Proof. remark When $0<s<1$ and $2\leqslant p<\infty$, (8.34) also follows from (4.19) and (8.2). Similarly, for the same ranges of indices, (8.35) is a consequence of (4.19) and (7.4). Note that, in this scenario, the hypothesis (1.27) on the metric tensor suffices. Thus, the main novel points addressed by Theorem 8.7 are $s=0$ and $s=1$.

Proof of Theorem 8.7. Our proof builds on an idea from [JK2]. Assume first that $s=0$. In this case, granted (1.20), proving (8.34) is the same as proving such a result with $\mathscr{D}$ replaced by $T:=\mathscr{D}\circ(\frac{1}{2}I+K)^{-1}$, in the range $2\leqslant p<\infty$. Based on [MMT], we have that

$$T\colon L^2(\partial\Omega) \to L^2_{1/2}(\Omega) \tag{8.36}$$

is bounded. Also, since T is the solution operator for the Dirichlet problem (1.4), the maximum principle gives that

$$T\colon L^\infty(\partial\Omega) \to L^\infty(\Omega) \tag{8.37}$$

is bounded. Now, the inclusion

$$[L^\infty(\Omega), L^2_{1/2}(\Omega)]_\theta \hookrightarrow L^p_{1/p}(\Omega), \qquad 0<\theta<1, \qquad p:=2/\theta, \tag{8.38}$$

together with (8.36), (8.37), and interpolation allows one to conclude that

$$\mathscr{D}\colon L^p(\partial\Omega) \to L^p_{1/p}(\Omega), \qquad 2\leqslant p<\infty, \tag{8.39}$$

is bounded. Observe that by interpolating (8.39) with $\mathscr{D}\colon L^2_1(\partial\Omega)\to L^2_{3/2}(\Omega)$ which, in turn, follows from (8.31), we arrive at the conclusion that

$$\mathscr{D}\colon L^p_s(\partial\Omega) \to L^p_{s+1/p}(\Omega), \qquad 2\leqslant p\leqslant 2/s, \qquad 0<s\leqslant 1, \tag{8.40}$$

is bounded.

Consider next $f\in L^p_1(\partial\Omega)$, $2\leqslant p<\infty$, and set $u:=\mathscr{D}f$. Our aim is to prove that $u\in L^p_{1+1/p}(\Omega)$ plus estimates. Introducing $v:=Xu$, where $X\in T^*M$ is an arbitrary vector field with smooth coefficients, it suffices to show that

$$v\in L^p_{1/p}(\Omega) \qquad \text{plus a natural estimate.} \tag{8.41}$$

By Proposition 8.6 we have $v^*\in L^p(\partial\Omega)$. If we next set $w:=v-\Pi_0(v)$ in Ω. it follows that $Lw=0$ in Ω and $w^*\in L^p(\partial\Omega)$, at least if p is large. Indeed, by Section 6, $\Pi_0(v) \in L^p_2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ if p is large. Thus, from the integral representation for the solution of the L^p-Dirichlet problem with $p \geqslant 2$ (established in [MT2]) and (8.39) we may conclude that $w \in L^p_{1/p}(\Omega)$ if p is large. Since $\Pi_0(v) \in L^p_2(\Omega)$, (8.41) follows. This proves (8.34) for $s=1$ and p large. Now, interpolating what we have just proved with (8.40) gives (8.34) for the full range of indices specified in the statement of the theorem.

The argument for (8.35) is similar. The starting point is to consider the solution operator $T := \mathscr{S} \circ S^{-1}$ plus the fact that $S^{-1}: L^p_1(\partial\Omega) \to L^p(\partial\Omega)$ is an isomorphism for $2 \leqslant p < \infty$ ([MT2]). Proceeding as before yields (8.35) for $s=1$. Finally, (8.35) with $s=0$ is handled as before by taking this time $u := \mathscr{S}f$, $f \in L^p(\partial\Omega)$, $2 \leqslant p < \infty$. This finishes the proof of the theorem. ∎

In closing, let us point out that (1.19) and Theorem 8.7 (or rather its proof) give that the solution u of the boundary problem (1.4) and (1.7) satisfies

$$\|u\|_{L^p_{1/p}(\Omega)} \leqslant C(\Omega, p)\, \|f\|_{L^p(\partial\Omega)} \qquad \text{if} \quad 2 \leqslant p < \infty. \tag{8.42}$$

On the other hand, by Theorem 8.5 and (1.20),

$$\|u\|_{B^{p,\, p \vee 2}_{1+1/p}(\Omega)} \leqslant C(\Omega, p)\, \|f\|_{L^p_1(\partial\Omega)} \qquad \text{if} \quad 1 < p \leqslant 2. \tag{8.43}$$

As for the solution u of the boundary problem (1.5) and (1.8), Theorem 7.4, in concert with (1.21)–(1.22), yields that

$$\|\nabla u\|_{L^p_{1/p}(\Omega)} \leqslant C(\Omega, p)\, \|g\|_{L^p(\partial\Omega)} \qquad \text{if} \quad 1 < p \leqslant 2. \tag{8.44}$$

9. AN ENDPOINT NEUMANN PROBLEM

Retaining the hypotheses of Section 1 on M, V and the metric, including (1.27), here we study a limiting case of the Neumann problem. Specifically, for a Lipschitz domain $\Omega \subset M$, we shall be concerned with the case when the boundary data belong to $B^1_{-s}(\partial\Omega)$ for $s \in (0, s_0)$, where $s_0 \in (0, 1)$ depends on the domain. First we need a couple of technical results.

LEMMA 9.1. *Let* $\psi \in B^{\infty}_s(\partial\Omega)$ *for some* $s \in (0, 1)$. *Then there exists* $\tilde{\psi} \in C^0(\bar{\Omega})$, *locally Lipschitz in* Ω, *so that* $\tilde{\psi}|_{\partial\Omega} = \psi$ *and* $\operatorname{dist}(\cdot, \partial\Omega)^{1-s}\, \nabla\tilde{\psi} \in L^\infty(\Omega)$.

Proof. The problem localizes and, via a bi-Lipschitz change of variables can be transported to the Euclidean upper-half space. There, the Poisson extension does the job; see, e.g., [St]. ∎

LEMMA 9.2. *There exists* $s_0 = s_0(\Omega) > 0$ *so that whenever* $0 < s < s_0$,

$$\|\mathrm{dist}(x, \partial\Omega)^{s-1} \, |\nabla u| \, \|_{L^1(\Omega)} \leqslant C(\Omega, s) \, \|u\|_{B^1_{2-s}(\Omega)} \tag{9.1}$$

uniformly for $u \in B^1_{2-s}(\Omega)$ *with* $Lu = 0$ *in* Ω.

The proof of this lemma is postponed until the next section; cf. (10.7).

Assume that $0 < s < s_0$, where s_0 is as in Lemma 9.2. We will be concerned with the Neumann boundary problem with boundary data in $B^1_{-s}(\partial\Omega)$, i.e.,

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f \in B^1_{-s}(\partial\Omega), \\ u \in B^1_{2-s}(\Omega). \end{cases} \tag{9.2}$$

The boundary condition in (9.2) is interpreted as the equivalent of

$$\int_\Omega \langle \nabla u, \nabla \tilde{\psi} \rangle \, d\,\mathrm{Vol} = \langle f, \psi \rangle, \qquad \forall \psi \in B^\infty_s(\partial\Omega), \tag{9.3}$$

where tilde is the extension operator introduced in Lemma 9.1, and $\langle \cdot, \cdot \rangle$ in the right side stands for the natural pairing between $B^1_{-s}(\partial\Omega) \subseteq (B^\infty_s(\partial\Omega))^*$ and $B^\infty_s(\partial\Omega)$. It is to be noted that, by Lemmas 9.1–9.2, the integral in the left side of (9.3) is absolutely convergent.

An observation made first (in the flat Euclidean setting) in [FMM] and which also applies to the present context is that the space of natural boundary data in (9.2) is indeed $B^1_{-s}(\partial\Omega)$ and not the larger space $(B^\infty_s(\partial\Omega))^*$. This is supported by the observation that even though $-\frac{1}{2}I + K^*$ is, as we shall see momentarily, an isomorphism of the larger space $(B^\infty_s(\partial\Omega))^*$ for small s, and even though $\mathscr{S}$ maps the latter space boundedly into $B^\infty_{2-s}(\Omega)$, the natural jump formula

$$\partial_\nu \mathscr{S} f = (-\tfrac{1}{2}I + K^*) \, f \tag{9.4}$$

necessarily fails for general $f \in (B^\infty_s(\partial\Omega))^*$. This is because, as will be shown in Theorem 10.1, the normal derivative of $\mathscr{S} f \in B^1_{2-s}(\Omega)$ always belongs to a smaller subspace of $(B^\infty_s(\partial\Omega))^*$, namely $B^1_{-s}(\partial\Omega)$.

Our main result in this section is the theorem below, which extends [FMM, Theorem 7.2]. Recall that $\mathscr{C}$ stands for the collection of all constant functions on $\partial\Omega$.

THEOREM 9.3. *There exists* $s_0 \in (0, 1)$ *depending only on* Ω *so that, if* $0 < s < s_0$ *and* $f \in B^1_{-s}(\partial\Omega)$ (*with the extra condition* $\langle f, \mathscr{C} \rangle = 0$ *imposed if*

$V=0$ in Ω) then there exists a unique (modulo constants, if $V=0$ in Ω) solution u to the Neumann problem (9.2). *Moreover, u satisfies the estimate*

$$\|u\|_{B^1_{2-s}(\Omega)} + \|\mathrm{dist}(\cdot, \partial\Omega)^{s-1}\, |\nabla u|\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \leqslant C(\Omega, s)\, \|f\|_{B^1_{-s}(\partial\Omega)}. \tag{9.5}$$

In particular, when $V>0$ on a set of positive measure in Ω,

$$\|\mathrm{Tr}\, u\|_{B^1_{1-s}(\partial\Omega)} \leqslant C(\Omega, s)\, \|f\|_{B^1_{-s}(\partial\Omega)}. \tag{9.6}$$

When $V=0$ in Ω, then Tr u *in* (9.6) *should be considered modulo constants.*

Note that this result implies that, under the same hypotheses, the Neumann-to-Dirichlet operator for L is bounded from $B^1_{-s}(\partial\Omega)$ into $B^1_{1-s}(\partial\Omega)$.

In proving Theorem 9.3, the following result is very useful.

Lemma 9.4. *There exists $s_0 \in (0, 1)$ such that, when $V>0$ on a set of positive measure in Ω,*

$$-\tfrac{1}{2}I + K^*\colon B^1_{-s}(\partial\Omega) \to B^1_{-s}(\partial\Omega) \tag{9.7}$$

is an isomorphism for $0<s<s_0$.

When $V=0$ in Ω, then the same conclusion holds if we restrict to the subspace of $B^1_{-s}(\partial\Omega)$ consisting of functionals that annihilate $\mathscr{C}$.

Proof. Granted that $V>0$ on a set of positive measure in Ω, it has been shown in [MT2, Theorems 7.6–7.7] that there exists $\varepsilon>0$ so that $-\frac{1}{2}I+K^*$ is an isomorphism of $\mathfrak{h}^p(\partial\Omega)$ for $1-\varepsilon<p\leqslant 1$. Our result then follows from this, Lemma 5.1 and Proposition 5.2. The case when $V=0$ in Ω follows by a minor variation of this argument. ∎

We are now ready to present the

Proof of Theorem 9.3. We give the proof when $V>0$ on a set of positive measure in Ω; the easy modifications needed when $V=0$ in Ω are left to the reader.

Let s_0 be as in Lemma 9.4, and $s\in(0, s_0)$. Then Lemma 9.4 gives that $g := (-\frac{1}{2}I+K^*)^{-1}f$ exists in $B^1_{-s}(\partial\Omega)$ and $\|g\|_{B^1_{-s}(\partial\Omega)} \leqslant C\,\|f\|_{B^1_{-s}(\partial\Omega)}$. If we now set $u := \mathscr{S}g$, it follows that $Lu=0$ in Ω and, by invoking Theorem 7.1 and (7.25), $u\in B^1_{2-s}(\Omega)$ and (9.5) holds. To see that u actually solves (9.2) we shall prove that u satisfies (9.3). Indeed, this is a consequence of the general identity

$$\int_\Omega \langle \nabla\mathscr{S}h, \nabla\tilde{\psi}\rangle \, d\,\mathrm{Vol} = \langle (-\tfrac{1}{2}I+K^*)\, h, \psi\rangle, \qquad \forall\, \psi\in B^\infty_s(\partial\Omega), \tag{9.8}$$

which we claim is valid for arbitrary $h \in B^1_{-s}(\partial\Omega)$ (recall that the tilde operator is as in Lemma 9.1). In turn, this is easily checked on $B^1_{-s}(\partial\Omega)$-atoms and, hence, extends by density to the whole $B^1_{-s}(\partial\Omega)$ thanks to (7.25). This completes the proof of the existence part.

Turning to uniqueness, assume that $u \in C^1_{\mathrm{loc}}(\Omega)$ solves the homogeneous version of (9.2) so that, in particular,

$$\int_\Omega \langle \nabla u, \nabla \widetilde{\psi} \rangle \, d\,\mathrm{Vol} = 0, \qquad \forall \psi \in B^\infty_s(\partial\Omega). \tag{9.9}$$

Let $\Omega_j \nearrow \Omega$ be a sequence of C^∞ subdomains approximating Ω and, for some fixed point $x_0 \in \Omega$, consider the Neumann function N_j for L in Ω_j with pole at x_0, i.e.,

$$N_j(x) := E(x_0, x) - \mathscr{S}_j((-\tfrac{1}{2}I + K_j^*)^{-1} (\partial_{\nu_j} E(x_0, \cdot)|_{\partial\Omega_j}) \qquad \text{for} \quad x \in \Omega_j. \tag{9.10}$$

Also, hereafter, $\mathscr{S}_j$, K_j^*, etc., will denote operators similar to $\mathscr{S}$, K^*, etc., but constructed in connection with $\partial\Omega_j$ rather than $\partial\Omega$. Take a smooth function $0 \leqslant \phi \leqslant 1$ which vanishes identically near x_0 and is identically 1 near $\partial\Omega$. Green's formula and an integration by parts then give

$$\begin{aligned} u(x_0) &= \int_{\partial\Omega_j} N_j \frac{\partial u}{\partial \nu_j} \, d\sigma_j = \int_{\partial\Omega_j} \phi N_j \frac{\partial u}{\partial \nu_j} \, d\sigma_j \\ &= \int_{\Omega_j} \langle \nabla(\phi N_j), \nabla u \rangle \, d\,\mathrm{Vol}. \end{aligned} \tag{9.11}$$

The key step is to prove that, as $j \to \infty$,

$$\int_{\Omega_j} \langle \nabla(\phi N_j), \nabla u \rangle \, d\,\mathrm{Vol} \to \int_\Omega \langle \nabla(\phi N), \nabla u \rangle \, d\,\mathrm{Vol} = 0, \tag{9.12}$$

where N is the Neumann function for L in Ω with pole at x_0. Then, passing to the limit in (9.11) will give $u \equiv 0$ in Ω as desired.

Turning our attention to (9.12), first we shall prove that the second integral vanishes. Indeed, it has been proved in [MT2] that $N_j \in C^\alpha(\overline{\Omega}_j \setminus \{x_0\})$ and $N \in C^\alpha(\overline{\Omega} \setminus \{x_0\})$ for some $\alpha = \alpha(\Omega) > 0$ independent of j. Consequently,

$$\begin{gathered} \mathrm{dist}(\cdot, \partial\Omega_j)^{1-\alpha} |\nabla N_j|, \ \mathrm{dist}(\cdot, \partial\Omega)^{1-\alpha} |\nabla N| \leqslant C \\ \text{away from } x_0, \text{ uniformly in } j, \end{gathered} \tag{9.13}$$

by the Hölder theory in [MT2]. In particular,

$$\operatorname{dist}(\cdot, \partial\Omega_j)^{1-\alpha} |\nabla(\phi N_j)|, \operatorname{dist}(\cdot, \partial\Omega)^{1-\alpha} |\nabla(\phi N)| \leqslant C \quad \text{in } \Omega_j, \text{ uniformly in } j. \tag{9.14}$$

With this in hand, the second integral in (9.12) vanishes on account of (9.9) if $0<s<\alpha$ which we will assume for the remaining part of the proof.

At this point we are left with proving the convergence in (9.12) which we tackle next. For $j \geqslant k$ we write

$$\begin{aligned} \int_{\Omega_j} \langle \nabla(\phi N_j), \nabla u\rangle \, d\,\mathrm{Vol} &= \int_{\Omega_j \backslash \Omega_k} \langle \nabla(\phi N_j), \nabla u\rangle \, d\,\mathrm{Vol} \\ &\quad + \int_{\Omega_k} \langle \nabla(\phi N_j) - \nabla(\phi N), \nabla u\rangle \, d\,\mathrm{Vol} \\ &\quad + \int_{\Omega_k} \langle \nabla(\phi N), \nabla u\rangle \, d\,\mathrm{Vol} \\ &=: I_{j,k} + II_{j,k} + III_k. \end{aligned} \tag{9.15}$$

Now, by (9.9), (9.14), and Lebesgue's dominated convergence theorem, $\lim_{k\to\infty} III_k = 0$. Further,

$$\begin{aligned} |I_{j,k}| &\leqslant \Big(\sup_{x \in \Omega_j \backslash \Omega_k} \operatorname{dist}(x, \partial\Omega_j)^{1-s} |\nabla(\phi N_j)(x)| \Big) \\ &\quad \times \int_{\Omega_j \backslash \Omega_k} \operatorname{dist}(\cdot, \partial\Omega_j \cup \partial\Omega_k)^{s-1} |\nabla u| \, d\,\mathrm{Vol}. \end{aligned} \tag{9.16}$$

The first factor in the right side of (9.16) is bounded uniformly in j, k, by (9.14) and our assumption on s. Also, by Lemma 9.2, the second factor in (9.16) is $\leqslant C \|u\|_{B^1_{2-s}(\Omega_j \backslash \Omega_k)}$, i.e., small if j, k are large enough.

To conclude the proof, we only need to show that, for a fixed k, $|II_{j,k}|$ is small if j is large enough. Thus, if we set $f_j := (-\frac{1}{2}I + K_j^*)^{-1} (\partial_{\nu_j} E(x_0, \cdot)|_{\partial\Omega_j})$ and $f := (-\frac{1}{2}I + K^*)^{-1} (\partial_\nu E(x_0, \cdot)|_{\partial\Omega_j})$, it suffices to prove that

$$\nabla(\phi \mathscr{S}_j f_j)|_{\Omega_k} \to \nabla(\phi \mathscr{S} f)|_{\Omega_k} \qquad \text{in } L^2(\Omega_k) \text{ as } j \to \infty. \tag{9.17}$$

This, in turn, follows from Lebesgue's dominated convergence theorem. Somewhat more specifically, if $\Lambda_j : \partial\Omega \to \partial\Omega_j$ is a natural bi-Lipschitz homeomorphism, it can be proved that $f_j \circ \Lambda_j \to f$ in $L^2(\partial\Omega)$. This gives pointwise convergence. The domination is trivially given by $|\nabla(\phi \mathscr{S}_j f_j)| \leqslant C$ on Ω_k uniformly in j. The proof of uniqueness is therefore complete.

Note that (9.5) and (9.6) follow from Lemma 7.2, Lemma 9.4 and the integral representation of the solution. ∎

10. AN ENDPOINT DIRICHLET PROBLEM

We continue to assume our standard hypotheses on M, Ω, V, and the metric tensor, including (1.27). The aim of this section is to study the Dirichlet problem with boundary data in the Besov space $B^1_{1-s}(\partial\Omega)$. Besides establishing the well posedness of this problem for small s, our approach also shows that the solution has a normal derivative in $B^{-1}_{-s}(\partial\Omega)$. Moreover, this is accompanied by a natural estimate. Specifically, we have the following extension of Theorem 5.8 of [JK2] and Theorem 7.1 of [FMM].

THEOREM 10.1. *There exists* $s_0 = s_0(\Omega) > 0$ *so that for* $0 < s < s_0$ *and* $f \in B^1_{1-s}(\partial\Omega)$, *the Dirichlet problem*

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ \operatorname{Tr} u = f & \text{on } \partial\Omega, \\ u \in B^1_{2-s}(\Omega), \end{cases} \tag{10.1}$$

has a unique solution. The solution satisfies

$$\|\operatorname{dist}(\cdot, \partial\Omega)^{s-1}\, |\nabla u|\,\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \approx \|u\|_{B^1_{2-s}(\Omega)} \approx \|f\|_{B^1_{1-s}(\partial\Omega)} \tag{10.2}$$

for constants that depend only on Ω *and s. Furthermore,* $\partial_\nu u \in B^{-1}_{-s}(\partial\Omega)$.

More specifically, there exists $g \in B^{-1}_{-s}(\partial\Omega)$ *such that u is also a solution of the Neumann problem* (9.2) *with boundary datum g. In addition, there holds*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{B^{-1}_{-s}(\partial\Omega)} \leqslant C(\Omega, s)\, \|f\|_{B^1_{1-s}(\partial\Omega)}. \tag{10.3}$$

As a corollary of this and Theorem 9.3 (cf. also the remark following its statement), we see that under the hypotheses of the above theorem the Dirichlet-to-Neumann operator for L is actually an isomorphism of $B^1_{1-s}(\partial\Omega)$ onto $B^{-1}_{1-s}(\partial\Omega)$ if $V > 0$ on a set of positive measure in Ω (in fact a similar result also holds when $V \equiv 0$ in $\bar{\Omega}$).

Proof of Theorem 10.1. We start with the existence part. As in [FMM], we develop an approach which will eventually give us information about the normal derivative of the solution.

Choose s_0 as in Theorem 9.3 and fix an arbitrary $s \in (0, s_0)$. First, we shall prove a technical result to the effect that there exists $C = C(\Omega) > 0$ such that, if $p \in ((n-1)/n, 1)$ is so that $s = (n-1)(\frac{1}{p} - 1)$, then

$$\|S^{-1}a\|_{\mathfrak{h}^p(\partial\Omega)} \leqslant C, \qquad \forall a B^1_{1-s}(\partial\Omega)\text{-atom}. \tag{10.4}$$

In the above, $B^1_{1-s}(\partial\Omega)$-atoms are regarded as elements in $L^2_1(\partial\Omega)$ so that $S^{-1}a$ belongs to $L^2(\partial\Omega) \subset \mathfrak{h}^p(\partial\Omega)$. In order to prove (10.4), the key step is to establish the estimate

$$\|f\|_{\mathfrak{h}^p(\partial\Omega)} \leqslant C\, \|\nabla_{\tan} Sf\|_{\mathfrak{h}^p(\partial\Omega)} + C\, \|Sf\|_{L^1(\partial\Omega)}, \tag{10.5}$$

for $1 - \varepsilon < p \leqslant 1$; here $C = C(\Omega) > 0$ is independent of $f \in \mathfrak{h}^p(\partial\Omega)$. Indeed, choosing $f := S^{-1}a$ in (10.5) yields (10.4) at once in view of (4.5). Note that s small guarantees that $|p-1| < \varepsilon$.

As for (10.5), we note that, for $p = 1$, this follows from [MT2, Theorem 6.3] with Proposition 3.2 of [MT2] via the usual jump-relations. Also, $\nabla_{\tan} S$ is a bounded mapping of the complex interpolation scale consisting of $\mathfrak{h}^p(\partial\Omega)$ for $(n-1)/n < p \leqslant 1$ and $L^p(\partial\Omega)$ for $1 < p < \infty$ into itself (cf. [MT2, Proposition B.6]). Consequently, by [KM], an estimate like (10.5) is stable under small perturbations of the parameter p near 1. This proves (10.5) and, hence, concludes the proof of (10.4).

Returning to the main line of reasoning, fix an arbitrary $f \in B^1_{1-s}(\partial\Omega)$. By Proposition 4.3, there exists a sequence of scalars $(\lambda_j)_j \in \ell^1$ and a sequence $(a_j)_j$ of $B^1_{1-s}(\partial\Omega)$-atoms such that

$$f = \sum_{j \geqslant 0} \lambda_j a_j \qquad \text{and} \qquad \sum_{j \geqslant 0} |\lambda_j| \leqslant 2\, \|f\|_{B^1_{1-s}(\partial\Omega)}. \tag{10.6}$$

If we now set $u_j := \mathscr{S}(S^{-1}a_j)$ in Ω then, so we claim, $u := \sum_{j \geqslant 0} \lambda_j u_j$ in Ω solves (10.1). To justify this observe that $Lu = 0$ and

$$\begin{aligned} \|u\|_{B^1_{2-s}(\Omega)} &+ \|\operatorname{dist}(\cdot, \partial\Omega)^{s-1}\, |\nabla u|\|_{L^1(\Omega)} + \|u\|_{L^1(\Omega)} \\ &\leqslant C \sum_{j \geqslant 0} |\lambda_j| \leqslant C\, \|f\|_{B^1_{1-s}(\partial\Omega)} = C\, \|\operatorname{Tr} u\|_{B^1_{1-s}(\partial\Omega)} \\ &\leqslant C\, \|u\|_{B^1_{2-s}(\Omega)}, \end{aligned} \tag{10.7}$$

by Theorem 7.1, (7.25), and (10.4). Let us point out that the estimate (10.7) takes care of Lemma 9.2, stated without proof in the previous section.

It is also implicit in the calculation above that $\sum_{j=0}^{m} \lambda_j u_j \to u$ in $B^1_{2-s}(\Omega)$ as $m \to \infty$ so that, by the continuity of the trace operator,

$$\operatorname{Tr} u = \sum_{j \geqslant 0} \lambda_j \operatorname{Tr} u_j = \sum_{j \geqslant 0} \lambda_j a_j = f, \tag{10.8}$$

in $B^1_{1-s}(\partial\Omega)$, which concludes the proof of the existence part.

Turning to uniqueness, let us assume that u solves the homogeneous version of (10.1). Adapting an idea from [JK2] consider $\Omega_j \nearrow \Omega$ an approximating sequence of smooth subdomains of Ω and, since $u \in B^1_{2-s,0}(\Omega)$, take $u_j \in C^\infty_{\mathrm{comp}}(\Omega_j)$ approximating u in the norm of $B^1_{2-s}(\Omega)$. Thus, with Tr_j standing for the trace operator on $\partial\Omega_j$,

$$\|\operatorname{Tr}_j u\|_{B^1_{1-s}(\partial\Omega_j)} = \|\operatorname{Tr}_j (u - u_j)\|_{B^1_{1-s}(\partial\Omega_j)} \leqslant C\, \|u - u_j\|_{B^1_{2-s}(\Omega)} \to 0, \tag{10.9}$$

as $j \to \infty$. Let us next assume for a moment that Ω is a smooth domain and denote by $G_j(x, y)$ is the Green function for L in Ω_j. Then $\|\nabla_y G_j(x, \cdot)\|_{L^p(\partial\Omega_j)} \leqslant C(x, p) < +\infty$ uniformly in j, for each $p > 1$. This, (10.9) and the integral representation formula

$$u(x) = \int_{\partial\Omega_j} \partial_{\nu_{j,y}} G_j(x, y)(\operatorname{Tr}_j u)(y)\, d\sigma_j(y), \qquad x \in \Omega_j, \tag{10.10}$$

allow us to conclude, upon letting $j \to \infty$, that $u(x) = 0$. Thus, since x was arbitrary, u vanishes in Ω and this concludes the proof of the uniqueness part when Ω is smooth.

Returning now to the general case of an arbitrary Lipschitz domain, by what we have proved so far (i.e., existence and estimates in Lipschitz domains plus uniqueness in smooth domains) we deduce that, in each Ω_j, the estimate

$$\|u\|_{B^1_{2-s}(\Omega_j)} \leqslant C\, \|\operatorname{Tr}_j u\|_{B^1_{1-s}(\partial\Omega_j)} \tag{10.11}$$

holds with a constant independent of j. Now the desired conclusion follows from (10.11) by passing to the limit in j and invoking (10.9).

In order to show that u constructed above solves a Neumann problem (in the sense of Section 9) with an appropriate boundary datum in $B^{1}_{-s}(\partial\Omega) = \hat{\mathfrak{h}}^p(\partial\Omega)$ it suffices to check that

$$\frac{\partial u_j}{\partial \nu} \in \mathfrak{h}^p(\partial\Omega) \qquad \text{and} \qquad \left\| \frac{\partial u_j}{\partial \nu} \right\|_{\mathfrak{h}^p(\partial\Omega)} \leqslant C, \qquad \text{uniformly in } j. \tag{10.12}$$

However, this easily follows from the integral representation of each u_j, the results in Section 6 and the boundedness of K^* on $\mathfrak{h}^p(\partial\Omega)$ (cf. [MT2, Proposition B.6]). Finally, (10.3) also follows in light of (10.12) and (10.6). ∎

11. INVERTIBILITY OF BOUNDARY INTEGRAL OPERATORS ON BESOV SPACES

Again, we assume the standard hypotheses on M, Ω, V and the metric tensor, including (1.27). To state the main result of this section, recall that $\mathscr{C}$ stands for the collection of all constant functions on $\partial\Omega$ and set

$$\widetilde{B}^p_{-s}(\partial\Omega) := \{f \in B^p_{-s}(\partial\Omega) : \langle f, \chi\rangle = 0, \forall \chi \in \mathscr{C}\}, \tag{11.1}$$

for $1 \leqslant p \leqslant \infty$, $0 < s < 1$.

THEOREM 11.1. *There exists* $\varepsilon \in (0, 1]$ *with the following significance. Let* $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$ *be so that one of the conditions* (I)–(III) *below are satisfied*:

$$\begin{aligned} &\text{(I)} && \frac{2}{1+\varepsilon} < p < \frac{2}{1-\varepsilon} && \textit{and} && 0 < s < 1;\\ &\text{(II)} && 1 \leqslant p < \frac{2}{1+\varepsilon} && \textit{and} && \frac{2}{p} - 1 - \varepsilon < s < 1;\\ &\text{(III)} && \frac{2}{1-\varepsilon} < p \leqslant \infty && \textit{and} && 0 < s < \frac{2}{p} + \varepsilon. \end{aligned} \tag{11.2}$$

Also, let $q \in [1, \infty]$ *denote the conjugate exponent of p. Then the operators listed below are invertible*:

(1) $\frac{1}{2}I + K$: $B^p_s(\partial\Omega) \to B^p_s(\partial\Omega)$;

(2) $\frac{1}{2}I + K^*$: $B^q_{-s}(\partial\Omega) \to B^q_{-s}(\partial\Omega)$;

(3) S: $B^q_{-s}(\partial\Omega) \to B^q_{1-s}(\partial\Omega)$.

If $V = 0$ *in* Ω, *then*

(4) $\pm\frac{1}{2}I + K$: $B^p_s(\partial\Omega)/\mathscr{C} \to B^p_s(\partial\Omega)/\mathscr{C}$;

(5) $\pm\frac{1}{2}I + K^*$: $\widetilde{B}^q_{-s}(\partial\Omega) \to \widetilde{B}^q_{-s}(\partial\Omega)$;

(6) S: $\widetilde{B}^q_{-s}(\partial\Omega) \to B^q_{1-s}(\partial\Omega)/\mathscr{C}$,

are also invertible. Finally, when $V>0$ *on a set of positive measure in* Ω, *then*

(7) $-\frac{1}{2}I+K\colon B^p_s(\partial\Omega)\to B^p_s(\partial\Omega)$;

(8) $-\frac{1}{2}I+K^*\colon B^q_{-s}(\partial\Omega)\to B^q_{-s}(\partial\Omega)$,

are invertible.

These results are sharp in the class of Lipschitz domains. However, if $\partial\Omega\in C^1$ *then we may take* $1\leqslant p\leqslant\infty$ *and* $0<s<1$.

For each $0\leqslant\varepsilon\leqslant 1$ consider the region $\mathscr{R}_\varepsilon\subset\mathbb{R}^2$ which is the interior of the hexagon $OABCDE$, where $O=(0,0)$, $A=(\varepsilon,0)$, $B=(1,\frac{1-\varepsilon}{2})$, $C=(1,1)$, $D=(1-\varepsilon,1)$, $E=(0,\frac{1+\varepsilon}{2})$. Then the "invertibility" region described in (11.2) simply says that $(s,1/p)$ belongs to $\mathscr{R}_\varepsilon$ or, possibly, to the (open) segments OA, CD. Note that the region encompassed by the parallelogram with vertices at $(0,0)$, $(1,\frac{1}{2})$, $(1,1)$, and $(0,\frac{1}{2})$ is common for all Lipschitz domains, and that $\mathscr{R}_\varepsilon$ can be thought of as an enhancement of it. Also, for $\varepsilon=1$, $\mathscr{R}_\varepsilon$ simply becomes the standard (open) unit square in the plane. The sense in which this result is optimal is that for each $\varepsilon>0$ and for each point $(s,\frac{1}{p})\in(0,1)\times(0,1)\backslash\mathscr{R}_\varepsilon$, there exists a Lipschitz domain Ω such that (1)–(8) in Theorem 11.1 fail.

The proof of Theorem 11.1 uses interpolation and several special cases of interest are singled out below.

PROPOSITION 11.2. *There exists* $s_0=s_0(\Omega)>0$ *such that for* $0<s<s_0$ *the operators*

$$S\colon B^1_{-s}(\partial\Omega)\to B^1_{1-s}(\partial\Omega), \tag{11.3}$$

$$S\colon B^\infty_{s-1}(\partial\Omega)\to B^\infty_s(\partial\Omega), \tag{11.4}$$

$$\tfrac{1}{2}I+K\colon B^1_{1-s}(\partial\Omega)\to B^1_{1-s}(\partial\Omega), \tag{11.5}$$

are isomorphisms.

Proof. Note that, by the results in Sections 7 and 8, they are well-defined and bounded. Consider some $f\in B^1_{1-s}(\partial\Omega)$ which has an atomic decomposition of the form $f=\sum\lambda_i a_i$, where $(\lambda_i)_i\in\ell^1$ and the a_i's are $B^1_{1-s}(\partial\Omega)$-atoms. By (10.4), we can find $h_i\in\mathfrak{h}^p(\partial\Omega)$ so that $\|h_i\|_{\mathfrak{h}^p(\partial\Omega)}\leqslant C$ and $Sh_i=a_i$. Then $\sum\lambda_i h_i$ converges in $\hat{\mathfrak{h}}^p(\partial\Omega)$ to an element g which S should send into f. By Proposition 5.2, $f\in B^1_{-s}(\partial\Omega)$. Thus, S in (11.3) is onto.

To see that S is also one-to-one, take some $f\in B^1_{-s}(\partial\Omega)=\hat{\mathfrak{h}}^p(\partial\Omega)$ so that $Sf=0$. It follows from the uniqueness for the Dirichlet problem in $\Omega_\pm$ with data in $B^1_{1-s}(\partial\Omega)$ that $\mathscr{S}f$ must vanish identically both in $\Omega_+:=\Omega$ and in $\Omega_-:=\mathbb{R}^n\backslash\bar{\Omega}$. See Section 10. Now, since f is the jump of $\partial_\nu\mathscr{S}f$ across $\partial\Omega$, we infer that $f=0$. This proves that the operator in (11.3) is invertible.

The invertibility of the operator (11.4) follows by duality from what we have just proved. Finally, the isomorphism (11.3) and the fact that $SK^* = KS$ (itself a simple consequence of Green's integral representation formula; cf. [MT, (7.41)]) imply that the operator (11.5) is also an isomorphism, given that

$$\tfrac{1}{2}I + K^*\colon B^1_{-s}(\partial\Omega) \to B^1_{-s}(\partial\Omega) \tag{11.6}$$

is an isomorphism. The proof of this is similar to, but simpler than, the proof of (9.7). ∎

Let us digress for a moment and point out that a proof of Proposition 11.2 can also be given based on the atomic theory from [MT2] plus a recent functional analytic result from [MM2]. In order to be more specific, we need some notation.

Call f an atom for $\mathfrak{H}^p_1(\partial\Omega)$, for $(n-1)/n < p \leqslant 1$, if it is supported in a surface ball of radius $r \in (0, \operatorname{diam} \Omega]$ and $\|\nabla_{\tan} f\|_{L^2(\partial\Omega)} \leqslant r^{(n-1)(1/2-1/p)}$. Then $\mathfrak{H}^p_1(\partial\Omega)$ is defined as the ℓ^p-span of such atoms (and is equipped with the natural quasi-norm). Also, for $(n-1)/n < p \leqslant 1$, introduce the local version $\mathfrak{h}^p_1(\partial\Omega) := L^q_1(\partial\Omega) + \mathfrak{H}^p_1(\partial\Omega)$, some $q > 1$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ so that

$$S\colon \mathfrak{h}^p(\partial\Omega) \to \mathfrak{h}^p_1(\partial\Omega), \qquad \tfrac{1}{2}I + K\colon \mathfrak{h}^p_1(\partial\Omega) \to \mathfrak{h}^p_1(\partial\Omega) \tag{11.7}$$

are isomorphisms for $1 - \varepsilon < p \leqslant 1$. Indeed, when $p = 1$ this follows as in [DK], granted the results of [MT2]. It then further extends to a small interval about $p = 1$ by general stability results for complex interpolation scales of quasi-Banach spaces from [KM].

The second ingredient we need is a result from [MM2] to the effect that

$$\begin{gathered} \widehat{F^{p,q}_\alpha(\mathbb{R}^{n-1})} = B^{1,1}_{\alpha+(n-1)(1-1/p)}(\mathbb{R}^{n-1}), \\ \forall p \in (0, 1), \qquad q \in (0, \infty), \qquad \alpha \in \mathbb{R}. \end{gathered} \tag{11.8}$$

Here $F^{p,q}_\alpha(\mathbb{R}^n)$ is the class of Triebel–Lizorkin spaces in $\mathbb{R}^n$ (cf., e.g., [Tr]), and hat denotes the Banach envelope (cf. Section 5). Our interest in the Triebel–Lizorkin scale $F^{p,q}_\alpha$ stems from the identifications $\mathfrak{H}^p = F^{p,2}_0$ and $\mathfrak{H}^p_1 = F^{p,2}_1$ valid for $0 < p \leqslant 1$ (cf. the discussion in [MM2]). When $(n-1)/n < p < 1$ and $0 \leqslant \alpha \leqslant 1$, the same results remain true with $\mathbb{R}^{n-1}$ replaced by the boundary of a (n-dimensional) Lipschitz domain Ω. Thus, applying the "hat" to (11.7) (in effect, invoking Lemma 5.1) yields (11.3) and (11.5) at once.

We now return to the task of presenting the

Proof of Theorem 11.1. For simplicity, assume that $V>0$ on a set of positive measure in Ω; the remaining cases require only minor alterations and are left to the reader.

We deal first with the operator $\frac{1}{2}I+K$. The segments (E, O) and (B, C) corresponding to invertibility on $L^p(\partial\Omega)$ for $0<1/p<(1+\varepsilon)/2$, and on $L^p_1(\partial\Omega)$ for $(1-\varepsilon)/2<1/p<1$, respectively, have been treated in [MT2]. Also, the segment (O, A), corresponding to invertibility on $B^{\infty}_s(\partial\Omega)$ for s small, has been taken care of in [MT2], while invertibility on the segment (C, D) is covered by Proposition 11.2. Notice that the interior of the convex hull of these segments is precisely the region $\mathscr{R}_\varepsilon$. Now, the desired result follows by repeated applications of the real and complex methods of interpolation together with a routine check that the inverses $T_{s,p}$ of $\frac{1}{2}I+K$ coincide on the intersections of the various spaces just considered, so that both $\frac{1}{2}I+K$ and $T_{s,p}$ can simultaneously be interpolated.

By duality, we obtain results for $\frac{1}{2}I+K^*$ on the corresponding dual scales. Similar interpolation arguments apply to yield the statements made about $-\frac{1}{2}I+K$ and $-\frac{1}{2}I+K^*$ from what has been established before. Furthermore, a similar reasoning applies to the operator S, given Proposition 11.2 and the results in [MT2, Sect. 7] on S, namely

$$S\colon L^p(\partial\Omega) \simeq L^p_1(\partial\Omega), \qquad 1<p<2+\varepsilon, \tag{11.9}$$

plus, by duality,

$$S\colon L^q_{-1}(\partial\Omega) \simeq L^q(\partial\Omega), \qquad 2-\varepsilon<q<\infty. \tag{11.10}$$

The only novel point here is to check that $B^1_{-s}(\partial\Omega)$ interpolates "well" with the scale $\{B^p_{-s}(\partial\Omega)\}$, $p\in(1,\infty)$, $s\in(0,1)$. That is, we need

$$[B^1_{-s_0}(\partial\Omega), B^p_{-s_1}(\partial\Omega)]_\theta = B^{p_\theta}_{-s_\theta}(\partial\Omega), \qquad \theta\in(0,1), \tag{11.11}$$

for $s_\theta := (1-\theta)\,s_0+\theta s_1$ and $1/p_\theta := (1-\theta)-\theta/p$, whenever $0<s_0, s_1<1$, $1<p<\infty$. Note that the space in the left side is reflexive, as the intermediate space between two Banach spaces one of which is reflexive. Thus, by the duality theorem for the complex interpolation method [BL],

$$[B^1_{-s_0}(\partial\Omega), B^p_{-s_1}(\partial\Omega)]_\theta = ([B^{\infty}_{s_0}(\partial\Omega), B^q_{s_1}(\partial\Omega)]_\theta)^* = (B^{q/\theta}_{s_\theta}(\partial\Omega))^*, \tag{11.12}$$

where $1/p+1/q=1$. From this, the desired conclusion follows.

The fact that these results are sharp in the class of Lipschitz domains is discussed in [FMM]. Finally, that $\partial\Omega\in C^1$ allows us to take $1\leqslant p\leqslant\infty$ and $0<s<1$ follows from the results in [MT, MMT, MT2] via the same interpolation patterns. ∎

Remark. We mention that the result (1) in Theorem 11.1, on the invertibility of $\frac{1}{2}I+K$, was established in (7.16) of [MT2] for $(s, 1/p)$ in the smaller region $\mathscr{P}_\varepsilon$, the interior of the parallelogram with vertices *OBCE*. There, the metric tensor was assumed to be Lipschitz. In view of [MT3, Theorem 7.1] this result holds even when the metric tensor is Hölder continuous. Similarly the other results in Theorem 11.1 hold for $(s, 1/p) \in \mathscr{P}_\varepsilon$, when the metric tensor is Hölder continuous.

12. THE GENERAL POISSON PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

Again, retain the standard hypotheses on M, Ω, V and the metric tensor, including (1.27). The first order of business is to properly formulate the Poisson problem for the operator L with Neumann boundary conditions in the Lipschitz domain $\Omega \subset M$. We commence by defining the normal component of any 1-form F with components in $L^q_{-s+1/q}(\Omega)$ for $0<s<1$ and $1<p, q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$.

Concretely, for an (arbitrary) extension $f \in (L^p_{s+1/p}(\Omega))^* = L^q_{-s-1/p,\,0}(\Omega)$ of the distribution $\delta F \in (C^\infty_{\rm comp}(\Omega))'$ (as usual, $\langle \delta F, \phi\rangle = \langle F, d\phi\rangle$, for each $\phi \in C^\infty_{\rm comp}(\Omega)$), we denote by $\nu_f \cdot F$ the (scalar) normal component of F, with respect to the extension f. This is defined as the linear functional in $B^q_{-s}(\partial\Omega) = (B^p_s(\partial\Omega))^*$ given by

$$\langle \nu_f \cdot F, \phi\rangle := \langle f, \tilde{\phi}\rangle + \langle F, d\tilde{\phi}\rangle, \qquad \forall \phi \in B^p_s(\partial\Omega), \tag{12.1}$$

where $\tilde{\phi} \in L^p_{s+1/p}(\Omega)$ is an extension (in the trace sense) of ϕ. The second pairing in the right side of (12.1) is understood in the sense of (4.15) and is well defined since $d\tilde{\phi} \in L^p_{s+1/p-1}(\Omega)$. In turn, this membership is a consequence of our assumptions and (4.16).

It is not difficult to check that the definition is correct and that

$$\|\nu_f \cdot F\|_{B^q_{-s}(\partial\Omega)} \leqslant C\,\|F\|_{L^q_{-s+1/q}(\Omega)} + C\,\|f\|_{(L^p_{s+1/p}(\Omega))^*}. \tag{12.2}$$

Note that for any function $u \in L^q_{1-s+1/q}(\Omega)$ and any extension f of $Lu = \delta\, du - Vu$, considered first as a distribution in Ω to an element in $L^q_{-1-s+1/q,\,0}(\Omega)$, the "normal derivative" $\partial^f_\nu u$ can be defined (with respect to the extension f), in the sense of (12.1), as $\nu_{f+Vu} \cdot du$.

For further reference we also note the integral formulas

$$u = \Pi_0 f + \mathscr{D}(\operatorname{Tr} u) - \mathscr{S}(\partial^f_\nu u), \tag{12.3}$$

valid for arbitrary $u \in L^q_{1-s+1/q}(\Omega)$ with Lu extendible to $f \in (L^p_{s+1/p}(\Omega))^*$. Here, $1<p<\infty$, $0<s<1$ and Π_0 is just the Newtonian potential on scalar-valued functions. Also,

$$\langle \partial^f_\nu \Pi_0(f), \phi \rangle = \langle f, \mathscr{D}\phi \rangle, \qquad \forall \phi \in B^p_s(\partial\Omega), \tag{12.4}$$

is valid for any $f \in (L^p_{s+1/p}(\Omega))^*$ for $1<p<\infty$, $0<s<1$. They can be easily justified starting from (12.1), using a limiting argument and invoking the mapping properties of Π_0, $\mathscr{S}$, $\mathscr{D}$ established in Sections 6–8.

The main focus of this section is the boundary problem

$$\begin{cases} Lu = f \in L^q_{1/q-s-1,\,0}(\Omega), \\ \partial^f_\nu u = g \in B^q_{-s}(\partial\Omega), \\ u \in L^q_{1-s+1/q}(\Omega). \end{cases} \tag{12.5}$$

When $V=0$ in Ω, this is subject to the (necessary) compatibility condition

$$\langle f, 1 \rangle = \langle g, 1 \rangle. \tag{12.6}$$

In this regard, our main result is the following.

THEOREM 12.1. *Assume that the metric tensor satisfies* (1.27). *There exists* $\varepsilon = \varepsilon(\Omega) > 0$ *having the following property.*

Suppose that $p \in (1, \infty)$ *and* $s \in (0, 1)$ *are such that one of the conditions* (I)–(III) *in* (11.2) *is satisfied. Also, let* q *be the conjugate exponent of* p. *Then, if* $V>0$ *on a subset of positive measure in* Ω, *the Poisson problem with Neumann boundary condition* (12.5) *has a unique solution. In fact,*

$$u = \Pi_0(f) + \mathscr{S}\left(-\tfrac{1}{2}I + K^*\right)^{-1}\left(g - \partial^f_\nu \Pi_0(f)\right) \tag{12.7}$$

and there exists a positive constant C *which depends only on* Ω, p, s, *such that*

$$\|u\|_{L^q_{1-s+1/q}(\Omega)} \leqslant C\,\|f\|_{L^q_{1/q-s-1,\,0}(\Omega)} + C\,\|g\|_{B^q_{-s}(\partial\Omega)}. \tag{12.8}$$

A similar set of results is valid when $V=0$ *in* Ω. *In this case, the compatibility condition* (12.6) *is assumed and the solution is unique modulo additive constants* (*also,* (12.8) *must be modified accordingly*).

Finally, if $\partial\Omega \in C^1$ *then we may take* $\varepsilon = 1$. *That is to say, the conclusions hold for all* $p \in (1, \infty)$, $s \in (0, 1)$.

Remark. It should be noted that similar results are valid for the scales of Besov spaces, i.e., when $f \in (B^p_{s+1/p}(\Omega))^*$. In this case the solution u belongs to $B^q_{1-s+1/q}(\Omega)$ and the second pairing in the right side of (12.1) remains meaningful because of [Gr, Theorem 1.4.4.6 and Corollary 1.4.4.5].

Proof of Theorem 12.1. Assume first that V does not vanish in Ω; the other case is handled similarly. In view of Proposition 6.1 and (12.4), subtracting $\Pi_0(f)$ reduces the problem to solving (12.5) with $f=0$ and $\tilde{g} := g - \partial_\nu^f \Pi_0(f) \in B^q_{-s}(\partial\Omega)$. For this latter problem, a solution is given by $\mathscr{S}[(-\frac{1}{2}I+K^*)^{-1}\tilde{g}]$; cf. Theorem 7.1 and Theorem 11.1. This finishes the proof of the existence part. Note that (12.8) follows from the integral representation formula (12.7) of the solution and mapping properties of layer potentials.

It remains to establish uniqueness. To this end, if $u \in L^q_{1-s+1/q}(\Omega)$ solves the homogeneous version of (12.5), then taking the boundary trace in (12.3) readily gives that $(-\frac{1}{2}I+K)(\mathrm{Tr}\, u)=0$. The important thing is that the region $\mathscr{R}_\varepsilon$ is invariant to the transformation $(s, \frac{1}{p}) \mapsto (1-s, 1-\frac{1}{p})$ and that $\mathrm{Tr}\, u \in B^q_{1-s}(\partial\Omega)$. Thus, on account of Theorem 11.1, $\mathrm{Tr}\, u \equiv 0$. Utilizing this back in (12.6) yields $u \equiv 0$ in Ω, as desired.

The argument for C^1 domains is similar and, hence, omitted. ∎

An important particular case, corresponding to $s=-\frac{1}{p}$, is singled out below.

COROLLARY 12.2. *There exists $\varepsilon = \varepsilon(\Omega) > 0$ so that, if $\frac{3}{2}-\varepsilon < p < 3+\varepsilon$, then for any $f \in L^p_{-1,0}(\Omega)$ and any $g \in B^p_{-1p}(\partial\Omega)$, satisfying the compatibility condition* (12.6) *if $V=0$ in Ω, the Neumann problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \partial_\nu^f u = g & \text{on } \partial\Omega, \\ u \in L^p_1(\Omega), \end{cases} \tag{12.9}$$

has a unique (*modulo additive constants, if $V=0$ in Ω*) *solution u. Moreover, ∇u satisfies the estimate*

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C(\Omega, p)(\|f\|_{L^p_{-1,0}(\Omega)} + \|g\|_{B^p_{-1/p}(\partial\Omega)}). \tag{12.10}$$

If $\partial\Omega \in C^1$, this holds for all $p \in (1, \infty)$.

Proof. One only needs to observe that $(\frac{1}{p}, 1-\frac{1}{p}) \in \mathscr{R}_\varepsilon$ for p in a neighborhood of the interval $[\frac{3}{2}, 3]$. ∎

In view of the counterexamples in [FMM], the results in this section are sharp in the class of Lipschitz domains.

Proof. remark Since $\partial_\nu^f u$ is defined for a class of functions u for which the notion of the trace of $\partial_\nu u$ is utterly ill defined, it is appropriate to explain that $\partial_\nu^f u$ is *not an extension* of the operation of taking the trace of $\partial_\nu u$; perhaps it is useful to regard it as a "renormalization" of this trace, in a fashion that depends strongly on the choice of f. Recall that, for

$u \in L^q_{1-s+1/q}(\Omega)$, Lu is naturally defined as a linear functional on the space $L^p_{s+1/p,0}(\Omega)$, which for $s \in (0,1)$ coincides with the closure of $C_0^\infty(\Omega)$ in $L^p_{s+1/p}(\Omega)$. The choice of f is the choice of an extension of this linear functional to an element of $(L^p_{s+1/q}(\Omega))^* = L^q_{1/q-s-1,0}(\Omega)$.

As an example, consider $u \in L^2_1(\Omega)$ and suppose that actually $u \in L^2_2(\Omega)$, so $\partial_\nu u \in L^2_{1/2}(\Omega)$ is well defined. In this case, $Lu \in L^2(\Omega)$ has a "natural" extension $f_0 \in L^2_{-1,0}(\Omega)$. Any other extension $f_1 \in L^2_{-1,0}(\Omega)$ differs from f_0 by an element of $L^2_{-1}(M)$ supported on $\partial\Omega$. We have

$$\partial_\nu^{f_0} u = \partial_\nu u, \tag{12.11}$$

but if $f_1 = f_0 + \omega$ and $\omega \neq 0$ is supported on $\partial\Omega$, then $\partial_\nu^{f_1} u$ is not equal to $\partial_\nu u$; we leave its computation as an exercise.

13. THE GENERAL POISSON PROBLEM WITH DIRICHLET BOUNDARY CONDITIONS

We once again retain the usual set of hypotheses on M, Ω, V and the metric tensor, including (1.27). In this section we shall deal with the Poisson problem for L with Dirichlet boundary conditions and data in Sobolev–Besov spaces. As such, this extends previous work for constant coefficient operators in [JK2, FMM]. Sharpness of the range of (s, p) for which the following theorem holds was established, for the class of Lipschitz domains in Euclidean space, in [JK2].

THEOREM 13.1. *Assume that the metric tensor satisfies* (1.27). *Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following property. If $p \in (1, \infty)$ and $s \in (0,1)$ are such that one of the conditions* (I)–(III) *in* (11.2) *are satisfied, then for any $f \in L^p_{s+1/p-2}(\Omega)$ and any $g \in B^p_s(\partial\Omega)$ the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \operatorname{Tr} u = g & \text{on } \partial\Omega, \\ u \in L^p_{s+1/p}(\Omega), \end{cases} \tag{13.1}$$

has a unique solution. Also, there exists $C > 0$ depending only on Ω, p, s, such that the solution satisfies the estimate

$$\|u\|_{L^p_{s+1/p}(\Omega)} \leqslant C \|f\|_{L^p_{1/p+s-2}(\Omega)} + C \|g\|_{B^p_s(\partial\Omega)}. \tag{13.2}$$

Moreover, if Π denotes the Newtonian potential for L on M, we can write

$$\begin{aligned} u &= \Pi(\tilde{f})|_{\Omega} + \mathscr{D}((\tfrac{1}{2}I+K)^{-1}(g - \operatorname{Tr} \Pi(\tilde{f}))) \\ &= \Pi(\tilde{f})|_{\Omega} + \mathscr{S}(S^{-1}(g - \operatorname{Tr} \Pi(\tilde{f}))) \qquad \text{in } \Omega, \end{aligned} \tag{13.3}$$

where $\tilde{f}$ is an extension of f to an element in $L^p_{s+1/p-2}(M)$.

In particular, if $f \in L^p_{1/p+s-2,0}(\Omega)$, then the solution has a normal derivative in $B^p_{s-1}(\partial\Omega)$ (in the sense discussed in Section 12) and

$$\|\partial^f_\nu u\|_{B^p_{s-1}(\partial\Omega)} \leqslant C(\|f\|_{L^p_{1/p+s-2,0}(\Omega)} + \|g\|_{B^p_s(\partial\Omega)}). \tag{13.4}$$

Similar results are valid on the scale of Besov spaces, i.e., when $f \in B^p_{1/p+s-2}(\Omega)$. In this case, the solution u belongs to $B^p_{s+1/p}(\Omega)$.

Finally, if $\partial\Omega \in C^1$ then we can actually take $\varepsilon = 1$, i.e., the conclusions hold for all $p \in (1, \infty)$, $s \in (0, 1)$.

Proof. Fix an arbitrary $f \in L^p_{1/p+s-2}(\Omega) = (L^q_{1+1/q-s,0}(\Omega))^*$. Since $L^q_{1+1q-s,0}(\Omega)$ can be identified (via extension by zero outside the support and restriction to Ω) with $\{\psi \in L^q_{1+1q-s}(M) : \operatorname{supp} \psi \subseteq \bar{\Omega}\}$, we may invoke the Hahn–Banach extension theorem to produce $\tilde{f} \in L^p_{1/p+s-2}(M)$ so that $\tilde{f}|_\Omega = f$ and the norm of $\tilde{f}$ is controlled by that of f. Now, clearly, (13.3) solves (13.1). Note that (13.2) and (13.4) also follow from (13.3) and the mapping properties of the operators involved. Uniqueness can be established by mimicking the argument already utilized in the proof of Theorem 12.1.

Finally, invoking [MT, MMT] and proceeding as before, it is clear that we may take $\varepsilon = 1$ if $\partial\Omega \in C^1$. ∎

COROLLARY 13.2. *There exists $\varepsilon = \varepsilon(\Omega) > 0$ so that if $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ then for any $f \in L^p_{-1}(\Omega)$ and any $g \in B^p_{1-1/p}(\partial\Omega)$ the Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \operatorname{Tr} u = g & \text{on } \partial\Omega, \\ u \in L^p_1(\Omega), \end{cases} \tag{13.5}$$

has a unique solution. This satisfies

$$\|u\|_{L^1_p(\Omega)} \leqslant C\,\|f\|_{L^p_{-1}(\Omega)} + C\,\|g\|_{B^p_{1-1/p}(\partial\Omega)} \tag{13.6}$$

and, if $f \in L^p_{-1,0}(\Omega)$,

$$\|\partial^f_\nu u\|_{B^p_{-1/p}(\partial\Omega)} \leqslant C(\|f\|_{L^p_{-1,0}(\Omega)} + \|g\|_{B^p_{1-1p}(\partial\Omega)}). \tag{13.7}$$

Similar results are valid on the scale of Besov spaces. Finally, if $\partial\Omega \in C^1$ then we can actually take $1 < p < \infty$.

In the last part of this section, we elaborate on the connection between Poisson problems and Helmholtz decompositions (for the latter topic see also [FMM, MT2]). The observation we wish to make is contained in the proposition below.

PROPOSITION 13.3. *Let Ω be an arbitrary (connected) Lipschitz subdomain of M and fix $1<p, q<\infty$ with $1/p+1/q=1$. Then the following are equivalent:*

(i) *The L^p-Helmholtz decomposition*

$$L^p(\Omega, \Lambda^1 TM) = dL^p_1(\Omega) \oplus \{\omega \in L^p(\Omega, \Lambda^1 TM);\ \delta\omega = 0,\ \nu \vee \omega = 0\} \tag{13.8}$$

holds (where the direct sum is topological).

(ii) *The L^p_1-Poisson problem for the Laplace–Beltrami operator with homogeneous Neumann boundary conditions*

$$\begin{cases} \Delta u = f \in L^p_{-1,0}(\Omega), & \langle f, 1\rangle = 0, \\ \partial^f_\nu u = 0 & \text{on } \partial\Omega, \\ u \in L^p_1(\Omega)/\mathbb{R}, \end{cases} \tag{13.9}$$

is well posed.

(iii) *The L^q-Helmholtz decomposition (analogous to (13.8)) holds.*

(iv) *The L^q_1-Poisson problem for the Laplace–Beltrami operator with homogeneous Neumann boundary conditions (analogous to (13.9)) holds.*

In particular, the range of p's for which the L^p-Helmholtz decomposition (13.8) as well as the L^p_1-Poisson problem (13.9) are valid is always an interval which is invariant under taking conjugate exponents.

Proof. We proceed to show (ii) assuming that (i) is valid. To this end, let q denote the conjugate exponent of p and let $f \in L^p_{-1,0}(\Omega)$ be such that $\langle f, 1\rangle = 0$, otherwise arbitrary; thus, $f \in (L^q_1(\Omega)/\mathbb{R})^*$. Note that, by Poincaré's inequality, d maps $L^q_1(\Omega)/\mathbb{R}$ isomorphically onto a closed subspace of $L^q(\Omega, \Lambda^1 TM)$. From the Riesz and Hahn–Banach theorems we may then conclude that there exists $w \in L^p(\Omega, \Lambda^1 TM)$ so that

$$\langle f, v\rangle = \int_\Omega \langle w, dv\rangle \, d\,\mathrm{Vol}, \qquad \forall v \in L^q_1(\Omega). \tag{13.10}$$

Next, decompose $w = du + \omega$, according to (1). Then, based on (13.10), it is easy to check that the just constructed function u solves (13.9). Uniqueness and estimates for (13.9) follow from the corresponding uniqueness and estimates for (13.10).

Conversely, assume now that (ii) is well posed; our goal is to show that (i) holds. Indeed, if $w \in L^p(\Omega, \Lambda^1 TM)$ is arbitrary and fixed then $v \mapsto \int_\Omega \langle w, dv \rangle \, d\,\mathrm{Vol}$ defines a linear functional on $L^q_1(\Omega)/\mathbb{R}$. Denoting by $f \in (L^q_1(\Omega)/\mathbb{R})^*$ this functional it follows that (13.10) holds. Let now $u \in L^p_1(\Omega)$ solve the Poisson problem (13.9) for this datum f and set $\omega := w - du \in L^p(\Omega, \Lambda^1 TM)$. Then it follows that $\int_\Omega \langle \omega, dv \rangle \, d\,\mathrm{Vol} = 0$ for each $v \in L^q_1(\Omega)$. In turn, this readily implies that, first, $\delta\omega = 0$ in Ω and, second, that $\nu \vee \omega = 0$ on $\partial\Omega$. Thus, $w = du + \omega$ is the desired decomposition. Everything else is as before.

Going further, let

$$T\colon (L^2_1(\Omega))^* = L^2_{-1,0}(\Omega) \to L^2_1(\Omega) \tag{13.11}$$

be the (well defined, linear and bounded) solution operator for the problem (13.10) with $p = 2$, i.e., $T(f) = u$. Since, by Green's formula, T is self-adjoint it follows that T extends as a bounded mapping of $L^p_{-1,0}(\Omega)$ into $L^p_1(\Omega)$ if and only if T extends to a bounded mapping of $L^q_{-1,0}(\Omega)$ into $L^q_1(\Omega)$, for $1/p + 1/q = 1$. This proves that (ii) and (iv) are equivalent.

The fact that (iv) is, in turn, equivalent to (iii) is already contained in (i) $\Leftrightarrow$ (ii). Finally, the last part in the statement of the proposition follows from what we have proved so far and interpolation. ∎

In closing, let us point out that a similar result is valid for the Helmholtz decomposition

$$L^p(\Omega, \Lambda^1 TM) = dL^p_{1,0}(\Omega) \oplus \{\omega \in L^p(\Omega, \Lambda^1 TM); \delta\omega = 0\} \tag{13.12}$$

and the Poisson problem for the Laplace–Beltrami operator with homogeneous Dirichlet boundary conditions

$$\begin{cases} \Delta u = f \in L^p_{-1}(\Omega), \\ u \in L^p_{1,0}(\Omega). \end{cases} \tag{13.13}$$

We omit the details.

Of course, counterexamples to the well posedness of the Poisson problems (13.9) and (13.13) translate in the failure of the Helmholtz decompositions (13.8) and (13.12), respectively.

14. COMPLEX POWERS OF THE LAPLACE–BELTRAMI OPERATOR

If $\Delta_0 = \sum_{j=1}^{n} \partial^2/\partial x_j^2$ stands for the usual space-flat Laplacian in $\mathbb{R}^n$ then $\nabla(-\Delta_0)^{-1/2}$ corresponds, modulo a normalization constant, precisely to $(R_j)_{j=1}^{n}$, the system of Riesz transforms in $\mathbb{R}^n$. In particular, from the classical Calderón–Zygmund theory, the (vector Riesz transform) operator

$$\nabla(-\Delta_0)^{-1/2} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \tag{14.1}$$

is bounded for any $1 < p < \infty$. See [St1].

In this section, our aim is to study the analogous problem when $\mathbb{R}^n$ is replaced by Ω, a connected Lipschitz domain in a Riemannian manifold M. In this context, we shall work with the associated Laplace–Beltrami operator Δ, although similar results hold for the Schrödinger operator $L := \Delta - V$. The hypotheses that we make on M and Ω are those of Section 1.

It is natural to impose boundary conditions and we shall consider Δ_D and Δ_N, the Laplace–Beltrami operators equipped, respectively, with homogeneous Dirichlet and Neumann conditions in Ω. Thus, the natural question is whether

$$(-\Delta_D)^{-1/2} : L^p(\Omega) \to L^p_{1,0}(\Omega) \tag{14.2}$$

and

$$(-\Delta_N)^{-1/2} : \left\{ f \in L^p(\Omega) : \int_\Omega f = 0 \right\} \to L^p_1(\Omega) \tag{14.3}$$

are bounded operators. For arbitrary domains, it has been recently shown in [CD, DMc] that this is indeed the case for any $1 < p \leqslant 2$. At the other extreme, if Ω has a smooth boundary, then well known techniques based on pseudo-differential operators and Calderón–Zygmund theory allow one to take $1 < p < \infty$. For Lipschitz domains in the flat Euclidean space, the optimal range of p's turns out to be $(1, 3+\varepsilon)$; see the discussion in [JK2, JK3, MM]. Here we present a variable coefficient extension of such results.

In fact, we shall deal with more general complex powers of

$$A := (-\Delta_D)^{1/2} \qquad \text{and} \qquad B := (-\Delta_N)^{1/2}. \tag{14.4}$$

In order to state our main results, let us introduce the region $\mathscr{R} \subset \mathbb{R}^2$ given by

$$\mathscr{R} := \{(r, t) : \max\{r/3, r-1\} \leqslant t < 1, 0 \leqslant r < 2\}. \tag{14.5}$$

THEOREM 14.1. *Assume that the metric tensor satisfies* (1.27). *There exists a neighborhood* $\tilde{\mathscr{R}}$ *of* $\mathscr{R}$ *in* $\{(r, t): 0 \leqslant r < 2, 0 < t < 1\}$, *depending on* Ω, *such that for any* $(r, 1/q) \in \tilde{\mathscr{R}}$ *and* $\delta \in \mathbb{R}$, *the operators*

$$A^{-r+i\delta}: L^q(\Omega) \xrightarrow{\approx} L^q_{r,0}(\Omega), \tag{14.6}$$

$$B^{-r+i\delta}: \widehat{L^q(\Omega)} \xrightarrow{\approx} L^q_r(\Omega)/\mathbb{R} \tag{14.7}$$

are isomorphisms on the indicated spaces.

Here, for a space X of functions in Ω, $\hat{X} := \{f \in X : \int_\Omega f = 0\}$ and $X/\mathbb{R}$ is the space X modulo constants.

Proof. The proof rests on three basic ingredients: the estimates for $(-\Delta_D)^{-1}$, $(-\Delta_N)^{-1}$ from Sections 12–13, Stein's complex interpolation theorem (cf. [SW]) and L^p-bounds for purely imaginary powers of the operators $(-\Delta_D)^{1/2}$, $(-\Delta_N)^{1/2}$ (cf. [St2]). It parallels arguments in [JK2, Sect. 7], supplemented by arguments in [MM], where this program is carried out in detail in the flat Euclidean setting. To give some of the flavor, we sketch the opening arguments for (14.6). Basic Hilbert space theory gives

$$A: L^2_{1,0}(\Omega) \xrightarrow{\approx} L^2(\Omega), \qquad A: L^2(\Omega) \xrightarrow{\approx} L^2_{-1}(\Omega). \tag{14.8}$$

We then have

$$\begin{aligned} A^{s+i\gamma}&: L^2(\Omega) \xrightarrow{\approx} L^2_{-s}(\Omega), && 0 \leqslant s \leqslant 1, \\ A^{-3/2+\delta+i\gamma}&: L^2(\Omega) \xrightarrow{\approx} L^2_{3/-\delta,0}(\Omega), && 0 < \delta \leqslant \tfrac{1}{2}, \end{aligned} \tag{14.9}$$

the first by Stein interpolation, the next by applying A^{-2} and using the $p = 2$ case of Theorem 13.1. A very general result of Stein gives

$$A^{i\gamma}: L^p(\Omega) \xrightarrow{\approx} L^p(\Omega), \qquad 1 < p < \infty, \tag{14.10}$$

with an exponential bound. Then another application of Stein interpolation gives

$$A^{-1+i\gamma}: L^p(\Omega) \xrightarrow{\approx} L^p_{1,0}(\Omega), \qquad \tfrac{3}{2} < p < 3. \tag{14.11}$$

The argument proceeds with further interpolation arguments and applications of Theorem 13.1. In particular, (14.11) is extended to

$$A^{-1+i\gamma}: L^p(\Omega) \xrightarrow{\approx} L^p_{1,0}(\Omega), \qquad 1 < p < 3 + \varepsilon. \tag{14.12}$$

We refer to [JK2, MM] for further details. ∎

The next corollary follows from Theorem 14.1 much as in [JK2, JK3, MM], so we shall state it here without further proof. For the flat-space Euclidean Laplacian, similar results have also been obtained (via a different approach which in the case of (14.15)–(14.16) yields smaller intervals of p's) in [AT].

COROLLARY 14.2. *Assume that the metric tensor satisfies* (1.27). *There exists* $\varepsilon = \varepsilon(\Omega) > 0$ *so that*

$$A^{-1}: L^q(\Omega) \simeq L^q_{1,0}(\Omega) \tag{14.13}$$

and

$$B^{-1}: \widehat{L^q(\Omega)} \simeq L^q_1(\Omega)/\mathbb{R} \tag{14.14}$$

are isomorphisms for $1 < q < 3 + \varepsilon$. *In particular,*

$$\int_\Omega |\nabla f|^q \, d\,\mathrm{Vol} \leqslant C_q \int_\Omega |\sqrt{-\Delta_D}\, f|^q \, d\,\mathrm{Vol}, \qquad \forall\, f \in L^q_{1,0}(\Omega), \quad q \in (1, 3+\varepsilon), \tag{14.15}$$

and

$$\int_\Omega |\nabla g|^q \, d\,\mathrm{Vol} \leqslant C_q \int_\Omega |\sqrt{-\Delta_N}\, g|^q \, d\,\mathrm{Vol}, \qquad \forall\, g \in L^q_1(\Omega), \quad q \in (1, 3+\varepsilon). \tag{14.16}$$

Furthermore, the operators

$$A: L^q_{1,0}(\Omega) \to L^q(\Omega) \tag{14.17}$$

and

$$B: L^q_1(\Omega) \to L^q(\Omega) \tag{14.18}$$

are bounded for $1 < q < \infty$. *That is, there exists* $C = C(\Omega, q) > 0$ *so that*

$$\int_\Omega |\sqrt{-\Delta_D}\, f|^q \, d\,\mathrm{Vol} \leqslant C_q \int_\Omega |\nabla f|^q \, d\,\mathrm{Vol}, \qquad \forall f \in L^q_{1,0}(\Omega), \quad q \in (1, \infty), \tag{14.19}$$

and

$$\int_{\Omega} |\sqrt{-\Delta_N}\, g|^q \, d\,\mathrm{Vol} \leqslant C_q \int_{\Omega} |\nabla g|^q \, d\,\mathrm{Vol}, \qquad \forall\, g \in L^q_1(\Omega), \quad q \in (1, \infty). \tag{14.20}$$

Finally, this result is sharp in the class of Lipschitz domains. If, however, $\partial\Omega \in C^1$, *then* (14.15)–(14.16) *are also valid for* $1 < q < \infty$.

The last corollary contains a variable coefficient extension of a result of B. Dahlberg [Da2] concerning estimates for Green potentials in Lipschitz domains. In addition, we also treat the case of the Neumann potential.

COROLLARY 14.3. *Assume that the metric tensor satisfies* (1.27), *and consider* $G(x, y)$ *and* $N(x, y)$, *the Green and Neumann functions of the Laplace–Beltrami operator in an arbitrary Lipschitz domain* $\Omega \subset M$. *Also, denote by* G *and* N *the operators sending* f, *respectively, to* $\int_\Omega G(\cdot, y) f(y)\, d\,\mathrm{Vol}(y)$ *and* $\int_\Omega N(\cdot, y) f(y)\, d\,\mathrm{Vol}(y)$.

Then, for some $\varepsilon = \varepsilon(\Omega) > 0$,

$$\left(\int_{\Omega} |\nabla G f(x)|^q \, d\,\mathrm{Vol}(x)\right)^{1/q} \leqslant C \left(\int_{\Omega} |f(x)|^p \, d\,\mathrm{Vol}(x)\right)^{1/p}, \qquad f \in L^p(\Omega), \tag{14.21}$$

and

$$\left(\int_{\Omega} |\nabla N g(x)|^q \, d\,\mathrm{Vol}(x)\right)^{1/q} \leqslant C \left(\int_{\Omega} |g(x)|^p \, d\,\mathrm{Vol}(x)\right)^{1/p}, \qquad g \in \widehat{L^p(\Omega)}, \tag{14.22}$$

provided $1 < p < q < 3 + \varepsilon$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. *This result is sharp in the class of Lipschitz domains.*

Proof. To prove (14.21), we note that $G = (-\Delta_D)^{-1} = A^{-2}$, and

$$A^{-1}\colon L^p(\Omega) \to L^p_{1,0}(\Omega) \hookrightarrow L^q(\Omega), \qquad A^{-1}\colon L^q(\Omega) \to L^q_1(\Omega), \tag{14.23}$$

under the given hypotheses on p and q, by Theorem 14.1, indeed by (14.12). The proof of (14.22) is similar. ∎

Remark. It is perhaps worth explaining why (14.21) is not an immediate corollary of Theorem 13.1. That is, given $f \in L^p(\Omega)$, write

$u = \varDelta_D^{-1} f$ as $u = v + w$, where $v \in L_2^p(M) \subset L_1^q(M)$ solves $\varDelta v = f$ on a neighborhood of $\bar{\Omega}$ (with f extended by 0 off $\bar{\Omega}$) and w solves

$$\varDelta w = 0 \qquad \text{on } \Omega, \qquad w|_{\partial\Omega} = h = -v|_{\partial\Omega}.$$

We have $h \in B_{1-1/q}^q(\partial\Omega)$, and hence Theorem 13.1 applies, but only for $\frac{3}{2} - \varepsilon < q < 3 + \varepsilon$. This argument fails to treat the case $1 < q \leqslant \frac{3}{2} - \varepsilon$.

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