An Application of Neutrix Calculus to Quantum Field Theory

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Abstract

Neutrices are additive groups of negligible functions that do not contain any constants except 0. Their calculus was developed by van der Corput and Hadamard in connection with asymptotic series and divergent integrals. We apply neutrix calculus to quantum field theory, obtaining *finite* renormalizations in the loop calculations. For renormalizable quantum field theories, we recover all the usual physically observable results. One possible advantage of the neutrix framework is that effective field theories can be accommodated. Quantum gravity theories appear to be more manageable.

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I. INTRODUCTION

Quantum field theory is the proud product of quantum mechanics and special relativity. It is difficult to contemplate modern theoretical physics without it, for the very simple reason that it is extraordinarily successful. Coleman, a master of quantum field theory, compares the triumph of quantum field theory to "a glorious triumph parade, full of wonderful things" that make "the spectator gasp with awe and laugh with joy."[1] But there is a fly in the ointment: in general, infinities pop up in loop corrections in quantum field theory. In renormalizable theories, these infinities can be renormalized away in a process that Feynman likened to sweeping dirt under the rug. Even quantum electrodynamics, arguably the most phenomenologically successful theory in physics, has this "defect." Dirac was quoted to have said, "Quantum electrodynamics is almost right." He was bemoaning the divergent integrals that plague loop calculations in QED. And it is probably not wrong to think that Schwinger put forth his source theory approach to replace the conventional quantum field theory partly because of the ultraviolet divergences in the latter.[2] To put it simply, quantum field theorists are confronted with the challenge to rid quantum field theory of infinities while preserving its many spectacular successes in the process.

In another development, Dyson showed that the power series in the electron charge e for QED cannot be a convergent series.[3] His argument went roughly as follows. If the series is convergent, then such a series has a finite radius of convergence in the complex plane around $e = 0$. This means that one can replace e by ie without encountering a discontinuity. But the physics under this replacement is actually very discontinous: electrons would attract each other, likewise for positrons; on the other hand, opposite charges would repel each other so that electrons would move away as far as possible from positrons. Then every physical state would be unstable against the spontaneous creation of large numbers of particles. Thus the series for QED cannot be a convergent series. Dyson went on to suggest that the series is an asymptotic series. Taking his argument at face value, one should look for a proper tool to handle asymptotic series, for perturbative QED in particular and perturbation calculations in quantum field theory in general.

Better yet, we should look for a mathematical tool that can handle asymptotic series in quantum field theory and rid it of infinities at the same time. We[4] have suggested that such a tool is already available in the neutrix calculus developed by van der Corput[5] and

Hadamard. But instead of beginning with the axioms of neutrix calculus, in this paper we follow the path suggested to us by Prof. E. M. de Jager. In the next section, we discuss the close relation between asymptotic series and the finite parts of divergent integrals, and then introduce neutrix calculus to regularize divergent integrals. To make the presentation easy to follow, we use various examples to illustrate the approach. In Section III, we apply neutrix calculus to one-loop calculations in QED and ϕ^4 theories. In Section IV, switching to dimensional regularization, we apply neutrix calculus to the pure Yang-Mills system and "rederive" asymptotic freedom in one-loop. In both these two sections, there is actually very little new calculation involved, but they do serve to illustrate how neutrix calculus converts infinite renormalizations to mathematically well-defined finite renormalizations. As far as we can tell, all finite physically meaningful results of renormalizable quantum field theories are recovered in the neutrix framework. Section V is devoted to conclusions and a short discussion on the application of neutrix calculus to quantum gravity and effective quantum field theories in general. For completeness, we include two appendices to further discuss some mathematical properties of neutrices. In Appendix A, we consider more general Hadamard neutrices and a more general type of asymptotic series. To do quantum field theory in configuration space, one multiplies operator-valued distributions of quantum fields. In Appendix B, we show that the use of neutrix calculus allows one to put these products on a mathematically sound basis.

II. FROM ASYMPTOTIC SERIES AND DIVERGENT INTEGRALS TO NEUTRIX CALCULUS

Neutrix calculus gives us a framework to deal with asymptotic series and divergent integrals. To motivate its introduction we follow de Jager's suggestion to begin with a standard example of asymptotic series. Consider

$$f(x) = \int_0^\infty \frac{e^{-t}}{x+t} dt, \tag{1}$$

for $x \to \infty$. The "normal" procedure is to expand the denominator and write formally

$$\begin{aligned} f(x) &= \int_0^\infty \left(\frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - ... \right) e^{-t}\, dt \\ &= \frac{1}{x} - \frac{1}{x^2} + ... + (-1)^{n-1} \frac{(n-1)!}{x^n} + ... \end{aligned} \tag{2}$$

The right hand side is clearly a divergent series for any finite value of x. This is not surprising because the expansion of the denominator is allowed only for $t < x$, but the integration over t extends to ∞.

Nevertheless one can distill from Eq. (1) a different type of expansion which avoids the rigid rules of the above power series approach. Here we find the asymptotic series by repeatedly integrating by parts:

$$f(x) = -\frac{e^{-t}}{x+t}\bigg|_0^\infty - \int_0^\infty \frac{e^{-t}}{(x+t)^2}dt, \tag{3}$$

giving

$$f(x) = \frac{1}{x} - \frac{1}{x^2} + ... + (-1)^{n-1}\frac{(n-1)!}{x^n} + (-1)^n n! \int_0^\infty \frac{e^{-t}}{(x+t)^{n+1}}\,dt. \tag{4}$$

This equation is exact and is valid for any finite n. If we stop at the nth term, the remnant is given by

$$(-1)^n n! \int_0^\infty \frac{e^{-t}}{(x+t)^{n+1}}dt = \mathcal{O}\left(\frac{n!}{x^{n+1}}\right). \tag{5}$$

We get a satisfactory approximation for a sufficiently large x ($x = n$, e.g., would do). This is an asymptotic series, not a convergent series. If we stop at the nth term, we will have subtracted from the series the infinite part which, formally, is given by $\Sigma_n^\infty (-1)^j j! x^{-(j+1)}$. Note that in the series expansion the coefficients grow beyond bound for $n \to \infty$. This is overcome by going to a larger x, a freedom not allowed for the "normal" (convergent) power series expansion in x. (The phenomenon of the coefficients growing with n happens often for asymptotic series, but it is not a definition of such a series.)

Another example of asymptotic series is provided by the Stieltjes series[6] for the Stieltjes integral

$$\begin{aligned} y(x) &= \int_0^\infty \frac{e^{-t}}{1+xt}dt \\ &= 1 - x + 2!x^2 + ... + (-1)^n n! x^n + \epsilon_n(x), \end{aligned} \tag{6}$$

where

$$\epsilon_n(x) = (-1)^{n+1}(n+1)!x^{n+1}\int_0^\infty (1+xt)^{-n-2}e^{-t}dt, \tag{7}$$

and

$$|\epsilon_n(x)| \leq (n+1)!x^{n+1} << x^n, \quad x \to 0_+. \tag{8}$$

Next we move on to divergent series and their finite parts (i.e., Hadamard's "Parti Fini"). Here we encounter a similar situation like that treated above for asymptotic series. For this discussion, we use examples that will be relevant to the one-loop field theory calculations in the next section. Let us consider

$$F' = \int_0^\infty \frac{x}{1+x} dx. \tag{9}$$

Partitioning the integration into two parts, we can write

$$F' = \int_0^a \frac{x}{1+x} dx + \int_a^\infty \frac{x}{1+x} dx, \tag{10}$$

for any finite a. The second term contains the infinite part and the approximation of F' by the finite part is improved by increasing a. But we cannot take $a = \infty$. We notice that here a corresponds to the n in Eq. (4). Doing the integration for the finite part, we obtain

$$F' = a - \log(1+a) + \int_a^\infty \frac{x}{1+x} dx, \tag{11}$$

valid for all finite values of a. Increasing a indefinitely and subtracting the infinite part we get

$$F' = \int_0^\infty \frac{x}{1+x} dx \longrightarrow 0. \tag{12}$$

As another example, we consider

$$F = \int_0^\infty \frac{x}{(1+x)^2} dx, \tag{13}$$

which is also relevant to the physics we discuss in the next section. Proceeding as in the calculation of F' we have

$$\begin{aligned} F &= \int_0^a \frac{dx}{(1+x)} - \int_0^a \frac{dx}{(1+x)^2} + \int_a^\infty \frac{x}{(1+x)^2} dx \\ &= \log a + \frac{1}{1+a} - 1 + \int_a^\infty \frac{x}{(1+x)^2} dx. \end{aligned} \tag{14}$$

Subtracting the infinite part with $a \to \infty$ yields

$$F = \int_0^\infty \frac{x}{(1+x)^2} dx \longrightarrow -1. \tag{15}$$

For our purposes, the above two examples illustrate and encapsulate the essence of neutrix calculus developed by J. G. van der Corput (partly based on Hadamard's earlier work) in connection with asymptotic series and divergent integrals. By definition, a neutrix is a class of neglgible functions which satisfy the following two conditions:

(1). A neutrix is an additive group;

(2). It does not contain any constant except 0.

Following van der Corput's lead[5], we illustrate the concepts of neutrix calculus with the help of various examples.

Example 1. The neutrix consists of functions which go to 0 as $x \to \infty$. We call this neutrix $\epsilon(x)$.

Example 2. This concerns the class of polynomials in x with real coefficients. The neutrix is the subgoup of those polynomials divisible by $x^2 + 1$. The procedure gives the complex numbers (and this is apparently how Cauchy defined complex numbers).

Eample 3. The neutrix H_∞ consists of functions

$$\nu(\xi) = U(\xi) + \epsilon(\xi), \tag{16}$$

where $\epsilon(\xi) \to 0$ as $\xi \to \infty$ as defined in Example 1, and

$$U(\xi) = c_1\xi + c_2\xi^2 + ... + c_0 \log \xi + ... \tag{17}$$

This neutrix is denoted by H_∞.

Example 4. The following example was considered by Hadamard:

$$C \int_\xi^2 x^{s-1}\, dx = \begin{cases} Cs^{-1}2^s - Cs^{-1}\xi^s & \text{for } s \neq 0 \\ C \log 2 - C \log \xi & \text{for } s = 0 \end{cases} \tag{18}$$

Here $\xi > 0$, s is a real number, and C is complex. For $s > 0$, the integral converges. For $s < 0$, $s^{-1}\xi^s$ grows without bound as $\xi \to 0$, and so does $-\log \xi$ for $s = 0$. Hadamard called $Cs^{-1}2^s$ and $C \log 2$ the finite part ("parti fini"), and $Cs^{-1}\xi^s$ and $C \log \xi$ the infinite part ("parti infini"). He neglected the infinite part. For $\xi \to 0$, van der Corput wrote the integral as

$$C \int_{H_0}^2 x^{s-1}\, dx = \begin{cases} Cs^{-1}2^s & \text{for } s \neq 0 \\ C \log 2 & \text{for } s = 0. \end{cases} \tag{19}$$

Note the analytic extension in the complex s plane of the answer for Re $s > 0$ to the entire complex plane with the exclusion of $s = 0$.

In terms of the neutrix H_∞ defined in Example 3, we can rewrite Eqs. (12) and (15) as

$$\int_0^{H_\infty} \frac{x}{1+x} dx = 0, \tag{20}$$

and

$$\int_0^{H_\infty} \frac{x}{(1+x)^2} dx = -1, \tag{21}$$

respectively.

We conclude this section by pointing out that neutrices are essential to find the coefficients of an asymptotic series of the type

$$f(x) = ... + c_2 x^2 + c_1 x + c_0 + c_{0l} \log x + ... + c_{-1} x^{-1} + c_{-2} x^{-2} + ... + c_{-1l} x^{-1} \log x + ... \quad (22)$$

To find the coefficient c_{-1}, for example, we multiply both sides by x and take the Hadamard H_0 limit at $x = 0$. On the right hand side of Eq. (22), the terms to the left of c_{-1} are part of $\epsilon(0)$ and hence converge to zero. On the other hand, the terms to the right of c_{-1} are all in H_0 and hence are neglected. We conclude

$$c_{-1} = \lim_{H_0} x f(x). \quad (23)$$

More mathematical properties of neutrices and asymptotic series can be found in the two appendices.

III. NEUTRIX CALCULUS APPLIED TO 1-LOOP QED AND ϕ^4 THEORY

As the first example in the application of neutrices to quantum field theory, let us consider QED:

$$\mathcal{L} = -\bar{\psi}[\gamma \cdot (\frac{1}{i}\partial - eA) + m]\psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2. \quad (24)$$

We begin with the one-loop contribution to the electron's self energy in the Feynman gauge,

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu[-\gamma \cdot (p-k) + m]\gamma^\mu}{[k^2 + \lambda^2][(p-k)^2 + m^2]}, \quad (25)$$

where m is the electron bare mass and we have given the photon a fictitious mass λ to regularize infrared divergences. Expanding $\Sigma(p)$ about $\gamma \cdot p = -m$,

$$\Sigma(p) = \mathcal{A} + \mathcal{B}(\gamma \cdot p + m) + \mathcal{R}, \quad (26)$$

one finds (cf. results found in Ref.[7])

$$\mathcal{A} = -\frac{\alpha}{2\pi} m \left(\frac{3}{2}D + \frac{9}{4}\right), \quad (27)$$

$$\mathcal{B} = -\frac{\alpha}{4\pi} \left(D - 4\int_{\frac{\lambda}{m}}^{1} \frac{dx}{x} + \frac{11}{2}\right), \quad (28)$$

where $\alpha = e^2/4\pi$ is the fine structure constant and D is given by

$$D = \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2+m^2)^2} = \int_0^\infty \frac{k^2 dk^2}{(k^2+m^2)^2}, \tag{29}$$

with the second expression of D obtained after a Wick rotation. We note that $\mathcal{R}$, the last piece of $\Sigma(p)$ in Eq. (26), is finite. Mass renormalization and wavefunction renormalization are given by $m_{ren} = m - \mathcal{A}$ and $\psi_{ren} = Z_2^{-1/2}\psi$ respectively with $Z_2^{-1} = 1 - \mathcal{B}$. Now, introducing dimensionless variable $q = k^2/m^2$, we bring in the neutrix H_∞ to write D as

$$D = \int_0^{H_\infty} \frac{q dq}{(q+1)^2} = -1, \tag{30}$$

where, for the last step, we have used Eq. (21). Since $D = -1$ is finite, it is abundantly clear that the renormalizations are *finite* in the framework of neutrix calculus. There is no need for a separate discussion of the electron vertex function renormalization constant Z_1 due to the Ward identity $Z_1 = Z_2$.

The one-loop contribution to vacuum polarization is given by

$$\Pi_{\mu\nu}(k) = ie^2 \int \frac{d^4p}{(2\pi)^4} Tr \left(\gamma_\mu \frac{1}{\gamma\cdot(p+\frac{k}{2})+m} \gamma_\nu \frac{1}{\gamma\cdot(p-\frac{k}{2})+m} \right). \tag{31}$$

A standard calculation[7] shows that $\Pi_{\mu\nu}$ takes on the form

$$\Pi_{\mu\nu} = \delta m^2 \eta_{\mu\nu} + (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2), \tag{32}$$

where $\eta_{\mu\nu}$ is the flat metric (+++-),

$$\delta m^2 = \frac{\alpha}{2\pi}(m^2 D + D'), \tag{33}$$

and

$$\Pi(k^2) = -\frac{\alpha}{3\pi}(D + \frac{5}{6}) + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\, \log\left(1 + \frac{k^2}{m^2}x(1-x)\right), \tag{34}$$

with

$$D' = \frac{1}{i\pi^2}\int \frac{d^4p}{p^2+m^2}, \tag{35}$$

and D given by Eq. (29). Just as D is rendered finite upon invoking neutrix calculus (see Eq. (30)), so is D':

$$D' = m^2 \int_0^{H_\infty} \frac{q dq}{q+1} = 0, \tag{36}$$

where, for the last step, we have used Eq. (20). Thus neutrix calculus yields a finite renormalization for both the photon mass and the photon wavefunction $A^\mu_{ren} = Z_3^{-1/2} A^\mu$ (and consequently also for charge $e_{ren} = Z_3^{1/2} e$) where $Z_3^{-1} = 1 - \Pi(0)$. In electron-electron scattering by the exchange of a photon with energy-momentum k, vacuum polarization effects effectively replace e^2 by $e^2/(1-\Pi(k^2))$, i.e.,

$$\begin{aligned} e^2 \to e^2_{eff} &= \frac{e^2}{1-\Pi(k^2)} \\ &= \frac{e^2_{ren}}{Z_3(1-\Pi(k^2))} \\ &= \frac{e^2_{ren}}{1-(\Pi(k^2)-\Pi(0))}. \end{aligned} \tag{37}$$

Eq. (34) can be used, for $k^2 \gg m^2$, to show that

$$\alpha_{eff}(k^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{k^2}{\exp(5/3)\, m^2}\right)}. \tag{38}$$

Thus we have obtained the correct running of the coupling[8] with energy-momentum in the framework of neutrices. In fact, the *only* effect of neutrix calculus, when applied to QED (and other renormalizable theories), is to convert *infinite* renormalizations (obtained without using neutrix calculus) to mathematically well-defined *finite* renormalizations. As far as we can tell, *all* (finite) physically observable results of QED are recovered. In passing we mention that the use of neutralized integrals does not affect the results of axial triangle anomalies.

Let us now apply neutrix calculus to the ϕ^4 theory,

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{39}$$

The one-loop self-energy is given by

$$\Sigma(p) = i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}, \tag{40}$$

which, by Eqs. (35) and (36), vanishes! Thus there is no mass renormalization ($m^2_{ren} = m^2 - \Sigma(0) = m^2$) and no wave-function renormalization (wave-function renormalization constant $Z_\phi = 1$) in the one-loop approximation. The four-point function $\Gamma(s,t,u)$, for incoming particle momenta p_i, receives the following one-loop contributions

$$\Gamma(s,t,u) = \frac{\lambda^2}{2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+m^2}\frac{1}{(p-q)^2+m^2} + (2 \text{ crossed terms}). \tag{41}$$

Here $q = p_1 + p_2$, and we have used the Mandelstam variables $s = -q^2$, $t = -(p_1 + P_3)^2$, and $u = -(p_1 + p_4)^2$. A straight-forward calculation yields

$$\Gamma(s,t,u) = i\frac{\lambda^2}{32\pi^2}\left\{3D - \int_0^1 dz \left(\log(1 - \frac{s}{m^2}z[1-z]) + (s \to t) + (s \to u)\right)\right\}. \tag{42}$$

Thus, the coupling receives a finite renormalization

$$\begin{aligned}\lambda_{ren} &= \lambda\left(1 - \frac{3D\lambda}{32\pi^2}\right) \\ &= \lambda\left(1 + \frac{3\lambda}{32\pi^2}\right),\end{aligned} \tag{43}$$

where we have defined λ_{ren} by $\Gamma(s,t,u)$ at $s = t = u = 0$, and have used $D = -1$, given by Eq. (30), for the last step.

IV. DIMENSIONAL REGULARIZATION AND PURE YANG-MILLS THEORY

As shown by the appearance of photon mass in the above discussion of vacuum polarization in QED, the application of neutrix calculus to the energy-momentum cutoff regularization scheme is not too convenient to use for more complicated theories like those involving Yang-Mills fields. For those theories, one should use other regularization schemes that manifestly preserve the Ward identity. In this regard, we note that already in 1961, van der Corput suggested that, instead of finding the appropriate neutrices, one can continue analytically in any variable (presumably including the dimension of integrations) contained in the problem of tackling apparent divergences to calculate the coefficients of the corresponding asymptotic series. It so happened that this was the approach taken by 't Hooft and Veltman who spearheaded the use of dimensional regularizations[8]. Let us now explore using neutrix calculus in conjunction with the dimensional regularization scheme. In that case, negligible functions will include $1/\epsilon$ where $\epsilon = 4 - n$ is the deviation of spacetime dimensions from 4. In the calculations, the internal energy-momentum integration is now over n dimensions. The one-loop contributions to the electron's self-energy in QED is given by

$$\Sigma(p) = -\frac{e^2}{(4\pi)^{n/2}}\Gamma(2 - \frac{n}{2})\int_0^1 dx \frac{(n-2)\gamma\cdot p(1-x) + nm}{[p^2x(1-x) + m^2x + \lambda^2(1-x)]^{2-n/2}}. \tag{44}$$

Again we expand $\Sigma(p)$ about $\gamma \cdot p = -m$ as in Eq. (26). Using the approximation for the gamma function

$$\lim_{\epsilon\to 0}\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \tag{45}$$

where $\gamma \simeq .577$ is the Euler-Mascheroni constant, and the approximation

$$f^\epsilon \simeq 1 + \epsilon \log f, \tag{46}$$

for $\epsilon \ll 1$, one finds

$$\begin{aligned} \mathcal{A} &= \frac{\alpha m}{4\pi}[3(\gamma - \log 4\pi) + 1] + \frac{\alpha}{2\pi} m \int_0^1 dx (1+x) \log D_0, \\ \mathcal{B} &= \frac{\alpha}{4\pi}[1 + \gamma - \log 4\pi] + \frac{\alpha m^2}{\pi} \int_0^1 dx \frac{x(1-x^2)}{m^2x^2 + \lambda^2(1-x)} + \frac{\alpha}{2\pi} \int_0^1 dx (1-x) \log D_0, \end{aligned} \tag{47}$$

where $D_0 = m^2x^2 + \lambda^2(1-x)$, [1] and we have invoked neutrix calculus in dropping $-3\alpha m/(2\pi\epsilon)$ and $-\alpha/(2\pi\epsilon)$ from $\mathcal{A}$ and $\mathcal{B}$ respectively. (In passing we note that $\mathcal{R}$ in the expansion of $\Sigma(p)$ in the dimensional regularization scheme is the same as in the energy-momentum regularization used in the last section.)

The one-loop vacuum polarization takes on the same form as given by Eq. (32), but now with $\delta m^2 = 0$, and

$$\Pi(k^2) = -8e^2 \frac{\Gamma(2 - \frac{n}{2})}{(4\pi)^{n/2}} \int_0^1 dx \frac{x(1-x)}{[k^2x(1-x) + m^2]^{2-n/2}}. \tag{48}$$

In the limit $\epsilon = 4 - n \to 0$ we obtain

$$\begin{aligned} \Pi(k^2) &= \frac{\alpha}{3\pi}[\gamma - \log 4\pi] + \frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x)\, \log[m^2 + x(1-x)k^2], \\ \frac{1}{Z_3} &= 1 - \frac{\alpha}{3\pi}[\gamma - \log 4\pi] - \frac{\alpha}{3\pi} \log m^2, \end{aligned} \tag{49}$$

where we have followed neutrix calculus in dropping $-e^2/(6\pi^2\epsilon)$ from $\Pi(k^2)$. By design, the generalized neutrix calculus renders all the renormalizations *finite*. Again, *all* physically measurable results of QED appear to be recovered.

Next we consider pure non-Abelian gauge theroy for a group G with structure constants C_{abc}

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g_0 C_{abc} A_b^\mu A_c^\nu)^2. \tag{50}$$

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[1] Since D_0 is dimensionful, the appearance of $\log D_0$ is very strange. This is due to the fact that the e^2 in Eq. (44) is in fact also dimensionful for $\epsilon \neq 0$. If we replace e^2 there by $e^2\mu^{4-n}$ with a certain mass μ so that the e^2 will then be dimensionless, D_0 will be replaced by dimensionless D_0/μ^2. For simplicity we have not written this dimensional dependence out explicitly. Hereafter this dependence will be left implicit.

The one-loop contribution to the self-energy of the gauge field, in the Feynman gauge, is given by

$$\Pi^{cd}_{\mu\nu}(k) = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi^{cd}(k), \tag{51}$$

with

$$\Pi^{cd}(k) = C^2_{cd}\frac{g_0^2}{2(4\pi)^{n/2}}\Gamma(2-\frac{n}{2})\int_0^1 dx\,\mathcal{D}^{\frac{n}{2}-2}[(4-2n)+(6n-4)x+(8-4n)x^2], \tag{52}$$

where the symbol C^2_{cd} stands for $C_{abc}C_{abd}$ and $\mathcal{D} = k^2x(1-x)$.

With the aid of Eqs. (45) and (46), we get

$$\Pi^{cd}(k) = \frac{C^2_{cd}g_0^2}{32\pi^2}\left(\frac{20}{3}\frac{1}{\epsilon} - \frac{10}{3}\log k^2 + \frac{10}{3}[-\gamma + \log 4\pi] + 7 - \frac{1}{9}\right). \tag{53}$$

Invoking neutrix calculus to remove the negligible ϵ^{-1} term and specializing to the $SU(N)$ group where $C^2_{cd} = N\delta_{cd}$, we obtain

$$\Pi^{cd}(k) = \delta_{cd}\Pi(k^2), \tag{54}$$

with

$$\Pi(k^2) \to \frac{Ng_0^2}{32\pi^2}\left(-\frac{10}{3}\log k^2 + ...\right), \tag{55}$$

where ... stands for the finite k-indepentdent terms in Eq. (53). For QED we have used the on-shell renormalizations with massive electrons. For pure Yang-Mills theory we will do the renormalizations for massless gauge fields at space-like energy-momentum μ^2. The renormalization constant Z_3 is then given by

$$Z_3 = \frac{1}{1-\Pi(\mu^2)} \simeq 1 - \frac{10}{3}\frac{Ng_0^2}{16\pi^2}\log\mu + ... \tag{56}$$

Similarly the renormalization constant for the three-point function in the Feynman gauge can be calculated to yield

$$Z_1 \simeq 1 - \frac{4}{3}\frac{Ng_0^2}{16\pi^2}\log\mu + ... \tag{57}$$

where ... stands for finite μ-independent terms. The Callan-Symanzik function for the renormalized coupling $g = Z_1^{-1}Z_3^{3/2}g_0$ in the one-loop approximation is then given by

$$\begin{aligned}\beta(g) &= \mu\frac{\partial g}{\partial\mu}\\ &= -\frac{11}{3}\frac{Ng^3}{16\pi^2}.\end{aligned} \tag{58}$$

Thus we have recovered the negative β function for $SU(N)$ Yang-Mills theory.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we have proposed to apply neutrix calculus, in conjunction with Hadamard integrals, developed by J.G. van der Corput [5] in connection with asymptotic series, to quantum field theory, to obtain finite results for the coefficients in perturbation series. The replacement of regular integrals by Hadamard integrals in quantum field theory appears to make good mathematical sense, as van der Corput observed that Hadamard integrals are the proper tool to calculate the coefficients of an asymptotic series. (Actually Hadamard integrals work equally well for convergent series.)

For renormalizable quantum field theories like QED, ϕ^4 theory, and pure Yang-Mills, we have demonstrated (even if not adequately) that we recover all the usual physically observable results. The only effect of neutrix calculus appears to just change the "amount" of renormalizations —from infinite renormalizatins to finite renormalizations. But what about non-renormalizable field theories like quantum gravity? Using dimensional regularization, 'tHooft and Veltman[9] found that pure gravity is one-loop renormalizable, but in the presence of a scalar field, renormalization was lost. For the latter case, they found that the counterterm evaluated on the mass shell is given by $\sim \epsilon^{-1}\sqrt{g}R^2$ with R being the Ricci scalar. Similar results for the cases of Maxwell fields and Dirac fields etc (supplementing the Einstien field) were obtained[10]. It is natural to inquire whether the application of neutrix calculus could improve the situation. Our preliminary result is that now essentially the divergent ϵ^{-1} factor is replaced by $-\gamma+$ finite terms.

Since neutrix calculus does not tolerate infinities, we conjecture that it can also be used to ameliorate other problems that can be traced to ultraviolet divergences. Two well-known problems come to mind. The hierarchy problem in particle physics is due to the fact that the Higgs scalar self-energies diverge quadratically. It is conceivable that neutrix calculus may help. The cosmological constant problem can be traced to the quartic divergences in zero-point fluctuations from all quantum fields. Perhaps neutrix calculus can help with this problem too. Indeed, for a theory of gravitation with a cosmological constant term, the cosmological constant receives at most a finite renormalization from the quantum loops in the framework of neutrix calculus.

By ridding quantum field theories of their ultraviolet divergences, neutrix calculus may appear to have fallen victim to its own successes, for now we have lost renormalizability as a physical restrictive criterion in the choice of sensible theories. However, we believe that this is actually not as big a loss as it may first appear. Quite likely, all realistic theories now in our possession are actually effective field theories.[2, 11] They appear to be renormalizable field theories because, at energies now accessible, or more correctly, at sufficiently low energies, all the non-renormalizable interactions are highly suppressed. By tolerating non-renormalizable theoriess, neutrix calculus has provided us with a more flexible framework to study all kinds of particle interactions.

To the extent that it is relevant for asymptotic series and lessens the divergences in quantum field theories, neutrix calculus is a powerful tool that is, in our opinion, too valuable not to make use of. How much it can really help to ameliorate problems related to ultraviolet divergences in quantum field theory remains to be seen. But based on our study so far, we tentatively argue that neutrix calculus has banished infinities from quantum field theory.

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Appendix A: The Hadamard neutrix H_a and Hadamard series expansion

In Section II, we considered some examples of asymtotic series. Essentially, the series $f(x) = a_0 + a_1(x-b) + a_2(x-b)^2 + ...$ for finite b is an asymptotic series [6] if and only if there exists an $n_0 > 0$, such that for $n > n_0$,

$$\lim_{x\to b} \frac{1}{(x-b)^n} \left| f(x) - a_0 - a_1(x-b) - \ldots - a_n(x-b)^n \right| = 0, \tag{59}$$

with the remnant being at most $\sim (x-b)^{n+1}$.

In this appendix, we consider the Hadamard series expansion and a generalized definition of asymptotic series. Let κ be a region in the complex plane with a finite limit point a which does not belong to κ. A crucial property is that a point ξ in κ can approach a in such a way that $\arg(\xi - a)$ has a finite limit. We say that $U(\xi)$ in κ has a Hadamard expansion in powers of $\xi - a$ if $U(\xi)$ is defined in κ, and for points ξ near a, has an asymptotic expansion of the kind

$$U(\xi) \sim \Sigma_{h=0}^{\infty} \chi_h \, (\xi - a)^{\Psi_h} \, \log^{k_h}(\xi - a). \tag{60}$$

Here χ_h, Ψ_h and integers $k_h \geq 0$ are independent of ξ, Re $\Psi_h \to \infty$ as $h \to \infty$, and $\log^{k_h}(\xi)$ stands for $(\log(\xi))^{k_h}$.

The definition of asymptotic series here is more powerful than the one we used in Section II in that, for every number q (independent of ξ), there exists an integer $m_0 \geq 0$ with the property that, for every number $m \geq m_0$, there are two positive numbers c_m and ϵ_m such that, for every point ξ in κ with $|\xi - a| < \epsilon_m$,

$$\left| U(\xi) - \Sigma_{h=0}^{m-1} \chi_h \, (\xi - a)^{\Psi_h} \, \log^{k_h}(\xi - a) \right| < c_m |\xi - a|^q. \tag{61}$$

The new definition of asymptotic series is essential in deriving the analytic properties of functions represented by neutralized integrals. Note that this means that there is a radius of convergence which goes to zero as m goes to infinity. If, in Eq. (61), there is no term with $\Psi_h = k_h = 0$, then we speak of a Hadamard series expansion without constant term.

We now define the Hadamard neutrix H_a. It consists of all functions $\nu(\xi)$, which in the neighborhood of a can be written as

$$\nu(\xi) = U(\xi) + \epsilon(\xi). \tag{62}$$

Here $U(\xi)$ has a Hadamard series expansion in powers of $(\xi - a)$ without a constant term and $\epsilon(\xi)$ approaches zero when ξ in κ approaches a. This neutrix is called H_a; it has a domain κ, a variable ξ, and a is called the carrier. Thus every negligible functions in H_a can be written as

$$\Sigma_{h=0}^{\infty} \chi_h \, (\xi - a)^{\Psi_h} \, \log^{k_h}(\xi - a) + \epsilon(\xi), \tag{63}$$

with no $\Psi_h = k_h = 0$ term.

We end Appendix A with three examples.

Example 1. If two different points a and b are connected by a rectifiable curve in the complex plane, which has a tangent in a but does not contain a, then for every integer $k \geq 0$ and for every complex s

$$\int_{H_a}^{b} (z-a)^{s-1} \log^k(z-a)\, dz = \begin{cases} \frac{1}{k+1} \log^{k+1}(b-a) & \text{for } s=0 \\ \left(\frac{\partial}{\partial s}\right)^k \frac{(b-a)^s}{s} & \text{for } s \neq 0 \end{cases} \tag{64}$$

Example 2. For every complex s, except $0, -1, -2, ...,$

$$\int_{H_0}^{\infty} x^{s-1}\, e^{-x}\, dx = \Gamma(s). \tag{65}$$

Example 3. For $-\pi < \arg l < \pi$, s not an integer ≤ 0, and $s+t$ not an integer ≥ 0, one has

$$\int_{H_0}^{H_\infty} x^{\lambda s-1}\, (l+x^\lambda)^t\, \log^k x\, dx = \frac{1}{\lambda^{k+1}} \left(\frac{\partial}{\partial s}\right)^k \frac{\Gamma(s)\Gamma(-s-t)}{\Gamma(-t)} l^{s+t}. \tag{66}$$

Here the integral is along the positive axis. This integral contains the special case (with $\lambda = 2, k = 0, t = -\alpha$, and $2s = \beta + 1$)

$$\int_{H_0}^{H_\infty} \frac{x^\beta}{(x^2+l)^\alpha} dx = \frac{1}{2} \frac{\Gamma\left(\frac{\beta+1}{2}\right) \Gamma\left(\alpha - \frac{\beta+1}{2}\right)}{\Gamma(\alpha) l^{\alpha-(\beta+1)/2}}, \tag{67}$$

an expression encountered in dimensional regularization calculations for which $\beta = 3 - \epsilon$ with a positive infinitesimally small ϵ.

Appendix B: Neutrices and products of singular functions

It has often been claimed that infinities find their way into quantum field theory because one works with products of singular functions (like Feynman propagators) in the theory. In what used to be called axiomatic field theory, one had to search for the definitions of the products of operator-valued distributions. But whereas generalized functions usually cannot be multiplied in the theory of distributions developed by Schwartz, the Hadamard approach adopted in this paper allows the multiplication for a wide class of distributions as we will show in this Appendix. In fact this is the reason that one obtains finite rather than infinite renormalizations in the framework of neutrix calculus.

Lighthill[12] has already defined and listed the products for some distributions in his book. Van der Corput[5] has been able to simplify and extend Lighthill's list by doing what one usually does in quantum field theory, namely, by using Fourier transformed (FT) functions

$$FT[f(x)] = F(p) = \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx. \tag{68}$$

Then he defines the FT of the product of two functions by folding (as usual), but using Hadamard integtrals (instead of the "normal" integrals):

$$FT[f_1(x)\, f_2(x)] = \int_{H_{-\infty}}^{H_{\infty}} \frac{dk}{2\pi} F_1(k)\, F_2(p-k). \tag{69}$$

Consider, for example, the one-dimensional Dirac delta function multiplying itself $\delta(x) \times \delta(x)$. In Schwartz' approach, this product is not mathematically meaningful because its Fourier transform diverges:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, 1 \times 1 \;\longrightarrow \infty, \tag{70}$$

where we have used the convolution rule in Fourier transform and have noted that the Fourier transform of the Dirac delta function is 1. In contrast, the Hadamard-neutralized Fourier transform of the product $\delta(x) \times \delta(x)$,

$$\int_{H_{-\infty}}^{H_{\infty}} \frac{dk}{2\pi}\, 1 \times 1 \; = 0, \tag{71}$$

yields

$$\delta(x) \times \delta(x) = 0, \tag{72}$$

a mathematically meaningful (though somewhat counter-intuitive) result! One can also show that (for non-negative integers a and b)

$$(\partial^a \delta(x)) \left(\partial^b \delta(x)\right) = 0, \tag{73}$$

because

$$\int_{H_{-\infty}}^{H_{\infty}} k^a (p-k)^b\, dk = 0. \tag{74}$$

Similarly one can show that, for non-negative integers m and a, the Fourier transform of $x^{-m}\partial^a\delta(x)$ vanishes, and the Fourier transform of $x^m\partial^a\delta(x)$ is zero for $m > a$ and $i^{m+a}a(a-1)...(a-m+1)p^{a-m}$ for $m \leq a$ respectively.

Lastly let us generalize the above discussion for products of two 1-dimensional Dirac delta functions to the case of (3 + 1)-dimensional Feynman propagators

$$\begin{aligned}\Delta_+(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 + m^2 - i\epsilon} \\ &= \frac{1}{4\pi}\delta(x^2) - \frac{m}{8\pi\sqrt{-x^2 - i\epsilon}} H_1^{(2)}(m\sqrt{-x^2 - i\epsilon}),\end{aligned} \tag{75}$$

where $H^{(2)}$ is the Hankel function of the second kind and we use the (+++-) metric. The Fourier transform of $\Delta_+(x) \times \Delta_+(x)$ (which appears in certain quantum loop calculations, e.g., in Eq. (42) in the text) is given by

$$\int d^4x\, e^{-ip\cdot x}\, \Delta_+(x)\, \Delta_+(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2-i\epsilon} \frac{1}{(p-k)^2+m^2-i\epsilon}$$

$$= \frac{i}{4(2\pi)^2} D - \frac{i}{4(2\pi)^2} \int_0^1 dz\, \log\left(1 + \frac{p^2}{m^2} z(1-z)\right), \quad (76)$$

where D is given by Eq. (29). But D is logarithmically divergent. Hence $\Delta_+(x) \times \Delta_+(x)$ is not mathematically well defined. On the other hand, this product is mathematically meaningful in the Hadamard-van der Corput approach. To wit, $D = -1$ in the neutralized version, hence $\Delta_+(x) \times \Delta_+(x) \sim \delta^{(4)}(x)+$ regular part, where $\delta^{(4)}(x)$ is the 4-dimensional Dirac delta function.

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