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Product Evaluations of Lefschetz Determinants for Grassmannians and of Determinants of Multinomial Coefficients

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A general result which produces product evaluations of determinants of certain raising operators for sl(2) representations is obtained. The most combinatorially interesting cases occur for self-dual raising operators of Peck posets. Applications include the following: A nice product expression is found for the determinant of the Lefschetz duality linear transformation on the cohomology of a Grassmannian. Known product expressions for the cardinalities of two sets of plane partitions are re-derived. The appearance of rising factorials for the hooks in one of these product expressions is "explained" by the appearance of rising factorials in sl(2) determinants. A higher dimensional generalization in a certain sense of MacMahon's famous product enumeration result for Ferrers diagrams contained in a box is stated in the context of nonintersecting lattice paths. © 1990 Academic Press, Inc.

1. INTRODUCTION

Recently there has been much interest in the enumeration of certain sets of plane partitions. There are many such sets of objects for which elegant and mysterious product enumeration formulas have been proved or conjectured. Often the enumeration process proceeds in two stages. First, combinatorial or recursive arguments are used to show that a determinant enumerates the set of objects in question. The entries of these determinants tend to be simple expressions involving binomial coefficients. Second, the determinant is evaluated to produce the desired product formula. Sometimes the second stage is very difficult.

Here we develop a method of obtaining product evaluations of determinants arising from certain raising operators of sl(2) representations. The more combinatorially interesting cases occur for self-dual raising operators of Peck posets. We will explicitly state the result of applying the method to the poset which is a product of chains, and to the distributive lattice

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$L(n, m)$. Each of these two cases evaluates a generalization of a determinant which enumerates plane partitions.

Let $G_{n,n+m}$ denote the Grassmann manifold of n-dimensional subspaces of an $(m+n)$-dimensional vector space. This can be realized as a complex projective variety of dimension nm under the Plucker embedding. The elements of $L(n, m)$ index the Schubert varieties in $G_{n,n+m}$, which in turn provide a basis for the cohomology of $G_{n,n+m}$. The kth Lefschetz transformation is a linear transformation from $H^{2k}(G_{n,n+m})$ to $H^{2nm-2k}(G_{n,n+m})$ which is obtained by intersecting with a generic hyperplane section $nm-2k$ times. Corollary 2 gives a product expression for the determinant of this linear transformation with respect to the natural Schubert cocycle basis for the cohomology.

There is a famous product enumeration formula due to MacMahon for the number of 3-dimensional Ferrers diagrams fitting in a rectangular parallelepiped. This formula actually makes sense and works in dimensions 1, 2, and 3. The analogous enumeration problem in dimension 4 does not seem to have a nice product answer. However, there is a viewpoint of Gessel by which the 3-dimensional Ferrer diagrams can be interpreted as non-intersecting tuples of lattice paths in the plane. These tuples are enumerated by the determinant of Corollary 1 when $n=2$. We will indicate how the general determinant identity of this corollary can be interpreted as a higher-dimensional analog in a certain sense of MacMahon's identity.

The product enumeration formulas for the number of plane partitions of rectangular shape and for the number of standard Young tableaux of rectangular shape have hook products in the denominator which are expressible with rising factorials. In fact, rising factorials are very common in the subject of product enumerations of plane partitions. Rising factorials arise naturally in the context of representations of sl(2). The connection between these two phenomenon will become apparent below for rectangular shapes.

In a paper [Pr3] contemporaneous with this one we use closely related sl(2) methods to confirm an old Sperner conjecture of Stanley's concerning the lattice of bounded column strict plane partitions of fixed shape.

2. Definitions and Background

A partition λ is a sequence of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The shape (or Ferrers diagram) λ is a left-justified diagram with λ_i boxes in the ith row. Let $L(n, m)$ denote the distributive lattice of all shapes contained in an $n \times m$ box, ordered by containment. I.e., $L(n, m)$ consists of weakly decreasing n-tuples $(\lambda_1, ..., \lambda_n)$ such that $0 \leqslant \lambda_i \leqslant m$, with order given by component-wise comparison. A plane partition T of shape contained in λ with parts bounded by n is an array of non-negative integers $T_{ij} \leqslant n$

satisfying $T_{ij} \geqslant T_{i,j+1}$ and $T_{ij} \geqslant T_{i+1,j}$. A column strict plane partition T of shape λ with parts bounded by n is an array of positive integers $T_{ij} \leqslant n$ satisfying $T_{ij} \geqslant T_{i,j+1}$ and $T_{ij} > T_{i+1,j}$.

A *ranked poset P of length R* is a partially ordered set P together with a partition $P = \bigcup_{i=0}^{R} P_i$ into $R+1$ non-empty ranks P_i, $0 \leqslant i \leqslant R$, such that elements in P_i cover only elements in P_{i-1}. A ranked poset P is *strongly Sperner* if for every $k \geqslant 1$ no union of k antichains contains more elements than the union of the k largest ranks of P does. A ranked poset is *rank symmetric* if $|P_i| = |P_{R-i}|$ for $0 \leqslant i < R/2$. It is *rank unimodal* if $|P_0| \leqslant |P_1| \leqslant \cdots \leqslant |P_k| \leqslant |P_{k+1}| \leqslant \cdots \leqslant |P_R|$ for some $0 \leqslant k \leqslant R$. It is *Peck* if it is rank symmetric, rank unimodal, and strongly Sperner.

If $a \in P_k$ *and* $b \in P_{k+h}$, then a saturated chain in P from a to b is a sequence of elements $a = c_0 < c_1 < c_2 < \cdots < c_h = b$. A rank symmetric unimodal poset P is said to have *Property T* if there exist $|P_k|$ disjoint saturated chains in P from elements in P_k to elements in P_{R-k}.

Associate to any ranked poset $P = \bigcup_{i=0}^{R} P_i$ a graded complex vector space $\tilde{P} = \bigoplus_{i=0}^{R} \tilde{P}_i$, where $\tilde{P}_i$ is the complex vector space freely generated by vectors $\tilde{a}$ corresponding to elements of P_i. A linear operator X on $\tilde{P}$ is a *lowering operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i-1}$. It is a *raising operator* if $X\tilde{P}_i \subseteq \tilde{P}_{i+1}$. A raising operator defined by

$$X\tilde{a} = \sum \Theta(a, b)\tilde{b}$$

is an *order raising operator* if $\Theta(a, b) \neq 0$ implies b covers a. The *unitary* raising operator is the raising operator with $\Theta(a, b) = 1$ whenever b covers a. Define a particular linear operator H on $\tilde{P}$ by

$$H\tilde{a} = (2i - R)\tilde{a}$$

when $a \in P_i$.

The Lie algebra $\mathrm{sl}(2) = \mathrm{sl}(2, \mathbb{C})$ consists of all 2×2 trace zero complex matrices with Lie algebra multiplication given by $[u, v] = uv - vu$. The basis usually taken for sl(2) is [Hum, p. 31]

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relations $[x, y] = h$, $[h, x] = 2x$, and $[h, y] = -2y$ completely describe the algebra structure of sl(2). A representation of sl(2) on a complex vector space V is a choice of three linear operators X, Y, and H on V such that $XY - YX = H$, $HX - XH = 2X$, and $HY - YH = -2Y$. A ranked poset P *carries a representation* of sl(2) if there exist a lowering operator Y and an order raising operator X on $\tilde{P}$ such that $XY - YX = H$.

EQUIVALENCE LEMMA [Gri], [St1], [Pr1]. *The following are equivalent for a ranked poset P*:

(i) *P is Peck.*

(ii) *P is rank symmetric, rank unimodal, and has Property T.*

(iii) *P carries a representation of* sl(2).

Let SL(2) (or $SL(2)^{\pm}$) be the subgroup of GL(2) = GL(2, $\mathbb{C}$) consisting of elements which have determinant 1 (or ± 1). Any representation of sl(2) (or SL(2)) determines a representation of SL(2) (or sl(2)) since SL(2) is simply connected. To further specify a representation of $SL(2)^{\pm}$, one need only describe an action σ of $\sigma' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ such that $\sigma X = Y\sigma$.

An *anti-involution* σ of a poset P is an order reversing bijection of the elements of P such that $\sigma^2 = e$, the identity bijection. Let the same symbol σ also denote the obvious linear operator induced on $\tilde{P}$. A ranked poset P *carries a self-dual representation of* sl(2) if, in addition to its carrying a representation of sl(2), there is an anti-involution σ of P such that $\sigma X = Y\sigma$. If σ is also allowed to possibly be a bijection from the poset basis vectors for $\tilde{P}_k$ to minus the poset basis vectors for $\tilde{P}_{R-k}$, then one has a *signed* self-dual representation of sl(2). (Here there is one choice of sign allowed for each pair of sister levels.)

The irreducible representations of $SL(2)^{\pm}$ are "strings" or "chains" as described in the Lemma of Section 5. Take a direct sum of irreducible representations all of which have even dimension, or all of which have odd dimension. Then certain changes of basis will produce Peck posets (with order defined by the support of the image of X) which carry representations of sl(2) in a (signed) self-dual fashion. Hence there are an unlimited supply of Peck posets to which Theorem 1 can be applied.

3. MAIN RESULTS

Theorem 1 really concerns concrete (i.e., with respect to specified bases) representations of $SL(2)^{\pm}$ in which the image of σ' is a bijection (up to overall sign) of bases elements between weight spaces of weights λ and $-\lambda$. Although the partial order structure does not come into play, posets do provide a nice setting psychologically.

Let $\delta_i = |P_i| - |P_{i-1}|$ be the first difference of the rank sizes of P. Suppose that P has length R, and that $k < \frac{1}{2}R$. Restrict the linear operator X^{R-2k} on $\tilde{P}$ to $\tilde{P}_k$ to obtain a linear transformation L_k from $\tilde{P}_k$ to $\tilde{P}_{R-k}$. We will call this the kth *Lefschetz transformation* of P. This linear transformation could be regarded as a linear operator if σ were used to identify $\tilde{P}_k$ with $\tilde{P}_{R-k}$. Instead we will refer to the square matrix M_k for L_k with respect to the bases of poset elements for $\tilde{P}_k$ and $\tilde{P}_{R-k}$. Define rising factorial $\langle x \rangle_n := x(x+1)\cdots(x+n-1)$. Consult Section 5 for proofs. This is the main result of the paper:

THEOREM 1. *Let P be a ranked poset of length R. Fix $k < \frac{1}{2}R$ and set $D = R - 2k$. Suppose that P carries a representation of* $\mathrm{SL}(2)^{\pm}$, *and that the determinant of σ with respect to the poset bases for $\tilde{P}_k$ and $\tilde{P}_{R-k}$ is ± 1. Then the absolute value of the determinant of the Lefschetz matrix is*

$$|\mathrm{Det}(M_k)| = \prod_{h=0}^{k} \langle h+1 \rangle_D^{\delta_{k-h}}.$$

A finer result is: Suppose that P carries a representation of sl(2) *in a (possibly signed) self-dual fashion. Then there are δ_{k-h} eigenvalues of M_k equal to $\langle h+1 \rangle_D$ in absolute value.*

Now fix $n \geqslant 1$ and $m_1, m_2, ..., m_n \geqslant 1$. Consider the poset $\mathscr{P}(n; m)$ which is defined to be the product of n chains (total orders) of lengths $m_i + 1$. The kth rank of P consists of the set $C(n, m, k)$ of compositions $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ of k (i.e., $k = \alpha_1 + \cdots + \alpha_n$) such that $0 \leqslant \alpha_i \leqslant m_i$. If β is also an n-tuple, then define

$$\alpha! = \alpha_1!\, \alpha_2! \cdots \alpha_n!$$

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1}\binom{\beta_2}{\alpha_2}\cdots\binom{\beta_n}{\alpha_n}$$

$$\beta - \alpha = (\beta_1 - \alpha_1, ..., \beta_n - \alpha_n).$$

In Corollaries 1 and 2 the signs of the determinants are found with more detailed versions of the proof of Theorem 1. Fix some ordering of the elements of $C(n, m, k)$ and use this order to order both the rows (which are indexed by α's) and the columns (which are indexed by $\beta = m - \alpha$'s) of M_k. Set $S_k = |P_0| + |P_1| + \cdots + |P_k|$, and let k' be the largest odd integer $\leqslant k$.

COROLLARY 1. *Consider $P = \mathscr{P}(n; m)$, which has length $R = m_1 + \cdots + m_n$. Fix $k < \frac{1}{2}R$ and set $D = R - 2k$. Then applying the proof of Theorem* 1 *to a certain self-dual* sl(2) *representation on P produces an identity which can be slightly rewritten as an evaluation of a determinant of n multinomial coefficients*:

$$\left|\binom{D}{\beta - \alpha}\right| = (-1)^{S_{k'}} \frac{\Pi_\alpha \alpha!}{\Pi_\beta \beta!} \prod_{h=0}^{k} \langle h+1 \rangle_D^{\delta_{k-h}}.$$

Here α and β run over $C(n, m, k)$ and $C(n, m, R-k)$, respectively. The *matrix is symmetric. This identity can be further rewritten as an evaluation of a determinant of products of n binomial coefficients*:

$$\left|\binom{\beta}{\alpha}\right| = (-1)^{S_k} \prod_{h=0}^{k} \binom{D+h}{D}^{\delta_{k-h}}.$$

Note that the (α, β)-matrix entry in the first form is simply the number of saturated chains in P from α to β. Trace identities nearly analogous to the determinant identities in Corollaries 1 and 2 can be formed by referring to the original self-dual raising operator matrices M_k given in the proofs. The eigenvalues of the matrices M_k are as described in Theorem 1.

We have also found a naive triangularization proof of Corollary 1 using induction on n. This proof was longer than the proof of Theorem 1 and was also expressed in terms of strings. A naive proof of Corollary 2 below would probably be much more difficult to construct.

Next consider the poset $P = L(n, m)$. Its elements are partitions rather than compositions. The length of P is nm. A saturated chain in $L(n, m)$ from $\mu \in P_k$ to $\lambda \in P_{nm-k}$ is a standard Young tableau of skew shape λ/μ with $D = nm - 2k$ squares: Start with the empty shape for μ. After each chain step fill in the added square to the shape with the number of the step, where the steps are numbered from 1 to D. Let $f_{\lambda/\mu}$ be the number of such standard skew Young tableaux. The following determinant formula for $f_{\lambda/\mu}$ is well known (e.g., [G–V]):

$$D! \left| \frac{1}{(\lambda_i - \mu_j + j - i)!} \right|_{1 \leqslant i, j \leqslant n}$$

For any $\mu \in P_k$, define $\mu^c \in P_{mn-k}$ to be $(m - \mu_n, ..., m - \mu_1)$. For each $k < mn/2$, fix some ordering of P_k. Also use this ordering to order the partitions λ in P_{mn-k} via the correspondence $\lambda = \mu^c$.

The elements of the kth rank of $L(n, m)$ index Schubert cocycles which are basis elements of $H^{2k}(G_{n,n+m}, \mathbb{C})$. Intersecting a Schubert variety with a generic hyperplane section corresponds exactly to acting with the unitary order operator X in P [St1]. Start with the Schubert variety μ and intersect with a generic hyperplane section D times. Then the multiplicity of λ in the result is $f_{\lambda/\mu}$, i.e., the number of saturated chains from μ to λ in P. These multiplicities are the matrix entries in the first determinant evaluation of the following result. The cohomology groups H^{2k} and $H^{2(nm-k)}$ are often paired together in geometry.

COROLLARY 2. *Consider* $P = L(n, m)$, *which has length* nm. *Fix* $k < \frac{1}{2}nm$ *and set* $D = nm - 2k$. *Then applying the proof of Theorem* 1 *to a certain* *signed self-dual* sl(2) *representation on P produces a compound determinant evaluation which can be slightly rewritten as follows*:

$$|f_{\lambda/\mu}| = \left| D! \left| \frac{1}{(\lambda_i - \mu_j + j - i)!} \right| \right|$$
$$= (-1)^{S_{k'}} \frac{\Pi_\mu(\mu+\rho)!}{\Pi_\lambda(\lambda+\rho)!} \prod_{h=0}^{k} \langle h+1 \rangle_D^{\delta_{k-h}}.$$

Here the outer indices μ *and* λ *run over all Ferrers diagrams fitting in an* $n \times m$ *box with* k *and* $nm-k$ *squares respectively. The inner indices* i *and* j *run from* 1 *to* n. *The* n*-tuple* ρ *is* $(n-1, n-2, ..., 0)$. *With respect to the Schubert cocycle basis, this is the determinant of the* kth *Lefschetz transformation on the cohomology of the Grassmannian* $G_{n,n+m}$. *(In this view* $\delta_h = \beta_{2h} - \beta_{2h-2}$, *the first difference of the even Betti numbers.) The outer matrices are symmetric. This identity can also be rewritten as*

$$\left| \left| \binom{\lambda_i + n - i}{\mu_j + n - j} \right| \right| = (-1)^{S_{k'}} \prod_{h=0}^{k} \binom{D+h}{D}^{\delta_{k-h}}.$$

4. Applications

Three-dimensional Ferrers diagrams fitting inside an $r \times p \times q$ box are equivalent to ordinary plane partitions bounded by q which are contained in a rectangular shape with r rows and p columns or to column strict plane partitions of exactly this rectangular shape which are bounded by $q+r$. Suppose that we want to count these. The column strict plane partitions can be re-interpreted as r-tuples of non-intersecting lattice paths in $\mathbb{N}^2$ from the sources $(r-1, 0)$, $(r-2, 1)$, ..., $(0, r-1)$ to the respective terminals $(p+r-1, q)$, $(p+r-2, q+1)$, ..., $(p, q+r-1)$: The jth entry in the ith row of the plane partition is just $q+r$ minus the y-coordinate of the jth horizontal step in the ith path. Embed the Hasse diagram for the Peck poset $P = \mathscr{P}(2; p+r-1, q+r-1)$ in $\mathbb{N}^2$. Set $R = p+q+2r-2$. The lattice paths become saturated chains from elements of P_{r-1} to elements of P_{R-r+1}. By Gessel's Theorem 2.7.1 of [St4] (or Theorem 14 of [G–V]), the number of non-intersecting r-tuples of paths is given by a determinant whose (i, j)th entry is the number of paths from the ith source to the jth terminal. So this determinant is the determinant appearing in the first identity of Corollary 1 when $n = 2$, $m_1 + 1 = p + r$, $m_2 + 1 = q + r$, and $k = r - 1$. (Also we must reverse the order of the columns; this gets rid of the sign on the right hand side.) Easy cancellation in the right hand side given by the corollary leads to the well known hook-content product enumeration formula [St2] for column strict plane partitions.

APPLICATION 1a. The number of 3-dimensional Ferrers diagrams fitting in an $r \times p \times q$ box is

$$\left| \binom{p+q}{p+j-i} \right|_{1 \leqslant i, j \leqslant r} = \prod_{i=1}^{r} \frac{\langle q+i \rangle_p}{\langle i \rangle_p}.$$

As noted in Macdonald's book [Mac, Example I.5.13], MacMahon's product formula above for the number of 3-dimensional Ferrers diagrams fitting in an $r \times p \times q$ box can be rewritten as a product running over the rpq boxettes contained in the large box. We will refer to this as the $n' = 3$ case; the analogous formulas hold true for $n' = 1$ and $n' = 2$. But there does not seem to be any sort of product enumeration formula in the $n' = 4$ case. If the goal is to obtain a nice product enumeration formula, then here we will show that the determinant identity of Corollary 1 arising from $\mathscr{P}(n, m)$ forms the basis for a generalization of MacMahon's result.

Left hand side—what is being counted?: We will be counting certain sets of non-intersecting lattice paths in $\mathbb{N}^n$ in a certain fashion. The $n = 2$ case is the smallest non-trivial case in this context; as seen in Application 1a it coincides with the $n' = 3$ case of MacMahon. The $n' = 1$ and $n' = 2$ cases of the Ferrers diagram viewpoint can actually be thought of as special cases of the $n' = 3$ case by taking one or two box side lengths to be 1. For general n and $k \geqslant 0$ and any $p_i \geqslant 0$ consider $P = \mathscr{P}(n; p_1 + k, ..., p_n + k)$. Embed the Hasse diagram for P in $\mathbb{N}^n$. The rank P_k is an integral $(n-1)$-simplex with $k+1$ points on an edge; it has $\rho := \binom{n+k-1}{k}$ elements. Fix orderings of P_k and P_{R-k} as before. Consider non-intersecting ρ-tuples of lattice paths from all of the sources α in P_k to all of the terminals β in P_{R-k}, paired up in any fashion. List the paths in the order given by their sources. Call such a ρ-tuple of lattice paths *even* if the terminals are an even permutation of the original order on terminals; otherwise call the ρ-tuple *odd*. Define the *net cardinality* of the set of all such non-intersecting ρ-tuples of lattice paths to be equal to the number of even ρ-tuples minus the number of odd ρ-tuples. By the reflection argument of Theorem 2.7.1 of [St4], it is obvious that the net cardinality is given by the determinant whose (α, β)-entry is the number of lattice paths from α to β for $\alpha \in P_k$ and $\beta \in P_{R-k}$. When $n = 2$, list both the sources and the terminals in reverse lexicographic order—for the terminals this is the reverse of the complementary order specified above. It is easy to see that this is the only ordering of the terminals for which non-intersecting ρ-tuples occur. So no cancellation of non-intersecting paths occurs, and the net cardinality is up to sign the total number of non-intersecting ρ-tuples of lattice paths.

Right hand side—what is the "correct" form? The product over the boxettes form of the formula for 3-dimensional Ferrers diagrams has an enormous amount of trivial cancellation. The hook content form given in Application 1a for $n=2$ has no such obvious cancellation. This form does not readily generalize to higher n as stated; as with binomial coefficients versus multinomial coefficients some cancellation unique to $n=2$ has been performed. Instead write the right hand side when $n=2$ as

$$\prod_{\alpha_1=0}^{k} \frac{\langle \alpha_1+1\rangle_{p+q}}{\langle \alpha_1+1\rangle_p \langle k-\alpha_1+1\rangle_q}.$$

Note that each lattice path has a total of $p+q$ steps. The product here runs over the elements of P_k, and there are $p+q$ factors apiece in the numerator and the denominator for each index of the product. Hence the product has the form which often occurs in the subject: Altogether the number of factors top or bottom is equal to the number of subobjects in one of the objects being counted. Here a subobject is one edge in one of the lattice paths, and an object is a $(k+1)$-tuple of paths. In order to come up with such a form in the general n case, use the implication (iii) $\rightarrow$ (ii) of the equivalence lemma. Hence there are $|P_k|$ disjoint saturated chains starting in rank P_k and ending in rank P_{R-k}. (Or use the fact that P has a symmetric chain decomposition.) So to any $\alpha \in P_k$ we can assign a $\beta \in P_{R-k}$ such that $\alpha \leqslant \beta$. Fix such an assignment. Here is the "generalization" of MacMahon's box theorem:

APPLICATION 1b. Fix $n \geqslant 1$, $k \geqslant 0$, and $p_i \geqslant 0$ for $i=1$ to n. Set $m_i = p_i + k$. Also set $R = p_1 + p_2 + \cdots + p_n + nk$ and $D = R - 2k$. Then the net cardinality of all $\binom{n+k-1}{k}$-tuples of non-intersecting lattice paths in $\mathscr{P}(n; m)$ from the $(n-1)$-simplex $C(n, m, k)$ to the $(n-1)$-simplex $C(n, m, R-k)$ is

$$(-1)^{\binom{n+k'}{k'}} \prod_{\alpha_1} \prod_{\alpha_2} \cdots \prod_{\alpha_{n-1}} \frac{\langle \alpha_1+1\rangle_D}{\langle \alpha_1+1\rangle_{\beta_1-\alpha_1} \cdots \langle \alpha_n+1\rangle_{\beta_n-\alpha_n}}.$$

Here the ith product runs from $\alpha_i = 0$ to $\alpha_i = k - \alpha_1 - \cdots - \alpha_{i-1}$. The β_i are the coordinates of the element β assigned to α as fixed above. And k' is the largest odd integer $\leqslant k$. When $n=3$, the net cardinality is always positive. When $n=2$, the net cardinality is positive when $k \equiv 0$ or 3 (mod 4) and negative when $k \equiv 1$ or 2 (mod 4).

Again altogether the total number of factors top or bottom is equal to the total number of edges appearing in any ρ-tuple of non-intersecting lattice paths. The factors appearing in the denominator are just the coefficients of the sl(2) representation corresponding to the edges in the lattice path matching α to β. When $n=2$ the original hook-content product had only factors corresponding to edges parallel to one of the two axes. Then these surviving sl(2) coefficients in the denominator were the hook lengths.

Disjoint $|P_k|$-tuples of saturated chains from P_k to P_{R-k} have previously played an important role in the theory of Peck posets. The central objects in [St3] are *unitary* Peck posets, which are ranked posets that are Peck by virtue of possessing a unitary raising operator X. As above, the determinant for the kth Lefschetz transformation of a unitary Peck poset with respect to the poset bases is equal to the net cardinality of the set of $|P_k|$-tuples of disjoint saturated chains matching P_k to P_{R-k}. (Now one must also specify at the outset some ordering of P_{R-k}.) Corollaries 1 and 2 obtained product expressions for the net cardinality by converting the determinant for a self-dual X to the determinant for the unitary X.

The degree of the Grassmannian in the Plucker embedding is equal to the multiplicity of $\lambda = (m, m, ..., m)$ in the result of intersecting $\mu = (0, 0, ..., 0)$ with a hyperplane section nm times. This is the number of standard Young tableaux on an $n \times m$ rectangular shape, which is the sole entry in the 1×1 outer determinant occurring in the $k=0$ case of Corollary 2. Hence the right hand side automatically evaluates the inner $n \times n$ determinant. Trivial cancellations give the familiar hook length product formula. Again it can be seen that the rising factorial hook lengths come directly from sl(2) rising factorials.

APPLICATION 2a. The degree of the Grassmannian $G_{n,n+m}$, i.e., the number of standard Young tableaux in an $n \times m$ rectangle, is

$$\frac{(nm)!}{\prod_{i=1}^{n} \langle i \rangle_m}.$$

Next we count bounded plane partitions contained in a staircase shape. Embed the Hasse diagram for $P = L(2, 2r+p-1)$ in $\mathbb{N}^2$. Consider r-tuples of disjoint lattice paths from the sources $(2r-1, 0)$, $(2r-2, 1)$, ..., $(r, r-1)$ to the respective terminals $(2r+p-1, p)$, $(2r+p-2, p+1)$, ..., $(r+p, r+p-1)$. Using the same viewpoint as in Application 1a, these r-tuples translate into column strict plane partitions of $r \times p$ rectangular shape, but now with added restrictions. After subtracting from $2r+p$, the jth entry of the last row cannot be less than $r+p+1-j$. This is a lower bound on all of the entries in the jth column. Convert to ordinary plane partitions by subtracting $r-i+1$ from the ith row. We now have plane partitions bounded by $r+p$ contained in an $r \times p$ rectangle with column lower bounds $r+p-j$. Complement these in 3 dimensions in the containing $r \times p \times (r+p)$ box. Rotate the 3-dimensional Ferrers diagrams and re-project to obtain ordinary plane partitions bounded by r which are contained in the staircase shape $(p, p-1, ..., 1)$. Choose $k = 2r-1$. Then the determinant appearing in the first statement of Corollary 2 is the determinant needed to count the non-intersecting r-tuples of lattice paths described above. In the right hand side cancel $(\mu+\rho)!$ for the ith source into $(\lambda+\rho)!$ for the ith terminal to obtain rising factorials in the denominator. Then the right hand side is now of the form advocated in Application 1b; this is the product expression on the left below. In [Pr2] we noted that these plane partitions correspond to the weights of certain symplectic group representations. There we gave an enumeration formula with $\binom{p+1}{2}$ factors which corresponded to roughly half of the positive roots of the Lie group $\mathrm{Sp}(2p)$. (This set of plane partitions has also been counted using more elementary techniques in [Kra] and [G–V].) King has given a hook-content type formula [Pr2] for the dimensions of symplectic group representations; the product expression on the right below is what one obtains in this case. The conversion from the left hand side to the right hand side involves simple changes in the products and cancellations.

APPLICATION 2b. A hook-content type formula for the number of ordinary plane partitions bounded by r which are contained in $(p, p-1, ..., 1)$ is

$$\prod_{i=1}^{r} \frac{\langle 2i\rangle_{2p}}{\langle r+i+1\rangle_p \langle r-i+1\rangle_p} = \prod_{i=1}^{r} \frac{\langle i+1\rangle_i \langle p+2r+2-i\rangle_{p-i}}{\langle i\rangle_p}.$$

5. PROOFS OF DETERMINANT EVALUATIONS

LEMMA. *Suppose that* $\mathrm{SL}(2)^{\pm}$ *acts irreducibly on a* $(d+1)$*-dimensional vector space. Then there are basis vectors* v_i *for* $0 \leqslant i \leqslant d$ *such that* $Xv_i = (i+1)v_{i+1}$, $Yv_i = (d-i+1)v_{i-1}$, *and* $Hv_i = (2i-d)v_i$. *And either* $\sigma v_i = v_{d-i}$ *for all i or else* $\sigma v_i = -v_{d-i}$ *for all i.*

Proof. The sl(2) statement is Lemma 7.2 of [Hum]. Now $\sigma Y v_0 = X\sigma v_0 = 0$ implies that $\sigma v_0 = \alpha_0 v_d$. And $\sigma X v_i = Y \sigma v_i$ together with induction on i implies $\sigma v_i = \alpha_0 v_{d-i}$. Note that σ^2 is the identity and so $\alpha_0^2 = 1$. ▮

Proof of Theorem 1. The vector space $\tilde{P}$ carries a representation of sl(2). Express this space as a direct sum of sl(2) irreducibles. For $0 \leqslant h \leqslant [R/2]$ there are δ_h irreducible representations of dimension $R-2h+1$. Let $V(h)$ be the direct sum of these. Set $V_i(h) = V(h) \cap \tilde{P}_i$. It is well known that $V_h(h) = \mathrm{Kernel}(Y|_{\tilde{P}_h})$ and that $V_{R-h}(h) = \mathrm{Kernel}(X|_{\tilde{P}_{R-h}})$. Hence $\sigma X = Y\sigma$ implies that $V(h)$ is stable under σ. It is easy to re-decompose $V(h)$ into $\mathrm{SL}(2)^{\pm}$ irreducible representations. Apply the lemma to each of these irreducible representations to get a new basis for all of $\tilde{P}$. Fix k and set $D = R-2k$. From the lemma, we can see that within one string the eigenvalue for the 1×1 matrix for $X^p|_{V_i}$ is $\langle i+1\rangle_p$. So a new basis vector in $\tilde{P}_k$ is an eigenvalue for σX^D with eigenvector $\pm\langle k-h+1\rangle_D$, if its string started in $\tilde{P}_h$. For each $h \leqslant k$, there are δ_h basis strings passing from $\tilde{P}_k$ to $\tilde{P}_{R-k}$ which started in $\tilde{P}_h$. So in the new basis, the matrix E_k for σX^D is diagonal with δ_h entries equal to $\langle k-h+1\rangle_D$ in absolute value. Let S_k be the matrix for σ with respect to the poset bases for $\tilde{P}_k$ and $\tilde{P}_{R-k}$, and let B_k be the change of basis (from string to poset) matrix for $\tilde{P}_k$. Then $B_k^{-1} S_k M_k B_k = E_k$. Now if Det $S_k = \pm 1$ the first statement of the theorem follows. If the representation on P was signed self-dual, then $S_k = \pm I_k$ (identity matrix). In this case the eigenvalues of M_k are clearly as claimed. ∎

Proof of Corollary 1. Let v be the defining representation of GL(2) or sl(2) on column vectors of length 2. Then the mth symmetric power $S^m v$ of this representation is $m+1$ dimensional and can be realized in a self-dual fashion as in the lemma. Take n such representations of dimensions m_i+1, and form their tensor product. Label the basis elements with elements of $\mathscr{P}(n; m)$. The induced action of σ' is $\sigma\alpha = \beta$, where $\beta = m - \alpha$. With respect to this σ, the induced tensor action of sl(2) is self-dual. Embed the Hasse diagram for P in $\mathbb{N}^n$ with the minimal element at the origin. View the coefficients of X as being assigned to lattice path edges in $\mathbb{N}^n$. An edge from α to β, where $\beta_i = \alpha_i + 1$, has the coefficient β_i assigned to it. Any lattice path from $\alpha \in P_k$ to $\beta \in P_{R-k}$ has the same collection of edge labels as any other, viz. the product over i of $\beta_i!/\alpha_i!$. There are $\binom{R-2k}{\beta-\alpha}$ lattice paths from α to β; multiply by $\beta!/\alpha!$ to obtain the (α, β)th determinant entry of M_k. Ignore the sign for now and apply Theorem 1. The first identity stated in the corollary is gotten by dividing each column by $\beta!$ and multiplying each row by $\alpha!$. To get the second form, divide each row of the original form by $(R-2k)!$ and note that the number of rows is $|P_k| = \delta_1 + \cdots + \delta_k$.

To get the sign, note that the GL(2) character with respect to the group element $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is

$$\prod_{i=1}^{n} \frac{u^{m_i+1} - v^{m_i+1}}{u-v} = \sum_{h=0}^{[R/2]} \delta_h (uv)^h \left[u^{R-2h} + u^{R-2h-1} v + \cdots + v^{R-2h}\right].$$

This describes the decomposition of $\tilde{P}$ into irreducible GL(2) and hence irreducible $\mathrm{SL}(2)^{\pm}$ representations. Each is one of the possibilities described in the lemma. The two possibilities in the lemma can only arise as the restrictions respectively of GL(2) representations $\mathrm{Det}^{\mathrm{even}} S^d v$ and $\mathrm{Det}^{\mathrm{odd}} S^d v$. But the character of Det is uv. Hence when $h \leqslant k$ is odd the entry in E_k for this string is negative. Here the matrix $S_k = I$. Therefore the sign of the determinant is $(-1)^{\delta_1 + \delta_3 + \cdots + \delta_{k'}} = (-1)^{S_{k'}}$. ∎

Proof of Corollary 2. Consider the representation $\bigwedge^n [S^{m+n-1} v]$ of GL(2). Realize $S^{m+n-1} v$ with the lemma. The basis vectors of the exterior product are indexed with strictly decreasing n-tuples with entries between 0 and $m+n-1$. Subtract $n-i$ from the ith component to obtain a correspondence with the basis for $\tilde{P}$. The induced action of σ' is $\sigma\lambda = (-1)^{\binom{n}{2}} \lambda^c$. With respect to this σ, the induced action of sl(2) is signed self-dual on $\tilde{P}$. Any lattice path in $L(n, m)$ from μ to λ will collect the same product of coefficients, viz., the product over i of $(\lambda_i + n - i)!/(\mu_i + n - i)!$. The number of such paths is $f_{\lambda/\mu}$, and so the (μ, λ)th determinant entry is $f_{\lambda/\mu}(\lambda + \rho)!/(\mu + \rho)!$. Apply Theorem 1 and manipulate as before to obtain the two identities. The determinant in the first form has the correct entries for the unitary order raising operator on $L(n, m)$ and hence gives the determinant of the Lefschetz operator for the Grassmannian.

The representation $\bigwedge^n [S^{m+n-1}v]$ can be viewed as a subspace of $\otimes^n[S^{m+n-1}v]$. Hence the GL(2) character of a string starting in $\tilde{P}_h$ has $\binom{n}{2} + h$ factors of Det. The overall factor of $(-1)^{\binom{n}{2}}$ for the action of σ with respect to the poset rank bases cancels the other $\binom{n}{2}$ factors of -1, and so the signs work out the same as in Corollary 1. ∎

Note added in proof. More recently, Stanley has obtained some results related to Theorem 1 in the latter part of Section 2 of [St5].

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