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Intersection Theory for Graphs*

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DEDICATED TO PROFESSOR WILLIAM TUTTE ON THE OCCASION OF HIS SIXTIETH BIRTHDAY

An intersection theory developed by the author for matroids embedded in uniform geometries is applied to the case when the ambient geometry is the lattice of partitions of a finite set so that the matroid is a graph. General embedding theorems when applied to graphs give new interpretations to such invariants as the dichromate of Tutte. A polynomial in $n + 1$ variables, the *polychromate*, is defined for graphs with n vertices. This invariant is shown to be strictly stronger than the dichromate, it is edge-reconstructible and can be calculated for proper graphs from the polychromate of the complementary graph. By using Tutte's construction for codichromatic graphs (*J. Combinatorial Theory* 16 (1974), 168–174), copolychromatic (and therefore codichromatic) graphs of arbitrarily high connectivity are constructed thereby solving a problem posed in Tutte's paper.

1. INTRODUCTION

In a previous paper [2], we put the theory of intersection numbers of subsets in a finite projective space (a topic of great interest in finite geometry) into the framework of matroid theory and showed, for example, the usefulness of the cardinality–corank matrix (and polynomial) for calculating these numbers. The context in which these results were proved was sufficiently general that intersection theorems for matroids embedded into structures other than projective spaces could also be stated. Applications of this general theory, however, were focused primarily on linear representations. In the present paper we concentrate on applications to graphical matroids.

In the following section we apply the general theory of [2] to the case when the ambient lattice for a matroid embedding is the lattice of partitions

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of a finite set, i.e., when the matroid is a graph. Although these embedding theorems all have more general, matroid–theoretic proofs, all results in Section 2 are stated in the language of graph theory and the reader need have no knowledge of matroids.

The combinatorial properties of the partition lattices are sufficiently similar to those of projective geometries to allow us to give "π-analogs" of many "q-representation" theorems. For example, in Section 3 we develop a formula for a two-variable chromatic polynomial (derived differently in [5]) by analogy with a similar two-variable codeweight polynomial derived in [2].

Two major properties of projective geometries not enjoyed by partition lattices are modularity and the fact that flats of equal rank have equal cardinality. Lack of modularity does not seem to be important, especially as partition lattices are "supersolvable"—a property which gives them combinatorial traits (like a complete factoring of their characteristic polynomial over the integers) similar to those of modular lattices. The other property (homogeneity) seems more essential. For projective spaces it allows one to compute intersection numbers for complements. To do this for graphs we need a more refined invariant than the matrix of intersection numbers: the "polychromatic matrix," M_G. This matrix (or corresponding polynomial, $\chi(G)$, the "polychromate") is shown to be calculable for G from the corresponding matrix of the complement G^c, as well as from the sum of the matrices of the deck of edge deletions,[1] $\{G-e : e \in G\}$. Properties of M_G are proved in Section 4 and illustrated in Section 5. We show that certain pairs of Tutte's "rotor graphs" constructed in [5] are polychromatically (as well as dichromatically) equivalent, and we modify them (for example, by joining new vertices to every vertex in each graph of the pair) to get new non-2-isomorphic codichromatic graphs with arbitrarily high connectivity (solving a problem posed by Tutte in [5]). Research problems are then given.

Finally, happy birthday to Professor William Tutte, whose study of chromatic-type invariants for graphs continues to suggest much of this author's research.

2. Definitions and General Intersection Theorems

The results of this section are proved for general matroids in [2]. The interpretation for graphs follows from standard matroid theory [3, 7, 9].

Throughout, a *graph* G is finite and can have *loops* or *multiple edges*. When it has neither it is denoted *simple.* We denote by V or $V(G)$ the vertices of G and by E or $E(G)$ its edges. Usually, $|V(G)| = n$ and

[1] In a recent paper [10] the author gives an explicit formula for M_G from the deck of vertex-deleted subgraphs of G. Thus, all remarks above apply to vertex-reconstruction as well as edge-reconstruction.

$|E(G)| = K$. We use the *embedding map* $\bar{f}: E(G) \to E(K_n) \cup 0$. Here K_n is a *complete graph* whose n vertices are identified with $V(G)$, $\bar{f}(e) = 0$ if e is a loop. Otherwise, if e joins vertices i and j in G, then f maps e to the edge joining i and j in K_n.

A *subgraph* $G(E')$ of G is a graph which consists of the vertices $V(G)$ and the edges $E' \subseteq E(G)$ with the induced incidences. Any subgraph $G(E')$ gives a vertex partition $\Pi = \Pi_{E'}(V)$, where two distinct vertices are in the same block of the partition if they are in the same connected component of $G(E')$. The partition Π is said to be of *type* $\pi(\Pi) = 1^{a_1}2^{a_2} \cdots n^{a_n}$ if there are a_i blocks of size i (so that $\sum ia_i = n$). The *weight* of a subset E' or of its partition type $\pi(\Pi_{E'})$ is defined to be the number of blocks of its partition: $w(E') = w(\pi(\Pi_{E'})) = \sum a_i$. Thus, $1 \leqslant w(E') \leqslant n$, $w(E') = 1$ if and only if E' contains a spanning tree, and $w(E') = n$ if and only if E' consists entirely of loops.

The (cycle) *matroid* structure on the set E induced from G is given by the *rank function* $r: 2^E \to \mathbb{N}$ and thus by the *corank function* $c: 2^E \to \mathbb{N}$, where $c(E') + 1 = w(E') = n - r(E')$. (Note that the rank function r^* of the cocycle matroid induced from G is given by $r^*(E') = |E'| - c(E - E')$.) If Π is a partition of V, the *partition-induced subgraph* $\Pi(G)$ consists of all the edges of G which join two vertices in the same block of Π. Thus, if $\Pi(G) = G(E')$, then $\Pi_{E'}$ refines Π. Note that $\Pi_{\Pi(G)}$ is equal to Π for all vertex partitions Π if and only if $\bar{f}$ is onto $E(K_V)$ (i.e., G contains an n-clique).

The number of partition-induced subgraphs of K_n of weight c (i.e., the number of set-partitions of an n-element set with c blocks) is the *Stirling number of the second kind*, $W^2(n, c)$. The *Stirling matrix of the second kind*, W^2_n, is the lower triangular matrix which tabulates the Stirling numbers $W^2(i, j)$ $(1 \leqslant i, j \leqslant n)$. The *Stirling number of the first kind*, $W^1(i, j)$, is the coefficient of λ^j in the polynomial $\lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - i + 1)$. If we define the *Stirling matrix of the first kind*, W^1_n, as above, and if I_n is the $n \times n$ identity matrix, then

$$W^1_n \cdot W^2_n = I_n. \tag{2.1}$$

We also have the matrix identity

$$T^+ \cdot T^- = I_{K+1}, \tag{2.2}$$

where $T^-(i, j) = (-1)^{i+j}\, T^+(i, j) = (-1)^{i+j} \binom{j}{i}$ for all $0 \leqslant i, j \leqslant K = |E(G)|$.

The *cardinality–corank polynomial* $S_{KC}(G; y, z)$ equals $\sum_{E' \subseteq E} y^{|E'|} z^{w(E')}$ or $\sum_{i,j} S^{KC}_{ij} y^i z^j$, where S^{KC}_{ij} is the number of subgraphs of G with i edges and weight j. These coefficients S^{KC}_{ij} are tabulated in the *cardinality–corank matrix* P_G, whose rows are indexed by the integers $[0, K]$ and whose columns are indexed by $[1, n]$. We then have the polynomial equation

$$S_{KC}(G; y, z) = zy^{n-1} f(G; (y + z)/y, y + 1) \tag{2.3}$$

with corresponding matrix equation

$$P_G = {}^t(T^+ \cdot F_G \cdot {}^tT^+)', \tag{2.4}$$

where $f(G; x, y)$, the *Tutte polynomial* of G, is called the *dichromate* in [5, 6] and is denoted by $t(G; x, y)$ in [2, 3] (and $\chi(G)$ in [5, 6]). The *Tutte matrix* of G, F_G, has as its (i, j)-entry the coefficient of $y^i z^j$ in the polynomial $f(G; y, z)$. Further, ${}^tQ$ is the matrix transpose of Q, and ${}^tQ'$ is the matrix ${}^tQ$ with its ith column preceded by $n - i$ zeros.

The *intersection matrix* of G, I_G, has its rows indexed by $[0, K]$ and its columns by $[1, n]$. $I_G(i, j)$ is the number of (partition-induced) subgraphs of G with i edges which come from a vertex partition of weight j. Then, the major theorem of [2], when applied to graphs, gives the formulas

$$T^- \cdot P_G \cdot W_n^2 = I_G, \tag{2.5}$$

$$T^+ \cdot I_G \cdot W_n^1 = P_G. \tag{2.6}$$

The polynomial $S_{KC}(G)$ (and thus the matrices P_G and I_G) is easily computed for certain graph-theoretic operations. For example, if G^* is dual to the connected planar graph G, then formulas in [2] yield the identities

$$S_{KC}(G^*; y, z) = \frac{y^K}{z^{n-1}} S_{KC}\left(G; \frac{z}{y}, z\right), \tag{2.7}$$

$$P_{G^*}(i, j) = P_G(K - i, i + j + n - K - 1). \tag{2.8}$$

If G_1 and G_2 are graphs, we denote by $G_1 \cup G_2$ their *union*, and by $G_1 \wedge G_2$ their *wedge* ($G_1 \cup G_2$ with a vertex of G_1 identified with a vertex of G_2). Then,

$$S_{KC}(G_1 \cup G_2; y, z) = S_{KC}(G_1; y, z) \cdot S_{KC}(G_2; y, z), \tag{2.9}$$

$$S_{KC}(G_1 \wedge G_2; y, z) = (1/z)\, S_{KC}(G_1; y, z) \cdot S_{KC}(G_2; y, z). \tag{2.10}$$

(Note that duals and wedges are not unique up to graphical isomorphism but are unique up to matroid isomorphism and therefore the above formulas are valid for any dual or wedge.)

Other theorems of [2] concern reembeddings. For example, when a new isolated vertex is added to G, giving the new graph G', then

$$P_{G'} = [\bar{0} P_G], \qquad \text{where } \bar{0} \text{ is a column of } (K + 1) \text{ zeros} \tag{2.11}$$

and $I_{G'} = I_{G'}Q$, where for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n+1$,

$$Q(i,j) = \begin{cases} 1 & \text{if } j = i+1 \\ i & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

If $V = \{v_1, v_2, ..., v_n\}$, let each loop of G be associated with the zero vector of $(F_q)^{n-1}$, and each edge $e = \{v_i, v_j\}$ be associated with the vector $\bar{v}_e(k) = \delta(i,k) - \delta(j,k)$ for all $k \in [1, n-1]$. If S is the set of vectors associated with $E(G)$ and there are $I_S(i,j)$ subspaces of $(F_q)^{n-1}$ of codimension $i-1$ (i.e., rank $n-i$) which contain j vectors in S (with multiplicity), then

$$I_S = I_G \cdot W_n^1 \cdot W_{n,q}^2,$$

where

$$W_{n,q}^2(i,j) = \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}_q = \frac{(q^{i-1}-1)(q^{i-2}-1)\cdots(q^{i-j+1}-1)}{(q^{j-1}-1)(q^{j-2}-1)\cdots(q-1)} \quad \text{for all} \quad 1 \leqslant i,j \leqslant n. \tag{2.13}$$

The other major operation in [2] applicable to graphs is the complement in K_n of a simple graph (one without loops or multiple edges). In the language of [2], the above invariants such as S_{KC} are not directly computable for complements because the lattice of partitions is not homogeneous. This problem will be handled in Section 4.

3. Calculations and the Chromatic Polynomial

We now give some examples of the intersection matrix and the cardinality–corank matrix for several classes of graphs.

Examples 3.1. 1. If G_1 and G_2 are 2-isomorphic [8] and have the same number of vertices, then $P_{G_1} = P_{G_2}$ so that $I_{G_1} = I_{G_2}$.

2. If T is any tree with n vertices, then for $0 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$,

$$P_T(i,j) = \binom{n-1}{i} \delta(i, n-j),$$

$$I_T(i,j) = \binom{n-1}{i} W^2(n-1-i, j-1) \text{ (where } W^2(0,i) = W^2(i,0) = \delta(i,0)).$$

3. If C is the circuit with n vertices, then

$$P_C(i,j) = \binom{n}{i}\delta(i, n-j) \qquad \text{for} \quad 0 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n$$
$$= \delta(1,j) \qquad \text{for} \quad i = n,\ 1 \leqslant j \leqslant n,$$
$$I_C(i,j) = \delta(i,n), \qquad 0 \leqslant i \leqslant n, j = 1,$$
$$= \binom{n}{i}\sum_{k=i}^{n-i}(-1)^{i+j+k}\binom{n-i}{k}W^2(k,j), \qquad 0 \leqslant i \leqslant n,\ 2 \leqslant j \leqslant n.$$

4. If K_n is the complete graph on n vertices, then its cardinality–corank polynomial is given by

$$S_{KC}(K_n; y, z) = \sum_{a_1,\ldots,a_n} \frac{n!}{a_1!\,(1!)^{a_1}\,a_2!\,(2!)^{a_2}\cdots a_n!\,(n!)^{a_n}} \cdot (y+1)^{\sum a_i\binom{i}{2}}\,(z)_{\sum a_i},$$

where the first sum is over all $a_i \geqslant 0$ such that $\sum ia_i = n$, and where the falling factorial $(z)_m$ is equal to $z(z-1)\cdots(z-m+1)$.

Proof. 1. This is a consequence of the fact that P_G is a matroid invariant and that 2-isomorphic graphs have isomorphic matroid structures.

2, 3. For any graph G in either class, P_G is easily computed and we use (2.5) to compute I_G. Alternately, one can compute I_T for a rooted tree of height 1 and use Example 3.1.1.

4. $$I_{K_n}(i,j) = \sum_{a_k \geqslant 0} \delta\left(\sum a_k, j\right)\delta\left(\sum ka_k, n\right)\delta\left(\sum\binom{k}{2}a_k, i\right) \cdot \frac{n!}{\prod a_k!\,(k!)^{a_k}}$$

and the calculation for S_{KC} is then the polynomial restatement of matrix equation (2.6).

We now apply intersection theory to generalize to two variables the chromatic polynomial of a graph. The resulting two-variable chromatic polynomial was defined in [5], where an identity equivalent to Theorem 3.2.2 was derived. This theorem should perhaps be compared to its q-analog, Theorem 7.4 of [2].

THEOREM 3.2. *Let $c_\lambda(i)$ be the number of ways to color the vertices of the graph G with (at most) λ colors so that exactly i edges of G join two vertices of the same color.*

1. $\bar{c}_\lambda = I_G \bar{u}_\lambda$, *where* $\bar{u}_\lambda(j) = (\lambda)_j = \lambda(\lambda-1)\cdots(\lambda-j+1)$ *for* $1 \leqslant j \leqslant n$.

2. $S_{KC}(G; y-1, \lambda) = \sum_i c_\lambda(i)\, y^i$.

Proof. 1. Any coloring gives a vertex partition whose blocks are the subsets of vertices assigned the same color. A given partition $\prod$ with j blocks comes from $(\lambda)_j$ colorings and for each of these colorings, the edges joining two vertices of the same color are the edges in the Π-induced subgraph. Thus, $c_\lambda(i) = \sum_j I_G(i,j)\, u_\lambda(j)$.

2. This is the polynomial interpretation of the matrix equation $T^- \cdot P_G \cdot \bar{v}_\lambda = \bar{c}_\lambda$, where $\bar{v}_\lambda(i) = \lambda^i$. But $\bar{v}_\lambda = W_n^2 \cdot \bar{u}_\lambda$ since the ith row of both sides count the total number of ways to color K_i with λ colors. Thus we may apply (2.5) so that Part 2 of Theorem 3.2 reduces to part 1.

4. The Polychromate

Definition 4.1. Let G be a graph with n vertices and K edges. The *polychromatic matrix* M_G has its rows indexed by $[0, K]$, its columns indexed by the $p(n)$ integer partitions of n, and $M_G(i, \pi)$ equals the number of vertex partitions of type π whose partition-induced subgraph contains i edges.

The *polychromate* $\chi(G; y, z_1, z_2, \ldots, z_n)$ is equal to $\sum_{i,\pi} M_G(i,\pi)\, y^i \bar{z}^\pi$, where if $\pi = 1^{a_1} 2^{a_2} \cdots n^{a_n}$ then $\bar{z}^\pi = z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}$. The polychromate is, of course, a multivariable generalization of the intersection polynomial (and thus of $S_{KC}(G)$):

Proposition 4.2. *$M_G \cdot D = I_G$, where D is a $p(n) \times n$ matrix with $D(\pi, i) = \delta(w(\pi), i)$. Furthermore,*

$$\chi(G; y, z, z, \ldots, z) = \sum_{i,j} I_G(i,j)\, y^i z^j.$$

Proposition 4.3. *If G is a simple graph and G^c is its complement, then*

$$M_{G^c}(i, 1^{a_1} 2^{a_2} \cdots n^{a_n}) = M_G\left(\left(\sum_k \binom{k}{2} a_k\right) - i,\ 1^{a_1} 2^{a_2} \cdots n^{a_n}\right),$$

and

$$\chi(G^c; y, z_1, \ldots, z_n) = \chi(G; 1/y, z_1, y^{\binom{2}{2}} z_2, y^{\binom{3}{2}} z_3, \ldots, y^{\binom{n}{2}} z_n).$$

The following proposition gives two linear identities for the columns of M_G given the coefficient of $\bar{z}^\pi$ in $\chi(G; 1, z_1, \ldots, z_n)$ and $\partial/\partial y\ \chi(G; 1, z_1, \ldots, z_n)$, respectively.

PROPOSITION 4.4. *Let G be a graph with n vertices, L loops, and K − L other edges.*

1. $\sum_i M_G(i, 1^{a_1}2^{a_2} \cdots n^{a_n}) = \dfrac{n!}{\prod (k!)^{a_k} a_k!}$.
2. $\sum_i iM_G(i, 1^{a_1}2^{a_2} \cdots n^{a_n})$

$$= \frac{(K-L)(n-2)!\,(2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 + \cdots + n(n-1)\,a_n) + (L)\,n!}{\prod (k!)^{a_k} a_k!}$$

Proof. 1. Both sides count the number of vertex partitions of type $1^{a_1} \cdots n^{a_n}$.

2. Both sides count the total number of pairs (e, Π), where Π is of type $\pi = 1^{a_1} \cdots n^{a_n}$, and the edge e is in the Π-induced subgraph. The left-hand side sums first over all partitions Π of type π. The right-hand side sums first over all edges: if e is a loop it is in every partition-induced subgraph and so is multiplied by $n(\pi) = n!/\prod (k!)^{a_k} a_k!$. If e is an edge, the probability over the uniform sample space of all partition-induced subgraphs of type π that e is in a given such subgraph is

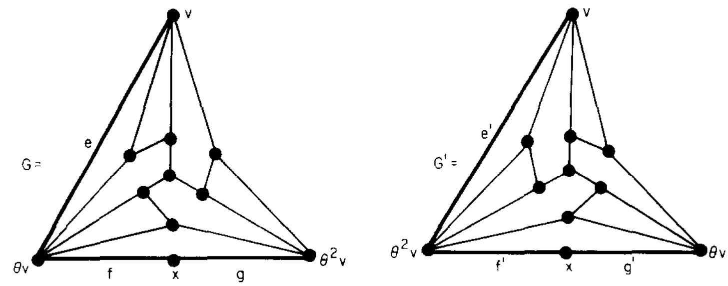
$$\frac{a_2 \binom{2}{2} + a_3 \binom{3}{2} + \cdots + a_n \binom{n}{2}}{\binom{n}{2}},$$

so we may multiply this latter quantity by $n(\pi)$ and Proposition 4.4.2 follows.

DEFINITION 4.5. A *k-rotor R* [6] is a graph whose automorphism group contains a k-cycle $(v, \theta v, \ldots, \theta^{k-1}v)$ called the *border B*, but does not contain the reflection $(v)(\theta v, \theta^{k-1}v) \cdots (\theta^i v, \theta^{k-i}v) \cdots$. A *k-rotor graph G* is a graph composed of a k-rotor and a *frame* or *back-graph*, the latter consisting of vertices and edges connected to the rotor at the border vertices. An *L-frame* consists of the edges $\{v, \theta v\}$, $\{\theta v, x\}$, and $\{x, \theta^2 x\}$.

We now form a new graph G' which consists of the vertices of G, the edges of R, and a frame edge $\{u', w'\}$ for every edge $\{u, w\}$ in the frame of G, where $u' = u$ if u is not on the border; and otherwise, if $u = \theta^i v$, then $u' = \theta^{k-i}v$. Similarly for w'. G' may be thought of as taking the rotor R out of its frame in G, reflecting it and then reattaching it.

A pair of rotor graphs is called *special* if they are non-2-isomorphic (and therefore non-isomorphic) and if they are related as G and G' above. An example of a pair of special three-rotor graphs with an L-frame follows. The rotor itself is the set of light edges in either graph and the border is the three-cycle $(v, \theta v, \theta v^2)$.



THEOREM 4.6. *If G and G' are a special pair of three-rotor graphs with an L-frame, then $M_G = M_{G'}$.*

Proof. We adapt Tutte's arguments in [6]. Let Π be a partition of $V(G)$. Then Π, when restricted to the border, induces Π_B, one of the five partitions of the border vertices $\{v, \theta v, \theta^2 v\}$. Depending on Π_B, we define the bijection f_Π: $V(G) \to V(G')$ which induces a partition Π' on $V(G')$ such that $\pi(\Pi) = \pi(\Pi')$ and $|\Pi(G)| = |\Pi'(G')|$. The reader should check that the collection of functions defined below is bijective on partitions of $V(G)$ to partitions of $V(G')$. For all Π, $f_\Pi(x) = x$.

Case 1. $\Pi_B = \langle v\rangle\langle \theta v, \theta^2 v\rangle$ or $\langle v\rangle\langle\theta v\rangle\langle\theta^2 v\rangle$: Let $f_\Pi(u) = u$ for all $u \in R$. This is an isomorphism from $G - e$ to $G' - e'$ so that $|\Pi(G-e)| = |\Pi'(G'-e')| = i$. But $e \notin \Pi(G)$ and $e' \notin \Pi(G')$ so that $|\Pi(G)| = |\Pi'(G')| = i$.

Case 2. $\Pi_B = \langle v, \theta v, \theta^2 v\rangle$: Let $f_\Pi(u) = u$. As in case 1, $|\Pi(G-e)| = |\Pi'(G'-e')| = i$. But $e \in \Pi(G)$ and $e' \in \Pi'(G')$, so that $\Pi(G) = \Pi'(G') = i + 1$.

Case 3. $\Pi_B = \langle v, \theta v\rangle\langle\theta^2 v\rangle$: Let $f_\Pi(u) = \theta^2 u$. This is an isomorphism from $G - x$ to $G' - x$, so that $|\Pi(G - \{f, g\})| = |\Pi'(G' - \{f', g'\})| = i$. Then $|\Pi(G)| = |\Pi'(G')| = i + 1$ if x is in a Π-block with either θv or $\theta^2 v$ (for example, if x is in a Π-block with θv, then $f \in \Pi(G)$ and x is in a Π'-block with v and $\theta^2 v$, so $f' \in \Pi'(G')$). If x is not in either block, then $|\Pi(G)| = |\Pi'(G')| = i$.

Case 4. $\Pi_B = \langle v, \theta^2 v\rangle\langle\theta v\rangle$. Let $f_\Pi(u) = \theta u$ and apply the same arguments as in case 3.

We can now extend Tutte's construction of five-connected dichromatically inequivalent graphs [5] to pairs with higher connectivity.

COROLLARY 4.7. *There exist non-isomorphic (and non-2-isomorphic) polychromatically equivalent (and thus dichromatically equivalent) graphs of arbitrarily high connectivity.*

Proof. Take the pair of rotor graphs above, add isolated vertices to the frame, and then take their respective graph complements.

We remark that L-type frames can be constructed for other k-rotors. For example, we may frame a four-rotor in the edges $\{v, \theta^2 v\}$, $\{v, \theta v\}$ producing copolychromatic graphs. However, as Example 5.2 below shows, unlike the dichromatic case, not all frames produce polychromatically equivalent graphs.

The following theorem shows that the polychromatic matrix is an edge-reconstructible invariant[1] [1]. Thus our previous results show that not only is the chromatic polynomial of G edge reconstructible from its deck of edge deletions, but also the chromatic polynomial of the complement of G is reconstructible. This observation might prove helpful as a theorem in [1] states, using results from [4], that for any simple graph G, G or G^c is edge-reconstructible. In the following theorem, $G - e$ denotes the subgraph $G(E - \{e\})$, and (H) denotes the polychromate $(H; y, x_1, \ldots, x_n)$.

THEOREM 4.8. *The polychromatic polynomial may be computed for the polynomials of its single-edge[1] deletions. In particular, if G is a graph with K edges, then,*

1. $\sum_{e \in E(G)} M_{G-e} = NM_G$, *where N is a $K \times K+1$ matrix with*

$$\begin{aligned} N(i,j) &= K - i && \text{for} \quad j = i \\ &= i + 1 && \text{for} \quad j = i + 1 \\ &= 0 && \text{otherwise} \end{aligned}$$
$$\text{for} \quad 0 \leqslant i \leqslant K - 1, \quad 0 \leqslant j \leqslant K.$$

2. $\sum_{e \in E(G)} \chi(G - e) = K \cdot \chi(G) + (1 - y)\, \partial/\partial y\, \chi(G)$.

3. *If G is connected, then column $\pi = n^1$ of M_G is given by $M_G(i, n^1) = \delta(i, K)$. When this column is deleted from M_H giving the matrix M'_H, we get*

$$M'_G = N' \cdot \left(\sum_{e \in E(G)} M'_{G-e} \right),$$

where for $0 \leqslant i \leqslant K$, $0 \leqslant j \leqslant K - 1$,

$$\begin{aligned} N'(i,j) &= 0, && j < i, \\ &= \frac{(-1)^{i+j}}{K - j} \frac{\binom{j}{j-i}}{\binom{K-i}{j-i}}, && i \leqslant j. \end{aligned}$$

Proof. 1. Fix a partition Π of $V(G)$. If the partition-induced subgraph $\Pi(G)$ contains i edges, then there are i subgraphs of the "deck" $\{G - e : e \in E(G)\}$ for which $\Pi(G - e)$ has cardinality $i - 1$, and $K - i$ Π-

induced subgraphs which have cardinality i. Thus, for any integer partition π, summing over all Π of type π, we get

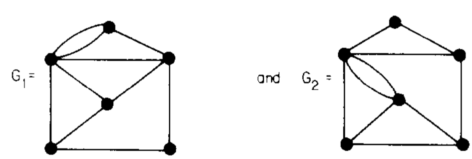
$$\sum_{e \in E(G)} M_{G-e}(i, \pi) = (K - i) \cdot M_G(i, \pi) + (i + 1) \cdot M_G(i + 1, \pi).$$

2. This is the polynomial restatement of the above recursion.

3. The formula for column $\pi = n^1$ is obvious. If G is connected, then $M_G(K, \pi) = 0$ for all $\pi \neq n^1$ so that the matrix equation of Theorem 4.8.3 is valid for row K (since row K of N' is the zero vector). If $\bar{M}'_G$ is the matrix M'_G with row K removed, and $\bar{N}$ is N with column K deleted, then Theorem 4.8.1 becomes $\bar{N}\bar{M}'_G = \sum M'_{G-e}$ (again since row K of M'_G is the zero vector). But $\bar{N}$ is invertible, and one can easily check that its inverse is given by $\bar{N}'$ (N' with row K removed) so that $\bar{M}'_G = \bar{N}' \cdot (\sum M'_{G-e})$ and Theorem 4.8.3 is valid for rows $i = 0$ to $K - 1$.

5. Examples and Computations

Example 5.1 (The Gray Graphs). We illustrate some of the previous results for two graphs attributed in [6] to M. C. Gray. These graphs are dichromatically but not polychromatically equivalent. Let



Then, one calculates P_{G_i} directly or uses formula (2.4) and the calculation of F_G in [6] to get

$$P_{G_1} = P_{G_2} = P = \begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 10 & 0 \\
2 & 0 & 0 & 0 & 44 & 1 & 0 \\
3 & 0 & 0 & 108 & 12 & 0 & 0 \\
4 & 0 & 151 & 58 & 1 & 0 & 0 \\
5 & 98 & 142 & 12 & 0 & 0 & 0 \\
6 & 151 & 58 & 1 & 0 & 0 & 0 \\
7 & 108 & 12 & 0 & 0 & 0 & 0 \\
8 & 44 & 1 & 0 & 0 & 0 & 0 \\
9 & 10 & 0 & 0 & 0 & 0 & 0 \\
10 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}.$$

One notes that the dual of G_1, G_1^*, is isomorphic to G_2 so that by using Eq. (2.8), we obtain $P(i, j) = P(10 - i, i + j - 5)$. Further, (2.5) becomes

$$T^{-1} \cdot P_{G_i} \cdot W_n^2 = I_{G_i} = \begin{bmatrix} 0 & 0 & 2 & 8 & 6 & 1 \\ 0 & 0 & 10 & 24 & 8 & 0 \\ 0 & 2 & 28 & 24 & 1 & 0 \\ 0 & 4 & 24 & 8 & 0 & 0 \\ 0 & 7 & 19 & 1 & 0 & 0 \\ 0 & 8 & 6 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$W_n^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{bmatrix} = (W_n^1)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}^{-1}.$$

A further computation yields

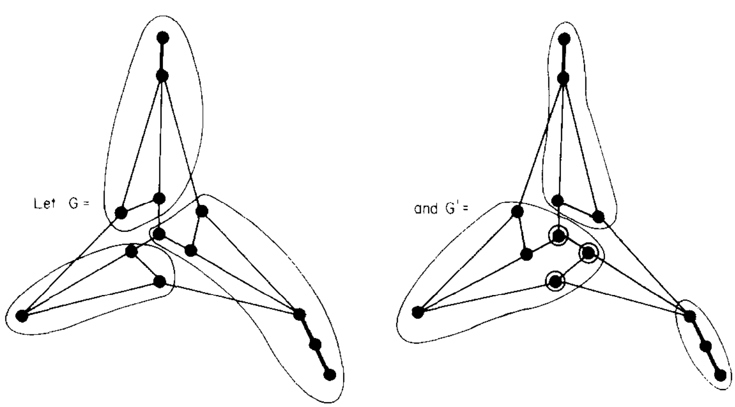
$$M_{G_1} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 6^1 & 5^1 1^1 & 4^1 2^1 & 3^1 3^1 & 4^1 1^2 & 3^1 2^1 1^1 & 2^3 & 3^1 1^3 & 2^2 1^2 & 2^1 1^4 & 1^6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 7 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 7 & 3 & 4 & 20 & 8 & 0 \\ 0 & 0 & 0 & 2 & 1 & 19 & 8 & 10 & 14 & 1 & 0 \\ 0 & 0 & 3 & 1 & 4 & 19 & 1 & 4 & 4 & 0 & 0 \\ 0 & 0 & 4 & 3 & 5 & 12 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 3 & 4 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

One is encouraged to use this matrix to check Eq. (4.2) and those in parts 1 and 2 of Proposition 4.4. We also remark that some rows and columns of M_{G_1} are not unimodal.

The columns 6^1, $5^1 1^1$, $4^1 1^2$, $2^1 1^4$, and 1^6 of M_{G_2} are identical with those of M_{G_1}. However, the other columns are different:

$$M_{G_2} = \begin{array}{c} \begin{array}{cccccc} 4^1 2^1 & 3^1 3^1 & 3^1 2^1 1^1 & 2^3 & 3^1 1^3 & 2^2 1^2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 9 & 1 & 7 & 17 \\ 1 & 1 & 19 & 8 & 7 & 17 \\ 1 & 3 & 17 & 3 & 5 & 3 \\ 4 & 3 & 13 & 1 & 1 & 0 \\ 6 & 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

EXAMPLE 5.2 (Noncopolychromatic Rotor Graphs).



One readily checks that $M_G(13, 6^1 4^1 3^1) = 1$ (for the partition pictured), while $M_{G'}(13, 6^1 4^1 3^1) = 2$ (for the partition pictured and one in which the three circled vertices are joined to the three-block).

6. RESEARCH PROBLEMS

Problem 6.1. Does there exist a definable set of graph–theoretic operations, $\mathcal{O}$, so that for any pair of polychromatically equivalent graphs

G_1 and G_2, G_1 can be transformed into G_2 by a succession of operations from $\mathscr{O}$? (For example, $\mathscr{O}$ will include certain rotor reflections and complementation. The idea behind using polychromatic instead of dichromatic equivalence is suggested by the fact that it might eliminate such accidentally (?) dichromatically equivalent graphs as the Gray graphs.)

Problem 6.2. What classes of graphs are reconstructible or recognizable (in the language of [1]) from their polychromates? (These classes would then, of course, be (edge)[1] reconstructible or recognizable, respectively, since the polychromatic matrix is reconstructible.)

Problem 6.3. Is G edge-reconstructible[1] when either G or G^c is a Tutte rotor graph? (If this question could be answered affirmatively, and if $\mathscr{O}$ above were shown to consist solely of rotor reflections and complementation, the edge-reconstruction problem would be solved!)

Problem 6.4. For what graphs and graph–theoretic operations is the polychromate quickly computable? It is easily computable for complete graphs, and although we do not give the formula (it is a bit messy and involves summing over partial matchings in complete bipartite graphs), one can compute $\chi(G_1 \cup G_2)$ from $\chi(G_1)$ and $\chi(G_2)$. This result, with our formula in Section 4 for complements, will yield $\chi(G)$ for any complete p-partite graph.

Problem 6.5. Are the rows and columns of the intersection matrix for a graph (or indeed any matroid embedding) unimodal (or logarithmically concave)?

REFERENCES

1. J. A. BONDY AND R. L. HEMMINGER, "Graph reconstruction–A survey," Research Report CORR 76–49, Dept. of Comb. and Opt., University of Waterloo, Waterloo, Ontario, Canada, 1976.
2. T. BRYLAWSKI, Intersection theory for embeddings of matroids into combinatorial geometries, *Studies Appl. Math.* **61** (1979), 211–244..
3. T. BRYLAWSKI AND D. KELLY, Matroids and combinatorial geometries, M.A.A. Studies in Combinatorics (G.-C. Rota, Ed.), Math. Ass. of America, 1977.
4. L. LOVASZ, A note on the line reconstruction problem, *J. Combinatorial Theory Ser. B* **13** (1972), 309–310.
5. W. TUTTE, On dichromatic polynomials, *J. Combinatorial Theory* **2** (1967), 301–320.
6. W. TUTTE, Codichromatic graphs, *J. Combinatorial Theory* **16** (1974), 168–174.
7. D. J. A. WELSH, "Matroid Theory," Academic Press, London, 1976.
8. H. WHITNEY, 2-isomorphic graphs, *Amer. J. Math.* **55** (1933), 245–254.
9. T. BRYLAWSKI AND D. KELLY, "Matroids and Combinatorial Geometries," UNC Lecture Note Series, Mathematics Department, University of North Carolina, Chapel Hill, N. C., 1980.
10. T. BRYLAWSKI, Hyperplane reconstruction of the Tutte polynomial of a geometric lattice, *Ann. Discrete Math.*, in press.