Discrete Mathematics 31 (1980) 161–170

SEVERAL IDENTITIES FOR THE CHARACTERISTIC POLYNOMIAL OF A COMBINATORIAL GEOMETRY

Tom BRYLAWSKI* and James OXLEY**
Department of Mathematics, University of North Carolina, Chapel Hill, NC 27514, USA

Received 20 June 1978

In this paper we explore a research problem of Greene: to find inequalities for the Möbius function which become equalities in the presence of modularity. We replace these inequalities with identities and give combinatorial interpretations for the difference.

1. Introduction

The purpose of this paper is to continue the study initiated in [2] of the broken-circuit complex of a combinatorial geometry. In the present paper we use this complex and its simplex numbers in order to explain and generalize identities due to Brylawski, Greene, Oxley, Stanley, and Zaslavsky involving the characteristic polynomial (and Möbius function) of a geometry. In particular, we study the quotient and remainder when the characteristic polynomial $\chi(G)$ of a geometry G is divided by $\chi(x)$, the characteristic polynomial of a flat (or, more generally, a subgeometry).

The broken-circuit complex is used here as a tool, the ordering on the points being chosen to suit our purpose. In particular, the points of G are identified with the interval $\boldsymbol{n}=\{0, 1, \ldots, n\}$ and the flat x is labeled with the initial segment $\boldsymbol{m}$. It is the subject of another paper [4] to explore what happens to the complex under arbitrary orderings.

If $\boldsymbol{T}_x(G)$ is the complete Brown truncation of the geometry G with respect to a flat x, then whenever x is modular, $\chi(G)=\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)$ [1]. We prove the converse of this identity by interpreting the remainder $\chi(G)-\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)$ in a number of ways when x is not modular (introducing the concept of d-nonmodularity, the degree of the difference polynomial, as a measure of how far x is from being modular).

2. Definitions and preliminary results

We assume that the reader is familiar with the basic concepts of matroid theory, especially that of the characteristic polynomial of a combinatorial geometry and

* Partially supported by N.S.F. Grant No. MCS 78-01149.
** Partially supported by a CSIRO (Australia) Postgraduate Studentship.

its many applications (see [3, 5, 8, 9, 11, and 12]). The results for the broken-circuit complex needed in (2.1) below are introduced in [2] while the relevant theory for modular flats comes from [1].

2.1. *The broken-circuit complex*

Let G be a (*combinatorial*) *geometry* (i.e. a matroid without loops or multiple points) of rank r on the set $\boldsymbol{n}=\{0, 1, \ldots, n\}$. The *broken-circuits* of G are subsets of $\boldsymbol{n}$ of the form $C-\{p\}$ where C is a circuit of G, $p \in C$, and $p<q$ for all $q \in C-\{p\}$. A *χ-independent set* is a subset containing no broken-circuit. The collection of all χ-independent sets forms a pure simplicial complex $\mathscr{C}(G)$ [2] whose facets (maximal simplices) are certain of the bases of G called *Z-bases* (all of which contain 0). A basis B is a Z-basis if and only if for all $p \in \boldsymbol{n}-B$, there is point $q \in B$ such that $q<p$ and $(B-\{q\}) \cup \{p\}$ is a basis. Similarly, an independent set is χ-independent if and only if for all $p \in \boldsymbol{n}-I$, either $\{p\} \cup I$ is independent or $(I-\{q\}) \cup \{p\}$ is independent for some $q \in I$, $q<p$. Equivalently, I is χ-independent in G if and only if I is a Z-basis of the flat $\bar{I}$ which it spans.

When 0 is deleted from every simplex which contains it, we get the *reduced broken-circuit complex* $\mathscr{C}'(G)$. The simplices and facets of $\mathscr{C}'$ are called *reduced χ-independent sets* and *reduced Z-bases* respectively. The *Whitney polynomial* $w(\mathscr{C})$ of $\mathscr{C}(G)$ is defined by $w(\mathscr{C})=\sum w_i \lambda^i$ where w_i is the number of χ-independent sets of size $r-i$. Then,

$$w(\mathscr{C})=(\lambda+1) w(\mathscr{C}')=(-1)^{r(G)} \chi(G,-\lambda)$$

where $\chi(G, \lambda)$ is the *characteristic polynomial* of G. The constant term of $w(\mathscr{C})$ (or) $w(\mathscr{C}')$) is the (absolute) Möbius function $|\mu(0,1)|$ computed in the lattice of flats of G. Thus, $|\mu(0,1)|$ is the number of Z-bases of G.

Results in [2] are proved by deleting or contracting the greatest point where loops can never be deleted or contracted and isthmuses never deleted. Combining a number of these operations we obtain that if A and B are disjoint subsets of $\boldsymbol{n}-\boldsymbol{m}$, if $|C|>|A|$ for all circuits C of G, and if $r((G/A)-B)=r(G)-r(A)$, then $\mathscr{C}((G/A)-B)$ consists of those subsets D of $\boldsymbol{n}-(A \cup B)$ such that $D \cup A$ is in $\mathscr{C}(G)$.

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are simplicial complexes on disjoint sets, their *join* $\mathscr{C}_1 \vee \mathscr{C}_2$ is the complex whose simplices are those subsets of the form $C_1 \cup C_2$ where C_i is a simplex of $\mathscr{C}_i$ $(i=1,2)$.

2.2. *The complete Brown truncation*

Let $x=G(\boldsymbol{m})$ be a flat of G of rank k whose ground set is the initial segment $\boldsymbol{m}$ of $\boldsymbol{n}$. The *complete Brown truncation* of G by x, denoted $\boldsymbol{T}_x(G)$, is the geometry on the points $(\boldsymbol{n}-\boldsymbol{m})\cup\{0\}=\{0, m+1, m+2, \ldots, n\}$ whose bases are subsets of the form B_i or $B_j'\cup\{0\}$ where B_i and B_j' are independent subsets (in G) of $\boldsymbol{n}-\boldsymbol{m}$ with $|B_i|=n-k+1$, $|B_j'|=n-k$, and $r(B_i\cup x)=r(B_j'\cup x)=r$.

Alternatively, $\boldsymbol{T}_x(G)$ is obtained from G by putting a set P of $k-1$ independent points freely on the flat x, contracting P, and then identifying the multiple point $\boldsymbol{m}$ as the single point 0. Since P is free in x, $(I-I_x)\cup P$ is independent in G for any independent set I where $I_x=I\cap x$ and I_x has size $k-1$. The bases of G/P may then be partitioned into those which contain some $p\in x$ (and give a basis of $\boldsymbol{T}_x(G)$ of the form $B_j'\cup\{0\}$), and those which contain no such point. A subset I of $\boldsymbol{n}-\boldsymbol{m}$ is a subset of the set B_j' (where B_j' (where $B_j'\cup\{0\}$ is a basis of $\boldsymbol{T}_x(G)$) if and only if

$$r(I\cup x)=r(x)+|I|.$$

I is then said to be *independent of* x.

When x is not a flat, the complete Brown truncation of G by the subgeometry x is defined to be $\boldsymbol{T}_{\bar{x}}(G)$, where $\bar{x}$ is the closure of x in G. The construction of the previous paragraph also gives $\boldsymbol{T}_x(G)$ in this case (when multiple points are identified), however there is little loss of generality in the theory below when x is assumed to be a flat.

2.3. *Modular flats*

A flat x is *modular* when $r(x)+r(y)=r(x\vee y)+r(x\wedge y)$ for every flat y in the geometric lattice of G. A subset A of $\boldsymbol{n}$ is a modular flat if and only if either of the following (equivalent) conditions holds:

2.3.1. For every circuit C of G which intersects $\boldsymbol{n}-A$, there is a point $p\in A$ such that $(C-A)\cup\{p\}$ is dependent in G.

2.3.2. Every subset I which is independent of each point of A is independent of A. That is, if $I\subseteq\boldsymbol{n}-A$ and $r(I\cup\{p\})=|I|+1$ for all $p\in A$, then $r(I\cup A)=|I|+r(A)$.

Proposition 2.4. *Let B_1 be a Z-basis of the rank k flat x whose points form the initial segment $\boldsymbol{m}$ and let B_2 be a reduced Z-basis of $\boldsymbol{T}_x(G)$. Then, the (disjoint) union $B_1\cup B_2$ is a Z-basis of G which intersects x in k points. Conversely, if B is a Z-basis of G which intersects x in k points, then $B\cap x$ is a Z-basis of x and $B-x$ is a reduced Z-basis of $\boldsymbol{T}_x(G)$.*

The reduced χ-independent sets of $\boldsymbol{T}_x(G)$ are those subsets of $\boldsymbol{n}-x$ which are independent of x and χ-independent in G.

Proof. The reduced Z-bases of $\boldsymbol{T}_x(G)$ are certainly included among the sets B_j' of (2.2). Thus, by remarks in (2.2), $B=B_1\cup B_2$ is a basis of G. If $p\in G-B$, then either $p \in \boldsymbol{m} - B_1$ or $p \in (\boldsymbol{n} - \boldsymbol{m}) - B_2$. In the former case, by (2.1), there is a $q < p$ such that $B_1' = (B_1 - \{q\}) \cup \{p\}$ is a basis of x. But, by (2.2), $B_1' \cup B_2$ is a basis of G. If, on the other hand, $p \in (\boldsymbol{n} - \boldsymbol{m}) - B_2$, there is a q in $B_2 \cup \{0\}$ such that $q < p$ and $B_2' = (B_2 \cup \{0, p\}) - \{q\}$ is a basis of $\boldsymbol{T}_x(G)$. If $q \neq 0$, then $B_1 \cup (B_2' - \{0\}) = (B \cup \{p\}) - \{q\}$ is a basis of G. If $q = 0$, then $B_2 \cup \{p\}$ is a basis of $\boldsymbol{T}_x(G)$ so that $B_1 \cup B_2 \cup \{p\}$ spans G and therefore there is a point $q' \in B_1$ (with $q' < p$ since $q' \in \boldsymbol{m}$ and $p \in \boldsymbol{n} - \boldsymbol{m}$) such that $(B - \{q'\}) \cup \{p\}$ is a basis of G. Since basis exchange exists with a lower point in either case, B is a Z-basis whose intersection with x has cardinality $|B_1| = k$.

To prove the converse, suppose B is a Z-basis of G intersecting x in k points. Then B contains no broken-circuit so that $B \cap x$ is χ-independent and spans x and is thus a Z-basis for x. Similarly, $B' = B - x$ is a reduced Z-basis for $\boldsymbol{T}_x(G)$. To see this note that $B' \cup \{0\}$ is a basis of $\boldsymbol{T}_x(G)$ by (2.2). Moreover, if $p \in \boldsymbol{n} - \boldsymbol{m}$, there is a $q \in B$ with $q < p$ such that $(B - \{q\}) \cup p$ is a basis of G. If $q \notin x$, then $(B' \cup \{0, p\}) - \{q\}$ is a basis of $\boldsymbol{T}_x(G)$, while if $q \in x$, then, using (2.2), $B' \cup \{p\} = ((B' \cup \{0\}) - \{0\}) \cup \{p\}$ is a basis of $\boldsymbol{T}_x(G)$.

The final statement follows since I is a reduced χ-independent set in $\boldsymbol{T}_x(G)$ is and only if it is a subset of a Z-basis B_2 of $\boldsymbol{T}_x(G)$. But this occurs if and only if I is in a Z-basis B of G which intersects x maximally. We may now apply the arguments of the previous paragraph to B.

Proposition 2.5. *Let $G(A)$ be a subgeometry of G and let s equal the size of the smallest independent subset of $\boldsymbol{n} - A$ which is independent of each point of A but is not independent of A itself. If no such set exists, set $s = r$. Then*

2.5.1. *$s = \min(|C - A|, r)$ where the minimum is over all circuits C of G such that $C \nsubseteq A$ and $(C - A) \cup \{p\}$ is independent for every point $p \in A$.*

Further, if $A = \boldsymbol{m}$,

2.5.2. *$s = \min(|I|, r)$ where the minimum is over all χ-independent subsets I of $\boldsymbol{n}$ (or, equivalently, of $\boldsymbol{n} - \boldsymbol{m}$) which are not contained in a Z-basis B of G with $|B \cap A| = r(A)$.*

Proof. Since G is a geometry, it is easy to see that A is not closed if and only if $s = 1$ for all three conditions. Thus, we may assume that A is a flat. Property (2.5.1) is easily seen to be equivalent to the definition of s where the independent set I of (2.5) is the set $C - A$ of (2.5.1). For example, if I is independent of each point of A but is not independent of A, then there is an independent subset J of A such that $J \cup I$ is dependent. If $|J \cup I|$ is minimal with this property, then $J \cup I$ is a circuit while $I \cup \{p\}$ is independent for all $p \in A$.

To show (2.5.2), let I be a χ-independent subset of G and assume that $I - A$ can be extended to a Z-basis B which intersects A maximally. If B_1 is a Z-basis of the flat A which contains the χ-independent set $I \cap A$, then by (2.4), $(B - A) \cup B_1$ is a Z-basis which contains I and intersects A maximally. Thus, the minimal nonextendable χ-independent set is a subset of $\boldsymbol{n} - \boldsymbol{m}$.

If I is a χ-independent set, then, since $\boldsymbol{m}$ is least labeled, I must be independent of each point of $\boldsymbol{m}$ while if I were independent of $\boldsymbol{m}$, I would be contained in a Z-basis which maximally intersects A by (2.4). Conversely, among all minimal independent sets satisfying (2.5), let I be the lexicographic minimum. Then it is χ-independent since if it contained a broken-circuit C (where $C \cup \{p\}$ is a circuit), then $p \in \boldsymbol{n} - \boldsymbol{m}$ since otherwise I would not be independent of p. However, if $p \in \boldsymbol{n} - \boldsymbol{m}$, then, for any $q \in C$, $(I \cup \{p\}) - \{q\}$ is a lexicographically smaller subset which, by a straightforward application of circuit exchange, satisfies (2.5). But as I is not independent of A, I is not contained in *any* basis which intersects A maximally, let alone a Z-basis.

3. Identities for the characteristic polynomial and Möbius function

In this section we present our principal results giving, via arguments for the broken-circuit complex, combinatorial interpretations for the quotient and remainder when $\chi(G)$ is divided by $\chi(x)$ for a flat (or subgeometry) x.

In [1, 2, and 10] the modularity of a flat x is shown to produce a factorization of $\chi(G)$. We first give a broken-circuit theoretic proof of the converse of this result. The rest of this section deals with the remainder when x is not modular. An example to illustrate the main results is given at the end of the section.

Lemma 3.1. *Let G be a geometry of rank r on the ground set $\boldsymbol{n}$ and let the initial segment $\boldsymbol{m}$ form a flat x of rank k. Then the simplicial complex $\mathscr{C}(x) \vee \mathscr{C}'(\boldsymbol{T}_x(G))$ consists of all χ-independent sets contained in a Z-basis of G which intersects x in k points.*

Proof. This is essentially the broken-circuit complex restatement of (2.4).

Theorem 3.2. *Let x be a subgeometry of G and let d be the degree of the polynomial $\chi(G) - \chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda - 1)$ (or the degree of the polynomial $w(\mathscr{C}(G)) - w(\mathscr{C}(x))w(\mathscr{C}'(\boldsymbol{T}_x(G)))$) where the degree of the zero polynomial is defined to be* 0. *Then $d = r - s$ where s is defined in* (2.5) *to be the size of the smallest independent subset of $S - x$ which is independent of each point of x but is not independent of x.*

Proof. If x is not a flat, then $s = 1$ in (2.5) while the degree of $\chi(G) - \chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda - 1)$ equals $r - 1$ since the coefficient of λ^{r-1} in the difference equals the number of points in $\bar{x} - x$. Thus, we may assume that x is a flat in which case by (2.5.2), if x is labeled by an initial segment, $r - s$ equals the maximal codimension of a χ-independent subset of G which is not contained in any Z-basis which intersects x maximally. Now $w(\mathscr{C}(G))$ tabulates all χ-independent sets and (3.1) shows that $w(\mathscr{C}(x))w(\mathscr{C}'(\boldsymbol{T}_x(G)))$ tabulates those which intersect x maximally. Thus d, the degree of $w(\mathscr{C}(G)) - w(\mathscr{C}(x))w(\mathscr{C}'(\boldsymbol{T}_x(G)))$, equals $r - s$.

We may interpret the d in (4.2) as a gauge of the nonmodularity of the subgeometry x defining x to be *d-nonmodular* if the difference polynomial has degree d. Note that x is $(r-1)$-nonmodular if and only if it is not closed, and a nonmodular flat of rank k is at least $(k-1)$-nonmodular. Therefore, for example, a nonmodular hyperplane is always $(r-2)$-nonmodular. Also, using (2.3), x is 0-nonmodular if and only if it is modular. This fact is expressed in the following corollary.

Corollary 3.3. *$|\mu(G)| \geq |\mu(x)\mu(\boldsymbol{T}_x(G))|$ with equality if and only if x is a modular flat.*

Corollary 3.4. *If x is a flat of rank k and is $(k-1)$-nonmodular, then $\chi(x)$ cannot divide $\chi(G)$.*

The following theorem was first proved for the hyperplane case ($c=1$) in [9] (where the result was shown to hold for an arbitrary matroid) using an inductive argument. We give the present generalization and proof as we believe it gives more insight into how the simplices of the broken-circuit complex can be partitioned in the presence of certain flats. We also remark that many corollaries are given in [9] based on the fact, which also holds under our more general hypotheses below, that the difference polynomial $\chi(G)-\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)$ or its negative can be expressed as a sum of characteristic polynomials of minors of G.

Theorem 3.5. *Let G be a geometry on the set $\boldsymbol{n}$ and let x be a flat of G of corank c whose ground set is the initial segment $\boldsymbol{m}$. Assume all the circuits of G have cardinality at least $c+2$. Then the Möbius function satisfies the following identity:*

$$|\mu(G)| = |\mu(x)\mu(\boldsymbol{T}_x(G))| + \sum |\mu(G[T])| \tag{3.5.1}$$

the sum being taken over all subsets T of $\boldsymbol{n}-\boldsymbol{m}$ such that $|T|=c+1$ and $r(G[T])=r-c-1$ where if $T=\{i_{r-c}<i_{r-c+1}<\cdots<i_r\}$, then $G[T]=G/T-\{i : i>i_{r-c}, i\neq i_{r-c+1},\ldots,i_r\}$. Further, if x is d-nonmodular with $d\leq r-c-1$, then

$$\chi(G) = \chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1) + (-1)^{c+1}\sum \chi(G[T]) \tag{3.5.2}$$

where the summation is over the same subsets as in (3.5.1).

Proof. We will prove (3.5.1) by showing the right-hand side counts the Z-bases of G. By (2.4), $|\mu(x)\mu(\boldsymbol{T}_x(G))|$ is the number of Z-bases which intersect x maximally (i.e. in $r-c$ points). All other Z-bases must contain at least $c+1$ points from $\boldsymbol{n}-\boldsymbol{m}$. Let $B=\{i_0,\ldots,i_{r-c},\ldots,i_r\}$ where $i_0<\cdots<i_{r-c}<\cdots<i_r$. Then, results of (2.2) show that B is a Z-basis of G if and only if $B-T$ is a Z-basis of $G[T]$ where $T=\{i_{r-c},\ldots,i_r\}$. (This can be seen by contracting and deleting the indicated points of G in reverse order starting with n and finishing with i_{r-c}.

The hypothesis that no circuit has size less than or equal to $c+1$ guarantees that no loop is deleted or contracted while the condition that $r(G[T])=r-c-1$ ensures that no isthmus is deleted.)

The identity (3.5.2) is proved similarly noting that the additional hypothesis on low nonmodularity implies that every χ-independent set I which is not in a Z-basis counted in $|\chi(x)\chi(T_x(G))|$ has at least $c+1$ points in $\boldsymbol{n}-\boldsymbol{m}$. If the $c+1$ greatest points of I are represented by T, I is counted in the unique summand $\chi(G[T])$ of (3.5.2) with the same exponent of λ (viz. $r-|I|$).

Corollary 3.6. *If $G(\boldsymbol{n})$ is a geometry of rank r and x is a hyperplane having ground set $\boldsymbol{m}$, then*

$$\chi(G)=(\lambda-n+m)\chi(x)+\sum_{\{p,q\}:m<p<q}\chi((G/\{p,q\})-\{i:i>p,i\neq q\}).$$

Proof. The hypotheses of (3.5.2) are all satisfied for $c=1$ since a geometry has no one-point or two-point circuits. Further, a hyperplane is $(r-2)$-nonmodular (by remarks following (3.2)) while, it is easy to check that $r(G[T])=r-2$.

Corollary 3.7. *If G is a geometry of rank r and x is a hyperplane, then $\chi(G)-\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)$ has degree $r-2$ and the coefficient of λ^{r-2} is equal to $\sum(|l|-1)$ where the sum is over all lines l which do not intersect x.*

Proof. Every subset $\{p,q\}$ in the right-hand summation of (3.5.2) spans a line l. If l intersects x, a loop will be produced in x when $\{p,q\}$ is contracted and so $\chi(G[T])$ is zero (the broken-circuit complex is empty). On the other hand, if l is parallel to x, a loop will occur (and not be deleted) unless p is the least-labeled point of l, in which case a monic polynomial of degree $r-2$ will contribute to the summand for each of the $|l|-1$ choices for q.

Another application of (3.5) is in the case $c=0$ where the following corollary can be given an elementary deletion-contraction proof.

Corollary 3.8. *If $\boldsymbol{m}$ spans a loopless matroid G, then*

$$\chi(G)=\chi(G(\boldsymbol{m}))-\sum_{p:p>m}\chi((G/p)-\{q:q>p\}).$$

Theorem 3.9. *If x is a flat of rank k in a geometry G of rank r, then*

$$\chi(\boldsymbol{T}_x(G))=(\lambda-1)\sum\mu(0,y)\lambda^{r-k-r(y)}$$

where the sum is over all flats y of G such that $r(x\vee y)=r(y)+k$.

Proof. The above identity, when interpreted for simplex polynomials, states that every reduced χ-independent set in $\mathscr{C}'(\boldsymbol{T}_x(G))$ corresponds to a Z-basis of some flat y which forms a modular pair with x and is disjoint from it (where the exponent of λ adjusts the codimension of each such χ-independent set I in $\mathscr{C}(y)$ to the appropriate codimension of I in $\mathscr{C}'(\boldsymbol{T}_x(G))$).

Let I be a subset of $\boldsymbol{n}-\boldsymbol{m}$. Then, as we have seen in (2.4), I is a reduced χ-independent set of $\boldsymbol{T}_x(G)$ if and only if I is independent of x and I is χ-independent in G. But the latter condition is equivalent by (2.1) to the property that I is a Z-basis of $\bar{I}$ (so that I contributes to $\mathscr{C}(y)$) while the former condition is equivalent to the fact that

$$r(I)=r(\bar{I})=r(y)=r(I\cup x)-r(x)=r(y\vee x)-k.$$

The inequality in the following corollary is proved by Greene in [6] by the Möbius algebra and in [7] by induction. In addition [7] shows how most of the known theorems regarding nonmodular inequalities for $\mu(G)$ follow from (3.9).

Corollary 3.10. *$|\mu(G)|\geqslant|\mu(x)|\,|\sum_{y:y\perp x}\mu(0,y)|$ where the sum is over all flats which are modular complements of x (i.e. all y such that $r(x)+r(y)=r(x\vee y)=r$).*

Further, $|\mu(G)|-|\mu(x)|\sum_{y:y\perp x}|\mu(0,y)|=|\mu(G)|-|\mu(x)\sum_{y:y\perp x}\mu(0,y)|$ is equal to the number of Z-bases of G which do not intersect x maximally and this number is zero if and only if x is modular.

Proof. When λ is set equal to 0 in (3.9) only those flats whose exponent is zero (i.e. those for which $r(y)+r(x)=r$) contribute. Thus $|\sum_{y:y\perp x}\mu(0,y)|=|\mu(\boldsymbol{T}_x(G))|$. We may now apply (3.4).

Corollary 3.11.

$$\beta(\boldsymbol{T}_x(G))=(-1)^{r(G)-r(x)}\sum_{y:r(x\vee y)=r(y)+r(x)}\mu(0,y)$$

where for any geometry H, $\beta(H)=(-1)^{r(H)+1}[\mathrm{d}\chi(H)/\mathrm{d}\lambda]_{\lambda=1}$ [3, 8, 11, *or* 12].

Proof.

$$\begin{aligned}\beta(\boldsymbol{T}_x(G))&=(-1)^{r(\boldsymbol{T}_x(G))+1}\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\chi(\boldsymbol{T}_x(G))\right]_{\lambda=1}\\&=(-1)^{r(G)-r(x)}\sum_{y:r(x\vee y)=r(x)+r(y)}\mu(0,y)\end{aligned}$$

by the product rule for derivatives and (3.9).

If the flat x in (3.11) is a point p of G, then $\boldsymbol{T}_p(G)=G$ and we have the following result of [12, p. 76].

Corollary 3.12. *If p is a point of a geometry G,*

$$\beta(G)=(-1)^{r-1}\sum_{y:p\nleq y}\mu(0,y).$$

Example 3.13. We now illustrate some of the previous results for the geometry whose points are the vertices of a triangular prism with lateral faces (circuits) 0123, 0145, and 2345.

Let x be the line 01. Then $\chi(x)=(\lambda-1)^2$ and $\boldsymbol{T}_x(G)$ is the geometry consisting of the two intersecting lines 023 and 045. Its reduced Z-bases are 24, 25, 34, and 35, the (Z-bases of the) modular complements of x (3.10).

$$\chi(\boldsymbol{T}_x(G))=(\lambda-1)(\lambda-2)^2$$

and

$$\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)=(\lambda-1)^2(\lambda-2)^2=\lambda^4-6\lambda^3+13\lambda^2-12\lambda+4$$

where the latter coefficients are the simplex numbers of the subcomplex of $\mathscr{C}(G)$ whose facets are the Z-bases 0124, 0125, 0134, and 0135, all of which intersect x maximally (i.e. in $r(x)=2$ points).

The other Z-bases, all of which contain at least $c+1=3$ points greater than 1 are 0234, 0235, and 0245. Thus, (3.5.1) is satisfied where $G[345]$ contains the loop 2.

The subset 23 is χ-independent but is in no Z-basis which intersects x maximally. In fact 23 is independent of both 0 and 1 but not of 01 (it forms the circuit 0123 with x). Thus $s=2$ in (2.5) so that x is d-nonmodular where $d=4-2=2$. Therefore the hypotheses of (3.5.2) are not satisfied and

$$\begin{aligned}&\chi(G)-(\chi(x)\chi(\boldsymbol{T}_x(G))/(\lambda-1)+(-1)^3\textstyle\sum\chi(G[T]))\\&=(\lambda^4-6\lambda^3+15\lambda^2-17\lambda+7)-((\lambda-1)^2(\lambda-2)^2-3(\lambda-1))\\&=2\lambda^2-2\lambda.\end{aligned}$$

The coefficients of this difference count the four χ-independent sets 23, 45, 023, and 045 which are neither contained in a Z-basis intersecting x maximally nor intersect 2345 in at least three points.

References

[1] T. Brylawski, Modular constructions for combinatorial geometries, Trans. Amer. Math. Soc. 203 (1975) 1–44.

[2] T. Brylawski, The broken-circuit complex, Trans. Amer. Math. Soc. 234 (1977) 417–433.

[3] T. Brylawski and D.G. Kelly, Matroids and combinatorial geometries, in: G.-C. Rota, ed., Studies in Combinatorial Theory (Mathematical Association of America, 1978).

[4] T. Brylawski and J. Oxley, The broken-circuit complex: its structure and factorizations (to appear).

[5] H.H. Crapo and G.-C. Rota, Combinatorial Geometries (M.I.T. Press, Cambridge, MA, 1970).

[6] C. Greene, On the Möbius algebra of a partially ordered set, Adv. in Math. 10 (1973) 177–187.

[7] C. Greene, An inequality for the Möbius function of a geometric lattice, Studies in Appl. Math. 54 (1975) 71–74.

[8] C. Greene and D. Kennedy, Matroid Theory, Lecture Notes, Bowdoin College, Brunswick, Maine, 1971.

[9] J.G. Oxley, Colouring, packing and the critical problem, Quart. J. Math. Oxford (2) 29 (1978) 11–22.
[10] R.P. Stanley, Modular elements of geometric lattices, Alg. Univ. 1 (1971) 214–217.
[11] D.J.A. Welsh, Matroid Theory (Academic Press, New York, 1976).
[12] T. Zaslavsky, Facing up to arrangements: face count formulas for partitions of space by hyperplanes, Mem. Amer. Math. Soc. 154 (1975).