RANDOM PERMUTATIONS AND UNIQUE FULLY SUPPORTED ERGODICITY FOR THE EULER ADIC TRANSFORMATION

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ABSTRACT. There is only one fully supported ergodic invariant probability measure for the adic transformation on the space of infinite paths in the graph that underlies the Eulerian numbers. This result may partially justify a frequent assumption about the equidistribution of random permutations.

1. INTRODUCTION

We give a new proof of the main result of [?], ergodicity of the symmetric measure for the Euler adic transformation. In fact we prove a stronger result by a different method: the symmetric measure is the *only* fully supported ergodic invariant probability measure for the Euler adic system.

This result has a probabilistic interpretation. For each $n = 0, 1, 2, \ldots, k = 0, 1, \ldots n$, the *Eulerian number* $A(n, k)$ is the number of permutations $i_1 i_2 \ldots i_{n+1}$ of $\{1, \ldots, n+1\}$ with exactly k *rises* (indices $j = 1, 2, \ldots, n$ with $i_j < i_{j+1}$) and $n - k$ *falls* (indices $j = 1, 2, \ldots, n$ with $i_j > i_{j+1}$). (See, for example, [?] for basic facts about the Eulerian numbers.) Besides their combinatorial importance, these numbers are also of interest in connection with the statistics of rankings: see, for example, [?],[?],[?],[?], and [?]. In studying random permutations, it is often assumed that all permutations are equally likely, each permutation of length $n + 1$ occurring with probability $1/(n + 1)!$. Our main result implies that in a sense (see Remark 1) there is a unique way to choose permutations at random so that (1) consistency holds: the distribution on permutations of $\{1, \ldots, n+m\}$ induces, upon deletion of $\{n+1, \ldots, m\}$, the distribution on $\{1, \ldots, n\}$; (2) any two permutations of the same length which have the same number of rises are equally likely; and (3) every permutation has positive probability. This unique way is to make all permutations of the same length, *no matter the number of rises*, equally likely.

This conclusion is achieved by means of ergodic theory, studying invariant measures for the adic (Bratteli-Vershik) transformation on the space of infinite paths on the Euler graph. We do not see how to prove this result by a straightforward combinatorial argument, because it is very difficult to make comparative asymptotic estimates of Eulerian numbers. Indeed, the key to our argument is to finesse this difficulty (in the proof of Proposition 3) by setting up a situation in which the dominant terms of the two expressions being compared are identical.

Not having the space here to survey all the literature on adic systems, we remark just that in certain senses they represent all dynamical systems, also have strong connections with group representations, C^* algebras, probability theory, and combinatorics, and thus facilitate the study of important problems in those areas. See

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[**?**, **?**, **?**, **?**, **?**, **?**] and the references that these works contain for background. Basic attributes of adic systems are the various kinds of dimension groups, which provide invariants, and the states or traces, which correspond to adic-invariant, or "central", measures—see, for example, [**?**]. These have been determined for many stationary and essentially simple adic systems and some classes of nonstationary and nonsimple ones [**?**, **?**], but not yet for many key individual examples. For the Pascal adic, the identification of the ergodic invariant measures as the one-parameter family of Bernoulli measures on the 2-shift is closely connected with de Finetti's Theorem, the Hewitt-Savage 0,1 Law, and uniform distribution of the Kakutani interval splitting procedure—see the discussion in [**?**]. Identifying the ergodic measures for new natural examples such as the Euler adic will also have consequences in other areas: mentioned below are connections with random permutations (Remark 1) and reinforced random walks (Remark 4).

2. The Euler adic system

The *Euler graph* Γ is a Bratteli diagram with levels $n = 0, 1, 2, \ldots$. There are $n + 1$ vertices of each level n, labelled (n, k), $0 \leq k \leq n$, and each vertex (n, k) is connected to vertex $(n+1, k)$ by $k+1$ edges and to vertex $(n+1, k+1)$ by $n-k+1$ edges. The number of paths into any vertex (n, k) is the Eulerian number $A(n, k)$. The Eulerian numbers have the recursion

$$(1) \qquad A(n, k) = (n - k + 1)A(n - 1, k - 1) + (k + 1)A(n - 1, k),$$

where $A(0, 0) = 1$ and by convention $A(n, k) = 0$ for $k \notin \{0, 1, ..., n\}$.

Level

0

1

2

3

4

5

$k = 0 \qquad k = 1 \qquad k = 2 \qquad k = 3 \qquad k = 4 \qquad k = 5$
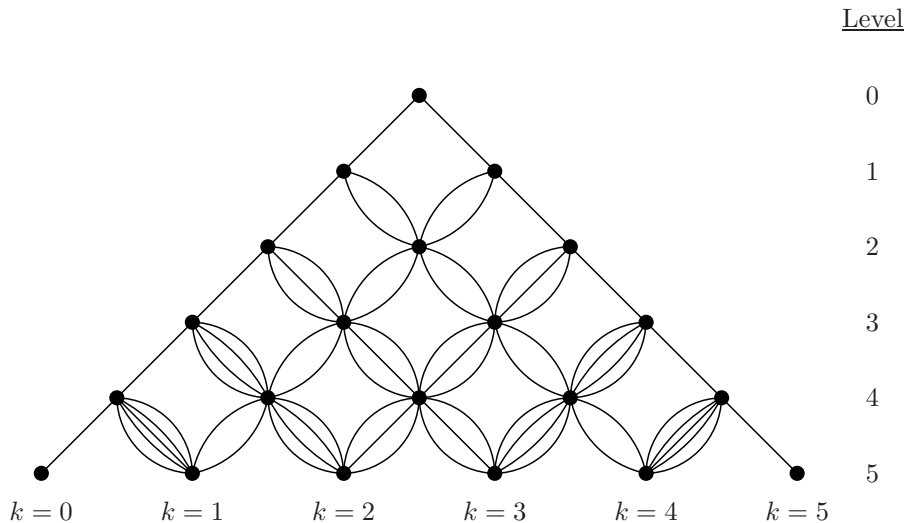
FIGURE 1. The Euler graph down to level 5

Define X to be the space of infinite edge paths in this graph: $X = \{x = (x_n) | n = 0, 1, 2, \ldots \}$, each x_n being an edge from level n to level $n + 1$. The vertex through which x passes at level n is denoted by $(n, k_n(x))$. We say that an edge is a *left turn* if it connects vertices (n, k) and $(n + 1, k)$ and a *right turn* if it connects vertices

(n, k) and $(n+1, k+1)$. X is a metric space in the usual way: $d(x, y) = 2^{-j}$, where $j = \inf\{i | x_i \neq y_i\}$. The cylinder sets, where a finite number of edges are specified, are clopen sets that generate the topology of X.

As with other Bratteli diagrams, we define a partial order on the edges in the diagram which extends to the entire space of paths. Two edges are comparable if they terminate in the same vertex. For each vertex, totally order the set of edges that terminate at that vertex. This is pictured by agreeing that the minimal edge between two vertices is the leftmost edge, and the edges increase from left to right. Two paths, x, y, are comparable if they agree after some level. Then x is less than y if the last edges of x and y that do not agree, x_i and y_i, are such that x_i is less than y_i in the edge ordering. We define the *adic transformation* T on the set of non-maximal paths in X to map a path to the next largest path according to this induced partial order. Then two paths are in the same orbit if and only if they are comparable. Figure 2 shows a particular path x and its image $y = Tx$. There are a countable number of paths for which there is no next largest path, we denote the set of such paths by $X_{\max}$. In particular for each k in $\mathbb{N} \cup \{\infty\}$, we define $x_{\max(k)}$ to be the infinite edge path that follows the unique finite path from $(0,0)$ to (k, k) and for $n \geq k$, follows the right most edge connecting vertices (n, k) and $(n+1, k)$. In a similar fashion there are a countable number of paths for which there is no next smaller path, denoted by $X_{\min}$, and for every k in $\mathbb{N} \cup \{\infty\}$ a unique path $x_{\min(k)} \in X_{\min}$. The transformation T maps $x_{\max k}$ to $x_{\min(k)}$.
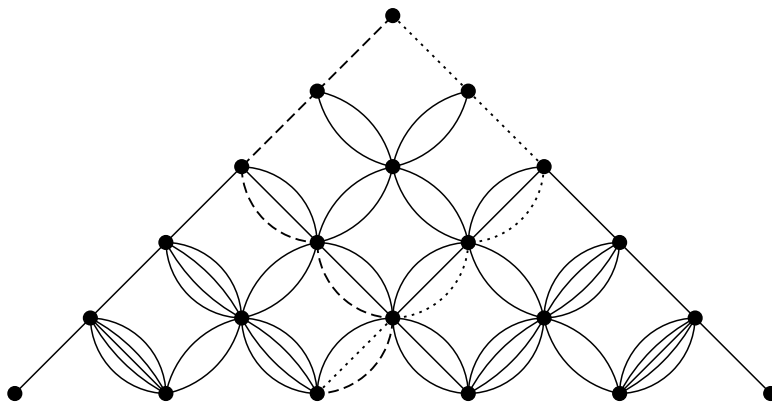

FIGURE 2. The dotted path maps to the dashed path

The *symmetric measure*, η, on the infinite path space X of the Euler graph is the Borel probability measure that for each n gives every cylinder of length n starting at the root vertex the same measure. Clearly η is T-invariant. The measure of any cylinder set can be computed by multiplying *weights* on the edges, each weight on an edge connecting level n to level $n+1$ being $1/(n+2)$. We can think of the weights as assigning equal probabilities to all the allowed steps for a random walker who starts at the root and descends step by step to form an infinite path $x \in X$. The
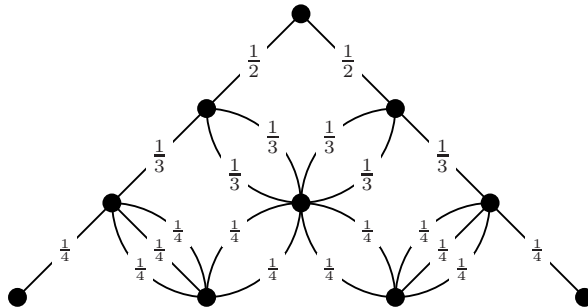
FIGURE 3. The symmetric measure given by weights on edges

main result of [**?**], proved by probabilistic methods, is that the symmetric measure is ergodic.

There is a bijective correspondence between paths (or cylinders) of length n_0 starting at the root vertex and terminating at vertex (n_0, k_0) and permutations of $\{1, 2, \ldots, n_0 + 1\}$ with k_0 rises. This correspondence is crucial for our proof of Proposition 3, which is the essential component of our main result. Consider the cylinder set defined by the single edge connecting the vertex $(0, 0)$ to the vertex $(1, 0)$. This cylinder set is of length 1 with 1 left turn, and we assign to it the permutation 21, which has one fall. Likewise, the cylinder set defined by the single edge connecting the vertex $(0, 0)$ to the vertex $(1, 1)$ is of length 1 with one right turn, and we assign to it the permutation 12, which has one rise. When a cylinder F of length n, corresponding to the permutation $\pi(F)$ of $\{1, 2, \ldots, n+1\}$, is extended by an edge from level n to level $n + 1$, we extend $\pi(F)$ in a unique way to a permutation of $\{1, 2, \ldots, n + 2\}$, as follows. If F is extended by a left turn down the i'th edge connecting (n, k) to $(n+1, k)$, insert $n + 2$ into $\pi(F)$ in the i'th place that adds an additional fall to the total number of falls of $\pi(F)$. Likewise, if F is extended by a right turn down the i'th edge connecting (n, k) to $(n + 1, k + 1)$, insert $n + 2$ into $\pi(F)$ in the i'th place that adds an additional rise to the total number of rises of $\pi(F)$.

Example 1. We shall determine which cylinder F corresponds to the permutation $\pi(F) = 2341$. When we delete 34, we have the permutation 21, which means the first edge of F is the unique edge connecting vertex (0,0) to (1,0). When the 3 is added, it is added into the middle of 21, creating the permutation 231, which has one rise and one fall. Therefore a rise has been added, and there is no earlier space that the 3 could have been placed to add a rise to 21. Therefore, the second edge of F is the first edge connecting vertex (1,0) to vertex (2,1). Since $\pi(F)$ has 2 rises and one fall, the addition of 4 into 2341 adds another rise. It is added in the first place it could be added in order to add a rise, meaning that the third edge of F must be the first edge connecting vertex (2,1) to (3,2). See Figure 4.

This correspondence produces a labeling of infinite paths in the Euler graph starting at the root; then the path space X corresponds to the set of all linear

orderings of $\mathbb{N} = \{1, 2, 3, \dots\}$, and the adic transformation T can be thought of as moving from an ordering to its successor.
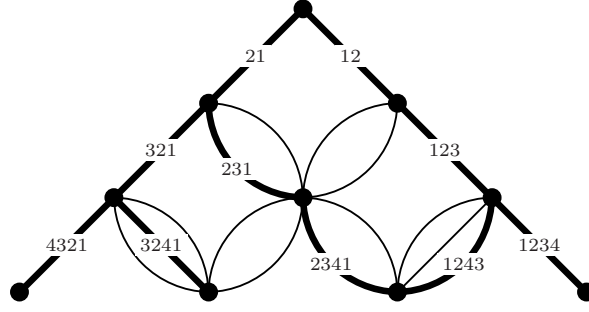

FIGURE 4. Some cylinders and their corresponding permutations

3. PROOF OF THE MAIN RESULT—UNIQUENESS OF THE SYMMETRIC MEASURE

The *dimension* $\dim(n, k)$ of a vertex (n, k) is defined to be the number of paths connecting the root vertex $(0, 0)$ to the vertex (n, k). For any cylinder F define $\dim(F, (n, k))$ to be the number of paths in F that connect the root vertex $(0, 0)$ to the vertex (n, k). If F is a cylinder starting at the root vertex $(0, 0)$, then $\dim(F, (n, k))$ is the number of paths from the terminal vertex of F to (n, k). Denote by $\mathcal{I}$ the σ-algebra of T-invariant Borel measurable subsets of X. The following is a well-known result about the measures of cylinders in an adic situation.

Lemma 1 (Vershik [?],[?]). *Let μ be an invariant probability measure for the Euler adic transformation. Then for every cylinder set F and μ-almost every $x \in X$,*

$$\lim_{n \to \infty} \frac{\dim(F, (n, k_n(x)))}{\dim((n, k_n(x)))} = E_\mu(\chi_F | \mathcal{I}).$$

It is clear from the Markov property and the patterns of weights on the edges that with respect to η almost every path $x \in X$ has infinitely many left and right turns.

Lemma 2. *Let μ be an invariant fully-supported ergodic probability measure for the Euler adic transformation. For μ-almost every $x \in X$, there are infinitely many left and right turns (i.e., $k_n(x)$ and $n - k_n(x)$ are unbounded a.e.).*

Proof. For each $K = 1, 2, \dots$, let

$$A_K = \{x \in X | x \text{ has no more than } K \text{ right turns}\}.$$

Then A_K is a proper closed T-invariant set. Since μ is ergodic and fully supported, $\mu(A_K) = 0$. Similarly, the measure of the set of paths with a bounded number of left turns is 0. $\square$

Proposition. *If μ is an invariant probability measure for the Euler adic transformation such that $k_n(x)$ and $n - k_n(x)$ are unbounded a.e., then $\mu = \eta$.*

Proof. For any string w on an ordered alphabet denote by $r(w)$ the number of rises in w and by $f(w)$ the number of falls in w (defined as above). Let F and F' be cylinder sets in X specified by fixing the first n_0 edges, and let $\pi(F)$ and $\pi(F')$ be the permutations assigned to them by the correspondence described in the preceding section. Suppose that the paths corresponding to F and F' terminate in the vertices (n_0, k_0) and (n_0, k_0') respectively. Fix $n \gg n_0$ and $k \gg k_0$.

We will show that $\mu(F) = \mu(F')$, and hence $\mu = \eta$. If $k_0 = k_0'$, then $\mu(F) = \mu(F')$ by the T-invariance of μ, so there is nothing to prove. We will deal with the case $k_0 \neq k_0'$.

Example 2. Let F be the dotted cylinder, and F' the dashed. Then $\pi(F) = 213$ and $\pi(F') = 132$. For purposes of following an example through, we will let $(n, k) = (8, 4)$.

$$(n_0, k_0) = (2, 1)$$

$$(n, k) = (8, 4)$$
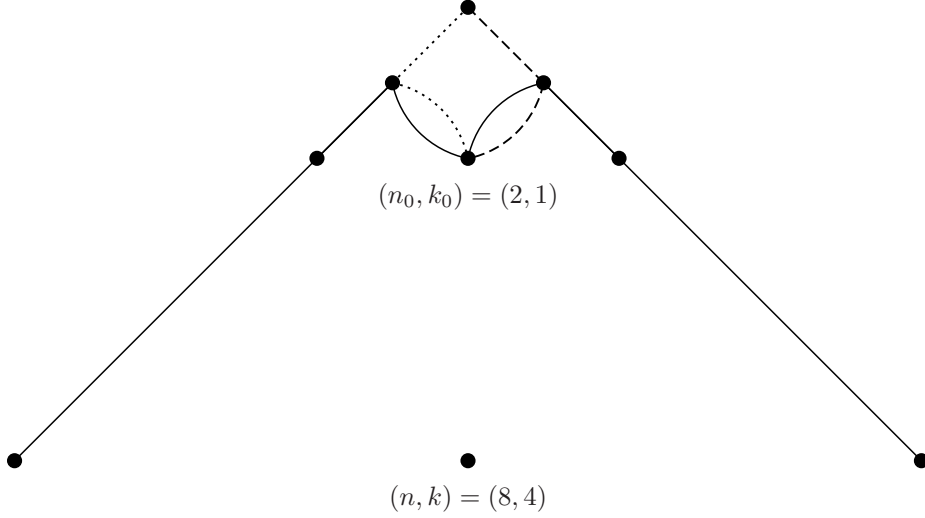
Each path s in Γ from (n_0, k_0) to (n, k) corresponds to a permutation σ_s of $\{1, 2, \ldots, n+1\}$ with k rises in which $1, 2, \ldots, n_0 + 1$ appear in the order $\pi(F)$. Counting $\dim(F, (n, k))$ is equivalent to counting the number of distinct such σ_s. Each such permutation σ_s has associated to it a permutation $t(\sigma_s)$ of $\{n_0+2, \ldots, n+1\}$ obtained by deleting $1, 2, \ldots, n_0 + 1$ from σ_s, see Example 3. Taking a reverse view, one obtains σ_s from $\rho = t(\sigma_s)$ by inserting $1, 2, \ldots, n_0 + 1$ from left to right, in the order prescribed by $\pi(F)$, into ρ.

We define a *cluster* in σ_s to be a subset of $\{1, 2, \ldots, n_0 + 1\}$ whose members are found consecutively in σ_s, with no elements of $\{n_0 + 2, \ldots, n + 1\}$ separating them, in the order prescribed by $\pi(F)$. The set M_s of clusters in σ_s is an ordered partition of the permutation $\pi(F)$, and we define

$$r(M_s) = \sum_{c \in M_s} r(c).$$

In general, $1 \leq |M_s| \leq n_0 + 1$ and $0 \leq r(M_s) \leq k_0$.

Example 3. $\pi(F) =$ **213**

$$\sigma_{s_1} = \mathbf{2}97\mathbf{1}4\mathbf{6}385 \qquad\qquad \sigma_{s_2} = 96\mathbf{2}47\mathbf{13}58$$
$$t(\sigma_{s_1}) = 974685 \qquad\qquad t(\sigma_{s_2}) = 964758$$
$$M_{s_1} = \{2,1,3\} \qquad\qquad M_{s_2} = \{2,13\}$$
$$r(M_{s_1}) = 0 \qquad\qquad r(M_{s_2}) = 1$$

Given a permutation ρ of $\{n_0 + 2, \ldots, n + 1\}$, $0 \le m \le |\rho| + 1$, and an ordered partition M of $\pi(F)$ with $|M| = m$, there are $C(|\rho|+1, m)$ (the binomial coefficient) choices for how to insert the members of M as clusters into the permutation ρ in order to form a permutation σ_s. But not all of these choices yield valid permutations σ_s, which have exactly k rises. Looking more closely, we see that placing a cluster $c \in M$ at the tail end of ρ or into a rise in ρ produces a permutation $\overline{\rho}$ whose number of rises is $r(\overline{\rho}) = r(\rho) + r(c)$, while placing c at the beginning or into a fall produces $\overline{\rho}$ with $r(\overline{\rho}) = r(\rho) + r(c) + 1$. So we must have

$$k = r(\sigma_s) = r(\rho) + r(M) +$$

$$\#\{c \in M | c \text{ is placed into a fall or at the beginning of } \rho\}.$$

In order to count the number of valid ways to place the members of M into ρ, we will first determine how many ways there are to place the clusters that will create a new rise. There are $n - n_0 - (k - k_0) + 1$ possible places in ρ to place a cluster of M and create a new rise (the $n - n_0 - (k - k_0)$ falls and at the end of ρ), and since $|\{c \in M | c \text{ is placed into a fall or at the beginning of } \rho\}|$ is $k - r(\rho) - r(M)$ we must choose $l(r(\rho), M) = k - r(\rho) - r(M)$ of them. For each of these possibilities we must then choose places to place the remaining clusters of M. There are $k - k_0 + 1$ places, and $m - (k - r(\rho) - r(M))$ remaining clusters. Therefore the number of ways (given m, M, and $r = r(\rho)$) to place the members of M into ρ in such a way as to form a valid permutation σ_s, with k rises and $n - k$ falls, is

$$C(n - n_0 - (k - k_0) + 1, k - r(\rho) - r(M))\, C(k - k_0 + 1, m - (k - r(\rho) - r(M))).$$

For each $m = 1, 2, \ldots, n_0 + 1$ denote by $P_m(F)$ the set of ordered partitions M of $\pi(F)$ such that $|M| = m$. For each $r = 0, 1, \ldots, n - n_0 - 1$ denote by $Q(n, n_0, r)$ the set of permutations of $\{n_0 + 2, \ldots, n + 1\}$ with exactly r rises, so that

$$|Q(n, n_0, r)| = A(n - n_0 - 1, r).$$

Example 4. $\pi(F) = 213$, $(n, k) = (8, 4)$.

$$P_1(F) = \{\{213\}\}$$
$$P_2(F) = \{\{2, 13\}, \{21, 3\}\}$$
$$P_3(F) = \{\{2, 1, 3\}\}$$
$$Q(8, 2, 0) = \{987654\} \qquad\qquad |Q(8, 2, 0)| = A(5, 0) = 1$$
$$Q(8, 2, 5) = \{456789\} \qquad\qquad |Q(8, 2, 5)| = A(5, 5) = 1$$

In order to compute $\dim(F, (n, k))$ we will partition the set of permutations σ_s of $\{1, 2, \ldots, n + 1\}$ which have exactly k rises and a subpermutation $\pi(F)$ in the following manner. First, partition the set of all such permutations according to the cardinality $m \in \{1, \ldots, n_0 + 1\}$ of the set of clusters. Now partition further by grouping the σ_s by their corresponding $M_s \in P_m(F)$. Partition yet again by grouping the σ_s by the number of rises, r, in their corresponding permutations $t(\sigma_s)$ of $\{n_0 + 2, \ldots, n + 1\}$. Since the total number of rises in each σ_s must be k, and there are $r(M)$ rises in M, the minimum such r is $k - r(M) - m$ and the maximum

is $k - r(M)$. Finally, partition these groups by the distinct permutations $t(\sigma_s)$ of $\{n_0 + 2, \ldots, n + 1\}$ which meet the foregoing criteria. Then for each r there are $|Q(n, n_0, r)| = A(n - n_0 - 1, r)$ of these groups (the same number for each F, m, M), and each group has cardinality

$$C(n - n_0 - (k - k_0) + 1, k - r - r(M))\, C(k - k_0 + 1, m - (k - r - r(M))).$$

Example 5. Fix $\pi(F) = 213$ and $(n, k) = (8, 4)$.
  1. Partition by m.

$$
\begin{aligned}
m = 1 &\quad : \quad \sigma_s = \ldots 213 \ldots \\
m = 2 &\quad : \quad \sigma_s = \ldots 2 \ldots 13 \ldots \text{ or } \sigma_s = \ldots 21 \ldots 3 \ldots \\
m = 3 &\quad : \quad \sigma_s = \ldots 2 \ldots 1 \ldots 3 \ldots
\end{aligned}
$$

  2. Fix $m = 2$, partition by M.

$$
\begin{aligned}
\ldots 2 \ldots 13 \ldots &\qquad \text{appears in each } \sigma_s \\
\ldots 21 \ldots 3 \ldots &\qquad \text{appears in each } \sigma_s
\end{aligned}
$$

  3. Fix $m = 2$, $M = \{2, 13\}$, then $r(M) = 1$, partition over $r(t(\sigma_s))$.

$$
\begin{aligned}
r(t(\sigma_s)) &= 1 \\
r(t(\sigma_s)) &= 2 \\
r(t(\sigma_s)) &= 3
\end{aligned}
$$

  4. Fix $m = 2$, $M = \{2, 13\}$, $r(M) = 1$, $r(t(\sigma_s)) = 3$, then partition over $t(\sigma_s)$ in $Q(8, 2, 3)$. There are $A(5, 3) = 302$ sets in this partition, and each one contains $C(4, 0)C(4, 2) = 6$ elements. (To each permutation of 4,5,6,7,8 we can now add only one rise, and $r(M)$ is already 1, so all the clusters must be inserted into rises or at the end.)

For ease of notation, for each r, let $l = l(r, M) = \inf\{|M|, k - r - r(M)\}$ and $p = p(M) = k - r(M) - |M|$. Then $\dim(F, (n, k)) =$

$$\sum_{m=1}^{n_0+1} \sum_{M \in P_m(F)} \sum_{r=p}^{k-r(M)} A(n - n_0 - 1, r)C(n - n_0 - (k - k_0) + 1, l)\, C(k - k_0 + 1, m - l).$$

Regrouping the terms, rewrite the sum as

$$(2) \qquad \dim(F, (n, k)) = \sum_{r=k-(n_0+1)}^{k} \left(\sum_{m=1}^{n_0+1} \alpha(F, r, m) \right) A(n - n_0 - 1, r),$$

where we have

$$\alpha(F, r, m) = \sum_{M \in P_m(F)} C(n - n_0 - (k - k_0) + 1, l)C(k - k_0 + 1, m - l).$$

In each term in this sum, the numerator of the first binomial coefficient factor simplifies to having l factors chosen from $[n - n_0 - (k - k_0) - m, n - n_0 - (k - k_0)]$, and the numerator of the second has $m - l$ factors chosen from $[k - k_0 - m, k]$. Combining the two, the numerator of the binomial factors in each individual term in this sum consists of m factors chosen from $[n - k - m, n - k] \cup [k - m, k]$, and the product of the denominators is bounded above by $(m!)^2$. Since $m \le n_0 + 1$, each of the m factors is comparable to (i.e., between two constant multiples of) either k or $n - k$. Then we see that

$$\beta(F,r) = \sum_{m=1}^{n_0+1} \alpha(F,r,m)$$

is a polynomial in k and $n-k$ of degree n_0+1.

Clearly the dominant term of each $\beta(F,r)$ as k and $n-k$ grow large is the one of maximum degree, $m = n_0 + 1$. This is exactly the term $\alpha(F,r,n_0+1)$, which corresponds to the partition M of $\pi(F)$ into singletons. Then $r(M) = 0$, and all the elements of $\{1,\ldots,n_0+1\}$ are placed, as singleton clusters, into a permutation ρ of $\{n_0+2,\ldots,n+1\}$ which has exactly r rises. Since every element of $\{1,\ldots,n_0+1\}$ is less than every element of the set $\{n_0+2,\ldots,n\}$, the order of the permutation $\pi(F)$ has no effect on $\alpha(F,r,n_0+1)$. The *main point* is that this term is the same for $\pi(F)$ as for any other permutation $\pi(F')$ of $\{1,2,\ldots,n_0+1\}$:

$$\alpha(F,r,n_0+1) = \alpha(F',r,n_0+1)$$

for all r and all F'. When $k-r$ of $\{1,\ldots,n_0+1\}$ are put into falls in ρ or at the beginning, and the rest are put into rises or at the end, *no matter which elements are placed in which slots* we always produce a permutation σ_s with exactly k rises.

Let us now consider the ratio $\dim(F,(n,k))/\dim(F',(n,k))$ when n and k are both very large. Divide top and bottom by the sum on r of the dominant terms (taking maximum degree coefficients in k and $n-k$ for each r),

$$\sum_{r=k-(n_0+1)}^{k} \alpha(F,r,n_0+1)A(n-n_0-1,r),$$

which is the *same* for F and F'. This shows that the ratio is very near 1 when k and $n-k$ are both very large.

Thus if $k_n(x)$, $n-k_n(x) \to \infty$ a.e. $d\mu$, we have for any two cylinders F and F' starting at the root vertex and of the same length that

$$E_\mu(X_F|\mathcal{I})(x) = E_\mu(X_{F'}|\mathcal{I})(x) \quad \text{a.e. } d\mu.$$

Integrating gives $\mu(F) = \mu(F')$, so that $\mu = \eta$. $\qquad\square$

Theorem. *The symmetric measure η is ergodic and is the only fully supported invariant ergodic Borel probability measure for the Euler adic transformation.*

Proof. If we show that there is an ergodic measure μ which has $k_n(x)$ and $n-k_n(x)$ unbounded a.e., then it will follow from the Proposition that $\mu = \eta$, and hence η is ergodic and is the only fully-supported T-invariant ergodic measure on X.

If an ergodic measure has, say, $k_n(x)$ bounded on a set of positive measure, then $k_n(x)$ is bounded a.e., since each set $\{x|k_n(x) \leq K\}$ is T-invariant. Let $\mathcal{E}_0 = \emptyset$, and for each $K = 1,2,\ldots$ let $\mathcal{E}_K$ be the set of ergodic measures supported on either $\{x \in X | k_n(x) \leq K \text{ for all } n\}$ or $\{x \in X | n-k_n(x) \leq K \text{ for all } n\}$. If no ergodic measure has $k_n(x)$ and $n-k_n(x)$ unbounded a.e., then the set of ergodic measures is

$$\mathcal{E} = \bigcup_K \mathcal{E}_K.$$

Form the ergodic decomposition of η:

$$\eta = \int_{\mathcal{E}} e\, dP_\eta(e) = \sum_{K=1}^{\infty} \int_{\mathcal{E}_K \setminus \mathcal{E}_{K-1}} e\, dP_\eta(e).$$

If S is the set of paths $x \in X$ for which both $k_n(x)$ and $n - k_n(x)$ are unbounded, then, from the remark before Lemma 2, $\eta(S) = 1$; but, for each K, $e(S) = 0$ for all e in $\mathcal{E}_K$. Hence there is an ergodic measure for which $k_n(x)$ and $n - k_n(x)$ are unbounded a.e.. $\qquad\square$

4. Concluding remarks

Remark 1. The connection of this Theorem with the statements made in the Abstract and Introduction about random permutations can be seen as follows. As noted above, the space X of infinite paths in the Euler graph is in correspondence with the set $\mathcal{L}$ of linear orderings of $\mathbb{N}$. A cylinder set in X determined by fixing an initial path of length n corresponds, as explained above, to a permutation π_{n+1} in the group S_{n+1} of permutations of $\{1, 2, \ldots, n+1\}$, and thus to the set $\mathcal{L}(\pi_{n+1})$ of all elements of $\mathcal{L}$ for which $1, 2, \ldots, n+1$ appear in the order specified by π_{n+1}. This family of clopen cylinder sets defines a compact metrizable topology and a Borel structure on $\mathcal{L}$. One way to speak about "random permutations" would be to give a Borel probability measure on $\mathcal{L}$. A Borel probability measure on X is T-invariant if and only if the corresponding measure on $\mathcal{L}$ assigns, for each n and $0 \leq k \leq n$, equal measure to all cylinders determined by permutations $\pi_{n+1} \in S_{n+1}$ which have the *same fixed number k of rises.* According to the Theorem, the only such fully supported measure is the one determined by the symmetric measure η on X, which assigns to each basic set $\mathcal{L}(\pi_{n+1})$, for all $\pi_{n+1} \in S_{n+1}$, *no matter the number of rises* in π_{n+1}, the *same* measure, $1/(n+1)!$.

Remark 2. It is easy to find measures for the Euler adic which are not fully supported, just by restricting to closed T-invariant sets. For example, if we restrict to the subgraph consisting of paths x with vertices $(n, k_n(x))$ with $k_n(x) = 0$ or 1 for all n, then there are invariant measures determined by the edge weights in Figure 5, provided that $\alpha_{n+1} = \alpha_n/(2 - 2n\alpha_n)$, for example $\alpha_n = 1/2(n+1)$. All such systems are of finite rank (see [**?**] for the definition). Since measures which are not fully supported do not involve the full richness of the Euler graph, we regard them as less interesting than fully supported ergodic measures.
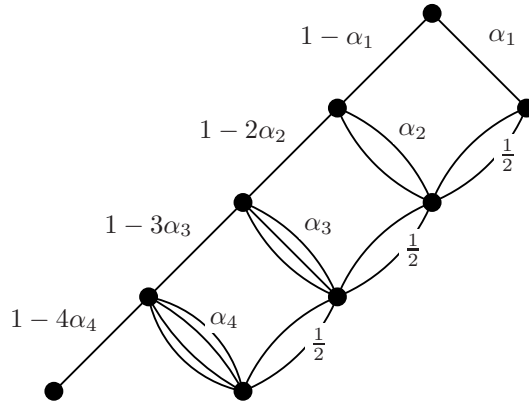

FIGURE 5. The invariant measures when T is restricted to the paths x with vertices $(n, k_n(x))$ with $k_n(x) = 0$ or 1 for all n.

Remark 3. For dynamic properties of the Euler adic with its symmetric measure beyond ergodicity, so far we know that this system is totally ergodic (there are no roots of unity among its eigenvalues)[**?**], has entropy 0, and is loosely Bernoulli [**?**] (see [**?**] for the definition). Determining whether the system is weakly or even strongly mixing and whether it has infinite rank are important, challenging, open questions.
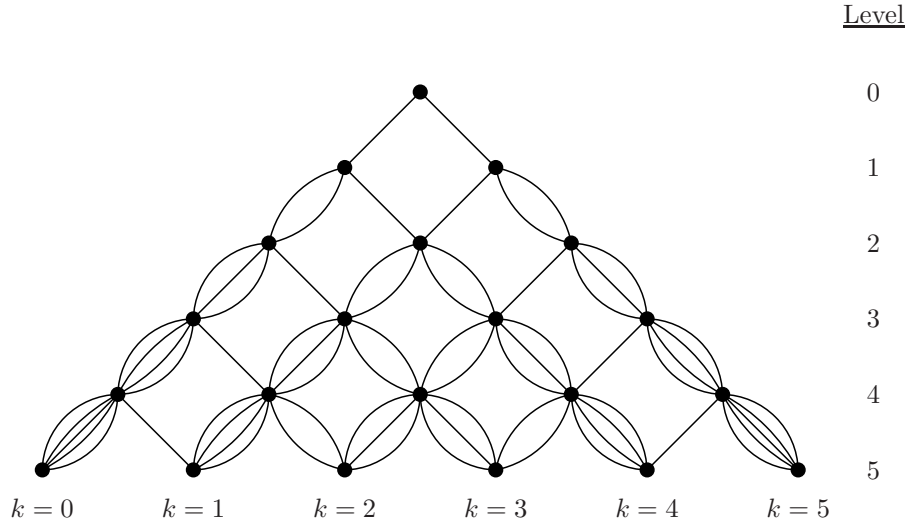

FIGURE 6. The Reverse Euler graph

Remark 4. The Euler adic system models a negatively reinforced random walk on the finite graph consisting of two loops based at a single vertex. In other words, each time an individual loop is followed the probability that it will be followed at the next stage decreases. The adic system based on the Bratteli diagram in Figure 6, which we call the *reverse Euler system*, models the positively reinforced random walk (each time an individual loop is followed the probability that it will be followed at the next stage increases) on this graph. Identification of the ergodic measures for such systems and determination of their dynamical properties has implications for the properties of the random walks. While the Euler adic has only one fully supported ergodic measure as we have shown, there is an infinite (one-parameter) family of fully supported ergodic measures for the reverse Euler adic. The connection between adic systems of this kind and reinforced random walks will be explored in a future publication [**?**].

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