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Reflection and Algorithm Proofs of Some More Lie Group Dual Pair Identities

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Recently developed reflection techniques of Gessel and Zeilberger and Schensted algorithm techniques of Benkart and Stroomer are used to give new proofs of some dual pair (or Cauchy-type) symmetric function identities first found by A. O. Morris long ago and recently found anew by Hasegawa in the context of dual pairs of representations of Lie algebras. © 1993 Academic Press, Inc.

1. INTRODUCTION

Although the identities considered here involve character of Lie groups, this paper has been written for combinatorialists. The algorithm part requires no Lie theoretic knowledge at all, and the reflection part uses only elementary facts about Weyl groups. All quantities with the arguments λ or μ are defined in Section 2.

The full fledged versions of the identities considerd here first came to our attention in a physics motivated mathematics paper [Has] by Koji Hasegawa which considers several similar "double centralizer" constructions. One goes as follows: A big representation space is constructed upon which both sp$(2n)$ and sp$(2r)$ act. It is shown that the linear span of the image of each of these Lie algebras centralizes the linear span of the image of the other. Hasegawa decomposes the big representation with respect to the action of sp$(2n)\oplus$sp$(2r)$. The (Laurent) polynomial identity $(\mathrm{C}_x\mathrm{C}_y)$ below results when the character of the big representation is equated with the sum of the characters of the irreducible constituent representations. Many other identities arising from double centralizer situations have already been studied by combinatorialists; often the adjective "Cauchy" has been attached because of an increasingly tenuous tradition. While working in a number theoretic context, Howe introduced [How] the terminology

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dual pair for a pair of algebras which centralize each other in a given situation.

An *n-partition* λ is a weakly decreasing *n-tuple* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ of integers. The corresponding *shape* λ is a diagram which has λ_i boxes in the ith row. The *rectangle* $n \times r$ consists of n rows with r boxes apiece. Fix λ such that $\lambda_1 \leqslant r$ and place the shape λ in the upper left-hand corner of $n \times r$. Notationally omitting its dependence on λ, n, and r, define μ to be the r-partition $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$, where μ_j is the number of boxes in the $(r-j+1)$st column of $n \times r$ which are not in λ. See Fig. 1 for the example $n=4$, $r=6$, $\lambda=(5, 4, 2, 0)$,and $\mu=(4, 3, 2, 2, 1, 1)$. We say that λ and μ are *complementary* in $n \times r$ and write $\lambda \cup \mu = n \times r$.

Recently Gessel and Zeilberger generalized the n-dimensional André reflection principle to finite and affine Weyl groups [GZ]. As an application, Gessel (personal communication) obtained an identity closely related to the following identity:

$$\left[\prod_{i=1}^{n} (x_i^{+1/2} + x_i^{-1/2})\right]^t = \sum \mathrm{sp}_{t-1}(\mu)\, \widetilde{\mathrm{SO}}_{2n+1}(\lambda(+1/2); x). \qquad (\mathsf{B}_x\mathsf{C})$$

Here the sum is over all $\lambda \cup \mu = n \times r$ if $t = 2r$ or $2r+1$, and the "$+1/2$" is to be used exactly when $t = 2r+1$. We first obtained this form of the identity by converting certain tableaux constructed by Gessel to "symplectic" tableaux. We then discovered that Benkart and Stroomer had recently obtained exactly the same identity with a Schensted-type algorithm [BS2] based upon work of Berele and Sundaram. However, the dual pair construction of [Has] immediately implies a closely related more general two-variable identity:

$$\prod_{j=1}^{r} \prod_{i=1}^{n} (x_i^{+1/2} y_j^{+1/2} + x_i^{-1/2} y_j^{-1/2})(x_i^{+1/2} y_j^{-1/2} + x_i^{-1/2} y_j^{+1/2})$$

$$= \prod_{j=1}^{r} \prod_{i=1}^{n} (x_i + y_j + x_i^{-1} + y_j^{-1}) = \sum \mathrm{Sp}_{2n}(\lambda; x)\, \mathrm{Sp}_{2r}(\mu; y), \qquad (\mathsf{C}_x\mathsf{C}_y)$$

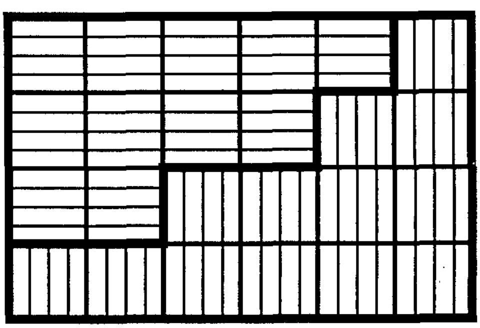

FIGURE 1

where the sum is over all $\lambda \cup \mu = n \times r$. Here we indicate how the algorithm of Benkart and Stroomer can be slightly modified to prove $(\mathsf{C}_x\mathsf{C}_y)$. We believe that this identity provides a more natural and elegant setting for their construction. Itaru Terada [Ter] and Arun Ram (personal communication) have each idependently discovered algorithmic proofs of $(\mathsf{C}_x\mathsf{C}_y)$.

We also provide a direct proof of the $y=1$ specialization of $(\mathsf{C}_x\mathsf{C}_y)$ using the Gessel–Zeilberger reflection principle. In addition we show how several specializations of some related orthogonal identities can be derived with the reflection method. These reflection constructions will affirm the naturalness of three peculiar aspects of the combinatorial descriptions of orthogonal characters which arose in [Pr1].

The following well-known Schur function identity is proved with both methods:

$$\prod_{j=1}^{r} \prod_{i=1}^{n} (x_i + y_j) = \sum s_\lambda(x)\, s_\mu(y), \tag{$\mathsf{A}_x\mathsf{A}_y$}$$

where the sum is as for $(\mathsf{C}_x\mathsf{C}_y)$. (Gessel already knew these proofs.) Using notation more consistent with the other identities, the summand could be rewritten as $\mathrm{GL}_n(\lambda; x)\, \mathrm{GL}_r(\mu; y)$. This identity also arises in the study of characters of the Lie superalgebra $\mathrm{pl}(n, r)$.

In this paper the various identities provide a beautiful setting in which the reflection and algorithm constructions can be illustrated. The reflection proofs of the identities $(\mathsf{D}_x^p\,\mathsf{BD}^p)$ and $(\mathsf{D}_x^{\mathrm{S}}\,\mathsf{BD}^+)$ return the favor by showing that it *really* is a good idea to keep in mind *all three* of the subtly different possible choices of families of orthogonal characters described in Section 5 of [Pr1]; this was a major theme of that paper.

Orthogonal identities closely related to $(\mathsf{C}_x\mathsf{C}_y)$ were first found by A. O. Morris in 1958, as is noted in more detail in Section 3. Morris used straightforward determinant calculations to establishes the identities, and $(\mathsf{C}_x\mathsf{C}_y)$ then quickly follows by a well-known auxiliary identity.

Several papers [Ber; Ok1; Pr2; Pr3; Sun] have derived symplectic or orthogonal character sums for quantities such as $(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^k$ using generalizations of Schensted's algorithm. Some of these identities can also be proved with the Gessel–Zeilberger technique, but we do not present such proofs.

Section 2 contains most definitions. On a first reading, one should skip ahead to Section 3 after the first half Section 2. There we list all of the identities which we consider and make comments concerning their proofs. Section 4 contains the algorithmic proof of $(\mathsf{C}_x\mathsf{C}_y)$. Section 5 gives background information for root systems and Weyl groups. The Gessel–Zeilberger reflection method is presented in Section 6. The reflection proofs are in Section 7.

We thank Ira Gessel for generously sharing his work with us at an early stage, and Sheila Sundaram and Itaru Terada for providing us with historical references. We also thank the referee for several helpful comments.

2. Most Definitions

Consult Sections 1 and 5 for any definitions not found in this section. Given an n-partition λ, an *n-semistandard tableau of shape λ* is a filling of the boxes in the shape λ with entries from the set $\{1, 2, ..., n\}$ such that the entries weakly increase across each row and strictly increase down each column. The *Schur function* $s_\lambda(x)$ is the sum over all n-semistandard tableaux P of shape λ of $x_1^{\# 1\text{'s}(P)} x_2^{\# 2\text{'s}(P)} \cdots x_n^{\# n\text{'s}(P)}$, where $\# k\text{'s}(P)$ is the number of times that the entry k appears in P. Given an n-partition λ, an *N-symplectic tableaux of shape λ* is a semistandard tableau with entries from $\{1, 2, ..., N\}$ such that the entries in the ith row are no smaller than $2i$-1. Given a $2n$-symplectic tableaux P, the weight monomial $x(P)$ is defined to be $x_1^{\# 2\text{'s}(P) - \# 1\text{'s}(P)} x_2^{\# 4\text{'s}(P) - \# 3\text{'s}(P)} \cdots x_n^{\# 2n\text{'s}(P) - \# (2n-1)\text{'s}(P)}$. Then the *symplectic character* $\mathrm{Sp}_{2n}(\lambda; x)$ is defined to be the sum of $x(P)$ over all $2n$-symplectic tableaux P of shape λ. The non-negative integer $\mathrm{sp}_N(\lambda)$ is defined to be the number of N-symplectic tableaux of shape λ. As noted in [Pr1], $\mathrm{sp}_N(\lambda)$ can be interpreted as the dimension of a representation of Sp_N for even *or odd N*. Given an n-partition λ, a *$(2n+1)$-Sundaram tableau of shape λ* is filling of λ with entries from $\{1, 2, ..., 2n, \infty\}$ such that the entries $\leqslant 2n$ form a symplectic tableau. The entries ∞ can occur only at the ends of rows and there cannot be more than one ∞ in a row, but one ∞ can be directly above another [Sun].

Suppose that $\lambda \cup \mu = n \times r$ and that P and Q are respectively $2n$- and $2r$-symplectic tableaux of shapes λ and μ. We often need to visualize P and Q placed simultaneously in $n \times r$. Then P is simply placed in the upper left-hand corner of $n \times r$. Before placing Q in the lower right-hand corner, it must be "flipped" about the $i = j$ main diagonal of μ and then rotated 180° in the plane. After this has been done, we still refer to the original rows (columns) of Q as rows (columns) with their original numbering, and so the usual semistandard adjectives "row weak and column strict" are preserved with respect to these nouns. However, since we are visualizing manipulations in the $n \times r$ framework, all prepositions such as "above" and "to the left" are with respect to $n \times r$, i.e., with respect to the paper. So the entries in the first column of Q end up in the last row of $n \times r$, and we say that "the entries in the first column of Q strictly decrease from left to right."

Our strongest result concering symplectic characters is the algorithm of Section 4, and by defining the $\mathrm{Sp}_{2n}(\lambda; x)$ as above it is possible for that work to take place entirely in the "catagory" of tableaux. However, we obtain only reflection results for the orthogonal characters, and so it is more natural to define these quantities with the Weyl character formula. As noted below, each case of the Weyl character formula which we use can be written as a quotient of two determinants (i.e., as a *bideterminant*), or as the sum of two such quotients. Section 5 reviews Weyl group and dominant weight terminology following [Hum]. Let W be a Weyl group acting on a Euclidean space $\mathbb{E}^n$ according to a root system R of rank n; the set of dominant weights is denoted Λ^+. Given some $\lambda \in \Lambda^+$, define

$$H(R, \lambda; x) := \frac{\sum_{\sigma \in W} (-1)^{l(\sigma)} x^{\sigma(\lambda+\delta)}}{\sum_{\sigma \in W} (-1)^{l(\sigma)} x^{\sigma(\delta)}},$$

where δ is half of the sum of the positive roots and $l(\sigma)$ is the length of σ. Once a basis for $\mathbb{E}^n$ has been fixed, x^α means $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{E}^n$. A *half-integer* is an odd integer divided by two. An *n-half partition* $\lambda = \rho + 1/2$ is a weakly decreasing sequence of positive half-integers. In Section 5 the root systems of types B_n, C_n, and D_n are described with respect to the usual axis basis for $\mathbb{E}^n$. The coordinates for the vectors δ are listed there. The set of dominant weights Λ^+ for B_n consists of all n-partitions together with all n-half partitions. Then for $\lambda \in \Lambda^+$, we define the *odd orthogonal character* $\widetilde{\mathrm{SO}}_{2n+1}(\lambda; x) := H(R(\mathsf{B}_n), \lambda; x)$. A *signed n-partition* λ^* is an n-tuple of integers satisfying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant |\lambda_n| \geqslant 0$. If λ is an n-partition, then define λ^+ to be λ, and λ^- to be the same as λ except for $\lambda_n^- = -\lambda_n$. This definition and notation is to be repeated for *signed n-half partitions* λ^*. The set of dominant weights Λ^+ for D_n consists of all signed n-partitions together with all signed n-half partitions. Then for $\lambda^* \in \Lambda^+$, we define the *even orthogonal character* $\widetilde{\mathrm{SO}}_{2n}(\lambda^*; x) := H(R(\mathsf{D}_n), \lambda^*; x)$. For either $N = 2n$ or $2n+1$, we often omit the $\sim$ when we know that λ or λ^* is integral; but note that the presence of $\sim$ does *not* imply that λ or λ^* is half-integral. The set of dominant weights Λ^+ for C_n consists of all n-partitions.

The identities listed in Section 3 have their most succinct interpretations when the characters are defined with bideterminants such as

$$\mathrm{s}_\lambda(x) = \frac{|x_j^{\lambda_i + n - i}|}{|x_j^{n-i}|}.$$

Given an n-partition λ, the symplectic character $\mathrm{Sp}_{2n}(\lambda; x)$ can also be expressed as $H(R(\mathsf{C}_n), \lambda; x)$. This quotient and the ones for $\widetilde{\mathrm{SO}}_{2n+1}(\lambda; x)$ and $\widetilde{\mathrm{SO}}_{2n}(\lambda^*; x)$ can be easily written with bideterminants; see e.g., Appendix 2 of [Pr1]. For a direct proof of the equivalence of the tableau and bideterminant definitions for $\mathrm{Sp}_{2n}(\lambda; x)$, see [Pr4]. Enough of the flavor of the definitions has now been conveyed for the reader to skip ahead to Section 3 for a first reading.

As is explained in Section 5 of [Pr1], there are actually two other useful sets of orthogonal characters $\tilde{O}_N^p(\lambda; x)$ and $\tilde{O}_N^+(\lambda; x)$ in addition to the $\widetilde{SO}_N(\lambda; x)$. Let $N=2n$ or $2n+1$. The $\tilde{O}_N^p(\lambda; x)$ are indexed by n-partitions λ or n-half partitions λ, and are defined to be just $\widetilde{SO}_N(\lambda; x)$ except when $N=2n$ and $\lambda_n \neq 0$. Then we define $\tilde{O}_N^p(\lambda; x) := \widetilde{SO}_{2n}(\lambda^+; x) + \widetilde{SO}_{2n}(\lambda^-; x)$. An *N-orthogonal partition* is an N-partition λ which does not have more than a total of N squares in the first two columns of its shape. These partitions λ index the set of characters $O_N^+(\lambda; x)$. (In this paper we do not need the spin characers $\tilde{O}_N^+(\lambda; x)$ with half-integral λ.) Let λ be N-orthogonal and let α be the number of squares in its first column. The *associate partition* λ^a is defined to have the same shape as λ except in the first column, where it is to have $N-\alpha$ squares. (It is easy to check that λ^a is a partition, and is in fact also N-orthogonal.) If $\alpha \leqslant n$, then define $O_N^+(\lambda; x) := O_N^p(\lambda; x)$. If $\alpha \geqslant n$, then define $O_N^+(\lambda; x) := O_N^p(\lambda^a; x)$, where clearly λ^a is an n-partition.

Let λ^* be a signed n-partition with $\lambda_n < 0$, and place the first $n-1$ rows of λ in the upper left-hand corner of $n \times r$. Visualize λ_n with $|\lambda_n|$ boxes sticking out to the left of the last row of $n \times r$. Now define μ^* not only to consists of the boxes in $n \times r$ which do not belong to λ^*, but also have the $|\lambda_n|$ boxes sticking out to the left as well. We describe this situation with $\lambda^* \cup \mu^* = n \times r$, which can also indicate the usual $\lambda \cup \mu = n \times r$, if $\lambda_n \geqslant 0$. See Fig. 2 for the example $n=4$, $r=6$, $\lambda=(5, 4, 2, -2)$, and $\mu=(4, 3, 2, 2, 1, 1, 1, 1)$. Let $\alpha = r + |\lambda_n|$ be the number of boxes in the first column of μ and let β be the number of boxes in the second column of μ. Note that $|\lambda_n| \leqslant \lambda_{n-1}$ and $\lambda_{n-1} + \beta = r$. Hence $\alpha + \beta = 2r - \lambda_{n-1} + |\lambda_n| \leqslant 2r$, and so we conclude that μ^* is a $2r$-orthogonal partition.

Let $N=2n$ or $2n+1$ and let λ be an n-partition. In this paper an *O_N^p-tableau Q of shape λ* is an N-symplectic tableau Q of shape λ which satisfies the additional requirement $Q_{m,2} \geqslant 2m$ (i.e., $Q_{m,2} \neq 2m-1$) on the second entry of the mth row. Given an O_N^p-tableau Q define its *weight* to be 2^*, where $*$ is the number of values of m for which $Q_{m,1} = 2m$ and

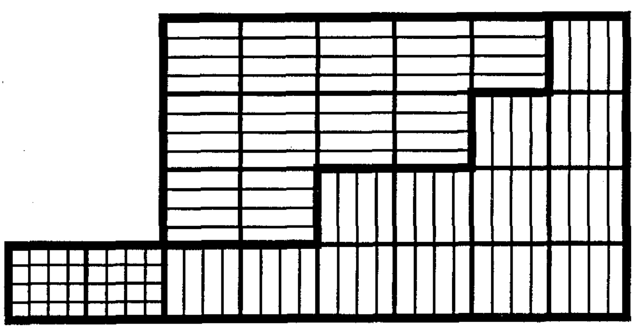

FIGURE 2

$Q_{m-1,1} = 2m-1$, as m runs from 2 to n. For a fixed n-partition λ, let $\mathrm{o}_N^p(\lambda)$ be the sum of the weights of all O_N^p-tableau of shape λ. These new O_N^p-tableaux are related to tableaux of [Kg1; KT] by a simple non-weight preserving bijection. Given a semistandard tableau Q, for each $M \geqslant 1$ let α_M (respectively β_M) be the number of entries in the first (respectively second) column of Q which are $\leqslant M$. Given an N-orthogonal partition λ, in this paper an O_N^+-*tableau* Q is an N-semistandard tableau Q such that $\alpha_M + \beta_M \leqslant M$ for each $1 \leqslant M \leqslant N$. Let $\mathrm{o}_N^+(\lambda)$ be the number of O_N^+-tableau of shape λ.

If one were concerned with only our algorithmic Section 4 and [BS2], it would be simplest to define the quantity $\widetilde{\mathrm{SO}}_{2n+1}(\lambda + 1/2; x)$ ":=" $\prod_{i=1}^n (x_i^{+1/2} + x_i^{-1/2}) \, \mathrm{Sp}_{2n}(\lambda; x)$. Since we also want this quantity in the reflection context, let us stay with the quotient expression as the definition; then the well-known identity ":=" can easily be proved using the simple identity (ii) of Appendix A2 of [Pr1] to relate the corresponding bideterminant expressions. Another definition conflict brought up by [BS2] is as follows. The orthogonal character $\mathrm{SO}_{2n+1}(\lambda; x)$ could be defined to be the sum of $x(P)$ over all $(2n+1)$-Sundaram tableaux P shape λ, where $x(P)$ is defined as in Section 2. (Just ignore the ∞'s.) See either Theorem 3.8 of [Sun] or Theorem 8.2 of [Pr1] to relate the two definitions.

The integer quantities $\mathrm{sp}_N(\lambda)$, $\mathrm{o}_{2n}^p(\lambda)$, and $\mathrm{o}_N^+(\lambda)$ were given combinatorial definitions above. The notational convention being followed here is: If we set all $x_i = 1$ in a character, then drop the "x" and switch to all lowercase letters. (In representation theory, this gives the dimension of the corresponding representation.) We now check that the orthogonal definitions are consistent under this notational convention with the quotient definitions of the corresponding characters. The $\mathrm{o}_N^+(\lambda)$ consistency is confirmed by Theorem 3.1 of [Pr1]. For $\mathrm{o}_{2n}^p(\lambda)$, let λ be an n-partition and consider the "$2n$-multiorthogonal tableaux" Q of shape λ of Theorem 8.4 of [Pr1], which come from [Kg1] (or [Ok1]). Given such a Q, for each $1 \leqslant m \leqslant n$ such that $Q_{m,1} = Q_{m,2} = \cdots = Q_{m,k} = 2m-1$, change $k-1$ of these values as follows: $Q_{m,2} = \cdots = Q_{m,k} = 2m$. This converts those multitableaux to the O_N^p-tableaux defined above.

3. List of Identities and Overview of Proofs

Fix $n \geqslant 1$ and $r \geqslant 1$. Define two quantities as follows:

$$\Delta(x) := \prod_{i=1}^{n} (x_i^{+1/2} + x_i^{-1/2}),$$

and

$$P(x, y) := \prod_{j=1}^{r} \prod_{i=1}^{n} (x_i^{+1/2} y_j^{+1/2} + x_i^{-1/2} y_j^{-1/2})(x_i^{+1/2} y_j^{-1/2} + x_i^{-1/2} y_j^{+1/2}).$$

Aside from the algorithm proof of $(\mathsf{C}_x\mathsf{C}_y)$, the most interesting aspect of this paper is the beautiful and surprising way in which the reflection derivation of the identity $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$ below perfectly explains and relates three peculiar aspects of combinatorial descriptions of orthogonal representations. (By "peculiar" we mean relative to the nicer analogous symplectic descriptions.) While the closely related identity $(\mathsf{D}_x^p\mathsf{BD}^p)$ might be preferred by some people because of the simpler nature of $\lambda\cup\mu=n\times r$ instead of $\lambda^*\cup\mu^*=n\times r$, the aspect of possible negative nth row length of highest weight partitions for $\widetilde{\mathrm{SO}}_{2n}$ fits together perfectly via the $\lambda^*\cup\mu^*=n\times r$ construction with the "long first column tail" aspect of t-orthogonal partitions for O_t. Although t-orthogonal partitions have been largely ignored since Weyl's time, they have recently arisen [Pr2] in an algorithm model for tensor representations of orthogonal groups. Even more recently tableaux of this shape (which are hybrids of the tableaux of Theorems 6.1 and 6.2 of [Pr1]) have been used by King and Welsh to construct [KW] representations of O_t. The third peculiar orthogonal aspect explained by the reflection derivation of $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$ is the $\alpha_r+\beta_r\leqslant r$ requirement for O_t^+-tableaux. It is interesting to note that of the three families of orthogonal characters, the somewhat synthetic choice of $\tilde{\mathrm{O}}_{2n}^p$ ends up paired with itself (viz. $\tilde{\mathrm{O}}_t^p$) in $(\mathsf{D}_x^p\mathsf{BD}^p)$ while the opposite natural choices of $\widetilde{\mathrm{SO}}_{2n}$ and O_t^+ end up paired with each other in $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$.

Here are all of the identities which we consider. Except for identities $(\mathsf{D}_x^{\mathrm{S}}\mathsf{D}_y^+)$ and $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$, the sums are over all $\lambda\cup\mu=n\times r$, which was defined in Section 1. For those two identities, the sums are over all $\lambda^*\cup\mu^*=n\times r$, which was defined in Section 2. In identities $(\mathsf{B}_x\mathsf{CJ})$, $(\mathsf{D}_x^p\mathsf{BD}^p)$, and $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$, let $t=2r$ or $2r+1$: then the $+1/2$ is to be used exactly when $t=2r+1$:

$$\prod_{j=1}^{r}\prod_{i=1}^{n}(x_i+y_j)=\sum s_\lambda(x)\,s_\mu(y). \qquad (\mathsf{A}_x\mathsf{A}_y)$$

$$P(x,y)=\sum \mathrm{Sp}_{2n}(\lambda;x)\,\mathrm{Sp}_{2r}(\mu;y) \qquad (\mathsf{C}_x\mathsf{C}_y)$$

$$\Delta(x)\,P(x,y)=\sum \tilde{\mathrm{O}}_{2n}^p(\lambda+1/2;x)\,\mathrm{SO}_{2r+1}(\mu;y) \qquad (\mathsf{D}_x^p\mathsf{B}_y)$$

$$P(x,y)=\sum \mathrm{O}_{2n}^p(\lambda;x)\,\mathrm{O}_{2r}^p(\mu;y) \qquad (\mathsf{D}_x^p\mathsf{D}_y^p)$$

$$\Delta(x)\,\Delta(y)\,P(x,y)=\sum \widetilde{\mathrm{SO}}_{2n+1}(\lambda+1/2;x)\,\widetilde{\mathrm{SO}}_{2r+1}(\mu+1/2;y) \qquad (\mathsf{B}_x\mathsf{B}_y)$$

$$P(x,y)=\sum \mathrm{SO}_{2n}(\lambda^*;x)\,\mathrm{O}_{2r}^+(\mu^*;y) \qquad (\mathsf{D}_x^{\mathrm{S}}\mathsf{D}_y^+)$$

$$\Delta(x)\,P(x,y)=\sum \widetilde{\mathrm{SO}}_{2n+1}(\lambda+1/2;x)\,\mathrm{Sp}_{2r}(\mu;y) \qquad (\mathsf{B}_x\mathsf{C}_y)$$

$$\Delta(x)^{2r}=\sum \mathrm{sp}_{2r}(\mu)\,\mathrm{Sp}_{2n}(\lambda;x) \qquad (\mathsf{C}_x\mathsf{C})$$

$$\Delta(x)^t = \sum \mathrm{sp}_{t-1}(\mu)\, \widetilde{\mathrm{SO}}_{2n+1}(\lambda(+1/2); x) \qquad (\mathsf{B}_x\mathsf{CJ})$$

$$\Delta(x)^t = \sum \mathrm{o}_t^p(\mu)\, \tilde{\mathrm{O}}_{2n}^p(\lambda(+1/2); x) \qquad (\mathsf{D}_x^p\mathsf{BD}^p)$$

$$\Delta(x)^t = \sum \mathrm{o}_t^+(\mu^*)\, \widetilde{\mathrm{SO}}_{2n}(\lambda^*(+1/2); x). \qquad (\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$$

In identities with only one set of variables, the order of the products has been reversed to keep with the usual way of writing a polynomial times an integer. The acronyms for the identities use the following naming conventions : 'A' for general linear, 'C' for even symplectic, 'J' for odd symplectic, 'B' for odd orthogonal, and 'D' for even orthogonal. The use of superscripts ^{p}, $^{\mathrm{S}}$, and $^+$ for the orthogonal cases follows the definitions of the three families of orthogonal characters made in Section 2.

Historic Occurrences. The variant of $(\mathsf{A}_x\mathsf{A}_y)$ presented in Section 4 below is very old. Morris obtained $(\mathsf{D}_x^p\mathsf{B}_y)$, $(\mathsf{D}_x^p\mathsf{D}_y^p)$, and $(\mathsf{B}_x\mathsf{B}_y)$ as Theorems III, II and IV of [Mor] with short bideterminant character calculations. King later conjectured these three identities and $(\mathsf{C}_x\mathsf{C}_y)$ as Eqs. (6.16)–(6.19) of [Kg2]. He then observed (personal communication) that $(\mathsf{C}_x\mathsf{C}_y)$ is an easy consequence (details below) of $(\mathsf{B}_x\mathsf{B}_y)$. We believe the first occurence of $(\mathsf{B}_x\mathsf{CJ})$ was in [BS2] and that the variations $(\mathsf{B}_x\mathsf{C}_y)$, $(\mathsf{D}_x^{\mathrm{S}}\mathsf{D}_y^+)$, and $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$ are new here.

Representation Constructions. In Theorem 3.2 of [Has], Hasegawa constructs certain big representations and then proves that they decompose in manners corresponding to the cases $(\mathsf{A}_x\mathsf{A}_y)$, $(\mathsf{C}_x\mathsf{C}_y)$, $(\mathsf{D}_x^p\mathsf{B}_y)$, and $(\mathsf{D}_x^p\mathsf{D}_y^p)$. The actual mechanics of the proof consist of a slightly general character theoretic argument posed in the language of abstract weights. He then explicitly notes that the variant of $(\mathsf{A}_x\mathsf{A}_y)$ presented in Section 4 below is a consequence in Section 3.3.1. Here is how to deduce the other three identities above from [Has], as illustrated for case $(\mathsf{C}_x\mathsf{C}_y)$: Knowledgeable representation theorists will recognize that the use of groups here instead of the Lie algebras used in [Has] is just a matter of taste. Let the eigenvalues of the groups Sp_{2n} and Sp_{2r} be $x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}$ and $y_1, ..., y_r, y_1^{-1}, ..., y_r^{-1}$, respectively. Then the eigenvalues for the subgroup $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2r}$ of the group O_{4nr} are $x_1y_1, ..., x_ny_r, x_1y_1^{-1}, ..., x_ny_r^{-1}$ and their inverses. Hence the spin character for the overall representation of $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2r}$ which Hasegawa constructs is $P(x,y)$. His Theorem 3.2 decomposes this representation into the sum of representations indicated by the right-hand sides above. Section 3.3.2 of [Has] presents a bideterminant manipulation proof of $(\mathsf{B}_x\mathsf{B}_y)$ which looks very similar to Morris' proof.

Algorithm Proofs. A variant of $(\mathsf{A}_x\mathsf{A}_y)$ is proved algorithmically in [Knu]; we give details of the relationship in Section 4. Then we present an algorithmic proof of $(\mathsf{C}_x\mathsf{C}_y)$.

Reflection Proofs. The identies $(\mathsf{A}_x\mathsf{A}_y)$, $(\mathsf{C}_x\mathsf{C})$, $(\mathsf{B}_x\mathsf{CJ})$, $(\mathsf{D}_x^p\mathsf{BD}^p)$, and $(\mathsf{D}_x^S\mathsf{BD}^+)$ are proved with the reflection technique in Section 7.

Interrelationships. The identities $(\mathsf{B}_x\mathsf{C}_y)$ and $(\mathsf{B}_x\mathsf{B}_y)$ can be deduced from $(\mathsf{C}_x\mathsf{C}_y)$ immediately by using the identity $\widetilde{\mathrm{SO}}_{2n+1}(\lambda+1/2;x)=\Delta(x)\,\mathrm{Sp}_{2n}(\lambda;x)$ noted in Section 2.

One can immediately rewrite $(\mathsf{D}_x^p\mathsf{D}_y^p)$ as $(\mathsf{D}_x^S\mathsf{D}_y^+)$ using defintions and facts from Section 2: Whenever $\lambda_n\neq 0$, replace $\mathrm{O}_{2n}^p(\lambda;x)$ by $\mathrm{SO}_{2n}(\lambda^+;x)+\mathrm{SO}_{2n}(\lambda^-;x)$. For the first of these two terms, change the corresponding factor $\mathrm{O}_{2r}^p(\mu;y)$ to $\mathrm{O}_{2r}^+(\mu;y)$. Since $\lambda\cup\mu=n\times r$, we have $\lambda_n=r-\alpha$, where α is the length of the first column of μ. Define μ^- to be the same as μ, except with a first column of length $\alpha+2\lambda_n=2r-\alpha$. So $\mu^-=\mu^a$, and hence $\mathrm{O}_{2r}^p(\mu;y)=\mathrm{O}_{2r}^+(\mu^-;y)$. Make this replacement for the cofactor of the second term above to change the sum over $\lambda\subseteq n\times r$ of $\mathrm{SO}_{2n}(\lambda^+;x)\,\mathrm{O}_{2r}^+(\mu;y)+\mathrm{SO}_{2n}(\lambda^-;x)\,\mathrm{O}_{2r}^+(\mu^-;y)$ to the sum over $\lambda^*\cup\mu^*=n\times r$

The identity $(\mathsf{C}_x\mathsf{C})$, the odd t cases of $(\mathsf{B}_x\mathsf{CJ})$ and $(\mathsf{D}_x^p\mathsf{BD}^p)$, and the even t cases of $(\mathsf{D}_x^p\mathsf{BD}^p)$ and $(\mathsf{D}_x^S\mathsf{BD}^+)$ follow immediately from the identities $(\mathsf{C}_x\mathsf{C}_y)$, $(\mathsf{B}_x\mathsf{C}_y)$, $(\mathsf{D}_x^p\mathsf{B}_y)$, $(\mathsf{D}_x^p\mathsf{D}_y^p)$, and $(\mathsf{D}_x^S\mathsf{D}_y^+)$, respectively, by setting. $y=1$. (For the odd t cases of $(\mathsf{D}_x^p\mathsf{BD}^p)$ one uses the equality $\mathrm{SO}_{2r+1}(\mu;y)=\mathrm{O}_{2r}^p(\mu;y)$ at the dimension level.) The odd t case of $(\mathsf{D}_x^S\mathsf{BD}^+)$ can be deduced from the odd t case of $(\mathsf{D}_x^p\mathsf{BD}^p)$ in a manner similar to the derivation of $(\mathsf{D}_x^S\mathsf{D}_y^+)$ from $(\mathsf{D}_x^p\mathsf{D}_y^p)$. When $t=2r$, the identity $(\mathsf{B}_x\mathsf{CJ})$ can be obtained from $(\mathsf{C}_x\mathsf{C})$ by changing each "$2t$" in a symplectic tableau for $\mathrm{sp}_{2r}(\mu)$ to an "∞" for a Sundaram-type $\mathrm{SO}_{2n+1}(\lambda';x)$ tableau. As the union runs over all $\lambda\subseteq n\times r$, this describes a simple bijection between the weighted multisets corresponding to the union of the terms $\mathrm{sp}_{2r}(\mu)\times\mathrm{Sp}_{2n}(\lambda;x)$ on the one hand and the union of terms $\mathrm{sp}_{2r-1}(\mu')\times\mathrm{SO}_{2n+1}(\lambda';x)$ on the other hand.

Orthogonal Algorithms? The only identities above which are not given any proof in this paper are $(\mathsf{D}_x^p\mathsf{B}_y)$, $(\mathsf{D}_x^p\mathsf{D}_y^p)$, and $(\mathsf{D}_x^S\mathsf{D}_y^+)$. We believe that it should be possible to give algorithmic proofs of these identities if one is willing to work hard enough. This would be done by using some kind of orthogonal tableaux from [Pr1], [Ok1], or [Sun] together with a modification of the algorithm in [Pr2] or [Ok1] made in the spirit of [BS2] and our Section 4. Also see the recent [Ok2].

4. Algorithmic Proofs

The following identity was proved by Knuth with a variation of Schensted's algorithm [Knu]:

$$\prod_{j=1}^{r}\prod_{i=1}^{n}(1+x_iz_j)=\sum s_\lambda(x)\,s_{\lambda'}(z).$$

The sum is over all $\lambda \subseteq n \times r$ and λ' is the conjugate shape of λ. It is easy to get $(\mathsf{A}_x\mathsf{A}_y)$ from this by setting $y_j = z_j^{-1}$ and then multiplying both sides by $(y_1 \cdots y_r)^n$. (Use the fact that $(y_1 \cdots y_r)^n \, s_{\lambda'}(y_1^{-1}, ..., y_r^{-1}) = s_\mu(y)$, where $\lambda \cup \mu = n \times r$.) There are several known direct algorithmic proofs of $(\mathsf{A}_x\mathsf{A}_y)$, including at least one published one [Rem]. We now sketch one such proof as an introduction to our proof of $(\mathsf{C}_x\mathsf{C}_y)$. This proof (known at least to Gessel) is easily obtained from Knuth's proof by watching the effect of applying this conversion. With more space a complete direct presentation from scratch could be made.

Rewrite the left-hand side of $(\mathsf{A}_x\mathsf{A}_y)$ to get

$$\prod_{j=1}^{r} \prod_{i=n}^{1} (y_j + x_i) = \sum s_\lambda(x) \, s_\mu(y),$$

where the inner product runs "backwards" and the sum as before is over all $\lambda \cup \mu = n \times r$. Corresponding to the expansion terms of the product, we form input "sentences" for the algorithm. Each *sentence* consists of r "words," where each *word* $i_1 i_2 \cdots i_k$ is such that $i_1 > i_2 > \cdots > i_k$. If this word is the jth word of the sentence, then its *weight* is defined to be $y_j^{n-k} x_{i_1} \cdots x_{i_k}$. The left-hand side of the identity is the generating function for the set of the 2^{nr} possible input sentences. The proof uses induction on r with n fixed; suppose we have so far constructed a weight preserving bijection proving this identity up to the $r-1$ case. Corresponding to the $(r-1)$-value of the right-hand side, consider the set of all ordered pairs (F, G) such that F and G are respectively n-semistandard of shape λ and $(r-1)$-semistandard of shape μ with $\lambda \cup \mu = n \times (r-1)$. Place F in the upper left-hand corner of an enlarged rectangular $n \times r$ and G in the lower right-hand corner of $n \times r$ (as in Section 2), leaving one empty square in each row of $n \times r$ between F and G. Use Schensted insertion to insert the decreasing letters $i_1 i_2 \cdots i_k$ of the rth word into F. By [Knu] it is known that the shape λ' of the resulting tableau P will have one more square than the shape of F in k distinct rows. Place an entry "r" in each of the remaining in-between $n-k$ squares and adjoin these squares to G to form a tableau Q of shape μ'. Now $\lambda' \cup \mu' = n \times r$ with P and Q being n-semistandard, respectively. This process is bijective by [Knu], and so the induction step is complete.

Now we prove $(\mathsf{C}_x\mathsf{C}_y)$ in a similar spirit, by revamping an algorithm of Benkart and Stroomer [BS2]. Ideally one will have [BS2] or its precursor [BS1] at hand while reading the following, but at least one should be familiar with either Berele's original algorithm [Ber] or one of its descendants [Ok1; Pr2; Pr3; Sun]. It would be much more elegant to present the entire proof from scratch. Then it would not be necessary to mention orthogonal or odd symplectic tableaux. However, such a use of journal space cannot be justified since very little effort is needed to translate the proof of [BS2] to a symplectic context (while at the same time introducing a second set of variables).

In this section the ordered alphabets $\{1, \bar{1}, 2, \bar{2}, ..., n, \bar{n}\}$ (or $\{1, \bar{1}, 2, \bar{2}, ..., r, \bar{r}\}$) are used instead of $\{1, 2, 3, 4, ..., 2n-1, 2n\}$ (or $\{1, 2, 3, 4, ..., 2r-1, 2r\}$) for the symplectic tableaux. This means that symplectic tableaux can be characterized as semistandard tableaux in which the letters i and $\bar{i}$ occur no "lower" than the ith row (i.e., these letters cannot occur in rows numbered $\geqslant i+1$). The weights assigned to these two letters are x_i and x_i^{-1} (or y_i and y_i^{-1}), respectively.

Corresponding to terms from the expansion of the left-hand side

$$\prod_{j=1}^{r} \prod_{i=n}^{1} (x_i^{+1/2} y_j^{+1/2} + x_i^{-1/2} y_j^{-1/2})(x_i^{+1/2} y_j^{-1/2} + x_i^{-1/2} y_j^{+1/2})$$

of the identity, we again form input sentences for the algorithm consisting of r words. Now each *word* further consists of an ordered pair of "syllables." Each *syllable* is an element of the product set $\{n, \bar{n}\} \times \cdots \times \{2, \bar{2}\} \times \{1, \bar{1}\}$. The x-weights of syllables are formed as in the example: $x((3, \bar{2}, 1)) = x_3^{+1/2} x_2^{-1/2} x_1^{+1/2}$. We assign the weight y_j^* to the first syllable of the jth word, where * is $n/2 - \#$, with # being the number of barred letters occuring in that syllable. Similarly assign the weight y_j^{-*} to the second syllable of the jth word. Let the xy-weight of a syllable be the product of its x-weight with its y-weight, and extend this definition multiplicatively to words and sentences. Now the left-hand side above is the generating function for the set of all possible 2^{2nr} input sentences. As we input one word of a given input sentence at a time, the algorithm will progressively construct a pair (P, Q) of symplectic tableaux. To each such pair we assign the weight $x(P)\, y(Q)$, where $x(P)$ is as Section 2 and $y(Q)$ is defined in the same way, with y_j replacing x_j.

THEOREM. *The algorithm described below gives a weight preserving bijection from the set of all sentences with r words to the set of all pairs (P, Q) of tableaux such that P and Q are respectively $2n$- and $2r$-symplectic and have complementary shapes in $n \times r$. Taking xy-weights, this bijection immediately implies the identity* $(\mathsf{C}_x \mathsf{C}_y)$.

The proof of this theorem is as above: Fix $n \geqslant 1$ and assume that it has been verified up to $r-1$. Let F and G be $2n$- and $2(r-1)$-symplectic tableaux of complementary shapes in $n \times (r-1)$. Given such a pair (F, G) and an input word w, use the following algorithm to create a new pair (P, Q) with P and Q being $2n$- and $2r$-symplectic, respectively, and complementary shapes in $n \times r$.

Algorithm. Let the two syllables of the input word w be denoted $u=\{u_1, ..., u_n\}$ and $v=\{v_1, ..., v_n\}$, where each $u_i=i$ or $\bar{\imath}$ and each $v_i=i$ or $\bar{\imath}$. Form a strictly decreasing sequence z of letters according to Definition 3.4 of [BS2]: Start with the empty sequence and let i run from n to 1. If both $u_i=v_i=i$ (or both $u_i=v_i=\bar{\imath}$), place i (or $\bar{\imath}$) at the end of z. If $u_i=i$ and $v_i=\bar{\imath}$, place nothing in z for that value of i. If $u_i=\bar{\imath}$ and $v_i=i$, place $\bar{\imath}$ followed by i on the end of z. In the sequel we will refer to these four possibilities for the ith iteration with (a)–(d), respectively. Insert the strictly decreasing sequence z of letters into F with the usual Berele procedure to create a tableau P', with the following simple modification: As annihilations occur, after sliding the empty square (or "puncture") out to the southeastern boundary of the tableau, place an $\bar{r}$ in that square, rather than simply forgetting about it as in [Ber] or placing an ∞ in it as in [BS2]. Now place P' in the upper left-hand corner of $n\times r$ and G in the lower right-hand corner of $n\times r$. Place entries r in any remaining unoccupied squares of $n\times r$, and transfer all entries $\bar{r}$ from P' to the new lower right-hand tableau which will called Q. Use P to denote the tableau resulting from removing the $\bar{r}$'s from P'. We now have a pair (P, Q) of tableaux of completementary shape in $n\times r$.

In the following lemma, the x-weight of a pair of tableaux (F, G) is defined to be $x(F)$.

LEMMA. *Fix $n\geqslant 1$ and $r\geqslant 1$. There is a simple x-weight preserving bijection between the set of all pairs (P', Q') of $(2n+1)$-Sundaram and $(2r-1)$-symplectic tableaux of complementary shape in $n\times r$ and the set of all pairs (P, Q) of $2n$-symplectic and $2r$-symplectic tableaux of complementary shape in $n\times r$.*

Proof of Lemma. Place P' and Q' in $n\times r$ in the usual way. Change each ∞ in P' to an $\bar{r}$ and move each of these to Q'. Call the resulting tableaux P and Q. Clearly P and Q have complementary shapes. The x-weight is preserved since the ∞'s contributed nothing to $x(P')$. Clearly P is $2n$-symplectic. Since there was no more than one ∞ per row P', there will be no more than one $\bar{r}$ per column of Q. Since the $\bar{r}$'s appear at the bottoms of the columns of Q, we see that Q is semistandard. Since Q has no more than r rows, it is $2r$-symplectic. ■

Proof of Theorem. Assume the theorem is true up to the value $r-1$. Use the above lemma in the reverse direction to convert each of the complementary pairs (F, G) of $2n$- and $2(r-1)$-symplectic tableaux to complementary pairs (F', G') of $(2n+1)$-Sundaram and $(2r-3)$-symplectic tableaux. Now refer to the $(2r-1)$th and $2r$th steps of the analogous procedure in [BS2]. Note that their $(2r-1)$th step really just temporarily "parks" their $(2r-1)$th input syllable S' in the column of half-boxes. Identify our second syllable v of the rth input word w with their S' syllable of the latter part of Definition 3.5 which is being "unparked" from the column of half-boxes. Identify our first syllable u of w with their $2r$th input syllable S of the latter part of Definition 3.5. So the procedure of forming and inserting z specified above is (almost) exactly the same as prescribed by Definition 3.4 and the latter part of Definition 3.5 of [BS2]. (We have harmlessly reversed the order of the pair syllables above in our version of applying Definition 3.4 since as a matter of taste we think that they should have interchanged the roles of S and S' in their application of Definition 3.4.) However, [BS2] places ∞'s in their new left tableau P' whereas we place $\bar{r}$'s in our new right tableau Q. The description of our algorithm above is the result of applying the lemma to the output of their algorithm. The placement of r's in the remaining empty boxes above is exactly the same as their placement of q's in part (ii) of Definition 4.1. Since applying the lemma is bijective in the forward direction, we can deduce the bijectiveness of our procedure from the bijectiveness of their procedure. Since their procedure produces a pair (P', Q') of $(2n+1)$-Sundaram and $(2n-1)$-symplecitc tableaux, the lemma implies that (P, Q) will be a pair of $2n$- and $2r$-symplectic tableaux. The preservation of x-weight is inherited from [BS2] via the lemma. During the n steps of forming the sequence z, suppose that each of the cases (a), (b), (c), (d) occurred respectively a, b, c, d times. Note that $a+b+c+d=n$. The y-weight of the input word w was $c-d$. Suppose that e annihilations occurred during the insertion of the $a+b+2d$ letters of z. Then P has $a+b+2d-2e$ letters more than F. Therefore Q has $n-(a+b+2d-2e)$ letters more than G. But e of these letters are $\bar{r}$'s. Therefore the y-weight of Q is y_r^* times the y-weight of G, where $* = \#\, r\text{'s} - \#\, \bar{r}\text{'s} = (n-a-b-2d+e)-e = n-a-b-2d = c-d$. Since this was the exponent of y_r for the y-weight of w, the procedure preserves y-weights as well. ■

5. Weyl Groups and Weight Lattices

First we review the general terminology [Hum]. Suppose that a finite set of vectors R in the Euclidean space $\mathbb{E}^n$ forms a *root system*. Define the modified inner product $\langle \gamma, \alpha \rangle := 2(\gamma, \alpha)/(\alpha, \alpha)$. The *reflection of* γ *with respect to* α is defined to be $r_\alpha(\gamma) := \gamma - \langle \gamma, \alpha \rangle \alpha$. The *Weyl group* W is the finite group generated by the reflections with respect to the roots. Fix a choice of *positive roots* R^+. This determines the set of *simple roots* $\Delta = \{\alpha_i\}_{i=1}^n$. The *length* $l(\sigma)$ of $\sigma \in W$ is the minimum number of reflections with respect to simple roots with which σ can be expressed as a product. The *lattice of weights* Λ consists of all vectors λ such that $\langle \lambda, \alpha \rangle$ is an integer for all $\alpha \in R$. It is invariant under W. The *dominant weights* Λ^+ consist for all $\lambda \in \Lambda$ such that $\langle \lambda, \alpha \rangle$ is nonnegative for all α, and the *strictly dominant weights* Λ^{++} consist of all λ such that $\langle \lambda, \alpha \rangle$ is always positive. Let δ be the unique vector such that $\langle \delta, \alpha_i \rangle = 1$ for $1 \leqslant i \leqslant n$. Then $\Lambda^{++} = \Lambda^+ + \delta$. The *dominant chamber* is $\mathscr{C} := \{\gamma : (\gamma, \alpha) > 0 \ \forall \alpha \in R\}$. A *wall* of the dominant chamber consists of the intersection of the closure $\bar{\mathscr{C}}$ of the dominant chamber with one of the *root hyperplanes* $\{\gamma : (\gamma, \alpha) = 0\}$.

In this paper we are concerned primarily with the root systems of types B_n, C_n, and D_n. The positive roots can be specified with respect to some fixed orthogonal basis $\{e_i\}$ of $\mathbb{E}^n$. First define four sets of vectors, where $1 \leqslant i \neq j \leqslant n$: $R_1 := \{e_i - e_j\}$, $R_2 := \{e_i + e_j\}$, $R_3 := \{e_i\}$, and $R_4 := \{2e_i\}$. Then $R = R_1 \cup R_2 \cup R_3$ for B_n, $R = R_1 \cup R_2 \cup R_4$ for C_n, and $R = R_1 \cup R_2$ for D_n. The respective reflections with respect to these roots have the following effects on the n-tuple of coordinates $(\gamma_1, \gamma_2, ..., \gamma_n)$ of a vector γ: reflection r_{ij} interchanges γ_i with γ_j, reflection $\bar{r}_{ij}$ interchanges γ_i with γ_j and then multiplies each of those two coordinates by -1, and corresponding to the roots in either R_3 or R_4 the reflections r_i multiply γ_i by -1. Hence the Weyl groups for the root systems B_n and C_n are the same and are generated by r_{ij}, $\bar{r}_{ij}$, and r_i, while the Weyl group of type D_n is generated by r_{ij} and $\bar{r}_{ij}$. Next define $\mathbb{Z}^n + 1/2$ to be the set of all n-tuples of half-integers. Then the set of weight Λ is $\mathbb{Z}^n$ for C_n and $\mathbb{Z}^n \cup (\mathbb{Z}^n + 1/2)$ for B_n and D_n. The sets of dominant weights Λ^+ were described in Section 2. The vector δ is $(n, n-1, ..., 2, 1)$ for C_n and $((n-1)/2, (n-3)/2, ..., 3/2, 1/2)$ for B_n and $(n-1, n-2, ..., 1, 0)$ for D_n. Hence we see that Λ^{++} consists of strictly decreasing n-tuples λ of the following quantities: positive integers for C_n; for B_n also allow all positive half-integers; and for D_n further also allow such n-tuples but with $\lambda_n < 0$ as long as $|\lambda_n| < \lambda_{n-1}$.

One other geometric context is needed for the identity $(\mathsf{A}_x\mathsf{A}_y)$. The following setup is closely related to the A_{n-1} version of the above, which would normally be described in $\mathbb{E}^{n-1}$. We call the following context S_n, since the reflection group W is be nth symmetric group (as it would be for A_{n-1}). This version gives a nicer picture which also has cleaner coordinates. Let Λ be $\mathbb{P}^n$, the set of all n-tuples of positive integers. Let Λ^+ be the set of all n-partitions with nonzero components, and let Λ^{++} be the subset of Λ^+ with strictly decreasing components. Let $\delta = (n, n-1, ..., 2, 1)$. The reflections r_{ij} still act by interchanging two coordinates, and they generate the symmetric group S_n.

6. The Gessel–Zeilberger Reflection Method

In 1887 André used a reflection argument to count the number of north-easterly lattice paths in the plane from $(2, 1)$ to $(m+2, n+1)$ with $m \geqslant n$

which never touch the line $x=y$. Zeilberger generalized this to n dimensions in 1983 [Zei]. Recently Gessel and Zeilberger further generalized this counting procedure to settings where the reflections come from Weyl groups or affine Weyl groups. The proposition below is a version due to Gessel of the methods in [GZ].

Let $\mathscr{P}$ be a lattice invariant under W, and let $\mathscr{S}$ be a subset of vectors β which is also invariant under W and which generates $\mathscr{P}$. We say that $\mathscr{S}$ is a *step set* for $\mathscr{P}$ and R if for each $\alpha \in R$ there is a constant $c(\alpha) \in \mathbb{R}$ such that $\langle \beta, \alpha \rangle = 0$ or $\pm c(\alpha)$ as β runs over $\mathscr{S}$. Let $\mathscr{P}^{++}$ be the intersection $\mathscr{P} \cap \mathscr{C}$. Given two points φ, $\psi \in \mathscr{P}^{++}$, we want to count the number $F(\varphi, \psi, t)$ of paths φ, $\varphi + \beta_{i_1}$, $\varphi + \beta_{i_1} + \beta_{i_2}, \ldots, \psi = \varphi + \sum_{j=1}^{t} \beta_{i_j}$ from φ to $\psi \in \mathscr{P}^{++}$ in t steps which never leave $\mathscr{P}^{++}$. Such paths will be called *good.*

PROPOSITION. *Let $\varphi \in \mathscr{P}^{++}$, and let $\mathscr{S}$ be a step set for $\mathscr{P}$ and R. Then*

$$\left[\sum_{\beta \in \mathscr{S}} x^{\beta}\right]^{t} \sum_{\sigma \in W} (-1)^{l(\sigma)} x^{\sigma(\varphi)} = \sum_{\psi \in \mathscr{P}^{++}} F(\varphi, \psi, t) \sum_{\sigma \in W} (-1)^{l(\sigma)} x^{\sigma(\psi)}.$$

Proof. Theorem 3 of [GZ] states that $F(\varphi, \psi, t)$ is the coefficient of x^{ψ} in the left-hand side for $\psi \in \mathscr{P}^{++}$. Since $\mathscr{S}$ is invariant under W, the left-hand side is skew-symmetric under W. Hence the rest of the right-hand side must be as stated. ∎

Further Proof Remarks. The constant inner product requirement on $\mathscr{S}$ is needed for the reflection argument in order to ensure that a path can never "jump" one of the walls of the dominant chamber: A wall $\{\gamma: (\gamma, \alpha') = 0\}$ can be jumped with a step $\beta \in \mathscr{S}$ only if there is some $\gamma \in \mathscr{P}^{++}$ such that $(\gamma, \alpha) > 0$ for all α but with $(\gamma + \beta, \alpha') < 0$. The condition implies $(\gamma, \alpha') = mc(\alpha') > 0$ for some $m \in \mathbb{Z}$ while $(\beta, \alpha') = c(\alpha')$. Hence $(\gamma + \beta, \alpha') < 0$ is impossible.

For the proof of the main result Theorem 1 of [GZ], a total order should be put on the set of positive roots R^{+}. Then when one is reflecting the beginning portion of a path to ψ up to its last point γ on a wall of $\mathscr{C}$, one would reflect with respect to the minimal positive root α for which $(\gamma, \alpha) = 0$. This resolves any ambiguity in the method caused by having γ on more than one root hyperplane. The implicit assumption of [GZ] that γ can only be on a wall corresponding to a simple root is unnecessary.

7. REFLECTION RESULTS

For the identity $(\mathsf{A}_x\mathsf{A}_y)$, the proposition above is used in the S_n context. The step set is $\mathscr{S} = \{\sum_{i \in T} e_i \colon T \subseteq \{1, 2, \ldots, n\}\}$, which consists of 2^n vectors, including 0. The jth step is given the weight $y_j^{n-|T|}$.

For the other identities listed in the following theorem, the proposition is used with the Weyl group BC_n or D_n. When either of these groups acts on $\mathbb{E}^n$ as in Section 5, it is not hard to show (Gessel, personal communication) that the only possible choice for $\mathscr{S}$ is either the $2n$ vectors $\{\pm e_i: 1 \leqslant i \leqslant n\}$ or the 2^n vectors $\{(\pm d, \pm d, ..., \pm d)\}$ for some choice of c or d. (Actually 0 can be adjoined to either choice.) We use the latter possibility with $d = 1/2$, i.e., $\mathscr{S} = \{(\pm 1/2, \pm 1/2, ..., \pm 1/2)\}$, and the lattice $\mathscr{P} = \Lambda(\mathsf{BD}_n)$. If the first choice for $\mathscr{S}$ is taken, then very similar work to that described here would prove the orthogonal and symplectic character sums for $[x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1}]^t$ which appear in [Ok1; Pr2; Pr3].

Is it possible to modify the reflection technique in order to obtain proofs of the two variable forms of the identities other than $(\mathsf{A}_x\mathsf{A}_y)$?

THEOREM. *The identities* $(\mathsf{A}_x\mathsf{A}_y)$, $(\mathsf{C}_x\mathsf{C})$, $(\mathsf{B}_x\mathsf{CJ})$, $(\mathsf{D}_x^p\mathsf{BD}^p)$, *and* $(\mathsf{D}_x^{\mathrm{S}}\mathsf{BD}^+)$ *can be proved with the reflection method.*

Proofs. Roughly speaking, we apply the reflection proposition with φ chosen to be the δ for the root system appearing with subscript x in the identity label and with $\psi = \lambda + \delta$. In each case we need to construct a set of tableaux for each $\lambda \in \mathscr{P}^{++}$ which label each possible good path counted by $F(\delta, \lambda + \delta, t)$. Then dividing both sides by the alternating Weyl denominator will give the desired identity, after we easily check that the sum condition accurately describes the set of all $\lambda + \delta$ that can be reached from δ in t steps. In each case the paths start at δ with $\lambda = 0$ being described with the empty shape $\lambda = \varnothing$ and the single tableau Q being the null tableau of shape $\varnothing$. Then using induction on t we assume that a pair (λ, Q) as in the statement of the identity has been constructed for each good path with $t-1$ steps from δ to $\delta + \lambda$, where the shape of Q is determined by λ.

For $(\mathsf{A}_x\mathsf{A}_y)$ take $W = S_n$, $\delta = \delta(S_n)$, $\mathscr{P} = \Lambda(S_n)$, and $\mathscr{P}^{++}(S_n)$ as at the end of Section 5, and $\mathscr{S}$ as above. Assume that after $t-1$ steps we are at a $\lambda + \delta$ such that λ is an n-partition, and that for each good path to $\lambda + \delta$ we have a $(t-1)$-semistandard tableau Q of shape μ such that $\lambda \cup \mu = n \times (t-1)$. Place λ in the upper left-hand corner of an enlarged rectangle $n \times t$ and Q in the lower right-hand corner of $n \times t$ (as in Section 2), leaving one unassigned square of $n \times t$ in each row between λ and Q. The only steps β from $\lambda + \delta$ which stay in $\mathscr{P}^{++}$ are those such that $\lambda + \beta$ is still an n-partition. If such a $\beta = \sum_{i \in T} e_i$, adjoin a square to λ at the end of the ith row for each $i \in T$, thereby creating λ'. The remaining unassigned squares occur at the bottoms of the jth columns of Q for each $j \notin T$. Place a "t" in each of these to create Q'. Clearly Q' is t-semistandard of shape μ' such that $\lambda' \cup \mu' = n \times t$. The weights $y_j^{n-|T|}$ are preserved under the action of W, and so the reflection proposition can be extended in this case from an integer coefficient $F(\delta, \lambda+\delta, t)$ to a polynomial coefficient $F(\delta, \lambda+\delta, t; y)$ $(=s_\mu(y)$ when $\lambda \cup \mu = n \times t)$.

For $(\mathsf{C}_x\mathsf{C})$ take $W=W(\mathsf{BC}_n)$, $\delta=\delta(\mathsf{C}_n)$, $\mathscr{P}=\Lambda(\mathsf{BD}_n)$, $\mathscr{P}^{++}=\Lambda^{++}(\mathsf{B}_n)$, and $\mathscr{S}$ as above. Assume that after $2r-2$ steps we are at a $\lambda+\delta$ such that $\lambda \subseteq n \times (r-1)$, and that for each good path to $\lambda+\delta$ we have a $(2r-2)$-symplectic tableau Q of shape μ such that $\lambda \cup \mu = n \times (r-1)$. Consider the next two steps from $\mathscr{S}$ together. Do the following procedure for each extension of (λ, Q) to a new λ' via a good path extension with two steps. Start the accounting update by placing λ in the upper left-hand corner of $n \times r$ and Q in the lower left-hand corner of $n \times r$. The length of each row of λ will change by -1, 0, or $+1$ when passing from λ to λ'. Give this new square to λ' if the row change was $+1$. If the row change was -1, place $2r$ and $2r-1$ in the two empty squares (left to right) of $n \times r$ in that row and give these two squares to Q'. If a row change of 0 resulted from a $+1/2$ at the $(2r-1)$th steps followed by a $-1/2$ at the $2r$th step, place $2r$ in the empty square and give it to Q'. Otherwise, if the 0 change came from a $-1/2$ followed by a $+1/2$, place a $2r-1$ in the square to be adjoined to Q'. This process is clearly injective from the set of good extensions of the paths to a set of (possibly bad) extensions of Q; we must check that this process produces good $(2r)$-symplectic extensions of Q and is bijective with all such good extensions. When checking the symplectic conditions for Q keep in mind its flipped and rotated position and the conventions regarding prepositions and nouns given in Section 2. Both λ and λ' are good shapes, and so by construction we have $\lambda \cup \mu = n \times (r-1)$ and $\lambda' \cup \mu' = n \times r$ with Q and Q', respectively, of good shapes μ and μ'. Hence the shape μ'/μ is that of a good skew tableau, and the new entries $2r-1$ and $2r$ are bigger than all existing entries of Q. By constuction, the entries strictly decrease left to right within a column of Q'. Given the procedure for filling in two new squares within a row, and given that μ/μ' is a good skew shape, the only conceivable problem could be one new square in each of rows i and $i+1$ directly above and underneath each other, with the respective new entries being $2r-1$ and $2r$. However, this would mean that the $(2r-1)$th step subtracted $1/2$ from the ith coordinate while adding $1/2$ to the $(i+1)$st coordinate, which is impossible since the two coordinates of λ were equal before the $(2r-1)$th step. (The impossibility arises because $\lambda_i+\delta_i-1/2=\lambda_i+n-i+1/2$ and $\lambda_{i+1}+\delta_{i+1}+1/2=\lambda_{i+1}+n-i+1/2$ and so $\lambda_i=\lambda_{i+1}$ would imply that these two coordinates are equal after the $(2r-1)$th step, which is not allowed in $\mathscr{P}^{++}$.) There are no special problems with the last row; it is actually possible for the last coordinate of λ to be $-1/2$ temporarily after the $(2r-1)$th step, since then $\lambda+\delta$ would still have its last coordinate positive. If all of the preceding is understood, then it is clear that this accounting process is bijective with *all* possible good symplectic extensions Q' of Q. Hence $F(\delta, \lambda+\delta, 2r)=\mathrm{sp}_{2r}(\mu)$ when $\lambda \cup \mu = n \times r$.

The $t = 2r$ case of $(\mathsf{B}_x\mathsf{CJ})$ is proved in the last paragraph. There are two ways to arrive at the $t = 2r + 1$ case. In the first way, one chooses $\delta = \delta(\mathsf{C}_n)$ as above, but then realizes that one cannot stop the path after an odd number of steps since one will not then be in $\Lambda(\mathsf{C}_n)$, which is needed to end up with a C_n character. So one simply multiplies $(\mathsf{C}_x\mathsf{C})$ by $\Delta(x)$ on both sides and uses the simple identity $\widetilde{\mathrm{SO}}_{2n+1}(\lambda + 1/2; x) = \Delta(x)\,\mathrm{Sp}_{2n}(\lambda; x)$. In the second way, one chooses $\delta = \delta(\mathsf{B}_n) = ((n-1)/2, (n-3)/2, ..., 3/2, 1/2)$. Note that the only first step possible is to choose $\beta = (1/2, 1/2, ..., 1/2)$ and go to $(n, n-1, ..., 2, 1) = \delta(\mathsf{C}_n)$.There is no need to "account" a forced step, so one just proceeds from $\delta(\mathsf{C}_n)$ as in the preceding paragraph. Then divide by the Weyl denominator alternating sum for B_n at the end to get $(\mathsf{B}_x\mathsf{C})$.

For the $t = 2r$ case of $(\mathsf{D}_x^p\mathsf{BD}^p)$, take $W = W(\mathsf{D}_n)$, $\delta = \delta(\mathsf{D}_n)$, $\mathscr{P} = \Lambda(\mathsf{BD}_n)$, $\mathscr{P}^{++} = \Lambda^{++}(\mathsf{D}_n)$, and $\mathscr{S}$ as above. The paths start at δ, which has $\delta_n = 0$. Let k be the number of later points γ on the path, possibly including the end $\lambda + \delta$, which have $\gamma_n = 0$. Map every good path to a path in $\Lambda^+(\mathsf{B}_n)$ be replacing γ_n by $|\gamma_n|$ for every point γ in path. Paths in $\Lambda^+(\mathsf{B}_n)$ which touch the $\gamma_n = 0$ hyperplane k times after δ have 2^k paths mapped onto them by this process. Hence if $\lambda + \delta \in \Lambda^+(\mathsf{B}_n)$, the count $F(\delta, \lambda + \delta, 2r)$ is obtained by summing the quantity 2^k over the set of all good paths which lie entirely in $\Lambda^+(\mathsf{B}_n)$. Note that if $\lambda_n \neq 0$, we must group together the two D_n Weyl quotients with $\pm\lambda_n$ in order to obtain the character $\mathrm{O}_{2n}^p(\lambda; x)$ in the identity. Nothing need be done if $\lambda_n = 0$. Now follow the proof of $(\mathsf{C}_x\mathsf{C})$. Normally in $\mathscr{P}^{++} = \Lambda^{++}(\mathsf{D}_n)$ the last coordinate of a point can be negative. Here, however, replace λ_n by $|\lambda_n|$ whenever this happens as the pairs (λ, Q) are being constructed as with $(\mathsf{C}_x\mathsf{C})$. The portion of the path following the last incidence of $\gamma_n = 0$ (which can be possibly empty) need not be given special accounting for the replacing of λ_n by $|\lambda_n|$ because of the character grouping noted above. We only need to count how many times we return to $\gamma_n = 0$ in order to account for the replacing of λ_n by $|\lambda_n|$ elsewhere. In terms of the construction of Q', this happens whenever the last row length λ_n of λ is reduced from 1 to 0 and two boxes containing $2r$ and $2r - 1$ are adjoined to Q to from Q'. However, this factor of 2 in the weight 2* is exactly one of the ways in which the combinatorial definition of $\mathrm{o}_{2n}^p(\mu)$ differs from the combinatorial definition of $\mathrm{sp}_{2n}(\mu)$. We now only need to check that the other special condition $Q'_{r,2} \neq 2r - 1$ arises in the reflection construction of Q'. In order for $Q'_{r,2} = 2r - 1$ to happen, we would have had to have $\lambda_{n-1} = 0$ and hence $\lambda_n = 0$ as well. Also, the $(2r-1)$th and $2r$th increments in the $(n-1)$th coordinates must have been $-1/2$ followed by $+1/2$. But it is impossible to have $\lambda_{n-1} + \delta_{n-1} - 1/2 = 1/2 > |\lambda_n + \delta_n + \beta_n| = |\beta_n| = 1/2$ for any value of β_n, as is required to be in $\mathscr{P}^{++} = \Lambda^{++}(\mathsf{D}_n)$. We leave it to the reader to confirm that there are no other differences between this case and $(\mathsf{C}_x\mathsf{C})$. If so, then $F(\delta, \lambda + \delta, 2r) = \mathrm{o}_{2n}^p(\mu)$ when $\lambda \cup \mu = n \times r$.

For the $t=2r$ case of $(\mathsf{D}_x^s\mathsf{BD}^+)$, take $W=W(\mathsf{D}_n)$, $\delta=\delta(\mathsf{D}_n)$, $\mathscr{P}=\Lambda(\mathsf{BD}_n)$, $\mathscr{P}^{++}=\Lambda^{++}(\mathsf{D}_n)$, and $\mathscr{S}$ as above. Now allow the last coordinate of λ^* to become negative. If the same construction of Q' is used as for $(\mathsf{C}_x\mathsf{C})$, we pass from $\lambda^* \cup \mu^* = n\times(r-1)$ to $\lambda^{*\prime} \cup \mu^{*\prime} = n\times r$. Drop the *'s for the rest of this paragraph. No problems are presented by the possible negative row length λ_n in the allocation of the new square in the last row. The only new aspect here are the O_{2r}^+-tableau conditions that the sum of the first two column lengths $\alpha'_{2r}+\beta'_{2r}$ of Q' cannot exceed $2r$, and also that $\alpha'_{2r-1}+\beta'_{2r-1}\leqslant 2r-1$ for Q'. The first condition is obviously satisfied if $\lambda'_n\geqslant 0$. So assume that $\alpha'_{2r}+\beta'_{2r}\leqslant 2$ is violated with $\lambda'_n<0$. Then $r+|\lambda'_n|+(r-\lambda_{n-1})>2r$, implying that $|\lambda'_n|>\lambda_{n-1}$, which would be a violation of the condition for $\mathscr{P}^{++}$. Since $\alpha_{2r-2}+\beta_{2r-2}\leqslant 2r-2$ for Q, the other condition $\alpha'_{2r-1}+\beta'_{2r-1}\leqslant 2r-1$ can be violated only if $\alpha_{2r-2}+\beta_{2r-2}\leqslant 2r-2$ and a $2r-1$ occurs in each of the last two rows of $n\times r$. When $\lambda_n\geqslant 0$ this can happen only when $Q'_{r,2}=2r-1$; the argument against this is as in the preceding paragraph. So suppose that the second condition is violated with $\lambda_n<0$. Then in each of the last two rows of λ, the $(2r-1)$th and $2r$th increments were $-1/2$ followed by $+1/2$. But $\alpha_{2r-2}+\beta_{2r-2}=2r-2$ (i.e., the first two columns of Q had a total of $2r-2$ squares), and so $(r-1+|\lambda_n|)+(r-1-\lambda_{n-1})=2r-2$. Thus $|\lambda_n|=\lambda_{n-1}$, which means $|\lambda_n-1/2|>\lambda_{n-1}-1/2$ after the $(2r-1)$th step, a violation of the condition for $\mathscr{P}^{++}$. So the second O_{2n}^+-tableau condition is satisfied as well and $F(\delta,\lambda+\delta,2r)=\mathrm{o}_{2n}^+(\mu^*)$ when $\lambda^*\cup\mu^*=n\times r$.

The $t=2r$ case of $(\mathsf{B}_x\mathsf{CJ})$ and the $t=2r-1$ cases of $(\mathsf{D}_x^p\mathsf{BD}^p)$ and $(\mathsf{D}_x^s\mathsf{BD}^+)$ are now confirmed. In each case, start with the $t-1$ version of the same identity and take one more step. Account this step as follows. Add a half square to each row between λ and Q. If the ith coordinate of λ increases by 1/2, give the half square to λ. If the ith coordinate of λ decreases by 1/2 remove a half square from λ and give Q' a full square with entry t. Now remove a half square from the end of each row of λ', creating λ''. Note that $\lambda''\cup\mu'=n\times(r-1)$ (and also analogously $\lambda^{*\prime\prime}\cup\mu^{*\prime}=n\times(r-1)$). Also note that the difference between Q and Q' is just a "horizontal strip" of t's, which is exactly the difference in the first case between sp_{2r-2} tableaux and sp_{2r-1} tableaux and in the last two cases between $(2r-2)$-tableaux and $(2r-1)$-tableaux. ■

Note added in proof. Two closely related recent papers are [Ok2] and [GM]. Our assertion at the beginning of Section 7 that there are only two possible step sets of type D_n is wrong; see Section 4.5 of [GM] for some other possibilities.

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