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# New Symmetric Plane Partition Identities from Invariant Theory Work of De Concini and Procesi

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Nine ($=2\times 2\times 2+1$) product identities for certain one-variable generating functions of certain families of plane partitions are presented in a unified fashion. The first two of these identities are originally due to MacMahon, Bender, Knuth, Gordon and Andrews and concern symmetric plane partitions. All nine identities are derived from tableaux descriptions of weights of especially nice representations of Lie groups, eight of them for the 'right end node' representations of $\widetilde{\mathrm{SO}}(2n+1)$ and $\mathrm{Sp}(2n)$. The two newest identities come from a tableaux description which originally arose in work of De Concini and Procesi on classical invariant theory. All of the identities are of the most interest when viewed in the context of plane partitions with symmetries contained in three-dimensional boxes.

## 1. INTRODUCTION

In 1898 MacMahon conjectured a product expression for the one-variable generating function for symmetric plane partitions for which the three-dimensional Ferrers diagrams are contained in a box. In 1972 Bender and Knuth conjectured a similar product expression for the generating function of a closely related set of plane partitions [4]. (Gordon actually formulated and proved this second identity around 1969, but he did not publish his proof until much later [8].) In 1977 Andrews confirmed MacMahon's conjecture [1] and then showed it to be equivalent to the Bender–Knuth identity [2]. This author and Stanley have shown [14] how both of these results follow from a Young tableaux basis result of Seshadri [21] for certain representations of $\widetilde{\mathrm{SO}}_{2n+1}$. (Macdonald has also given unified proofs of these two identities [13].) The main results of this paper are two more closely related identities which are found in an analogous fashion using certain representations of $\mathrm{Sp}_{2n}$. These results were first announced in [20]. The Young tableaux basis result which is used here is due to De Concini and Procesi [6], which in turn was inspired by the work of the combinatorialists Doubilet, Rota and Stein. (And [6] could also have been used instead of [21] to prove the MacMahon and Bender–Knuth identities.) In an addendum at the end of this paper, we briefly describe how these two new identities have also been found by other workers.

At the same time we will present four more generating function identities in the variable $q$ for some other families of plane partitions. These will be obtained from the Gelfand pattern descriptions of the same representations. (Since the Gelfand patterns needed can be derived with just determinant manipulations, no representation theory is really needed for the proofs of these identities.) Hence we have a total of eight ($=2\times 2\times 2$) closely related identities. When $q=1$ these reduce to four identities: there are two distinct numbers (one an orthogonal dimension and one a symplectic dimension) for each choice of the size of the containing box, each of which enumerates both a family of ordinary plane partitions and a family of shifted plane partitions. (This will be depicted in Figure 2.) Also included in the same framework is a ninth identity, the archetypal identity of MacMahon for plane partitions for which the three-dimensional Ferrers diagrams are contained in a fixed box.

Before stating our results in Section 3, we will start with a prologue describing



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some nice combinatorial interpretations of the representation dimensions in the most elementary cases. Sections 4 and 5 contain combinatorial and representation theoretic definitions respectively. Proofs are given in Section 6. In Section 7 we discuss the relationship of this work to Stanley's view of bounded plane partitions with symmetries [22].

## 2. Binomial Coefficients, Catalan Numbers and Dimensions of Representations

Aside from the natural numbers, perhaps the most famous single parameter combinatorial quantities are the number $2^n$ of subsets of an $n$-element set and the number of paths $C(n)$ in the plane from $(0, 0)$ to $(n+1, n+1)$ which never rise above the line $x=y$ (viz. Catalan numbers):

$$\begin{array}{llllll} 1, & 2, & 4, & 8, & 16, & 32, \ldots \\ 1, & 2, & 5, & 14, & 42, & 132, \ldots. \end{array}$$

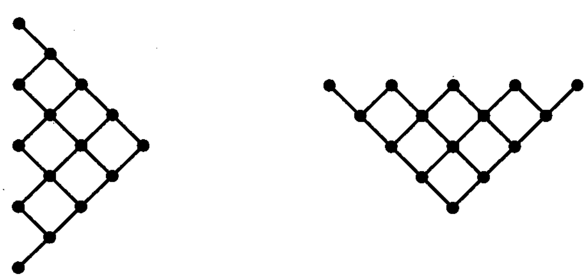
The powers of two can be further broken down into the most famous two parameter family of combinatorial quantities $\binom{n}{k}$:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1. \end{array}$$

The sequence of Catalan numbers can be broken down into a two-parameter family of quantities $C(n, k)$ for $0 \leqslant k \leqslant n$ by counting the number of right-hand turns that a path makes while going from $(0, 0)$ to $(n+1, n+1)$:

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 3 && 1 && \\ & 1 && 6 && 6 && 1 & \\ 1 && 10 && 20 && 10 && 1. \end{array}$$

We will see below (case 'BY' with $M=1$) that $2^n$ is both the dimension of the representation $\widetilde{\mathrm{SO}}_{2n+1}(\omega_n)$ and the number of order ideals (sets closed below with respect to the order) in the partially ordered set shown on the left in Figure 1 (e.g. $n=5$). We will also see (case 'CG' with $m=1$) that $C(n)$ is both the dimension of the representation $\mathrm{Sp}_{2n}(\omega_n)$ and the number of order ideals in the poset shown on the right in Figure 1 (e.g. $n=5$). In addition, we note here that on the one hand when the representation $\widetilde{\mathrm{SO}}_{2n+1}(\omega_n)$ is restricted to the subgroup $\widetilde{\mathrm{GL}}_n$, there result $n+1$



'BY' 'CG'

Figure 1

irreducible representations of $GL_n$ of dimensions $\binom{n}{k}$, for $0 \leqslant k \leqslant n$. On the other hand, when $Sp_{2n}(\omega_n)$ is restricted to $GL_n$, there result $n+1$ irreducible representations of dimensions $C(n, k)$.

## 3. Results

The $2 \times 2 \times 2 + 1$ cases of this paper are labelled as follows: 'B' or 'C' for either the representation $\widetilde{SO}_{2n+1}(M\omega_n)$ or $Sp_{2n}(m\omega_n)$; 'Y' or 'G' for either Young tableaux or Gelfand patterns; and 'I' or 'H' for either the integral or half-integral 'principal' specializations. The ninth case is simply labelled 'A' for the representation $SL_N(M\omega_{n_1})$. This terminology will be explained as we go along, and in Section 5, wherein Figure 3 (which should be glanced at now) will also be explained.

Theorem 1 below is the main result of this paper. It is a direct consequence of Theorem 2. Cases 'BYI' and 'BYH' were conjectured by MacMahon and by Bender and Knuth respectively. In addition to the historical remarks made earlier for these cases, we note that our proof [14] of 'BYH' followed from combining the [14] proof of 'BYI' with the insight of [13, Ex. I.5.19 and I.5.17]. The representation theoretic result needed for the $q = 1$ version of identities 'BGI/H' appeared in [7], and the result needed for the identities 'CGI' and 'CGH' appeared in [24]. Generalizations of the two 'CG' cases appear in [15] and [19]. The alternating weighted diagonal sum interpretation of the $q$-weight in the 'G' cases is due to Stanley (personal communication). To our knowledge, the four $q$-identities in the 'G' cases are new. The two 'BG' cases depend upon our case 'BG' of Theorems 2 and 3 below. When this paper was first written we thought that the identities 'CYI' and 'CYH' were entirely new; see the addendum for independent and later work. The $m = 1$ cases of 'CYI' and 'CYH' give $q$-analogs of Catalan numbers; various $q$-analogs have been studied elsewhere, e.g. [3]. With $q = 1$ the objects of 'BY' are equinumerous with the objects of 'BG'; and similarly for 'CY' and 'CG'. Can explicit bijections be found?

Define $[k] := (1 - q^k)$ and $\langle k \rangle := (q^{-k} - q^{+k})$. Refer to Figure 2 (e.g. $n = 5$), where '$e$' means 'even' and '$p$' means '$\equiv$M (mod 2)'.

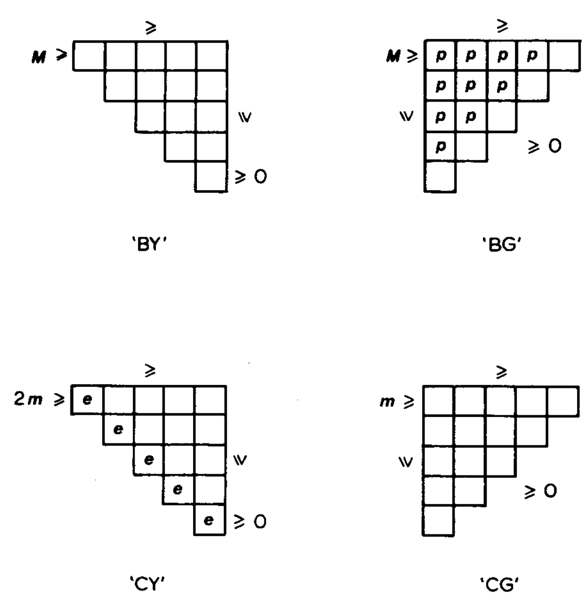



Figure 2

THEOREM 1. *Let $n$, $M \geq 1$ be integers and set $m = \frac{M}{2}$. The following quotients of products (for $1 \leq i, j \leq n$)*

(BYI) $\displaystyle\prod_i \frac{[M+2i-1]}{[2i-1]} \prod_{i<j} \frac{[M+i+j-1]}{[i+j-1]}$

(BGI) $\displaystyle\prod_i \frac{\langle M+2i-1\rangle}{\langle 2i-1\rangle} \prod_{i<j} \frac{\langle M+i+j-1\rangle}{\langle i+j-1\rangle}$

(BYH) $\displaystyle\prod_i \frac{[M+2i-1]}{[2i-1]} \prod_{i<j} \frac{[2M+2i+2j-2]}{[2i+2j-2]}$

(BGH) $\displaystyle\prod_i \frac{\left\langle \frac{M+2i-1}{2}\right\rangle}{\left\langle \frac{2i-1}{2}\right\rangle} \prod_{i<j} \frac{\langle M+i+j-1\rangle}{\langle i+j-1\rangle}$

(CYI) $\displaystyle\prod_i \frac{[2m+2i]}{[2i]} \prod_{i<j} \frac{[2m+i+j]}{[i+j]}$

(CGI) $\displaystyle\prod_i \frac{\langle m+i\rangle}{\langle i\rangle} \prod_{i<j} \frac{\left\langle m+\frac{i+j}{2}\right\rangle}{\left\langle \frac{i+j}{2}\right\rangle}$

(CYH) $\displaystyle\prod_i \frac{[2m+2i]}{[2i]} \prod_{i<j} \frac{[4m+2i+2j]}{[2i+2j]}$

(CGH) $\displaystyle\prod_i \frac{\left\langle \frac{m+i}{2}\right\rangle}{\left\langle \frac{i}{2}\right\rangle} \prod_{i<j} \frac{\left\langle m+\frac{i+j}{2}\right\rangle}{\left\langle \frac{i+j}{2}\right\rangle}$

***are one-variable (Laurent) polynomial generating functions for the following families of (shifted) plane partitions P contained in the specified shapes:***

(BYI) Shape: shifted $n$-staircase
Bound: $M$
Entries: nothing special

(BGI) Shape: $n$-staircase
Bound: $M$
Entries: non-anti-diagonal entries $\equiv M \pmod 2$

(BYH) Shape: $n$-square
Bound: $M$
Entries: symmetric

(BGH) Shape: $n$-staircase
Bound: $M$
Entries: non-anti-diagonal entries $\equiv M \pmod 2$

(CYI) Shape: shifted $n$-staircase
Bound: $2m$
Entries: even on diagonal

(CGI) Shape: $n$-staircase
Bound: integral $m$
Entries: nothing special

(CYH) Shape: $n$-square
Bound: $2m$
Entries: symmetric and even on diagonal

(CGH) Shape: $n$-staircase
Bound: integral $m$
Entries: nothing special

*The weight of P is the usual sum of all parts for the 'Y' cases. In the following formulas take $\mu = M$ or $m$ for cases 'B' or 'C' respectively. For the 'GH' cases the weight of P is the weighted alternating sum of the diagonal sums $-d_1 + 2d_2 - 3d_3 + \cdots - (2n-1)d_{2n-1}(+\mu n/2$ if n is odd). For the 'GI' cases the weight of P is $-2d_1 + 3d_2 - 4d_3 + \cdots - (2n)d_{2n-1}(+\mu(n+1)/2$ if n is odd).*

Here is MacMahon's original result in the subject:

THEOREM 1A. *Let $n_1$, $n_2$, $M \geq 1$. The polynomial on the left below is the (usual weight) generating function for the set of plane partitions described on the right:*

| | |
|---|---|
| (A) $\prod_i \prod_j \frac{[M+i+j-1]}{[i+j-1]}$ | (A) Shape: $n_1 \times n_2$ rectangle<br>Bound: $M$<br>Entries: nothing special |

*Here $1 \leq i \leq n_1$ and $1 \leq j \leq n_2$.*

A plane partition $P$ contained in an $(n+1)$-square and bounded by $M$ is said to be *transpose self-complementary* if $P_{ij} = M - P_{n+2-j,n+2-i}$. (When rotated about the line $i+j = n+2$, $z = \frac{M}{2}$, the three-dimensional Ferrers diagram of such a $P$ will coincide with its complement in the containing $(n+1) \times (n+1) \times M$ box.) It is necessary to have $M$ even, say $M = 2m$, for such plane partitions $P$ to exist. Stanley observed [22] that such $P$ are equivalent to the plane partitions $P'$ of cases 'CG' above with $q = 1$. (For $i+j \leq n+1$ set $P_{ij} = P'_{ij} + m$; for $i+j = n+2$ set $P_{ij} = m$; for $i+j \geq n+3$ set $P_{ij} = m - P'_{n+2-j,n+2-1}$.) Suppose that in order to handle odd $M = 2m+1$ we allow 'cubies' lying along the line $i+j = n+2$, $z = m + \frac{1}{2}$ be sliced in half. This leads to an interpretation of the parity condition in the 'BG' cases above, after one divides all entries by 2.

COROLLARY 1. *Let $M$ be odd or even and set $m = \frac{M}{2}$. Suppose anti-diagonal entries of $P$ in the $(n+1)$-square are allowed to be half-integral. Then from 'CG' the number of transpose self-complementary plane partitions $P$ contained in a $(n+1) \times (n+1) \times M$ box is:*

$$\prod_i \frac{m+i}{i} \prod_{i<j} \frac{M+i+j}{i+j},$$

*where $i$ and $j$ are integers $1 \leq i, j \leq n$. Now suppose that sub- and super-anti-diagonal entries are also allowed to be half-integral. Then from case 'BG' the number of transpose self-complementary plane partitions $P$ is:*

$$\prod_i \frac{m+i}{i} \prod_{i<j} \frac{M+i+j}{i+j},$$

*where the only difference is that now $i$ and $j$ are half integers $\frac{1}{2} \leq i, j \leq n - \frac{1}{2}$.*

If the identities of type 'G' are viewed in this transpose self-complementary light, then the strangeness of the combinatorial interpretation of the $q$-weight no longer seems so unfortunate: the conventional $q$-weight of transpose self-conjugate plane partitions is constant, and hence of no interest.

Call a plane partition contained in a $(n+1) \times (n+1) \times M$ box *symmetric self-complementary* if it is symmetric and transpose self-complementary. Stanley enumerated such plane partitions in [22] by using the main result of [16], which came from the branching rule $SL_N \downarrow Sp_{N-1}$. Results similar to the corollary above can be obtained for this symmetry by applying these viewpoints to orthogonal Gelfand patterns and branchings such as $SL_{2n+1} \downarrow \widetilde{SO}_{2n+1}$. For example, it can be shown that the number of symmetric self-complementary plane partitions contained in an $(2n+1) \times (2n+1) \times 2p$ box with even diagonal entries and sub- and super-anti-diagonal entries possibly half-integral is equal to the number of ordinary (i.e. arbitary integral entries) symmetric self-conjugate plane partitions contained in the same box. This is also the number of unrestricted plane partitions fitting in an $n \times (n+1) \times p$ box.

The generating functions in Theorem 1 are one variable specializations of the

$n$-variate generating functions in Theorem 2 below. Each case of Theorem 2 gives rise to two cases of Theorem 1, depending upon which principal specialization is chosen. In passing from Theorem 2 to Theorem 1, column strict plane partitions with $m$ or $M$ columns are first converted into shifted plane partitions bounded by $m$ or $M$ with the three-dimensional conversion described below in Section 4. The costripps in the 'Y' cases are called semi-standard Young tableaux in representation theory; the costripps in the 'G' cases are converted Gelfand patterns. Nice generalizations of the 'G' cases of Theorems 2 and 3 to (almost) arbitrary shapes (i.e. arbitrary representations) are known (e.g. [17]); but generalizations of the 'Y' cases are considerably more complicated and less explicit [11] (and sequels thereof) [5].

THEOREM 2. *Let $n$, $M \geqslant 1$ be integers and set $m = \frac{M}{2}$. The following quotients of $n \times n$ determinants*

$$\text{(BY)}\quad x_1^m \cdots x_n^m \frac{|x_j^{-m-i+\frac{1}{2}} - x_j^{m+i-\frac{1}{2}}|}{|x_j^{-i+\frac{1}{2}} - x_j^{i-\frac{1}{2}}|} \qquad \text{(BG)}\quad \frac{|x_j^{-M-2i+1} - x_j^{M+2i-1}|}{|x_j^{-2i+1} - x_j^{2i-1}|}$$

$$\text{(CY)}\quad x_1^m \cdots x_n^m \frac{|x_j^{-m-i} - x_j^{m+i}|}{|x_j^{-i} - x_j^{i}|} \qquad \text{(CG)}\quad \frac{|x_j^{-m-i} - x_j^{m+i}|}{|x_j^{-i} - x_j^{i}|}$$

*are $n$-variate (Laurent) polynomial generating functions for the following respective families of column strict plane partitions $P$:*

(BY) Shape: contained in $n \times M$
Bound: $n$

(BG) Shape: exactly $n \times M$
Row bounds: $P_{ij} \leqslant 2n - 2i + 2$, and any entry $\leqslant 2n - 2i$ must occur an even number of times in the $i$th row

(CY) Shape: contained in $n \times 2m$ with even row lengths
Bound: $n$

(CG) Shape: exactly $n \times m$, with $m$ integral
Row bounds: $P_{ij} \leqslant 2n - 2i + 2$

*The weight monomial of $P$ for* 'BY' *and* 'CY' *is $x_1^{\#1\text{'s}} \cdots x_n^{\#n\text{'s}}$ whereas the weight monomial of $P$ for* 'BG' *and* 'CG' *is $x_1^{\#2\text{'s}-\#1\text{'s}} \cdots x_n^{\#2n\text{'s}-\#(2n-1)\text{'s}}$.*

The following result is a special case of the well known description of costripps with Schur functions, e.g. [18].

THEOREM 2A. *Let $n_1$, $n_2$, $M \geqslant 1$ be integers and set $N = n_1 + n_2$. The quotient of $N \times N$ determinants on the left is the usual $N$-variate generating function for the family of column strict plane partitions on the right:*

$$\text{(A)}\quad \frac{\begin{vmatrix} x_j^{i-1} \\ x_j^{M+i-1} \end{vmatrix}}{|x_j^{i-1}|}$$

(A) Shape: exactly $n_1 \times M$
Bound: $N$

*The numerator determinant has $n_2$ rows with entries $x_j^{i-1}$ and $n_1$ rows with entries $x_j^{M+i-1}$.*

Theorem 2 is immediately equivalent to Theorem 3 below, which describes the characters of the representations at hand with the same costripps. The equivalence occurs because the quotients of determinants (with $x_i = t_i$ or $t_i^2$ are just the Weyl character formulas for the representations being considered [12, Th. 11.9.IX and 11.11.X; 23, Th 7.8.C]. For The 'G' cases Theorem 2 can be proved directly with determinant manipulations; hence Theorem 3 in these cases can be taken as a consequence. Our proof of the 'CY' case is via Theorem 3, with Theorem 2 then being taken as the consequence.

THEOREM 3. *Let $n$, $M \geq 1$ be integers and set $m = \frac{M}{2}$. The character $\widetilde{\mathrm{SO}}_{2n+1}(M\omega_n; t_1, \ldots, t_n)$ can essentially be described as the generating function for the column strict plane partitions either in Theorem* 2(BY) *(divide by $x_1^m \cdots x_n^m$ and set $x_i = t_i$) or in Theorem* 2(BG) *(set $x_i = t_i^{\frac{1}{2}}$). Let $m$ be integral. The character $\mathrm{Sp}_{2n}(m\omega_n; t_1, \ldots, t_n)$ can essentially be described as the generating function for the column strict plane partitions either in Theorem* 2(CY) *(divide by $x_1^m \cdots x_n^m$ and set $x_i = t_i$) or in Theorem* 2(CG) *(set $x_i = t_i$). Let $n_1$, $n_2 \geq 1$ and set $N = n_1 + n_2$. The character $\mathrm{GL}_N(M\omega_{n_1}; x_1, \ldots, x_n)$ can be described as the generating function for the column strict plane partitions in Theorem* 2A.

## 4. COMBINATORIAL DEFINITIONS

Given a sequence of numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$, the *shape* (or *Ferrers diagram*) $\lambda$ is a left justified array of boxes, there being $\lambda_i$ boxes in the $i$th row. Given a sequence $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$, the *shifted shape* $\lambda$ is an array of boxes with $\lambda_i$ boxes in the $i$th row placed in columns $i$ through $i + \lambda_i - 1$. A (shifted) shape $\mu$ is *contained in* a shape $\lambda$ if $\mu_i \leq \lambda_i$ for all $i$ such that $\mu_i > 0$. The *$n$-square* is the shape with $n$ rows each of length $n$. The *$n$-staircase* is the shape with $n$ rows of lengths $n, n-1, \ldots, 1$. The *shifted $n$-staircase* is the shifted shape with rows of lengths $n$, $n-1, \ldots, 1$ (see Figure 2). The *main diagonal* of a shape is the set of squares with co-ordinates $(i, i)$. The *anti-diagonal* of the $n$-square is the set of squares with co-ordinates $(i, n+1-i)$.

Given a shape $\lambda$, filling its boxes with positive integers $P_{ij} \leq M$ such that $P_{ij} \geq P_{i,j+1}$ and $P_{ij} \geq P_{i+1,j}$ yields a *plane partition of shape $\lambda$* bounded by $M$. Allowing some of the *parts* (or *entries*) $P_{ij}$ to be zero gives a plane partition with *shape contained in $\lambda$*. *Shifted plane partitions* are defined similarly, except now the entries are defined only for $i \leq j$. *Column strict plane partitions* (or *costripps*) satisfy $P_{ij} > P_{i+1,j}$. A plane partition is said to be *symmetric* if $P_{ij} = P_{ji}$ for all $i$ and $j$.

The *weight* $|P|$ of the (shifted) (column strict) plane partition $P$ is defined to be the sum of all of its parts. The usual *one-variable generating function* for a given set of (shifted) plane partitions is the sum of $q^{|P|}$ over all of the (shifted) plane partitions in the set. Let $\# k$'s be the number of $(i, j)$ for which $P_{ij} = k$. Then the usual *$n$-variate generating function* for a set of costripps bounded by $n$ is the sum of the weight monomials $x_1^{\#1\text{'s}} \cdots x_n^{\#n\text{'s}}$ over all of the costripps in the set. We also consider other one variable or $n$-variate generating functions which are sums of different powers of $q$ or weight monomials in $x$. An $n$-staircase has $2n - 1$ *diagonals* parallel to the main diagonal: Let $d_k$ be the sum of the entries lying along the $k$th diagonal, where the box at $(n, 1)$ is on the 1st diagonal. The diagonal sums of a shifted plane partition are defined similarly, except now the main diagonal is taken to be the 0th diagonal.

The *three-dimensional Ferrers diagram* of a (shifted) (column strict) plane partition is obtained by plotting $P_{ij}$ dots above the position $(i, j)$ in the $i, j$-plane, viz. in positions $(i, j, 1), \ldots, (i, j, P_{ij})$. There is a standard conversion of costripps to shifted plane partitions which goes as follows: shift the dots in the $i$th row of the three-dimensional Ferrers diagram of a costripp $P$ up $i - 1$ positions and project all of the dots to the $i$, $z$-plane. Relabel the $z$-axis as the new $j$-axis to obtain a shifted plane partition. The entries now along the main diagonal describe the original shape of the costripp.

A *Gelfand pattern for* $\mathrm{Sp}_{2n}$ is a shifted plane partition with row lengths $2n$, $2n-2, \ldots, 2$ which has been rotated in the plane 135° clockwise: the main diagonal becomes the first row. *Gelfand patterns for* $\widetilde{\mathrm{SO}}_{2n+1}$ are defined similarly, except now all entries in the shifted plane partition other than the last entry in each row may be all integral *or* all half-integral, and the last entries in each row may be integral or half-integral independently from the other entries.

## 5. Representation Theory: Definitions and Comments

The Lie groups used in this paper are taken over the complex numbers. The group $\widetilde{\mathrm{SO}}_{2n+1}$ is the simply connected covering group of the odd orthogonal group $\mathrm{SO}_{2n+1}$. If $N=2n$, then the symplectic group $\mathrm{Sp}_{2n}$ is a subgroup of $\mathrm{GL}_N$. If $\mu$ is the highest weight of a representation of the group $X_n$, then $X_n(\mu)$ will denote that representation. And $X_n(\mu; t_1, \ldots, t_n)$ will denote the character of the representation, where the variables $t_1, \ldots, t_n$ parameterize a maximal torus of $X$.

It is well known that the structure and representation theories of $\widetilde{\mathrm{SO}}_{2n+1}$ and $\mathrm{Sp}_{2n}$ are closely interrelated. The root systems of these two simple Lie groups are dual to each other, a fact that is reflected in the near identical appearance of the Dynkin diagrams for the two groups (see Figure 3). The weight of the fundamental representation of either group which corresponds to the right end node of the Dynkin diagram will be denoted $\omega_n$. In terms of the co-ordinate systems needed to apply the formulas of [12] and [23], we have $M\omega_n = (\frac{M}{2}, \frac{M}{2}, \ldots, \frac{M}{2})$ for $\widetilde{\mathrm{SO}}_{2n+1}$ and $m\omega_n = (m, m, \ldots, m)$ for $\mathrm{Sp}_{2n}$.

The principal specialization for $\widetilde{\mathrm{SO}}_{2n+1}$ characters is $t_i = q^i$, and the principal specialization for $\mathrm{Sp}_{2n}$ characters is $t_i = q^{i-\frac{1}{2}}$. (The representation theoretic meaning of such specializations is explained in Section 5 of [14].) Happily, one still obtains Laurent polynomials which factor completely if one applies one of these specializations to the numerator or denominator of a character for the 'wrong' group. In this paper we abuse terminology and refer to either specialization for either group as a principal specialization. Ignoring overall factors of $q$, we have altogether four Laurent polynomials in $q$; 'BI', 'BH', 'CI', and 'CH'. (The $q$'s appearing in cases 'BGI', 'BYH', 'BGI' and 'CYH' of Theorem 1 are 'really' $q^{\frac{1}{2}}$'s.)

The final factor of 2 which gives a total of eight identities arises as follows. The Young tableaux and Gelfand patterns of Theorem 3 can be viewed as keeping track of the results of repeatedly restricting the representations at hand to smaller and smaller subgroups. The Gelfand patterns arise from successive restrictions of the kind $\mathrm{Sp}_{2n} \downarrow \mathrm{Sp}_{2n-2} \times \mathrm{GL}_1 \downarrow \cdots \downarrow \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$. This type of restriction is depicted on the right-hand side of Figure 3. The 'Y' cases of Theorem 3 can be used to state and prove

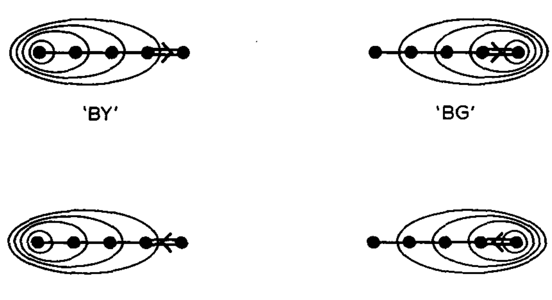



Figure 3

branching rules of $\widetilde{\mathrm{SO}}_{2n+1}(M\omega_n) \downarrow \widetilde{\mathrm{GL}}_n$ and $\mathrm{Sp}_{2n}(m\omega_n) \downarrow \mathrm{GL}_n$. For example, in the second rule one $\mathrm{GL}_n$ representation arises for each shape contained in the $n \times 2m$ rectangle with all row lengths even. Then repeatedly using the well known branching rule for $\mathrm{GL}_k \downarrow \mathrm{GL}_{k-1}$ corresponds to noting which entries of the costripps are $>n-k+1$ [18]. The two restrictions of the $\mathrm{Sp}_{2n} \downarrow \mathrm{GL}_n \downarrow \mathrm{GL}_{n-1} \times \mathrm{GL}_1 \downarrow \cdots \downarrow \mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1$ type are depicted on the left-hand side of Figure 3. Therefore the ordinary plane partitions of Theorem 1 arise from restricting from right to left on the Dynkin diagram, whereas the shifted plane partitions arise from restricting from left to right on the Dynkin diagram.

## 6. Proofs

Proofs of Theorems 2 and 3. The origins of the 'G' cases of Theorem 2 lie entirely in representation theory. In addition to references given in Section 3, [9] should be mentioned in connection with the 'CG' case. However, it is now possible to avoid using representation theory for these cases. The equivalence of the multivariable generating function for Gelfand patterns of type 'CG' with the 'column-wise' one-determinant character formula for $\mathrm{Sp}_{2n}(m\omega_n; x_1, \ldots, x_n)$ is proved with easy determinant manipulations in Theorem 4.2 of [17]. (Theorem 4.2 is stated with Young tableaux (increasing entries) instead of plane partitions (decreasing entries), but this difference is inconsequential.) The equivalence of the columnwise character formula with the quotient character formula given Theorem 2 can be proved with Cauchy series arguments as in [10], [12] and [23]. An analogous process for case 'BG' uses Theorems 8.1 and 7.1 of [17]. (The square root which appears in the statement of Theorem 8.1 is not needed here in Theorem 2, since each of the exponents in the usual quotient character formula for $\widetilde{\mathrm{SO}}_{2n+1}(m\omega_n; t_1, \ldots, t_n)$ have been doubled here.) We can actually pass directly from the quotient determinant formulas of Theorem 2 to the Gelfand pattern generating functions with a harder version of the 'division' proof used in [18]; see [19]. Case 'BY' was proved on page 340 of [14], using [21].

We now verify case 'CY' of Theorem 3 using Proposition 4.10 and Theorem 4.11 of [5]. (Theorem 5.2 of [11] could also be used.) There are two errors in the statement of 4.10: Replace '$\chi = \det$' with '$\chi = \alpha\det$' and replace 'lower' with 'upper'. Replace his $m$, $r$ and $\alpha$ with our $n$, $n$ and $m$, respectively. Refering to Theorem 4.11, we see that $H^0(\mathscr{I}_{n,n}, L_\chi)$ is an irreducible $\mathrm{Sp}_{2n}$ module with highest weight $m\omega_n$. Interchanging rows with columns, Proposition 4.10 states that a basis for this module is indexed by increasing chains of $m$ 'admissible minors', each described by a pair of two-part columns: $\substack{I\\J'}\ \substack{I'\\J}$. Here $I = \{i_1 < \cdots < i_s\}$, $J' = \{j'_1 < \cdots < j'_{n-s}\}$, $I' = \{i'_1 < \cdots < i'_s\}$ and $J = \{j_1 < \cdots < j_{n-s}\}$ are subsets of $\Delta = \{1, 2, \ldots, n\}$. Remark (1) on page 9 forces $J' = \Delta - I$ and $J = \Delta - I'$ in the special case of (his) $k = n$ (ours). So we will ignore $J'$ and $J$ for combinatorial purposes. Remark (2) states that $i_h \leqslant i'_h$ for $1 \leqslant h \leqslant s$. If the admissible minor $I_1, I'_1$ is followed by $I_2, I'_2$ in a chain, then $s_1 \geqslant s_2$ and $i_{1,h} \leqslant i_{2,h}$ for $1 \leqslant h \leqslant s_2$ by Definitions 2.3 and 1.2. Subtract all entries in the tableaux formed by all of the $I$ and $I'$ from $n+1$ to obtain the costripps described in case 'CY'. According to Proposition 4.10, the weight contributed by a pair of columns is

$$[(t_{i_1} \cdots t_{i_s} t_{j'_1}^{-1} \cdots t_{j'_{n-s}}^{-1})(t_{i'_1} \cdots t_{i'_s} t_{j_1}^{-1} \cdots t_{j_{n-s}}^{-1})]^{\frac{1}{2}} = (t_1 \cdots t_n)^{-1} t_{i_1} \cdots t_{i_s} t_{i'_1} \cdots t_{i'_s}.$$

The operation of subtracting all entries from $n+1$ does not affect the overall generating function: for any fixed shape determined by the $I$'s and $I'$'s, the sum of the terms of that shape is $(t_1 \cdots t_n)^{-m}$ times a Schur function, which is symmetric in $t_1, \ldots, t_n$. $\square$

The following two 'denominator' identities for $n \times n$ determinants are analogs of Vandermonde's identity for the root systems of types B and C respectively [15]:

$$\text{(I)}\ |x_i^{-j} - x_i^{+j}| = \prod_i (x_i^{-1} - x_i^{+1}) \prod_{i<j} (x_i^{-\frac{1}{2}} x_j^{+\frac{1}{2}} - x_i^{+\frac{1}{2}} x_j^{-\frac{1}{2}})(x_i^{-\frac{1}{2}} x_j^{-\frac{1}{2}} - x_i^{+\frac{1}{2}} x_j^{+\frac{1}{2}});$$

$$\text{(H)}\ |x_i^{-j+\frac{1}{2}} - x_i^{+j-\frac{1}{2}}| = \prod_i (x_i^{-\frac{1}{2}} - x_i^{+\frac{1}{2}}) \prod_{i<j} (x_i^{-\frac{1}{2}} x_j^{+\frac{1}{2}} - x_i^{+\frac{1}{2}} x_j^{-\frac{1}{2}})(x_i^{-\frac{1}{2}} x_j^{-\frac{1}{2}} - x_i^{+\frac{1}{2}} x_j^{+\frac{1}{2}}).$$

PROOF OF THEOREM 1. Convert the costripps of Theorem 2 to shifted plane partitions as described in Section 2. In the 'G' cases this just returns us to the Gelfand patterns refered to in the proof of Theorem 2. At first, ignore the $q$ weight. For the 'G' cases all of the entries in the left half of the inverted Aztec pyramid shape $(2n, 2n-2, \ldots, 2)$ will be $M$ (case 'B') or $m$ (case 'C'). Drop the left half to obtain ordinary plane partitions contained in $n$-staircases. For the 'YH' cases 'flip/duplicate' about the main diagonal to obtain plane partitions contained in $n$-squares. Do nothing to the 'YI' cases.

To obtain the product expressions, perform the following specializations: $x_i = q^i$ for the 'I' cases, $x_i = q^{i-\frac{1}{2}}$ for the 'GH' cases, and $x_i = q^{2i-1}$ for the 'YH' cases. Observe that $(q^j)^{-m-i} - (q^j)^{m+i} = (q^{m+i})^{-j} - (q^{m+i})^j$ and apply identity (I) above with $x_i = q^{m+i}$ for the 'CI' cases. The $\binom{n}{2}$ factors from $x_i^{-\frac{1}{2}} x_j^{\frac{1}{2}} - x_i^{\frac{1}{2}} x_j^{-\frac{1}{2}}$ cancel top and bottom. For 'CYI' bring in the additional $m\binom{n+1}{2}$ factors of $q$ and multiply top and bottom by $\binom{n+1}{2} + \sum_{i<j} (i+j)/2$ more factors of $q$. The other six cases are similar.

In the cases 'YI', the specialization $x_i = q^i$ yields the usual weight of the costripps, which is preserved by the conversion to shifted plane partitions. In the cases 'YH' the 'flip/duplicate' step causes an entry of $i$ in the costripp to contribute $2i-1$ to the weight of the symmetric plane partition. The total $q$-weight in the 'GI' cases is (e.g. 'CGI', $n = 3$): $1 \times (3m - 2(2m + d_1) + (m + d_2)) + 2 \times ((m + d_2) - 2d_3 + d_4) + 3 \times (d_4 - 2d_5) = -2d_1 + 3d_2 - 4d_3 + 5d_4 - 6d_5 + 2m$. In the 'GH' cases it is (e.g. 'CGH', $n = 3$): $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2} \times$ same factors as above $= -d_1 + 2d_2 - 3d_3 + 4d_4 - 5d_d + \frac{3}{2}m$. □

## 7. CONCLUSION: PLANE PARTITIONS WITH SYMMETRIES

Stanley has noted [22] that there are ten possible distinct symmetries which can be possessed by the three-dimensional Ferrers diagrams of plane partitions contained in an $r \times s$ rectangle and bounded by $t$. (For all but two of these one must require $r = s$; sometimes $s = t$ is required as well.) Each of these cases has a known or conjectured product formula (sometimes in $q$) which enumerates the total number of such plane partitions. (One of the ten cases is actually two cases, since it has two different $q$ generating functions.) Four of the eleven cases appear in Theorem 1, viz. 'A', 'BYI', 'BYH' and 'CG' (with $q = 1$). In addition, three closely related identities which allow half-integral entries or require some entries to be even also appear in Theorem 1, viz. 'CYI', 'CYH' and 'BG' (as explained in Corollary 1). Grouping two pairs of identities into the cases 'BG' and 'CG' by taking $q = 1$, we see that all seven identities of Theorem 1 can be so interpreted. Two other of the 11 cases can at present be solved only with representation theoretic (or symmetric function) techniques: transpose self-complementary [16] and self-complementary [22]. A close relative of the transpose self-complementary case was mentioned after Corollary 1. So 10 of the 15 cases mentioned in [22] or here are solvable with representation theory; seven of them in the unified setting of Theorem 1.

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## ADDENDUM

After this paper was written, we learned of some overlapping results by Desainte-Catherine and Viennot [A1] and by Desarmenien [A2]. The paper [A1] contains a mostly combinatorial lattice path proof of the $q = 1$ version of identities 'CYI/H'. The paper [A2] derives identities 'CY', 'CYI' and 'CYH', with methods similar to those of [13]. More recently, Stembridge [A4] also does this, but at the same time he also rederives some generalizations of the Rogers–Ramanujan identities with the theory of Hall–Littlewood functions. And Desarmenien has found [A3] an easy way to derive 'CYI' and 'CYH' from 'BYI' and 'BYH'. So although they are short, our proofs of the 'CY' type identities are now relatively undesirable from a combinatorial viewpoint. However, none of the four papers cited mention the 'G' identities. This paper places all nine $q$-identities in a uniform Lie theoretic setting, and via the De Concini–Procesi proof advertises the algebraic geometric work of those authors and Lakshmibai, Musili and Seshadri. It is natural to believe that there may be more material in that setting of interest to combinatorialists; this belief is supported by the recent development of a Littlewood–Richardson rule for all classical groups by Littelmann.

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