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# Max-balanced flows in oriented matroids

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**Abstract**

Let $M=(E,\mathcal{O})$ be an oriented matroid on the ground set $E$. A real-valued vector $x$ defined on $E$ is a max-balanced flow for $M$ if for every signed cocircuit $Y\in\mathcal{O}^{\perp}$, we have $\max_{e\in Y^+} x_e=\max_{e\in Y^-} x_e$. We extend the admissibility and decomposition theorems of Hamacher from regular to general oriented matroids in the case of max-balanced flows, which gives necessary and sufficient conditions for the existence of a max-balanced flow $x$ satisfying $l\leqslant x\leqslant u$. We further investigate the semilattice of such flows under the usual coordinate partial order, and obtain structural results for the minimal elements. We also give necessary and sufficient conditions for the existence of such a flow when we are allowed to reverse the signs on a subset $F\subseteq E$. The proofs of all of our results are constructive, and yield polynomial algorithms in case $M$ is coordinatized by a rational matrix $A$. In this same setting, we describe a polynomial algorithm that for a given vector $w$ defined on $E$, either finds a potential $p$ such that $w'=w+pA$ is max-balanced, or a certificate that $M$ has no max-balanced flow.

## 1. Introduction

A real-valued vector $x$ indexed on the ground set $E$ of an oriented matroid $M=(E,\mathcal{O})$ is called a *max-balanced matroid flow for M*, or simply a *max-balanced flow*, if

$$\max_{e\in Y^+} x_e=\max_{e\in Y^-} x_e \quad \text{for all } Y\in\mathcal{O}^{\perp}.$$

If $M$ is a graphic oriented matroid, then this definition is equivalent to the following: A real-valued vector $x$ indexed on the arc set $E$ of a directed graph $D=(V,E)$ is a max-balanced flow if

$$\max_{e\in\delta^+(W)} x_e=\max_{e\in\delta^-(W)} x_e \quad \text{for all } \emptyset\subset W\subset V.$$

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In this paper, we extend a number of results for max-balanced flows in directed graphs and regular oriented matroids to general oriented matroids. In particular, we extend the admissibility and decomposition theorems of Hamacher [8, (3.21), (3.24)] from regular to general oriented matroids in the case of max-balanced flows. The resulting necessary and sufficient conditions for the existence of a max-balanced flow $x$ satisfying $l \leqslant x \leqslant u$ also yield a good characterization.

We further investigate the semilattice of such flows under the usual coordinate partial order, and obtain structural results for the minimal elements which extend results in [10]. We also give necessary and sufficient conditions for the existence of such a flow when we are allowed to reverse the signs on a subset $F \subseteq E$, generalizing the result of Robbins [14] that an undirected graph is 2-edge connected if and only if it has an orientation which is strongly connected.

The proofs of all of our results are constructive, and yield polynomial algorithms in case $M$ is coordinatized by a rational matrix $A$. In this same setting, we describe a polynomial algorithm which for a given vector $w$ defined on $E$, either finds a potential $p$ such that $w' = w + pA$ is max-balanced, or a certificate that $M$ has no max-balanced flow. For a directed graph, this is known as the max-balancing problem, and has been studied by Schneider and Schneider [18, 20], Rothblum et al. [16] and Young et al. [21].

Max-balanced flows defined on digraphs have been studied by Schneider and Schneider [18–20]. See also [10] for a discussion of max-balanced flows satisfying lower and upper bounds, [17] for a discussion of a related algebraic matrix scaling problem, and [21] for a discussion of efficient algorithms for max-balancing. Related algebraic generalizations of network flow and linear programming problems have been considered by Hoffman [11], Cunningham-Green [4], Hamacher [5–7, 9], and Zimmerman [22, 23]. See also the survey paper by Burkard and Zimmermann [3] and the collection of papers in [2]. *Algebraic matroid flows* defined on regular matroids were introduced and studied by Hamacher [5–8]. If the matroid $M$ is regular, then a max-balanced matroid flow is an instance of an algebraic matroid flow with flows contained in the ordered semigroup of the reals together with the semigroup operation of max.

## 2. Preliminaries

Following the development in [1], we present some of the theory of oriented matroids needed for this paper. We will assume knowledge of some basic properties of matroids (see, for example, [13]). A *signed set* $X$ is a set $\underline{X}$, called the *underlying set* of $X$, together with a partition of $\underline{X}$ into possibly empty subsets $(X^+, X^-)$. For a signed set $X$, we use $-X$ to denote the signed set with underlying set $\underline{X}$ such that $(-X)^+ = X^-$ and $(-X)^- = X^+$. For a matroid $M$ with ground set $E$, a *circuit signature of* $M$ is a collection $\mathcal{C}$ of signed sets $X$ whose underlying sets $\underline{X}$ are the circuits of $M$, such that $-X \in \mathcal{C}$ whenever $X \in \mathcal{C}$. A *cocircuit signature of* $M$ is a circuit signature of the dual matroid $M^\perp$. The pair $(E, \mathcal{O})$ is an *oriented matroid* if $\mathcal{O}$ is a circuit signature of $M$ and there exists a cocircuit signature $\mathcal{O}^\perp$ of $M$ such that the following property holds.

**Orthogonality.** *For all* $X \in \mathcal{O}$ *and* $Y \in \mathcal{O}^\perp$ *with* $\underline{X} \cap \underline{Y} \neq \emptyset$, *both* $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \emptyset$ *and* $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \emptyset$.

Such a cocircuit signature is uniquely determined whenever it exists. Signed sets $X \in \mathcal{O}$ and $Y \in \mathcal{O}^\perp$ are called *(signed) circuits* and *cocircuits* of $(E, \mathcal{O})$.

We will also make use of two other properties of oriented matroids:

**Signed circuit elimination.** *For all* $X_1, X_2 \in \mathcal{O}$, $x \in (X_1^+ \cap X_2^-) \cup (X_1^- \cap X_2^+)$, *and* $y \in (X_1^+ \setminus X_2^-) \cup (X_1^- \setminus X_2^+)$ *there exists* $X_3 \in \mathcal{O}$ *such that* $X_3^+ \subseteq (X_1^+ \cup X_2^+) \setminus x$, $X_3^- \subseteq (X_1^- \cup X_2^-) \setminus x$ *and* $y \in \underline{X}_3$.

**Minty's painting lemma.** *For all* $e \in E$ *and all partitions of* $E$ *into (possibly empty) subsets* $R, G, B$, *and* $W$ *with* $e \in R \cup G$, *exactly one of the following holds:*

(i) *there exists* $X \in \mathcal{O}$ *such that*

$$e \in \underline{X} \subseteq R \cup G \cup B \quad \text{and} \quad X^+ \cap G = X^- \cap R = \emptyset$$

or,

(ii) *there exists* $Y \in \mathcal{O}^\perp$ *such that*

$$e \in \underline{Y} \subseteq R \cup G \cup W \quad \text{and} \quad Y^+ \cap G = Y^- \cap R = \emptyset.$$

An oriented matroid $M = (E, \mathcal{O})$ is said to be *coordinatizable* over a field $\mathcal{F}$ if there exists an $\mathcal{F}$-valued matrix $A$ whose columns are indexed on the elements $E$ such that the circuits and cocircuits of $M$ are the signed supports of elementary vectors of, respectively, the null space and row space of $A$. Recall for any field $\mathcal{F}$, an elementary vector of a subspace of the vector space $\mathcal{F}^E$ is a nonzero vector $x$ whose support is minimal. In the circuit $X$ corresponding to $x$, the sets $X^+$ and $X^-$ are, respectively, the coordinates of the positive and negative elements of $x$. In this case we say that $M = (E, \mathcal{O})$ is *coordinatized* by the matrix $A$. For oriented matroids coordinatized by a real matrix $A$, we will make use of the following lemma from [15].

**Harmonious decomposition.** *If* $Ax = 0$ *and* $x \geqslant 0$ *then there exist elementary vectors* $x^1, \ldots, x^n \geqslant 0$ *in the null space of* $A$ *and nonnegative numbers* $\mu_1, \ldots, \mu_n$ *such that* $x = \sum_{i=1}^n \mu_i x^i$. *If the vector* $x$ *has an element* $x_a > 0$, *then there exists an elementary vector* $\bar{x} \geqslant 0$ *in the null space of* $A$ *such that* $\bar{x}_a > 0$ *and* $\bar{x}_e = 0$ *whenever* $x_e = 0$.

This lemma implies that if $x$ is a max-balanced flow for the oriented matroid coordinatized by the real matrix $A$, then $\max_{e:(pA)_e > 0} x_e = \max_{e:(pA)_e < 0} x_e$ for any *potential* $p$ such that $pA \neq 0$.

## 3. Feasibility conditions

There is a simple condition for the existence of a max-balanced matroid flow for a given oriented matroid.

**Lemma 1.** *There exists a max-balanced flow for the oriented matroid $M=(E,\mathcal{O})$ if and only if there is no $Y\in\mathcal{O}^\perp$ with $Y^+=\underline{Y}$.*

**Proof.** Clearly if $Y$ is a cocircuit with $Y^+=\underline{Y}$, then we cannot have $\max_{e\in Y^+} x_e=\max_{e\in Y^-} x_e$ for any $x\in\mathbb{R}^E$. Conversely, if no such cocircuit exists, then the vector $x$ defined by $x_e=1$ for $e\in E$ is a max-balanced flow for $M$. □

As a consequence of Lemma 1 we obtain the following result for oriented matroids coordinatized over $\mathbb{R}^E$.

**Corollary 2.** *There exists a max-balanced flow for the oriented matroid $M=(E,\mathcal{O})$ coordinatized by the real matrix $A$ if and only if $Ax=0$, $x>0$, is feasible.*

**Proof.** If $x$ is a max-balanced flow for $M$, then it follows from Lemma 1 and Minty's painting lemma with $R=E$ and $G=B=W=\emptyset$ that for each $e\in E$ there exists $X\in\mathcal{O}$ with $e\in X^+=\underline{X}$. Because of the correspondence between circuits of $M$ and elementary vectors, for each $e\in E$ there exists a vector $x$ such that $x\geqslant 0$, $x_e>0$, and $Ax=0$. Summing these vectors gives the result.

Conversely, if $M$ has no max-balanced flow then by Lemma 1 there exists $Y\in\mathcal{O}^\perp$ with $Y^+=\underline{Y}$. Let $y=pA\geqslant 0$ be the elementary vector in the row space of $A$ corresponding to $Y$. Then if $Ax=0$ and $x>0$, we must have $pA=0$, a contradiction. □

Next, we give conditions for the existence of a max-balanced flow for an oriented matroid satisfying given lower and upper bounds.

**Theorem 3.** *For an oriented matroid $M=(E,\mathcal{O})$ and vectors $l,u\in\mathbb{R}^E$ satisfying $l\leqslant u$, the following are equivalent:*

(i) *There exists a max-balanced flow $x$ for $M$ satisfying $l\leqslant x\leqslant u$.*

(ii) *For every $Y\in\mathcal{O}^\perp$,*

$$\max_{e\in Y^+} l_e\leqslant \max_{e\in Y^-} u_e.$$

(iii) *For each $a\in E$ there exists $X_a\in\mathcal{O}$ such that $a\in X_a^+=\underline{X}_a$ and*

$$u_e\geqslant l_a \quad \text{for all } e\in\underline{X}_a.$$

**Proof.** (i) ⇒ (ii): This follows directly, since for a max-balanced $x$ satisfying $l\leqslant x\leqslant u$ we have

$$\max_{e\in Y^+} l_e\leqslant \max_{e\in Y^+} x_e=\max_{e\in Y^-} x_e\leqslant \max_{e\in Y^-} u_e.$$

(ii) ⇒ (iii): Suppose that (ii) holds. Then it follows from Minty's painting lemma by setting $R=\{e\in E \mid u_e \geqslant l_a\}$, $W=\{e\in E \mid u_e < l_a\}$, and $G=B=\emptyset$ that either the desired circuit exists or there exists $Y\in\mathcal{O}^{\perp}$ such that $a\in Y^+$ and $u_e<l_a$ for $e\in Y^-$. In the latter case,

$$\max_{e\in Y^+} l_e \geqslant l_a > \max_{e\in Y^-} u_e,$$

which violates (ii), and so the desired circuit exists.

(iii) ⇒ (i): Suppose that (iii) holds. Then define $x\in\mathbb{R}^E$ by

$$x_a=\max\{l_e \mid e\in E \text{ and } a\in X_e^+\}.$$

Note that for $a\in E$, we have $l_a\leqslant x_a\leqslant u_a$ since $a\in X_a^+$ and since $a\in X_e^+$ implies that $l_e\leqslant u_a$. To see that $x$ is max-balanced, let $Y\in\mathcal{O}^{\perp}$ and choose $a\in Y^+$ so that $x_a=\max_{e\in Y^+} x_e$. Now choose $b$ so that $a\in X_b^+$ and $x_a=l_b$. Since $a\in X_b^+\cap Y^+$ it follows from orthogonality that there exists $c\in X_b^+\cap Y^-$. Therefore, it follows that

$$\max_{e\in Y^+} x_e = x_a = l_b \leqslant x_c \leqslant \max_{e\in Y^-} x_e.$$

Since $-Y\in\mathcal{O}^{\perp}$ for all $Y\in\mathcal{O}^{\perp}$, the reverse inequality holds also, and therefore (i) holds. □

Setting $l=u=x$, we obtain the following characterization for max-balanced flows in oriented matroids.

**Corollary 4.** *A vector $x\in\mathbb{R}^E$ is a max-balanced flow for the oriented matroid $M=(E,\mathcal{O})$ if and only if for each $a\in E$ there exists $X_a\in\mathcal{O}$ such that $a\in X_a^+=\underline{X}_a$ and*

$$x_e\geqslant x_a \quad \text{for all } e\in\underline{X}_a. \tag{1}$$

We will use $mbf(l,u)$ to denote the set of max-balanced flows $x$ satisfying $l\leqslant x\leqslant u$. Parts (ii) and (iii) of Theorem 3 give a good characterization for the feasibility of the set $mbf(l,u)$. The following procedure can be used to find a max-balanced flow in $mbf(l,u)$ or show that no such flow exists.

## Feasible flow algorithm

*Input*: An oriented matroid $M=(E,\mathcal{O})$ and vectors $l,u\in\mathbb{R}^E$ satisfying $l\leqslant u$.

*Output*: Either a max-balanced flow $x$ satisfying $l\leqslant x\leqslant u$, or an element $a\in E$ and a cocircuit $Y\in\mathcal{O}^{\perp}$ such that $a\in Y^+$ and

$$u_e<l_a \quad \text{for all } e\in Y^-.$$

**(0)** Set $x_e=-\infty$ for all $e\in E$.

**(1)** If $l\leqslant x$, return $x$ and **STOP**.

(**2**) Choose any $a \in E$ with $x_a < l_a$. Find $X \in \mathcal{O}$ such that $a \in X^+ = \underline{X}$ and $u_e \geqslant l_a$ for all $e \in \underline{X}$, and update $x_e$ for $e \in X^+$ by

$$x_e = \max\{x_e, l_a\}.$$

Return to (**1**).

(**3**) Otherwise, find $Y \in \mathcal{O}^\perp$ such that $a \in Y^+$ and $u_e < l_a$ for all $e \in Y^-$. Return $a$ and $Y$, and **STOP**.

This simple algorithm is a paradigm for other algorithms which give constructive proofs for most of the results in this paper. If we process the elements in decreasing order of $l_a$, then the algorithm can be thought of as trying to increase the subset of arcs that satisfy (1) in a top–down fashion, which guarantees once a value is set by the algorithm it is not changed in subsequent iterations. If $M$ is the graphic oriented matroid associated with a directed graph $D = (V, E)$, then we can find either a directed circuit or a directed cocircuit containing $a$ in $O(|E|)$ time. If we contract the directed circuits found in (**2**), the feasible-flow algorithm can be modified to have an $O(|V||E|)$ running time in this case.

If for a given partition $E = R \cup W$ with $a \in R$ we can find in polynomial time either $X \in \mathcal{O}$ such that $a \in X^+ = \underline{X} \subseteq R$ or $Y \in \mathcal{O}^\perp$ such that $a \in Y^+$ and $Y^- \subseteq W$, then the feasible-flow algorithm is polynomial. In particular, if $M$ is coordinatized by a rational matrix $A$, then such $X$ or $Y$ can be obtained from corresponding basic optimal solutions $x^*$ and $p^*$, $\lambda^*$ to the primal and dual linear programs:

$$\begin{aligned} &\text{maximize} && x_a \\ &\text{subject to} && Ax = 0, \\ &&& x_e = 0 \ \text{ for } e \in W, \\ &&& x_a \leqslant 1, \\ &&& x \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} &\text{minimize} && \lambda \\ &\text{subject to} && (pA)_e \geqslant 0 \ \text{ for } e \in R \backslash a, \\ &&& (pA)_a + \lambda \geqslant 1, \\ &&& \lambda \geqslant 0. \end{aligned}$$

*If* $\lambda^* = 1$, then $X$ is the circuit corresponding to $x^*$. If $\lambda^* = 0$, then $Y$ is the cocircuit corresponding to $y^* = p^* A$.

Similarly, if $M$ is coordinatized by a rational matrix $A$, then for a given partition $E = R \cup G \cup B \cup W$ with $a \in R \cup G$, we can obtain either $X \in \mathcal{O}$ such that $a \in \underline{X} \subseteq E \backslash W$ and $X^+ \cap G = X^- \cap R = \emptyset$ or $Y \in \mathcal{O}^\perp$ such that $a \in \underline{Y} \subseteq E \backslash B$ and $Y^+ \cap G = Y^- \cap R = \emptyset$ from corresponding basic optimal solutions to a primal–dual pair of linear programming problems. As a consequence, our constructive proofs based on the alternatives in

Minty's painting lemma yield polynomial algorithms when $M$ is coordinatized by a rational matrix.

## 4. Sign reversal properties of max-balanced flows

For an oriented matroid $M=(E,\mathcal{O})$ and a subset $F\subseteq E$, the matroid obtained from $M$ by reversing the signs on $F$, which we denote by $rev(M,F)$, is defined as follows: For each circuit $X\in\mathcal{O}$, $rev(M,F)$ contains the circuit $Z$ where $Z^+=(X^+\backslash F)\cup(X^-\cap F)$ and $Z^-=(X^-\backslash F)\cup(X^+\cap F)$. Note that the underlying sets $\underline{X}$ and $\underline{Z}$ coincide. It is straightforward to show that $rev(M,F)$ is an oriented matroid and that $(rev(M,F))^{\perp}=rev(M^{\perp},F)$. We will use $\mathcal{O}_F$ and $\mathcal{O}_F^{\perp}$ to denote the circuit and cocircuit signatures of $rev(M,F)$. We will also call a diagonal matrix $\Lambda$ a $\pm 1$-*diagonal matrix* if each diagonal entry $\lambda_{ii}\in\{-1,+1\}$.

The following sign-reversal problem was shown by Itai [12] to be NP-Complete.

**Problem 1.** Given an undirected graph $G=(V,E)$ and vectors $l,u\in\mathbb{R}^E$ satisfying $l\leqslant u$, is there an orientation of $G$ which admits a circulation $x$ satisfying $l\leqslant x\leqslant u$?

In this section, we give good characterizations for analogous problems for max-balanced matroid flows, which yield polynomial algorithms when $M=(E,\mathcal{O})$ is coordinatized by a rational matrix $A$. We start by describing two fundamentally different approaches.

Let $D$ be a fixed orientation of the graph $G=(V,E)$, and let $A$ be the node-arc incidence matrix of $D$. Then Problem 1 asks for the existence of a $\pm 1$-diagonal matrix $\Lambda$ and a vector $x\in\mathbb{R}^E$ such that

$$A\Lambda x=0 \quad\text{and}\quad l\leqslant x\leqslant u. \tag{2}$$

By making the change of variables $y=\Lambda x$, it is easy to see that (2) has a solution if and only if the system

$$Ax=0 \quad\text{and}\quad l\leqslant\Lambda x\leqslant u \tag{3}$$

has a solution. In the context of max-balanced flows, the requirements analogous to (2) and (3) are not equivalent, and we show that each of the corresponding problems for max-balanced flows has a good characterization.

The following theorem generalizes the result of Robbins [14] that an undirected graph is 2-edge connected if and only if it has an orientation which is strongly connected.

**Theorem 5.** *Let $M=(E,\mathcal{O})$ be an oriented matroid, and let $l,u\in\mathbb{R}$ satisfy $l\leqslant u$. Then the following are equivalent:*

(i) *There exists a subset $F\subseteq E$ such that $rev(M,F)$ has a max-balanced flow $x$ satisfying $l\leqslant x\leqslant u$.*

(ii) *For every $Y\in\mathcal{O}^{\perp}$ and $f\in\underline{Y}$ there exists $e\in\underline{Y}\backslash f$ such that $l_f\leqslant u_e$.*

**Proof.** (i) $\Rightarrow$ (ii): This implication follows directly from part (iii) of Theorem 3 since if $rev(M,F)$ has a feasible max-balanced flow and $f\in \underline{Y}$ for some $Y\in\mathcal{O}_F^\perp$, there exists $X\in\mathcal{O}_F$ such that $f\in X^+=\underline{X}$ and $l_f\leqslant u_e$ for all $e\in\underline{X}$. Since orthogonality implies that $|\underline{X}\cap\underline{Y}|\neq 1$, there must be some $e\in(\underline{X}\cap\underline{Y})\setminus f$ which satisfies $l_f\leqslant u_e$.

(ii) $\Rightarrow$ (i): To prove this implication, we show that the following procedure terminates either with the set $F$, as required in (i), or with a cocircuit $Y$ violating (ii):

**(0)** Set $E'=E$ and $F=\emptyset$.
**(1)** If $E'=\emptyset$, return $F$ satisfying (i) and **STOP**.
**(2)** Select $f\in E'$ satisfying

$$l_f=\max_{e\in E'} l_e,$$

and let $X\in\mathcal{O}_F$ be such that $f\in X^+$, $X^-\subseteq E'$, and

$$u_e\geqslant l_f \quad \text{for all } e\in\underline{X}.$$

Set $F=F\cup X^-$, $E'=E'\setminus\underline{X}$, and return to **(1)**.
**(3)** If there is no such $X\in\mathcal{O}$, then **STOP**—(ii) is violated.

Consider the following condition:

For all $a\in E\setminus E'$, there exists $X_a\in\mathcal{O}_F$ with $a\in X_a^+=\underline{X}_a\subseteq E\setminus E'$ such that $u_e\geqslant l_a$ for all $e\in X_a^+$. Further, $u_e\geqslant l_a$ for all $e\in E\setminus E'$ and $a\in E'$.

We claim that, if this condition is satisfied at the beginning of an execution of **(2)** and the required circuit $X$ in **(2)** exists, then it is satisfied at the end of **(2)** with respect to the new values of $E'$ and $F$, which we denote by $\hat{E}'=E'\setminus\underline{X}$ and $\hat{F}=F\cup X^-$, where $X\in\mathcal{O}_F$ is the circuit selected in **(2)**. To see this, note that for $a\in E\setminus E'$, the circuit $X_a\in\mathcal{O}_F$ also satisfies $a\in X_a^+=\underline{X}_a\subseteq E\setminus\hat{E}'$ with respect to $\mathcal{O}_{\hat{F}}$ since $F\subseteq\hat{F}$ and $\hat{E}'\subset E'$. For $a\in E'$, it follows from the selection of $f$ and $X$ in **(2)** that

$$u_e\geqslant l_f\geqslant l_a \quad \text{for all } e\in\underline{X},$$

and therefore $u_e\geqslant l_a$ for all $e\in E\setminus\hat{E}'$ and $a\in\hat{E}'$ and the circuit formed from $X$ by reversing the signs on $X^-$ is the required $X_a$ in the condition.

As a consequence, if the procedure terminates in **(1)**, then the condition is precisely the characterization of part (iii) of Theorem 3 for the existence of a max-balanced flow $x$ for $rev(M,F)$ satisfying $l\leqslant x\leqslant u$.

If no such $X$ exists in **(2)**, then since $u_e\geqslant l_f$ for all $e\in E\setminus E'$ it follows from Minty's painting lemma with $G=\emptyset$ and

$$R=\{f\}\cup E\setminus E',\quad B=\{e\in E'\setminus f\,|\,u_e\geqslant l_f\},\quad W=\{e\in E'\,|\,u_e<l_f\}$$

that there exists $Y=\mathcal{O}_F^\perp$ such that $Y^-\subseteq W$ and $f\in Y^+\subseteq R\cup W$. We claim that $Y^+\cap R=\{f\}$. To see this, suppose that $a\in Y^+\cap R$, $a\neq f$. Then $a\in E\setminus E'$, and therefore there exists $X_a\subseteq E\setminus E'$ with $a\in X_a^+=\underline{X}_a$. Therefore $Y^+\cap X_a^+\neq\emptyset$ and it follows from orthogonality that $X_a^+ \cap Y^- \neq \emptyset$, which is a contradiction since $Y^- \subseteq W$. Thus $Y$ satisfies $u_e < l_f$ for all $e \in \underline{Y}$, $e \neq f$, and therefore $Y$ violates (ii). $\square$

The next theorem is an analogue of the sign reversal property described in (3).

**Theorem 6.** *Let $M = (E, \mathcal{O})$ be an oriented matroid, and let $l, u \in \mathbb{R}^E$ satisfy $-u \leqslant l \leqslant u$. Then the following are equivalent:*

(i) *There exists a $\pm 1$-diagonal matrix $\Lambda$ such that there is a max-balanced flow $x$ for $M$ with $l \leqslant \Lambda x \leqslant u$.*

(ii) *There is no subset $F \subseteq E$ and $f \in F$ such that*

  (a) *for each $a \in F \setminus f$ there exists $Y_a \in \mathcal{O}^\perp$ such that $a \in Y_a^+$ and for each $e \in Y_a^-$ either $u_e < l_a$ or $e \in F$ and $-l_e < l_a < l_e$;*

  (b) *there exists $Y_f \in \mathcal{O}^\perp$ with $f \in Y_f^+$, $-l_e < -u_f$ for all $e \in Y_f^- \cap F$, and $u_e < -u_f$ for all $e \in Y_f^- \setminus F$.*

**Proof.** (i) $\Rightarrow$ (ii): Suppose that for a $\pm 1$-diagonal matrix $\Lambda$, $x$ is a max-balanced flow for $M$ satisfying $l \leqslant \Lambda x \leqslant u$, and there is a subset $F \subseteq E$ and $f \in F$ violating (ii). First note that $-u \leqslant l \leqslant u$ and $l \leqslant \Lambda x \leqslant u$ implies that $-u \leqslant x \leqslant u$.

We claim that $\lambda_a = -1$ for all $a \in F \setminus f$. To see this, suppose that for some $a \in F \setminus f$ every $e \in F \setminus f$ with $l_e > l_a$ has $\lambda_e = -1$ but that $\lambda_a = +1$. Consider the cocircuit $Y_a$ and let $e \in Y_a^-$. Using part (ii) (a), if $-l_e < l_a < l_e$ and $e \in F$, then by assumption $\lambda_e = -1$ and $l_e \leqslant -x_e \leqslant u_e$, which implies that $x_e \leqslant -l_e < l_a$. Otherwise we must have $u_e < l_a$ and thus $x_e \leqslant u_e < l_a$. Therefore, since $\lambda_a = +1$ and $a \in Y_a^+$, it follows that

$$\max_{e \in Y_a^+} x_e \geqslant x_a \geqslant l_a > \max_{e \in Y_a^-} x_e,$$

which contradicts the assumption that $x$ is max-balanced.

Now consider $Y_f$. Since $\lambda_e = -1$ for all $e \in F \setminus f$, it follows from part (ii) (b) that $x_e \leqslant -l_e < -u_f$ for $e \in Y_f^- \cap F$ and that $x_e \leqslant u_e < -u_f$ for $e \in Y_f^- \setminus F$. Therefore $f \in Y_f^+$ implies that

$$\max_{e \in Y_f^+} x_e \geqslant x_f \geqslant -u_f > \max_{e \in Y_f^-} x_e,$$

again contradicting the assumption that $x$ is max-balanced.

(ii) $\Rightarrow$ (i): We show that the following procedure terminates either with a $\pm 1$-diagonal matrix $\Lambda$, as required in (i), or with a subset $F \subseteq E$ and $f \in F$ violating (ii):

**(0)** Set $E' = E$ and $F = \emptyset$, $l' = l$, $u' = u$, and $\Lambda = I$.

**(1)** If $E' = \emptyset$, return $\Lambda$ and **STOP**.

**(2)** Select $f \in E'$ satisfying

$$l'_f = \max_{e \in E'} l'_e,$$

and let $X \in \mathcal{O}$ be such that $f \in X^+ = \underline{X}$ and

$$l'_f \leqslant u'_e \quad \text{for all } e \in \underline{X}.$$

If there is no such $X \in \mathcal{O}$ and $f \in F$, then return $F$ and $f$, and **STOP**.

**(3)** If there is no such $X \in \mathcal{O}$ and $f \notin F$, then set $F = F \cup \{f\}$, $l'_f = -u_f$, $u'_f = -l_f$, $\lambda_f = -1$ and return to **(2)**. Otherwise, set $E' = E' \setminus \underline{X}$, and return to **(1)**.

This procedure must terminate, since each execution of **(3)** either increases $|F|$ or decreases $|E'|$. Consider the following condition:

For all $a \in E \setminus E'$ there exists $X_a \in \mathcal{O}$ with $a \in X_a^+ = \underline{X}_a \subseteq E \setminus E'$ such that $l'_a \leqslant u'_e$ for all $e \in \underline{X}_a$, and for each $a \in F$ there exists $Y_a \in \mathcal{O}^\perp$ such that $a \in Y_a^+$, and for each $e \in Y_a^-$ either $e \in F$ and $-l_e < l_a < l_e$, or $u_e < l_a$. Further, $l_a \geqslant l'_e$ for all $a \in F$ and $e \in E'$.

We claim that, if this condition is satisfied at the beginning of an execution of **(2)** and the procedure does not terminate in **(2)**, then it is satisfied at the end of **(3)** with respect to the new values of $E'$ and $F$, which we denote by $\hat{E}'$ and $\hat{F}$.

First, suppose that no such $X \in \mathcal{O}$ exists in **(2)** and $f \notin F$. If there exists no $Y_f \in \mathcal{O}^\perp$ such that $f \in Y_f^+$, and for each $e \in Y_f^-$ either $u_e < l_f$ or $e \in F$ and $-l_e < l_f < l_e$, then setting $G = B = \emptyset$,

$$W = \{e \in F \mid -l_e < l_f < l_e\} \cup \{e \in E \mid u_e < l_f\},$$

and $R = E \setminus W$, then it follows from Minty's painting lemma that there exists $X \in \mathcal{O}$ such that $f \in X^+ = \underline{X}$ and for all $e \in \underline{X}$, $u_e \geqslant l_f$ and either $e \notin F$ or $-l_e \geqslant l_f$ or $l_e \leqslant l_f$. There must be some $a \in \underline{X} \cap F$ with $l_a \leqslant l_f$, since otherwise $u'_e \geqslant l'_f$ for all $e \in \underline{X}$. Since $l_a \geqslant l'_f = l_f$ for all $a \in F$, it must be the case that $l_a = l_f$. Since $a \in X^+ \cap Y_a^+$, it follows from orthogonality that $X^+ \cap Y_a^- \neq \emptyset$, which is a contradiction since $X^+ \subseteq R$ and $Y_a^- \subseteq W$. Therefore since $l_f = l'_f \geqslant l'_e$ for all $e \in E'$, the condition remains satisfied for $\hat{F} = F \cup \{f\}$ and $\hat{E}' = E'$ at the end of (3).

Next, if $X \in \mathcal{O}$ in **(2)** exists, then since $u'_e \geqslant l'_f \geqslant l'_a$ for all $e \in \underline{X}$ and $a \in E' \cap \underline{X}$, taking $X_a = X$ for all $a \in E' \cap \underline{X}$, the condition holds for $\hat{F} = F$ and $\hat{E}' = E' \setminus \underline{X}$ at the end of **(3)**. As a consequence, we now argue that the algorithm terminates as stated.

If the algorithm terminates in **(1)**, then the condition is precisely the cycle cover characterization in part (iii) of Theorem 3 for the existence of a max-balanced flow $x$ for $M$ satisfying $l' \leqslant x \leqslant u'$, or equivalently $l \leqslant \Lambda x \leqslant u$. If in **(2)** there is no such $X \in \mathcal{O}$ and $f \in F$ and so the algorithm terminates, then by Minty's painting lemma there exists $Y \in \mathcal{O}^\perp$ with $f \in Y^+$ and $u'_e < l'_f$ for all $e \in Y^-$. It follows that $-l_e < -u_f$ for $e \in Y^- \cap F$ and that $u_e < -u_f$ for $e \in Y^- \setminus F$, and so part (ii) is violated. □

A similar result also holds when the sign reversal properties described in (2) and (3) are combined. We omit the proof, which is similar to that of Theorem 6.

**Theorem 7.** *Let $M=(E,\mathcal{O})$ be an oriented matroid, and let $l,u\in\mathbb{R}^E$ satisfy $-u\leqslant l\leqslant u$. Then the following are equivalent*:

(i) *There exists a subset $F\subseteq E$ and a $\pm 1$-diagonal matrix $A$ such that $rev(M,F)$ has a max-balanced flow $x$ satisfying $l\leqslant Ax\leqslant u$.*

(ii) *There is no subset $G\subseteq E$ and $g\in G$ such that*

(a) *for each $a\in G\backslash g$ there exists $Y_a\in\mathcal{O}^\perp$ such that $a\in \underline{Y}_a$ and for each $e\in \underline{Y}_a\backslash a$ either $u_e<l_a$ or $e\in G$ and $-l_e<l_a<l_e$;*

(b) *there exists $Y_g\in\mathcal{O}^\perp$ with $g\in \underline{Y}_g$, $-l_e<-u_g$ for all $e\in \underline{Y}_g\cap G\backslash g$, and $u_e<-u_g$ for all $e\in \underline{Y}_g\backslash G$.*

## 5. Structure of the set $mbf(l,u)$

In this section, we extend results from [10] on the structure of the set $mbf(l,u)$ in digraphs to oriented matroids. Throughout we will assume that $l$ and $u$ are such that $mbf(l,u)\neq\emptyset$. First of all, we note that if $x,y\in mbf(l,u)$ then $z\in mbf(l,u)$, where $z_e=\max\{x_e,y_e\}$ for $e\in E$. This implies that $mbf(l,u)$ is a semilattice under the usual coordinate partial order. From part (iii) of Theorem 3, we conclude that for an $a\in E$

$$\max_{x\in mbf(l,u)} x_a=\max_{a\in X^+=\underline{X}}\left\{\min_{e\in\underline{X}} u_e\right\}, \tag{4}$$

where the maximum on the right is taken over $X\in\mathcal{O}$. If in fact we can test feasibility of the set $mbf(l,u)$ in polynomial time, then we can use binary search over the values $u_e$ for $e\in E$ to find the largest value $l_a$ for which $mbf(l,u)$ is nonempty for each $a\in E$ and thus find the maximal element in $mbf(l,u)$ in polynomial time. A similar approach can be used to find a minimal element in $mbf(l,u)$, but the analogue of (4) is somewhat more complicated.

Following [10], we say that $b\in E$ is *forcing* for $a$ if $l_b\geqslant l_a$ and every $X\in\mathcal{O}$ with $b\in X^+=\underline{X}\subseteq E$ and $u_e\geqslant l_b$ for all $e\in\underline{X}$ has $a\in\underline{X}$. We will denote the set of elements which are forcing for $a$ by *force* $(a;l,u)$ and omit the dependence on $l$ and $u$ when the meaning is clear from the context. Note that by definition $a\in force(a)$. The significance of forcing elements is a result of the following lemma.

**Lemma 8.** *Let $M=(E,\mathcal{O})$ be an oriented matroid, and let $l,u\in\mathbb{R}^E$ be such that $mbf(l,u)$ is nonempty. Then the following are true*:

(i) *$b\in force(a)$ if and only if $b=a$ or $l_b\geqslant l_a$ and there exists $Y\in\mathcal{O}^\perp$ such that $b\in Y^+$ and $u_e<l_b$ for all $e\in Y^-\backslash a$.*

(ii) *if $c\in force(b)$ and $b\in force(a)$ then $c\in force(a)$.*

(iii) *For each $a\in E$,*

$$\min_{x\in mbf(l,u)} x_a=\max_{b\in force(a)} l_b.$$

(iv) *If $y$ is a minimal element in the set $mbf(l,u)$ then $y_b=l_b$ for each $b\in force(a;l,y)$ with $y_a=l_b$.*

**Proof.** (i) This follows from the definition of *force*$(a)$ and Minty's painting lemma with $R=\{e\in E\backslash a \mid u_e \geqslant l_b\}$, $W=E\backslash R$ and $B=G=\emptyset$.

(ii) Suppose that $b\neq a$ and $b\in force(a)$. From part (i), there exists $Y\in\mathcal{O}^{\perp}$ such that $b\in Y^+$ and $u_e<l_b$ for all $e\in Y^-\backslash a$. Since $c\in force(b)$, we also have $l_c\geqslant l_b$. If $c\notin force(a)$, then since $l_c\geqslant l_a$ there must exist $X\in\mathcal{O}$ such that $c\in X^+=\underline{X}\subseteq E\backslash a$ and $u_e\geqslant l_c$ for all $e\in\underline{X}$. But then since $l_c\geqslant l_b$, $X^+\cap Y^-=\emptyset$ so that by orthogonality $X^+\cap Y^+=\emptyset$. Therefore $\underline{X}\subseteq E\backslash b$, which contradicts the fact that $c\in force(b)$.

(iii) If $x\in mbf(l,u)$, then clearly $x_a\geqslant l_a$. Suppose that $b$ is forcing for $a$, and $b\neq a$. From part (i), there exists $Y\in\mathcal{O}^{\perp}$ such that $b\in Y^+$ and $u_e<l_b$ for all $e\in Y^-\backslash a$. If $x_a<l_b$, then $\max_{e\in Y^-} x_e<l_b\leqslant\max_{e\in Y^+} x_b$ so that $x$ is not max-balanced. Therefore it follows that

$$\min_{x\in mbf(l,u)} x_a\geqslant \max_{b\in force(a)} l_b.$$

Define $u'\in\mathbb{R}^E$ by $u'_e=u_e$ for $e\in E\backslash a$ and $u'_a=\max_{b\in force(a)} l_b$. If the set $mbf(l,u')$ is empty, then by part (ii) of Theorem 3 there exists $Y\in\mathcal{O}^{\perp}$ such that $\max_{e\in Y^+} l_e>\max_{e\in Y^-} u'_e$. Since $mbf(l,u)$ is nonempty, by part (ii) of Theorem 3, $\max_{e\in Y^+} l_e\leqslant\max_{e\in Y^-} u_e$, so we must have $a\in Y^-$. Let $b\in Y^+$ satisfy $l_b=\max_{e\in Y^+} l_e$. Since $a\in force(a)$, we have $l_b>u'_a\geqslant l_a$ and thus $b\in force(a)$ by part (i). By definition of $u'_a$, this implies that $u'_a\geqslant l_b$, a contradiction.

(iv) Suppose that $b\in force(a;l,y)$ has $y_b>l_b=y_a$. Since $y$ is minimal, $y_b=\min\{x_b\colon x\in mbf(l,\ y)\}$ and hence by part (iii), $y_b=l_c$ for some $c\in force(b;l,y)$. But then $c\in force(a;l,y)$ by part (ii), so that by part (iii) we have $y_a\geqslant l_c=y_b>l_b=y_a$, a contradiction. $\square$

Part (iii) of Lemma 8 shows that we can use binary search over the values $l_e$ for $e\in E$ to find the minimal value $u_a$ for which $mbf(l,u)$ is nonempty. This leads to the following algorithm for finding a minimal element.

### Minimal element algorithm

*Input*: An oriented matroid $M=(E,\mathcal{O})$, vectors $l,u\in\mathbb{R}^E$ such that $mbf(l,u)$ is nonempty, and a vector $z\in\mathbb{R}^E$.

*Output*: A minimal element in $mbf(l,u)$.

**(0)** Set $E'=\{e\in E \mid u_e>l_e\}$ and $u'_e=u_e$ for all $e\in E$.

**(1)** If $E'=\emptyset$, then **STOP** and return $u'$.

**(2)** Select $a\in E'$ satisfying

$$z_a=\min_{e\in E'} z_e,$$

set $E'=E'\backslash a$,

$$u'_a=\min_{b\in force(a;l,u')} l_b,$$

and return to **(1)**.

The following corollary of Lemma 8 shows that every minimal element is the output of the minimal element algorithm for some vector $z \in \mathbb{R}^E$ (see Corollary 13 for a stronger result).

**Corollary 9.** *If $y$ is a minimal element in $mbf(l,u)$, then $y$ is the output of the minimal element algorithm with $z=y$.*

**Proof.** Suppose that $a \in E'$ is selected in (**2**) and that $u'_e = y_e$ if $y_e < y_a$ and $y_e \leqslant u'_e \leqslant u_e$ if $y_e \geqslant y_a$. Since $u' \geqslant y$ and $y$ is a minimal element, by part (iii) of Lemma 8 we have

$$u'_a = \min_{x \in mbf(l,u')} x_a \leqslant \min_{x \in mbf(l,y)} x_a = y_a.$$

Now suppose that $u'_a < y_a$. By part (iv) of Lemma 8 there must be some $b \in force(a; l, y)$ with $y_a = l_b = y_b$. Since $l_a \leqslant u'_a < y_a \leqslant u_a$, we must have $b \neq a$ so by part (i) of Lemma 8 there exists $Y \in \mathcal{O}^\perp$ such that $b \in Y^+$ and $y_e < l_b = y_a$ for all $e \in Y^- \backslash a$. Thus $u'_e = y_e < l_b$ for all $e \in Y^- \backslash a$, so $b \in force\,(a; l, u')$ and hence $u'_a \geqslant l_b = y_a$ by part (iii) of Lemma 8. Since this is a contradiction, the result follows. $\square$

Next, we prove a necessary condition for a vector $y$ to be a minimal element in $mbf(l,u)$. This is an extension of Lemma 8 in [10], whose proof extends to regular oriented matroids, but not to general oriented matroids.

**Lemma 10.** *Let $M=(E,\mathcal{O})$ be an oriented matroid and let $l, u \in \mathbb{R}^E$ be such that $mbf(l,u)$ is nonempty. If $y$ is a minimal element in $mbf(l,u)$ and $C \in \mathcal{O}$, then there exists $c \in \underline{C}$ such that*

$$l_c = y_c = \min_{e \in \underline{C}} y_e.$$

**Proof.** Suppose not. Let

$$L = \left\{ a \in \underline{C}\colon\ y_a = \min_{e \in \underline{C}} y_e \right\}$$

and let $F$ be a *maximal* subset of $L$ such that for some $a \in F$, every $X \in \mathcal{O}$ with $a \in X^+ = \underline{X}$ and $y_e \geqslant y_a$ for all $e \in \underline{X}$ has $F \subseteq \underline{X}$. Clearly $F \neq \emptyset$, since we can take $F = \{a\}$ for any $a \in L$. We claim that for every $f \in \underline{C} \backslash F$, there exists $X_f \in \mathcal{O}$ such that $f \in X_f^+ = \underline{X}_f \subseteq E \backslash a$ and $y_e \geqslant y_a$ for all $e \in \underline{X}_f$. To see this note if $y_f > y_a$, then since $y$ is max-balanced, there exists $X \in \mathcal{O}$ such that $f \in X^+ = \underline{X}$ and $y_e \geqslant y_f$ for all $e \in \underline{X}$. Since $y_f > y_a$, we must have $a \notin \underline{X}$. On the other hand, if $f \in L \backslash F$ but every $X \in \mathcal{O}$ with $f \in X^+ = \underline{X}$ and $y_e \geqslant y_a$ for all $e \in \underline{X}$ has $a \in \underline{X}$, then every $X \in \mathcal{O}$ with $f \in X^+ = \underline{X}$ and $y_e \geqslant y_f$ for all $e \in \underline{X}$ has $F \subseteq \underline{X}$, so that $F$ is not maximal.

Let $b \in force\,(a; l, y)$ with $y_a = l_b = y_b$, as guaranteed by parts (iii) and (iv) of Lemma 8. This means that every $X \in \mathcal{O}$ with $b \in X^+ = \underline{X}$ and $y_e \geqslant l_b$ for all $e \in \underline{X}$ has $a \in \underline{X}$, and hence $F \subseteq \underline{X}$. Because $y_e > l_e$ for all $e \in L$, $b \neq a$ and further $b \notin \underline{C}$.

Now without loss of generality we may assume that $a \in C^-$ (if $a \in C^+$, then we can replace $C$ by $-C$). If there exists $f \in C^- \setminus F$, then since $f \in C^- \cap X_f^+$ and $a \in C^- \setminus X_f^+$, the signed elimination property implies that there exists $C_1 \in \mathcal{O}$ with $C_1^+ \subseteq (C^+ \cup X_f^+) \setminus f$, $C_1^- \subseteq (C^- \cup X_f^-) \setminus f = C^- \setminus f$, and $a \in \underline{C}_1$. We must have $a \in C_1^-$ since $a \notin C^+ \cup X_f^+$, but $b \notin C_1^-$ since $b \notin C^-$. Clearly $y_e \geqslant y_a$ for all $e \in \underline{C}_1$. Now if there exists $g \in C_1^- \setminus F$, we again apply the signed elimination property to $C_1$ and $X_g$ to obtain $C_2 \in \mathcal{O}$ with $a \in C_2^-$, $b \notin C_2^-$, $C_2^- \subseteq C_1^- \setminus g$, and $y_e \geqslant y_a$ for all $e \in \underline{C}_2$. Since $|C_i^- \setminus F|$ is decreasing, we can repeat this until we obtain $C_k \in \mathcal{O}$ with $a \in C_k^-$, $b \notin C_k^-$, $C_k^- \subseteq F$, and $y_e \geqslant y_a$ for all $e \in \underline{C}_k$.

Since $y$ is max-balanced, by Corollary 4 there exists $X \in \mathcal{O}$ such that $b \in X^+ = \underline{X}$ and $y_e \geqslant y_b$ for all $e \in \underline{X}$. Since $b \in force\,(a;\, l, y)$ we must have $a \in \underline{X}$ and hence $F \subseteq \underline{X}$.

Now by construction, $C_k^- \neq \emptyset$. So let $f \in C_k^- \subseteq X^+$. Since $f \in X^+ \cap C_k^-$ and $b \in X^+ \setminus C_k^-$, by the signed elimination property there exists $C_{k+1} \in \mathcal{O}$ with $C_{k+1}^+ \subseteq (C_k^+ \cup X^+) \setminus f$, $C_{k+1}^- \subseteq (C_k^- \cup X^-) \setminus f = C_k^- \setminus f$, and $b \in \underline{C}_{k+1}$. We must have $b \in C_{k+1}^+$ since $b \notin C_k^-$, but here $f \notin \underline{C}_{k+1}$. Clearly $y_e \geqslant y_a$ for all $e \in \underline{C}_{k+1}$. Now if there exists $g \in C_{k+1}^-$, we again apply the signed elimination property to $X$ and $C_{k+1}$ to obtain $C_{k+2} \in \mathcal{O}$ with $b \in C_{k+2}^+$, $g \notin \underline{C}_{k+2}$, and $y_e \geqslant y_a$ for all $e \in \underline{C}_2$. Since $|C_i^-|$ is decreasing, we can repeat this until we obtain $C_n \in \mathcal{O}$ with $b \in C_n^+ = \underline{C}_n$, $e \notin \underline{C}_n$ for some $e \in C_k^- \subseteq F$, and $y_e \geqslant y_a$ for all $e \in \underline{C}_n$.

This contradicts the fact that every $X \in \mathcal{O}$ with $b \in X^+ = \underline{X}$ and $y_e \geqslant l_b$ for all $e \in \underline{X}$ has $F \subseteq \underline{X}$. $\square$

The following theorem shows that if $y$ is a minimal element of $mbf\,(l, u)$, then the set $\{e \in E \mid y_a = l_a\}$ must be maximal.

**Theorem 11.** *Let $M = (E, \mathcal{O})$ be an oriented matroid, and let $l, u \in \mathbb{R}^E$ such that $mbf\,(l, u)$ is nonempty. If $y$ is a minimal element of $mbf\,(l, u)$, and $x \in mbf\,(l, u)$ with $x \neq y$, then there exists $c \in E$ such that $y_c = l_c < x_c$ and $y_c < y_e$ for all $e \in E$ such that $x_e < y_e$.*

**Proof.** First, suppose that $x_e < y_e$ for some $e \in E$. Let $y_a = \min\{y_e \mid x_e < y_e\}$ and $x_a < y_a$. It follows from part (iii) of Lemma 8 that

$$\max_{b \in force(a;\, l, x)} l_b \leqslant x_a < y_a = \max_{b \in force(a;\, l, y)} l_b.$$

Therefore there must be some $b \in force(a;\, l, y) \setminus force(a;\, l, x)$. Since $b \notin force(a;\, l, x)$, there exists $X \in \mathcal{O}$ with $b \in X^+ = \underline{X} \subseteq E \setminus a$ with $x_e \geqslant l_b$ for all $e \in \underline{X}$. Since $b \in force(a;\, l, y)$ we must have $y_e < l_b$ for some $e \in \underline{X}$. By Lemma 10 there is an element $c \in \underline{X}$ such that $l_c = y_c \leqslant y_e < l_b \leqslant x_c$. This satisfies the requirements of the theorem, since $y_e \geqslant y_a \geqslant l_b > y_c$ for all $e \in E$ such that $x_e < y_e$.

Next, suppose that $y \leqslant x$ and that $y_a < x_a$ for some $a \in E$. By Corollary 4, there exists $X \in \mathcal{O}$ with $a \in X^+ = \underline{X}$ and $x_e \geqslant x_a$ for all $a \in \underline{X}$. By Lemma 10 there exists $c \in \underline{X}$ such that $l_c = y_c \leqslant y_e$ for all $e \in \underline{X}$. Since $a \in \underline{X}$, it follows that $l_c = y_c \leqslant y_a < x_a \leqslant x_c$. $\square$

The following results are immediate consequences of Theorem 11.

**Corollary 12.** *Let $M=(E,\mathcal{O})$ be an oriented matroid, and let $l,u\in\mathbb{R}^E$ such that $mbf(l,u)$ is nonempty. If $x$ and $y$ are minimal elements of $mbf(l,u)$ and $x_e=y_e$ whenever $y_e=l_e$, then $x=y$.*

**Corollary 13.** *Let $M=(E,\mathcal{O})$ be an oriented matroid, and let $l,u\in\mathbb{R}^E$ such that $mbf(l,u)$ is nonempty. If $z\in\mathbb{R}^E$ satisfies $z_a<z_e$ whenever $l_a=y_a<y_e$ and $y_e>l_e$, then $y$ is the output of the minimal element algorithm.*

In [10] it was shown that the problem of minimizing a nonnegative linear function $c^{\mathrm{T}}x$ over $x\in mbf(l,u)$ is NP-hard, even for the special case of 0–1 vectors $c,l,u$ and graphic oriented matroids. However, we can use Lemma 10 to obtain an a priori upper bound for this problem. First of all, note that we may assume that the optimal solution $x^*$ is a minimal element in $mbf(l,u)$. Then by Lemma 10, the set of $e\in E$ such that $x_e^*>l_e$ must be an independent set in the matroid underlying $M$. Therefore if $I$ is an independent set in this matroid which maximizes $\sum_{e\in I}c_e(u_e-l_e)$ then $\sum_{e\in I}c_eu_e+\sum_{e\in E\setminus I}c_el_e$ is an upper bound. In fact, by part (iii) of Lemma 8 we may replace $l_a$ by $l'_a=\max_{b\in force(a)}l_b$ and $u_a$ by $u'_a=\max_{a,b\in X^+=\underline{X}}l_b$ for all $a\in E$, although there appears to be no good characterization for the latter value.

## 6. Max-balancing in coordinatized oriented matroids

In this section we present an algorithm for the following problem.

**Problem 2** (*Oriented matroid max-balancing*). Given an oriented matroid $M=(E,\mathcal{O})$ coordinatized by the matrix $A$ and a weight vector $\omega\in\mathbb{R}^E$, find a potential $p$ such that $\omega'=\omega+pA$ is max-balanced, or conclude that no such $p$ exists.

First, we show that the resulting max-balanced flow $\omega'$ is unique.

**Theorem 14.** *Let $M=(E,\mathcal{O})$ be the oriented matroid coordinatized by the matrix $A$ and let $\omega\in\mathbb{R}^E$. If $p$ and $q$ are vectors such that $\omega^p=\omega+pA$ and $\omega^q=\omega+qA$ are max-balanced, then $\omega^p=\omega^q$.*

**Proof.** Suppose that $\omega^p\neq\omega^q$ and without loss of generality that $\omega_a^p>\omega_a^q$. If there exists $X\in\mathcal{O}$ with $a\in X^+$ such that $\omega_e^p>\omega_e^q$ for all $e\in X^+$ and $\omega_e^p\leqslant\omega_e^q$ for all $e\in X^-$, then letting $x$ be the elementary vector corresponding to $X$ we have $(\omega^p)^{\mathrm{T}}x>(\omega^q)^{\mathrm{T}}x$. Since $Ax=0$ this cannot be the case, so by Minty's painting lemma with $R=\{e\in E\mid\omega_e^p>\omega_e^q\}$, $G=\{e\in E\mid\omega_e^p\leqslant\omega_e^q\}$ and $B=W=\emptyset$ there exists $Y\in\mathcal{O}^\perp$ with $a\in Y^+$ such that $\omega_e^p>\omega_e^q$ for all $e\in Y^+$ and $\omega_e^p\leqslant\omega_e^q$ for all $e\in Y^-$. Therefore

$$\max_{e\in Y^+}\omega_e^q<\max_{e\in Y^+}\omega_e^p=\max_{e\in Y^-}\omega_e^p\leqslant\max_{e\in Y^-}\omega_e^q,$$

contradicting the fact that $\omega^q$ is max-balanced. $\square$

Consider the following algorithm.

**Matroid balancing algorithm**

*Input*: A matrix $A$ and a vector $\omega \in \mathbb{R}^E$.

*Output*: A potential $p$ such that $\omega' = \omega + pA$ is a max-balanced flow for the oriented matroid coordinatized by $A$ or a potential $q$ such that $qA \leqslant 0$ and $qA \neq 0$.

**(0)** Set $E' = E$, $\omega' = \omega$ and $p = 0$.

**(1)** If $E' = \emptyset$, then **STOP** and output $p$; $\omega'$ is max-balanced.

**(2)** Solve the primal linear program

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E'} \omega'_e x_e \\ \text{subject to} \quad & Ax = 0, \\ & \sum_{e \in E'} x_e = 1, \\ & x \geqslant 0, \end{aligned}$$

whose dual is

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{subject to} \quad & (pA)_e \geqslant 0 \ \text{ for } e \in E \setminus E', \\ & (pA)_e + \lambda \geqslant \omega'_e \ \text{ for } e \in E'. \end{aligned}$$

**(3)** If the primal problem is infeasible, **STOP** and output the phase-one dual solution $q^*$; $M$ has no max-balanced flow.

**(4)** Let $x^*$ and $(\lambda^*, p^*)$ be optimal solutions to the primal and dual problems, respectively. Set

$$E' = E' \setminus \{e \in E \mid x^*_e > 0\}, \quad \omega' = \omega' - p^* A, \quad p = p - p^*$$

and return to **(1)**.

Note, that $|E'|$ strictly decreases each time $E'$ is updated in **(4)**, since the constraint $\sum_{e \in E'} x_e = 1$ implies that $\{e \in E' \mid x_e > 0\}$ is nonempty.

**Theorem 15.** *If $M = (E, \mathcal{O})$ is the oriented matroid coordinatized by $A$ and $\omega \in \mathbb{R}^E$, then either the matroid balancing algorithm terminates in* **(1)** *with a vector $p$ such that $\omega' = \omega + pA$ is max-balanced, or the matroid balancing algorithm terminates in* **(2)** *with a vector $q$ such that $qA \leqslant 0$ and $qA \neq 0$, and $M$ has no max-balanced flow.*

**Proof.** If there is a max-balanced flow for $M$, then by Corollary 2 the primal linear programming problem in **(2)** is feasible for any $E' \neq \emptyset$. If the primal linear programming problem is ever infeasible, then the optimal value of the phase-one problem

$$\begin{aligned} &\text{minimize} && z \\ &\text{subject to} && Ax=0, \\ &&& \sum_{e\in E'} x_e+z=1, \\ &&& x\geqslant 0,\ z\geqslant 0 \end{aligned}$$

and its dual

$$\begin{aligned} &\text{maximize} && w \\ &\text{subject to} && (qA)_e\leqslant 0 \ \text{ for } e\in E\setminus E', \\ &&& (qA)_e+w\leqslant 0 \ \text{ for } e\in E', \\ &&& w\leqslant 1 \end{aligned}$$

are both 1, so the optimal dual solution $(q^*, w^*)$ has $q^*A\leqslant 0$ and $q^*A\neq 0$. By the harmonious decomposition lemma, there would then exist $Y\in\mathcal{O}^{\perp}$ such that $Y^-=\underline{Y}$, and so $M$ has no max-balanced flow.

Now consider the following condition:

For each $a\in E\setminus E'$ there exists $X\in\mathcal{O}$ with $a\in X^+=\underline{X}\subseteq E\setminus E'$ such that $\omega'_e\geqslant\omega'_a$ for all $e\in\underline{X}$. Further, $\omega'_e\leqslant\omega'_a$ for all $e\in E'$ and $a\in E\setminus E'$.

We claim that, if this condition is satisfied at the beginning of an execution of **(2)** and the primal linear programming problem is feasible, then it is satisfied at the end of **(4)** with respect to the new values of $E'$ and $\omega'$, which we denote by $\hat{E}'$ and $\hat{\omega}'$. First note that because of the correspondence between circuits of $M$ and elementary vectors there must be a vector $x$ such that $Ax=0$, $x_e=0$ for $e\in E'$ and $x_e>0$ for $e\in E\setminus E'$. Therefore since $(p^*A)_e\geqslant 0$ for $e\in E\setminus E'$ it follows that $(p^*A)_e=0$ for $e\in E\setminus E'$. Thus $\hat{\omega}'_e=\omega'_e$ for all $e\in E\setminus E'$. By complementary slackness, if $x^*_e>0$ and $e\in E'$ (that is, $e\in E'\setminus\hat{E}'$), then

$$\hat{\omega}'_e=\omega'_e-(p^*A)_e=\lambda^*.$$

If $x^*_e=0$ and $e\in E'$ (that is, $e\in\hat{E}'$), then

$$\hat{\omega}'_e=\omega'_e-(p^*A)_e\leqslant\lambda^*. \tag{5}$$

Now since $\lambda^*$ is the optimal value of the primal linear programming problem, for each $a\in E\setminus E'$ the condition implies that we have

$$\lambda^*=\sum_{e\in E'}\omega'_e x^*_e\leqslant\sum_{e\in E'}\omega'_a x^*_e=\omega'_a=\hat{\omega}'_a.$$

This shows that $\hat{\omega}'_e\leqslant\lambda^*\leqslant\hat{\omega}'_a$ for all $e\in\hat{E}'$ and $a\in E\setminus\hat{E}'$. It remains to show that for each $a\in E'\setminus\hat{E}'$ there exists $X\in\mathcal{O}$ with $a\in X^+=\underline{X}\subseteq E\setminus\hat{E}'$ such that $\hat{\omega}'_e\geqslant\lambda^*$ for all $e\in\underline{X}$, but this follows from the harmonious decomposition lemma applied to $x^*$.

Now if the matroid balancing algorithm stops in **(1)**, then the condition is precisely the characterization of Corollary 4 for $\omega'$ to be max-balanced. $\square$

In Appendix A of [16], Rothblum et al. present a related approach for graphic oriented matroids based on a closely related dual linear programming problem. Because they do not explicitly consider the primal problem, their approach requires computing an optimal dual solution for which the inequality in (5) is strict. Without the additional directed graph structure, their approach would necessarily be more time consuming for the case of an arbitrary rational matrix $A$.

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