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Note

Equivalence of the Combinatorial and the Classical Definitions of Schur Functions

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A short elementary proof of the title is presented.

Fix $n \geqslant 1$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The λth Schur function $s_\lambda(x_1, ..., x_n)$ is an important n-variate polynomial which arises in many different areas of mathematics. The following "classical" definition as a quotient of two $n \times n$ determinants is probably the most common definition. It must be the oldest, going back to at least 1841 when used by Jacobi:

$$s_\lambda(x_1, x_2, ..., x_n) := \frac{|x_j^{\lambda_i + n - i}|_{1 \leqslant i, j \leqslant n}}{|x_j^{n-i}|_{1 \leqslant i, j \leqslant n}}.$$

Another definition of Schur function, which is very much in style with combinatorialists now, views s_λ as a generating function for a certain kind of tableaux. Use the symbol λ to also denote the left justified shape with λ_i boxes in the ith row. Here a *tableau T of shape λ* will be a placement of numbers from $\{1, 2, ..., n\}$ (repetition allowed) in the boxes of λ such that the numbers weakly increase across the rows and strictly increase down the columns. If the number of 1's, 2's, 3's, ... occuring in T is $p, q, r, ...$, then associate the monomial $x^{w(T)} = x_1^p x_2^q x_3^r \cdots$ to T. It is well known [Lit, p. 191] that

PROPOSITION. *$s_\lambda(x_1, ..., x_n) = \sum_T x^{w(T)}$, where the sum is over all tableaux of shape λ.*

Here we present a short proof of this proposition which uses only high school algebra. The British mathematical physicist R. C. King and some of his associates are familiar with this proof, as other people at other times

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also may have been. However, we know of no explicit appearances in print, and most American combinatorialists seem to believe that representation theory or symmetric function theory is necessarily involved in the proof. The main step used here is a lemma which can be found on page 391 of [Wey], where it describes the "branching rule" $\mathrm{GL}(n)\downarrow\mathrm{GL}(n-1)$. However, the $\mathrm{GL}(n)$ connection is irrelevant since the lemma comes *after* the reduction of the theory of characters of finite-dimensional representations to the theory of Schur functions. Much more difficult analogs of this approach have been used to obtain elementary derivations of the Gelfand patterns for the symplectic and orthogonal groups [Pro].

For the sake of exposition, we will state and prove the lemma in the case $n=3$ and $\lambda_1=4$, $\lambda_2=2$, $\lambda_3=1$.

LEMMA.

$$\frac{\begin{vmatrix} x^{4+2} & y^{4+2} & z^{4+2} \\ x^{2+1} & y^{2+1} & z^{2+1} \\ x^{1+0} & y^{1+0} & z^{1+0} \end{vmatrix}}{\begin{vmatrix} x^2 & y^2 & z^2 \\ x^1 & y^1 & z^1 \\ x^0 & y^0 & z^0 \end{vmatrix}} = \sum_{\substack{4\geqslant\mu_1\geqslant 2\\ 2\geqslant\mu_2\geqslant 1}} \frac{\begin{vmatrix} x^{\mu_1+1} & y^{\mu_1+1} \\ x^{\mu_2+0} & y^{\mu_2+0} \end{vmatrix}}{\begin{vmatrix} x^1 & y^1 \\ x^0 & y^0 \end{vmatrix}} z^{\lambda_1-\mu_1+\lambda_2-\mu_2+\lambda_3}.$$

Proof. Set $z=1$ in the left-hand side and in each determinant subtract the last column from all other columns. In each determinant divide the first and second columns by $x-1$ and $y-1$, respectively. So far we have

$$\frac{\begin{vmatrix} x^5+x^4+\cdots+x+1 & y^5+y^4+\cdots+y+1 & 1 \\ x^2+x+1 & y^2+y+1 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x+1 & y+1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}}.$$

Now in each determinant subtract the second row from the first, the third row from the second, etc. We are left with

$$\frac{\begin{vmatrix} x^5+x^4+x^3 & y^5+y^4+y^3 \\ x^2+x^1 & y^2+y^1 \end{vmatrix}}{\begin{vmatrix} x^1 & y^1 \\ x^0 & y^0 \end{vmatrix}},$$

which is the claimed sum, except for the power of z. This is gotten by noting that the original left-hand side was homogeneous of degree $\lambda_1+\lambda_2+\lambda_3$, whereas the right-hand side so far has degree $\mu_1+\mu_2$. ∎

In general, the exponent $\mu_i+(n-1)-i$ in the ith row of the resulting numerator will range from $\lambda_i+n-i-1=\lambda_i+(n-1)-i$ down to $\lambda_{i+1}+n-(i+1)=\lambda_{i+1}+(n-1)-i$. Hence, in general, the lemma states that

$$s_\lambda(x_1, ..., x_n)=\sum_\mu s_\mu(x_1, ..., x_{n-1})\, x_n^{|\lambda|-|\mu|},$$

where $|\lambda|=\lambda_1+\cdots+\lambda_n$ and the sum is over all μ such that $\lambda_1\geqslant\mu_1\geqslant\lambda_2\geqslant\mu_2\geqslant\cdots\geqslant\mu_{n-1}\geqslant\lambda_n$.

The proof of the proposition can be finished with the following now standard construction [Mac, pp. 41–42].

Proof. Start with a shape $\lambda^{(0)}=\lambda$ and apply the lemma once. For each resulting $\lambda^{(1)}=\mu$, begin to construct a tableau by placing an "n" in each box of $\lambda^{(0)}$ which lies outside of $\lambda^{(1)}$. Repeat this procedure $n-1$ more times with $n-1$'s, $n-2$'s, ..., and 1's. After n steps we are left with $\lambda^{(n)}=\varnothing$, so we should have defined $s_\varnothing(-)=1$ to start the induction. We have expressed $s_\lambda(x_1, ..., x_n)$ as a sum of monomials which are indexed by a collection of tableaux of appropriate weights. The entries obviously weakly increase across the rows and down the columns. Note that $\lambda_i^{(k+1)}\geqslant\lambda_{i+1}^{(k)}$ and $\lambda_{n-k+1}^{(k)}=0$ imply that two $(n-k)$'s will not be placed in the same column at step $k=0, 1, ..., n-1$. This implies that the entries are actually column strict. ∎

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