# Inverse Harish-Chandra transform and difference operators 

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In the paper we calculate the images of the operators of multiplication by Laurent polynomials with respect to the Harish-Chandra transform and its non-symmetric generalization due to Opdam. It readily leads to a new simple proof of the Harish-Chandra inversion theorem in the zonal case (see [HC,He1]) and the corresponding theorem from [O1]. We assume that $k>0$ and restrict ourselves to compactly supported functions, borrowing the growth estimates from [O1].

The Harish-Chandra transform is the integration of symmetric functions on the maximal real split torus $A$ of a semi-simple Lie group $G$ multiplied by the spherical zonal function $\phi(X, \lambda)$, where $X \in A, \lambda \in\left(\operatorname{Lie} A \otimes_{\mathbf{R}} \mathbf{C}\right)^{*}$. The measure is the restriction of the invariant measure on $G$ to the space of double cosets $K \backslash G / K \subset A / W$ for the maximal compact subgroup $K \subset G$ and the restricted Weyl group $W$. The function $\phi$ is a symmetric ( $W$-invariant) eigenfunction of the radial parts of the $G$-invariant differential operators on $G / K ; \lambda$ determines the set of eigenvalues. The parameter $k$ is given by the root multiplicities ( $k=1$ in the group case). There is a generalization to arbitrary $k$ due to Calogero, Sutherland, Koornwinder, Moser, Olshanetsky, Perelomov, Heckman, and Opdam. See [HO1,H1,O2] for a systematic theory. In the non-symmetric variant due to Opdam [O1], the operators from [C5] replace the radial parts of $G$-invariant operators and their $k$-generalizations. The problem is to define the inverse transforms for various classes of functions.

In the papers [C1-C4], a difference counterpart of the Harish-Chandra transform was suggested, which is also a deformation of the Fourier transform in the $p$-adic theory of spherical functions. Its kernel (a $q$-generalization of $\phi$ ) is defined as an eigenfunction of the $q$-difference "radial parts" (Macdonald's operators and their generalizations). There are applications in combinatorics (the Macdonald polynomials), the representation theory (say, quantum groups at roots of unity), and the mathematical physics (the Knizhnik-Zamolodchikov equations and more). The difference Fourier transform is self-dual, i.e. the

[^0]kernel is $x \leftrightarrow \lambda$ symmetric for $X=q^{x}$. This holds in the differential theory only for the so-called rational degeneration with the tangent space $T_{e}(G / K)$ instead of $G / K$ (see [He1,D,J]) and for special $k=0,1$. In the latter cases, the differential and difference transforms coincide up to a normalization.

The differential case is recovered when $q$ goes to 1 . At the moment the analytic methods are not mature enough to manage the limiting procedure in detail. So we develop the corresponding technique without any reference to the difference Fourier transform, which is not introduced and discussed in the paper, switching from the double affine Hecke algebra [C9] to the degenerate one from [C7]. The latter generalizes Lusztig's graded affine Hecke algebra [L] (the $G L_{n}$-case is due to Drinfeld). Mainly we need the intertwiners and creation operators from [C2] (see [C9] and [KS] for $G L_{n}$ ). It is worth mentioning that the $p$-adic theory corresponds to the limit $q \rightarrow 0$, which does not violate the standard difference convergence condition $|q|<1$ and hardly creates any analytic problems.

The key result is that the Opdam transforms of the operators of multiplication by the coordinates coincide with the operators from [C6] (see (4.11) below). Respectively, the Harish-Chandra transforms of the multiplications by $W$-invariant Laurent polynomials are the difference operators from (4.14). It is not very surprising that the transforms of these important operators haven't been found before. The calculation, not difficult by itself, involves the following ingredients new in the harmonic analysis on symmetric spaces:
a) The Dunkl-type operators [C5], their relation to degenerate (graded) affine Hecke algebras [C8], and the Opdam transform [O1].
b) The Macdonald operators, difference counterparts of Dunkl operators, and double affine Hecke algebras [M1,M2,C6,C9].
We begin the paper with some basic facts about double affine Hecke algebras and their polynomial representations which are necessary to produce difference Dunkl-type operators. Then we discuss the two degeneration procedures leading respectively to the operators from [C5](differential) and from [C6](difference). Both are governed by the same degenerate double affine Hecke algebra (introduced in [C7]). It eventually results in the inversion theorem.

Hopefully this approach can be extended to any $k$ and, moreover, may help to (re)establish the analytic properties of the direct and inverse transforms. I believe that this paper will stimulate a systematic renewal of the HarishChandra theory and related representation theory on the basis of the differenceoperator methods.

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## 1. Affine Weyl groups

Let $R=\{\alpha\} \subset \mathbf{R}^{n}$ be a root system of type $A, B, \ldots, F, G$ with respect to a euclidean form $\left(z, z^{\prime}\right)$ on $\mathbf{R}^{n} \ni z, z^{\prime}$, normalized by the standard condition that $(\alpha, \alpha)=2$ for long $\alpha$. Let us fix the set $R_{+}$of positive roots, the corresponding simple roots $\alpha_{1}, \ldots, \alpha_{n}$, and their dual counterparts $a_{1}, \ldots, a_{n}, a_{i}=\alpha_{i}^{\vee}$, where $\alpha^{\vee}=2 \alpha /(\alpha, \alpha)$. The dual fundamental weights $b_{1}, \ldots, b_{n}$ are determined by the relations $\left(b_{i}, \alpha_{j}\right)=\delta_{i}^{j}$ for the Kronecker delta. We will also use the dual root system $R^{\vee}=\left\{\alpha^{\vee}, \alpha \in R\right\}, R_{+}^{\vee}$, and

$$
A=\oplus_{i=1}^{n} \mathbf{Z} a_{i} \subset B=\oplus_{i=1}^{n} \mathbf{Z} b_{i}, B_{+}=\oplus_{i=1}^{n} \mathbf{Z}_{+} b_{i} \text { for } \mathbf{Z}_{+}=\{m \geq 0\}
$$

In the standard notations, $A=Q^{\vee}, B=P^{\vee}$ - see [B]. Later on,

$$
\nu_{\alpha}=(\alpha, \alpha), \nu_{i}=\nu_{\alpha_{i}}, \nu_{R}=\left\{\nu_{\alpha}, \alpha \in R\right\} \subset\{2,1,2 / 3\}
$$

The vectors $\tilde{\alpha}=[\alpha, k] \in \mathbf{R}^{n} \times \mathbf{R}=\mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the affine root system $R^{a} \supset R\left(z \in \mathbf{R}^{n}\right.$ are identified with $\left.[z, 0]\right)$. We add $\alpha_{0} \stackrel{\text { def }}{=}[-\theta, 1]$ to the simple roots for the maximal root $\theta \in R$. The corresponding set $R_{+}^{a}$ of positive roots coincides with $R_{+} \cup\{[\alpha, k], \alpha \in R, k>0\}$. Let $a_{0}=\alpha_{0}$.

We will use the affine Dynkin diagram $\Gamma^{a}$ with $\left\{\alpha_{j}, 0 \leq j \leq n\right\}$ as the vertices $\left(m_{i j}=2,3,4,6\right.$ if $\alpha_{i}$ and $\alpha_{j}$ are joined by $0,1,2,3$ laces respectively). The set of the indices of the images of $\alpha_{0}$ by all the automorphisms of $\Gamma^{a}$ will be denoted by $O\left(O=\{0\}\right.$ for $\left.E_{8}, F_{4}, G_{2}\right)$. Let $O^{*}=\{r \in O, r \neq 0\}$. The elements $b_{r}$ for $r \in O^{*}$ are the so-called minuscule coweights $\left(\left(b_{r}, \alpha\right) \leq 1\right.$ for $\alpha \in R_{+}$).

Given $\tilde{\alpha}=[\alpha, k] \in R^{a}, b \in B$, let

$$
\begin{equation*}
s_{\tilde{\alpha}}(\tilde{z})=\tilde{z}-\left(z, \alpha^{\vee}\right) \tilde{\alpha}, \quad b^{\prime}(\tilde{z})=[z, \zeta-(z, b)] \tag{1.1}
\end{equation*}
$$

for $\tilde{z}=[z, \zeta] \in \mathbf{R}^{n+1}$.
The affine Weyl group $W^{a}$ is generated by all $s_{\tilde{\alpha}}$ (simple reflections $s_{j}=$ $s_{\alpha_{j}}$ for $0 \leq j \leq n$ are enough). It is the semi-direct product $W \ltimes A$, where the non-affine Weyl group $W$ is the span of $s_{\alpha}, \alpha \in R_{+}$. Here and further we identify $b \in B$ with the corresponding translations. For instance,

$$
\begin{equation*}
\alpha^{\vee}=s_{\alpha} s_{[\alpha, 1]}=s_{[-\alpha, 1]} s_{\alpha} \text { for } \alpha \in R \tag{1.2}
\end{equation*}
$$

The extended Weyl group $W^{b}$ generated by $W$ and $B$ is isomorphic to $W \ltimes B$ :

$$
\begin{equation*}
(w b)([z, \zeta])=[w(z), \zeta-(z, b)] \text { for } w \in W, b \in B \tag{1.3}
\end{equation*}
$$

For $b_{+} \in B_{+}$, let

$$
\begin{equation*}
\omega_{b_{+}}=w_{0} w_{0}^{+} \in W, \pi_{b_{+}}=b_{+}\left(\omega_{b_{+}}\right)^{-1} \in W^{b}, \omega_{i}=\omega_{b_{i}}, \pi_{i}=\pi_{b_{i}} \tag{1.4}
\end{equation*}
$$

where $w_{0}$ (respectively, $w_{0}^{+}$) is the longest element in $W$ (respectively, in $W_{b_{+}}$ generated by $s_{i}$ preserving $b_{+}$) relative to the set of generators $\left\{s_{i}\right\}$ for $i>0$.

The elements $\pi_{r} \stackrel{\text { def }}{=} \pi_{b_{r}}, r \in O^{*}$, and $\pi_{0}=\mathrm{id}$ leave $\Gamma^{a}$ invariant and form a group denoted by $\Pi$, which is isomorphic to $B / A$ by the natural projection $b_{r} \mapsto \pi_{r}$. As to $\left\{\omega_{r}\right\}$, they preserve the set $\left\{-\theta, \alpha_{i}, i>0\right\}$. The relations $\pi_{r}\left(\alpha_{0}\right)=\alpha_{r}=\left(\omega_{r}\right)^{-1}(-\theta)$ distinguish the indices $r \in O^{*}$. Moreover (see e.g. [C6]):
(1.5) $W^{b}=\Pi \ltimes W^{a}$, where $\pi_{r} s_{i} \pi_{r}^{-1}=s_{j}$ if $\pi_{r}\left(\alpha_{i}\right)=\alpha_{j}, 0 \leq i, j \leq n$.

The length $l(\hat{w})$ of $\hat{w}=\pi_{r} \tilde{w} \in W^{b}$ is the length of a reduced decomposition $\tilde{w} \in W^{a}$ with respect to $\left\{s_{j}, 0 \leq j \leq n\right\}:$

$$
\begin{align*}
l(\hat{w})=|\ell(\hat{w})|, \quad \ell(\hat{w}) & =R_{+}^{a} \cap \hat{w}^{-1}\left(-R_{+}^{a}\right) \\
& =\left\{\tilde{\alpha} \in R_{+}^{a}, l\left(\hat{w} s_{\tilde{\alpha}}\right)<l(\hat{w})\right\} \tag{1.6}
\end{align*}
$$

For instance, given $b_{+} \in B_{+}$,

$$
\begin{align*}
& \ell\left(b_{+}\right)=\left\{\tilde{\alpha}=[\alpha, k], \quad \alpha \in R_{+} \text {and }\left(b_{+}, \alpha\right)>k \geq 0\right\} \\
& l\left(b_{+}\right)=2\left(b_{+}, \rho\right), \quad \text { where } \rho=(1 / 2) \sum_{\alpha \in R_{+}} \alpha \tag{1.7}
\end{align*}
$$

We will also use the dominant affine Weyl chamber

$$
\begin{equation*}
C_{+}^{a}=\left\{z \in \mathbf{R}^{n} \text { such that }\left(z, \alpha_{i}\right)>0 \text { for } i>0,(z, \theta)<1\right\} \tag{1.8}
\end{equation*}
$$

## 2. Double affine Hecke algebras

We put $m=2$ for $D_{2 l}$ and $C_{2 l+1}, m=1$ for $C_{2 l}, B_{l}$, otherwise $m=|\Pi|$. We consider $q,\left\{t(\nu), \nu \in \nu_{R}\right\}, X_{1}, \ldots, X_{n}$ as independent variables, setting

$$
\begin{align*}
& t_{\tilde{\alpha}}=t(\nu(\alpha)), t_{j}=t_{\alpha_{j}}, \text { where } \tilde{\alpha}=[\alpha, k] \in R^{a}, 0 \leq j \leq n \\
& X_{\tilde{b}}=\prod_{i=1}^{n} X_{i}^{k_{i}} q^{k} \text { if } \tilde{b}=[b, k] \text { for } b=\sum_{i=1}^{n} k_{i} b_{i} \in B \tag{2.1}
\end{align*}
$$

Later on $k \in(1 / 2 m) \mathbf{Z}, \mathbf{C}_{q, t}$ is the field of rational functions in terms of $q^{1 / 2 m},\left\{t(\nu)^{1 / 2}\right\}, \mathbf{C}_{q, t}[X]=\mathbf{C}_{q, t}\left[X_{b}\right]$ means the algebra of polynomials in terms of $X_{i}^{ \pm 1}$ with the coefficients from $\mathbf{C}_{q, t}$. We set $\alpha_{r^{*}} \stackrel{\text { def }}{=} \pi_{r}^{-1}\left(\alpha_{0}\right)$, so $\omega_{r} \omega_{r^{*}}=1=\pi_{r} \pi_{r^{*}}$.

DEFINITION [C9] 2.1. The double affine Hecke algebra $\mathfrak{H}$ is generated over the field $\mathbf{C}_{q, t}$ by the elements $\left\{T_{j}, 0 \leq j \leq n\right\}$, pairwise commutative $\left\{X_{b}, b \in B\right\}$ satisfying (2.1), and the group $\Pi$ where the following relations are imposed:
(o) $\left(T_{j}-t_{j}^{1 / 2}\right)\left(T_{j}+t_{j}^{-1 / 2}\right)=0,0 \leq j \leq n$;
(i) $T_{i} T_{j} T_{i} \ldots=T_{j} T_{i} T_{j} \ldots, m_{i j}$ factors on each side;
(ii) $\pi_{r} T_{i} \pi_{r}^{-1}=T_{j}$ if $\pi_{r}\left(\alpha_{i}\right)=\alpha_{j}$;
(iii) $T_{i} X_{b} T_{i}=X_{b} X_{a_{i}}^{-1}$ if $\left(b, \alpha_{i}\right)=1,1 \leq i \leq n$;
(iv) $T_{0} X_{b} T_{0}=X_{s_{0}(b)}=X_{b} X_{\theta} q^{-1} \quad$ if $(b, \theta)=-1$;
(v) $T_{i} X_{b}=X_{b} T_{i}$ if $\left(b, \alpha_{i}\right)=0(i>0),(b, \theta)=0(i=0)$;
(vi) $\pi_{r} X_{b} \pi_{r}^{-1}=X_{\pi_{r}(b)}=X_{\omega_{r}^{-1}(b)} q^{\left(b_{r^{*},}, b\right)}, r \in O^{*}$.

Given $\tilde{w} \in W^{a}, r \in O$, the product

$$
\begin{equation*}
T_{\pi_{r} \tilde{w}} \stackrel{\text { def }}{=} \pi_{r} \prod_{k=1}^{l} T_{i_{k}}, \quad \text { where } \quad \tilde{w}=\prod_{k=1}^{l} s_{i_{k}}, l=l(\tilde{w}) \tag{2.2}
\end{equation*}
$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$
\begin{equation*}
T_{\hat{v}} T_{\hat{w}}=T_{\hat{v} \hat{w}} \text { whenever } l(\hat{v} \hat{w})=l(\hat{v})+l(\hat{w}) \text { for } \hat{v}, \hat{w} \in W^{b} \tag{2.3}
\end{equation*}
$$

In particular, we arrive at the pairwise commutative elements

$$
\begin{equation*}
Y_{b}=\prod_{i=1}^{n} Y_{i}^{k_{i}} \text { if } b=\sum_{i=1}^{n} k_{i} b_{i} \in B, \quad \text { where } \quad Y_{i} \stackrel{\text { def }}{=} T_{b_{i}} \tag{2.4}
\end{equation*}
$$

satisfying the relations

$$
\begin{align*}
& T_{i}^{-1} Y_{b} T_{i}^{-1}=Y_{b} Y_{a_{i}}^{-1} \text { if }\left(b, \alpha_{i}\right)=1 \\
& T_{i} Y_{b}=Y_{b} T_{i} \text { if }\left(b, \alpha_{i}\right)=0,1 \leq i \leq n \tag{2.5}
\end{align*}
$$

The following basic anti-involution * [C11] plays the key role for the socalled polynomial (basic) representation of $\mathfrak{H}$ and difference counterparts of the Dunkl operators:

$$
\begin{align*}
& X_{i}^{*}=X_{i}^{-1}, \quad Y_{i}^{*}=Y_{i}^{-1}, \quad T_{i}^{*}=T_{i}^{-1}, \pi_{r}^{*}=\pi_{r}^{-1} \\
& t(\nu)^{*}=t(\nu)^{-1}, q^{*}=q^{-1}, 0 \leq i \leq n,(A B)^{*}=B^{*} A^{*}, A, B \in \mathfrak{H} \tag{2.6}
\end{align*}
$$

The $Y$-intertwiners (see [C2]) are introduced as follows:

$$
\begin{aligned}
& \Phi_{j}=T_{j}+\left(t_{j}^{1 / 2}-t_{j}^{-1 / 2}\right)\left(Y_{a_{j}}^{-1}-1\right)^{-1}, 1 \leq j \leq n \\
& \Phi_{0}=X_{\theta} T_{s_{\theta}}-\left(t_{0}^{1 / 2}-t_{0}^{-1 / 2}\right)\left(q^{-1} Y_{\theta}^{-1}-1\right)^{-1} \\
& P_{r}=X_{r} T_{\omega_{r}^{-1}}, r \in O^{*}
\end{aligned}
$$

They belong to the extension $\mathfrak{H}$ by the field $\mathbf{C}_{q, t}(Y)$ of rational functions in $\{Y\}$. The elements $\left\{\Phi_{j}, P_{r}\right\}$ satisfy the relations for $\left\{T_{j}, \pi_{r}\right\}$. Hence the elements

$$
\begin{equation*}
\Phi_{\hat{w}}=P_{r} \Phi_{j_{l}} \cdots \Phi_{j_{1}}, \quad \text { where } \hat{w}=\pi_{r} s_{j_{l}} \cdots s_{j_{1}} \in W^{b} \tag{2.7}
\end{equation*}
$$

are well-defined for reduced decompositions of $\hat{w}$, and $\Phi_{\hat{w}} \Phi_{\hat{u}}=\Phi_{\hat{w} \hat{u}}$ whenever $l(\hat{w} \hat{u})=l(\hat{w})+l(\hat{u})$.

The following property of $\{\Phi\}$ fixes them uniquely up to left or right multiplications by functions of $Y$ :

$$
\begin{equation*}
\Phi_{\hat{w}} Y_{b}=Y_{\hat{w}(b)} \Phi_{\hat{w}}, \hat{w} \in W^{b} \tag{2.8}
\end{equation*}
$$

where $Y_{[b, k]} \stackrel{\text { def }}{=} Y_{b} q^{-k}$.
The $\left\{\Phi_{\hat{w}}\right\}$ are exactly the images of the $X$-intertwiners (see [C2], (2.12)) with respect to the involution $\varepsilon$ from (2.6) ibidem.

Note that

$$
\begin{equation*}
\Phi_{\hat{w}}^{*}=\Phi_{\hat{w}^{-1}}, \hat{w} \in W^{b} \tag{2.9}
\end{equation*}
$$

Use the quadratic relation for $T_{j}$ to check it for $\left\{\Phi_{j}, 0 \leq j \leq n\right\}$.
In the case of $G L_{n}$, the element $P_{1}$ (it is of infinite order) was used by Knop and Sahi (see $[\mathrm{Kn}]$ ) to construct the non-symmetric Macdonald polynomials and eventually confirm the Macdonald integrality conjecture. They introduced it independently of [C9] and checked the "intertwining" property directly. Generally speaking, a direct verification of (2.8) without the $X$-intertwiners and the involution $\varepsilon$ is possible but much more difficult.

## 3. Polynomial representation

We will identify $X_{\tilde{b}}$ with the corresponding multiplication operators:

$$
\begin{equation*}
X_{\tilde{b}}(p(X))=X_{\tilde{b}} p(X), p(X) \in \mathbf{C}_{q, t}[X] \tag{3.1}
\end{equation*}
$$

and use the action (1.1):

$$
\begin{align*}
& \hat{w}\left(X_{b}\right)=X_{\hat{w}(b)}, \text { for instance } \\
& s_{0}\left(X_{b}\right)=X_{b} X_{\theta}^{-(b, \theta)} q^{(b, \theta)}, a\left(X_{b}\right)=q^{-(a, b)} X_{b} \text { for } a, b \in B \tag{3.2}
\end{align*}
$$

The Demazure-Lusztig operators

$$
\begin{equation*}
\hat{T}_{j}=t_{j}^{1 / 2} s_{j}+\left(t_{j}^{1 / 2}-t_{j}^{-1 / 2}\right)\left(X_{a_{j}}-1\right)^{-1}\left(s_{j}-1\right), 0 \leq j \leq n \tag{3.3}
\end{equation*}
$$

are well-defined on $\mathbf{C}_{q, t}[X]$.
We note that only $\hat{T}_{0}$ depends on $q$ :

$$
\begin{equation*}
\hat{T}_{0}=t_{0}^{1 / 2} s_{0}+\left(t_{0}^{1 / 2}-t_{0}^{-1 / 2}\right)\left(q X_{\theta}^{-1}-1\right)^{-1}\left(s_{0}-1\right) \tag{3.4}
\end{equation*}
$$

ThEOREM [C9] 3.1. The map $T_{j} \mapsto \hat{T}_{j}, X_{b} \mapsto X_{b}, \pi_{r} \mapsto \pi_{r}$ can be extended to a faithful representation $H \mapsto \hat{H}$ of $\mathfrak{H}$ (fixing $q, t)$. The operators $\mathcal{L}_{p}=p\left(\hat{Y}_{1}, \ldots, \hat{Y}_{n}\right)$ are $W$-invariant for $W$-invariant polynomials $p$ and preserve $\mathbf{C}_{q, t}[X]^{W}$.

The exact formulas for the restrictions $L_{p}$ of $\mathcal{L}_{p}$ to $W$-invariant functions are known only for rather simple $p$.

Proposition [C11] 3.2. Given $r \in O^{*}$, let $m_{r}=\sum_{w \in W / W_{r}} X_{w\left(-b_{r}\right)}$, where $W_{r}$ is the stabilizer of $b_{r}$ in $W$. Then $\ell_{r}=\ell\left(b_{r}\right) \subset R_{+}$and

$$
\begin{equation*}
L_{r}=L_{m_{r}}=\sum_{w \in W / W_{r}} \prod_{\alpha \in \ell_{r}} \frac{t_{\alpha}^{1 / 2} X_{w\left(\alpha^{\vee}\right)}-t_{\alpha}^{-1 / 2}}{X_{w\left(\alpha^{\vee}\right)}-1} w(-b), \tag{3.5}
\end{equation*}
$$

where $w(-b)=-w(b) \in B$ acts as in (3.2).
The operators $L_{r}$ were introduced by Macdonald [M1,M2]. For $G L_{n}$, they were found independently by Ruijsenaars $[R]$. Later we will use these formulas in the "rational" limit.

The coefficient of $X^{0}=1$ (the constant term) of a polynomilal $f \in \mathbf{C}_{q, t}[X]$ will be denoted by $\langle f\rangle$. Let

$$
\begin{equation*}
\mu=\prod_{a \in R_{+}^{\vee}} \prod_{i=0}^{\infty} \frac{\left(1-X_{a} q_{a}^{i}\right)\left(1-X_{a}^{-1} q_{a}^{i+1}\right)}{\left(1-X_{a} t_{a} q_{a}^{i}\right)\left(1-X_{a}^{-1} t_{a} q_{a}^{i+1}\right)} \tag{3.6}
\end{equation*}
$$

where $q_{a}=q^{(a, a) / 2}=q^{2 /(\alpha, \alpha)}$ for $a=\alpha^{\vee}$.
The coefficients of $\mu_{0} \stackrel{\text { def }}{=} \mu /\langle\mu\rangle_{0}$ are from $\mathbf{C}(q, t)$, and $\mu_{0}^{*}=\mu_{0}$ with respect to the involution

$$
X_{b}^{*}=X_{-b}, t^{*}=t^{-1}, q^{*}=q^{-1}
$$

Hence

$$
\begin{equation*}
\{f, g\}_{\mu} \stackrel{\text { def }}{=}\left\langle f g^{*} \mu_{0}\right\rangle=\left(\{g, f\}_{\mu}\right)^{*} \text { for } f, g \in \mathbf{C}(q, t)[X] . \tag{3.7}
\end{equation*}
$$

Here and further see [C3,C11].
Proposition 3.3. For any $H \in \mathfrak{H}$ and the anti-involution* from (2.6), $\{\hat{H}(f), g\}_{\mu}=\left\{f, \hat{H}^{*}(g)\right\}_{\mu}$ for $f, g \in \mathbf{C}_{q, t}[X]$. The operators $\left\{X_{b}, Y_{b}, T_{j}, \pi_{r}\right\}$ and $q, t$ are unitary with respect to the form $\{,\}_{\mu}$.

Actually this proposition is entirely algebraic. It holds for other inner products. For instance, the inner product in terms of Jackson integrals considered in [C2] can be taken, which is expected to have applications to negative $k$.

## 4. Degenerations

Let us fix $\kappa=\left\{\kappa(\nu) \in \mathbf{C}, \nu \in \nu_{R}\right\}, \kappa_{\alpha}=\kappa((\alpha, \alpha)), \kappa_{j}=\kappa_{\alpha_{j}}$ and introduce $\rho_{\kappa}=(1 / 2) \sum_{\alpha \in R_{+}} \kappa_{\alpha} \alpha$. One has: $\left(\rho_{\kappa}, \alpha_{j}^{\vee}\right)=\kappa_{j}$ for $1 \leq j \leq n$.

Definition 4.1. The degenerate (graded) double affine Hecke algebra $\mathfrak{H}^{\prime}$ is generated by the group algebra $\mathbf{C}\left[W^{b}\right]$ and the pairwise commutative

$$
\begin{equation*}
y_{\tilde{b}} \stackrel{\text { def }}{=} \sum_{i=1}^{n}\left(b, \alpha_{i}\right) y_{i}+u \text { for } \tilde{b}=[b, u] \in \mathbf{R}^{n} \times \mathbf{R}, \tag{4.1}
\end{equation*}
$$

satisfying the following relations:

$$
\begin{align*}
& s_{j} y_{\tilde{b}}-y_{s_{j}(\tilde{b})} s_{j}=-\kappa_{j}\left(b, \alpha_{j}\right),\left(b, \alpha_{0}\right)=-(b, \theta), \\
& \pi_{r} y_{\tilde{b}}=y_{\pi_{r}(\tilde{b})} \pi_{r} \text { for } 0 \leq j \leq n, r \in O \tag{4.2}
\end{align*}
$$

Without $s_{0}$ and $\pi_{r}$, we arrive at the defining relations of the graded affine Hecke algebra from [L] (see also [C8]). It is a natural degeneration of the double affine Hecke algebra when $q \rightarrow 1, t \rightarrow 1$.

We will also use the parameters $k_{\alpha} \stackrel{\text { def }}{=}(\alpha, \alpha) \kappa_{\alpha} / 2, k_{i}=k_{\alpha_{i}}$, and the derivatives of $\mathbf{C}[X]$ :

$$
\partial_{a}\left(X_{b}\right)=(a, b) X_{b}, a, b \in B .
$$

Note that $w\left(\partial_{b}\right)=\partial_{w(b)}, w \in W$.
Differential degeneration. Following [C2,C10], let

$$
\begin{equation*}
q=\exp (h), t_{j}=q_{j}^{k_{j}}=q^{\kappa_{j}}, Y_{b}=1-h y_{b} \tag{4.3}
\end{equation*}
$$

and $h$ tend to zero. The terms of order $h^{2}$ are ignored (without touching $\left.X_{b}, \kappa_{j}\right)$. We will readily arrive at the relations (4.2) for $y_{b}$.

Setting $X_{b}=e^{x_{b}}$, the limit of $\mu$ from (3.6) is

$$
\begin{equation*}
\tau \stackrel{\text { def }}{=} \prod_{\alpha \in R_{+}}\left|2 \sinh \left(x_{\alpha^{\vee}} / 2\right)\right|^{2 k_{\alpha}} \tag{4.4}
\end{equation*}
$$

Proposition 4.2. a) The following operators acting on Laurent polynomials $f \in \mathbf{C}[X]$

$$
\begin{equation*}
D_{b} \stackrel{\text { def }}{=} \partial_{b}+\sum_{\alpha \in R_{+}} \frac{\kappa_{\alpha}(b, \alpha)}{\left(1-X_{\alpha^{\vee}}^{-1}\right)}\left(1-s_{\alpha}\right)-\left(\rho_{\kappa}, b\right) \tag{4.5}
\end{equation*}
$$

are pairwise commutative, and $y_{[b, u]}=D_{b}+u$ satisfy (4.2) for the following action of the group $W^{b}$ :

$$
w^{x}(f)=w(f) \text { for } w \in W, b^{x}(f)=X_{b} f \text { for } b \in B
$$

For instance, $s_{0}^{x}(f)=X_{\theta} s_{\theta}(f), \pi_{r}^{x}(f)=X_{r} \omega_{r}^{-1}(f)$.
b) The operators $D_{b}$ are formally self-adjoint with respect to the inner product

$$
\begin{equation*}
\{f, g\}_{\tau} \stackrel{\text { def }}{=} \int f(x) g(-x) \tau d x \tag{4.6}
\end{equation*}
$$

i.e. $\tau^{-1} D_{b}^{+} \tau=D_{b}$ for the anti-involution ${ }^{+}$sending $\hat{w}^{x} \mapsto\left(\hat{w}^{x}\right)^{-1}, \partial_{b} \mapsto-\partial_{b}$, where $\hat{w} \in W^{b}$.

Proof. First, $y_{b}=\lim _{h \rightarrow 0}\left(1-\hat{Y}_{b}\right) / h$ is precisely (4.5). Degenerate the product formulas for $Y_{i}=T_{b_{i}}$ in the polynomial representation (cf. [C6,C11], and [C10]). As to the term $-\left(\rho_{\kappa}, b\right)$, use (1.7). The condition $q^{*}=q^{-1}$ formally leads to $h^{*}=-h$, and $k^{*}=k$. Similarly, $x_{b}^{*}$ must be equal to $-x_{b}$, and all these give that $D_{b}^{*}=D_{b}$ for the inner product (4.6). Use Proposition 3.3. Actually the claim can be checked directly without the limiting procedure. However we want to demonstrate that the pairing (4.6) is an exact counterpart of the difference one. We mention that Opdam uses somewhat different pairing and involution in [O1].

Degenerating $\{\Phi\}$, we get the intertwiners of $\mathfrak{H}^{\prime}$ :

$$
\begin{align*}
& \Phi_{i}^{\prime}=s_{i}+\frac{k_{i}}{y_{\alpha_{i}}}, \Phi_{0}^{\prime}=X_{\theta} s_{\theta}+\frac{k_{0}}{1-y_{\theta}}, \\
& P_{r}^{\prime}=X_{r} \omega_{r}^{-1}, \text { for } 1 \leq i \leq n, r \in O^{*} . \tag{4.7}
\end{align*}
$$

The operator $P_{1}^{\prime}$ in the case of $G L_{n}$ (it is of infinite order) plays the key role in [KS]. The formulas for $\Phi_{i}^{\prime}$ when $1 \leq i \leq n$ are well-known in the theory of degenerate (graded) Lusztig algebras. See [L,C5] and [O1], Definition 8.2. However the main applications (say, the raising operators) require affine ones.

Let us mention that equating renormalized $\Phi^{\prime}$ and $\Phi([\mathrm{C} 5]$, Corollary 2.5 and [C2], Appendix) one comes to a Lusztig-type isomorphism [L] between proper completions of the algebras $\mathfrak{H}^{\prime}$ and $\mathfrak{H}$.

Difference-rational limit. We follow the same procedure (4.3), but now assume that

$$
\begin{equation*}
X_{b}=q^{\lambda_{b}}, \lambda_{a+b}=\lambda_{a}+\lambda_{b}, \lambda_{[b, u]}=\lambda_{b}+u . \tag{4.8}
\end{equation*}
$$

It changes the result drastically. Degenerating $\left\{\hat{T}_{j}\right\}$ from (3.3), we come to the rational Demazure-Lusztig operators from [C6]:

$$
\begin{equation*}
S_{j}=s_{j}^{\lambda}+\frac{k_{j}}{\lambda_{\alpha_{j}}}\left(s_{j}^{\lambda}-1\right), \quad 0 \leq j \leq n, \tag{4.9}
\end{equation*}
$$

where by $\hat{w}^{\lambda}$ we mean the action on $\left\{\lambda_{b}\right\}$ : $\hat{w}^{\lambda}\left(\lambda_{b}\right)=\lambda_{\hat{w}(b)}$. For instance, $S_{0}=s_{0}^{\lambda}+\frac{k_{0}}{1-\lambda_{\theta}}\left(s_{0}^{\lambda}-1\right)$.

We set

$$
\begin{align*}
& S_{\hat{w}} \stackrel{\text { def }}{=} \pi_{r}^{\lambda} S_{i_{1}} \ldots S_{i_{l}} \text { for } \hat{w}=\pi_{r} s_{i_{1}} \ldots s_{i_{l}},  \tag{4.10}\\
& \Delta_{b} \stackrel{\text { def }}{=} S_{b} \text { for } b \in B, \Delta_{i}=\Delta_{b_{i}} . \tag{4.11}
\end{align*}
$$

The definition does not depend on the particular choice of the decomposition of $\hat{w} \in W^{b}$, and the map $\hat{w} \mapsto S_{\hat{w}}$ is a homomorphism. The operators $\Delta_{b}$ are pairwise commutative.

The limit of $\mu$ is the asymmetric Harish-Chandra function:

$$
\begin{equation*}
\sigma=\prod_{a \in R_{+}} \frac{\Gamma\left(\lambda_{\alpha}+k_{a}\right) \Gamma\left(-\lambda_{\alpha}+k_{\alpha}+1\right)}{\Gamma\left(\lambda_{\alpha}\right) \Gamma\left(-\lambda_{\alpha}+1\right)} \tag{4.12}
\end{equation*}
$$

where $k_{\alpha}=\kappa_{\alpha}(\alpha, \alpha) / 2, \Gamma$ is the classical $\Gamma$-function.
Proposition 4.3. a) The operators $S_{\hat{w}}$ are well-defined and preserve the space of polynomials $\mathbf{C}[\lambda]$ in terms of $\lambda_{b}$. The map $\hat{w} \mapsto S_{\hat{w}}, y_{b} \mapsto \lambda_{b}$ is a representation of $\mathfrak{H}^{\prime}$.
b) The operators $S_{\hat{w}}$ are formally unitary with respect to the inner product

$$
\begin{equation*}
\{f, g\}_{\sigma} \stackrel{d e f}{=} \int f(\lambda) g(\lambda) \sigma d \lambda \tag{4.13}
\end{equation*}
$$

i.e. $\quad \sigma^{-1} S_{\hat{w}}^{+} \sigma=S_{\hat{w}}^{-1}$ for the anti-involution ${ }^{+}$sending $\lambda_{b} \mapsto \lambda_{b}, \hat{w} \mapsto \hat{w}^{-1}$, where $\hat{w} \in W^{b}$.

Proof. Statement a) is a straightforward degeneration of the polynomial representation of $\mathfrak{H}$. Since $\left(X_{b}\right)^{*}=\left(q^{\lambda_{b}}\right)^{*}=\left(X_{b}^{-1}\right)^{*}$ and $q^{*}=q^{-1}$, we get the *-invariance of $\lambda$ and come to (4.13).

Following Theorem 3.1 (and [C6]), the operators

$$
\begin{equation*}
\Lambda_{p}=p\left(\Delta_{1}, \ldots, \Delta_{n}\right) \text { for } p \in \mathbf{C}\left[X_{1}^{ \pm 1}, \ldots, X_{n}^{ \pm 1}\right]^{W} \tag{4.14}
\end{equation*}
$$

are $W$-invariant and preserve $\mathbf{C}[\lambda]^{W}$ (usual symmetric polynomials in $\lambda$ ).
Let us degenerate Proposition 3.2 (the notations $r, m_{r}, \ell_{r}$ remain the same). The restriction of $\Lambda_{m_{r}}\left(\Delta_{b}\right)$ onto $\mathbf{C}[\lambda]^{W}$ is as follows:

$$
\begin{equation*}
\Lambda_{r}=\sum_{w \in W / W_{r}} \prod_{\alpha \in \ell_{r}} \frac{\lambda_{w(\alpha)}+k_{\alpha}}{\lambda_{w(\alpha)}} w(-b) \tag{4.15}
\end{equation*}
$$

In the differential case the formulas (and the corresponding limiting procedure) are more complicated.

## 5. Opdam transform

From now on we assume that $\mathbf{R} \ni k_{\alpha}>0$ for all $\alpha$ and keep the notation from the previous section; $i$ is the imaginary unit, $\Re, \Im$ the real and imaginary parts.

THEOREM 5.1. There exists a solution $G(x, \lambda)$ of the eigenvalue problem

$$
\begin{equation*}
D_{b}(G(x, \lambda))=\lambda_{b} G(x, \lambda), \quad b \in B, G(0, \lambda)=1 \tag{5.1}
\end{equation*}
$$

holomorphic for all $\lambda$ and for $x$ in $\mathbf{R}^{n}+i U$ for a neighbourhood $U \subset \mathbf{R}^{n}$ of zero. If $x \in \mathbf{R}^{n}$ then $|G(x, \lambda)| \leq|W|^{1 / 2} \exp \left(\max _{w}(w(x), \Re \lambda)\right)$, so $G$ is
bounded for $x \in \mathbf{R}^{n}$ when $\lambda \in i \mathbf{R}^{n}$. The solution of (5.1) is unique in the class of continuously differentiable functions on $\mathbf{R}^{n}$ (for a given $\lambda$ ).

This theorem is from [O1] (Theorem 3.15 and Proposition 6.1). Opdam uses that

$$
\begin{equation*}
F \stackrel{\text { def }}{=}|W|^{-1} \sum_{w \in W} G(w(x), \lambda) \tag{5.2}
\end{equation*}
$$

is a generalized hypergeometric function, i.e a $W$-symmetric eigenfunction of the restrictions $L_{p}^{\prime}$ of the operators $p\left(D_{b_{1}}, \ldots, D_{b_{n}}\right)$ to symmetric functions:

$$
L_{p}^{\prime} F(x, \lambda)=p\left(\lambda_{1}, \ldots, \lambda_{n}\right) F(x, \lambda),
$$

where $p$ is any $W$-invariant polynomial of $\lambda_{i}=\lambda_{b_{i}}$. The operators $L_{p}^{\prime}$ generalize the radial parts of Laplace operators on the corresponding symmetric space (see Introduction). The normalization is the same: $F(0, \lambda)=1$. It fixes $F$ uniquely. So it is $W$-invariant with respect to $\lambda$ as well.

A systematic algebraic and analytic theory of $F$-functions is due to Heckman and Opdam (see [HO1,H1,O2,HS]). There is a formula for $G$ in terms of $F$ (at least for generic $\lambda$ ) via the operators $D_{b}$ from (4.5). The positivity of $k$ implies that it holds for all $\lambda \in \mathbf{C}^{n}$. See [O1] for a nice and simple argument (Lemma 3.14). This formula and the relation to the affine KnizhnikZamolodchikov equation [C5,Ma,C8,O1] are applied to establish the growth estimates for $G$. Actually it gives more than was formulated in the theorem (see Corollary 6.5, [O1]).

We introduce the Opdam transform (the first component of what he called "Cherednik's transform") as follows:

$$
\begin{equation*}
\mathcal{F}(f)(\lambda) \stackrel{\text { def }}{=} \int_{\mathbf{R}^{n}} f(x) G(-x, \lambda) \tau d x \tag{5.3}
\end{equation*}
$$

for the standard measure $d x$ on $\mathbf{R}^{n}$.
Proposition 5.2. a) Let us assume that $f(x)$ are taken from the space $\mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$ of $\mathbf{C}$-valued compactly supported $\infty$-differentiable functions on $\mathbf{R}^{n}$. The inner product

$$
\begin{equation*}
\left\{f, f^{\prime}\right\}_{\tau} \stackrel{\text { def }}{=} \int_{\mathbf{R}^{n}} f(x) f^{\prime}(-x) \tau d x \tag{5.4}
\end{equation*}
$$

satisfies the conditions of part b), Proposition 4.2. Namely, $\left\{D_{b}\right\}$ preserve $\mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$ and are self-adjoint with respect to the pairing (5.4).
b) The Opdam transforms of such functions are analytic in $\lambda$ on the whole $\mathbf{C}^{n}$ and satisfy the Paley-Wiener condition. A function $g(\lambda)$ is of PW-type $\left(g \in P W\left(\mathbf{C}^{n}\right)\right)$ if there exists a constant $A=A(g)>0$ such that for any $N>0$

$$
\begin{equation*}
g(\lambda) \leq C(1+|\lambda|)^{-N} \exp (A|\Re \lambda|) \tag{5.5}
\end{equation*}
$$

for a proper constant $C=C(N ; g)$.
Proof. The first claim is obvious. The Paley-Wiener condition follows from Theorem 8.6 [O1]. The transform under consideration is actually the first component of Opdam's transform from Definition 7.9 (ibid.). We also omit the complex conjugation in his definition and change the sign of $x$ (instead of $\lambda$ ). The estimates remain the same.

## 6. Inverse transform

We are coming to the inversion procedure (for positive $k$ ). The inverse transform is well-defined for Paley-Wiener functions $g(\lambda)$ on $\mathbf{C}^{n}$ by the formula

$$
\begin{equation*}
\mathcal{G}(g)(x) \stackrel{\text { def }}{=} \int_{i \mathbf{R}^{n}} g(\lambda) G(x, \lambda) \sigma d \lambda \tag{6.1}
\end{equation*}
$$

for the standard measure $d \lambda$. The transforms of such $g$ belong to $\mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$.
The existense readily follows from (5.5) and known properties of the Ha-rish-Chandra $c$-function (see below). The embedding $\mathcal{G}\left(P W\left(\mathbf{C}^{n}\right)\right) \subset \mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$ is due to Opdam. It is similar to the classical one from [He1] (see also [GV]).

Let us first discuss the shift of integration contour in (6.1). There exists an open neighborhood $U_{+}^{a} \subset \mathbf{R}^{n}$ of the closure $\bar{C}_{+}^{a} \in \mathbf{R}^{n}$ of the affine Weyl chamber $C_{+}^{a}$ from (1.8) such that

$$
\begin{equation*}
\mathcal{G}(g)(x)=\int_{\xi+i \mathbf{R}^{n}} g(\lambda) G(x, \lambda) \sigma d \lambda \tag{6.2}
\end{equation*}
$$

for $\xi \in U_{+}^{a}$. Indeed, $k_{\alpha}>0$ and $\sigma$ has no singularities on $U_{+}^{a}+i \mathbf{R}^{n}$. Then we use the classical formulas for $|\Gamma(x+i y) / \Gamma(x)|$ for real $x, y, x>0$. It gives that (cf. [He1,O1])

$$
\begin{equation*}
|\sigma(\lambda)| \leq C(1+|\lambda|)^{K}, \text { where } K=2 \sum_{\alpha>0} k_{\alpha}, \lambda \in U_{+}^{a}+i \mathbf{R}^{n}, \tag{6.3}
\end{equation*}
$$

for sufficiently big $C>0$. So the products of PW-functions by $\sigma$ tend to 0 for $|\lambda| \mapsto \infty$, and we can switch to $\xi$. Actually we can do this for any integrand analytic in $U_{+}^{a}+i \mathbf{R}^{n}$ and approaching 0 at $\infty$. We come to the following:

Proposition 6.1. The conditions of part b), Proposition 4.3 are satisfied for

$$
\begin{equation*}
\left\{g, g^{\prime}\right\}_{\sigma} \stackrel{\text { def }}{=} \int_{i \mathbf{R}^{n}} g(\lambda) g^{\prime}(\lambda) \sigma d \lambda \tag{6.4}
\end{equation*}
$$

in the class of $P W$-functions, i.e. the operators $S_{\hat{w}}$ are well-defined on such functions and unitary.

Proof. It is sufficient to check the unitarity for the generators $S_{j}=$ $S_{s_{j}}(0 \leq j \leq n)$ and $\pi_{r}^{\lambda}\left(r \in O^{*}\right)$. For instance, let us consider $s_{0}$. We
will integrate over $\xi+i \mathbf{R}$, assuming that $\xi^{\prime} \stackrel{\text { def }}{=} s_{\theta}(\xi)+\theta \in U_{+}^{a}$ and avoiding the wall $(\theta, \xi)=1$.

We will apply (4.13), the formula

$$
\int_{\xi+i \mathbf{R}^{n}} s_{0}(g(\lambda) \sigma) d \lambda=\int_{\xi^{\prime}+i \mathbf{R}^{n}} g(\lambda) \sigma d \lambda=\int_{\xi+i \mathbf{R}^{n}} g(\lambda) \sigma d \lambda,
$$

and a similar formula for $\left(1-\lambda_{\theta}\right)^{-1} g \sigma$, where $g$ is of PW-type on $U_{+}^{a}+i \mathbf{R}^{n}$. Note that $\left(1-\lambda_{\theta}\right)^{-1} \sigma$ is regular in this domain. One has:

$$
\begin{align*}
& \int_{\xi+i \mathbf{R}^{n}} S_{0}(g(\lambda)) g^{\prime}(\lambda) \sigma d \lambda \\
= & \int_{\xi+i \mathbf{R}^{n}}\left(s_{0}+k_{0}\left(1-\lambda_{\theta}\right)^{-1}\left(s_{0}-1\right)\right)(g(\lambda)) g^{\prime}(\lambda) \sigma d \lambda \\
= & \int_{\xi+i \mathbf{R}^{n}} g(\lambda)\left(s_{0}+k_{0}\left(s_{0}-1\right)\left(1-\lambda_{\theta}\right)^{-1}\right)\left(g^{\prime}(\lambda) \sigma\right) d \lambda \\
= & \int_{\xi+i \mathbf{R}^{n}} g(\lambda) S_{0}^{-1}\left(g^{\prime}(\lambda)\right) \sigma d \lambda . \tag{6.5}
\end{align*}
$$

Since (6.5) holds for one $\xi$ it is valid for all of them in $U_{+}^{a}$ including 0 . The consideration of the other generators is the same.

Main Theorem 6.2. Given $\hat{w} \in W^{b}, b \in B, f(x) \in \mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right), g(\lambda) \in$ PW $\left(\mathbf{C}^{n}\right)$,

$$
\begin{align*}
& \hat{w}^{x}(G(x, \lambda))=S_{\hat{w}}^{-1} G(x, \lambda), \text { e.g. } X_{b} G=\Delta_{b}^{-1}(G),  \tag{6.6}\\
& \mathcal{F}\left(\hat{w}^{x}(f(x))\right)=S_{\hat{w}} \mathcal{F}(f(x)), \mathcal{G}\left(S_{\hat{w}}(g(\lambda))\right)=\hat{w}^{x}(\mathcal{F}(f(x))),  \tag{6.7}\\
& \mathcal{F}\left(X_{b} f(x)\right)=\Delta_{b}(\mathcal{F}(f(x))), \mathcal{G}\left(\Delta_{b}(g(\lambda))\right)=X_{b} \mathcal{G}(g(\lambda)),  \tag{6.8}\\
& \mathcal{F}\left(D_{b}(f(x))\right)=\lambda_{b} \mathcal{F}(f(x)), \mathcal{G}\left(\lambda_{b} g(\lambda)\right)=D_{b}(\mathcal{G}(g(\lambda))) . \tag{6.9}
\end{align*}
$$

Proof. Applying the intertwiners from (4.7) to $G(x, \lambda)$, we see that

$$
\begin{align*}
& \left(1+\frac{k_{i}}{\lambda_{\alpha_{i}}}\right)^{-1}\left(s_{i}+\frac{k_{i}}{\lambda_{\alpha_{i}}}\right)(G)=s_{i}^{\lambda}(G), \text { for } 1 \leq i \leq n,  \tag{6.10}\\
& \left(1+\frac{k_{0}}{1-\lambda_{\theta}}\right)^{-1}\left(X_{\theta} s_{\theta}+\frac{k_{0}}{1-\lambda_{\theta}}\right)(G)=s_{0}^{\lambda}(G), \\
& P_{r}^{\prime}(G)=X_{r} \omega_{r}^{-1}(G)=\left(\pi_{r}^{\lambda}\right)^{-1}(G) \text { for } \quad r \in O^{*} .
\end{align*}
$$

The scalar factors on the left are necessary to preserve the normalization $G(0, \lambda)=1$, so we can use the main property of the intertwiners (2.8) and the uniqueness of $G(x, \lambda)$ (Theorem 5.1). Expressing $s_{j}^{x}$ in terms of $s_{j}^{\lambda}$ (when applied to $G!$ ), we get (6.6) for $\hat{w}=s_{j}$. It is obvious for $\hat{w}=\pi_{r}$. Using the commutativity of $\hat{w}^{x}$ and $S_{\hat{u}}$ for $\hat{w}, \hat{u} \in W^{b}$, we establish this relation in the general case. For $w \in W$ it is due to Opdam.

Formula (6.7) results directly from (6.6) because we already know that $\hat{w}^{x}$ are unitary for $\left\{f, f^{\prime}\right\}_{\tau}$ and $S_{\hat{w}}$ are unitary with respect to $\left\{g, g^{\prime}\right\}_{\sigma}$ for the considered classes of functions. See Theorems 4.2, 4.3 and (5.4),(6.4).

For instance, let us check (6.8) for $\mathcal{F}$, which is a particular case of (6.7):

$$
\begin{gather*}
\mathcal{F}\left(X_{b} f(x)\right)=\int_{\mathbf{R}^{n}} f(-x) X_{b}^{-1} G(x, \lambda) \tau d x= \\
\int_{\mathbf{R}^{n}} f(-x) \Delta_{b}(G(x, \lambda)) \tau d x=\Delta_{b}(\mathcal{F}(f(x))) . \tag{6.11}
\end{gather*}
$$

Thus the Opdam transforms of the "multiplications by the coordinates" $X_{b}$ are the operators $\Delta_{b}$.

Since $D_{b}$ and $\lambda_{b}$ are self-adjoint for the corresponding inner products (see Theorem 4.2), we get (6.9), which is in fact the defining property of $\mathcal{F}$ and $\mathcal{G}$.

Corollary 6.3. The compositions $\mathcal{G F}: C_{c}^{\infty}\left(\mathbf{R}^{n}\right) \rightarrow C_{c}^{\infty}\left(\mathbf{R}^{n}\right)$, and $\mathcal{F G}: P W\left(\mathbf{C}^{n}\right) \rightarrow P W\left(\mathbf{C}^{n}\right)$ are multiplications by nonzero constants. The transforms $\mathcal{F}, \mathcal{G}$ establish isomorphisms between the corresponding space identifying $\{,\}_{\tau}$ and $\{,\}_{\sigma}$ up to proportionality.

Proof. The first statement readily follows from the Main Theorem. The composition $\mathcal{G F}$ sends $\mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$ into itself, is continuous (due to Opdam), and commutes with the operators $X_{b}$ (multiplications by $X_{b}$ ). So it is the multiplication by a function $u(x)$ of $\mathbf{C}^{\infty}$-type. It must also commute with $D_{b}$. Hence $G(x, \lambda) u(x)$ is another solution of the eigenvalue problem (5.1) and $u(x)$ has to be a constant.

Let us check that $\mathcal{F G}$, which is a continuous operator on $P W\left(\mathbf{C}^{n}\right)$ (for any fixed $A$ ) commuting with multiplications by any $\lambda_{b}$, has to be a multiplication by an analytic function $v(\lambda)$. Indeed, the image of $\mathcal{F G}$ with respect to the standard Fourier transform $(k=0)$ is a continuous operator on $C_{c}^{\infty}\left(\mathbf{R}^{n}\right)$ commuting with the derivatives $\partial / \partial x_{i}$. So it is the convolution with some function and its inverse Fourier transform is the multiplication by a certain $v(\lambda)$. Since $\mathcal{F G}(g)=$ Const $g$ for any $g(\lambda)$ from $\mathcal{F}\left(C_{c}^{\infty}\left(\mathbf{R}^{n}\right)\right), v$ is constant.

The claim about the inner products is obvious because $\{\mathcal{F}(f), g\}_{\sigma}=$ $\{f, \mathcal{G}(g)\}_{\tau}$ for $f \in C_{c}^{\infty}\left(\mathbf{R}^{n}\right), g \in P W\left(\mathbf{C}^{n}\right)$.

The corrolary is due to Opdam ([O1],Theorem 9.13 (1)). He uses Peetre's characterization of differential operators (similar to what Van Den Ban and Schlichtkrull did in [BS]). Anyway a certain nontrivial analytic argument is involved to check that $\mathcal{G F}$ is multiplication by a constant.

The symmetric case. The transforms can be readily reduced to the symmetric level. If $f$ is $W$-invariant then

$$
\begin{equation*}
\mathcal{F}(f(x))=\int_{\mathbf{R}^{n}} f(x) F(-x, \lambda) \tau d x \tag{6.12}
\end{equation*}
$$

for $F$ from (5.2). Here we applied the $W$-symmetrization to the integrand of the (5.3) and used that $\tau$ is $W$-invariant. So $\mathcal{F}$ coincides with the $k$-deformation of the Harish-Chandra transform on $W$-invariant functions up to a minor technical detail. The $W$-invariance of $F$ in $\lambda$ results in the $W$-invariance of $\mathcal{F}(f)$.

As to $\mathcal{G}$, we $W$-symmetrize the integrand in the definition with respect to $x$ and $\lambda$, replacing $G$ by $F$ and $\sigma$ by by its $W$-symmetrization. The latter is the genuine Harish-Chandra "measure"

$$
\begin{equation*}
\sigma^{\prime}=\prod_{a \in R_{+}} \frac{\Gamma\left(\lambda_{\alpha}+k_{a}\right) \Gamma\left(-\lambda_{\alpha}+k_{\alpha}\right)}{\Gamma\left(\lambda_{\alpha}\right) \Gamma\left(-\lambda_{\alpha}\right)} \tag{6.13}
\end{equation*}
$$

up to a coefficient of proportionality.
Finally, given a $W$-invariant function $f \in \mathbf{C}_{c}^{\infty}\left(\mathbf{R}^{n}\right)$,

$$
\begin{equation*}
f(x)=\text { Const } \int_{i \mathbf{R}^{n}} g(\lambda) F(x, \lambda) \sigma^{\prime} d \lambda \text { for } g=\mathcal{F}(f) \tag{6.14}
\end{equation*}
$$

A similar formula holds for $\mathcal{G}$. See [HC,He2] and [GV], Ch. 6 for the classical theory.

As an application, we are able to calculate the Fourier transforms of the operators $p(X) \in \mathbf{C}\left[X_{b}\right]^{W}$ (symmetric Laurent polynomials acting by multiplication) in the Harish-Chandra theory and its $k$-deformation. They are exactly the operators $\Lambda_{p}$ from Section 4. In the minuscule case, we get formulas (4.15). We mention that in [O1] and other papers the pairings serving the Fourier transforms are hermitian. Complex conjugations can be added to ours.

We hope that the method used in this paper can be generaized to negative $k$ and to other classes of functions (cf. [BS,HO2]). Relations (6.6) considered as difference equations for $G(x, \lambda)$ with respect to $\lambda$ may help with the growth estimates via the theory of difference equations and the equivalence with difference Knizhnik-Zamolodchikov equations. However I believe that the main progress here will be connected with the difference Fourier transform.

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