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Dyadic Sampling Approximations for Non-Sequency-Limited Signals

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Dyadic sampling approximations, as well as error estimates, are derived for non-random signals which are Walsh–Stieltjes transforms and for dyadic-stationary and Walsh-harmonizable random signals. Also derived are inversion formulae for Walsh-Stieltjes transforms, which are used in this paper.

1. INTRODUCTION

Recently, Walsh functions have been increasingly used in digital communication systems: they are easily generated, their pulse shape $(1, -1)$ conforms with operations of digital computers, and they play, for signals with possible discontinuities at the dyadic rationals, the role complex exponential functions (and Fourier analysis) play for continuous signals. In addition, they have been used in experimental sequency-multiplex systems, image coding and enhancement, etc. (see, e.g., Harmuth, 1977).

Dyadic sampling representations for sequency-limited (non-random) signals have been obtained by several authors, among them Pichler (1968) and Kak (1970). The concept of a sequency-limited signal is the (Walsh) analogue of the (Fourier) concept of a band-limited signal. However, as Kak pointed out, the class of sequency-limited functions contains only step functions and is therefore a rather small class (this is in sharp contrast to the richness of the class of band-limited functions). Butzer and Splettstösser (1978) obtained dyadic sampling approximations, as well as error estimates, for time-limited non-random signals. Maqusi (1980) derived a dyadic sampling representation for sequency-limited non-stationary random signals.

In Section 2 we derive dyadic sampling approximations, as well as error estimates, for non-random signals which are Walsh transforms of finite measures and for dyadic-stationary and Walsh-harmonizable random signals. These signals are not necessarily sequency-limited or time-limited. These

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sampling approximations are of the sample and hold type, and while such step function approximations are well known for continuous signals, it is remarkable that they remain valid for classes of signals that are not everywhere continuous. Similar results for the classical sampling series were established by Cambanis and Habib (1982). In Section 3 we derive inversion formulae for Walsh–Stieltjes transforms which are needed in Section 2.

2. Sampling Approximations

The following notation and definitions will be used in the sequel. $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}$ is the set of all integers, $\mathbb{N}_+$ is the set of all non-negative integers, and D_+ is the set of all non-negative dyadic rationals ($k2^{-n}$; $k \in \mathbb{N}_+$, $n \in \mathbb{N}$). Each $t > 0$ has the dyadic expansion

$$t = \sum_{j=-N(t)}^{\infty} t_j 2^{-j}, \qquad t_j \in \{0, 1\}, \tag{2.1}$$

for all j, where $N(t)$ is such that $2^{N(t)} \leqslant t < 2^{N(t)+1}$, and we put $t_j = 0$ for $j < -N(t)$. If $t \in D_+$, there are two expansions and the finite one is chosen. The component-wise addition modulo 2 (dyadic addition) of $t, s \in \mathbb{R}_+$ is defined by $t \oplus s = \sum_{j=-\infty}^{\infty} |t_j - s_j| \, 2^{-j}$ (see Butzer and Splettstösser, 1978). The set of Walsh functions $\{\psi_n(t)\}_{n \in \mathbb{N}_+}$ on $[0, 1)$ is defined for each non-negative integer $n = \sum_{n=-N(n)}^{0} n_j 2^{-j}$ by

$$\psi_n(t) = \exp\left\{\pi i \sum_{j=1}^{N(n)+1} n_{1-j} t_j\right\},$$

and is orthonormal and complete in $L^2[0, 1)$. The Walsh functions are extended by Fine (1950) to $\{\psi_u(t)\}_{u,t \geqslant 0}$ by

$$\psi_t(u) = \psi_u(t) = \exp\left\{\pi i \sum_{j=-N(t)}^{N(u)+1} u_{1-j} t_j\right\},$$

and they have the property that, for all $u \geqslant 0$ whenever $t \oplus s \notin D_+$,

$$\psi_u(t \oplus s) = \psi_u(t)\, \psi_u(s).$$

A function f on $\mathbb{R}_+$ is called W-continuous (W is for Walsh) if f is continuous on $\mathbb{R}_+ \backslash D_+$ and right continuous on D_+. If $f \in L^1(\mathbb{R}_+)$, its Walsh transform $\hat{f}$ is defined by

$$\hat{f}(u) = \int_0^{\infty} \psi_u(t) f(t)\, dt, \qquad u \geqslant 0,$$

and $\hat{f}$ is bounded and W-continuous. If $f, \hat{f} \in L^1(\mathbb{R}_+)$ and f is W-continuous, then the Walsh transform of f can be inverted to give

$$f(t) = \int_0^\infty \psi_t(u) \hat{f}(u)\, du, \qquad t \geqslant 0,$$

(Butzer and Splettstösser, 1978). The Walsh transform of a finite (signed or complex) measure μ on the Borel subsets of $\mathbb{R}_+$ is defined by

$$\hat{\mu}(u) = \int_0^\infty \psi_u(t)\, d\mu(t), \qquad u \geqslant 0,$$

and inversion formulae are established in Section 3. Both inversion formulae of Walsh transforms, of integrable functions on $\mathbb{R}_+$ and of finite measures, have the usual multidimensional analogues.

A complex function f on $\mathbb{R}_+$ of the form

$$f(t) = \int_0^{2^n} \psi_t(u)\, F(u)\, du, \qquad t \geqslant 0,$$

for some $n \in \mathbb{N}$ and some $F \in L^1(0, 2^n)$ is called sequency-limited to 2^n. A sequency-limited function in $L^1(\mathbb{R}_+)$ has a dyadic sampling expansion of the form

$$f(t) = \sum_{k=0}^{\infty} f\left(\frac{k}{2^n}\right) J(1; (2^n t) \oplus k), \qquad t \geqslant 0,$$

where $J(v; t) = \int_0^v \psi_t(u)\, du$, $t, v \geqslant 0$ is the Walsh–Dirichlet kernel (Fine, 1950). As it was pointed out by Kak (1970) and Butzer and Splettstösser (1978),

$$J(1; (2^n t) \oplus k) = 1_{[2^{-n}k, 2^{-n}(k+1))}(t),$$

and thus (under the stated conditions) the functions that are sequency-limited to 2^n, for some $n \in \mathbb{N}$, are precisely the functions that are constant on each interval $[2^{-n}k, 2^{-n}(k+1))$, a rather small class of functions.

A dyadic sampling approximation for W-continuous time-limited non-random functions was derived by Butzer and Splettstösser (1978) along with error estimates. We first derive a finite dyadic sampling approximation as well as error estimates for functions which are Walsh transforms of finite (signed or complex) measures and thus not necessarily time-limited or sequency-limited.

The following notation will be needed. The dyadic modulus of continuity of a function $f \in L^1(\mathbb{R}_+)$ is defined for $\delta > 0$ by

$$\omega(f; \delta) = \sup_{0 \leqslant h < \delta} \|f(\cdot) - f(\cdot \oplus h)\|_1,$$

where $\|f\|_1 = \int_0^\infty |f(t)|\,dt$. The Lipschitz class $\mathrm{Lip}(\alpha, L)$, $0 < \alpha$, $L < \infty$, is defined by

$$\mathrm{Lip}(\alpha, L) = \{f \in L^1(\mathbb{R}_+) \colon \omega(f; \delta) \leqslant L\delta^\alpha, \delta > 0\}.$$

If should be noted that $\alpha > 1$ does not imply that $\mathrm{Lip}(\alpha, L)$ contains only constant functions (as in the case where the usual rather than the dyadic addition is used in defining the modulus of continuity).

THEOREM 2.1. *Let f be the Walsh transform of a finite (signed or complex) measure μ on the Borel subsets of $\mathbb{R}_+$:*

$$f(t) = \int_0^\infty \psi_t(u)\, d\mu(u), \qquad t \geqslant 0. \tag{2.2}$$

Then for every $t \geqslant 0$ and integer n,

$$f_n(t) := \sum_{k=0}^{\infty} f\left(\frac{k}{2^n}\right) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) = \int_0^\infty {}_n\psi_t(u)\, d\mu(u), \tag{2.3}$$

where for each fixed $t \geqslant 0$, ${}_n\psi_t(u)$ is the 2^n-periodic extension of the function $\psi_t(u)$, $0 \leqslant u < 2^n$, to $[0, \infty)$,

$$|f(t) - f_n(t)| \leqslant 2\, |\mu| \{[2^n, \infty)\}, \tag{2.4}$$

and thus

$$f(t) = \lim_{n \to \infty} f_n(t). \tag{2.5}$$

If, in addition, $f \in \mathrm{Lip}(\alpha, L)$ for some $\alpha > 1$, $0 < L < \infty$, then for every $t \geqslant 0$ and for large values of n we have

$$|f(t) - f_n(t)| \leqslant \frac{L}{\alpha - 1} \cdot \frac{1}{2^{n(\alpha-1)}}. \tag{2.6}$$

Proof. Fix $t \geqslant 0$ and $n \in \mathbb{N}$. Since ${}_n\psi_t \in L^1[0, 2^n)$ is periodic with period 2^n, W-continuous, and of bounded variation on $[0, 2^n)$, then the partial sums of its Walsh–Fourier series converge everywhere to ${}_n\psi_t$ (Chrestenson, 1955, Theorem 2), i.e.,

$${}_n\psi_t(u) = \sum_{k=0}^{\infty} a_{n,k}(t)\, \psi_k(2^{-n}u), \qquad u \geqslant 0, \tag{2.7}$$

where

$$\begin{aligned} a_{n,k}(t) &= \frac{1}{2^n} \int_0^{2^n} \psi_t(v)\, \psi_k(2^{-n}v)\, dv \\ &= J(1; (2^n t) \oplus k) = 1_{[2^{-n}k, 2^{-n}(k+1))}(t). \end{aligned} \tag{2.8}$$

Let

$$\varepsilon_K(t; n) = \left| \sum_{k=0}^{K} f\left(\frac{k}{2^n}\right) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) - \int_0^\infty {}_n\psi_t(u)\, d\mu(u) \right|$$
$$\leqslant \int_0^\infty \left| {}_n\psi_t(u) - \sum_{k=0}^{K} \psi_k(2^{-n}u)\, 1_{[2^{-n}k, 2^{-n}(k+1))}(t) \right| d|\mu|(u). \quad (2.9)$$

The integrand in (2.9) is bounded by 2 and tends to zero as $K \to \infty$ by (2.7) and (2.8). It follows that $\varepsilon_K(t; n) \to 0$ as $K \to \infty$, proving (2.3). Now by (2.2) and (2.3),

$$|f(t) - f_n(t)| \leqslant \int_{2^n}^\infty |\psi_t(u) - {}_n\psi_t(u)|\, d|\mu|(u) \leqslant 2|\mu|\{[2^n, \infty)\},$$

hence (2.4) and (2.5). To prove (2.6), notice that, by Corollary 3.1, if $f \in L^1(\mathbb{R}_+)$ then μ is absolutely continuous with respect to Lebesgue measure, and we put $d\mu(u)/du = \hat{f}(u)$ (and consider a W-continuous version of $\hat{f}$). Thus if $f \in \text{Lip}(\alpha, L)$, it follows from (the proof of) a lemma on page 102 of Butzer and Splettstösser (1978) that for any $\varepsilon > 0$,

$$|\hat{f}(u)| \leqslant \frac{1}{2}\,\omega\left(f; \frac{1+\varepsilon}{u}\right) \leqslant \frac{L}{2}\left(\frac{1+\varepsilon}{u}\right)^\alpha, \qquad u > 0,$$

and thus $|\hat{f}(u)| \leqslant L/(2u^\alpha)$, $u > 0$. (2.6) then follows from (2.4). ∎

Even though the approximating function f_n is defined through a series, since it is of the sample and hold type (cf. (2.3)), it can be used in the case of practical interest where only a finite number of samples is available, $\{f(k2^{-n})\}_{k=1}^N$, to approximate f over the interval $[0, N2^{-n})$. Notice that in this case the interval where the approximation is feasible depends on the number N of samples available, while the bound on the approximation error given by (2.4) depends only on the frequency of sampling.

We now derive dyadic sampling approximations for certain random signals which are not necessarily mean square continuous. A second order process $\{x(t),\ t \geqslant 0\}$ with (not necessarily continuous) correlation function $R(t, s)$ is called Walsh-harmonizable if for all $t, s \geqslant 0$,

$$R(t, s) = \int_0^\infty \int_0^\infty \psi_t(u)\, \psi_s(v)\, d\mu(u, v), \quad (2.10)$$

where μ is a finite (signed or complex) measure on the Borel subsets of $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$, called its spectral measure, or equivalently if for all $t \geqslant 0$,

$$x(t) = \int_0^\infty \psi_t(u)\, dZ(u), \quad (2.11)$$

where Z is a random measure on the Borel subsets of $\mathbb{R}_+$ such that for all Borel sets $A, B \in \mathbb{R}_+$, $\mu(A \times B) = E[Z(A)\bar{Z}(B)]$.

We also need the fllowing notation. A function R on $\mathbb{R}_+^2$ is called W-continuous if R is continuous on $\mathbb{R}_+^2 \backslash D_+^2$ and continuous from above-right on D_+^2. If $R \in L^1(\mathbb{R}_+^2)$, its dyadic modulus of continuity is defined for $\delta, \lambda > 0$ by

$$\omega(R; \delta, \lambda) = \sup\{\|\Delta_{h,g} R\|_1, 0 \leqslant h < \delta, 0 \leqslant g < \lambda\}$$

where $(\Delta_{h,g}R)(t, s) = R(t \oplus h, s \oplus g) - R(t \oplus h, s) - R(t, s \oplus g) + R(t, s)$ and $\|R\|_1 = \int_0^\infty \int_0^\infty |R(t, s)|\, dt\, ds$. Also the Lipschitz class $\mathrm{Lip}(\alpha, L)$, $0 < \alpha$, $L < \infty$, is defined by

$$\mathrm{Lip}(\alpha, L) = \{R \in L^1(\mathbb{R}_+^2): \omega(R; \delta, \lambda) \leqslant L\delta^\alpha \lambda^\alpha, \delta > 0, \lambda > 0\}.$$

THEOREM 2.2. *Let $\{x(t), t \geqslant 0\}$ be a second order Walsh-harmonizable process with random measure Z and spectral measure μ. Then for each $t \geqslant 0$ and $n \in \mathbb{N}$ with probability one,*

$$x_n(t) := \sum_{k=0}^{\infty} x\left(\frac{k}{2^n}\right) 1_{[2^{-n}k, 2^{-n}(k+1))}(t) = \int_0^\infty {}_n\psi_t(u)\, dZ(u), \qquad (2.12)$$

and

$$E|x(t) - x_n(t)|^2 \leqslant 4|\mu| \{[2^n, \infty) \times [2^n, \infty)\} \qquad (2.13)$$

so that

$$x(t) = \lim_{n \to \infty} x_n(t) \qquad (2.14)$$

in quadratic mean. If, in addition, $R \in \mathrm{Lip}(\alpha, L)$ for some $\alpha > 1$, $L < \infty$, then for every $t \geqslant 0$ and for large values of n we have

$$E|x(t) - x_n(t)|^2 \leqslant \frac{L}{(\alpha - 1)^2} \cdot \frac{1}{2^{2n(\alpha-1)}}. \qquad (2.15)$$

Proof. The proofs of (2.12) through (2.14) are similar to those of Theorem 2.1 and are thus omitted. To show (2.15) observe that since $R \in L^1(\mathbb{R}_+^2)$, then by Corollary 3.1, μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+^2$. Put $d\mu(u, v)/du\, dv = \hat{R}(u, v)$ and choose a W-continuous version of $\hat{R}$. The existence of such a version follows from the two-dimensional Walsh transform (established in the one-dimensional case by Butzer and Splettstösser (1978)):

$$\hat{R}(u, v) = \int_0^\infty \int_0^\infty \psi_u(t)\, \psi_v(s)\, R(t, s)\, dt\, ds, \qquad u, v \geqslant 0.$$

Now for any $0 < u \neq 2^n$, $n \in \mathbb{N}_+$, $\psi_u(u^{-1}) = -1$ (Butzer and Splettstösser, 1978, page 102), so that $\psi_u(t \oplus u^{-1}) = \psi_u(t)\,\psi_u(u^{-1}) = -\psi_t(u)$ whenever $t \oplus u^{-1} \notin D_+$. Thus for any $0 < u \neq 2^n$, $n \in \mathbb{N}_+$, we have

$$\hat{R}(u, v) = -\int_0^\infty \int_0^\infty R(t, s)\,\psi_u(t \oplus u^{-1})\,\psi_v(s)\,dt\,ds$$

$$= -\int_0^\infty \int_0^\infty R(t \oplus u^{-1}, s)\,\psi_u(t)\,\psi_v(s)\,dt\,ds,$$

and similarly for the other terms. It follows that for all $u, v \notin \{2^n, n \in \mathbb{N}_+\}$ we have

$$\hat{R}(u, v) = \tfrac{1}{4} \int_0^\infty \int_0^\infty [R(t \oplus u^{-1}, s \oplus v^{-1}) - R(t \oplus u^{-1}, s) - R(t, s \oplus v^{-1}) + R(t, s)]\,\psi_u(t)\,\psi_v(s)\,dt\,ds.$$

It follows for all $u, v > 0$ and $\varepsilon > 0$ that

$$|\hat{R}(u, v)| \leqslant \frac{1}{4}\,\omega\left(R; \frac{1+\varepsilon}{u}, \frac{1+\varepsilon}{v}\right) \leqslant \frac{1}{4} L \left(\frac{1+\varepsilon}{u}\right)^\alpha \left(\frac{1+\varepsilon}{v}\right)^\alpha,$$

so that putting $\varepsilon = 0$ we obtain $|\hat{R}(u, v)| \leqslant L/\{4(uv)^\alpha\}$. The proof is completed as that of Theorem 2.1. ∎

The following corollary shows that the approximating sequence $x_n(t)$ frequently converges to $x(t)$ with probability one (cf. (2.14)) and gives the rate of convergence.

COROLLARY 2.1. *Let x be as in Theorem 2.1 and assume that $\alpha > \frac{3}{2}$. Then for each $t \geqslant 0$, as $n \to \infty$,*

$$2^{\gamma n}\{x(t) - x_n(t)\} \to 0, \tag{2.16}$$

with probability one where $0 < \gamma < \alpha - \frac{3}{2}$.

Proof. For each fixed $t \geqslant 0$, define X_u, $0 \leqslant u \leqslant 1$, by

$$X_u = x(t) \qquad \text{for } u = 0$$

$$= x_n(t) \qquad \text{for } \frac{1}{2^n} < u \leqslant \frac{1}{2^{(n-1)}}, \qquad n \geqslant 1.$$

Then X is separable in u, and from (2.15) we have (with n such that $2^{-n} < u \leqslant 2^{-(n-1)}$)

$$E\,|X_0 - X_u|^2 = E\,|x(t) - x_n(t)|^2 \leqslant \text{Const}\,2^{-2n(\alpha-1)} < \text{Const}\,u^{1+\beta},$$

where $\beta = 2\alpha - 3 > 0$. Thus, by Kolmogorov's theorem (Neveu, 1965, page 97), as $h \downarrow 0$,

$$\frac{1}{h^\gamma} \sup_{0<u<h} |X_0 - X_u| \to 0, \qquad 0 < \gamma < \frac{\beta}{2},$$

with probability one, and (2.16) follows by putting $h = 2^{-n}$. ∎

When the random measure Z of a Walsh-harmonizable process x is orthogonal, or, equivalently, when its spectral measure μ is supported by the diagonal of $\mathbb{R}_+^2$, then x is called dyadic-stationary and its correlation function $R(t, s)$ is a function of $t \oplus s$:

$$R(t, s) = \int_0^\infty \psi_{t\oplus s}(\lambda)\, d\mu(\lambda)$$

(where, with the usual abuse of notation, we denote by μ the measure on $\mathbb{R}_+$ which represents the spectral measure μ on D_+). (See Morettin, 1974, and Nagai, 1977, for the discrete time case.) For a dyadic-stationary process the bound (2.15) simplifies to

$$E\,|x(t) - x_n(t)|^2 \leqslant \frac{2L}{\alpha - 1} \cdot \frac{1}{2^{n(\alpha-1)}};$$

therefore, γ in Corollary 2.1 satisfies $0 < \gamma < \alpha/2 - 1$ and $\alpha > 2$.

We conclude by considering certain Walsh-harmonizable (but not dyadic-stationary) stable processes which are the analogues of real dyadic-stationary Gaussian processes. Since stable processes have infinite second moments, in this case the approximating sampling sequence converges in the pth mean for appropriate values of p strictly less than 2. The following notation and facts are needed.

A random variable X is symmetric α-stable ($S\alpha S$), $0 < \alpha \leqslant 2$, if its characteristic function is of the form $E(e^{itX}) = \exp(-\|X\|^\alpha\, |t|^\alpha)$, where for $1 < \alpha \leqslant 2$, the positive constant $\|X\|$ defines a norm on a linear space of $S\alpha S$ random variables (Schilder, 1970). Also $\mathscr{E}\,|X|^p < \infty$ for all $0 < p < \alpha$, and in fact

$$\mathscr{E}\,|X|^p = C(p, \alpha)\, \|X\|^p$$

for some universal constant $C(p, \alpha)$ depending only on p and α and not on X (Cambanis and Miller, 1981). A stochastic process $\{x(t), t \geqslant 0\}$ is called $S\alpha S$ if every finite linear combination of its random variables is $S\alpha S$.

The following can be found in Schilder (1970). If the process $Z(\lambda)$, $\lambda \geqslant 0$, is $S\alpha S$ with independent increments, and $1 < \alpha \leqslant 2$, then the function $F(\lambda) = \|Z(\lambda)\|^\alpha$, $\lambda \geqslant 0$, is non-decreasing and thus defines a Lebesgue–Stieltjes measure μ on the Borel sets of $[0, \infty)$. If the family of functions $\{f(t, \cdot),$

$t \geqslant 0\}$ belongs to $L^{\alpha}(\mu)$, then the integral $\int_0^\infty f(t,\lambda)\,dZ(\lambda)$, $t \geqslant 0$, defines an $S\alpha S$ process, and for every $t \geqslant 0$,

$$\left\| \int_0^\infty f(t,\lambda)\,dZ(\lambda) \right\|^{\alpha} = \int_0^\infty |f(t,\lambda)|^{\alpha}\,d\mu(\lambda).$$

COROLLARY 2.2. *Consider the $S\alpha S$ process*

$$x(t) = \int_0^\infty \psi_t(\lambda)\,dZ(\lambda), \qquad t \geqslant 0,$$

where $1 < \alpha < 2$ and Z is a $S\alpha S$ process with independent increments and finite (spectral) measure μ. Then for each $t \geqslant 0$, (2.12) holds, and for $0 < p < \alpha$,

$$\mathscr{E}\,|x(t) - x_n(t)|^p \leqslant C(p,\alpha)\,2^p \mu^{p/\alpha}\{[2^n, \infty)\},$$

and thus if $n \to \infty$,

$$x(t) = \lim_{n\to\infty} x_n(t)$$

in the pth *mean.*

The proof is straightforward and thus omitted.

3. INVERSION FORMULAE FOR WALSH–STIELTJES TRANSFORMS

In this section, inversion formulae are derived for Walsh transforms of finite measures, and it is shown that if the Walsh transform of a finite measure is integrable, then the measure is absolutely continuous with respect to Lebesgue measure. While for simplicity the univariate case is considered, the natural multivariate analogues are similarly valid. These results are used in Section 2 and, to the best of our knowledge, they are not available in the Walsh transform literature.

THEOREM 3.1. *If μ is a finite (signed or complex) measure on the Borel subsets of $\mathbb{R}_+$, and if f is its Walsh transform,*

$$f(t) = \int_0^\infty \psi_t(\lambda)\,d\mu(\lambda), \qquad t \geqslant 0,$$

then for all $0 \leqslant a < b < \infty$,

$$\mu((a,b)) + \mu(\{a\}) + \mu(\{b\}) = \lim_{n\to\infty} \frac{1}{2^n} \int_0^{2^n} \Psi_{a,b}(t) f(t)\,dt, \tag{3.1}$$

where $\Psi_{a,b}(t)=\int_a^b \psi_t(u)\,du$, *and for all* $a>0$,

$$\mu(\{a\})=\lim_{n\to\infty}\frac{1}{2^n}\int_0^{2^n}\psi_a(t)f(t)\,dt. \tag{3.2}$$

Proof. We have

$$\begin{aligned}\int_0^{2^n}\Psi_{a,b}(t)f(t)\,dt&=\int_0^{2^n}\left(\int_a^b\psi_t(u)\,du\right)\left(\int_0^\infty\psi_t(\lambda)\,d\mu(\lambda)\right)dt\\&=\int_0^\infty d\mu(\lambda)\int_a^b du\int_0^{2^n}dt\psi_t(u)\,\psi_t(\lambda)\\&=2^n\int_0^\infty d\mu(\lambda)\int_0^\infty 1_{(a,b)}(u)\,1_{[0,2^{-n})}(u\oplus\lambda)\,du,\end{aligned} \tag{3.3}$$

where we used Fubini's theorem and the properties $\psi_t(u)\,\psi_t(\lambda)=\psi_t(u\oplus\lambda)$ whenever $u\oplus\lambda\notin D_+$ and

$$\int_{0.}^{2^n}\psi_t(v)\,dt=2^n 1_{[0,2^{-n})}(v) \tag{3.4}$$

(Fine, 1950). If $\lambda=\sum_{K=-N(\lambda)}^{\infty}\lambda_k 2^{-k}$, we put

$$\lambda^{(n)}=\sum_{k=-N(\lambda)}^{n}\lambda_k 2^{-k},$$

and notice that for fixed n and λ, and all $u\geqslant 0$,

$$1_{[0,2^{-n})}(u\oplus\lambda)=1_{[\lambda^{(n)},\lambda^{(n)}+2^{-n})}(u). \tag{3.5}$$

Thus, (3.3) may be written as

$$\int_0^{2^n}\Psi_{a,b}(t)f(t)\,dt=\int_0^\infty A_n(\lambda)\,d\mu(\lambda), \tag{3.6}$$

where

$$A_n(\lambda)=2^n\int_0^\infty 1_{(a,b)}(u)\,1_{[\lambda^{(n)},\lambda^{(n)}+2^{-n})}(u)\,du.$$

A straightforward calculation shows that

$$\begin{aligned}A_n(\lambda)&=0 &&\text{for}\quad \lambda>a^{(n)},\\&=1-2^n(a-a^{(n)}) &&\text{for}\quad a^{(n)}\leqslant\lambda\leqslant a^{(n)}+2^{-n},\\&=1 &&\text{for}\ a^{(n)}+2^{-n}<\lambda<b^{(n)},\\&=1-2^n(b-b^{(n)}) &&\text{for}\quad b^{(n)}\leqslant\lambda\leqslant b^{(n)}+2^{-n},\\&=0 &&\text{for}\quad \lambda>b^{(n)}+2^{-n},\end{aligned}$$

and using the dominated convergence theorem, we have from (3.6) that

$$\lim_{n\to\infty} \int_0^{2^n} \Psi_{a,b}(t) f(t)\, dt = \mu((a, b)) + \mu(\{a\}) + \mu(\{b\}),$$

proving (3.1). Equation (3.2) is proven in a similar way:

$$\begin{aligned}\frac{1}{2^n}\int_0^{2^n} \psi_t(a) f(t)\, dt &= \frac{1}{2^n}\int_0^{\infty} d\mu(u) \int_0^{2^n} dt\, \psi_t(u \oplus a)\\ &= \int_0^{\infty} d\mu(u)\, 1_{[0,2^{-n})}(u \oplus a)\\ &= \int_0^{\infty} 1_{[a^{(n)}, a^{(n)}+2^{-n})}(u)\, d\mu(u)\end{aligned}$$

by (3.4) and (3.5). Since $1_{[a^{(n)},a^{(n)}+2^{-n})}(u) \to 1_{\{a\}}(u)$ as $n \to \infty$, the dominated convergence theorem implies

$$\lim_{n\to\infty} \frac{1}{2^n}\int_0^{2^n} \psi_a(t) f(t)\, dt = \mu(\{a\}). \quad ∎$$

Corollary 3.1. *Let f and μ be defined as in Theorem* 3.1. *If $f \in L^1(\mathbb{R}_+)$, then μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ with Radon–Nikodym derivative*

$$F(u) = \int_0^{\infty} \psi_u(t) f(t)\, dt, \qquad u \geqslant 0.$$

Proof. Since $f \in L^1(\mathbb{R}_+)$, $\int_0^{2^n} \psi_a(t) f(t)\, dt$ converges to the finite number $\int_0^{\infty} \psi_a(t) f(t)\, dt$ as $n \to \infty$, and by (3.2), $\mu(\{a\}) = 0$ for all $a \geqslant 0$. Then (3.1) gives, for all $0 \leqslant a < b < \infty$,

$$\mu((a, b)) = \int_0^{\infty} \Psi_{a,b}(t) f(t)\, dt = \int_a^b \left(\int_0^{\infty} \psi_u(t) f(t)\, dt\right) du,$$

where we used Fubini's theorem, and the result follows. ∎

For the sake of completeness, we include and prove the following result, whose counterpart in Fourier analysis is due to Wiener.

Corollary 3.2. *With f and μ as in Theorem* 3.1, *we have*

$$\lim_{n\to\infty} \frac{1}{2^n}\int_0^{2^n} |f(t)|^2\, dt = \sum_{u \geqslant 0} |\mu(\{u\})|^2. \tag{3.7}$$

Proof. Notice that the right-hand side of (3.7) is meaningful since μ has at most a countable number of atoms. Now, if $\mu \geqslant 0$, then

$$\begin{aligned}
\frac{1}{2^n}\int_0^{2^n} f^2(u)\,du &= \frac{1}{2^n}\int_0^{2^n} dt \int_0^\infty \int_0^\infty d\mu(u)\,d\mu(v)\,\psi_t(u)\,\psi_t(v) \\
&= \frac{1}{2^n}\int_0^\infty \int_0^\infty d\mu(u)\,d\mu(v) \int_0^{2^n} dt\psi_t(u \oplus v) \\
&= \int_0^\infty \left(\int_0^\infty 1_{[0,2^{-n})}(u \oplus v)\,d\mu(v)\right) d\mu(u) \\
&= \int_0^\infty \left(\int_0^\infty 1_{[u^{(n)},u^{(n)}+2^{-n})}(v)\,d\mu(v)\right) d\mu(u) \\
&= \int_0^\infty \mu([u^{(n)}, u^{(n)} + 2^{-n}) \cap [0, \infty))\,d\mu(u). \qquad (3.8)
\end{aligned}$$

Since $\mu\{[u^{(n)}, u^{(n)} + 2^{-n}) \cap [0, \infty)\}$ is integrable for each fixed n (by the integrability of the left-hand side of (3.8)), and for all $u \geqslant 0$ it converges to $\mu(\{u\})$, then by the dominated convergence theorem we have

$$\lim_{n\to\infty} \frac{1}{2^n}\int_0^{2^n} f^2(t)\,dt = \sum_{u\geqslant 0} \mu^2(\{u\}).$$

If μ is complex, the same argument gives (3.7). ∎

Finally, we notice from (3.7) that if $f \in L^2(\mathbb{R}_+)$, then μ is non-atomic.

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