Discrete Mathematics 35 (1981) 25–38
North-Holland Publishing Company

HYPERPLANE RECONSTRUCTION OF THE TUTTE POLYNOMIAL OF A GEOMETRIC LATTICE

Thomas H. BRYLAWSKI

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27514, USA

Received 21 October 1980

An explicit construction is given which produces all the proper flats and the Tutte polynomial of a geometric lattice (or, more generally, a matroid) when only the hyperplanes are known. A further construction explicitly calculates the polychromate (a generalization of the Tutte polynomial) for a graph from its vertex-deleted subgraphs.

1. Introduction

It was shown in [11] that geometric lattices of series-parallel networks satisfy no lattice identity not satisfied by the class of all lattices. Such is not the case for numerical invariants, since for all such connected (simple) lattices L, the invariant

$$\beta(L)=(-1)^{r(L)}\sum_{x\in L} r(x)\mu(0,x)$$

takes the value 1, and this, in fact, characterizes series-parallel lattices among all geometric lattices [2]. This latter invariant can be calculated from the Tutte polynomial of L, $t(L;x,y)$ (it is equal to $(\partial t/\partial x)(L;0,0)$). Other invariants (for appropriate geometric lattices) which can be evaluated from $t(L)$ have applications in the fields of graph colorings and orientations, coding theory, network flows, embeddings in projective and affine space, hyperplane dissection and separation in Euclidean space, zonotopes, percolation theory, designs, and packing. Surveys of these applications can be found, for example, in [1, 3, 7, 8, 10, 15].

The Tutte polynomial was introduced in [13] for graphs (it was later generalized by Crapo to arbitrary geometric lattices [9]), and Tutte later showed that for graphs one could reconstruct it from the deck of vertex-deleted induced subgraphs [14]. Tutte's proof, although adaptable to other invariants, is not easily implemented: it involves the calculation of a complicated and potentially infinite generating function.

The purpose of this paper is two-fold. First, it is shown that for an arbitrary geometric lattice L, when the deck of hyperplane isomorphism classes of L is given, the deck of isomorphism classes of all proper flats (with their multiplicities) can be computed. This in turn readily gives the Tutte polynomial of the lattice. (This generalizes vertex reconstruction of graphs, since for three-connected graphs, if L is the geometric lattice of G, the geometric lattices of the vertex-deleted subgraphs of G are precisely the simple hyperplane intervals of L [5].)

Secondly, the multichromatic polynomial of a graph G (an invariant introduced in [6]) is shown to be explicitly reconstructible from the multichromatic polynomials of the vertex-deleted subgraphs of G, and this in turn leads to an easy reconstruction of $t(G)$. The paper concludes with some examples and counterexamples.

2. Matroid lattices and invariants

A (finite) *geometric lattice* L is a semimodular point lattice (i.e. every lattice element is a supremum of atoms, and there is a rank function r such that for all x, $y \in L$,

$$r(x)+r(y) \geq r(x \vee y)+r(x \wedge y)).$$

For any such lattice the upper interval $[x, \hat{1}]$ is also a geometric lattice, as is the lower interval $[\hat{0}, x]$. A *matroid lattice* M is a geometric lattice with an integer weight assigned to its zero, $\hat{0}$, and to each atom $a \in A$, where $w(\hat{0}) \geq 0$ is the number of *loops* of M, and $w(a) \geq 1$ is the *multiplicity* of the atom a. We then say the atom a consists of $w(a)$ *points*, and the $m=|M|$ points of M are partitioned by the atoms (and perhaps $\hat{0}$) (Geometric lattices generalize the notion of a subset of projective space, while matroid lattices generalize subsets of vectors in a vector space allowing the zero vector and repetitions.)

General lattice elements are called *flats*, and for each flat x, its cardinality is given by

$$|x| = w(\hat{0}) + \sum_{a \leq x} w(a).$$

The case $w(\hat{0})>0$ is easily reduced to the loopless case, so we will usually assume in the following that M is loopless. The geometric lattice associated with M is denoted $\bar{M}$, and for each such lattice ($w(\hat{0})=0$ and) $w(a)=1$ for all atoms a. Thus, $|\bar{x}|$ will be the number of atoms contained in (i.e., less than or equal to) the flat x, and we will identify x with its matroid lattice $[\hat{0}, x]$. Flats x such that $r(x)=r(\hat{1})-1$ are termed *hyperplanes.*

For any point $p \in M$, the *deletion* $M-p$ is a matroid lattice where if p is a loop or part of a multiple atom a, $\overline{M-p} \simeq \bar{M}$ with $w(\hat{0})$ or $w(a)$ respectively reduced by one. If p is an atom, $\overline{M-p}$ is isomorphic to the supremum subsemilattice generated by the atoms $A-\{p\}$ with weights the same as in M. A point p is an *isthmus* if $r(M-p)<r(M)$.

The *contraction* M/p is isomorphic to $M-p$ if $p \in \hat{0}$. Otherwise, if $p \in a$, $\overline{M/p}$ is isomorphic to the interval $[a, \hat{1}]$ with all weights (cardinalities) in the interval reduced by one.

The *Tutte polynomial* $t(M; x, y)$ is defined recursively by:

$t(M) = y$ if M consists of a single loop,

$t(M) = x$ if M is the geometric lattice consisting of a single (atomic) point,

$t(M) = t(p)t(M-p)$ if p is a loop or isthmus,

and

$t(M) = t(M-p) + t(M/p)$ otherwise.

For a geometric lattice L, the *characteristic polynomial* of L is given by

$$\chi(L; \lambda) = \sum_{x \in L} \mu(\hat{0}, x)\lambda^{r(\hat{1})-r(x)} \tag{1}$$

where μ is the Möbius function of L. The *coboundary polynomial* of M is defined as

$$\begin{aligned} \bar{\chi}(M; z, \lambda) &= \sum_{x \in M} z^{|x|}\chi([x, \hat{1}]; \lambda) \\ &= \sum_{\hat{0} \leq x \leq y \leq \hat{1}} \mu(x, y) z^{|x|} \lambda^{r(\hat{1})-r(y)}. \end{aligned} \tag{2}$$

For a matroid M, if P is its set of points and $P' \subseteq P$, define

$$r(P') = r\left(\bigvee \{a : p \in a \text{ for some } p \in P'\}\right).$$

The *rank generating function* is then given by

$$S(M; u, v) = \sum_{P' \subseteq P} u^{|P'|} v^{r(P')}. \tag{3}$$

For any flat F of rank k, let

$$S^k(F; u) = \sum_i a_i u^i$$

where a_i is the number of (spanning) subsets of F of rank k. Therefore:

$$S(M; u, v) = \sum_{k=0}^{r} v^k \sum_{F : r(F)=k} S^k(F; u). \tag{4}$$

The three two-variable polynomial invariants defined above are all equivalent in the sense that any can be derived from any other [3, 8, 9]:

$$S(M; u, v) = (uv)^{r(\hat{1})} t\left(M; \frac{uv+1}{uv}, u+1\right), \tag{5}$$

$$\bar{\chi}(M; z, \lambda) = \lambda^{r(\hat{1})} S(M; z-1, \lambda^{-1}). \tag{6}$$

The reason for mentioning all three invariants is that the coboundary polynomial, being a sum over flats, is more lattice-theoretic, while the rank generating function is a sum over subsets of points and so is more in the spirit of matroids.

The Tutte polynomial on the other hand is generally easier to compute being defined recursively and is useful in applications (there being many classical invariants which obey the same recursion and which are therefore evaluations of $t(M)$).

3. Hyperplane reconstruction

In this section, M is a fixed matroid lattice of rank r with n atoms. For any $k \geq 0$, let us index the isomorphism classes of matroids of rank k by the indexing set $A^k = \{\alpha^k\}$. Thus F_{α^3} stands for a rank-three matroid isomorphism class, and it has $|F_{\alpha^3}|$ atoms. Let $n(F_{\alpha^k})$ denote the number of flats x of M isomorphic to F_{α^k}. More generally, call the sequence of isomorphism classes $(F_{\alpha^k}, F_{\alpha^{k+1}}, \ldots, F_{\alpha^m})$ the *flag* $\mathcal{F}_{\alpha^k,\ldots,\alpha^m}$ if F_{α^i} is (isomorphic to) a hyperplane of $F_{\alpha^{i+1}}$ for $i = k, k+1, \ldots, m-1$. Further, for any such sequence, let $n(\mathcal{F}_{\alpha^k,\ldots,\alpha^m})$ be the number of sequences of flats $(x_k, \ldots, x_m)$ in F_{α^m} with x_i isomorphic to F_{α^i} for all i. Thus, $n(\mathcal{F}_{\alpha^k,\ldots,\alpha^m}) = n(\mathcal{F}_{\alpha^k,\alpha^{k+1}}) \cdot n(\mathcal{F}_{\alpha^{k+1},\alpha^{k+2}}) \cdots n(\mathcal{F}_{\alpha^{m-1},\alpha^m})$.

Lemma 3.1. *Let M be a matroid with n atoms, and let F_{α^k} be an isomorphism class. Then:*

$$n(F_{\alpha^k})(n - |F_{\alpha^k}|) = \sum_{\alpha^{k+1} \in A^{k+1}} n(\mathcal{F}_{\alpha^k,\alpha^{k+1}}) n(F_{\alpha^{k+1}})(|F_{\alpha^{k+1}}| - |F_{\alpha^k}|) \tag{7}$$

where the sum is over all isomorphism classes of matroids of rank $k+1$ (and in fact need only be over the finite set of classes of rank-$(k+1)$ flats of M).

Proof. A geometric lattice is characterized by the fact that for any flat x and any atom a, $a \not\leq x$, $a \vee x$ covers x, and that, conversely, any cover of x is the supremum of x and some atom. Thus, both sides of (7) count the pairs (a, x_k) where $x_k \simeq F_{\alpha^k}$ and $a \not\leq x_k$. The left-hand side sums first over such flats x_k and the right-hand side sums over all flats x_{k+1} which have a hyperplane isomorphic to F_{α^k}.

Proposition 3.2. *For a matroid lattice M of rank r with $|\bar{M}| = n$ atoms,*

$$n(F_{\alpha^k}) = \sum_{\alpha^{r-1} \in A^{r-1}} n(F_{\alpha^{r-1}})$$
$$\times \sum_{\alpha^{k+1},\ldots,\alpha^{r-2}} \frac{n(\mathcal{F}_{\alpha^k,\ldots,\alpha^{r-1}})(|F_{\alpha^{k+1}}| - |F_{\alpha^k}|)(|F_{\alpha^{k+2}}| - |F_{\alpha^{k+1}}|) \cdots (|F_{\alpha^{r-1}}| - |F_{\alpha^{r-2}}|)}{(n - |F_{\alpha^k}|)(n - |F_{\alpha^{k+1}}|) \cdots (n - |F_{\alpha^{r-2}}|)}. \tag{8}$$

Proof. We use induction on $r-k$, the corank of F_{α^k}. If $r-k=1$, all products are empty while $n(\mathcal{F}_{\alpha^{r-1}}) = 1$, so that both sides of (8) give the number of hyperplanes of M which are isomorphic to $F_{\alpha^{r-1}}$. Now assume we have proved the proposition for isomorphism classes of corank $r-k-1$ and let F_{α^k} be some isomorphism class. Then by (7) we get:

$$n(F_{\alpha^k}) = \sum_{\alpha^{k+1} \in A^{k+1}} \frac{n(\mathscr{F}_{\alpha^k,\alpha^{k+1}})n(F_{\alpha^{k+1}})(|\overline{F_{\alpha^{k+1}}}| - |\overline{F_{\alpha^k}}|)}{(n - |\overline{F_{\alpha^k}}|)}. \tag{9}$$

Using (8) to calculate $n(F_{\alpha^{k+1}})$ we obtain the correct formula for $n(F_{\alpha^k})$ after noting that $n(\mathscr{F}_{\alpha^k,\alpha^{k+1}}) \cdot n(\mathscr{F}_{\alpha^{k+1},\ldots,\alpha^{r-1}}) = n(\mathscr{F}_{\alpha^k,\ldots,\alpha^{r-1}})$.

Since $n(\mathscr{F}_{\alpha^k,\ldots,\alpha^{r-1}})$ can be calculated within $F_{\alpha^{r-1}}$ as can $|\overline{F_{\alpha^i}}|$ and $|\overline{F_{\alpha^i}}| - |\overline{F_{\alpha^{i-1}}}|$ for all $i < r$, if $n(F_{\alpha^{r-1}})$ is known for all $\alpha^{r-1} \in A^{r-1}$ the right-hand side of (8) gives $n(F_{\alpha^k})$ for all indices α^k $(k \leq r-1)$ as a function of n, the number of atoms of M. But n is reconstructible as we now prove.

Theorem 3.3. *The number of flats x of M isomorphic to* F_{α^k} *is given for all* $k \leq r-1$ *and* $\alpha^k \in A^k$ *by* (8) *where n is the unique integer (greater than the size of any hyperplane) which satisfies the equation*:

$$1 = \sum_{\alpha^{r-1} \in A^{r-1}} n(F_{\alpha^{r-1}})$$

$$\times \sum_{\alpha^1,\ldots,\alpha^{r-2}} \frac{n(\mathscr{F}_{\alpha^0,\ldots,\alpha^{r-1}})(|\overline{F_{\alpha^2}}| - |\overline{F_{\alpha^1}}|) \cdots (|\overline{F_{\alpha^{r-1}}}| - |\overline{F_{\alpha^{r-2}}}|)}{n(n-1)(n - |\overline{F_{\alpha^2}}|) \cdots (n - |\overline{F_{\alpha^{r-2}}}|)} \tag{10}$$

where α^0 *indexes the matroid isomorphic to the number of loops in any hyperplane of M (so that* $(n - |\overline{F_{\alpha^0}}|) = n$ *and* $(|\overline{F_{\alpha^1}}| - |\overline{F_{\alpha^0}}|) = 1$).

Proof. By (8), the right hand side is the number of flats isomorphic to $\hat{0} \in M$. The solution is unique since for $n > \max_{\alpha^{r-1}} (|\overline{F_{\alpha^{r-1}}}|)$, the right-hand side decreases monotonically in n.

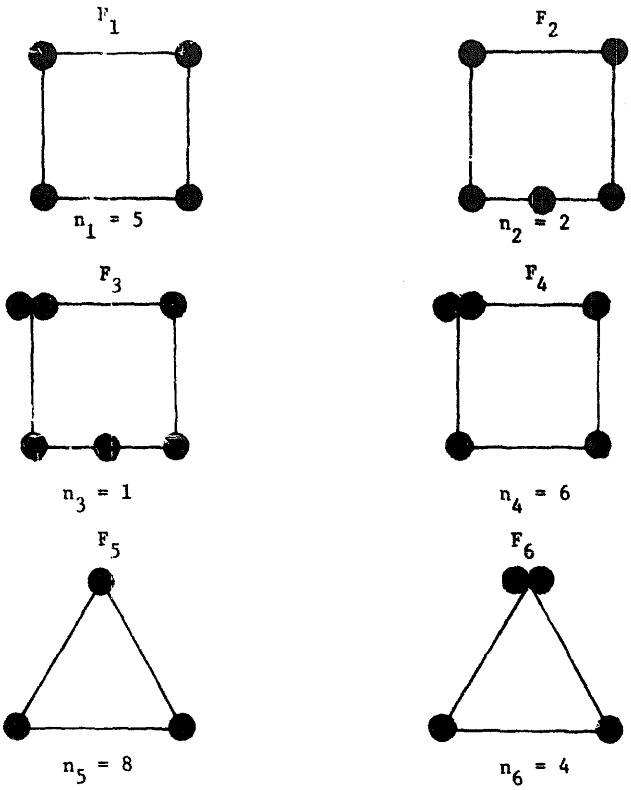
Corollary 3.4. *A deck of hyperplanes comes from a matroid M only if for all l,* $0 < l \leq r-1$,

$$\binom{n}{l} = \sum_{k=1}^{r-1} \sum_{\alpha^k \in A^k} n^*(F_{\alpha^k}) B_l(F_{\alpha^k}) \tag{10'}$$

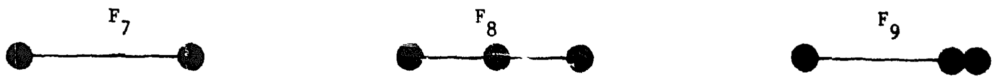
where n is the solution of (10), $n^*(F_{\alpha^k})$ *is given by the right-hand side of* (8), *and* $B_l(F)$ *is the number of l-element subsets of atoms whose supremum is F.*

Proof. Any l-element subset of atoms must span some flat of rank k $(0 < k \leq l)$. Thus (10′) partitions the l-subsets of atoms of M according to their supremum.

Example 3.5. Let the deck $\mathscr{H}$ of hyperplanes of a rank four matroid be as follows (where in the "affine picture," multiple points are represented by juxtaposed dots, $n_i = n(F_i)$ and $n_{ij} = n(\mathscr{F}_{i,j}) \cdot (|\overline{F_j}| - |\overline{F_i}|)$).



Then, isomorphism classes of lines are



and isomorphism classes of atoms are

(there are no loops).

Calculating the flag multiplicities we obtain:

$n_{7,1} = 6(4-2) = 12$

$n_{7,2} = 21$ $\qquad n_{8,2} = 2$

$n_{7,3} = 9$ $\qquad n_{8,3} = 2$ $\qquad n_{9,3} = 12$

$n_{7,4} = 6$ $\qquad n_{9,4} = 6$

$n_{7,5} = 3$

$n_{7,6} = 1$ $\qquad n_{9,6} = 2$

$n_{10,7} = 2$

$n_{10,8} = 6$

$n_{10,9} = 1$ $\qquad n_{11,9} = 1.$

Thus, by (8)

$$n_7=\frac{1}{n-2}(5\cdot 12+2\cdot 21+1\cdot 9+6\cdot 6+8\cdot 3+4\cdot 1)=\frac{175}{n-2}$$

$$n_8=\frac{6}{n-3},\qquad n_9=\frac{56}{n-2},$$

$$n_{10}=\frac{1}{n-1}\left(\frac{175}{n-2}\cdot 2+\frac{36}{n-3}\cdot 6+\frac{56}{n-2}\cdot 1\right),\qquad n_{11}=\frac{1}{n-1}\left(\frac{56}{n-2}\right)$$

Equation (10) becomes:

$$1=\frac{1}{n(n-1)}\left(\frac{350}{n-2}+\frac{6}{n-3}+\frac{56}{n-2}+\frac{56}{n-2}\right)$$

which for $n>5$ has the unique solution $n=9$.

Therefore, $n_7=25$, $n_8=1$, $n_9=8$, $n_{10}=8$, and $n_{11}=1$. (An example of such a matroid is AG(3, 2) with one double point and another point placed on one of the two-point lines.)

We now reconstruct the rank generating function (and therefore the Tutte polynomial or coboundary polynomial).

Theorem 3.6. *The rank generating function $S(M;u,v)$ can be reconstructed from the deck of proper flats (and therefore from the deck of hyperplanes) by the equation*:

$$S(M;u,v)=v^r(u+1)^m+\sum_{k<r}(v^k-v^r)\sum_{\alpha^k\in A^k}n(F_{\alpha^k})S^k(F_{\alpha^k};u) \tag{11}$$

where $r=r(M)=r(F_{\alpha^{r-1}})+1$, *and* $m=|M|=|F_{\alpha^0}|+\sum_{\alpha^1}n(F_{\alpha^1})\cdot|F_{\alpha^1}|$.

Proof. By (4),

$$S(M;u,v)=v^rS^r(M;u)+\sum_{k<r}v^k\sum_{\alpha^k}n(F_{\alpha^k})S^k(F_{\alpha^k};u). \tag{12}$$

Clearly, (11) and (12) agree except perhaps for terms involving v^r. But $S(M;u,1)=(u+1)^m$, and (11) also satisfies this identity. Thus both agree on the coefficient of u^iv^r for all i.

Example 3.7. For the matroid of Example 3.4, we have

$$\begin{aligned}S(M;u,v)=\;&v^4(u+1)^{10}+(v^3-v^4)\\&\times[5(u^4+4u^3)+2(u^5+5u^4+9u^3)\\&+(u^6+6u^5+15u^4+15u^3)\\&+6(u^5+5u^4+7u^3)+8u^3+4(u^4+2u^3)]\\&+(v^2-v^4)[25u^2+(u^3+2u^2)+8(u^3+2u^2)]\\&+(v-v^4)[8u+(u^2+2u)]+1-v^4.\end{aligned}$$

4. Reconstructing graphs

Let G be a graph with vertex set V, $|V|=r$. In [6] the *polychromatic polynomial* (or *polychromate*) of G, $c(G; y, z_1, \ldots, z_r)$ is defined:

$$c(G; y, z_1, \ldots, z_r) = \sum M_G(i, \pi) y^i \bar{z}^\pi \tag{13}$$

where π ranges over all partitions of the integer r, and if $\pi = 1^{a_1}2^{a_2}\cdots r^{a_r}$ (so that $\sum_i ja_i = r$), then $\bar{z}^\pi = z_1^{a_1}z_2^{a_2}\cdots z_r^{a_r}$, and $M_G(i, \pi)$ is the number of partitions of the vertex set of G of type π (that is with a_j blocks of j vertices each) such that there are precisely i edges of G each of which join two vertices within the same block. The polychromate $c(G)$ generalizes the Tutte polynomial in the sense that if $M(G)$ is the matroid lattice of G (where multiple points correspond to multiple edges), then

$$\bar{\chi}(M(G); z, \lambda) = c(G; z, w, w, \ldots, w)|_{w^i \to (\lambda-1)(\lambda-2)\cdots(\lambda-i+1)}. \tag{14}$$

The polychromate, however, gives more information about the graph G. For example, if G has no multiple edges or loops, then $c(G)$ allows one to compute $t(\bar{G})$ for the complementary graph $\bar{G}$ of G and also gives the size of a maximal matching.

We will assume in the following that G has no loops (for reconstruction, loops are easily handled). If $V' \subseteq V$, let $G(V-V')$ denote the induced subgraph of G obtained when the vertices V' (and all incident edges) are deleted. A classical result [12, 14] shows that (in analogy with Proposition 3.2) the deck of proper subgraphs $G(V-V')$ ($V' \subseteq V$, $V' \neq \emptyset$) can be obtained from the vertex deletions $\{G(V-v)\}$. Let $\{\beta^k\} = B^k$ index isomorphism classes of graphs with k vertices, and for any given isomorphism class let $n(G_{\beta^k})$ be the number of k-vertex subsets V' such that $G(V') \simeq G_{\beta^k}$. Thus, if $n(G_{\beta^{r-1}})$ is given for all indices $\beta^{r-1} \in B^{r-1}$, then $n(G_{\beta^k})$ is known for all $k<r$.

For the graph G, we now define the one-variable polynomial $p_k(G; y)$ by:

$$p_k(G) = \sum_{\substack{W \subseteq V \\ |W|=k}} y^{|G(W)|} = \sum_i M_G(i, \pi) y^i \tag{15}$$

where $|G(W)|$ is the number of edges in the subgraph induced by the vertex subset W, and π is the partition with $a_k = 1$ and $a_1 = |V|-k$. More generally, let

$$p_\pi(G) = \sum_i M_G(i, \pi) y^i \tag{16}$$

for a partition π of r.

Lemma 4.1. *Let G be a graph with r vertices. For all k, $p_k(G)$ is reconstructible:*

$$p_r(G) = y^m \quad \textit{where } m = \sum_{\beta^2 \in B^2} n(G_{\beta^2})\,|G_{\beta^2}|$$

and

$$p_k(G) = \sum_{\beta^k \in B^k} n(G_{\beta^k}) y^{|G_{\beta^k}|} \quad \textit{for all } k < r.$$

Proof. For $k < r$, the equation for $p_k(G)$ is immediate, while $p_r(G) = y^{|G|}$, and $|G|$ is the sum of the multiplicities of all edges.

We are now able to prove our principal result: that $c(G)$ is (explicitly) reconstructible. We do so by first showing how to get the polynomial contribution $\sum_i y^i z_1^{a_1} \cdots z_r^{a_r}$ for a fixed partition $\pi = 1^{a_1} \cdots r^{a_r}$. From Theorem 4.2, the formula for $c(G)$ in Corollary 4.3 is immediate, but we emphasize that it is Theorem 4.2 which provides the easiest calculation for the actual coefficients in $c(G)$.

Theorem 4.2. *Let G be a graph with r vertices and let $\pi = 1^{a_1} 2^{a_2} \cdots r^{a_r}$ be a partition of r. Then*

$$p_\pi(G) = \frac{1}{\prod_j a_j!} \sum_{k=0}^{r} (-1)^{r-k} \sum_{\beta^k \in B^k} n(G_{\beta^k}) \prod_j (p_j(G_{\beta^k}))^{a_j}. \tag{17}$$

Proof. Let π be represented by the sequence $(\pi_1, \ldots, \pi_r)$ where $\pi_i \geq \pi_{i+1} \geq 0$ and $|\{\pi_i : \pi_i = k\}| = a_k$. For any subgraph H of G with vertex set V_H, let $\mathcal{W}_\pi$ be the set of all sequences $(W_1, \ldots, W_r)$ of vertex subsets of H with $|W_i| = \pi_i$. The number $n_{m,\pi}(H)$ of such sequences such that $\sum_i |H(W_i)| = m$ is the coefficient of y^m in

$$\prod_i p_{\pi_i}(H) = \prod_j (p_j(H))^{a_j}.$$

Then the coefficient of y^m in the right-hand side of (17) is given by

$$\frac{1}{\prod_j a_j!} \sum_{V' \subseteq V} (-1)^{|V|-|V'|} n_{m,\pi}(G(V')). \tag{18}$$

For any subset $V' \subseteq V_H$, denote by $\tilde{n}_{m,\pi,V'}(H)$ the number of sequences in $\mathcal{W}_\pi$ such that $\sum_i |H(W_i)| = m$ and $\bigcup_i W_i = V'$. Then, clearly,

$$n_{m,\pi}(H) = \sum_{V' \subseteq V_H} \tilde{n}_{m,\pi,V'}(H). \tag{19}$$

Inverting (19) by inclusion-exclusion we get

$$\tilde{n}_{m,\pi,V_H}(H) = \sum_{V' \subseteq V_H} (-1)^{|V_H|-|V'|} n_{m,\pi}(H(V')). \tag{20}$$

But $\tilde{n}_{m,\pi,V_G}(G)$ is the coefficient of y^m in $\prod_j a_j!\, p_\pi(G)$, so combining (20) and (18), we get that both sides of (17) are equal to

$$\frac{1}{\prod_j a_j!}\sum_m \bar{n}_{m,\pi,V_G}(G)y^m.$$

When both sides of (17) are multiplied by $z_1^{a_1}\cdots z_r^{a_r}$ and then summed over all integer partitions of r, we obtain:

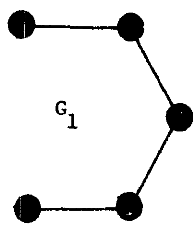
Corollary 4.3. *For any graph G, the polychromate is reconstructible by the formula:*

$$\begin{aligned} c(G; y, z_1, \ldots, z_r) &= \sum_{k=0}^{r}(-1)^{r-k}\sum_{\beta^k\in B^k} n(G_{\beta^k})\cdot\exp\Big(\sum_i z_ip_i(G_{\beta^k})\Big)\\ &= \sum_{k=0}^{r-1}(-1)^{r-k}\sum_{\beta^k} n(G_{\beta^k})\exp\Big(\sum_{i=0}^{r-1} z_ip_i(G_{\beta^k})\Big)\\ &\quad+\exp\Big(\sum_{i=0}^{r-1} z_i\Big(\sum_{\beta^i} n(G_{\beta^i})y^{|G_{\beta^i}|}\Big)+z_ry^{\sum n(G_{\beta^2})|G_{\beta^2}|}\Big) \qquad (21)\end{aligned}$$

where, as usual

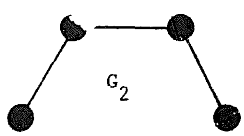
$$\exp\Big(\sum_{i=0}^{r-1} z_ip_i(H)\Big)=\Big(1+z_1p_1(H)+\frac{z_1^2p_1^2(H)}{2!}+\cdots\Big)\cdots\Big(1+z_{r-1}p_{r-1}(H)+\frac{z_{r-1}^2p_{r-1}^2(H)}{2!}+\cdots\Big).$$

Example 4.4. Let G be a six-edge circuit, and for $B^5=\{1\}$, $B^4=\{2,3,4\}$, $B^3=\{5,6,7\}$, and $B^2=\{8,9\}$, let the induced graphs G_i be pictured below along with their multiplicities and "edge-generating function" $p_2(G_i)$.



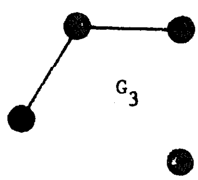
$n_1 = 6$

$P_2(G_1) = 4y + 6$



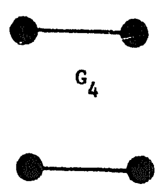
$n_2 = \frac{6\cdot 2}{2} = 6$

$P_2(G_2) = 3y + 3$



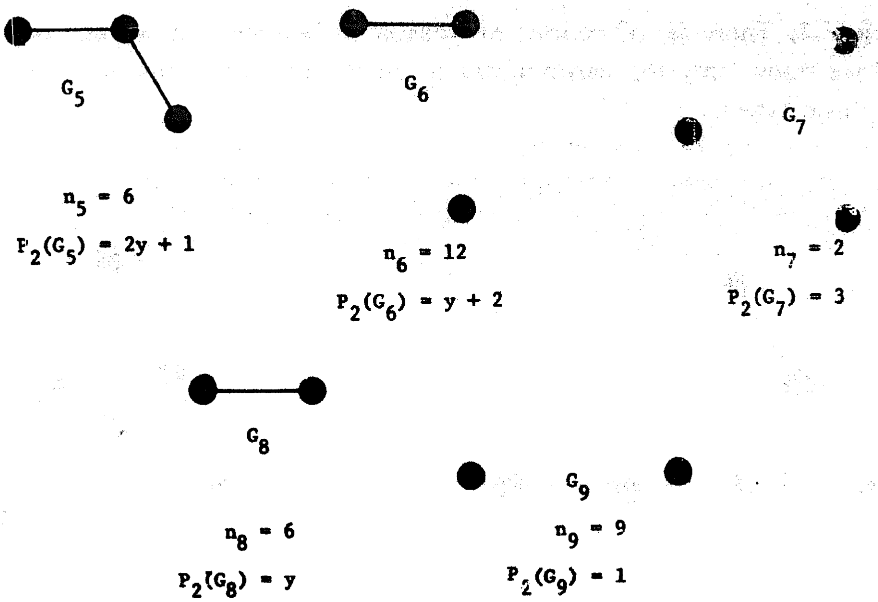
$n_3 = 6$

$P_2(G_3) = 2y + 4$



$n_4 = 3$

$P_2(G_4) = 2y + 4$



Then $p_2(G) = n_8 y + n_9 = 6y + 9$, and

$$\begin{aligned} p_{2^3}(G) &= \tfrac{1}{6}[(6y+9)^3 - 6(4y+6)^3 + 6(3y+3)^3 + 6(2y+4)^3 + 3(2y+4)^3 \\ &\quad - 6(2y+1)^3 - 12(y+2)^3 - 2(3)^3 + 6y^3 + 9] \\ &= 2y^3 + 3y^2 + 6y + 4 \end{aligned}$$

(the polynomial coefficient of z_2^3 in $c(G)$). Thus, for example, G has 2 perfect matchings, and its complement has 4.

5. Counterexamples

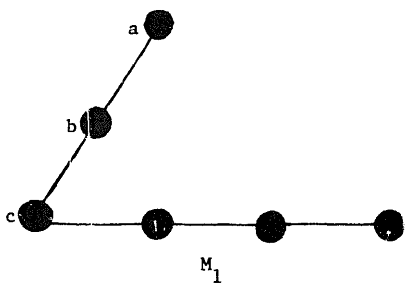
Generally, of course, the number of atoms of a lattice can not be determined from its deck of corank-one intervals. For example, within the class of order duals of geometric lattices, the following counterexample occurs.

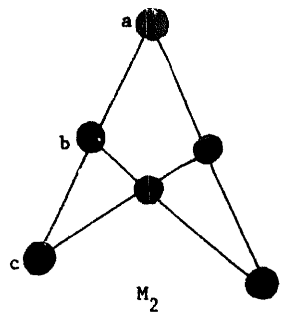
Example 5.1. Let L_1 be the geometric lattice of the affine geometry AG(2, 3) and let L_2 be the projective geometry PG(2, 3) with a quadrangle deleted. Then both L_1 and L_2 have nine atoms and every upper interval is isomorphic to the lattice of a four-point line (rank two, four atoms). However, L_1 has 12 hyperplanes and L_2 has 13 hyperplanes since in the former case a line is removed from PG(2, 3) and in the latter case, at most two points are deleted from any original line of PG(2, 3), and so each is spanned by atoms not in Q.

Results in [4] however show that for matroid lattices (i.e., when the atoms are assigned multiplicities in the contractions), all isomorphism classes of proper contractions and their multiplicities are reconstructible from single-point contractions.

Example 5.2. There is, of course, no reason to believe that we can reconstruct $t(G)$ if we know only the cardinalities of all the flats of L (as opposed to their isomorphism type).

For example, let M_1 and M_2 be the rank-three matroids pictured below:

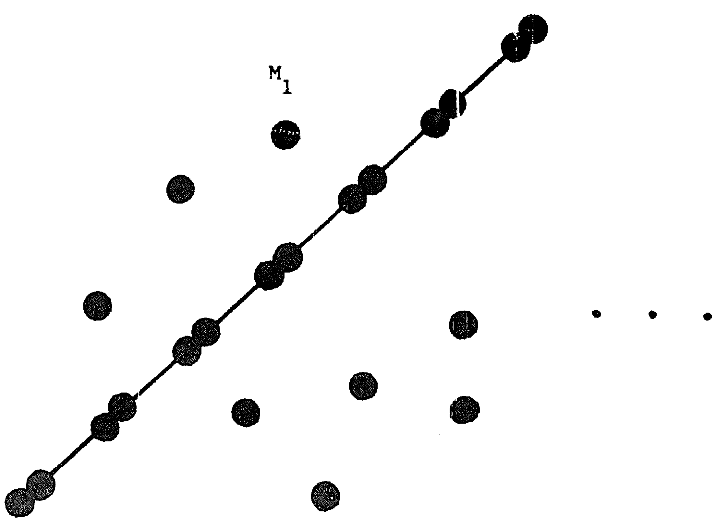


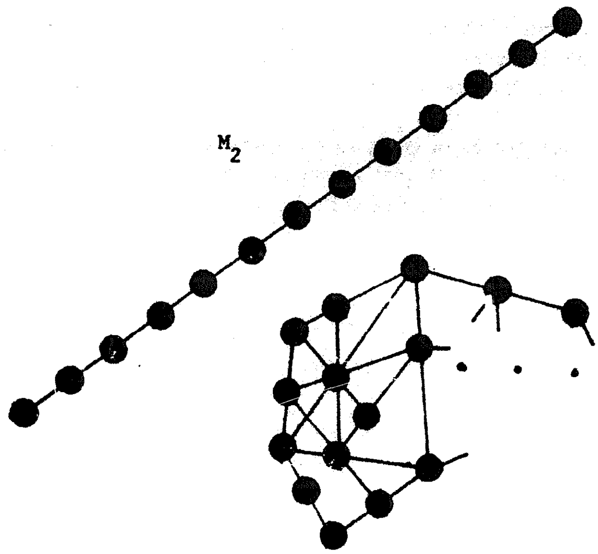


M_1 and M_2 each have ten points with nontrivial multiplicities $|a|=3$ and $|b|=|c|=2$. Further, let M_i' be the seven point matroid whose geometric lattice is isomorphic to that of M_i but with multiplicities $|a|=2$, $|b|=|c|=1$.

Finally, let $\bar{M}_1$ be the matroid on 34 points consisting of M_1, three copies of M_2', and a triangle placed freely in rank three and let $\bar{M}_2$ be the rank-three matroid consisting of M_2, three copies of M_1', and a three-point line. (Each lattice is the cartesian product of five lattices truncated to rank three.) The reader may check that $t(\bar{M}_1) \neq t(\bar{M}_2)$ but that $\bar{M}_1$ and $\bar{M}_2$ have the same number of i-point atoms and lines for all i.

Example 5.3. We present a matroid example to show that knowing the cardinalities of all the hyperplanes is insufficient to determine the number of atoms





(or points). Let M_1 be the rank-three matroid consisting of one line of seven double points and 99 other points in general position. Let M_2 consist of a line L of 14 points placed freely in rank three with respect to a matroid M_2' consisting of 105 atoms, 693 three-point lines, and 3381 two-point lines (such a matroid is possible since one may take a Steiner triple system of 105 points and destroy 1127 of the 1820 three-point subsets (lines).

One easily checks that M_1 and M_2 both have one line of 14 points, 4851 lines of two points each, and 693 three-point lines. However, M_1 has 106 atoms and 113 points, while M_2 has 119 atoms (with no multiple points).

References

[1] M. Aigner, Kombinatorik (Matroide und Transversaltheorie) (Springer-Verlag, Berlin, 1976).
[2] T. Brylawski, A combinatorial model for series-parallel networks, Trans. Amer. Math. Soc. 154 (1971) 1–22.
[3] T. Brylawski, A decomposition for combinatorial geometries, Trans. Am. Math. Soc. 171 (1972) 235–282.
[4] T. Brylawski, Reconstructing combinatorial geometries, Graphs and Combinatorics, Lecture Notes in Mathematics 406 (Springer-Verlag, Berlin, 1974) 226–235.
[5] T. Brylawski, On the nonreconstructibility of combinatorial geometries, Combinatorial Theory B 19 (1975) 72–76.
[6] T. Brylawski, Intersection theory for graphs, to appear in J. Combinatorial Theory B (1981).
[7] T. Brylawski and D. Kelly, Matroids and Combinatorial Geometries, U.N.C. Lecture Note Series, Mathematics Department, University of North Carolina, Chapel Hill, NC (1980).
[8] T. Brylawski, Numerical invariants of matroids, Lecture Notes, Third 1980 Centro Internazionale Matematico Estivo International Mathematical Summer Center.
[9] H. Crapo, The Tutte polynomial, Aequationes Math. 3 (1969) 211–229.
[10] H. Crapo, G.-C. Rota and N. White, ed., Combinatorial Geometries (to appear, 1982).

[11] A.P. Huhn, On contraction lattices of graphs, Abstract: Conference Proceedings of 23rd 1980 Mathematical Institute (Oberwolfach. Germany).
[12] P.J. Kelly, A congruence theorem for trees, Pacific J. Math. 7 (1957) 961–968.
[13] W. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. (1953), 80–91.
[14] W. Tutte, All the king's horses (a guide to reconstruction), in: J.A. Bondy and U.S.R. Murty, eds., Graph Theory and Related Topics (Academic Press, New York, 1979) 15–23.
[15] D.J.A. Welsh, Matroid Theory (Academic Press, New York and London, 1976).