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Facets of the p-cycle polytope

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Abstract

The purpose of this study is to provide a polyhedral analysis of the p-cycle polytope, which is the convex hull of the incidence vectors of all the p-cycles (simple directed cycles consisting of p arcs) of the complete directed graph K_n. We first determine the dimension of the p-cycle, polytope, characterize the bases of its equality set, and prove two lifting results. We then describe several classes of valid inequalities for the case $2 < p < n$, together with necessary and sufficient conditions for these inequalities to induce facets of the p-cycle polytope. We also briefly discuss the complexity of the associated separation problems. Finally, we investigate the relationship between the p-cycle polytope and related polytopes, including the p-circuit polytope. Since the undirected versions of symmetric inequalities which induce facets of the p-cycle polytope are facet-inducing for the p-circuit polytope, we obtain new classes of facet-inducing inequalities for the p-circuit polytope.

Keywords: p-cycle; p-median; Traveling salesman; p-circuit

1. Introduction

The purpose of this study is to provide a polyhedral analysis of the p-cycle polytope Q_p^n, which is the convex hull of the incidence vectors of all the p-cycles (simple directed cycles consisting of p arcs) of the complete directed graph K_n. In Hartmann and Özlük [16], we use these results as the basis for a branch-and-cut algorithm to solve the traveling circus problem, which combines the location aspects of the p-median problem with the routing aspects of the traveling salesman problem. The traveling circus problem has applications in the design of distributed computer networks and the construction of routes for traveling health care teams in developing countries (see [11]). Generalizations of the traveling salesman problem, such as the time-dependent

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traveling salesman and (equality) generalized traveling salesman problems, can also be modeled as instances of the traveling circus problem. The largest traveling circus problems we have been able to solve were symmetric Euclidean problems with $n = 100$ cities randomly distributed in the unit square.

The p-cycle polytope should not be expected to have a simple characterization, since it contains the asymmetric traveling salesman (ATS) polytope on p nodes as a face. Along the same lines, it is clear that the p-cycle problem (the problem of finding a minimum length p-cycle) is NP-hard. There do not appear to be any published studies of the p-cycle polytope for $p < n$ (see [23]); however, several closely related polytopes have received some attention in the literature. The cycle polytope Q^n, which is the convex hull of the incidence vectors of all simple directed cycles of the complete directed graph K_n, has been studied by Balas [3] and Balas and Oosten [5], who describe several classes of valid inequalities that induce facets of Q^n. Balas [2] also studies the prize collecting traveling salesman polytope, which can be loosely defined as the integer hull of the cycle polytope with an additional inequality constraint. The circuit polytope, which is the undirected analog of Q^n, has been studied by Coullard and Pulleyblank [10], and more recently by Bauer [6]. The p-circuit polytope, which is the undirected analog of Q^n_p, has been studied by Kovalev et al. [18] and Maurras and Nguyen [19] for $p = 3$ and by Nguyen and Maurras [21] for $2 < p < n$. Bauer et al. [7] also studied the cardinality constrained circuit polytope, which is the convex hull of all circuits with at most p nodes on a complete undirected graph, in order to solve the cardinality constrained circuit problem by branch-and-cut.

The rest of this paper is organized as follows. In Section 2, we determine the dimension of the p-cycle polytope, characterize the bases of its equality set, and prove two lifting results. In Section 3, we describe several classes of valid inequalities for $2 < p < n$. In each case, we give necessary and sufficient conditions (on n and p) under which inequalities induce facets, in most cases expressing those inequalities which do not induce facets as a consequence of facet-inducing inequalities. We also briefly discuss the complexity of the associated separation problems. In Section 4, we investigate the relationship between the p-cycle polytope and related polytopes, including the p-circuit polytope. Since the undirected versions of symmetric inequalities which induce facets of the p-cycle polytope are facet-inducing for the p-circuit polytope, we obtain new classes of facet-inducing inequalities for the p-circuit polytope.

2. Basic results

Based on the mathematical program given by Current and Schilling [11], the feasible region of the p-cycle problem can be defined as follows:

$$x(\delta^+(k)) - x(\delta^-(k)) = 0 \quad (k \in N), \tag{1}$$

$$x(N : N) = p, \tag{2}$$

$$x(S : S) \leqslant |S| - 1 \quad (s \subset N;\ 2 \leqslant |S| < p), \tag{3}$$

$$x_{ij} \in \{0, 1\} \quad (i \neq j),$$

where $x_{ij} = 1$ if the arc (i, j) is on the cycle and $x_{ij} = 0$ otherwise. Here, and throughout the paper, we will use the following conventions: $\delta^+(k)$ and $\delta^-(k)$ are the sets of arcs directed out of and into node k, respectively; a vector $\boldsymbol{a}$ whose components are indexed by the arcs of the complete graph defines a set-function via $a(F) = \sum_{(i,j)\in F} a_{ij}$; we abbreviate $x(S : T) = \sum_{i\in S} \sum_{j\in T} x_{ij}$ for any subsets $S, T \subseteq N$, where the summation does not extend over loops (i, i) for $i \in S \cap T$; and the symbol N is reserved for the node set $\{1, 2, \ldots, n\}$ of the complete graph K_n.

The cycle polytope Q^n and the p-cycle polytopes Q^n_p for $2 \leqslant p \leqslant n$ are closely related; in fact, Q^n_p is contained in the "slice" $Q^n \cap \{x(N : N) = p\}$. However, $Q^n_p = Q^n \cap \{x(N : N) = p\}$ only if $p = 2$ or $p = n$, as can be seen by taking a convex combination of the incidence vectors of a $(p - 1)$-cycle and a $(p + 1)$-cycle. So while facet-inducing inequalities for the cycle polytope will be valid for the p-cycle polytope, they may not induce facets of the p-cycle polytope. The cycle polytope has dimension $n^2 - 2n + 1$ (see [3]), but the dimension of the p-cycle polytope depends on both n and p, as can be seen in the following result.

Theorem 1. *Let Q^n_p be the convex hull of the incidence vectors of p-cycles in the complete directed graph K_n. Then*

$$\dim(Q^n_p) = \begin{cases} \binom{n}{2} - 1, & p = 2, \\ n^2 - 2n, & 2 < p < n \text{ and } n \geqslant 5, \\ n^2 - 3n + 1, & p = n \text{ and } n \geqslant 3 \end{cases}$$

and $\dim(Q^4_3) = 6$.

Proof. It is easy to determine the dimension of the 2-cycle polytope, which is a simplex whose vertices are the (linearly independent) incidence vectors of the $\binom{n}{2}$ 2-cycles of K_n. For $p = n$, the p-cycle polytope is the ATS polytope, which has dimension $(n - 1)^2 - n$ for $n \geqslant 3$ (see [15, Theorem 20]).

The fact that $\dim(Q^4_3) = 6$ can be verified using PORTA;[1] in this case, the subtour elimination constraints (3) also hold with equality. For $2 < p < n$ and $n \geqslant 5$, we will show that the lineality space of the p-cycle polytope is described by (1) and (2) using the following lemma.

Lemma 2. *The following are equivalent for a row vector $\boldsymbol{c}$ when $n \geqslant 5$:*

[1] PORTA is a collection of routines for analyzing polytopes and polyhedra (see [9]), available free of charge at the URL **http.//www.iwr.uni-heidelberg.de/iwr/comopt/soft/PORTA.**

(i) *There exists* c_0 *and* p *with* $2 < p < n$ *such that* $c(P) = c_0$ *for all* p*-cycles* P *of* K_n.
(ii) $c_{ij} + c_{jk} = c_{il} + c_{lk}$ *for all distinct* $i, j, k, l \in N$.
(iii) *There exists* λ *and* $\pi_1, \ldots, \pi_n$ *such that* $c_{ij} = \lambda + \pi_i - \pi_j$ *for all* $i \neq j \in N$.

Proof. (i) $\Rightarrow$ (ii) Since $2 < p < n$, there must be a p-cycle P that contains the arcs (i,j) and (j,k) but does not visit node l. Form the p-cycle P' by replacing node k by node l in P. The hypothesis implies that $c(P) = c(P')$ and thus $c_{ij} + c_{jk} = c_{il} + c_{lk}$ for all distinct $i, j, k, l \in N$.

(ii) $\Rightarrow$ (iii) First we will show that (i) holds for $p = 2$ and $p = 3$. Condition (ii) implies that $c_{ij} + c_{jk} + c_{ki} = c_{il} + c_{lk} + c_{ki}$ for all distinct nodes $i, j, k, l \in N$. Since $n \geqslant 4$, this implies that there exists α such that $c_{ij} + c_{jk} + c_{ki} = \alpha$ for all distinct nodes $i, j, k \in N$. Now if $i, j, k, l \in N$ are distinct, we must have

$$\begin{aligned} c_{ij} + c_{ji} &= c_{ij} + (c_{jl} + c_{lk} - c_{ik}) \\ &= (c_{ij} + c_{jl} - c_{ik}) + c_{lk} \\ &= c_{kl} + c_{lk}, \end{aligned}$$

where the first and last equalities follow from (ii). Since $n \geqslant 5$, this implies that there exists β such that $c_{ij} + c_{ji} = \beta$ for all $i \neq j \in N$.

Next, we show that (iii) holds for $\lambda = \alpha - \beta$, $\pi_1 = -\lambda$ and $\pi_j = -c_{1j}$ for $j \geqslant 2$, using the fact that

$$c_{ij} + c_{jk} - c_{ik} = (c_{ij} + c_{jk} + c_{ki}) - (c_{ik} + c_{ki}) = \alpha - \beta = \lambda$$

for all distinct $i, j, k \in N$.

If $i = 1$,

$$\lambda + \pi_1 - \pi_j = \lambda - \lambda + c_{1j} = c_{1j}.$$

If $j = 1$, then for some $k \neq 1$, i

$$\lambda + \pi_i - \pi_1 = 2\lambda - c_{1i} = (c_{i1} + c_{1k} - c_{ik}) + (c_{1i} + c_{ik} - c_{1k}) - c_{1i} = c_{i1}.$$

Finally, if $i, j \geqslant 2$ then

$$\lambda + \pi_i - \pi_j = \lambda - c_{1i} + c_{1j} = (c_{1i} + c_{ij} - c_{1j}) - c_{1i} + c_{1j} = c_{ij}.$$

(iii) $\Rightarrow$ (i) the condition clearly holds for any $2 \leqslant p \leqslant n$ with $c_0 = \lambda p$. $\square$

Since $n(n-1) - n = n^2 - 2n$, it suffices to show that there are n linearly independent equations in (1) and (2). Clearly, there are at most n, since summing the flow conservation constraints (1) yields $\mathbf{0x} = 0$. We will show that Eqs. (1) for $k \geqslant 2$ and Eq. (2) are linearly independent. Let $\boldsymbol{B}$ be the $n \times n$ submatrix of this linear system with columns corresponding to the arcs $(1,j)$ for $j \geqslant 2$ and $(n-1,n)$. Expanding the determinant of $\boldsymbol{B}$ about the rows corresponding to nodes $2,3,\ldots,n-2$ we see that

$$\det(\boldsymbol{B}) = \pm \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \pm 1. \qquad \square$$

The next theorem can be used to define a canonical form of valid inequalities for the p-cycle polytope. In the context of similar polyhedral analysis, canonical forms can be used to determine if two valid inequalities are equivalent (see [4]). We consider two valid inequalities $\boldsymbol{cx} \leqslant c_0$ and $\boldsymbol{c}'\boldsymbol{x} \leqslant c_0'$ to be *equivalent* if $\boldsymbol{c}' = \mu\boldsymbol{c} + \boldsymbol{\pi A} + \lambda\mathbf{1}$ and $c_0' = \mu c_0 + \lambda p$ for some scaling factor $\mu > 0$ and multipliers $\boldsymbol{\pi}$ and λ, where $\boldsymbol{A}$ is the matrix representing the flow conservation constraints (1) and $\mathbf{1}$ is the vector of all ones; equivalent inequalities determine the same face of the p-cycle polytope, and two valid inequalities for the p-cycle polytope determine the same facet if and only if they are equivalent (see [20, Theorem I.4.3.6]).

We will refer to a (not a necessarily directed) cycle that contains the same number of forward and backward arcs as *balanced* and a subgraph consisting of a spanning tree T plus an arc (k,l) whose fundamental cycle $C(k,l)$ is not balanced as an *unbalanced* 1-*tree*.

Theorem 3. *Let H be a subgraph of K_n for $n \geqslant 2$. The variables corresponding to the arcs of H form a basis for linear equality system* (1)–(2) *if and only if H is an unbalanced* 1-*tree*.

Proof. First suppose that H is an unbalanced 1-tree consisting of a spanning tree T and an additional arc (k,l). Let π_j be determined by $\pi_l = 0$ and $\pi_j = \pi_i + 1$ for all arcs (i,j) in T. It is easy to show that if P is the unique path in T from l to k, then π_k will be the number of forward arcs in P minus the number of backward arcs in P. Since the fundamental cycle $C(k,l)$ consists of the path P followed by the arc (k,l), it is balanced if and only if $\pi_k + 1 = 0$.

Let $\boldsymbol{B}$ be the $n \times n$ submatrix of linear system (1)–(2) whose first $n-1$ columns correspond to arcs in T and nth column corresponds to the arc (k,l), and whose first $n-1$ rows are from $\boldsymbol{A}$ except the one corresponding to node l and nth row corresponds to the cardinality constraint (2). Here

$$\boldsymbol{B} = \left[\begin{array}{c|c} \boldsymbol{T} & -\boldsymbol{e}^k \\ \hline \mathbf{1} & 1 \end{array}\right],$$

where $\boldsymbol{e}^k$ is the kth unit column vector. Now $\boldsymbol{T}$ is a nonsingular matrix, so there is a unique non-trivial linear combination of its rows that gives the row vector $\mathbf{1}$. The multipliers that give this vector are exactly the π_j values defined in the previous paragraph (since $\pi_l = 0$). Then $\pi_k + 1 \neq 0$ implies that the last row of $\boldsymbol{B}$ is linearly independent of the others; hence, the columns of $\boldsymbol{B}$ form a basis for linear system (1)–(2).

Next, suppose that the variables corresponding to the arcs of H form a basis for linear system (1)–(2), and note that if C is a cycle, then the "incidence" vector $\boldsymbol{x}$ of C defined by

$$x_{ij} = \begin{cases} +1 & \text{if } (i,j) \text{ is a forward arc in } C, \\ -1 & \text{if } (i,j) \text{ is a backward arc in } C, \\ 0 & \text{if } (i,j) \text{ does not lie on } C \end{cases} \tag{4}$$

satisfies $\boldsymbol{Ax}=\boldsymbol{0}$ and the integer $\boldsymbol{1x}$ is the difference between the number of forward and backward arcs in C. Thus if H contains a balanced cycle, then $\boldsymbol{1x}=0$ and the arcs of H cannot form a basis. If H contains two unbalanced cycles C and C' with "incidence" vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$, then for $\alpha = \boldsymbol{1x}/\boldsymbol{1x}'$ we have $\boldsymbol{A}(\boldsymbol{x} - \alpha\boldsymbol{x}') = \boldsymbol{0}$ and $\boldsymbol{1}(\boldsymbol{x} - \alpha\boldsymbol{x}') = 0$ so the arcs of H cannot form a basis. Since H consists of n arcs, H must contain exactly one cycle and that cycle must be unbalanced. □

The equality set for the p-cycle polytope is the same as that of the linear programming formulation of the minimum cycle mean problem. A primal simplex algorithm for the minimum cycle mean problem is described by Dantzig et al. [12], where a primal feasible basis corresponds to a rooted arborescence and an additional arc which induces a directed fundamental cycle, and a dual simplex algorithm for the minimum cycle mean problem is described by Karp and Orlin [17], where a dual feasible basis corresponds to a rooted arborescence and an additional arc (k,l) such that there are at least as many arcs in the path from the root to k as there are in the path from the root to l.

Corollary 4. *Let $\boldsymbol{cx} \leqslant c_0$ be a valid inequality for the p-cycle polytope, and let values $\boldsymbol{a}_H$ be specified for the arcs in an unbalanced 1-tree H. Then there is an equivalent inequality $\boldsymbol{c}'\boldsymbol{x} \leqslant c_0'$ for which $\boldsymbol{c}_H' = \boldsymbol{a}_H$.*

Proof. We must have $\boldsymbol{c}' = \mu\boldsymbol{c} + \boldsymbol{\pi A} + \lambda\boldsymbol{1}$ for some $\mu > 0$ and multipliers $\boldsymbol{\pi}$ and λ; setting $\mu = 1$, Theorem 3 implies that there exist $\boldsymbol{\pi}$ and λ such that $\boldsymbol{c}_H - \boldsymbol{\pi A}_H - \lambda\boldsymbol{1}_H = \boldsymbol{a}_H$. Explicitly, $\lambda = (\boldsymbol{cx} - \boldsymbol{ax})/\boldsymbol{1x}$, where $\boldsymbol{x}$ is the "incidence" vector of the unbalanced cycle C defined in (4). Once λ has been determined, solving the rest of the system reduces to finding node potentials $\boldsymbol{\pi}$ for the costs $\boldsymbol{c} - \boldsymbol{a} - \lambda\boldsymbol{1}$ and any spanning tree for H. □

Note: Setting $\boldsymbol{a}_H = \boldsymbol{0}$ yields an H-canonical form of the inequality $\boldsymbol{cx} \leqslant c_0$ which can be computed in $O(n^2)$ time, since λ and $\boldsymbol{\pi}$ can be computed in $O(n)$ time (see [1]).

Corollary 5. *For a row vector $\boldsymbol{c}$, if either condition* (i) *or* (ii) *of Lemma* 2 *holds when restricted to a subset S with $|S| \geqslant 5$ and $c_{ij} = \beta$ for all (i,j) in an unbalanced* 1*-tree H on S, then $c_{ij} = \beta$ for all $i,j \in S$.*

Proof. In either case, Lemma 2 implies that there exists λ and $\{\pi_j\colon j \in S\}$ such that $c_{ij} = \lambda + \pi_i - \pi_j$ for all $i \neq j \in S$. Without loss of generality, we can set $\pi_k = 0$ for some $k \in S$. Theorem 3 then implies that $\lambda = \beta$ and $\pi_j = 0$ for all $j \in S$. □

The following lemma will be used to show that certain classes of valid inequalities induce facets of the p-cycle polytope.

Lemma 6. *Let $R \subset N$ with $0 < r < |R|$ and let $s, t \in N \setminus R$. Suppose that $c(P) = \gamma$ for each s,t-path P whose internal nodes are all the nodes of R and $c(\Gamma) = \theta$ for each s,t-path Γ all r of whose internal nodes are in R. Then there exists λ, $\{\pi_j\colon j \in R\}$, π_s and π_t such that $c_{sj} = \lambda + \pi_s - \pi_j$, $c_{ij} = \lambda + \pi_i - \pi_j$ and $c_{it} = \lambda + \pi_i - \pi_t$ for $i, j \in R$.*

Proof. Identifying the nodes s and t in an s,t-path P whose internal nodes are all the nodes of R yields a Hamilton cycle on $|R| + 1 \geqslant 3$ nodes; since all such Hamilton cycles have cost γ, we must have

$$\begin{aligned} c_{sj} &= \alpha_s + \beta_j \quad (j \in R),\\ c_{ij} &= \alpha_i + \beta_j \quad (i, j \in R),\\ c_{it} &= \alpha_i + \beta_t \quad (i \in R) \end{aligned}$$

for some $\{\alpha_i,\ i \in S\}$, $\{\beta_j\colon j \in S\}$, α_s and β_t by Theorem 23 of Grötschel and Padberg [15]. Next, consider an s,t-path Γ all r of whose internal nodes are in R that contains the arcs (i, j) and (j, t) but does not visit node k. Replacing node j by node k yields another such s,t-path, and thus $c_{ij} + c_{jt} = c_{ik} + c_{kt}$ which implies that $\alpha_j + \beta_j = \alpha_k + \beta_k$ for all $j, k \in R$. Letting λ denote this common value yields the result with $\pi_s = \alpha_s$, $\pi_j = \alpha_j$ for $j \in R$ and $\pi_t = \lambda - \beta_t$. $\square$

When studying facet defining inequalities for the p-cycle polytope, it is also important to determine when facets of the p-cycle polytope defined on K_n can be "lifted" to facets of the p-cycle or $(p+k)$-cycle polytope defined on K_{n+k}. In Theorems 8 and 9, we show some conditions under which this lifting is possible for the p-cycle polytope.

Before proving the first lifting result, we cite an earlier result of Balas and Fischetti [4]. In their paper, the result is stated for the ATS polytope; however, the same arguments are valid for the p-cycle polytope. Throughout the rest of our presentation, we will refer to a facet as *regular* if it is induced by an inequality $\boldsymbol{ax} \leqslant a_0$ that is not equivalent to a non-negativity constraint $x_{ij} \geqslant 0$ or a *broom inequality*

$$x(\delta^+(i)) \geqslant x_{ij} + x_{ji} \tag{5}$$

for some $i \neq j \in N$ and we say that a p-cycle P is tight (with respect to $\boldsymbol{ax} \leqslant a_0$) if $a(P) = a_0$.

Lemma 7. *Let $\boldsymbol{ax} \leqslant a_0$ induce a regular facet of Q_p^n, where $2 < p < n$ and $n \geqslant 5$, and let $k \in N$. Then there exists a sequence of $2n-3$ tight p-cycles $P_1, \ldots, P_{2n-3}$, where each p-cycle P_t is associated with an arc $(i_t, j_t) \in \delta^-(k) \cup \delta^+(k)$ as follows: $(i_t, j_t) \in P_t$ but $(i_t, j_t) \notin P_s$ for $s < t$.*

Proof. We will show that the following procedure determines a sequence of tight p-cycles with the desired property.

1. Let $t=0$, $T=\delta^-(k)\cup\delta^+(k)$, and label all the arcs in T *unmarked*.
2. Choose any arc (i_t,j_t) from T and label it *marked*.
3. Find a tight p-cycle P^* containing exactly one *unmarked* arc (i^*,j^*) in T.
4. If no such p-cycle exists, STOP. Otherwise, let $t=t+1$, $P_t=P^*$, $(i_t,j_t)=(i^*,j^*)$, and label (i_t,j_t) *marked*.

Our claim is that the above algorithm stops only when all the arcs in T are marked, producing the sequence of $|T|-1=2(n-1)-1=2n-3$ tight p-cycles.

Assume that this is not the case, and let $T^*\subset T$ be the set of arcs labeled *marked* at the end of the procedure. First, we show that $\emptyset \neq T^*\cap\delta^-(k)\subset\delta^-(k)$ and $\emptyset\neq T^*\cap\delta^+(k)\subset\delta^+(k)$.

The first time Step 2 is performed, the arc (i_0,j_0) is marked, and since the inequality $\boldsymbol{ax}\leqslant a_0$ is not equivalent to a non-negativity constraint, there exists at least one tight p-cycle P^* that contains (i_0,j_0). So the procedure will not stop at the end of the first iteration, which implies that $T^*\cap\delta^-(k)\neq\emptyset$ and $T^*\cap\delta^+(k)\neq\emptyset$. Now if the procedure stops with $T^*\cap\delta^-(k)=\delta^-(k)$ but $(k,j)\notin T^*$ for some $j\neq k$, then there are no tight p-cycles P using arc (k,j), contradicting the hypothesis that $\boldsymbol{ax}\leqslant a_0$ is not equivalent to $x_{kj}\geqslant 0$. Similarly, it cannot be the case that $T^*\cap\delta^+(k)=\delta^+(k)$ but $(i,k)\notin T^*$ for some $i\neq k$, and hence we have shown that there exist marked and unmarked arcs in both $\delta^-(k)$ and $\delta^+(k)$.

Now since each extreme point of the facet induced by $\boldsymbol{ax}\leqslant a_0$ is the incidence vector of a p-cycle, every $\boldsymbol{x}\in Q_p^n$ with $\boldsymbol{ax}=a_0$ satisfies the equality

$$\sum_{(i,k)\in T^*} x_{ik} - \sum_{(k,j)\in T^*} x_{kj} = 0 \tag{6}$$

which must be equivalent to $\boldsymbol{ax}=a_0$. Then one of the inequalities obtained from (6) by replacing the equality sign with $\leqslant$ or $\geqslant$ is satisfied by all $\boldsymbol{x}\in Q_p^n$. This certainly does not hold when there exist both a p-cycle containing some arc $(i,k)\in T^*$ and some arc $(k,l)\notin T^*$ and a p-cycle containing some arc $(l,k)\notin T^*$ and some arc $(k,j)\in T^*$. On the other hand, if a p-cycle with the former property does not exist, then $T^*=\delta^+(k)\cup\{(l,k)\}\setminus\{(k,l)\}$, and non-existence of a p-cycle with the latter property implies $T^*=\delta^-(k)\cup\{(k,l)\}\setminus\{(l,k)\}$ for some node $l\neq k$ since there exist marked and unmarked arcs both in $\delta^-(k)$ and $\delta^+(k)$. In the first case, the equation above becomes

$$x_{lk}-x(\delta^+(k))+x_{kl}=0,$$

and in the second case

$$x(\delta^-(k))-x_{lk}-x_{kl}=0.$$

In both cases, the equation is equivalent to $x(\delta^+(k))=x_{kl}+x_{lk}$, which contradicts the assumption that $\boldsymbol{ax}\leqslant a_0$ is not equivalent to the broom inequality $x(\delta^+(k))\geqslant x_{kl}+x_{lk}$. $\square$

We will call a facet-inducing inequality *primitive* if it cannot be obtained by lifting another inequality, as described by (7) below. The following theorem will reduce the task of showing that a class of regular inequalities determine facets of the p-cycle polytope to showing that the primitive inequalities in the class determine facets. We define a *p-bowtie* to be a connected subgraph of K_n with p arcs composed of two simple cycles connected at exactly one node. The p-bowtie is said to be *tied at* node k if the intersection of the simple cycles is node k.

Theorem 8. *Suppose $\boldsymbol{ax} \leqslant a_0$ induces a regular facet of Q_p^n, where $2 < p < n$ and $n \geqslant 5$. Let k be a node such that $a(B) \leqslant a_0$ for all p-bowties B tied at node k (for $p = 3$, k can be chosen arbitrarily) and let δ_k be the maximum of $a(\Gamma)$ over all $(p-1)$-cycles Γ that visit node k. Then*

$$\boldsymbol{ax} + \sum_{i \neq k} a_{ik} x_{i,n+1} + \sum_{j \neq k} a_{kj} x_{n+1,j} + (a_0 - \delta_k) x_{k,n+1} + (a_0 - \delta_k) x_{n+1,k} \leqslant a_0 \qquad (7)$$

induces a facet of Q_p^{n+1}.

Proof. We first argue that (7) is valid for Q_p^{n+1}. Since node $n+1$ is essentially a copy of node k, the lifted inequality (7) will be satisfied by any p-cycle that does not contain both node k and node $n+1$. If a p-cycle contains either of the arcs $(k, n+1)$ or $(n+1, k)$, then contracting these arcs yields a $(p-1)$-cycle and the validity of (7) follows from the definition of δ_k. If a p-cycle visits both node k and node $n+1$ but does not contain either of the arcs $(k, n+1)$ or $(n+1, k)$, then contracting these arcs yields a p-bowtie B tied at node k, and the validity of (7) follows from $a(B) \leqslant a_0$.

Next, we show that (7) induces a facet of Q_p^{n+1}. In the following, we will loosely say that a collection of p-cycles is linearly independent when it is actually their incidence vectors that are linearly independent.

Since every tight p-cycle in N remains tight after lifting, we have $n(n-2)$ affinely independent p-cycles that are tight with respect to (7). To these, we must add $(n+1)(n-1) - n(n-2) = 2n-1$ more affinely independent tight p-cycles that are affinely independent of these $n(n-2)$ tight p-cycles.

By Lemma 7, there exist $2n-3$ affinely independent tight p-cycles that visit node k. For each such p-cycle, replace the arcs $(i,k),(k,j)$ with $(i,n+1),(n+1,j)$ to obtain $2n-3$ new affinely independent tight p-cycles that visit node $n+1$. Also, let Γ be a $(p-1)$-cycle with $a(\Gamma) = \delta_k$ that contains the arcs $(i,k),(k,j)$ for some $i,j \in N$. Construct the p-cycles P_{2n-2} and P_{2n-1} by replacing (i,k) by $(i,n+1),(n+1,k)$ and (k,j) by $(k,n+1),(n+1,j)$, respectively. From the definition of δ_k, P_{2n-2} and P_{2n-1} are both tight p-cycles, which are also affinely independent of the previously produced $n(n-2) + 2n - 3$ tight p-cycles.

This yields the desired number of affinely independent p-cycles so (7) induces a facet of Q_p^{n+1}. $\square$

Lifted inequality (7) is obtained by copying the coefficient structure of node k, so we refer to this process as "lifting by cloning node k". In addition to showing that $\boldsymbol{ax} \leqslant a_0$ is satisfied by all p-bowties tied at node k, we must also show that it induces a regular facet of Q_p^n. Note that if $\boldsymbol{ax} \leqslant a_0$ is equivalent to either $x_{ij} \geqslant 0$ or $x(\delta^+(i)) \geqslant x_{ij} + x_{ji}$, all p-cycles that do *not* visit node i are tight with respect to $\boldsymbol{ax} \leqslant a_0$. Thus it suffices to show that for *each* node $k \in N$, there is a non-tight p-cycle that does not visit node k. If a primitive inequality satisfies this condition, then so do all the inequalities obtained through lifting it, since none of the non-tight p-cycles on N visit the cloned node $n+1$.

The next theorem gives conditions under which it is possible to lift a family of facet-inducing inequalities for Q_p^n to a family of facet-inducing inequalities for Q_{p+k}^{n+k}. We will need the following definitions before the statement of the theorem: if F is a subset of the arcs of the complete graph K_n, then the *auxiliary graph* G_F is an undirected bipartite graph on $2n$ nodes with the property that $(i,j) \in F$ if and only if G_F contains the undirected arc $(i, n+j)$. We also define an equivalence relation on the arcs of the complete graph as the transitive closure of the following relation: two arcs (i,j) and (k,l) are related with respect to an inequality $\boldsymbol{ax} \leqslant a_0$, if $a_{ij} = a_{kl}$ and there exists a tight p-cycle that uses both of them.

Theorem 9. *Let $\boldsymbol{ax} \leqslant a_0$ with $\boldsymbol{a} \geqslant \boldsymbol{0}$ be facet-inducing for Q_p^n, where $2 < p < n$ and $n \geqslant 5$. Suppose that the auxiliary graph G_Z for the arc set $Z = \{(i,j)\colon a_{ij} = 0\}$ is connected, every tight p-cycle with respect to $\boldsymbol{ax} \leqslant a_0$ contains at least one arc $(i,j) \in Z$, and every arc $(i,j) \in Z$ belongs to the same equivalency class with respect to $\boldsymbol{ax} \leqslant a_0$. Let R be a set of nodes, let $q = p + |R|$, and let t be the smallest number such that*

$$\boldsymbol{ax} + t \sum_{j \in R} x(\delta^+(j)) \leqslant a_0 + |R|t \tag{8}$$

is valid for all q-cycles on $N \cup R$, and if $|R| \geqslant 2$ suppose further that at least one tight q-cycle with respect to (8) *visits exactly r nodes in R for some $0 < r < |R|$. Then* (8) *is facet-inducing for the q-cycle polytope on $N \cup R$.*

Proof. If $|R| \geqslant 2$, we will assume that the nodes in N are numbered so that there is a tight q-cycle Γ with respect to (8) that contains the arc $(1, v)$ for some $v \in R$ and does not visit some other node $u \in R$. If $|R| \geqslant 3$, we will assume further that *all* of the nodes in R visited by Γ are internal nodes on a path from 1 to j for some $j \in N$. If this is not the case, then Γ contains another path of length at least two, say from node i to node k, all of whose internal nodes are in R. Replacing this path by the arc (i,k) and adding its internal nodes to the path from 1 to j results in a q-cycle Γ' for which the left-hand side of (8) increases by $a_{ik} \geqslant 0$. Therefore, Γ' is also a tight q-cycle, and the process can be repeated until all of the nodes in R visited by Γ are internal nodes on a path from 1 to j.

Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_q^{n+|R|}$ that satisfies (8) with equality. Consider any p-cycle P on N which is tight with respect to $\boldsymbol{ax} \leqslant a_0$. By hypothesis, there exists an arc $(i,j) \in Z$ on P and replacing (i,j) by a path P_{ij} from i to j whose internal nodes are all of the nodes in R we obtain a q-cycle Γ which is tight with respect to (8). This implies that

$$c(P) + c(P_{ij}) - c_{ij} = c_0.$$

Now if (i,j) and (k,l) are two arcs in Z that are contained in the same tight p-cycle, we must have

$$c(P_{ij}) - c_{ij} = c(P_{kl}) - c_{kl}.$$

Then since every arc $(i,j) \in Z$ belongs to the same equivalence class with respect to $\boldsymbol{ax} \leqslant a_0$, for some β we have $c(P_{ij}) - c_{ij} = \beta$ for all arcs $(i,j) \in Z$. Consequently, $c(P) = c_0 - \beta$ for every tight p-cycle P on N. Theorem I.4.3.6 of Nemhauser and Wolsey [20] implies that $\boldsymbol{cx} = c_0$ is equivalent to an equation $\boldsymbol{dx} = d_0$ such that $d_{ij} = \mu a_{ij}$ for all $i \neq j \in N$. Using node potentials π_v for $v \in R$ we may further assume that $d_{1v} = 0$ for $v \in R$. Together with the fact that $c(P_{ij}) - c_{ij} = \beta$, this implies there exists γ such that $d(P_{ij}) = \gamma$ for all arcs $(i,j) \in Z$ and paths P_{ij} from i to j whose internal nodes are all of the nodes in R.

If $|R| = 1$, then the fact that G_Z contains a spanning tree and $d_{1v} = 0$ implies that $d_{iv} = 0$ and $d_{vj} = \gamma$ for all $i,j \in N$, where $R = \{v\}$. Using the fact that there must be a tight $(p+1)$-cycle with respect to (8) on N, we conclude that $d_0 = \mu(a_0 + t)$ and $\gamma = \mu t$.

Next suppose that $|R| \geqslant 2$. As in the case $|R| = 1$, the fact that G_Z contains a spanning tree and $d_{1v} = 0$ for $v \in R$ implies that $d_{iv} = 0$ for all $i \in N$ and $v \in R$. By assumption, there is a tight q-cycle for (8) that contains a path from 1 to j all r of whose internal nodes are in R for some $0 < r < |R|$, and hence every path Γ_{1j} from 1 to j all r of whose internal nodes are in R must have $d(\Gamma_{1j}) = \theta$ for some θ. Now G_Z is connected, so there must be an arc $(i,j) \in Z$ for some $i \neq j$. Since $d_{iv} = d_{1v} = 0$ for all $v \in R$, every path Γ_{ij} from i to j all r of whose internal nodes are in R must also have $d(\Gamma_{ij}) = \theta$. Since all paths P_{ij} from i to j whose internal nodes are all of the nodes in R have $d(P_{ij}) = \gamma$, Lemma 6 implies that there exists $\lambda, \{\pi_u\colon u \in R\}, \pi_i$ and π_j such that $d_{iv} = \lambda + \pi_i - \pi_v$, $d_{uv} = \lambda + \pi_u - \pi_v$ and $d_{uj} = \lambda + \pi_u - \pi_j$ for $u,v \in R$. Without loss of generality, we may set $\pi_i = 0$. Then the fact that $d_{iv} = 0$ for all $v \in R$ implies that $\pi_v = \lambda$ for all $v \in R$. Hence $d_{uv} = \lambda$ for all $u,v \in R$ and $d_{uj} = \gamma - (|R| - 1)\lambda$ for all $u \in R$ and $j \in N$.

The fact that these q-cycles are tight and $d_{ij} = \mu a_{ij}$ for all $i \neq j \in N$ then implies that $d_0 = \mu a_0 + \gamma$ and

$$d_0 = \mu(a_0 + |R|t - rt) + (r-1)\lambda + \gamma - (|R| - 1)\lambda = \mu a_0 + \gamma + (|R| - r)(\mu t - \lambda)$$

which implies that $\lambda = \mu t$. Finally, consider a tight p-cycle P with respect to $\boldsymbol{ax} \leqslant a_0$ that contains two arcs (i,j) and $(k,l) \in Z$. Replacing the arc (i,j) by the path $(i,v),(v,j)$ and the arc (k,l) by a path from k to l whose internal nodes are all of the nodes in R except v we obtain a q-cycle that is tight with respect to (8), and hence

$$\begin{aligned}\mu a_0 + \gamma &= \mu a_0 + \gamma - (|R| - 1)\mu t + (|R| - 2)\mu t + \gamma - (|R| - 1)\mu t \\ &= \mu a_0 + 2\gamma - |R|\mu t\end{aligned}$$

and hence $d_{vj} = \mu t$ for all $v \in R$ and $j \in N$. □

Table 1
Complete description of the 3-cycle polytope when $n=4$

$x(\delta^+(j)) - x(\delta^-(j)) = 0$	$(j \in N)$
$x(N : N) = 3$	
$x(S : N \setminus S) = 1$	$(S \subset N$ with $\|S\| = 2)$
$x(\delta^+(j)) \leqslant 1$	$(j \in N)$
$x_{ij} \geqslant 0$	$(i \neq j)$

Table 2
Complete description of the 3-cycle polytope when $n=5$

$x(\delta^+(j)) - x(\delta^-(j)) = 0$	$(j \in N)$
$x(N : N) = 3$	
$x(S : N \setminus S) \leqslant 1$	$(S \subset N$ with $\|S\| = 2)$
$x(S : T) + x_{ik} + x_{kj} - x_{ji} \leqslant 1$	(partitions $\langle S, T, \{k\}\rangle$ of N with $i \in S$, $j \in T$ and $\|S\| = \|T\| = 2)$
$x(\delta^+(k)) + x(S : T) \geqslant 1$	(partitions $\langle \{k\}, S, T\rangle$ of N with $\|S\| = \|T\| = 2)$
$x_{ij} \geqslant 0$	$(i \neq j)$

Note: Lifted inequality (8) cannot be a facet when $|R| \geqslant 2$ and every tight q-cycle that visits any node in R must visit all of them. (This can only happen when $t > 0$, since if $t = 0$ any tight p-cycle that contains two arcs in Z can be converted to a tight q-cycle that visits $|R| - 1$ nodes in R.) In this case, for any $k, l \in R$ the inequality

$$\boldsymbol{ax} + \varepsilon x(\delta^+(k)) - \varepsilon x(\delta^+(l)) + t \sum_{j \in R} x(\delta^+(j)) \leqslant a_0 + |R| t$$

is valid for $\varepsilon > 0$ sufficiently small. Clearly (8) is implied by this class of inequalities.

3. Facets and valid inequalities

In this section, we will describe classes of inequalities which induce facets of the p-cycle polytope Q_p^n; throughout, we assume that $2 < p < n$. To prove that a valid inequality $\boldsymbol{ax} \leqslant a_0$ is facet-inducing we generally use the "indirect method," which amounts to showing that any equation $\boldsymbol{cx} = c_0$ satisfied by every $\boldsymbol{x} \in Q_n^p$ with $\boldsymbol{ax} = a_0$ must have $\boldsymbol{c} = \mu \boldsymbol{a} + \boldsymbol{\pi A} + \lambda \mathbf{1}$ for some $\mu, \boldsymbol{\pi}$ and λ (see [20, Theorem I.4.3.6]). Inferences are made about the coefficients c_{ij} using the fact that $c(P) = c_0$ for every p-cycle P that is tight with respect to $\boldsymbol{ax} = a_0$.

One may hope to obtain a complete characterization of the p-cycle polytope Q_p^n for any fixed p, but such a description will be highly complex even for $p = 6$ (since the same is true for the ATS polytope on 6 nodes, see [15]). In Tables 1 and 2 we list the facet-inducing inequalities for the 3-cycle polytope when $n = 4$ and $n = 5$ as generated by PORTA. We also used PORTA to generate a complete description of the 4-cycle polytope when $n = 5$, but do not have compact representations for the 809 inequalities that induce facets of Q_4^5, so we will not be listing them here; however, 20 of these are non-negativity constraints (9), 5 are degree constraints (10), and 10 are min-cut inequalities (11) with $|S| = 2$.

3.1. Trivial inequalities

The most natural valid inequalities for the p-cycle polytope are the non-negativity and degree constraints. In what follows we will investigate the conditions under which these inequalities induce facets of the p-cycle polytope.

Theorem 10. *The non-negativity constraint*

$$x_{ij} \geqslant 0 \tag{9}$$

is valid for the p-cycle polytope Q_p^n and induces a facet of Q_p^n whenever $p \geqslant 3$ and $n \geqslant p + 1$.

Proof. When $n \leqslant 5$ and $p = 3$ or $p = 4$, (9) can be seen to determine a facet using PORTA, so assume that $p \geqslant 3$ and $n \geqslant 6$. Without loss of generality, we will show that $x_{12} \geqslant 0$ induces a facet. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ with $x_{12} = 0$. Using Corollary 4, we may assume that $c_{21} = 0$ and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on $S = \{2, 3, \ldots, n\}$.

Let i, j, k, l be distinct nodes in S, and let P be a p-cycle that contains the arcs $(i, k), (k, j)$ but does not visit node l or use the arc $(1, 2)$; such a p-cycle exists even if $i = 2$ because $|S| \geqslant 5$. Replacing node k by node l yields another p-cycle that does not use $(1, 2)$, and hence condition (ii) of Lemma 2 holds when restricted to S and Corollary 5 implies that $c_{ij} = 0$ for $i, j \geqslant 2$ which also implies that $c_0 = 0$. Each p-cycle that uses the arc $(2, 1)$ also satisfies (9) with equality, so $c_{21} + c_{1j} = 0$ for all $j \geqslant 3$. Here the fact that $c_{21} = 0$ implies that $c_{1j} = 0$ for all $j \geqslant 3$; similarly, $c_{i1} + c_{1j} = 0$ for all $i \neq j$ with $i \geqslant 3$ and therefore $c_{ij} = 0$ for all arcs $(i, j) \neq (1, 2)$. Hence $\boldsymbol{cx} = c_0$ is simply $c_{12}x_{12} = 0$. □

The degree constraints state that a p-cycle can visit each node at most once; in the following argument, they are shown to be facet-inducing for Q_p^n unless $p = 3$ and $n \geqslant 5$.

Theorem 11. *The degree constraint*

$$x(\delta^+(j)) \leqslant 1 \tag{10}$$

is valid for the p-cycle polytope Q_p^n and induces a facet of Q_p^n if and only if $p = 3$ and $n = 4$ or $p \geqslant 4$ and $n \geqslant p + 1$.

Proof. When $p = 3$ and $n = 4$, the degree constraint (10) can be seen to induce a facet using PORTA. When $p = 3$ and $n \geqslant 5$, the degree constraint (10) is can be obtained by summing the $n - 1$ one-sided min-cut inequalities (12) with $|S| = 2$ and $j \in S$ (which do not induce the same facet of Q_3^n), subtracting the flow conservation constraint $x(\delta^+(j)) - x(\delta^-(j)) = 0$ and the cardinality constraint $x(N : N) = 3$, and dividing by three.

When $p = 4$ and $n = 5$, degree constraint (10) can be seen to induce a facet using PORTA, so assume that $p \geqslant 4$ and $n \geqslant 6$. Without loss of generality let $j = 1$. First, note that every p-cycle that goes through node 1 satisfies (10) with equality. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ that satisfies (10) with equality. Using Corollary 4, we may assume that $c_{21} = 0$ and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on the subset $S = \{2, 3, \ldots, n\}$.

Let i, j, k, l be distinct nodes in S, and let P be a p-cycle that visits node 1 and contains the arcs $(i, k), (k, j)$ but does not visit node l. Replacing node k by node l yields another p-cycle that visits node 1, and hence condition (ii) of Lemma 2 holds when restricted to S. Since $|S| \geqslant 5$, Corollary 5 implies that $c_{ij} = 0$ for $i, j \geqslant 2$. Next by considering p-cycles that use the arc (2,1), we see that $c_{1j} = c_0$ for all $j \geqslant 3$. This in turn implies that $c_{i1} = 0$ for all $i \geqslant 2$ and hence also $c_{12} = c_0$. Thus $\boldsymbol{cx} = c_0$ is simply $c_0 x(\delta^+(1)) = c_0$. $\square$

3.2. Cut inequalities

In this section, we introduce several inequalities that give upper and lower bounds on the intersection of a p-cycle and a directed cutset, which depend on the size of the shores of the cutset. The first three families of inequalities presented here establish lower bounds. They can often be used to separate over points $\boldsymbol{x}^*$ whose support graph is disconnected (see [16]).

Theorem 12. *Let $S \subset N$ with $|S| \leqslant p - 1$ and $|N \setminus S| \leqslant p - 1$. The min-cut inequality*

$$x(S : N \setminus S) \geqslant 1 \tag{11}$$

is valid for the p-cycle polytope Q_p^n and induces a facet of Q_p^n if and only if $p \geqslant 4$ and $p + 1 \leqslant n \leqslant 2p - 2$.

Proof. To see that (11) is valid, note that no p-cycle can be contained in S or $N \setminus S$. When $p = 3$ and $n = 4$, min-cut inequalities (11) are implicit equations.

In order to apply Theorem 8, we must show that min-cut inequalities (11) are regular; to this end, assume without loss of generality that $|S| \leqslant |N \setminus S|$. If $|S| = 2$, then for each arc (i, j), it is easy to see that there is a tight p-cycle containing (i, j) so (11) is not equivalent to $x_{ij} \geqslant 0$. Also, for each pair of nodes i, j there is a tight p-cycle that contains node i but does not contain node j, so (11) is not equivalent to $x(\delta^+(i)) \geqslant x_{ij} + x_{ji}$. If $|S| \geqslant 3$, it is easy to see that for each node k there is a non-tight p-cycle that does not visit node k. When $|S| < p - 1$, (11) is satisfied by the incidence vectors of all p-bowties tied at node $k \in S$ (and symmetrically for $N \setminus S$). Hence, we can clone nodes in S and $N \setminus S$ to prove the result provided that primitive min-cut inequalities (11) induce facets of Q_p^n.

The only primitive inequalities are those with $p=n-1$. Consider such an inequality and without loss of generality let $1,2 \in S$ and $|S| \leqslant |N \setminus S|$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ with that satisfies (11) with equality. Using Corollary 4, we may assume that $c_{12}=0$ and $c_{i1}=0$ for all $i \geqslant 2$. Since $p \geqslant 4$, $|N \setminus S| \geqslant 3$ and so there must be tight p-cycles with respect to (11) that visit a node $k \in N \setminus S$ followed by all $|S|$ (or any $|S|-1$) nodes in S and a different node $l \in N \setminus S$. Since $|S| \geqslant 2$, Lemma 6 implies that

$$c_{kj} = \lambda + \pi_k - \pi_j \quad (j \in S),$$

$$c_{ij} = \lambda + \pi_i - \pi_j \quad (i,j \in S),$$

$$c_{il} = \lambda + \pi_i - \pi_l \quad (i \in S)$$

for some λ, $\{\pi_j\colon j \in S\}$, π_k and π_l. Without loss of generality, let $\pi_1 = 0$. Theorem 3 then implies that $\lambda = 0$ and $\pi_i = 0$ for all $i \in S$, and $c_{k1} = 0$ implies that $\pi_k = 0$. Therefore $c_{ij}=0$ for all $i,j \in S$ and $c_{kj}=0$ for all $k \in N \setminus S$ and $j \in S$. Next, consider a tight p-cycle that skips node $k \in N \setminus S$ and uses the arcs (i,j), $(j,1)$ for some $i \in N \setminus S$ and $j \in S$. Replacing these arcs by the arcs (i,k), $(k,1)$ we obtain another tight p-cycle, and thus $c_{ik} = c_{ij} + c_{j1} - c_{k1} = 0$ for all $i,k \in N \setminus S$. It is then easy to see that $\boldsymbol{cx} = c_0 x(S : N \setminus S)$. □

Note: Min-cut inequalities can be generalized to linear ordering inequalities (17), which are shown to be facet inducing under certain conditions in Theorem 16.

Next, we derive one- and two-sided min-cut constraints that allow the restrictive conditions on $|S|$ and $|N \setminus S|$ to be relaxed. The following theorem yields a class of facet-inducing inequalities that dominate the subtour elimination constraints (3).

Theorem 13. *Let $S \subset N$ with $2 \leqslant |S| \leqslant p-1$. The one-sided min-cut inequality*

$$x(S : N \setminus S) \geqslant x(\delta^+(j)) \tag{12}$$

is valid for the p-cycle polytope Q_p^n for all $j \in S$, and facet-inducing for Q_p^n if and only if $p=3$ and $n \neq 5$ or $p \geqslant 4$ and $|N \setminus S| \geqslant p$.

Proof. It is easy to see that (12) is valid, because no p-cycle can be contained in S.

If $|N \setminus S| \leqslant p-1$, then one-sided min-cut inequality (12) is implied by the degree constraint $x(\delta^+(j)) \leqslant 1$ and min-cut inequality (11); however, when $p=3$ and $n=4$, the min-cut inequality is an implicit equation, and the one-sided min-cut inequality (12) induces the same facet as degree constraint (10). When $p=3$ and $n=5$, the one-sided min-cut inequality (12) with $S=\{i,j\}$ can be obtained by summing the three max-cut inequalities $x(S : N \setminus S) \leqslant 1$ with $S=\{j,k\}$ and $k \neq i,j$ (which do not induce the same facet of Q_3^5) and subtracting the flow conservation constraint $x(\delta^+(j)) - x(\delta^-(j)) = 0$ and the cardinality constraint $x(N : N) = 3$.

So suppose that either ($p=3$, $|S|=2$ and $n \geqslant 6$) or ($p \geqslant 4$ and $|N \setminus S| \geqslant p$). Without loss of generality assume that $j=1$ and let $R = S \setminus \{1\}$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ that satisfies (12) with equality. Using Corollary 4, we may assume that $c_{i1} = 0$ for all $i \in R$ and $c_{ij} = 0$ for all arcs (i, j) in some unbalanced 1-tree on $N \setminus R$. Since $|N \setminus R| \geqslant 5$, $|N \setminus R| \geqslant p+1$ and every p-cycle on $N \setminus R$ is tight, Corollary 5 implies that $c_{ij} = 0$ for all $i, j \in N \setminus R$ which also implies that $c_0 = 0$.

Since every tight p-cycle P that contains the arcs (k, i), $(i, 1)$ for $k \in N \setminus S$ and $i \in R$ has cost $c(P) = 0$ and $c_{i1} = 0$, we see that $c_{ki} = 0$ for all $k \in N \setminus S$ and $i \in R$. The reversals of these p-cycles are also tight, so $c_{1i} + c_{ik} = 0$ for all $i \in R$ and $k \in N \setminus S$. If $|R| = 1$, this implies that $\boldsymbol{cx} = c_0$ is simply $c_{1i}x(\delta^+(1)) - c_{1i}x(S : N \setminus S) = 0$.

If $|R| \geqslant 2$, consider a tight p-cycle P that contains the arcs (k, i), (i, j), $(j, 1)$ for some $k \in N \setminus S$ and $i \neq j \in R$ (such p-cycles exist because $p \geqslant 4$). Since $c_{ki} = c_{j1} = 0$ and $c(P) = 0$, we see that $c_{ij} = 0$ for all $i \neq j \in R$. Now considering tight p-cycles that contain the arcs $(1, i)$, (i, j), (j, k) for some $i \neq j \in R$ and $k \in N \setminus S$, we see that $c_{1i} + c_{jk} = 0$ and hence $\boldsymbol{cx} = c_0$ is again $c_{1i}x(\delta^+(1)) - c_{1i}x(S : N \setminus S) = 0$ for some $i \in R$. $\square$

Notes:. The one-sided min-cut inequality (12) is equivalent to $x(S : S) \leqslant \sum_{i \in S \setminus \{j\}} x(\delta^+(i))$. Together with the degree constraints (10) for $i \in S \setminus \{j\}$ this yields subtour elimination constraint (3), thus showing that it determines a lower dimensional face. When $S = \{i, j\}$, one-sided min-cut inequality (12) is equivalent to broom inequality (5). In Özlük [22], broom inequalities are generalized to *single-tooth comb inequalities*,

$$x(S : N \setminus S) \geqslant x_{ij} + x_{ji} \tag{13}$$

which are valid for the p-cycle polytope Q_p^n for all $i \in S$, $j \in N \setminus S$ and $p \geqslant 3$. For $1 < |S| < n$, (13) induces a facet of Q_p^n if and only if ($p = 4$, $|S| \geqslant 5$ and $|N \setminus S| \geqslant 5$) or ($p \geqslant 5$, $|S| \geqslant p$, $|N \setminus S| \geqslant p$ and $n \geqslant 2p+1$).

In the following theorem, we consider a class of inequalities that are shown to be facet-inducing for the cycle polytope by Balas [2].

Theorem 14. *Let $S \subset N$ with $i \in S$ and $j \in N \setminus S$. The two-sided min-cut inequality*

$$x(S : N \setminus S) \geqslant x(\delta^+(i)) + x(\delta^+(j)) - 1 \tag{14}$$

is valid for the p-cycle polytope Q_p^n and facet inducing for Q_p^n if and only if $p \geqslant 4$, $|S| \geqslant p$ and $|N \setminus S| \geqslant p$.

Proof. The validity of (14) follows from the fact that $Q_p^n \subset Q^n$. When $p = 3$, the two-sided min-cut inequality is a consequence of the single-tooth comb inequality (13) and the max-cut inequality (16) with $S = \{i, j\}$, which can be stated as $x_{ij} + x_{ji} \geqslant x(\delta^+(i)) + x(\delta^+(j)) - 1$. If $|S| \leqslant p-1$, then (14) is a consequence of the one-sided min-cut inequality $x(S : N \setminus S) \geqslant x(\delta^+(i))$ and the degree constraint $x(\delta^+(j)) \leqslant 1$; if $|N \setminus S| \leqslant p-1$, then (14) is a consequence of the equation $x(S : N \setminus S) = x(N \setminus S : S)$, the one-sided min-cut inequality $x(N \setminus S : S) \geqslant x(\delta^+(j))$ and the degree constraint $x(\delta^+(i)) \leqslant 1$.

Without loss of generality, assume that $S = \{1, 2, \ldots, s\}$ and $N \setminus S = \{s+1, \ldots, n\}$ for some $p \leqslant s \leqslant n-p$, $i=1$ and $j=n$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ that satisfies (14) with equality. Now restricted to the node set $R = \{1\} \cup N \setminus S$, two-sided min-cut inequality (14) is equivalent to the degree constraint $x(\delta^-(n)) \leqslant 1$, which induces a facet of Q_p^n for $p \geqslant 4$ and $n \geqslant p+1$. Theorem I.4.3.6 of Nemhauser and Wolsey [20] implies that $\boldsymbol{cx} = c_0$ is equivalent to an equation $\boldsymbol{dx} = d_0$ such that $d_{in} = \mu$ for $i \in R$ with $i < n$ and $d_{ij} = 0$ for $i, j \in R$ with $j < n$. Using node potentials π_j for $j \notin R$ we may further assume that $d_{1j} = \mu$ for all $j \in S$ with $j > 1$.

Observe that every tight p-cycle visits either node 1 or node n, and crosses over the directed cutset exactly once if it visits both nodes. Since there are tight p-cycles on $N \setminus S$, we must have $d_0 = \mu$. Considering a tight p-cycle that uses arcs $(1, j)$, (j, k) where $j \in S$ and $k \neq n \in N \setminus S$, we get $d_{jk} = -\mu$ for all $j \in S \setminus \{1\}$ and $k \in N \setminus S$ with $k < n$. Considering a tight p-cycle that uses the arcs $(1, i)$, (i, n), (n, j) and $(k, 1)$ where $i \in S$ and $j, k \in N \setminus S$, we obtain $d_{in} = 0$ for all $i \in S \setminus \{1\}$. (Note that for $p = 4$, we have $j = k$.) Next, consider a tight p-cycle that uses arcs $(1, i)$, (i, j), (j, n) where $i, j \in S$, to get $d_{ij} = 0$ for all $i, j \in S \setminus \{1\}$.

From the tight p-cycles on S, we obtain $d_{i1} = 0$ for all $i \in S$. Next, consider a tight p-cycle that uses arcs (i, j), $(j, 1)$ where $i \in N \setminus S$ with $i < n$ and $j \in S$ to get $d_{ij} = 0$ for all $i \in N \setminus S$ with $i < n$ and $j \in S \setminus \{1\}$. Finally, considering the tight p-cycles that visit only node n and nodes from S, we obtain $d_{nj} = d_{jn} = 0$ for all $j \in S \setminus \{1\}$. Thus, for $p \geqslant 4$, $\boldsymbol{dx} = d_0$ is simply $\mu x(\delta^+(1)) - \mu x(S : N \setminus S) + \mu x(\delta^-(n)) = \mu$. □

We obtain another class of facet-inducing inequalities by generalizing the observation that the intersection of a p-cycle and a directed cutset can contain at most $\lfloor p/2 \rfloor$ arcs.

Theorem 15. *Let $\langle R, S, T \rangle$ be a partition of N. The generalized max-cut inequality*

$$x(S : T) + \sum_{i \in R} x(\delta^+(i)) \leqslant \lfloor (p + |R|)/2 \rfloor \tag{15}$$

is valid for the p-cycle polytope Q_p^n for $p \geqslant 3$ and facet-inducing for Q_p^n if and only if $n \geqslant 5$, $p + |R|$ is odd, $|R| \leqslant p-3$, $|S| > (p - |R|)/2$ and $|T| > (p - |R|)/2$.

Proof. We first show that (15) is valid. Since $x(N : N) = p$ and $x(S : T) \leqslant x(T : N \setminus T)$, we must have $2x(S : T) + \sum_{i \in R} x(\delta^+(i)) \leqslant p$. Adding the inequalities $x(\delta^+(i)) \leqslant 1$ for all $i \in R$, dividing by two and rounding down yields (15). When $p + |R|$ is even, we obtain (15) with no rounding, and hence the inequality does not induce a facet in this case.

It is easy to see that if $|S| \leqslant (p - |R|)/2$ or $|T| \leqslant (p - |R|)/2$, then (15) is a consequence of the degree constraints $x(\delta^+(i)) \leqslant 1$. If $p < |R|$, then (15) is a consequence of the cardinality constraint $x(N : N) = p$ and the non-negativity constraints. When $p = |R| + 1$, (15) is implied by the non-negativity constraints and any linear ordering inequality (17) with $T = N_1 \cup N_2 \cup \cdots \cup N_k$, $S = N_{k+1} \cup \cdots \cup N_{t-1}$ and $R = N_t$. Therefore $p \geqslant |R| + 3$ whenever (15) induces a facet of Q_p^n, since $p - |R|$ is odd. Also note that together $|S| > (p - |R|)/2$ and $|T| > (p - |R|)/2$ imply that $p < n$.

First, we prove that (15) is facet-inducing when $R=\emptyset$. In this case, $p=2q+1$ for some $q\geqslant 1$ and we call the resulting inequality

$$x(S : N\setminus S)\leqslant \lfloor p/2\rfloor = q \tag{16}$$

a *max-cut inequality*. If $p=3$ and $|S|=|T|=2$, (16) is actually an implicit equation. When $p=3$ and $n\geqslant 5$, we will show that (16) induces a facet using Theorem 8; the only primitive inequalities have $n=5$ and (without loss of generality) $|S|=2$. In this case (16) induces a facet of Q_3^5 (see Table 2), and is regular: it cannot be equivalent to a broom inequality, because broom inequalities do not include facets of Q_3^5; and for each arc (i,j), there is a tight 3-cycle that contains (i,j), so it cannot be equivalent to a non-negativity constraint. Therefore, we may assume that $q\geqslant 2$. Without loss of generality, assume $S=\{1,2,\ldots,s\}$ and $N\setminus S=\{s+1,\ldots,n\}$ for some $3\leqslant s\leqslant n-3$.

Suppose that $\boldsymbol{cx}=c_0$ is satisfied by every $\boldsymbol{x}\in Q_p^n$ that satisfies (16) with equality. Using Corollary 4, we may assume that $c_{1j}=1$ for all $j\in N\setminus S$, $c_{j1}=0$ for all $j\in S\setminus\{1\}$ and $c_{sn}=1$.

First, consider any $2q$-cycle Γ that alternates between nodes in S and nodes in $N\setminus S$, but does not visit node 1. Replacing any arc $(i,j)\in\Gamma$ with $i\in S$ and $j\in N\setminus S$ by the arcs $(i,1)$ and $(1,j)$ we obtain a tight p-cycle, and hence $c(\Gamma)-c_{ij}=c_0$. Since $s\geqslant 3$ and $c_{sn}=1$, this implies that $c_{ij}=1$ for all $i\in S\setminus\{1\}$ and $j\in N\setminus S$.

Now consider a tight p-cycle that uses the arcs (i,k), (k,j) form some $i,j\in S$ and $k\in N\setminus S$, but that does not visit node $l\in N\setminus S$. If we replace node k by node l, we obtain another tight p-cycle, and hence $c_{ik}+c_{kj}=c_{il}+c_{lj}$. Interchanging the roles of S and $N\setminus S$, we see that $c_{ki}+c_{il}=c_{kj}+c_{jl}$ and hence $c_{ik}+c_{ki}=c_{lj}+c_{jl}$ for all $i,j\in S$ and $k,l\in N\setminus S$. Since $s\geqslant 3$ and $c_{ik}=c_{jl}=1$, this implies that $c_{ki}=\sigma$ for all $k\in N\setminus S$ and $i\in S$, for some σ.

Finally, evaluating the cost of tight p-cycles yields $c_{ij}=c_0-q(1+\sigma)$ for $i,j\in S$ or $i,j\in N\setminus S$, which for $i=s$ and $j=1$ yields $c_0=q(1+\sigma)$. Adding σ times $x(S : N\setminus S)-x(N\setminus S : S)=0$, we see that $\boldsymbol{cx}=c_0$ is equivalent to $(1+\sigma)x(S : N\setminus S)=q(1+\sigma)$.

When $R\neq\emptyset$, we can prove that (15) is facet-inducing by verifying that the conditions of Theorem 9 hold for (16) when $R=\emptyset$. First of all, note that the auxilliary graph G_Z contains the arcs $(i,n+j)$ for all $i\in T$ and $j\in N$ and for each $k\in S$, G_Z contains the arc $(k,n+j)$ for some $j\in S$ with $j\neq k$. Hence G_Z is connected. Also, every tight p-cycle contains $q+1\geqslant 2$ arcs in Z. Since there are tight p-cycles that contain the arcs (i,j), (j,k) and (j,k), (k,l) for $i,j\in T$ and $k,l\in S$, the arcs (i,j) and (k,l) and hence every arc in Z belongs to the same equivalency class with respecet to (16). Since there are tight p-cycles with respect to (15) that visit $|R|-1$ of the nodes in R, Theorem 9 implies that (15) induces a facet of the p-cycle polytope unless $p=n-1$ and $|S|=|T|=2$.

Finally, suppose that $p=n-1$ and $|S|=|T|=2$. When $n=5$, (15) can be seen to be facet-inducing using PORTA. Without loss of generality, assume that $S=\{1,2\}$, $T=\{3,4\}$ and $R=\{5,6,\ldots,n\}$ for some $n\geqslant 6$. Suppose that $\boldsymbol{cx}=c_0$ is satisfied by every $\boldsymbol{x}\in Q_p^n$ that satisfies (15) with equality. Using Corollary 4, we may assume that $c_{nj}=1$

for all $j \in R \cup S$, $c_{3n} = c_{4n} = 0$ and $c_{n-1,n} = 1$. There are tight p-cycles that visit a node $l \in T$ followed by all $|R|$ (or any $|R|-1$) nodes in R and a node $m \in S$. Considering these tight p-cycles and applying Lemma 6, we see that

$$c_{lj} = \lambda + \pi_l - \pi_j \quad (j \in R),$$

$$c_{ij} = \lambda + \pi_i - \pi_j \quad (i, j \in R),$$

$$c_{im} = \lambda + \pi_i - \pi_m \quad (i \in R)$$

for some $\lambda, \pi_1, \pi_2, \dots, \pi_r, \pi_l$ and π_m. Without loss of generality, let $\pi_n = 0$. Theorem 3 then implies that $\lambda = 1$ and $\pi_j = 0$ for $j \in R$, $c_{ln} = 0$ implies that $\pi_l = -1$ and $c_{nm} = 1$ implies that $\pi_m = 0$. Therefore, $c_{ij} = 1$ for all $i \in R$, $j \in R \cup S$ and $c_{ij} = 0$ for all $i \in T$, $j \in R$.

Let $\sigma = c_{12}$ and consider a tight p-cycle that contains the arcs $(i,1)$, $(1,2)$, $(2,3)$ and $(3,j)$ for some $i, j \in R$. Replacing these arcs by the arcs $(i,2)$, $(2,3)$, $(3,4)$ and $(4,j)$ yields another tight p-cycle, and hence $c_{34} = \sigma$ since $c_{i1} = c_{i2} = 1$ and $c_{3j} = c_{4j} = 0$; similarly, $c_{21} = c_{43} = \sigma$. This also implies that $c_{13} = c_{14} = c_{23} = c_{24}$. Next consider a tight p-cycle that contains the arcs (i,j), $(j,1)$, $(1,2)$ for some $i, j \in R$. Replacing these arcs by the arcs $(i,1)$, $(1,j)$, $(j,2)$ yields another tight p-cycle, and hence $c_{1j} = \sigma$; similarly, $c_{ij} = \sigma$ for $i \in S$ and $j \in R$.

Then consider a tight p-cycle that contains the arcs $(3,4)$, $(4,i)$, (i,j) for some $i, j \in R$. Replacing these arcs by the arcs $(3,i)$, $(i,4)$, $(4,j)$ yields another tight p-cycle, and hence $c_{i4} = 1 + \sigma$; similarly, $c_{ij} = 1 + \sigma$ for all $i \in R$ and $j \in T$. Finally, consider a tight p-cycle that contains the arcs $(1,3)$, $(3,2)$, $(2,4)$, $(4,i)$, (i,j) for $i, j \in R$. Replacing these arcs by the arcs $(1,3)$, $(3,i)$, $(1,2)$, $(2,4)$, $(4,j)$ yields another tight p-cycle, and hence $c_{32} = 0$; similarly, $c_{31} = c_{42} = c_{41} = 0$. One can then deduce that $c_{13} = c_{14} = c_{23} = c_{24} = 1 + \sigma$ and $c_0 = n - 3 + 2\sigma$. Adding σ times the equation $x(T : N \setminus T) - x(N \setminus T : T) = 0$ and subtracting σ times the equation $x(N : N) = p$, we see that $\boldsymbol{cx} = c_0$ is equivalent to

$$(1-\sigma)x(S : T) + (1-\sigma)\sum_{i \in R} x(\delta^+(i)) = (1-\sigma)(p - 1 + |R|)/2. \qquad \square$$

3.3. Partition inequalities

In this section, we consider classes of valid inequalities for the p-cycle polytope that bound the intersection of p-cycles with cutsets based on more general partitions of the node set. The first class states that there must be at least one arc across a partition of N into subsets of size at most $p-1$.

Theorem 16. *Let $n \geqslant 5$ and let $\langle N_1, N_2, \dots, N_t \rangle$ be a partition of N with $|N_i| \leqslant p-1$ for $i = 1, 2, \dots, t$ for some $t \geqslant 3$. Then the linear ordering inequality*

$$\sum_{i<j} x(N_i : N_j) \geqslant 1 \tag{17}$$

is valid for the p-cycle polytope Q_p^n and induces a facet of Q_p^n if and only if $|N_t| + |N_1| \geqslant p$ *and* $|N_i| + |N_{i+1}| \geqslant p$ *for all* $i = 1, 2, \ldots, t-1$.

Proof. To see that (17) is valid, note that no p-cycle can be entirely contained in any subset N_i, so contracting the subset yields a directed Euler tour, which cannot be topologically sorted.

Note that if the subset $N_1, \ldots, N_t$ are instead ordered $N_k, \ldots, N_t, N_1, \ldots, N_{k-1}$ then the resulting inequality is equivalent to (17). To see this, note that because $\langle N_1, N_2, \ldots, N_t \rangle$ is a partition of N, the flow conservation constraints imply that for each k,

$$\sum_{i<k\leqslant j} x(N_i : N_j) = \sum_{i<k\leqslant j} x(N_j : N_i).$$

Next, we show that the linear ordering inequality (17) does not induce a facet unless the required conditions hold. We may assume without loss of generality that $|N_t| + |N_1| \leqslant p-1$. In this case, (17) is a consequence of the non-negativity constraints and the linear ordering inequality with $t' = t-1$, $N_1' = N_1 \cup N_t$ and $N_i' = N_i$ for $i = 2, 3, \ldots, t-1$.

If $|N_i| < p-1$, then (17) is satisfied by all p-bowties tied at node $k \in N_i$, so we can clone nodes in N_i to obtain facets provided we can show that the primitive linear ordering inequalities (17) are regular and facet inducing for Q_p^n.

Now if $p = 3$ and $n = 5$, there is a facet-inducing linear ordering inequality with $|N_1| = 1$ and $|N_2| = |N_3| = 2$ (see Table 2): this inequality cannot be equivalent to a broom inequality, because broom inequalities do not induce facets when $p = 3$ and $n = 5$; and for each arc (i, j), there is a tight p-cycle that contains (i, j), so it cannot be equivalent to a non-negativity constraint. In all other cases, we will show that (17) satisfies the sufficient condition for regularity. Because the ordering is "cyclic", it suffices to find a non-tight p-cycle that does not visit a node $k \in N_2$. If $t \geqslant 4$ or $t = 3$ and $|N_2| \geqslant 2$, then such a p-cycle is easy to find. If $t = 3$ and $N_2 = \{k\}$, then $|N_1|, |N_3| \geqslant p-1$ and there is a p-cycle that uses two arcs from N_1 to N_3 since $p \geqslant 4$ (or else $p = 3$ and $n = 5$). Thus, (17) is always regular under the stated conditions.

First of all, suppose that $t = 3$. We may assume without loss of generality that $|N_3| \leqslant |N_1|$ and $|N_3| \leqslant |N_2|$. The only primitive inequalities of this form have $|N_1| = |N_2| = q$ and $|N_3| = p - q$ for some $q \geqslant p/2$. Let $N_1 = \{1, 2, \ldots, q\}$, $N_2 = \{q+1, \ldots, 2q\}$ and $N_3 = \{2q+1, \ldots, p+q\}$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ with

$$x(N_1 : N_2) + x(N_1 : N_3) + x(N_2 : N_3) = 1. \tag{18}$$

First of all, suppose that $q > p/2$ so that $|N_1| + |N_2| > p$. If $q = 2$, then the linear ordering inequality with $|N_1| = |N_2| = 2$ and $|N_3| = 1$ can be seen to induce a facet of Q_3^5 using PORTA. If $q \geqslant 3$, the min-cut inequality $x(N_1 : N_2) \geqslant 1$ induces a facet of Q_p^{2q} and Theorem I.4.3.6 of Nemhauser and Wolsey [20] implies that $\boldsymbol{cx} = c_0$ is equivalent to an equation $\boldsymbol{dx} = d_0$ such that $d_0 = \mu$, $d_{ij} = \mu$ for $i \in N_1$ and $j \in N_2$ and $d_{ij} = 0$ for all other arcs with $i, j \leqslant 2q$. Using node potentials π_j for $j \in N_3$ we may further assume that $d_{j,2q} = 0$ for $j \in N_3$. Considering a tight p-cycle which includes the arcs (i, j), $(j, 2q)$ for some $i \in N_1$ and $j \in N_3$, we see that $d_{ij} = \mu$ for all $i \in N_1$ and $j \in N_3$. Next considering any tight p-cycle which includes the arcs $(1, j)$, (j, k)

for some $j \in N_3$ and $k \in N_2$ we see that $d_{jk} = 0$ for all $j \in N_3$ and $k \in N_2$. If $|N_3| > 1$, then $p \geqslant 4$ and considering any tight p-cycle that contains the arcs $(1, i)$, (i, j), $(j, 2q)$ for some $i, j \in N_3$ we see that $d_{ij} = 0$ for all $i, j \in N_3$. Then considering the tight p-cycles on $N_2 \cup N_3$, we see that $d_{ij} = \mu$ for all $i \in N_2$ and $j \in N_3$ and considering the tight p-cycles on $N_1 \cup N_3$ we see that $d_{ij} = 0$ for $i \in N_3$ and $j \in N_1$. Hence $\boldsymbol{dx} = d_0$ is equivalent to (18).

Next, suppose that $q = p/2$ so that $|N_1| = |N_2| = |N_3| = q$ and $p = 2q$. First of all, since $n \geqslant 5$ we may assume that $q \geqslant 2$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ with that satisfies (18) with equality. Using Corollary 4, we may assume that $c_{12} = 0$ and $c_{i1} = 0$ for all $i \geqslant 2$. Since $p \geqslant q + 2$, there are tight p-cycles with respect to (18) that visit a node $k \in N_2$ followed by all $|N_1|$ (or any $|N_1| - 1$) nodes in N_1 and a node $l \in N_3$. Since $|N_1| \geqslant 2$, Lemma 6 implies that

$$c_{kj} = \lambda + \pi_k - \pi_j \quad (j \in N_1),$$

$$c_{ij} = \lambda + \pi_i - \pi_j \quad (i, j \in N_1),$$

$$c_{il} = \lambda + \pi_i - \pi_l \quad (i \in N_1)$$

for some λ, $\{\pi_j : j \in N_1\}$, π_k and π_l. Without loss of generality, let $\pi_1 = 0$. Theorem 3 then implies that $\lambda = 0$ and $\pi_i = 0$ for all $i \in N_1$, and $c_{k1} = 0$ implies that $\pi_k = 0$. Therefore, $c_{ij} = 0$ for all $i, j \in N_1$ and $c_{kj} = 0$ for all $k \in N_2$ and $j \in N_1$. Next consider a tight p-cycle that skips node $k \in N_2$ and uses the arcs (i, j), $(j, 1)$ for some $i \in N_2$ and $j \in N_1$. Replacing these arcs by the arcs (i, k), $(k, 1)$ we obtain another tight p-cycle, and thus $c_{ik} = c_{ij} + c_{j1} - c_{k1} = 0$ for all $i, k \in N_2$. Then considering tight p-cycles on $N_1 \cup N_2$, it is easy to see that $c_{ij} = c_0$ for all $i \in N_1$ and $j \in N_2$.

Now consider a tight p-cycle on $N_1 \cup N_3$ that uses the arcs $(i, 1)$, $(1, j)$ for some $i \in N_3$ and $j \in N_1$. Replacing these arcs by the arcs (i, k), (k, j) for some $k \in N_2$ we obtain another tight p-cycle, and thus $c_{ik} = c_{i1} + c_{1j} - c_{kj} = 0$ for all $i \in N_3$ and $k \in N_2$. Considering tight p-cycles that use the arcs (i, j), (j, k) for some $i \in N_1$, $j \in N_3$ and $k \in N_2$ we see that $c_{ij} = c_0$ for all $i \in N_1$ and $j \in N_3$. Next considering tight p-cycles that use the arcs $(1, i)$, (i, j), (j, k) for some $i, j \in N_3$ and $k \in N_2$ we see that $c_{ij} = 0$ for all $i, j \in N_3$. Then considering the tight p-cycles on $N_2 \cup N_3$, we see that $c_{ij} = c_0$ for all $i \in N_2$ and $j \in N_3$ and considering the tight p-cycles on $N_1 \cup N_3$ we see that $c_{ij} = 0$ for $i \in N_3$ and $j \in N_1$. Hence $\boldsymbol{cx} = c_0$ is equivalent to (18).

For $t > 3$, we may assume without loss of generality that $|N_{t-1}| \geqslant |N_j|$ for $j \neq t - 1$. The only primitive inequalities of this form have $|N_{t-1}| = q$ and $|N_t| = p - q$ for some $q \geqslant p/2$. Suppose that $\boldsymbol{cx} = c_0$ is satisfied by every $\boldsymbol{x} \in Q_p^n$ that satisfies (17) with equality. Since $|N_{t-1}| + |N_1| \geqslant |N_t| + |N_1| \geqslant p$, we may assume inductively that the inequality (17) for $t - 1$ determines a facet of the p-cycle polytope with node set $N_1 \cup N_2 \cup \cdots \cup N_{t-1}$, and hence Theorem I.4.3.6 of Nemhauser and Wolsey [20] implies that $\boldsymbol{cx} = c_0$ is equivalent to an equation $\boldsymbol{dx} = d_0$ such that $d_0 = \mu$, $d_{ij} = \mu$ for $i \in N_k$ and $j \in N_l$ with $k < l < t$ and $d_{ij} = 0$ for all other arcs with $i, j \notin N_t$. Using node potentials π_j for $j \in N_t$ we may further assume that $d_{jr} = 0$ for all $j \in N_t$ for some node $r \in N_{t-1}$.

Suppose that $s \leqslant t-2$. Considering a tight p-cycle which includes the arcs (i,j) and (j,r) for some $i \in N_s$ and $j \in N_t$, we see that $d_{ij} = \mu$ for all $i \in N_s$ and $j \in N_t$. Next, considering any tight p-cycle which includes the arcs (i,j) and (j,k) for some $i \in N_s$, $j \in N_t$ and $k \in N_{s+1}$ we see that $d_{jk} = 0$ for all $j \in N_t$ and $k \in N_{s+1}$. If $|N_t| > 1$, then $p \geqslant 4$ and considering any tight p-cycle that contains the arcs (i,j), (j,k), (k,r) for some $i \in N_{t-2}$ and $j,k \in N_t$ we see that $d_{jk} = 0$ for all $j,k \in N_t$. Then considering the tight p-cycles on $N_{t-1} \cup N_t$, we see that $d_{ij} = \mu$ for all $i \in N_{t-1}$ and $j \in N_t$ and considering the tight p-cycles on $N_1 \cup N_t$, we see that $d_{ij} = 0$ for all $i \in N_t$ and $j \in N_1$. Hence $\boldsymbol{dx} = d_0$ is equivalent to (17). $\square$

Notes:. Linear ordering inequalities (17) generalize the inequalities $\sum_{i<j} x_{\pi(i)\pi(j)} \geqslant 1$ for permutations $\boldsymbol{\pi}$ of N, shown to be facet inducing for the cycle polytope by Balas and Oosten [5]. Both (16) and (17) have representations of the form $x(H) \leqslant p-1$ with maximal p-cycle free support graphs. This may lead one to believe that all classes of maximal p-cycle free graphs are associated with strong valid inequalities for Q_p^n, but there are also examples that induce low-dimensional faces.

In the following theorem, we consider the undirected multi-cut induced by the partition $\langle N_1, N_2, \ldots, N_t \rangle$ of N.

Theorem 17. *Let $\langle N_1, N_2, \ldots, N_t, R \rangle$ be a partition of N into $t+1$ subsets with $|N_1| \geqslant \cdots \geqslant |N_t| \geqslant 1$, such that $|N_1| + \cdots + |N_q| + |R| \leqslant p-1$ for some $1 \leqslant q \leqslant t-2$. Then the multi-cut inequality*

$$\sum_{i \neq j} x(N_i : N_j) + x(R : N \setminus R) \geqslant q+1 \tag{19}$$

is valid for the p-cycle polytope Q_p^n and facet-inducing for Q_p^n if and only if

(i) $|N_2| + \cdots + |N_{q+2}| + |R| \geqslant p$,
(ii) $|N_1| + \cdots + |N_{q-1}| + |N_{t-1}| + |N_t| + |R| \geqslant p$,
(iii) $|N_1| + \cdots + |N_q| + |N_t| + |R| \geqslant p+1$ (if $|N_t| > 1$),
$|N_1| + \cdots + |N_q| + |N_{t-1}| + |R| \geqslant p+1$ (if $|N_t| = 1$),

except when $|R| = 0$ and $q = 1$; $|R| = 0$, $q = 2$ and $t = 4$; $|R| = 0$, $q = 2$, $t \geqslant 5$ and ($|N_3| + |N_4| + |N_5| \leqslant p-1$ or $|N_2| + |N_3| + |N_t| \leqslant p-1$); or $|R| = 1$, $q = 1$ and ($|N_t| = 1$ or $|N_2| + |N_3| \leqslant p-1$).

Proof. To see that (19) is valid, let $S \subset N$ consist of $|N_r| - |N_q|$ nodes from each subset N_r with $r < q$. Then summing the cardinality constraint $x(N : N) = p$ and the one-sided min-cut inequalities $x(N_r : N \setminus N_r) \geqslant x(\delta^+(j))$ for all $j \in N \setminus (R \cup S)$ and subtracting the degree constraints $x(\delta^+(j)) \leqslant 1$ for all $j \in R \cup S$, we obtain

$$\sum_{r=1}^{q-1} |N_q| x(N_r : N \setminus N_r) + \sum_{r=q}^{t} |N_r| x(N_r : N \setminus N_r)$$
$$= q|N_q| + (p - |N_1| - \cdots - |N_q| - |R|).$$

Dividing by $|N_q|$ and rounding up yields $\sum_{r=1}^{t} x(N_r : N \setminus N_r) \geqslant q+1$, which is equivalent to the multi-cut inequality (19). Now, it is easy to see that the tight p-cycles for (19) visit exactly $q+1$ of the subsets $N_1, N_2, \ldots, N_t$; in addition, they must satisfy $x(N_r : N \setminus N_r) \leqslant 1$ for $r = 1, 2, \ldots, t$.

The multi-cut inequality does not induce a facet of Q_p^n when $t = q+1$, because in this case $|N \setminus N_r| \leqslant |N_1| + \cdots + |N_q| + |R| \leqslant p-1$ and hence (19) is a consequence of the min-cut inequalities $x(N_r : N \setminus N_r) \geqslant 1$ for $r = 1, 2, \ldots, t$. Next, we will show that (19) does not induce a facet unless conditions (i)–(iii) hold.

First suppose that condition (i) does not hold. If $|N_2| + \cdots + |N_t| + |R| < p$, then multi-cut inequality (19) is a consequence of the min-cut inequality $x(N_1 : N \setminus N_1) \geqslant 1$ and the inequality $\sum_{r=2}^{t} x(N_r : N \setminus N_r) \geqslant q$, which is an instance of (19) with $R' = R \cup N_1$. Otherwise, let $s \geqslant 1$ be the smallest integer such that $|N_2| + \cdots + |N_{q+s+2}| + |R| \geqslant p$ (hence $s \geqslant 1$). Then (19) is a consequence of

$$(s+1)x(N_1 : N \setminus N_1) + \sum_{r=2}^{t} x(N_r : N \setminus N_r) \geqslant q+s+1 \tag{20}$$

and $\sum_{r=2}^{t} x(N_r : N \setminus N_r) \geqslant q$. To see that (20) is valid, use the disjunction $x(N_1 : N \setminus N_1) \leqslant 0$ or $x(N_1 : N \setminus N_1) \geqslant 1$. In the former case, (20) follows from (19) applied to $N \setminus N_1$ since $|N_2| + \cdots + |N_{q+s+1}| + |R| \leqslant p-1$. In the latter case, (20) follows from (19).

Next suppose that condition (ii) does not hold. Then we can set $N_{t-1}' = N_{t-1} \cup N_t$ to obtain a dominating multi-cut inequality with $t' = t-1$.

Next, suppose that condition (iii) does not hold, but that conditions (i) and (ii) both hold. First consider the case when $|N_{t-1}| = |N_t| = 1$. If $|N_1| = 1$, then (19) is simply $\sum_{j \notin R} x(\delta^+(j)) \geqslant q+1$, which follows from the cardinality constraint $x(N : N) = p$ and the degree constraints for $j \in R$, since $|N_1| + \cdots + |N_q| + |R| \leqslant p-1$ implies that $|R| \leqslant p-q-1$. If $|N_1| \geqslant 2$, then subtracting $|N_1| + \cdots + |N_q| + |R| \leqslant p-1$ from condition (i) we see that $|N_{q+1}| + |N_{q+2}| \geqslant |N_1| + 1 \geqslant 3$, and hence $|N_{q+1}| \geqslant 2$, which allows us to set $N_{t-1}' = N_{t-1} \cup N_t$ to obtain a dominating multi-cut inequality with $t' = t-1$. Finally, suppose that (iii) does not hold for $|N_s| \geqslant 2$, where $s = t$ or $s = t-1$. Because condition (ii) holds, we must have $|N_1| + \cdots + |N_q| + |N_s| + |R| = p$. Then the one-sided min-cut inequality for some $j \in N_s$ and the multi-cut inequality for $N_s' = \{j\}$ and $R' = R \cup N_s \setminus \{j\}$ imply that

$$\sum_{r=1}^{t} x(N_r : N \setminus N_r) \geqslant \sum_{r \neq s} x(N_r : N \setminus N_r) + x(\delta^+(j)) \geqslant q+1.$$

Next we consider the exceptional cases. If $|R| = 0$ and $q = 1$, then it is easy to see that (19) is a consequence of the linear ordering inequalities with partition $\langle N_1, N_2, \ldots, N_t \rangle$ and orderings $N_1, N_2, \ldots, N_t$ and $N_t, N_{t-1}, \ldots, N_1$.

If $|R| = 0$, $q = 2$ and $t = 4$, then (19) is a consequence of the six min-cut inequalities $x(S : N \setminus S) \geqslant 1$ where $S = N_i \cup N_j$ for $i \neq j$.

If $|R| = 0$, $q = 2$ and $|N_3| + |N_4| + |N_5| \leqslant p - 1$, then every tight p-cycle visits $S = N_1 \cup N_2$ and multi-cut inequality (19) is a consequence of

$$\sum_{r=1}^{t} x(N_r : N \setminus N_r) + x(S : N \setminus S) \geqslant 4$$

and

$$\sum_{r=1}^{t} x(N_r : N \setminus N_r) - x(S : N \setminus S) \geqslant 2.$$

Note that $|N_3| + |N_4| + |N_5| \leqslant p - 1$ is only required for the first inequality to be valid.

If $|R| = 0$, $q = 2$ and $|N_2| + |N_3| + |N_t| \leqslant p - 1$, then every tight p-cycle that visits N_t also visits N_1 and multi-cut inequality (19) is a consequence of

$$2x(N_1 : N \setminus N_1) + \sum_{r=2}^{t} x(N_r : N \setminus N_r) - x(S : N \setminus S) \geqslant 3$$

and

$$\sum_{r=2}^{t} x(N_r : N \setminus N_r) + x(S : N \setminus S) \geqslant 3$$

with $S = N_1 \cup N_t$. Note that $|N_2| + |N_3| + |N_t| \leqslant p - 1$ is only required for the first inequality to be valid.

If $|R| = 1$, $q = 1$ and $|N_t| = 1$, then comparing condition (iii) with $|N_1| + \cdots + |N_q| + |R| \leqslant p - 1$, we obtain

$$|N_1| + 1 \leqslant p - 1 \leqslant |N_{t-1}| + |N_t|$$

which implies that $|N_1| = \cdots = |N_{t-1}|$ and $p = |N_1| + 2$. Then the broom inequality $x(\delta^+(i)) \geqslant x_{ij} + x_{ji}$ for $N_t = \{i\}$ and $R = \{j\}$ implies that

$$\begin{aligned} \sum_{r=1}^{t} x(N_r : N \setminus N_r) &= \sum_{r=1}^{t-1} x(N_r : N \setminus N_r) + x(\delta^+(i)) \\ &\geqslant \sum_{r=1}^{t-1} x(N_r : N \setminus N_r) + (x_{ij} + x_{ji}) \geqslant 2, \end{aligned}$$

where the last inequality can be shown to be facet inducing for Q_p^n.

If $|R| = 1$, $q = 1$, $|N_t| \geqslant 2$ and $|N_2| + |N_3| \leqslant p - 1$ then using condition (ii) we derive

$$|N_2| + |N_3| \leqslant p - 1 \leqslant |N_{t-1}| + |N_t|$$

which implies that either $t = 3$ and $|N_2| + |N_3| = p - 1$ or $|N_2| = \cdots = |N_t|$ and $p = 2|N_2| + 1$. In either case, every tight p-cycle visits $S = N_1 \cup R$ and the multi-cut inequality (19) is a consequence of

$$\sum_{r=1}^{t} x(N_r : N \setminus N_r) + x(S : N \setminus S) \geqslant 3$$

and

$$\sum_{r=1}^{t} x(N_r : N \setminus N_r) - x(S : N \setminus S) \geqslant 1.$$

Note that $|N_2| + |N_3| \leqslant p - 1$ is only required for the first inequality to be valid.

Suppose that $\boldsymbol{cx} = c_0$ for every $\boldsymbol{x} \in Q_p^n$ that satisfies (19) with equality. Without loss of generality, suppose that $\{1,2\} \subseteq N_1$. Using Corollary 4, assume that $c_{12} = 0, c_{i1} = 0$ for $i \in N_1$ and $c_{i1} = 1$ for $i \notin N_1$. We will first show that

$$\boldsymbol{cx} = \sum_{i \neq j} \gamma_{ij} x(N_i : N_j) + \sum_{i=1}^{t} \gamma_{i0} x(N_i : R) + \sum_{j=1}^{t} \gamma_{0j} x(R : N_j)$$

for some values γ_{ij} for $i \neq j = 0, 1, \ldots, t$.

We may assume that $p \geqslant |N_1| + 2$, since $|N_1| + \cdots + |N_q| + |R| \leqslant p - 1$ implies that $p \geqslant |N_1| + 1$ and $p = |N_1| + 1$ only if $q = 1$ and $R = \emptyset$, in which case (19) does not induce a facet. Then because condition (iii) holds and $|N_1| \geqslant 2$, if $k \in N \setminus (N_1 \cup N_t)$ there must be tight p-cycles that visit node k followed by all $|N_1|$ (or any $|N_1| - 1$) nodes in N_1 and a node $l \notin N_1$. Considering these tight p-cycles and applying Lemma 6, we see that

$$c_{kj} = \lambda + \pi_k - \pi_j \quad (j \in N_1),$$

$$c_{ij} = \lambda + \pi_i - \pi_j \quad (i, j \in N_1),$$

$$c_{il} = \lambda + \pi_i - \pi_l \quad (i \in N_1)$$

for some $\lambda, \{\pi_j : j \in N_1\}$, π_k and π_l. Without loss of generality, let $\pi_1 = 0$. Theorem 3 then implies that $\lambda = 0$ and $\pi_j = 0$ for $j \in N_1$, and $c_{k1} = 1$ implies that $\pi_k = 1$. Therefore $c_{ij} = 0$ for all $i, j \in N_1$ and $c_{kj} = 1$ for all $j \in N_1$ (and hence $\gamma_{i1} = 1$ for all $i \neq t$). Since the reversal of any tight p-cycle with respect to (19) is also tight, we similarly derive the fact that $c_{il} = c_{1l}$ for all $i \in N_1$ and $l \in N \setminus (N_1 \cup N_t)$.

Condition (ii) guarantees that there is a tight p-cycle that visits a node $k \in N_t$ followed by all $|N_1|$ nodes in N_1 and a node $l \in N \setminus (N_1 \cup N_t)$. Since $c_{il} = c_{1l}$ for all $i \in N_1$ and $c_{k1} = 1$, changing the order in which the nodes in N_1 are visited yields $c_{kj} = 1$ for $k \in N_t$ and $j \in N_1$ (and hence $\gamma_{t1} = 1$). Similarly, we see that $c_{il} = c_{1l}$ for all $i \in N_1$ and $l \in N_t$ and hence $c_{il} = c_{1l}$ for all $i \in N_1$ and $l \in N \setminus N_1$; if $|N_t| = 1$, this implies that $c_{il} = \gamma_{1t}$ for $i \in N_1$ and $l \in N_t$.

Next let $|N_s| \geqslant 2$ for some $s \geqslant 2$. Condition (iii) implies that there is a tight p-cycle that visits a node $i \in N_s$ followed by all the nodes in N_1 using arc $(i, 1)$ but does not visit node $j \in N_s$. Replacing the arc $(i, 1)$ by the arcs $(i, j), (j, 1)$ and skipping a node from N_1 we obtain another tight p-cycle, which implies that $c_{ij} = 0$ for $i, j \in N_s$. Condition (iii) also implies that there are tight p-cycles that visit a node $k \in N_1$ followed by all $|N_s|$ (or any $|N_s| - 1$) nodes in N_s and a node $l \in N \setminus (N_1 \cup N_s)$; further, if $R \neq \emptyset$ we can take $l \in R$. Considering these tight p-cycles and applying Lemma 6, we see that

$$c_{kj} = \lambda + \pi_k - \pi_j \quad (j \in N_s),$$

$$c_{ij} = \lambda + \pi_i - \pi_j \quad (i, j \in N_s),$$

$$c_{il} = \lambda + \pi_i - \pi_l \quad (i \in N_s)$$

for some λ, $\{\pi_j\colon j \in N_s\}$, π_k and π_l. Without loss of generality, let $\pi_i = 0$ for some $i \in N_s$. Theorem 3 then implies that $\lambda = 0$ and $\pi_j = 0$ for all $j \in N_s$. Hence $c_{ki} = c_{kj}$ for all $k \in N_1$ and $i, j \in N_s$, and thus $c_{kj} = \gamma_{1s}$ for $k \in N_1$ and $j \in N_s$. Note that if $R \neq \emptyset$, then we can take $l \in R$ and hence $c_{il} = c_{jl}$ for all $i, j \in N_s$ and $l \in R$. Since the reversals of all of these tight p-cycles are tight, we similarly obtain $c_{ki} = c_{kj}$ for all $k \in R$ and $i, j \in N_s$. If $|R| = 1$, this implies that $c_{il} = \gamma_{s0}$ for $i \in N_s$ and $l \in R$ and $c_{kj} = \gamma_{0s}$ for $k \in R$ and $j \in N_s$.

If $|R| \geqslant 2$, then Condition (iii) implies that there is a tight p-cycle that visits a node $i \in R$ followed by all the nodes in N_1 using arc $(i, 1)$ but does not visit node $j \in R$. Replacing the arc $(i, 1)$ by the arcs $(i, j), (j, 1)$ and skipping a node from N_1 we obtain another tight p-cycle, which implies that $c_{ij} = 0$ for $i, j \in R$. Condition (iii) also implies that for any $s \geqslant 2$ with $|N_s| \geqslant 2$, there is a tight p-cycle that visits a node $k \in N_1$, followed by all $|R|$ (or any $|R| - 1$) nodes in R, followed by a node $l \in N_s$. Considering these tight p-cycles and applying Lemma 6 we can derive that $c_{ki} = c_{kj}$ for all $k \in N_1$ and $i, j \in R$ and $c_{il} = c_{jl}$ for all $i, j \in R$ and $l \in N_s$. Thus $c_{kj} = \gamma_{10}$ for $k \in N_1$ and $j \in R$ and $c_{il} = \gamma_{0s}$ for $i \in R$ and $l \in N_s$. Condition (ii) implies that there is a tight p-cycle that visits node 1 followed by all the nodes in R and a node $l \in N_t$. Changing the order in which the nodes in R are visited, we obtain $c_{il} = c_{jl}$ for all $i, j \in R$ and $l \in N_t$; hence $c_{il} = \gamma_{0t}$ for $i \in R$ and $l \in N_t$. Condition (ii) also implies that there is a tight p-cycle that visits any node $k \in N \setminus (N_1 \cup R)$ followed by all the nodes in R and node 1. Changing the order in which the nodes in R are visited, we obtain $c_{ki} = c_{kj}$ for all $k \in N \setminus (N_1 \cup R)$ and $i, j \in R$; hence, $c_{kj} = \gamma_{s0}$ for $k \in N_s$ and $j \in R$, for $s = 1, 2, \ldots, t$.

Now suppose $r, s \geqslant 2$ with $r \neq s$. If $R \neq \emptyset$, condition (ii) implies that there are tight p-cycles that visit a node $k \in N_r$, followed by all the nodes in N_s and a node $l \in R$; if $R = \emptyset$, then $q \geqslant 2$ and condition (ii) implies that there are tight p-cycles that visit a node $k \in N_r$, followed by all the nodes in N_s and node 1. Changing the order in which the nodes in N_s are visited, we obtain $c_{ki} = c_{kj}$ for all $k \in N_r$ and $i, j \in N_s$. Since the reversals of these tight p-cycles are tight, interchanging the roles of N_r and N_s, we see that $c_{il} = c_{jl}$ for all $i, j \in N_r$ and $l \in N_s$. Therefore $c_{kj} = \gamma_{rs}$ for all $k \in N_r$ and $j \in N_s$.

Next, we consider the cases $q \geqslant 2$ and $R \neq \emptyset$, $q \geqslant 3$ and $R = \emptyset$, $q = 2$ and $R = \emptyset$, and $q = 1$ and $R \neq \emptyset$ separately.

First suppose that $q \geqslant 2$ and $R \neq \emptyset$. Condition (iii) implies that for $2 \leqslant s < t$ there exists a tight p-cycle that contains the arcs $(i, j), (j, 1)$ for some $i \in N_s$ and $j \in R$ but skips a node $k \in N_1$. Replacing the arcs $(i, j), (j, 1)$ by the arcs $(i, k), (k, 1)$ we obtain another tight p-cycle, and hence $\gamma_{s0} = 0$ for $2 \leqslant s < t$, since $\gamma_{01} = \gamma_{s1} = 1$ and $c_{k1} = 0$. Now consider the restriction of (19) to the subsets $N_1, N_2, \ldots, N_{q-1}, N_r, N_s$ and R. There are tight p-cycles that visit all $q + 2$ of these subsets in any order, so by Theorem 23 of Grötschel and Padberg [15] we must have $\gamma_{ij} = \alpha_i + \beta_j$ for all $i, j = 0, 1, \ldots, q - 1, r$

and s for some $\alpha_0, \alpha_1, \ldots, \alpha_{q-1}, \alpha_r$ and α_s, $\beta_0, \beta_1, \ldots, \beta_{q-1}, \beta_r$ and β_s. Without loss of generality, set $\beta_1 = 1$. Then $\gamma_{i1} = 1$ implies that $\alpha_i = 0$ for all $i \neq 1$, and the fact that either $\gamma_{s0} = 0$ or $\gamma_{r0} = 0$ implies that $\beta_0 = 0$. This implies that $\alpha_1 = \gamma_{10}$ and $\beta_j = \gamma_{0j}$ for $j = 2, 3, \ldots, q-1, r$ and s. Therefore $\alpha = \alpha_1$ and $\beta_2, \beta_3, \ldots, \beta_t$ are consistently defined for all choices of r and s; hence we may assume that $\gamma_{10} = \alpha$, $\gamma_{r0} = 0$ for $r \geqslant 2$, $\gamma_{0r} = \beta_r$ for $r \geqslant 2$, $\gamma_{1r} = \alpha + \beta_r$ for $r \geqslant 2$, and $\gamma_{rs} = \beta_s$ for $r \neq s \geqslant 2$.

Condition (i) guarantees that there exists a tight p-cycle that visits only nodes from subsets $N_2, N_3, \ldots, N_{q+2}$ and R. Replacing the nodes from a subset N_r with $2 \leqslant r \leqslant q+2$ by nodes from N_1, we obtain another tight p-cycle. This implies that $\beta_2 = \beta_3 = \cdots = \beta_{q+2} = \alpha + 1$. Condition (ii) guarantees that there exists a tight p-cycle that visits only nodes from $N_1, N_2, \ldots, N_{q-1}, N_{t-1}, N_t$ and R. Replacing the nodes from N_{t-1} or N_t by nodes from a subset N_r with $q \leqslant r \leqslant t-2$, we obtain another tight p-cycle. This implies that $\beta_t = \beta_{t-1} = \cdots = \beta_{q+1} = \beta_q$. Hence $\gamma_{0r} = \alpha + 1$ for $r \geqslant 2$, $\gamma_{1r} = 2\alpha + 1$ for $r \geqslant 2$, and $\gamma_{rs} = \alpha + 1$ for $r \neq s \geqslant 2$. Subtracting α times the equation $x(N_1 : N \setminus N_1) - x(N \setminus N_1 : N_1) = 0$, we see that $\boldsymbol{cx} = c_0$ is equivalent to

$$(\alpha + 1) \sum_{i \neq j} x(N_i : N_j) + (\alpha + 1) x(R : N \setminus R) = (\alpha + 1)(q + 1).$$

Next suppose that $q \geqslant 3$ and $R = \emptyset$. Consider the restriction of (19) to the subsets $N_1, N_2, \ldots, N_{q-1}, N_r$ and N_s. There are tight p-cycles that visit all $q+1$ of these subsets in any order, so by Theorem 23 of Grötschel and Padberg [15] we must have $\gamma_{ij} = \alpha_i + \beta_j$ for all $i, j = 1, 2, \ldots, q-1, r$ and s for some $\alpha_1, \alpha_2, \ldots, \alpha_{q-1}, \alpha_r$ and $\alpha_s, \beta_1, \beta_2, \ldots, \beta_{q-1}, \beta_r$ and β_s. Without loss of generality, set $\beta_1 = 1$. Then $\gamma_{i1} = 1$ implies that $\alpha_i = 0$ for all $i \neq 1$, and hence $\alpha_1 = \gamma_{1r} - \gamma_{2r} = \gamma_{1s} - \gamma_{2s}$, $\beta_2 = \gamma_{12} - \alpha_1$ and $\beta_j = \gamma_{2j}$ for $j = 3, 4, \ldots, q-1, r$ and s. The fact that $\gamma_{1r} - \gamma_{2r} = \gamma_{1s} - \gamma_{2s}$ for all choices of r and s implies that $\alpha = \alpha_1$ and $\beta_2, \beta_3, \ldots, \beta_t$ are consistently defined for all choices of r and s; hence, we may assume that $\gamma_{1r} = \alpha + \beta_r$ for $r \geqslant 2$ and $\gamma_{rs} = \beta_s$ for $r \neq s \geqslant 2$. The rest of the proof for the case $q \geqslant 3$ and $R = \emptyset$ is the same as the previous case.

Next consider the case $q = 2$ and $R = \emptyset$. Consider the restriction of (19) to the subsets $N_1, N_2, \ldots, N_5$. Since $|N_3| + |N_4| + |N_5| \geqslant p$, there are tight p-cycles that visit any three of these subsets in any order, so Lemma 2 implies that $\gamma_{ij} = \lambda + \pi_i - \pi_j$ for some λ and $\pi_1, \pi_2, \ldots, \pi_5$. Without loss of generality, set $\pi_1 = \lambda - 1$. Then $\gamma_{i1} = 1$ implies that $\pi_i = 0$ for $2 \leqslant i \leqslant 5$. Therefore $\gamma_{1i} = 2\lambda - 1$ for $i \geqslant 2$, $\gamma_{ij} = \lambda$ for $2 \leqslant i, j \leqslant 5$ and $c_0 = 3\lambda$. Condition (ii) implies that there is a tight p-cycle that visits N_2, N_3 and N_r (in that order) which implies that $\gamma_{3r} = c_0 - \gamma_{r1} - \gamma_{13} = \lambda$. Since $|N_2| + |N_3| + |N_t| \geqslant p$, there is a tight p-cycle that visits N_2, N_3 and N_r (in that order) which implies that $\gamma_{r2} = c_0 - \gamma_{23} - \gamma_{3r} = \lambda$. Condition (ii) implies that there is a tight p-cycle that visits N_2, N_1 and N_r (in that order) which implies that $\gamma_{1r} = c_0 - \gamma_{r2} - \gamma_{21} = 2\lambda - 1$ for $r \geqslant 6$. Condition (ii) also implies that there is a tight p-cycle that visits N_1, N_r and N_s (in that order) which implies that $\gamma_{rs} = c_0 - \gamma_{s1} - \gamma_{1r} = \lambda$ for all $r, s \geqslant 2$. Subtracting $\lambda - 1$ times the equation $x(N_1 : N \setminus N_1) - x(N \setminus N_1 : N_1) = 0$, we see that $\boldsymbol{cx} = c_0$ is equivalent to $\lambda \sum_{i \neq j} x(N_i : N_j) = 3\lambda$.

Finally, consider the case $q=1$ and $R \neq \emptyset$. Suppose that $\sigma=\gamma_{23}$. If $|R| \geqslant 2$, condition (ii) implies that there is a tight p-cycle that contains the arcs $(i,j),(j,1)$ for some $i \in N_s$ and $j \in R$. Replacing these arcs by the arc $(i,1)$ and visiting all the nodes in R between N_1 and N_s, we obtain another tight p-cycle. This implies that $\gamma_{s0}=0$ for $s \geqslant 2$, since $\gamma_{01}=\gamma_{s1}=1$ and $c_{jk}=0$ for all $j,k \in R$. If $|R|=1$, then condition (iii) implies that there is a tight p-cycle that contains the arcs $(i,j),(j,1)$ for some $i \in N_s$ and $j \in R$ but skips a node $k \in N_1$ (such a tight p-cycle exists even when $s=t$, since $|N_t| \geqslant 2$ in this case). Replacing the arcs $(i,j),(j,1)$ by the arcs $(i,k),(k,1)$ we obtain another tight p-cycle, and hence $\gamma_{s0}=0$ for $s \geqslant 2$, since $\gamma_{01}=\gamma_{s1}=1$ and $c_{k1}=0$.

Condition (ii) implies that there is a tight p-cycle that visits N_1, R and N_t (in that order). Replacing the nodes in N_t by nodes in a subset N_r with $2 \leqslant r \leqslant t-1$, we see that $\gamma_{0t}+\gamma_{t1}=\gamma_{0r}+\gamma_{r1}$. Since $\gamma_{t1}=\gamma_{r1}=1$, this implies that $\gamma_{0r}=\gamma_{0t}$ for all $2 \leqslant r \leqslant t-1$. Next consider a tight p-cycle that visits N_r, N_s and R (in that order) for $r,s \geqslant 2$. Since $\gamma_{s0}=0$, this implies that $\gamma_{rs}=c_0-\gamma_{0r}=c_0-\gamma_{0t}$, and hence $\gamma_{rs}=\gamma_{23}=\sigma$ for all $r,s \geqslant 2$. If $|R| \geqslant 2$, then because $p \geqslant |N_1|+2 \geqslant 4$, condition (ii) implies that there is a tight p-cycle that visits N_r, a node $k \in R$, N_s and $R \setminus \{k\}$ (in that order). This in turn implies that $c_0=\gamma_{0r}+\gamma_{0s}=2(c_0-\sigma)$, and hence $c_0=2\sigma$. If $|R|=1$, then since $|N_2|+|N_3| \geqslant p$, there is a tight p-cycle that visits only N_2 and N_3, and hence $c_0=2\sigma$. In either case, $\gamma_{or}=c_0-\gamma_{rs}=\sigma$ for all $r \geqslant 2$. Then considering tight p-cycles that visit subsets N_1, N_r and R (in that order) we see that $\gamma_{1r}=c_0-\gamma_{r0}-\gamma_{01}=2\sigma-1$, and considering the reversal of these tight p-cycles we see that $\gamma_{10}=c_0-\gamma_{0r}-\gamma_{r1}=\sigma-1$. Subtracting $\sigma-1$ times the equation $x(N_1 : N \setminus N_1)-x(N \setminus N_1 : N_1)=0$, we see that $\boldsymbol{cx}=c_0$ is equivalent to $\sigma \sum_{i \neq j} x(N_i : N_j)+\sigma x(R : N \setminus R)=2\sigma$. $\square$

Note: Applying Theorem 9 to the equivalent inequality

$$\sum_{r=1}^{t} x(N_r : N_r)+\sum_{j \in R} x(\delta^+(j)) \leqslant p-q-1$$

with $R=\emptyset$ would give the general form of (19), but this approach does not simplify the proof due to the large number of exceptional cases.

3.4. Separation problems

In order to use a class of valid inequalities for branch-and-cut, we must be able to solve the associated separation problem: given a point $\boldsymbol{x}^*$, find an inequality in the class that is violated by $\boldsymbol{x}^*$ or assert that $\boldsymbol{x}^*$ satisfies every inequality in the class. Next we summarize results from Hartmann and Özlük [16] on the complexity of the separation problems for the classes of inequalities based on cuts and partitions of the node set N.

The separation problems for the min-cut and one-sided min-cut inequalities are NP-hard, even if $\boldsymbol{x}^*$ is required to satisfy the flow conservation, cardinality, non-negativity and degree constraints (the reduction is from the EQUICUT problem); however, the problem can be simplified by contracting arcs (i,j) with $x^*_{ij}=1$ for the min-cut inequalities or $x^*_{ij} = \max_k x^*(\delta^+(k))$ for the one-sided min-cut inequalities. The separation problem for the two-sided min-cut inequalities can be solved by finding a minimum (i,j)-cut for each $j \neq i$, where $x^*(\delta^+(i)) = \max_k x^*(\delta^+(k))$, as observed in Fischetti et al. [14]. The separation problem for the max-cut inequalities is NP-hard, even if $\boldsymbol{x}^*$ is required to satisfy the flow conservation, cardinality, non-negativity and degree constraints (the reduction is from the SIMPLE MAX-CUT problem). We suspect that the separation problem for the generalized max-cut inequalities is also NP-hard; however, the problem can be simplified by contracting arcs (i,j) with $x^*_{ij} = 1$.

The separation problem for the class of linear ordering inequalities is NP-hard, even if $\boldsymbol{x}^*$ is required to satisfy the flow conservation, cardinality, non-negativity and degree constraints (the reduction is from the SIMPLE MAX-CUT problem); however, the problem can be simplified by contracting arcs (i,j) with $x^*_{ij} = 1$. We suspect that the separation problem for the multi-cut inequalities is also NP-hard, but we can solve the special case where $|N_r| \leqslant 2$ for $r = 1, 2, \dots, t$ as a maximum weighted matching problem. The separation problem for the multi-cut inequalities can be simplified by contracting arcs (i,j) with $x^*_{ij} + x^*_{ji} = 1$, which allows the matching-based approach to find violated multi-cut inequalities based on more general partitions of the node set.

4. Facets of related polytopes

In this section, we consider when facet-inducing inequalities for the p-cycle polytope induce facets of related polytopes. For example, relaxing the cardinality constraint to $x(N : N) \leqslant p$ or $x(N : N) \geqslant p$, we obtain two related polytopes. The *lower* (*upper*) *p-cycle polytope* is the convex hull of incidence vectors of all simple directed cycles of K_n with at most (at least) p arcs. The lower p-cycle polytope is the directed analog of the cardinality constrained circuit polytope. It is easy to see that the dimension of the both the lower and upper p-cycle polytopes defined on K_n is $(n-1)^2$ for $2 < p < n$, which leads to the following result.

Theorem 18. *Let $\boldsymbol{cx} \leqslant c_0$ induce a facet of the p-cycle polytope Q^n_p, where $2 < p < n$ and $n \geqslant 5$. If μ is the smallest (largest) value such that*

$$\mu x(N : N) + \boldsymbol{cx} \leqslant \mu p + c_0 \tag{21}$$

is valid for the lower (upper) p-cycle polytope, then (21) is facet inducing for the lower (upper) p-cycle polytope.

Corollary 19. *If $\langle N_1, N_2, \dots, N_t, R\rangle$ is a partition of N into $t+1$ subsets that satisfy the conditions of Theorem 17, then the inequality*

$$\mu x(N : N) - \sum_{r=1}^{t} x(N_r : N \setminus N_r) \leqslant \mu p - q - 1 \tag{22}$$

induces a facet of the lower p-cycle polytope, where $\mu = 1/(p - |N_1| - \cdots - |N_q| - |R|)$.

Proof. It suffices to show that (22) is valid for the lower p-cycle polytope, since there is a tight $(p-\mu^{-1})$-cycle that visits $N_1, N_2, \ldots, N_q$ and R (in that order). So let $p' < p$ and let k be the largest integer such that $|N_1| + \cdots + |N_k| + |R| \leqslant p' - 1$. If $k = q$, then every $\boldsymbol{x} \in Q^n_{p'}$ satisfies

$$\mu x(N : N) - \sum_{r=1}^{t} x(N_r : N \setminus N_r) \leqslant \mu p' - q - 1 \leqslant \mu p - q - 1.$$

If $k < q$, then

$$p - p' \geqslant |N_{k+2}| + \cdots + |N_q| + \mu^{-1} \geqslant (q-k)\mu^{-1}$$

because $\mu^{-1} < |N_{q+1}|$. Hence every $\boldsymbol{x} \in Q^n_{p'}$ satisfies

$$\mu x(N : N) - \sum_{r=1}^{t} x(N_r : N \setminus N_r) \leqslant \mu p' - k - 1 \leqslant \mu p - q - 1.$$

Since (22) is valid for the p'-cycle polytope for all $p' \leqslant p$, it is valid for the lower p-cycle polytope. $\square$

Next we consider the p-circuit polytope. We call a valid ineqality $\boldsymbol{cx} \leqslant c_0$ for the p-cycle polytope *symmetric* if it satisfies $c_{ij} = c_{ji}$ for all $i < j$. It is easy to see that the undirected counterpart $\bar{\boldsymbol{c}}\boldsymbol{y} \leqslant c_0$ of a symmetric inequality $\boldsymbol{cx} \leqslant c_0$ (obtained by setting $\bar{c}_{ij} = c_{ij} = c_{ji}$ for all $i < j$) is valid for the p-circuit polytope. An argument of Fischetti [13], originally stated for the ATS and the symmetric traveling salesman polytopes, shows that this also holds for facet-inducing inequalities: Suppose that the symmetric inequality $\boldsymbol{cx} \leqslant c_0$ is facet-inducing for the p-cycle polytope Q^n_p, but its undirected counterpart $\bar{\boldsymbol{c}}\boldsymbol{y} \leqslant c_0$ does not induce a facet of the p-circuit polytope. Then there exists a facet-inducing inequality $\bar{\boldsymbol{d}}\boldsymbol{y} \leqslant d_0$ for the p-circuit polytope such that $\{\boldsymbol{y}: \bar{\boldsymbol{c}}\boldsymbol{y} = c_0\} \subset \{\boldsymbol{y}: \bar{\boldsymbol{d}}\boldsymbol{y} = d_0\}$ with strict containment. But this implies that the directed counterpart $\boldsymbol{dx} \leqslant d_0$ of $\bar{\boldsymbol{d}}\boldsymbol{y} \leqslant d_0$ (obtained by setting $d_{ij} = d_{ji} = \bar{d}_{ij}$ for all $i < j$) is valid for the p-cycle polytope and induces a proper face that strictly contains the face of Q^n_p induced by $\boldsymbol{cx} \leqslant c_0$, a contradiction.

Because the flow conservation constraints are not themselves symmetric, it is necessary to determine if a given facet-inducing inequality is equivalent to a symmetric inequality. This turns out to be relatively straight forward: $\boldsymbol{cx} \leqslant c_0$ is equivalent to a symmetric inequality if and only if the system $t_i - t_j = c_{ij} - c_{ji}$ for $i < j$ is consistent (see [8] for related results). The degree constraints are symmetric, as are the min-cut, one-sided min-cut, two-sided min-cut, max-cut and multi-cut inequalities. Many of their undirected counterparts are known to be facet-inducing for the p-circuit polytope (see [21]); however, the one-sided min-cut and multi-cut inequalities yield new facet-inducing inequalities for the p-circuit polytope as corollaries of Theorems 13 and 17.

Corollary 20. *Let $S \subset N$ with $2 \leqslant |S| \leqslant p - 1$. The inequality*

$$y(S : N \setminus S) \geqslant y(\delta(j))$$

induces a facet of the p-circuit polytope if $p = 3$ and $n \neq 5$ or $p \geqslant 4$ and $|N \setminus S| \geqslant p$.

Corollary 21. *If $\langle N_1, N_2, \ldots, N_t, R\rangle$ is a partition of N into $t+1$ subsets that satisfy the conditions of Theorem* 17, *then the inequality*

$$\sum_{r=1}^{t} y(N_r : N \setminus N_r) \geqslant 2q + 2 \tag{23}$$

induces a facet of the p-circuit polytope.

Note: Nguyen and Maurras [21, Proposition 5] show that in most cases, (23) also induces a facet of the p-circuit polytope in the exceptional case $q = 1$ and $R = \emptyset$.

The argument of Fischetti can also be combined with Theorem 18 to obtain inequalities that induce facets of the cardinality constrained circuit polytope. Many of the resulting inequalities are known to be facet-inducing (see [7]); however, the max-cut and multi-cut inequalities yield new facet-inducing inequalities for the cardinality constrained circuit polytope as corollaries of Theorems 15 and 17.

Corollary 22. *Let $S \subset N$ with $p/2 < |S| < n - p/2$. The inequality*

$$y(S : N \setminus S) \leqslant p - 1$$

induces a facet of the cardinality constrained circuit polytope if $n \geqslant 5$ and p is odd.

Corollary 23. *If $\langle N_1, N_2, \ldots, N_t, R\rangle$ is a partition of N into $t+1$ subsets that satisfy the conditions of Theorem* 17, *then the inequality*

$$2\mu y(E) - \sum_{r=1}^{t} y(N_r : N \setminus N_r) \leqslant 2\mu p - 2q - 2$$

induces a facet of the cardinality constrained circuit polytope, *where* $\mu = 1/(p - |N_1| - \cdots - |N_q| - |R|)$.

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