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A Möbius Identity Arising from Modularity in a Matroid Bilinear Form

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DEDICATED TO THE MEMORY OF GIAN-CARLO ROTA

The matrix for the bilinear form of the flag space of a matroid has (with respect to an appropriate basis) a tensor product structure when the matroid has a modular flat K. When determinants are taken, an identity is obtained for the rho function (a certain product of the Möbius and beta functions) summed over flats with a fixed intersection with K. When the identity is interpreted for Dowling lattices and finite projective spaces, identities with similar combinatorial proofs are obtained for binomial and Gaussian coefficients, respectively.

1. INTRODUCTION AND DEDICATION

In 1966 Gian-Carlo Rota published the now celebrated first part of his herculean effort to find broad genera in the family of combinatorics and identify the species for each. In that paper [R], Rota showed how the Möbius function could be used to unify inversion formulae in such superficially diverse fields as number theory, enumeration (inclusion–exclusion), finite difference calculus, graph coloring, and vector space duality. I came from Dartmouth two years later to his MIT office to discuss this paper, and he gave me a preliminary version of the second "labor" to read [CR]. I was smitten.

E poi che la sua mano a la mia puose
Con lieto volto, ond'io mi confortai,
Mi mise dentro a le segrete cose.

—"*Inferno*," canto III, 7

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For several years after Foundations I, there was a flurry of activity exploring the so-called Möbius algebra, much of it by another of my nurturers, Henry Crapo, Rota's first combinatorial disciple, at the time expatriated to Waterloo. The proof in the following section is in the spirit of that time, using such classical tools as Weisner's theorem and Crapo's complementation theorem.

For the past quarter century most of the applications of the Möbius function and its related invariants have been for geometric lattices (or, as Rota preferred, combinatorial geometries) [S, GZ, BO2]. Methods have expanded to include topology, homological algebra [Bj], and, for the invariant explored here, bilinear and linear algebra [BV].

The invariant explored here we now denote $\rho_M(A)$ and call the rho function in honor of Rota. For a flat A of a geometry (O.K., matroid) M, $\rho_M(A)$ is defined as the product of Crapo's beta function $\beta(A)$ and the absolute value of the Möbius function $\mu^+(M/A)$. The function rho first appeared [V, p. 550, and called *discrete volume*] (in a non-Möbius form) as the exponent of each flat weight in a linear factorization of the determinant $D(\mathscr{H})$ of a matrix $B(\mathscr{H})$ for a real hyperplane arrangement $\mathscr{H}$. It was generalized to matroids in [BV]. Results in [BO2] or [GZ] lead to several interpretations of $\rho_M(A)$. If M is the matroid of a simple graph G with edge set S and vertex set V, then $\rho_M(A)$ is non-zero only when $A \subseteq S$ is a non-empty two-connected subset of edges induced by a vertex subset V', in which case $\rho_M(A)$ gives the number of acyclic orientations $O(G)$ of G such that a fixed edge e in A is oriented from the unique source v of $O(G)$ to the unique sink v' of $O(A)$, and all edges in the cut-set $V' \mid V - V'$ are oriented from V' to $V - V'$. For an affine (non-central) hyperplane arrangement $\mathscr{H}$, if F is a flat (hyperplane intersection) contained in a hyperplane H, then $\rho_{\mathscr{H}}(F)$ counts all regions R whose intersection with H is a bounded region of F.

Section 3 summarizes results in [BV], especially the structure of the matrix $B(M)$ whose determinant $D(M)$ gives the weighted exponential generating function for rho. When, for a fixed modular flat K of M, a fortuitous basis is chosen for the flag space of M which gives the bilinear form, and modifications are made in $B(M)$, the resulting matrix is shown to be a tensor product of matrices for K and for the complete principal truncation $\bar{T}_K(M)$. Then, when the determinant is evaluated and exponents are compared with linear factors in $D(M)$, we obtain our principal result: a closed form for $\sum \rho_M(K'')$ summed over all flats K'' whose intersection with K is a fixed flat K'.

On the one hand, there is a surprising singularity when $K' = \emptyset$: the $K' \neq \emptyset$ formula would give zero (the weight of $\emptyset$), whereas the correct answer is calculated in Section 2 using the Möbius algebra. On the other hand, such a factorization could have been anticipated as it parallels similar results for the Möbius function and its generating function, the characteric polynomial

$$\chi(M) := \sum_{\text{flats } x} \mu(\hat{0}, x)\, \lambda^{\text{corank}(x)}.$$

It was first noted [S] that, for a modular flat K, $\chi(K)$ divides $\chi(M)$. Then, the quotient was given a matroid interpretation [Br1], giving the factorization

$$\chi(M) = \frac{\chi(K) \cdot \chi(\bar{T}_K(M))}{\lambda - 1}.$$

Finally [Br2, BZ, BO1], the factorization was shown to reflect a factorization of the broken-circuit complex of M. The facets of this complex when interpreted in the Orlik–Solomon algebra (see [Bj]) are dual to the flag space basis which defines $B(M)$ [BV].

In the final section, we interpret our identity for matroids in which the sum can be interpreted with classical combinatorial invariants for Dowling lattices (in particular the partition lattice) and finite projective geometries. Then, the same Möbius identity gives an identity involving binomial coefficients (in the partition case), Newton binomial coefficients (in the Dowling case), and Gaussian coefficients (for projective geometries). Thus, we have two q-analogs (computing an analog for Dowling lattices suggested by Joe Bonin), and indeed the binomial and Gaussian identities are given analogous combinatorial proofs using binary sequences and matrices, respectively. This was another foundational idea of Rota's [GR]. In fact, the three foundational topics treated here—the Möbius function, combinatorial geometries, and q-analogs—constitute three chapters of the recent combinatorics textbook [LW]. I am afraid that today automatic identity proving techniques are used to find or verify such formulas, but the computer has still to find Möbius underpinnings for them.

I am proud that the article this work is based on appeared (notwithstanding the delay) in the journal Rota ran and that this one appears in the journal he co-founded.

2. MÖBIUS ALGEBRA AND THE RHO FUNCTION

Of course, there is no need to define the Möbius function $\mu(x, y)$ of [R] for elements x and y of a geometric lattice except to say that we will use, as is common in matroid theory, the one-variable version for a flat K of a combinatorial geometry G: $\mu(K) := \mu(\hat{0}, K)$, and, following [GZ], denote its absolute value by

$$\mu^+(K) = |\mu(K)| = (-1)^{r(K)} \mu(K). \tag{2.0}$$

We further make the usual extension to a matroid M and subset A of its groundset:

$$\mu^+(A) = \begin{cases} \mu^+(\bar{A}) & \text{if } A \text{ is loopless, and} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that, when M is loopless, $\mu^+(M/A) = |\mu(A, \hat{1})|$ when A is a flat and is 0 otherwise. Also, μ^+ has the (matroid) recursion for any $p \in A$ not an isthmus or loop:

$$\mu^+(A) = \mu^+(A - p) + \mu^+(A/p). \tag{2.1}$$

The one-variable beta function [C1] for a flat K can be defined by

$$\beta(K) = (-1)^{r(K)} \sum_{K' : K' \leqslant K} \mu(K')\, r(K') \tag{2.2}$$

and, like μ, extended to subsets A in a matroid. It also obeys the recursion (2.1) and is zero unless K is non-empty and connected, in which case it is positive.

Definition 2.3. For a matroid $M(S)$ and $A \subseteq S$ define the rho function

$$\rho_M(A) := \beta(A)\, \mu^+(M/A).$$

By the remarks above, $\rho_M(A) = 0$ unless A is a non-empty, connected flat of M. The recursion (2.4) below for rho was shown in [BV] (and holds for any function $g(A) = \beta(A)\, f(M/A)$ if f obeys (2.1) and is zero on loops),

$$\rho_M(A) + \rho_M(A \cup p) = \rho_{M-p}(A) + \rho_{M/p}(A), \tag{2.4}$$

where M is loopless and $p \in S - A$ is not an isthmus of M.

Interpretations for rho are given in the Introduction. We now prove an identity for rho using classical identities by Weisner [W] and Crapo [C2], respectively:

$$\sum_{x : x \wedge u = v} \mu(x, w) = \begin{cases} 0 & \text{if } \; w \not\leqslant u \\ \mu(v, w) & \text{otherwise} \end{cases} \tag{2.5}$$

$$\sum_{\substack{x, y : x \leqslant y, \\ x \wedge z = y \wedge z = \hat{0}, \\ x \vee z = y \vee z = \hat{1}}} \mu(\hat{0}, x)\, \mu(y, \hat{1}) = \mu(\hat{0}, \hat{1}). \tag{2.6}$$

Recall that z is a modular flat (element) in a combinatorial geometry M (geometric lattice L) if and only if

$$(y \wedge z) \vee x = y \wedge (z \vee x) \qquad \text{for all} \quad x \leqslant y \quad \text{in } L \tag{2.7.1}$$

or, equivalently, if

$$r(x) + r(z) = r(x \vee z) + r(x \wedge z) \qquad \text{for all} \quad x \in L \tag{2.7.2}$$

PROPOSITION 2.8. *If z is a modular element in a geometric lattice, then*

$$\sum_{x : x \wedge z = \hat{0}} \rho_M(x) = \operatorname{corank}(z) \cdot \mu^+(M).$$

Proof. Using lattice arguments:

$$\sum_{x : x \wedge z = \hat{0}} \rho_M(x) = \sum_{x : x \wedge z = \hat{0}} \beta(x)\, \mu(x, \hat{1})(-1)^{r(\hat{1}) - r(x)}$$

(from (2.0), (2.3))

$$= \sum_{x : x \wedge z = \hat{0}} (-1)^{r(x)} \Bigg(\sum_{y : y \leqslant x} \mu(\hat{0}, y)\, r(y) \Bigg) \mu(x, \hat{1})(-1)^{r(\hat{1}) - r(x)}$$

(from (2.2))

$$= (-1)^{r(\hat{1})} \sum_{y : y \wedge z = \hat{0}} \mu(\hat{0}, y)\, r(y) \Bigg(\sum_{\substack{x : x \geqslant y \\ x \wedge z = \hat{0}}} \mu(x, \hat{1}) \Bigg). \tag{$*$}$$

The latter parenthesized sum in $(*)$ is over all x such that $x \wedge (z \vee y) = y$ since, for all such x, $x \geqslant y$, and $z \wedge x = (z \wedge x) \wedge (z \vee y) = z \wedge (x \wedge (z \vee y)) = z \wedge y = \hat{0}$. Conversely, if $x \geqslant y$ and $x \wedge z = \hat{0}$, then $x \wedge (z \vee y) = (x \wedge z) \vee y$ (by (2.7.1)) $= \hat{0} \vee y = y$. Letting $u = z \vee y$, $v = y$, and $w = \hat{1}$ in (2.5), we obtain for the parenthesized sum in $(*)$:

$$\sum_{x : x \wedge (z \vee y) = y} \mu(x, \hat{1}) = \begin{cases} 0 & \text{if} \quad z \vee y \neq \hat{1}, \qquad \text{and} \\ \mu(y, \hat{1}) & \text{otherwise.} \end{cases}$$

Hence, $(*)$ becomes

$$(-1)^{r(\hat{1})} \sum_{\substack{y : y \wedge z = \hat{0} \\ y \vee z = \hat{1}}} \mu(\hat{0}, y)\, r(y)\, \mu(y, \hat{1}) \tag{$**$}$$

Since z is modular, all complements y of z have the same rank, $r(\hat{1})-r(z)$, and thus $(**)$ becomes

$$\begin{aligned}(r(\hat{1})-r(z))(-1)^{r(\hat{1})} & \sum_{\substack{y:\, y\wedge z=\hat{0}\\ y\vee z=\hat{1}}} \mu(\hat{0}, y)\, \mu(y, \hat{1}) \\ &= (r(\hat{1})-r(z))(-1)^{r(\hat{1})}\, \mu(\hat{0}, \hat{1}) \qquad \text{by (2.6)} \\ &= \operatorname{corank}(z)\, \mu^{+}(M). \quad \blacksquare\end{aligned}$$

3. THE DETERMINENTAL IDENTITY IN THE FLAG SPACE

We give a quick summary of the definitions and properties of the matrix $B(M)$ and its determinant $D(M)$ from [BV].

Let M be a combinatorial geometry of rank r on the ground set $S=\{p_1, ..., p_n\}$. Let S be totally ordered by $\prec$ (usually, $p_i \prec p_j$ if $i<j$).

3.1. *Weights*

We assign a *weight* to each p_i: $a(p_i) := a_i$. For any flat (more generally, subset) K of S, define the *weight of* K by

$$a(K) := \sum_{p_i \in K} a_i,$$

and, for any r-tuple $t=(p_{i_1}, ..., p_{i_r})$, define its weight by

$$a(t) = a_{i_1} a_{i_2} \cdots a_{i_r}.$$

3.2. *Flags*

For any (complete) flag $F=[K_0=\varnothing \lessdot K_1 = p_i \lessdot K_2 \lessdot \cdots \lessdot K_r = S]$ in M, *label* each of its flats K_i by

$$l(K_i) = \text{the minimum (under } \prec \text{) point it contains,}$$

and call the flag *standard* if its r labels are distinct.

Another way to describe (standard) flags is by the set *differences* of consecutive flats:

$$D_i(F) := K_i - K_{i-1} \qquad (1 \leqslant i \leqslant r).$$

From now on, we represent the flag F by its *difference partition* of S:

$$D(F) := D_1(F) \,|\, D_2(F) \,|\, \cdots \,|\, D_r(F).$$

If we label D_i as above, the flag is standard if and only if the labels decrease. There are $\mu^+(M)$ standard flags of M and the r-tuples of their labels give (in descending order) the bases of M containing no broken circuit, called *standard χ-bases.* Conversely, if $A = (p_{i_r}, p_{i_{r-1}}, ..., p_{i_1})$ is a standard χ-basis ($p_{i_j} \prec p_{i_{j+1}}$), then the flag

$$F(A) := [\varnothing \lessdot p_{i_r} \lessdot p_{i_r} \vee p_{i_{r-1}} \lessdot p_{i_r} \vee p_{i_{r-1}} \vee p_{i_{r-2}} \lessdot \cdots]$$

is a standard flag whose r ordered labels give back A.

3.3. *The Entries of $B(M)$*

For two flags F and F', define the function $\sigma_{F, F'}(i)$: $[1, r] \to [1, r]$ by

$$\sigma_{F, F'}(i) := \min(j : D_i(F) \cap D_j(F') \neq \varnothing).$$

This will be a permutation if and only if F and F' form a *modular flag pair*,

$$r(K_i) + r(K'_j) = r(K_i \vee K'_j) + r(K_i \wedge K'_j),$$

for all $K_i \in F$ and $K'_j \in F'$.

In the modular pair case, $\sigma_{F, F'}$ is the unique permutation σ such that $D_i(F) \cap D_{\sigma(i)}(F') \neq \varnothing$ for all i, and, further, in the non-modular case, no such permutation σ exists.

Now define a (symmetric) matrix B whose rows and columns are indexed by the standard flags and whose entries are given by

$$B(F, F') = \begin{cases} \operatorname{sgn}(\sigma_{F, F'}) \prod_{i=1}^{r} a(D_i(F) \cap D_{\sigma_{F, F'}(i)}(F')) \\ \qquad \text{if } F \text{ and } F' \text{ are a modular pair, and} \\ 0 \qquad \text{otherwise.} \end{cases} \tag{3.3.1}$$

Thus, $B(F, F')$ is a (possibly empty) signed sum of all monomials of the form $a_{k_1} a_{k_2} \cdots a_{k_r}$ such that $\{p_{k_1}, p_{k_2}, ..., p_{k_r}\}$ is a set of distinct representatives for both $D(F)$ and $D(F')$.

Remark 3.4. The permutation $\sigma := \sigma_{F, F'}$ occurs in the Bruhat decomposition for flag varieties which says that for any nonsingular matrix A, there are lower triangular matrices L and L' such that for a (unique) permutation matrix P^σ,

$$A = LP^\sigma L'.$$

In this case, the flags are given by $K_i =$ the subspace spanned by the first i rows of the identity matrix, while K'_j is spanned by the first j rows of A. One can obtain the decomposition by row reducing A from right to left producing an upper left triangular matrix U, pivoting when necessary to produce $LA = P^{\sigma'}U$. Then $A = L^{-1}P^{\sigma}L'$, with $\sigma = \sigma'\sigma_{\text{opp}}$, where $\sigma_{\text{opp}} := (1, r)(2, r-1)\cdots(\lfloor \frac{r}{2} \rfloor, \lceil \frac{r}{2} \rceil)$.

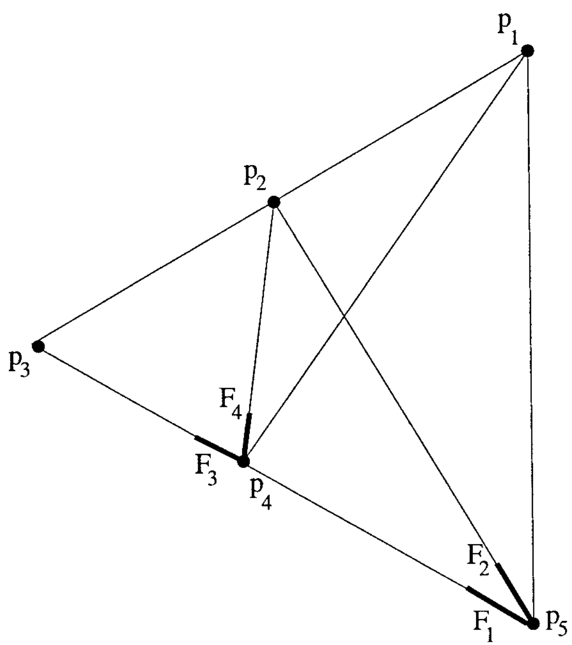
THEOREM 3.5 [BV]. *If $D(M)$ is the determinant of $B(M)$, then*

$$D(M) = \prod_K (a(K))^{\rho_M(K)},$$

the product being over all flats (necessarily non-empty and connected) of M.

$$\text{\textit{The determinant $D(M)$ is independent of the ordering $\prec$ and, as a polynomial, is homogeneous of degree $r(M)\,\mu^+(M)$ (the number of rows of $B(M)$ times the homogeneous degree of its entries).}} \tag{3.6}$$

EXAMPLE 3.7. Let M be the combinatorial geometry whose affine picture and standard flags are indicated below.



Its standard flag differences are

$$D(F_1) = p_5 \mid p_3, p_4 \mid p_1, p_2; \qquad D(F_2) = p_5 \mid p_2 \mid p_1, p_3, p_4;$$

$$D(F_3) = p_4 \mid p_3, p_5 \mid p_1, p_2; \qquad D(F_4) = p_4 \mid p_2 \mid p_1, p_3, p_5,$$

and we obtain the matrix $B(M)$ below, where the first monomial in $B(F, F)$ reflects the flag labels.

$$B(M) = \begin{array}{c} \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \begin{array}{c} \begin{array}{cccc} F_1 & F_2 & F_3 & F_4 \end{array} \\ \begin{bmatrix} a_5a_3a_1+a_5a_3a_2+a_5a_4a_1+a_5a_4a_2 & -a_5a_3a_2-a_5a_4a_2 & -a_5a_4a_1-a_5a_{43}a_2 & a_5a_4a_2 \\ -a_5a_3a_2-a_5a_4a_2 & a_5a_2a_1+a_5a_2a_3+a_5a_2a_4 & a_5a_2a_4 & -a_5a_2a_4 \\ -a_5a_4a_1-a_5a_4a_2 & a_5a_2a_4 & a_4a_3a_1+a_4a_3a_2+a_4a_5a_1+a_4a_5a_2 & -a_4a_3a_2-a_4a_5a_2 \\ a_5a_4a_2 & -a_5a_2a_4 & -a_4a_3a_2-a_4a_5a_2 & a_4a_2a_1+a_4a_2a_3+a_4a_2a_5 \end{bmatrix} \end{array}.$$

Then, $D(M) = a_1^2a_2^2a_3a_4^2a_5^2 \cdot (a_1+a_2+a_3) \cdot (a_3+a_4+a_5) \cdot (a_1+a_2+a_3+a_4+a_5)$.

EXAMPLE 3.8. If M is separable: $M = M_1 \oplus M_2$ (i.e., $M_1 = M(S')$ is a distributive flat with modular lattice complement $M_2 = M(S-S')$), then

$$B(M) = B(M_1) \otimes B(M_2),$$

and

$$D(M) = (D(M_1))^{\mu^+(M_2)} \cdot (D(M_2))^{\mu^+(M_1)}. \tag{3.4}$$

4. A MODULAR IDENTITY FOR THE FLAG MATRIX AND FOR THE MÖBIUS ALGEBRA

We now see what we can say about $D(M)$ and $B(M)$ when M has a (non-trivial) modular flat. To that end, we first factor and then examine $D(M)$.

DEFINITION 4.1. For a flat K of M and all flats K', define

$$D_K(M) = \prod_{K' : K' \cap K \neq \varnothing} (a(K' \cap K))^{\rho_M(K')},$$

and

$$\hat{D}_K(M) = \prod_{K' : K' \cap K = \varnothing} (a(K'))^{\rho_M(K')}.$$

Evidently,

$$\deg(D_K(M)) + \deg(\hat{D}_K(M)) = \deg(D(M)) = r(M)\ \mu^+(M) \qquad \text{by (3.6).} \tag{4.1.1}$$

When K is a direct-sum factor, all K' (necessarily connected) which give a positive exponent in $D_K(M)$ are contained in K, so that, in this case, $D(M) = D_K(M)\ \hat{D}_K(M)$. Further, (3.8) shows that $D_K(M) = (D(K))^{\mu^+(S-K)}$, and $\hat{D}_K(M) = D(M(S-K))^{\mu^+(K)}$. We now explore the structure of these polynomials when K is modular.

PROPOSITION 4.2. 1. *If K is a modular flat of M, then*

$$\deg(D_K(M)) = r(K)\ \mu^+(M),$$

and

$$\deg(\hat{D}_K(M)) = (r(M) - r(K))\ \mu^+(M).$$

2. *Conversely, if K is not modular, then*

$$\deg(D_K(M)) < r(K)\ \mu^+(M).$$

Proof. (1) The degree of $\hat{D}_K(M)$ was computed in (2.8), and the degree of $D_K(M)$ follows from (4.1.1).

(2) For any flat K,

$$\deg(D_K(M)) \leqslant r(K)\ \mu^+(K) \tag{4.2.3}$$

since any monomial in $B(M)$ has at most $r(K)$ factors a_i with $p_i \in K$, and each summand in the expansion of $D(M)$ is a product of $\mu^+(M)$ such monomials. If K is not modular, let K' be a non-modular complementary flat ($K \cap K' = \varnothing$ and $r(K') + r(K) > r$). Reorder S so that $K' = \{p_1, p_2, ..., p_{k'}\}$. Then the lexicographically minimum standard basis $A^{\min}$ for M (which is necessarily a χ-basis) intersects K' in $r(K')$ points, so that in the row of $B(M)$ indexed by $F(A^{\min})$, every monomial contains less than $r(K)$ factors a_i with $p_i \in K$. Thus, there is strict inequality in (4.2.3). ∎

We now describe $D_K(M)$ for a modular flat K as a power of $D(K)$, and $\hat{D}_K(M)$ as a power of $D_{\bar{p}}(M')$ for an associated matroid M' (where $\bar{p}$ is a point). The formulas come from a matrix tensor product similar to that for separable matroids. Again, we assume (with the appropriate weights on multiple points: assign a multiple point its weight as one would any flat and treat it as a single weight) that M (necessarily loopless) has been simplified to a combinatorial geometry. We review some results principally from [Br1] on modular flats and their complete principal truncations.

DEFINITION 4.3. The *complete principal truncation* $\bar{T}_K(M)$ (called the *Brown truncation* in [Br1]) is the matroid M/P, where, first $r(K)-1$ points P are put freely on the flat K (and, after the contraction, the resulting multiple point $\{K\}$ is simplified to $\bar{p}$). When K is modular, $\bar{T}_K(M)$ has as flats which do not contain $\bar{p}$ all flats K' of M disjoint from K and, as flats containing $\bar{p}$, the sets of the form $(K'-K)\cup\{\bar{p}\}$, where, in M, K' is a flat containing K [Br1, (5.14.3)]. Hence, the factors appearing in $\hat{D}_K(M)$ are those in $\hat{D}_{\bar{p}}(\bar{T}_K)$. In hyperplane arrangements $\mathscr{H}$, $T_K(\mathscr{H})$ is constructed by freely adding a flat $\bar{T}$ covering K and considering the arrangement in $\bar{T}$ of its intersections with the hyperplanes of $\mathscr{H}$.

We now order S so that the k points of K precede those of $S-K$ and give the latter points the same order in $\bar{T}_K(M)$ with $\bar{p}$ adjoined preceding $S-K$.

PROPOSITION 4.4. *Let* $K=\{p_1, ..., p_k\}$ *be a modular flat of the matroid* $M(S)$ *with* $S=\{p_1, ..., p_n\}$ *and, for a polynomial* $p(a_1, ..., a_n)$, *define its* K*-degree by* $\deg(p)|_{a_i=1 \text{ for } i>k}$.

1. $D_K(M)\,\hat{D}_K(M)$ *has homogeneous* K*-degree equal to* $r(K)\,\mu^+(M)$.
2. $D(M)-D_K(M)\cdot\hat{D}_K(M)$ *has* K*-degree less than* $r(K)\,\mu^+(M)$.
3. *If* $B_K(M):=B(M)$ *with all its monomials removed whose* K*-degree is less than* $r(K)$, *then*

$$\operatorname{Det}(B_K(M))=D_K(M)\cdot\hat{D}_K(M).$$

Proof. (1) This follows from (4.2.1).

(2) The monomials in the expansion of this polynomial have at least one factor a_i from a linear term $a(K')$ where some $p_j\in K'\cap K$, and $p_i\in K'-K$. Replacing such a_i's with an associated a_j creates a monomial in the expansion for $D(M)$ with greater K-degree.

(3) The monomials in the determinantal expansion of $B(M)$ with K-degree $<r(K)\,\mu^+(M)$ have at least one factor which is a monomial entry of $B(M)$ with K-degree $<r(K)$ and conversely. ∎

LEMMA 4.5 [BO1, (2.4)]. *A is a standard* χ*-basis of* M *if and only if*

$$A=\hat{A}_T A_K \qquad (\textit{by concatenation}),$$

where A_K is a standard χ-basis of K, and $\hat{A}_T$ is a standard χ-basis of $\overline{T}_K(M)$ with $\bar{p}$ (which it always contains) removed.

We are ready to prove our principal matrix result.

THEOREM 4.6. *Let K be a modular flat of M.*

1. $B_K(M) = B(K) \otimes \frac{1}{a(\bar{p})} B_{\bar{p}}(\overline{T}_K(M)) = \frac{1}{a(\bar{p})} B_{\bar{p}}(K \oplus \overline{T}_K(M))$.
2. $D_K(M) = (D(K))^{\mu^+(M)/\mu^+(K)}$.
3. $\hat{D}_K(M) = \hat{D}_{\bar{p}}(\overline{T}_K(M))^{\mu^+(K)}$.

Proof. To obtain $B_{\bar{p}}(\overline{T}_K(M))$, consider, in (3.3.1), only sets of distinct common representatives which include $\bar{p}$. The permutation σ in (3.3.1) must have $\sigma(r') = r'$, where $r' = r - r(K) + 1$ is the rank of $\overline{T}_K(M)$ since $\bar{p} \prec S - K$ and so occurs in $D_{K'}$ for any standard flag. Let $\sigma' \in \mathrm{Symm}(r'-1)$ denote σ restricted to $[1, r'-1]$. Then $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$. When $a(\bar{p})$ is removed from all such monomials, we obtain, in

$$B' := \frac{1}{a(\bar{p})} B_{\bar{p}}(\overline{T}_K(M)),$$

the entries

$$B'(F := F(\hat{A}_T), F' := F(\hat{A}'_T)) = \mathrm{sgn}\, \sigma'_{F,F'} \prod_{i=1}^{r'-1} a(D_i \cap D'_{\sigma'_{F,F'}}(i)). \tag{4.6.4}$$

By (4.5), flags of $B(M)$ are of the form $F(\hat{A}_T A_K)$. For all such flags, the modularity of K assures us that for $i \leqslant r' - 1$, the difference D_i in (3.3.1) has empty intersection with K since $r(F_{r'-1} \vee K) = r = r(F_{r'-1}) + r(K)$.

Thus, if $B_K(F(\hat{A}_T A_K), F(\hat{A}'_T, A'_K))$ is non-zero (i.e., has a monomial with K-degree $r(K)$), then $\sigma(F, F')$ is of the form $\sigma'\sigma''$, where σ' permutes the first $r'-1$ coordinates, and σ'' the last $r(K)$ coordinates (and, of course, $\mathrm{sgn}(\sigma) = (\mathrm{sgn}(\sigma')) \cdot (\mathrm{sgn}(\sigma''))$). Further, in $B_K(F, F')$, only representatives from K are chosen from the last $r(K)$ differences. We may effect this by intersecting all D_i, D'_j in (3.3.1) with K ($i, j \geqslant r - r(K)$). Then, restricting (3.3.1) to its last k factors and using $\mathrm{sgn}(\sigma'')$, we obtain an entry from $B'' := B(K)$, and

$$B_K(F(\hat{A}_T A_K), F(\hat{A}'_T, A'_K)) = B'(F(\hat{A}_T, \hat{A}'_T)) \cdot B''(F(A_K, F(A'_K)) \tag{4.6.5}$$

This proves the first equality in (4.6.1), and the other formulas follow from the formula for the determinant of a tensor product (see (3.8) and (4.4)). ∎

EXAMPLE 4.7 (3.7 continued). For the modular flat $L=\{p_1, p_2, p_3\}$ in the matroid M of (3.7), $\bar{T}_L(M)$ is the three-point line $\{\bar{p}, p_4, p_5\}$, and

$$B_L=\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ F_4 \end{array}\begin{bmatrix} a_5a_3a_1+a_5a_3a_2 & -a_5a_3a_2 & 0 & 0 \\ -a_5a_3a_2 & a_5a_2a_1+a_5a_2a_3 & 0 & 0 \\ 0 & 0 & a_4a_3a_1+a_4a_3a_2 & -a_4a_3a_2 \\ 0 & 0 & -a_4a_3a_2 & a_4a_2a_1+a_4a_2a_3 \end{bmatrix}$$

$$=B(L)\otimes\frac{1}{a(\bar{p})}B(\bar{T}_L(M)),$$

where

$$B(L)=\begin{array}{c} p_3 \mid p_1, p_2 \\ p_2 \mid p_1, p_3 \end{array}\begin{bmatrix} a_3a_1+a_3a_2 & -a_3a_2 \\ -a_3a_2 & a_2a_1+a_2a_3 \end{bmatrix},$$

and

$$B_{\bar{p}}(\bar{T}_L(M))=\begin{array}{c} p_5 \mid \bar{p}, p_4 \\ p_4 \mid \bar{p}, p_5 \end{array}\begin{bmatrix} a(\bar{p})\,a_5 & 0 \\ 0 & a(\bar{p})\,a_4 \end{bmatrix}.$$

Further

$$D_L(M)=(a_1a_2a_3(a_1+a_2+a_3))^{4/2}, \qquad \text{and} \qquad \hat{D}_L(M)=(a_4a_5)^2.$$

The above theorem should prove useful in Möbius calculations. We conclude with an identity (4.8) obtained by comparing the exponent of $a(K'')$ on each side of (4.6.2).

For $K'=\varnothing$, we could manipulate (4.6.3), but prefer instead to use (2.8). We leave it to the interested reader to devise a Möbius algebra proof of (4.8).

Also, note that (4.8) implies, for example, that if a separable flat K' is contained in a modular flat K, then, for all K'' with $K''\wedge K=K'$, K'' is separable (since $\beta(K')$ and hence the sum would be 0).

COROLLARY 4.8. *Let K be a modular flat of M. Then*

$$\sum_{K'' : K''\wedge K=K'} \rho_M(K'')$$
$$=\sum_{K'' : K''\wedge K=K'} \beta(K'')\,\mu^+(M/K'')$$
$$=\begin{cases} \dfrac{\rho_K(K')\,\mu^+(M)}{\mu^+(K)}=\dfrac{\beta(K')\,\mu^+(K/K')\,\mu^+(M)}{\mu^+(K)} & \textit{if } K'\neq\varnothing, \textit{ and} \\ \operatorname{corank}(K)\cdot\mu^+(M) & \textit{if } K'=\varnothing. \end{cases}$$

5. q-ANALOGS FROM A MÖBIUS IDENTITY

We apply the first Möbius identity in (4.8) to two important classes of geometric lattices and see that the resulting q-identities have similar combinatorial interpretations.

5.1. *Dowling Lattices*

For specificity, we consider linear Dowling lattices $D(r, q)$ with representing matrix $[ID]$, where the columns of D are all vectors of the form $\underline{e}_i + f\underline{e}_j$ where the $\underline{e}_k$'s are unit vectors from I, f is a non-zero element from the field with $q+1$ elements, and $1 \leqslant i < j \leqslant r$.

Using results from [D], we have

$$\chi(D(r, q)) = \prod_{i=0}^{r-1} (\lambda - 1 - qi) \tag{5.1.1}$$

for the characteristic polynomial (defined in Section 1). Then,

$$\mu^+(D(r, q)) = |\chi(0)| = \prod_{i=1}^{r-1} (1 + qi) = q^{r-1} \left(r - 1 + \frac{1}{q}\right)_{r-1}, \tag{5.1.2}$$

where $(x)_{r-1}$ is the falling factorial: $x(x-1) \cdots (x-(r-1)+1)$. Further,

$$\beta(D(r, q)) = \left|\frac{\chi(D)}{\lambda - 1}(1)\right| = (r-1)!\, q^{r-1}. \tag{5.1.3}$$

Let K be a connected flat of D which contains $\underline{e}_1$. If it contains a vector $\underline{e}_{ij}$ (one of the form $\underline{e}_i + f\underline{e}_j$ or $\underline{e}_j + f\underline{e}_i$), then, by connectivity, it must contain vectors $\underline{e}_{1i_1}$, $\underline{e}_{i_1 i_2}$, ..., $\underline{e}_{i_{n-1} i}$ which, with $\underline{e}_1$, when successive linear operations are applied, will span $\underline{e}_i$ and $\underline{e}_j$. Thus, such a connected flat is the set of all vectors whose supports are in a prescribed subset of m rows (including the first). This flat K is modular and isomorphic to $D(m, q)$.

Further, $D(r, q)/K$ is isomorphic to $D(r-m, q)$, so that, using (5.1.2) and (5.1.3), we obtain

$$\begin{aligned} \rho(K) &= q^{m-1} \cdot (m-1)! \cdot q^{r-m-1} \cdot \left(r - m - 1 + \frac{1}{q}\right)_{r-m-1} \\ &= q^{r-2} \cdot (m-1)! \cdot \left(r - m - 1 + \frac{1}{q}\right)_{r-m-1}. \end{aligned} \tag{5.1.4}$$

PROPOSITION 5.2. *In $D(1+k'+k+n, q)$, let K' be the flat supported by the first $k'+1$ rows of $[ID]$, and let K be the flat supported by the first $k'+k+1$ rows. Then,* (4.8) *becomes*

$$q^{n+k+k'-1} \sum_{m=0}^{n} \binom{n}{m} (m+k')! \left(n+k-m-1+\frac{1}{q}\right)_{n+k-m-1}$$
$$= \frac{q^{k'+k-1}(k')!\,(k-1+\frac{1}{q})_{k-1} \cdot q^{n+k+k'}(n+k+k'+\frac{1}{q})_{n+k+k'}}{q^{k'+k}(k+k'+\frac{1}{q})_{k+k'}}.$$

Proof. A connected flat K'' which intersects K in K' contains $\underset{\sim}{e}_1$, so by the above remarks, it must be the set of vectors whose supports are contained in the first k' rows and m of the last n rows of $[ID]$. There are $\binom{n}{m}$ such flats K'', all of rank $m+k'$. All other evaluations are from (5.1.2), (5.1.4). ∎

In the above formula, powers of q cancel. Further,

$$\frac{(n+k+k'+\frac{1}{q})_{n+k+k'}}{(k+k'+\frac{1}{q})_{k+k'}} = (n+k+k'+\tfrac{1}{q})_n,$$

and

$$\frac{(n+k-m-1-\frac{1}{q})_{n+k-m-1}}{(k-1+\frac{1}{q})_{k-1}} = \left(n-m+k-1+\frac{1}{q}\right)_{n-m}.$$

Using the above and rearranging the factorials in (5.2), we obtain the equivalent formula

$$\sum_{m=0}^{n} \binom{m+k'}{m} \binom{n-m+k-1+\frac{1}{q}}{n-m} = \binom{n+k'+k+\frac{1}{q}}{n}, \tag{5.2.1}$$

which is a standard (Newton) binomial coefficient identity. For the partition lattice, a favorite of Rota's, and a Dowling lattice for $q=1$, we have

$$\sum_{m=0}^{n} \binom{m+k'}{m} \binom{n-m+k}{n-m} = \binom{n+k'+k+1}{n}. \tag{5.2.2}$$

Remark 5.3. Formula (5.2.1) can be easily proved by expanding appropriate binomial power series and comparing coefficients, while (5.2.2) has the following combinatorial interpretation: partition all 0–1 sequences of length $n+k'+k+1$ with n 0's according to the largest m such that the initial sequence of length $m+k'$ has m 0's (i.e., such that m 0's precede the $(k'+1)$st 1). A q-analogous argument below interprets (4.8) for projective geometries.

5.4. *Projective Geometries*

For the projective geometry $PG(d, q)$ of dimension d over a field with q elements, Möbius invariant formulas are better known:

$$\chi(PG(d, q)) = \prod_{i=0}^{d} (\lambda - q^i). \tag{5.4.1}$$

Calculations as in (5.1) yield

$$\mu^+(PG(d, q)) = q^{\binom{d+1}{2}} \tag{5.4.2}$$

(a result which appears in [R]), while

$$\beta(PG(d, q)) = (q-1)(q^2-1)\cdots(q^d-1) := \beta_q(d). \tag{5.4.3}$$

All flats K of dimension d' are connected, modular, and isomorphic to $PG(d', q)$, while $PG/K \simeq PG(d-d'-1, q)$. Hence, if K is a flat of dimension d' we obtain

$$\rho(K) = (q-1)(q^2-1)\cdots(q^d-1)\cdot q^{\binom{d-d'}{2}}. \tag{5.4.4}$$

Proposition 5.5. *In $PG(k'+k+n, q)$, let K be a flat of dimension $k'+k$ containing a flat K' of dimension k'. Then* (4.8) *becomes*

$$\sum_{m=0}^{n} q^{mk} \begin{bmatrix} n \\ m \end{bmatrix}_q (q-1)\cdots(q^{m+k'}-1)\, q^{\binom{n+k-m}{2}}$$
$$= \frac{(q-1)\cdots(q^{k'}-1)\cdot q^{\binom{k}{2}}\cdot q^{\binom{n+k+k'+1}{2}}}{q^{\binom{k+k'+1}{2}}}.$$

Proof. To count the flats K'' of dimension $k'+m$ which intersect K in K', we first contract by K' and consider a flat K''/K' disjoint from K/K'. There are then $(q^{n+k}-q^k)(q^{n+k}-q^{k+1})\cdots(q^{n+k}-q^{k+m-1})$ possible ordered bases for such a K''/K', while each K''/K' has $(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})$ such bases. The quotient is $q^{mk}\left[{n \atop m}\right]_q$, where $\left[{n \atop m}\right]_q$, a Gaussian coefficient, equals $\beta_q(n)/\beta_q(m)\,\beta_q(n-m)$. All other evaluations are from (5.4.2), (5.4.4). ∎

Remark 5.6. Manipulating powers of q and terms $\beta_q(r)$, we obtain the equivalent identity

$$\sum_{m=0}^{n} \begin{bmatrix} k'+m \\ m \end{bmatrix}_q (q^n-1)\cdot(q^n-q)\cdot\,\cdots\,\cdot(q^n-q^{m-1})\, q^m\cdot q^{n(n-m-1)}$$
$$= q^{n(n+k')}. \tag{5.6.1}$$

A combinatorial interpretation which mimics, for matrices, that in (5.3) is obtained by partitioning all $q^{n(n+k')}$ matrices $A_{n\times n+k'}$ (over F_q) according to the largest m such that the first $m+k'$ columns have rank m. The number of such $A'_{n\times m+k'}$ is $\left[{m+k' \atop m}\right]_q (q^n-1)\cdot(q^n-q)\cdot \cdots \cdot(q^n-q^{m-1})$ since each such matrix is uniquely factored as $C\cdot E$, where E is an $m\times m+k'$ row–echelon matrix representing an $(m-1)$-dimensional flat in $PG(m+k'-1, q)$ while C is an $n\times m$ matrix with independent columns (the first m independent columns of A). The next column of $A_{n\times n+k'}$ then depends on the columns of $A'_{n\times m+k'}$, and there are q^m such columns.

For completeness, we give the exact q-analog of (5.2.2) although we know no Möbius generalization:

$$\sum_{m=0}^{n}\begin{bmatrix} m+k_1 \\ m\end{bmatrix}_q\begin{bmatrix} n-m+k_2 \\ n-m\end{bmatrix}_q q^{m(k_2+1)}=\begin{bmatrix} n+k_1+k_2+1 \\ n\end{bmatrix}_q. \tag{5.6.2}$$

It is obtained by partitioning all $n\times(n+k_1+k_2+1)$ row–echelon matrices C according to the largest m such that the first $m+k'$ columns have rank m: $\left[\begin{smallmatrix} C_1 & \underline{v} & A \\ \mathbf{O} & \underline{O} & C_2\end{smallmatrix}\right]$, where C_1 is an $m\times(m+k_1)$ row–echelon matrix, the column $\left(\begin{smallmatrix}\underline{v}\\ \underline{O}\end{smallmatrix}\right)$ depends on the columns of $\left[\begin{smallmatrix} C_1 \\ \mathbf{O}\end{smallmatrix}\right]$ ($\underline{v}$ is arbitrary), C_2 is an $(n-m)\times(n-m+k_2)$ row–echelon matrix, and A is an $m\times(n-m+k_2)$ matrix which has zero vectors above the $n-m$ echelon columns of C_2 but is otherwise arbitrary.

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