

DIFFERENCE EQUATIONS COMPATIBLE WITH TRIGONOMETRIC KZ DIFFERENTIAL EQUATIONS

V. TARASOV^{*,1} AND A. VARCHENKO^{◇,2}

* *St. Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St. Petersburg
191011, Russia,
vt@pdmi.ras.ru*

◇ *Department of Mathematics, University of North Carolina at Chapel Hill,
Chapel Hill, NC 27599-3250, USA,
av@math.unc.edu*

February, 2000

To the memory of Anatoly Izergin

ABSTRACT. The trigonometric KZ equations associated with a Lie algebra \mathfrak{g} depend on a parameter $\lambda \in \mathfrak{h}$ where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra. We suggest a system of dynamical difference equations with respect to λ compatible with the KZ equations. The dynamical equations are constructed in terms of intertwining operators of \mathfrak{g} -modules.

1. INTRODUCTION

The trigonometric KZ equations associated with a Lie algebra \mathfrak{g} depend on a parameter $\lambda \in \mathfrak{h}$ where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra. We suggest a system of dynamical difference equations with respect to λ compatible with the trigonometric KZ differential equations. The dynamical equations are constructed in terms of intertwining operators of \mathfrak{g} -modules.

Our dynamical difference equations are a special example of the difference equations introduced by Cherednik. In [Ch1, Ch2] Cherednik introduces a notion of an affine R-matrix associated with the root system of a Lie algebra and taking values in an algebra F with certain properties. Given an affine R-matrix, he defines a system of equations for an element of the algebra F .

In this paper we construct an example of an affine R-matrix and call the corresponding system of equations the dynamical equations. In our example, F is the algebra of functions of complex variables z_1, \dots, z_n and $\lambda \in \mathfrak{h}$ taking values in the tensor product of

¹ Supported in part by RFFI grant 99-01-00101.

² Supported in part by NSF grant DMS-9801582.

n copies of the universal enveloping algebra of \mathfrak{g} . The fact that our dynamical difference equations are compatible with the trigonometric KZ differential equations is a remarkable property of our affine R-matrix.

There is a similar construction of dynamical difference equations compatible with the qKZ difference equations associated with a quantum group. The dynamical difference equations in that case are constructed in the same way in terms of intertwining operators of modules over the quantum group. We will describe this construction in a forthcoming paper.

There is a degeneration of the trigonometric KZ differential equations to the standard (rational) KZ differential equations. Under this limiting procedure the dynamical difference equations constructed in this paper turn into the system of differential equations compatible with the standard KZ differential equations and described in [FMTV]. In [FMTV] we proved that the standard hypergeometric solutions of the standard KZ equations [SV, V] satisfy also the dynamic differential equations of [FMTV].

The trigonometric KZ differential equations also have hypergeometric solutions, see [Ch3, EFK]. We conjecture that the hypergeometric solutions of the trigonometric KZ differential equations also solve the dynamical difference equations of this paper.

In Section 2 we study relations between intertwining operators of \mathfrak{g} -modules and the Weyl group \mathbb{W} of \mathfrak{g} . For any finite dimensional \mathfrak{g} -module V and $w \in \mathbb{W}$ we construct a rational function $\mathbb{B}_{w,V} : \mathbb{C} \rightarrow \text{End}(V)$. The operators $\mathbb{B}_{w,V}(\lambda)$ are used later to construct an affine R-matrix and dynamical equations.

In Section 3 we define the dynamical difference equations for $\mathfrak{g} = sl_N$ in terms of operators $\mathbb{B}_{w,V}(\lambda)$ directly (without introducing affine R-matrices). For $\mathfrak{g} = sl_N$, we prove that the dynamical equations are compatible with the trigonometric KZ differential equations. We give a formula for the determinant of a square matrix solution of the combined system of KZ and dynamical equations.

In Section 4 we review [Ch1, Ch2] and construct the dynamical difference equations for any simple Lie algebra \mathfrak{g} . We show that the dynamical equations are compatible with the trigonometric KZ equations if the Lie algebra \mathfrak{g} has minuscule weights, i.e. is not of type E_8, F_4, G_2 . We conjecture that the dynamical difference equations and trigonometric KZ equations are compatible for any simple Lie algebra.

We thank I.Cherednik for valuable discussions and explanation of his articles [Ch1, Ch2] and P.Etingof who taught us all about the Weyl group and intertwining operators.

2. INTERTWINING OPERATORS

2.1. Preliminaries. Let \mathfrak{g} be a complex simple Lie algebra with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_{\alpha})$ where $\Sigma \subset \mathfrak{h}^*$ is the set of roots.

Fix a system of simple roots $\alpha_1, \dots, \alpha_r$. Let Γ be the corresponding Dynkin diagram, and Σ_{\pm} — the set of positive (negative) roots. Let $\mathfrak{n}_{\pm} = \oplus_{\alpha \in \Sigma_{\pm}} \mathfrak{g}_{\alpha}$. Then $\mathfrak{g} = \mathfrak{n}_{+} \oplus \mathfrak{h} \oplus \mathfrak{n}_{-}$.

Let $(,)$ be an invariant bilinear form on \mathfrak{g} . The form gives rise to a natural identification $\mathfrak{h} \rightarrow \mathfrak{h}^*$. We use this identification and make no distinction between \mathfrak{h} and \mathfrak{h}^* .

This identification allows us to define a scalar product on \mathfrak{h}^* . We use the same notation $(,)$ for the pairing $\mathfrak{h} \otimes \mathfrak{h}^* \rightarrow \mathbb{C}$.

We use the notation: $Q = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ - root lattice; $Q^+ = \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\alpha_i$; $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ - dual root lattice, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$; $P = \{\lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}\}$ - weight lattice; $P^+ = \{\lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0}\}$ - cone of dominant integral weights; $\omega_i \in P^+$ - fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$; $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha = \sum_{i=1}^r \omega_i$; $P^\vee = \bigoplus_{i=1}^r \mathbb{Z}\omega_i^\vee$ - dual weight lattice, where ω_i^\vee -dual fundamental weights: $(\omega_i^\vee, \alpha_j) = \delta_{ij}$.

Define a partial order on \mathfrak{h} putting $\mu < \lambda$ if $\lambda - \mu \in Q^+$.

Let $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$ denote a simple reflection, defined by $s_i(\lambda) = \lambda - (\alpha_i^\vee, \lambda)\alpha_i$; \mathbb{W} - Weyl group, generated by s_1, \dots, s_r . The following relations are defining:

$$s_i^2 = 1, \quad (s_i s_j)^m = 1 \quad \text{for} \quad m = 2, 3, 4, 6,$$

where $m = 2$ if α_i and α_j are not neighboring in Γ , otherwise, $m = 3, 4, 6$ if 1,2,3 lines respectively connect α_i and α_j in Γ . For an element $w \in \mathbb{W}$, denote $l(w)$ the length of the minimal (reduced) presentation of w as a product of generators s_1, \dots, s_r .

Let $U\mathfrak{g}$ be the universal enveloping algebra of \mathfrak{g} ; $U\mathfrak{g}^{\otimes n}$ - tensor product of n copies of $U\mathfrak{g}$; $\Delta^{(n)} : U\mathfrak{g} \rightarrow U\mathfrak{g}^{\otimes n}$ - the iterated comultiplication (in particular, $\Delta^{(1)}$ is the identity, $\Delta^{(2)}$ is the comultiplication); $U\mathfrak{g}_0^{\otimes n} = \{x \in U\mathfrak{g}^{\otimes n} \mid [\Delta^{(n)}(h), x] = 0 \text{ for any } h \in \mathfrak{h}\}$ - subalgebra of weight zero elements.

For $\alpha \in \Sigma$ choose generators $e_\alpha \in \mathfrak{g}_\alpha$ so that $(e_\alpha, e_{-\alpha}) = 1$. For any α , the triple

$$H_\alpha = \alpha^\vee, \quad E_\alpha = \frac{2}{(\alpha, \alpha)} e_\alpha, \quad F_\alpha = e_{-\alpha}$$

forms an sl_2 -subalgebra in \mathfrak{g} , $[H_\alpha, E_\alpha] = 2E_\alpha$, $[H_\alpha, F_\alpha] = -2F_\alpha$, $[E_\alpha, F_\alpha] = H_\alpha$.

A dual fundamental weight ω_i^\vee is called minuscule if (ω_i^\vee, α) is 0 or 1 for all $\alpha \in \Sigma_+$, i.e. for any positive root $\alpha = \sum_{i=1}^r m_i \alpha_i$, the coefficient m_i is either 0 or 1. For a root system of type A_r all dual fundamental weights are minuscule. There is no minuscule dual fundamental weight for E_8, F_4, G_2 . For a minuscule dual fundamental weight ω_i^\vee , define an element $w_{[i]} = w_0 w_0^i \in \mathbb{W}$ where w_0 (respectively, w_0^i) is the longest element in \mathbb{W} (respectively, in \mathbb{W}^i generated by all simple reflections s_j preserving ω_i^\vee).

Lemma 1. *Let α be a positive root. Then $w_{[i]}(\alpha) \in \Sigma_+$ if $(\omega_i^\vee, \alpha) = 0$ and $w_{[i]}(\alpha) \in \Sigma_-$ if $(\omega_i^\vee, \alpha) = 1$.*

Let \mathbb{G} be the simply connected complex Lie group with Lie algebra \mathfrak{g} , $\mathbb{H} \subset \mathbb{G}$ the Cartan subgroup corresponding to \mathfrak{h} , $N(\mathbb{H}) = \{x \in \mathbb{G} \mid x\mathbb{H}x^{-1} = \mathbb{H}\}$ the normalizer of \mathbb{H} . Then the Weyl group is canonically isomorphic to $N(\mathbb{H})/\mathbb{H}$. The isomorphism sends x to $\text{Ad}_x|_{\mathfrak{h}}$.

Let V be a finite dimensional \mathfrak{g} -module with weight decomposition $V = \bigoplus_{\mu \in \mathfrak{h}} V[\mu]$. \mathbb{G} acts on V so that \mathbb{H} acts trivially on $V[0]$. Thus the action of \mathbb{W} on $V[0]$ is well defined. For any n , the Weyl group in the same way acts also on $U\mathfrak{g}_0^{\otimes n}$.

Lemma 2. For $\alpha \in \Sigma$ and $k \in \mathbb{Z}_{\geq 0}$, consider $e_\alpha^k e_{-\alpha}^k \in U\mathfrak{g}_0$ and $e_\alpha \otimes e_{-\alpha} \in U\mathfrak{g}_0^{\otimes 2}$. Then for any $w \in \mathbb{W}$,

$$w(e_\alpha^k e_{-\alpha}^k) = e_{w(\alpha)}^k e_{-w(\alpha)}^k, \quad w(e_\alpha \otimes e_{-\alpha}) = e_{w(\alpha)} \otimes e_{-w(\alpha)}.$$

Proof. Let $x \in N(\mathbb{H})$ be a lifting of w . $\text{Ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ is an automorphism of \mathfrak{g} preserving the invariant scalar product and sending \mathfrak{g}_β to $\mathfrak{g}_{w(\beta)}$ for all β . Thus, $\text{Ad}_x e_\beta = c_{x,\beta} e_{w(\beta)}$ for suitable numbers $c_{x,\beta}$ and $c_{x,\alpha} c_{x,-\alpha} = 1$. \square

Let x_1, \dots, x_r be an orthonormal basis in \mathfrak{h} , set

$$\Omega^0 = \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i, \quad \Omega^+ = \Omega^0 + \sum_{\alpha \in \Sigma_+} e_\alpha \otimes e_{-\alpha}, \quad \Omega^- = \Omega^0 + \sum_{\alpha \in \Sigma_+} e_{-\alpha} \otimes e_\alpha.$$

Define the Casimir operator Ω and the trigonometric R-matrix $r(z)$ by

$$\Omega = \Omega^+ + \Omega^-, \quad r(z) = \frac{\Omega^+ z + \Omega^-}{z - 1}.$$

For any $x \in U\mathfrak{g}$, we have $\Delta(x)\Omega = \Omega\Delta(x)$. We will use a more symmetric form of the trigonometric R-matrix: $r(z_1/z_2)$.

The Weyl group acts on $r(z_1/z_2)$, $\Omega \in U\mathfrak{g}_0^{\otimes 2}$. Ω is Weyl invariant. For any $w \in \mathbb{W}$,

$$w(r(z_1/z_2)) = \frac{1}{z_1 - z_2} \left(\frac{z_1 + z_2}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Sigma_+} (z_1 e_{w(\alpha)} \otimes e_{-w(\alpha)} + z_2 e_{w(-\alpha)} \otimes e_{w(\alpha)}) \right).$$

Lemma 3. For a minuscule dual fundamental weight ω_i^\vee ,

$$z_1^{-(\omega_i^\vee)^{(1)}} z_2^{-(\omega_i^\vee)^{(2)}} r(z_1/z_2) z_1^{(\omega_i^\vee)^{(1)}} z_2^{(\omega_i^\vee)^{(2)}} = w_{[i]}^{-1}(r(z_1/z_2)).$$

Proof. Using Lemma 1 it is easy to see that both sides of the equation are equal to

$$\begin{aligned} \frac{1}{z_1 - z_2} \left(\frac{z_1 + z_2}{2} \sum_{i=1}^r x_i \otimes x_i + \sum_{\alpha \in \Sigma_+, (\alpha, \omega_i^\vee)=0} (z_1 e_\alpha \otimes e_{-\alpha} + z_2 e_{-\alpha} \otimes e_\alpha) + \right. \\ \left. \sum_{\alpha \in \Sigma_+, (\alpha, \omega_i^\vee)=1} (z_1 e_{-\alpha} \otimes e_\alpha + z_2 e_\alpha \otimes e_{-\alpha}) \right). \quad \square \end{aligned}$$

2.2. The Trigonometric KZ Equations. Let $V = V_1 \otimes \dots \otimes V_n$ be a tensor product of \mathfrak{g} -modules. For $\kappa \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$, introduce the KZ operators $\nabla_i(\lambda, \kappa)$, $i = 1, \dots, n$, acting on functions $u(z_1, \dots, z_n)$ of n complex variables with values in V and defined by

$$\nabla_i(\lambda, \kappa) = \kappa z_i \frac{\partial}{\partial z_i} - \sum_{j, j \neq i} r(z_i/z_j)^{(i,j)} - \lambda^{(i)}.$$

Here $r^{(i,j)}$, $\lambda^{(i)}$ denote r acting in the i -th and j -th factors of the tensor product and λ acting in the i -th factor.

The trigonometric KZ equations are the equations

$$(1) \quad \nabla_i(\lambda, \kappa)u(z_1, \dots, z_n, \lambda) = 0, \quad i = 1, \dots, n,$$

see [EFK]. The KZ equations are compatible, $[\nabla_i, \nabla_j] = 0$.

2.3. Intertwining Operators, Fusion Matrices, [ES, EV1]. For $\lambda \in \mathfrak{h}$, let M_λ be the Verma module over \mathfrak{g} with highest weight λ and highest weight vector v_λ . We have $\mathfrak{n}_+v_\lambda = 0$, and $hv_\lambda = (h, \lambda)v_\lambda$ for all $h \in \mathfrak{h}$. Let $M_\lambda = \bigoplus_{\mu \leq \lambda} M_\lambda[\mu]$ be the weight decomposition. The Verma module is irreducible for a generic λ . Define the dual Verma module M_λ^* to be the graded dual space $\bigoplus_{\mu \leq \lambda} M_\lambda^*[\mu]$ equipped with the \mathfrak{g} -action: $\langle u, av \rangle = -\langle au, v \rangle$ for all $a \in \mathfrak{g}$, $u \in M_\lambda$, $v \in M_\lambda^*$. Let v_λ^* be the lowest weight vector of M_λ^* satisfying $\langle v_\lambda, v_\lambda^* \rangle = 1$.

Let V be a finite dimensional \mathfrak{g} -module with weight decomposition $V = \bigoplus_{\mu \in \mathfrak{h}} V[\mu]$. For $\lambda, \mu \in \mathfrak{h}$ consider an intertwining operator $\Phi : M_\lambda \rightarrow M_\mu \otimes V$. Define its expectation value by $\langle \Phi \rangle = \langle \Phi(v_\lambda), v_\mu^* \rangle \in V[\lambda - \mu]$. If M_μ is irreducible, then the map $\text{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu \otimes V) \rightarrow V[\lambda - \mu]$, $\Phi \mapsto \langle \Phi \rangle$, is an isomorphism. Thus for any $v \in V[\lambda - \mu]$ there exists a unique intertwining operator $\Phi_\lambda^v : M_\lambda \rightarrow M_\mu \otimes V$ such that $\Phi_\lambda^v(v_\lambda) \in v_\lambda \otimes v + \bigoplus_{\nu < \mu} M_\mu[\nu] \otimes V$.

Let V, W be finite-dimensional \mathfrak{g} -modules and $v \in V[\mu]$, $w \in W[\nu]$. Consider the composition

$$\Phi_\lambda^{w,v} : M_\lambda \xrightarrow{\Phi_\lambda^v} M_{\lambda-\mu} \otimes V \xrightarrow{\Phi_{\lambda-\mu}^w} M_{\lambda-\mu-\nu} \otimes W \otimes V.$$

Then $\Phi_\lambda^{w,v} \in \text{Hom}_{\mathfrak{g}}(M_\lambda, M_{\lambda-\mu-\nu} \otimes W \otimes V)$. Hence, for a generic λ there exists a unique element $u \in (V \otimes W)[\mu + \nu]$ such that $\Phi_\lambda^u = \Phi_\lambda^{w,v}$. The assignment $(w, v) \mapsto u$ is bilinear, and defines an \mathfrak{h} -linear map

$$J_{WV}(\lambda) : W \otimes V \rightarrow W \otimes V.$$

The operator $J_{WV}(\lambda)$ is called the fusion matrix of W and V . The fusion matrix $J_{WV}(\lambda)$ is a rational function of λ . $J_{WV}(\lambda)$ is strictly lower triangular, i.e. $J = 1 + L$ where $L(W[\nu] \otimes V[\mu]) \subset \bigoplus_{\tau < \nu, \mu < \sigma} W[\tau] \otimes V[\sigma]$. In particular, $J_{WV}(\lambda)$ is invertible.

If V_1, \dots, V_n are \mathfrak{h} -modules and $F(\lambda) : V_1 \otimes \dots \otimes V_n \rightarrow V_1 \otimes \dots \otimes V_n$ is a linear operator depending on $\lambda \in \mathfrak{h}$, then for any homogeneous u_1, \dots, u_n , $u_i \in V_i[\nu_i]$, we define $F(\lambda - h^{(i)})(u_1 \otimes \dots \otimes u_n)$ to be $F(\lambda - \nu_i)(u_1 \otimes \dots \otimes u_n)$.

There is a universal fusion matrix $J(\lambda) \in U\mathfrak{g}_0^{\otimes 2}$ such that $J_{WV}(\lambda) = J(\lambda)|_{W \otimes V}$ for all W, V . The universal fusion matrix $J(\lambda)$ is the unique solution of the [ABRR] equation

$$J(\lambda) \left(1 \otimes (\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2) \right) = \left(1 \otimes (\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2) \right) + \sum_{\alpha \in \Sigma_+} e_{-\alpha} \otimes e_\alpha J(\lambda).$$

such that $(J(\lambda) - 1) \in \mathfrak{b}_-(U\mathfrak{b}_-) \otimes (U\mathfrak{b}_+)\mathfrak{b}_+$ where $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$.

We transform this equation to a more convenient form. The equation can be written as

$$J(\lambda) \left(\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2 \right)^{(2)} = \left(\left(\lambda + \rho - \frac{1}{2} \sum_{i=1}^r x_i^2 \right)^{(2)} - \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i + \Omega^- \right) J(\lambda).$$

We make a change of variables: $\lambda \mapsto \lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})$. Then the equation takes the form

$$\begin{aligned} & J\left(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})\right) \left(\lambda + \frac{1}{2}(h^{(1)} + h^{(2)}) - \frac{1}{2} \sum_{i=1}^r x_i^2 \right)^{(2)} = \\ & \left(\left(\lambda + \frac{1}{2}(h^{(1)} + h^{(2)}) - \frac{1}{2} \sum_{i=1}^r x_i^2 \right)^{(2)} - \frac{1}{2} \sum_{i=1}^r x_i \otimes x_i + \Omega^- \right) J\left(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})\right). \end{aligned}$$

Notice that $(h^{(1)} + h^{(2)})^{(2)} = \sum_{i=1}^r x_i^{(2)}(x_i^{(1)} + x_i^{(2)})$. Now the equation takes the form

$$(2) \ J\left(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})\right) (\lambda^{(2)} + \Omega^0) = (\lambda^{(2)} + \Omega^-) J\left(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})\right).$$

For $w \in \mathbb{W}$, let $w(J(\lambda))$ be the image of $J(\lambda)$ under the action of w . Let $x \in N(\mathbb{H})$ be a lifting of w . Let W, V be finite dimensional \mathfrak{g} -modules. Then

$$(3) \quad w(J(\lambda))|_{W \otimes V} = x J_{WV}(\lambda) x^{-1},$$

and RHS does not depend on the choice of x .

2.4. Main Construction, I. Introduce a new action of the Weyl group \mathbb{W} on \mathfrak{h} by

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Remind facts from [BGG].

Let M_μ, M_λ be Verma modules. Two cases are possible: a) $\text{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = 0$, b) $\text{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C}$ and every nontrivial homomorphism $M_\mu \rightarrow M_\lambda$ is an embedding.

Let M_λ be a Verma module with dominant weight $\lambda \in P^+$. Then $\text{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) = \mathbb{C}$ if and only if there is $w \in \mathbb{W}$ such that $\mu = w \cdot \lambda$.

Let $w = s_{i_k} \dots s_{i_1}$ be a reduced presentation. Set $\alpha^1 = \alpha_{i_1}$ and $\alpha^j = (s_{i_1} \dots s_{i_{j-1}})(\alpha_{i_j})$ for $j = 2, \dots, k$. Let $n_j = (\lambda + \rho, (\alpha^j)^\vee)$. For a dominant $\lambda \in P^+$, n_j are positive integers.

Lemma 4. *The collection of integers n_1, \dots, n_k and the product $(e_{-\alpha_{i_k}})^{n_k} \dots (e_{-\alpha_{i_1}})^{n_1}$ do not depend on the reduced presentation.*

Proof. It is known that $\alpha^1, \dots, \alpha^k$ are distinct positive roots and $\{\alpha^1, \dots, \alpha^k\} = \{\alpha \in \Sigma_+ \mid w(\alpha) \in \Sigma_-\}$. Hence, the collection n_1, \dots, n_k does not depend on the reduced presentation.

The vector $(e_{-\alpha_{i_k}})^{n_k} \dots (e_{-\alpha_{i_1}})^{n_1} v_\lambda$ is a singular vector in M_λ . If $w = s_{i'_k} \dots s_{i'_1}$ is another reduced presentation, then the vectors $(e_{-\alpha_{i_k}})^{n_k} \dots (e_{-\alpha_{i_1}})^{n_1} v_\lambda$ and $(e_{-\alpha_{i'_k}})^{n'_k} \dots (e_{-\alpha_{i'_1}})^{n'_1} v_\lambda$ are proportional. Since M_λ is a free \mathfrak{n}_- -module, we have

$(e_{-\alpha_{i'_k}})^{n'_k} \dots (e_{-\alpha_{i'_1}})^{n'_1} = c(e_{-\alpha_{i_k}})^{n_k} \dots (e_{-\alpha_{i_1}})^{n_1}$ in \mathfrak{n}_- for a suitable $c \in \mathbb{C}$. $c = 1$ since the monomials are equal when projected to the commutative polynomial algebra generated by $e_{-\alpha_1}, \dots, e_{-\alpha_r}$. \square

Define a singular vector $v_{w \cdot \lambda}^\lambda \in M_\lambda$ by

$$(4) \quad v_{w \cdot \lambda}^\lambda = \frac{(e_{-\alpha_{i_k}})^{n_k}}{n_1!} \dots \frac{(e_{-\alpha_{i_1}})^{n_1}}{n_k!} v_\lambda.$$

This vector does not depend on the reduced presentation by Lemma 4.

For all $\lambda \in P^+$, $w \in \mathbb{W}$, fix an embedding $M_{w \cdot \lambda} \hookrightarrow M_\lambda$ sending $v_{w \cdot \lambda}$ to $v_{w \cdot \lambda}^\lambda$.

Let V be a finite dimensional \mathfrak{g} -module, $V = \bigoplus_{\nu \in \mathfrak{h}} V[\nu]$ the weight decomposition, $P(V) = \{\nu \in \mathfrak{h} \mid V[\nu] \neq 0\}$ the set of weights of V . We say that $\lambda \in P^+$ is generic with respect to V if

- I. For any $\nu \in P(V)$ there exist a unique intertwining operator $\Phi_\lambda^\nu : M_\lambda \rightarrow M_{\lambda-\nu} \otimes V$ such that $\Phi_\lambda^\nu(v_\lambda) = v_{\lambda-\nu} \otimes v +$ lower order terms.
- II. For any $w, w' \in \mathbb{W}$, $w \neq w'$, and any $\nu \in P(V)$, the vector $w \cdot \lambda - w' \cdot (\lambda - \nu)$ does not belong to $P(V)$.

It is clear that all dominant weights lying far inside the cone of dominant weights are generic with respect to V .

Lemma 5. *Let $\lambda \in P^+$ be generic with respect to V . Let $v \in V[\nu]$. Consider the intertwining operator $\Phi_\lambda^\nu : M_\lambda \rightarrow M_{\lambda-\nu} \otimes V$. For $w \in \mathbb{W}$, consider the singular vector $v_{w \cdot \lambda}^\lambda \in M_\lambda$. Then there exists a unique vector $A_{w,V}(\lambda)(v) \in V[w(\nu)]$ such that*

$$\Phi_\lambda^\nu(v_{w \cdot \lambda}^\lambda) = v_{w \cdot (\lambda-\nu)}^{\lambda-\nu} \otimes A_{w,V}(\lambda)(v) + \text{lower order terms}.$$

Proof. $\Phi_\lambda^\nu(v_{w \cdot \lambda}^\lambda)$ is a singular vector in $M_{\lambda-\nu} \otimes V$. It has to have weight components of the form $v_{w' \cdot (\lambda-\nu)}^{\lambda-\nu} \otimes u$ for suitable $w' \in \mathbb{W}$ and $u \in V$. Since λ is generic, we have $w = w'$ and $\Phi_\lambda^\nu(v_{w \cdot \lambda}^\lambda)$ is of the required form for a suitable $A_{w,V}(\lambda)(v) \in V[w(\nu)]$. \square

For generic $\lambda \in P^+$, Lemma 5 defines a linear operator $A_{w,V}(\lambda) : V \rightarrow V$ such that $A_{w,V}(\lambda)(V[\nu]) \subset V[w(\nu)]$ for all $\nu \in P(V)$. It follows from calculations in Section 2.5 that $A_{w,V}(\lambda)$ is a rational function of $\lambda \in \mathfrak{h}$.

The following Lemmas are easy consequences of definitions.

Lemma 6. *If $w_1, w_2 \in \mathbb{W}$ and $l(w_1 w_2) = l(w_1) + l(w_2)$, then*

$$A_{w_1 w_2, V}(\lambda) = A_{w_1, V}(w_2 \cdot \lambda) A_{w_2, V}(\lambda).$$

Lemma 7. *Let W, V be finite dimensional \mathfrak{g} -modules. Let $w \in \mathbb{W}$. Then*

$$A_{w, W \otimes V}(\lambda) J_{WV}(\lambda) = J_{WV}(w \cdot \lambda) (A_{w, W}(\lambda - h^{(2)}) \otimes A_{w, V}(\lambda)).$$

Let $x_w \in N(\mathbb{H}) \subset \mathbb{G}$ be a lifting of $w \in \mathbb{W}$. For a finite dimensional \mathfrak{g} -module V , define an operator

$$B_{x_w, V}(\lambda) : V \rightarrow V, \quad v \mapsto x_w^{-1} A_{w, V}(\lambda) v.$$

$B_{x_w, V}$ preserves the weight of elements of V .

Lemma 7 implies

$$B_{x_w, W \otimes V}(\lambda) J_{WV}(\lambda) = (x_w^{-1} J_{WV}(w \cdot \lambda) x_w) (B_{x_w, W}(\lambda - h^{(2)}) \otimes B_{x_w, V}(\lambda)),$$

cf. (3).

The operator $B_{x_w, V}$ depends on the choice of x_w . If $x_w g$, $g \in \mathbb{H}$, is another lifting of w , then $B_{x_w g, V} = g^{-1} B_{x_w, V}$.

The operators $B_{x_w, V}(\lambda)$, $w \in \mathbb{W}$, are defined now for generic dominant λ and depend on the choice of liftings x_w . In the next two Sections we fix a normalization $B_{w, V}(\lambda)$ of $B_{x_w, V}(\lambda)$ so that $B_{w, V}(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$. We show that for any $w \in \mathbb{W}$, there is a universal $B_w(\lambda) \in U\mathfrak{g}_0$ such that $B_w(\lambda)|_V = B_{w, V}(\lambda)$ for every finite dimensional \mathfrak{g} -module V . For any $w \in \mathbb{W}$, we present $B_w(\lambda)$ as a suitable product of operators $B_{s_i}(\lambda)$ corresponding to simple reflections.

2.5. Operators $B_{x_w, V}(\lambda)$ for $\mathfrak{g} = sl_2$. Consider sl_2 with generators H, E, F and relations $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$. Let α_1 be the positive root. Identifying \mathfrak{h} and \mathfrak{h}^* , we have $\alpha_1 = \alpha_1^\vee = H$, $\omega_1 = \omega_1^\vee = H/2$, $\mathbb{W} = \{1, s_1\}$.

Let $\lambda = l\omega_1$, $l \in \mathbb{Z}_{\geq 0}$, be a dominant weight. Then $s_1 \cdot \lambda = -(l+2)\omega_1$. For any dominant weight λ , fix an embedding

$$M_{s_1 \cdot \lambda} \hookrightarrow M_\lambda, \quad v_{s_1 \cdot \lambda} \mapsto v_{s_1 \cdot \lambda}^\lambda = \frac{F^{(\lambda, \alpha_1) + 1} v_\lambda}{((\lambda, \alpha_1) + 1)!}$$

as in Section 2.4.

For $m \in \mathbb{Z}_{\geq 0}$, let L_m be the irreducible sl_2 module with highest weight $m\omega_1$. L_m has a basis v_0^m, \dots, v_m^m such that

$$Hv_k^m = (m - 2k)v_k^m, \quad Fv_k^m = (k + 1)v_{k+1}^m, \quad Ev_k^m = (m - k + 1)v_{k-1}^m.$$

For $\mathfrak{g} = sl_2$, we have $\mathbb{G} = SL(2, \mathbb{C})$. Then $\mathbb{H} \subset \mathbb{G}$ is the subgroup of diagonal matrices. Fix a lifting $x \in N(\mathbb{H})$ of s_1 , set $x = (x_{ij})$ where $x_{11} = x_{22} = 0$, $x_{12} = -1$, $x_{21} = 1$. Then the action of x in L_m is given by $v_k^m \mapsto (-1)^k v_{m-k}^m$ for any k . We have $x = \exp(-E) \exp(F) \exp(-E)$.

For $t \in \mathbb{C}$, introduce

$$(5) \quad p(t; H, E, F) = \sum_{k=0}^{\infty} F^k E^k \frac{1}{k!} \prod_{j=0}^{k-1} \frac{1}{(t - H - j)}.$$

$p(t; H, E, F)$ is an element of $U(sl_2)_0$.

Theorem 8. *Let λ be a dominant weight for sl_2 . Let L_m , x be as above. Let $B_{x, L_m}(\lambda) : L_m \rightarrow L_m$ be the operator defined in Section 2.4. Then for $k = 0, \dots, m$,*

$$(6) \quad B_{x, L_m}(\lambda) v_k^m = \frac{((\lambda, \alpha_1^\vee) + 2)((\lambda, \alpha_1^\vee) + 3) \cdots ((\lambda, \alpha_1^\vee) + k + 1)}{((\lambda, \alpha_1^\vee) - m + k + 1)((\lambda, \alpha_1^\vee) - m + k + 2) \cdots ((\lambda, \alpha_1^\vee) - m + 2k)} v_k^m$$

and

$$(7) \quad p((\lambda, \alpha_1^\vee); H, E, F)|_{L_m} = B_{x, L_m}(\lambda).$$

Corollary 9. $B_{x, L_m}(\lambda)$ is a rational function of (λ, α_1^\vee) . $B_{x, L_m}(\lambda)$ tends to 1 as (λ, α_1^\vee) tends to infinity.

The Theorem is proved by direct verification. First we calculate explicitly $\Phi_\lambda^{v_k^m}(v_\lambda)$, $\Phi_\lambda^{v_k^m}(\frac{F^{(\lambda, \alpha_1^\vee)+1}}{((\lambda, \alpha_1^\vee)+1)!} v_\lambda)$, and then get an expression for $B_{x, L_m}(\lambda)v_k^m$ as a sum of a hypergeometric type. Using standard formulas from [GR] we see that $B_{x, L_m}(\lambda)v_k^m$ is given by (6). Similarly we check that $p((\lambda, \alpha_1^\vee); H, E, F)v_k^m$ gives the same result. Thus we get (7). \square

Formula (6) becomes more symmetric if λ is replaced by $\lambda - \rho + \frac{1}{2}\nu$ where $\nu = m\omega_1 - k\alpha_1$ is the weight of v_k^m , then

$$(8) \quad p((\lambda + \frac{1}{2}\nu, \alpha_1^\vee) - 1; H, E, F)v_k^m = \prod_{j=0}^{k-1} \frac{(\lambda, \alpha_1^\vee) + \frac{m}{2} - j}{(\lambda, \alpha_1^\vee) - \frac{m}{2} + j} v_k^m.$$

Theorem 10.

$$p(-t - 2; -H, F, E) \cdot p(t; H, E, F) = \frac{t - H + 1}{t + 1}.$$

To prove this formula it is enough to check that RHS and LHS give the same result when applied to $v_k^m \in L_m$, which is done using (8). \square

Notice that $p(t; -H, F, E) = s_1(p(t; H, E, F))$.

Remark. Let $J(\lambda) = \sum_i a_i \otimes b_i$ be the universal fusion matrix of sl_2 . Following [EV2] introduce $S(Q)(\lambda) \in U(sl_2)_0$ as $S(Q)(\lambda) = \sum_i S(a_i)b_i$ where $S(a_i)$ is the antipode of a_i . The action of $S(Q)(\lambda)$ in L_m was computed in [EV2]. Comparing the result with Theorem 8, one sees that $p((\lambda, \alpha_1^\vee); H, E, F)$ is equal to $(S(Q)(\lambda))^{-1}$ up to a simple change of argument λ .

Corollary 11. Let $A_{s_1, L_m}(\lambda) : L_m \rightarrow L_m$ be the operator defined in Section 2.4. Then $A_{s_1, L_m}(\lambda) = x p((\lambda, \alpha_1^\vee); H, E, F)|_{L_m}$. $A_{s_1, L_m}(\lambda)$ is a rational function of (λ, α_1^\vee) . $A_{s_1, L_m}(\lambda)$ tends to x as (λ, α_1^\vee) tends to infinity.

2.6. Main Construction, II. Return to the situation considered in Section 2.4.

For any simple root α_i , the triple $H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}$ defines an embedding $sl_2 \hookrightarrow \mathfrak{g}$ and induces an embedding $SL(2, \mathbb{C}) \hookrightarrow \mathbb{G}$. Denote $x_i \in \mathbb{G}$ the image under this embedding of the element $x \in SL(2, \mathbb{C})$ defined in Section 2.5.

Lemma 12. For $i = 1, \dots, r$, we have $x_i \in N(\mathbb{H})$ and $Ad_{x_i} : \mathfrak{g} \rightarrow \mathfrak{g}$ restricted to \mathfrak{h} is the simple reflection $s_i : \mathfrak{h} \rightarrow \mathfrak{h}$.

Proof. Since $x_i = \exp(-E_{\alpha_i}) \exp(F_{\alpha_i}) \exp(-E_{\alpha_i})$, we have that $Ad_{x_i}(H_{\alpha_i}) = -H_{\alpha_i}$ and $Ad_{x_i}(h) = h$ for any $h \in \mathfrak{h}$ orthogonal to α_i . Hence $x_i \in N(\mathbb{H})$ and $Ad_{x_i}|_{\mathfrak{h}} = s_i$. \square

For $i = 1, \dots, r$ and $\lambda \in \mathfrak{h}$, set

$$B_{s_i}(\lambda) = p((\lambda, \alpha_i^\vee); H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i})$$

where $p(t; H, E, F)$ is defined in (5). Set

$$A_{s_i}(\lambda) = x_i B_{s_i}(\lambda).$$

For any $\nu \in P(V)$, we have $A_{s_i}(\lambda)(V[\nu]) \subset V[s_i(\nu)]$.

Let V be a finite dimensional \mathfrak{g} -module. For $w \in \mathbb{W}$, let $w = s_{i_k} \dots s_{i_1}$ be a reduced presentation. For a generic dominant $\lambda \in P^+$, consider the operator $A_{w,V}(\lambda) : V \rightarrow V$ defined in Section 2.4.

Lemma 13.

$$A_{w,V}(\lambda) = A_{s_{i_k}}((s_{i_{k-1}} \dots s_{i_1}) \cdot \lambda)|_V A_{s_{i_{k-1}}}((s_{i_{k-2}} \dots s_{i_1}) \cdot \lambda)|_V \dots A_{s_{i_1}}(\lambda)|_V.$$

Proof. See Corollary 11 and Lemma 6. □

Corollary 14. *The operator $A_{w,V}(\lambda)$ is a rational function of λ . $A_{w,V}(\lambda)$ tends to $x_{i_k} \dots x_{i_1}$ as λ tends to infinity in a generic direction. In particular, the product $x_{i_k} \dots x_{i_1}$ does not depend on the choice of the reduced presentation.*

Set $x_w = x_{i_k} \dots x_{i_1}$. $x_w \in N(\mathbb{H})$ is a lifting of w . Consider the operator $B_{x_w,V}(\lambda) : V \rightarrow V$ defined in Section 2.4 for this lifting x_w . Denote this operator $B_{w,V}(\lambda)$.

Corollary 15.

$$B_{w,V}(\lambda) = (s_{i_{k-1}} \dots s_{i_1})^{-1} (B_{s_{i_k}}((s_{i_{k-1}} \dots s_{i_1}) \cdot \lambda))|_V (s_{i_{k-2}} \dots s_{i_1})^{-1} (B_{s_{i_{k-1}}}((s_{i_{k-2}} \dots s_{i_1}) \cdot \lambda))|_V \dots B_{s_{i_1}}(\lambda)|_V.$$

$B_{w,V}(\lambda)$ is a rational function of λ . $B_{w,V}(\lambda)$ tends to 1 as λ tends to infinity in a generic direction.

For any nontrivial element $w \in \mathbb{W}$ and $\lambda \in \mathfrak{h}$, define an element $B_w(\lambda) \in U\mathfrak{g}_0$ by

$$B_w(\lambda) = (s_{i_{k-1}} \dots s_{i_1})^{-1} (B_{s_{i_k}}((s_{i_{k-1}} \dots s_{i_1}) \cdot \lambda)) (s_{i_{k-2}} \dots s_{i_1})^{-1} (B_{s_{i_{k-1}}}((s_{i_{k-2}} \dots s_{i_1}) \cdot \lambda)) \dots B_{s_{i_1}}(\lambda).$$

Set $B_w(\lambda) = 1$ if w is the identity in \mathbb{W} . We have $B_w(\lambda)|_V = B_{w,V}(\lambda)$, and $B_w(\lambda)$ does not depend on the choice of the reduced presentation of w .

Properties of $B_w(\lambda)$.

I. If $w_1, w_2 \in \mathbb{W}$ and $l(w_1 w_2) = l(w_1) + l(w_2)$, then

$$B_{w_1 w_2}(\lambda) = (w_2)^{-1} (B_{w_1}(w_2 \cdot \lambda)) B_{w_2}(\lambda).$$

II. Let $i = 1, \dots, r$, $\omega \in \mathfrak{h}$, and $(\alpha_i, \omega) = 0$, then

$$B_{s_i}(\lambda + \omega) = B_{s_i}(\lambda).$$

III. For $i = 1, \dots, r$,

$$s_i(B_{s_i}(s_i \cdot \lambda)) \cdot B_{s_i}(\lambda) = \frac{(\lambda, \alpha_i^\vee) - H_{\alpha_i} + 1}{(\lambda, \alpha_i^\vee) + 1}.$$

IV. Every relation $(s_i s_j)^m = 1$ for $m = 2, 3, 4, 6$ in \mathbb{W} is equivalent to a homogeneous relation $s_i s_j \dots = s_j s_i \dots$. Every such a homogeneous relation generates a relation for $B_{s_i}(\lambda), B_{s_j}(\lambda)$. Namely, for $m = 2$, the relation is

$$(s_j)^{-1}(B_{s_i}(s_j \cdot \lambda)) B_{s_j}(\lambda) = (s_i)^{-1}(B_{s_j}(s_i \cdot \lambda)) B_{s_i}(\lambda),$$

for $m = 3$, the relation is

$$\begin{aligned} (s_j s_i)^{-1}(B_{s_i}((s_j s_i) \cdot \lambda)) (s_i)^{-1}(B_{s_j}(s_i \cdot \lambda)) B_{s_i}(\lambda) = \\ (s_i s_j)^{-1}(B_{s_j}((s_i s_j) \cdot \lambda)) (s_j)^{-1}(B_{s_i}(s_j \cdot \lambda)) B_{s_j}(\lambda), \end{aligned}$$

and so on.

V.

$$\Delta(B_w(\lambda)) J(\lambda) = w^{-1}(J(w \cdot \lambda)) (B_w(\lambda - h^{(2)}) \otimes B_w(\lambda)).$$

The operators $B_w(\lambda)$ are closely connected with extremal projectors of Zhelobenko, see [Zh1, Zh2].

2.7. Operators $\mathbb{B}_{w,V}$. In order to study interrelations of operators $B_{w,V}(\lambda)$ with KZ operators it is convenient to change the argument λ .

Let V be a finite dimensional \mathfrak{g} -module. For $w_1, w_2 \in \mathbb{W}$ and $\lambda \in \mathfrak{h}$, define $w_1(\mathbb{B}_{w_2,V}(\lambda)) : V \rightarrow V$ as follows. For any $\nu \in P(V)$ and $v \in V[\nu]$, set

$$w_1(\mathbb{B}_{w_2,V}(\lambda)) v = w_1(B_{w_2}(\lambda - \rho + \frac{1}{2}\nu))|_V v.$$

In particular,

$$\mathbb{B}_{w,V}(\lambda)v = B_{w,V}(\lambda - \rho + \frac{1}{2}\nu)v.$$

$w_1(\mathbb{B}_{w_2,V}(\lambda))$ is a meromorphic function of λ , $w_1(\mathbb{B}_{w_2,V}(\lambda))$ tends to 1 as λ tends to infinity in a generic direction.

Properties of $\mathbb{B}_{w,V}(\lambda)$.

I. If $w_1, w_2 \in \mathbb{W}$ and $l(w_1 w_2) = l(w_1) + l(w_2)$, then

$$\mathbb{B}_{w_1 w_2, V}(\lambda) = w_2^{-1}(\mathbb{B}_{w_1, V}(w_2(\lambda))) \mathbb{B}_{w_2, V}(\lambda).$$

II. If $i = 1, \dots, r$, $w \in \mathbb{W}$, $v \in V[\nu]$, then

$$\mathbb{B}_{s_i, V}(\lambda) v = p((\lambda + \frac{1}{2}\nu, \alpha_i^\vee) - 1; H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}) v$$

and

$$w(\mathbb{B}_{s_i, V}(w^{-1}(\lambda))) v = p((\lambda + \frac{1}{2}\nu, w(\alpha_i^\vee)) - 1; H_{w(\alpha_i)}, E_{w(\alpha_i)}, F_{w(\alpha_i)}) v$$

where $p(t; H, E, F)$ is defined in (5).

For $\alpha \in \Sigma$, $\lambda \in \mathfrak{h}$, define a linear operator $\mathbb{B}_V^\alpha(\lambda) : V \rightarrow V$ by

$$\mathbb{B}_V^\alpha(\lambda)v = p\left(\left(\lambda + \frac{1}{2}\nu, \alpha^\vee\right) - 1; H_\alpha, E_\alpha, F_\alpha\right)v$$

for any $v \in V[\nu]$.

III.

$$\mathbb{B}_V^\alpha(\lambda) \mathbb{B}_V^{-\alpha}(\lambda)v = \frac{(\lambda - \frac{1}{2}\nu, \alpha^\vee)}{(\lambda + \frac{1}{2}\nu, \alpha^\vee)}v$$

for any $v \in V[\nu]$.

IV. Let $\alpha \in \Sigma$, $\omega \in \mathfrak{h}$, and $(\alpha, \omega) = 0$, then

$$\mathbb{B}_V^\alpha(\lambda + \omega) = \mathbb{B}_V^\alpha(\lambda).$$

V. Every relation $(s_i s_j)^m = 1$ for $m = 2, 3, 4, 6$ in \mathbb{W} is equivalent to a homogeneous relation $s_i s_j \dots = s_j s_i \dots$. Every such a homogeneous relation generates a relation for $\mathbb{B}_{s_i, V}(\lambda), \mathbb{B}_{s_j, V}(\lambda)$. Namely, for $m = 2$, the relation is

$$(s_j)^{-1}(\mathbb{B}_{s_i, V}(s_j(\lambda))) \mathbb{B}_{s_j, V}(\lambda) = (s_i)^{-1}(\mathbb{B}_{s_j, V}(s_i(\lambda))) \mathbb{B}_{s_i, V}(\lambda),$$

for $m = 3$, the relation is

$$\begin{aligned} (s_j s_i)^{-1}(\mathbb{B}_{s_i, V}((s_j s_i)(\lambda))) (s_i)^{-1}(\mathbb{B}_{s_j, V}(s_i(\lambda))) \mathbb{B}_{s_i, V}(\lambda) = \\ (s_i s_j)^{-1}(\mathbb{B}_{s_j, V}((s_i s_j)(\lambda))) (s_j)^{-1}(\mathbb{B}_{s_i, V}(s_j(\lambda))) \mathbb{B}_{s_j, V}(\lambda), \end{aligned}$$

and so on.

These relations can be written in terms of operators $\mathbb{B}_V^\alpha(\lambda)$.

VI. For $\alpha, \beta \in \Sigma$, denote $\mathbb{R}\langle\alpha, \beta\rangle$ the subspace $\mathbb{R}\alpha + \mathbb{R}\beta \subset \mathfrak{h}$. Then

$$\begin{aligned} \mathbb{B}_V^\alpha(\lambda) \mathbb{B}_V^\beta(\lambda) &= \mathbb{B}_V^\beta(\lambda) \mathbb{B}_V^\alpha(\lambda), \\ \mathbb{B}_V^\alpha(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^\beta(\lambda) &= \mathbb{B}_V^\beta(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^\alpha(\lambda), \\ \mathbb{B}_V^\alpha(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^{\alpha+2\beta}(\lambda) \mathbb{B}_V^\beta(\lambda) &= \mathbb{B}_V^\beta(\lambda) \mathbb{B}_V^{\alpha+2\beta}(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^\alpha(\lambda), \end{aligned}$$

$$\begin{aligned} \mathbb{B}_V^\alpha(\lambda) \mathbb{B}_V^{3\alpha+\beta}(\lambda) \mathbb{B}_V^{2\alpha+\beta}(\lambda) \mathbb{B}_V^{3\alpha+2\beta}(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^\beta(\lambda) = \\ \mathbb{B}_V^\beta(\lambda) \mathbb{B}_V^{\alpha+\beta}(\lambda) \mathbb{B}_V^{3\alpha+2\beta}(\lambda) \mathbb{B}_V^{2\alpha+\beta}(\lambda) \mathbb{B}_V^{3\alpha+\beta}(\lambda) \mathbb{B}_V^\alpha(\lambda) \end{aligned}$$

under the assumption that $\mathbb{R}\langle\alpha, \beta\rangle = \{\pm\gamma\}$ where γ runs over all indices in the corresponding identity.

VII.

$$\begin{aligned} \mathbb{B}_{w, W \otimes V}(\lambda) &= x_w^{-1} \left(J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)})) \right) x_w \cdot \\ &(\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} \end{aligned}$$

Lemma 16. *Let W, V be finite dimensional \mathfrak{g} -modules, $\lambda \in \mathfrak{h}$, $w \in \mathbb{W}$. Then*

$$\Omega \mathbb{B}_{w, W \otimes V}(\lambda) = \mathbb{B}_{w, W \otimes V}(\lambda) \Omega$$

and

$$(w^{-1}(\Omega^-) + \lambda^{(2)}) \mathbb{B}_{w, W \otimes V}(\lambda) = \mathbb{B}_{w, W \otimes V}(\lambda) (\Omega^- + \lambda^{(2)}).$$

Proof. The first equation holds since Ω commutes with the comultiplication. Now

$$\begin{aligned} \mathbb{B}_{w, W \otimes V}(\lambda) (\Omega^- + \lambda^{(2)}) &= x_w^{-1} (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) x_w \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} (\Omega^- + \lambda^{(2)}) &= \\ &= x_w^{-1} (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) x_w \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) (\Omega^0 + \lambda^{(2)}) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} &= \\ &= x_w^{-1} (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) x_w (\Omega^0 + \lambda^{(2)}) \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} &= \\ &= x_w^{-1} (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) (\Omega^0 + (w(\lambda))^{(2)}) x_w \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} &= \\ &= x_w^{-1} (\Omega^- + (w(\lambda))^{(2)}) (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) x_w \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} &= \\ &= (w^{-1}(\Omega^-) + \lambda^{(2)}) x_w^{-1} (J_{WV}(w(\lambda) - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))) x_w \cdot \\ (\mathbb{B}_{w, W}(\lambda - \frac{1}{2}h^{(2)}) \otimes \mathbb{B}_{w, V}(\lambda + \frac{1}{2}h^{(1)})) J_{WV}(\lambda - \rho + \frac{1}{2}(h^{(1)} + h^{(2)}))^{-1} &= \\ &= (w^{-1}(\Omega^-) + \lambda^{(2)}) \mathbb{B}_{w, W \otimes V}(\lambda). \end{aligned}$$

□

3. DIFFERENCE EQUATIONS COMPATIBLE WITH KZ EQUATIONS FOR $\mathfrak{g} = sl_N$

3.1. Statement of Results. Let $e_{i,j}$, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra gl_N ,

$$[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{j,k}.$$

sl_N is the Lie subalgebra of gl_N such that $sl_n = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ where

$$\mathfrak{n}_+ = \bigoplus_{1 \leq i < j \leq N} \mathbb{C} e_{i,j}, \quad \mathfrak{n}_- = \bigoplus_{1 \leq j < i \leq N} \mathbb{C} e_{i,j},$$

and $\mathfrak{h} = \{\lambda = \sum_{i=1}^N \lambda_i e_{i,i} \mid \lambda_i \in \mathbb{C}, \sum_{i=1}^N \lambda_i = 0\}$.

The invariant scalar product is defined by $(e_{i,j}, e_{k,l}) = \delta_{i,l} \delta_{j,k}$. The roots are $e_{i,i} - e_{j,j}$ for $i \neq j$. $\alpha^\vee = \alpha$ for any root. For a root $\alpha = e_{i,i} - e_{j,j}$, we have $H_\alpha = e_{i,i} - e_{j,j}$, $E_\alpha = e_{i,j}$, $F_\alpha = e_{j,i}$. The simple roots are $\alpha_i = e_{i,i} - e_{i+1,i+1}$ for $i = 1, \dots, N-1$. \mathbb{W} is the symmetric group S^N permutting coordinates of $\lambda \in \mathfrak{h}$. The (dual) fundamental weights are $\omega_i = \omega_i^\vee = \sum_{j=1}^i (1 - \frac{j}{N}) e_{j,j} - \sum_{j=i+1}^N \frac{j}{N} e_{j,j}$ for $i = 1, \dots, N-1$. All dual fundamental weights are minuscule. For $i = 1, \dots, N-1$, the permutation $w_{[i]}^{-1} \in S^N$ is $(\begin{smallmatrix} 1 & 2 & \dots & N-i & N-i+1 & \dots & N \\ i+1 & i+2 & \dots & N & 1 & \dots & i \end{smallmatrix})$.

For any finite dimensional sl_N -module V and $w \in S^N$ consider the operators $\mathbb{B}_{w,V}(\lambda) : V \rightarrow V$.

Let $V = V_1 \otimes \dots \otimes V_n$ be a tensor product of finite dimensional sl_N -modules. For $\kappa \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$, consider the trigonometric KZ equations with values in V ,

$$(9) \quad \nabla_j(\lambda, \kappa) u(z_1, \dots, z_n, \lambda) = 0, \quad j = 1, \dots, n.$$

Here $u(z_1, \dots, z_n, \lambda) \in V$ is a function of complex variables z_1, \dots, z_n and $\lambda \in \mathfrak{h}$.

Introduce *the dynamical difference equations* on a V -valued function $u(z_1, \dots, z_n, \lambda)$ as

$$(10) \quad u(z_1, \dots, z_n, \lambda + \kappa \omega_i^\vee) = K_i(z_1, \dots, z_n, \lambda) u(z_1, \dots, z_n, \lambda), \quad i = 1, \dots, N-1$$

where

$$K_i(z_1, \dots, z_n, \lambda) = \prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} \mathbb{B}_{w_{[i]}, V}(\lambda).$$

The operator $\prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}}$ is well defined if the argument of z_1, \dots, z_n is fixed. The dynamical difference equations are well defined on functions of (z, λ) where $\lambda \in \mathfrak{h}$ and z belongs to the universal cover of $(\mathbb{C}^*)^n$. Notice that the KZ equations are well defined for V -valued functions of the same variables.

The KZ operators $\nabla_j(\lambda, \kappa)$ and the operators $K_i(z_1, \dots, z_n, \lambda)$ preserve the weight decomposition of V .

Theorem 17. *The dynamical equations (10) together with the KZ equations (9) form a compatible system of equations.*

3.2. Proof. First prove that

$$\prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} \mathbb{B}_{w_{[i]}, V}(\lambda) \nabla_j(\lambda, \kappa) = \nabla_j(\lambda + \kappa \omega_i^\vee, \kappa) \prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} \mathbb{B}_{w_{[i]}, V}(\lambda)$$

for all i and j . Multiplying both sides from the left by $\prod_{k=1}^n z_k^{-(\omega_i^\vee)^{(k)}}$ and using Lemma 3, we reduce the equation to

$$\mathbb{B}_{w_{[i]}, V}(\lambda) \left(\sum_{k, k \neq j} r(z_j/z_k)^{(j,k)} + \lambda^{(j)} \right) = \left(\sum_{k, k \neq j} w_{[i]}^{-1}(r(z_j/z_k))^{(j,k)} + \lambda^{(j)} \right) \mathbb{B}_{w_{[i]}, V}(\lambda).$$

Lemma 18. For $j = 1, \dots, n$ and $w \in \mathbb{W}$, we have

$$\mathbb{B}_{w,V}(\lambda) \left(\sum_{k, k \neq j} r(z_j/z_k)^{(j,k)} + \lambda^{(j)} \right) = \left(\sum_{k, k \neq j} w^{-1}(r(z_j/z_k))^{(j,k)} + \lambda^{(j)} \right) \mathbb{B}_{w,V}(\lambda).$$

Proof. It is sufficient to check the equation for the residues of both sides at $z_j = z_k$, $k \neq j$, and for the limit of both sides as $z_j \rightarrow \infty$. The residue equation $[\mathbb{B}_{w,V}(\lambda), \Omega^{(j,k)}] = 0$ is true since the Casimir operator commutes with the comultiplication. The limit equation

$$\mathbb{B}_{w,V}(\lambda) \left(\sum_{k, k \neq j} (\Omega^+)^{(j,k)} + \lambda^{(j)} \right) = \left(\sum_{k, k \neq j} w^{-1}(\Omega^+)^{(j,k)} + \lambda^{(j)} \right) \mathbb{B}_{w_{[j]},V}(\lambda)$$

is a corollary of Lemma 16. \square

The Theorem is proved for sl_N , $N = 2$. For $N > 2$, it remains to prove that

$$(11) \quad K_i(z, \lambda + \kappa\omega_j^\vee) K_j(z, \lambda) = K_j(z, \lambda + \kappa\omega_i^\vee) K_i(z, \lambda)$$

for all i, j , $0 < i < j < N$. We prove (11) for $N = 3$. For arbitrary N the proof is similar. Another proof see in Section 4. For $N = 3$, $i = 1$, $j = 2$, equation (11) takes the form

$$(12) \quad \prod_{k=1}^n z_k^{(\omega_1^\vee)^{(k)}} \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_2^\vee) \mathbb{B}_V^{\alpha_1}(\lambda + \kappa\omega_2^\vee) \prod_{k=1}^n z_k^{(\omega_2^\vee)^{(k)}} \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_V^{\alpha_2}(\lambda) = \\ \prod_{k=1}^n z_k^{(\omega_2^\vee)^{(k)}} \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_1^\vee) \mathbb{B}_V^{\alpha_2}(\lambda + \kappa\omega_1^\vee) \prod_{k=1}^n z_k^{(\omega_1^\vee)^{(k)}} \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_V^{\alpha_1}(\lambda).$$

We have $\mathbb{B}_V^{\alpha_1}(\lambda + \kappa\omega_2^\vee) = \mathbb{B}_V^{\alpha_1}(\lambda)$ since $(\omega_2^\vee, \alpha_1) = 0$. We have $[\mathbb{B}_V^{\alpha_1}(\lambda), \prod_{k=1}^n z_k^{(\omega_2^\vee)^{(k)}}] = 0$ since $\mathbb{B}_V^{\alpha_1}(\lambda)$ is a power series in $E_{\alpha_1}, F_{\alpha_1}$. Similarly, $\mathbb{B}_V^{\alpha_2}(\lambda + \kappa\omega_1^\vee) = \mathbb{B}_V^{\alpha_2}(\lambda)$ and $[\mathbb{B}_V^{\alpha_2}(\lambda), \prod_{k=1}^n z_k^{(\omega_1^\vee)^{(k)}}] = 0$. Using these remarks and the relation

$$\mathbb{B}_V^{\alpha_2}(\lambda) \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_V^{\alpha_1}(\lambda) = \mathbb{B}_V^{\alpha_1}(\lambda) \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda) \mathbb{B}_V^{\alpha_2}(\lambda)$$

we reduce (12) to

$$\prod_{k=1}^n z_k^{(\omega_1^\vee - \omega_2^\vee)^{(k)}} \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_2^\vee) = \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_1^\vee) \prod_{k=1}^n z_k^{(\omega_1^\vee - \omega_2^\vee)^{(k)}}.$$

This equation holds since $\mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_2^\vee) = \mathbb{B}_V^{\alpha_1+\alpha_2}(\lambda + \kappa\omega_1^\vee)$, each of these operators is a power series in $E_{\alpha_1+\alpha_2}, F_{\alpha_1+\alpha_2}$, and $(\omega_1^\vee - \omega_2^\vee, \alpha_1 + \alpha_2) = 0$.

3.3. An Equivalent Form of Dynamical Equations for sl_N . For $j = 1, \dots, N$, set $\delta_j = \omega_j^\vee - \omega_{j-1}^\vee$ where $\omega_0^\vee = \omega_N^\vee = 0$. Then the system of equations (10) is equivalent to

the system

$$u(z_1, \dots, z_n, \lambda + \kappa \delta_i) = \left(\mathbb{B}_V^{e_{i-1, i-1} - e_{i, i}}(\lambda + \kappa \delta_i) \right)^{-1} \dots \left(\mathbb{B}_V^{e_{1, 1} - e_{i, i}}(\lambda + \kappa \delta_i) \right)^{-1} \times \prod_{k=1}^n z_k^{(\delta_i)^{(k)}} \mathbb{B}_V^{e_{i, i} - e_{n, n}}(\lambda) \dots \mathbb{B}_V^{e_{i, i} - e_{i+1, i+1}}(\lambda) u(z_1, \dots, z_n, \lambda)$$

where $i = 1, \dots, N$.

Notice that the inverse powers can be eliminated using property III in Section 2.7.

3.4. Application to Determinants. Let \mathfrak{g} be a simple Lie algebra, V a finite dimensional \mathfrak{g} -module, $V[\nu]$ a weight subspace. For a positive root α fix the sl_2 subalgebra in \mathfrak{g} generated by $H_\alpha, E_\alpha, F_\alpha$. Consider V as an sl_2 -module. Let $V[\nu]_\alpha \subset V$ be the sl_2 -submodule generated by $V[\nu]$,

$$V[\nu]_\alpha = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} W_k^\alpha \otimes L_{\nu+k\alpha}$$

the decomposition into irreducible sl_2 -modules. Here $L_{\nu+k\alpha}$ is the irreducible module with highest weight $\nu + k\alpha$ and W_k^α the multiplicity space. Let $d_k^\alpha = \dim W_k^\alpha$. Set

$$X_{\alpha, V[\nu]}(\lambda) = \prod_{k \in \mathbb{Z}_{\geq 0}} \left(\prod_{j=1}^k \frac{\Gamma\left(1 - \frac{(\lambda - \frac{1}{2}(\nu + j\alpha), \alpha)}{\kappa}\right)}{\Gamma\left(1 - \frac{(\lambda + \frac{1}{2}(\nu + j\alpha), \alpha)}{\kappa}\right)} \right)^{d_k^\alpha},$$

cf. formula (8). Here Γ is the standard gamma function.

Let $V = V_1 \otimes \dots \otimes V_n$ be a tensor product of finite dimensional \mathfrak{g} -modules. Set $\Lambda_k(\lambda) = \text{tr}_{V[\nu]} \lambda^{(k)}$, $\epsilon_{k, l} = \text{tr}_{V[\nu]} \Omega^{(k, l)}$, $\gamma_k = \sum_{l, l \neq k} \epsilon_{k, l}$. Set

$$(13) \quad D_{V[\nu]}(z_1, \dots, z_n, \lambda) = \prod_{k=1}^n z_k^{\frac{\Lambda_k(\lambda)}{\kappa} - \frac{\gamma_k}{2\kappa}} \prod_{1 \leq k < l \leq n} (z_k - z_l)^{\frac{\epsilon_{k, l}}{\kappa}} \prod_{\alpha \in \Sigma_+} X_{\alpha, V[\nu]}(\lambda).$$

Let $\mathfrak{g} = sl_N$, $V = V_1 \otimes \dots \otimes V_n$ a tensor product of finite dimensional sl_N -modules. Fix a basis v_1, \dots, v_d in a weight subspace $V[\nu]$. Suppose that $u_i(z_1, \dots, z_n, \lambda) = \sum_{j=1}^d u_{i, j} v_j$, $i = 1, \dots, d$, is a set of $V[\nu]$ -valued solutions of the combined system of KZ equations (9) and dynamical equations (10).

Corollary 19.

$$\det(u_{i, j})_{1 \leq i, j \leq d} = C_{V[\nu]}(\lambda) D_{V[\nu]}(z_1, \dots, z_n, \lambda)$$

where $C_{V[\nu]}(\lambda)$ is a function of λ (depending also on V_1, \dots, V_n and ν) such that

$$C_{V[\nu]}(\lambda + \kappa \omega) = C_{V[\nu]}(\lambda)$$

for all $\omega \in P^\vee$.

Proof. The Corollary follows from the following simple Lemma.

Lemma 20. For $i = 1, \dots, N - 1$, the operator $\mathbb{B}_{w_{[i]}, V}(\lambda)$ is the product in a suitable order of all operators $\mathbb{B}_V^\alpha(\lambda)$ with $\alpha \in \Sigma_+$ and $(\omega_i^\vee, \alpha) > 0$.

□

Notice that Lemma 20 in particular implies that operators $\mathbb{B}_{w_{[i]},V}(\lambda)$ and the dynamical equations are well defined in the tensor product of any highest weight sl_N -modules.

4. DYNAMICAL DIFFERENCE EQUATIONS

In this section we introduce dynamical difference equations for arbitrary simple Lie algebra. The compatibility of the dynamical equations follows from [Ch1]. We prove the compatibility of dynamical and KZ equations.

4.1. Affine Root Systems, [Ch1, Ch2]. Let \mathfrak{g} be a simple Lie algebra. The vectors $\tilde{\alpha} = [\alpha, j] \in \mathfrak{h} \times \mathbb{R}$ for $\alpha \in \Sigma, j \in \mathbb{Z}$ form the affine root system Σ^a corresponding to the root system $\Sigma \subset \mathfrak{h}$. We view Σ as a subset in Σ^a identifying $\alpha \in \mathfrak{h}$ with $[\alpha, 0]$. The simple roots of Σ^a are $\alpha_1, \dots, \alpha_r \in \Sigma$ and $\alpha_0 = [-\theta, 1]$ where $\theta \in \Sigma$ is the maximal root. The positive roots are $\Sigma_+^a = \{[\alpha, j] \in \Sigma^a \mid \alpha \in \Sigma, j > 0 \text{ or } \alpha \in \Sigma_+, j = 0\}$. The Dynkin diagram and its affine completion with $\{\alpha_i\}_{0 \leq i \leq n}$ as vertices are denoted Γ and Γ^a , respectively. The set of the indices of the images of α_0 with respect to all automorphisms of Γ^a is denoted O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = \{i \in O \mid i \neq 0\}$. For $i = 1, \dots, r$, the dual fundamental weight ω_i^\vee is minuscule if and only if $i \in O^*$.

Given $\tilde{\alpha} = [\alpha, j] \in \Sigma^a$ and $\omega \in P^\vee$, set

$$s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad t_\omega(\tilde{z}) = [z, \xi - (z, \omega)]$$

for $\tilde{z} = [z, \xi]$.

The affine Weyl group \mathbb{W}^a is the group generated by reflections $s_{\tilde{\alpha}}, \tilde{\alpha} \in \Sigma_+^a$. One defines the length of elements of \mathbb{W}^a taking the simple reflections $s_i = s_{\alpha_i}, i = 0, \dots, r$, as generators of \mathbb{W}^a . The group \mathbb{W}^a is the semidirect product $\mathbb{W} \ltimes Q_t^\vee$ of its subgroups $\mathbb{W} = \langle s_\alpha \mid \alpha \in \Sigma_+ \rangle$ and $Q_t^\vee = \{t_\omega \mid \omega \in Q^\vee\}$, where for $\alpha \in \Sigma$ we have $t_{\alpha^\vee} = s_\alpha s_{[\alpha, 1]} = s_{[-\alpha, 1]} s_\alpha$.

Consider the group $P_t^\vee = \{t_\omega \mid \omega \in P^\vee\}$. The *extended affine Weyl group* \mathbb{W}^b is the group of transformations of $\mathfrak{h} \times \mathbb{R}$ generated by \mathbb{W} and P_t^\vee . \mathbb{W}^b is isomorphic to $\mathbb{W} \ltimes P_t^\vee$ with action $(w, \omega)([z, \xi]) = [w(z), \xi - (z, \omega)]$.

Notice that for any $w \in \mathbb{W}^b$ and $\tilde{\alpha} \in \Sigma^a$, we have $w(\tilde{\alpha}) \in \Sigma^a$.

The extended affine Weyl group has a remarkable subgroup $\Pi = \{\pi_i \mid i \in O\}$, where $\pi_0 \in \Pi$ is the identity element in \mathbb{W}^b and for $i \in O^*$ we have $\pi_i = t_{\omega_i^\vee} w_{[i]}^{-1}$. The group Π is isomorphic to P^\vee/Q^\vee with the isomorphism sending π_i to the minuscule weight ω_i^\vee . For $i \in O^*$, the element $w_{[i]}$ preserves the set $\{-\theta, \alpha_1, \dots, \alpha_r\}$ and $\pi_i(\alpha_0) = \alpha_i = w_{[i]}^{-1}(-\theta)$. We have

$$\mathbb{W}^b = \Pi \ltimes \mathbb{W}^a, \quad \text{where} \quad \pi_i s_l \pi_i^{-1} = s_k \quad \text{if} \quad \pi_i(\alpha_l) = \alpha_k \quad \text{and} \quad 0 \leq k \leq r.$$

We extend the notion of length to \mathbb{W}^b . For $i \in O^*$, $w \in \mathbb{W}^a$, we set the length of $\pi_i w$ to be equal to the length of w in \mathbb{W}^a .

4.2. **Affine R-matrices**, [Ch1, Ch2]. Fix a \mathbb{C} -algebra F . A set $G = \{G^\alpha \in F \mid \alpha \in \Sigma\}$ is called a closed R-matrix if

$$\begin{aligned} G^\alpha G^\beta &= G^\beta G^\alpha, \\ G^\alpha G^{\alpha+\beta} G^\beta &= G^\beta G^{\alpha+\beta} G^\alpha, \\ G^\alpha G^{\alpha+\beta} G^{\alpha+2\beta} G^\beta &= G^\beta G^{\alpha+2\beta} G^{\alpha+\beta} G^\alpha, \\ G^\alpha G^{3\alpha+\beta} G^{2\alpha+\beta} G^{3\alpha+2\beta} G^{\alpha+\beta} G^\beta &= G^\beta G^{\alpha+\beta} G^{3\alpha+2\beta} G^{2\alpha+\beta} G^{3\alpha+\beta} G^\alpha \end{aligned}$$

under the assumption that $\alpha, \beta \in \Sigma$ and $\mathbb{R}\langle \alpha, \beta \rangle = \{\pm\gamma\}$ where γ runs over all indices in the corresponding identity.

A set $G^a = \{\tilde{G}^{\tilde{\alpha}} \in F \mid \tilde{\alpha} \in \Sigma^a\}$ is called a closed affine R-matrix if $\tilde{G}^{\tilde{\alpha}}$ satisfy the same relations where α, β are replaced with $\tilde{\alpha}, \tilde{\beta}$.

If G^a is an affine R-matrix, for any $w \in \mathbb{W}^b$ define an element $\tilde{G}_w \in F$ as follows. Given a reduced presentation $w = \pi_i s_{j_1} \dots s_{j_l}$, $i \in O$, $0 \leq j_1, \dots, j_l \leq r$, set $\tilde{G}_w = \tilde{G}^{\tilde{\alpha}^l} \dots \tilde{G}^{\tilde{\alpha}^1}$ where $\tilde{\alpha}^1 = \alpha_{j_1}$, $\tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2})$, $\tilde{\alpha}^3 = s_{j_1} s_{j_2}(\alpha_{j_3}), \dots$. The element \tilde{G}_w does not depend on the reduced presentation of w . We set $\tilde{G}_{\text{id}} = 1$.

The unordered set $\{\tilde{\alpha}^1, \dots, \tilde{\alpha}^l\}$ is denoted $\tilde{A}(w)$. There is a useful formula valid for any (not necessarily minuscule) dual fundamental weight ω_i^\vee , $i = 1, \dots, r$,

$$(14) \quad \tilde{A}(t_{\omega_i^\vee}) = \{[\alpha, j] \mid \alpha \in \Sigma_+, \text{ and } (\omega_i^\vee, \alpha) > j \geq 0\},$$

Prop. 1.4 [Ch2].

Introduce the following formal notation: for $w \in \mathbb{W}^b$, $\tilde{\alpha}, \tilde{\beta} \in \Sigma^a$, set $w(\tilde{G}^{\tilde{\alpha}}) = G^{w(\tilde{\alpha})}$, $w(\tilde{G}^{\tilde{\alpha}} \tilde{G}^{\tilde{\beta}}) = G^{w(\tilde{\alpha})} G^{w(\tilde{\beta})}, \dots$. Then the elements $\{\tilde{G}_w \mid w \in \mathbb{W}^b\}$ form a 1-cocycle:

$$\tilde{G}_{xy} = y^{-1} \tilde{G}_x \tilde{G}_y$$

for all $x, y \in \mathbb{W}^b$ such that $l(xy) = l(x) + l(y)$.

There is a way to construct a closed affine R-matrix if a closed nonaffine R-matrix $G = \{G^\alpha \in F \mid \alpha \in \Sigma\}$ is given. Namely, assume that the group P_t^\vee acts on the algebra F so that ${}^t_\omega(G^\alpha) = G^\alpha$ whenever $(\omega, \alpha) = 0$, $\omega \in P^\vee$, $\alpha \in \Sigma$. Then for $\tilde{\alpha} = [\alpha, j] \in \Sigma^a$, choose $\omega \in P^\vee$ so that $(\omega, \alpha) = -j$ and set $\tilde{G}^{\tilde{\alpha}} = {}^t_\omega(G^\alpha)$. The set $G^a = \{\tilde{G}^{\tilde{\alpha}} \in F \mid \tilde{\alpha} \in \Sigma^a\}$ is well defined and forms a closed affine R-matrix called the affine completion of the R-matrix G .

Assume that a closed affine R-matrix G^a is the affine completion of a closed nonaffine R-matrix G . Consider the system of equations for an element $\Phi \in F$:

$$(15) \quad {}^{t_{-\omega_i^\vee}}(\Phi) = \tilde{G}_{t_{\omega_i^\vee}} \Phi, \quad i = 1, \dots, r,$$

where $\omega_1^\vee, \dots, \omega_r^\vee$ are the dual fundamental weights.

Theorem 21. [Ch1] *The system of equations (15) is compatible,*

$${}^{t_{-\omega_i^\vee}}(\tilde{G}_{t_{\omega_j^\vee}}) \tilde{G}_{t_{\omega_i^\vee}} = {}^{t_{-\omega_j^\vee}}(\tilde{G}_{t_{\omega_i^\vee}}) \tilde{G}_{t_{\omega_j^\vee}}$$

for $1 \leq i < j \leq r$.

Example, [Ch1]. Let $\alpha = \alpha_1$, $\beta = \alpha_2$, $a = -\omega_1^\vee$, $b = -\omega_2^\vee$. Then the system for A_2 is

$${}^{t_a}(\Phi) = \tilde{G}^{\alpha+\beta} \tilde{G}^\alpha \Phi, \quad {}^{t_b}(\Phi) = \tilde{G}^{\alpha+\beta} \tilde{G}^\beta \Phi.$$

The system for B_2 is

$${}^{t_a}(\Phi) = \tilde{G}^{\alpha+2\beta} \tilde{G}^{\alpha+\beta} \tilde{G}^\alpha \Phi, \quad {}^{t_b}(\Phi) = \tilde{G}^{[\alpha+2\beta,1]} \tilde{G}^{\alpha+\beta} \tilde{G}^{\alpha+2\beta} \tilde{G}^\beta \Phi.$$

The system for G_2 is

$$\begin{aligned} {}^{t_a}(\Phi) &= \tilde{G}^{[3\alpha+2\beta,2]} \tilde{G}^{[3\alpha+\beta,2]} \tilde{G}^{[2\alpha+\beta,1]} \tilde{G}^{[3\alpha+2\beta,1]} \tilde{G}^{[3\alpha+\beta,1]} \times \\ &\quad \tilde{G}^{\alpha+\beta} \tilde{G}^{3\alpha+2\beta} \tilde{G}^{2\alpha+\beta} \tilde{G}^{3\alpha+\beta} \tilde{G}^\alpha \Phi, \\ {}^{t_b}(\Phi) &= \tilde{G}^{[3\alpha+2\beta,1]} \tilde{G}^{3\alpha+\beta} \tilde{G}^{2\alpha+\beta} \tilde{G}^{3\alpha+2\beta} \tilde{G}^{\alpha+\beta} \tilde{G}^\beta \Phi. \end{aligned}$$

4.3. Affine R-matrix for Dynamical Equations. Fix $\kappa \in \mathbb{C}$ and a natural number n . Let F be the algebra of meromorphic functions of $z_1, \dots, z_n \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$ with values in $U\mathfrak{g}_0^{\otimes n}$. Define an action of \mathbb{W} on F by

$${}^w f(z_1, \dots, z_n, \lambda) = w(f(z_1, \dots, z_n, w^{-1}(\lambda)))$$

and an action of P_t^\vee on F by

$${}^{t_\omega} f(z_1, \dots, z_n, \lambda) = \prod_{k=1}^n z_k^{\omega^{(k)}} f(z_1, \dots, z_n, \lambda - \kappa\omega) \prod_{k=1}^n z_k^{-\omega^{(k)}}$$

where $w \in \mathbb{W}$, $\omega \in P^\vee$, $f \in F$.

Lemma 22. *Those actions extend to an action of $\mathbb{W}^b = \mathbb{W} \times P_t^\vee$ on F , i.e. ${}^{t_\omega}({}^w f) = {}^{t_\omega(\omega)}({}^w f)$ for $w \in \mathbb{W}$, $\omega \in P^\vee$, $f \in F$. \square*

Define a closed nonaffine F -valued R-matrix $G_F = \{G_F^\alpha \mid \alpha \in \Sigma\}$ by

$$G_F^\alpha(\lambda) = \Delta^{(n)}(p((\lambda, \alpha^\vee) - 1; H_\alpha, E_\alpha, F_\alpha)).$$

Properties of operators \mathbb{B}_V^α described in Section 2.7 ensure that G_F is a closed R-matrix. The action of P_t^\vee on F defined above clearly has the property: ${}^{t_\omega}(G_F^\alpha) = G_F^\alpha$ whenever $(\omega, \alpha) = 0$, $\omega \in P^\vee$, $\alpha \in \Sigma$. This allows us to define a closed affine R-matrix $G_F^a = \{\tilde{G}_F^{\tilde{\alpha}} \in F \mid \tilde{\alpha} \in \Sigma^a\}$ as the affine completion of the R-matrix G_F . Namely, for $\tilde{\alpha} = [\alpha, j] \in \Sigma^a$, we choose $\omega \in P^\vee$ so that $(\omega, \alpha) = -j$ and set

$$\tilde{G}_F^{[\alpha, j]}(z_1, \dots, z_n, \lambda) = {}^{t_\omega}(G_F^\alpha) = \prod_{k=1}^n z_k^{\omega^{(k)}} G_F^\alpha(\lambda - \kappa\omega) \prod_{k=1}^n z_k^{-\omega^{(k)}}.$$

Let $V = V_1 \otimes \dots \otimes V_n$ be a tensor product of finite dimensional \mathfrak{g} -modules. Let F_V be the algebra of meromorphic functions of $z_1, \dots, z_n \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$ with values in $\text{End}(V)$. The closed affine R-matrix G_F^a induces a closed affine R-matrix $G_V^a = \{\tilde{G}_V^{\tilde{\alpha}}\}$ where

$$\tilde{G}_V^{\tilde{\alpha}}(z_1, \dots, z_n, \lambda) = \tilde{G}_F^{\tilde{\alpha}}(z_1, \dots, z_n, \lambda + \frac{1}{2} \sum_{k=1}^n h^{(k)})|_V.$$

In other words,

$$\tilde{G}_V^{[\alpha, j]}(z_1, \dots, z_n, \lambda) = \prod_{k=1}^n z_k^{\omega^{(k)}} \mathbb{B}_V^\alpha(\lambda - \kappa\omega) \prod_{k=1}^n z_k^{-\omega^{(k)}}$$

where $(\omega, \alpha) = -j$ and the operators \mathbb{B}_V^α are defined in Section 2.7. For any $w \in \mathbb{W}^b$ and $\tilde{\alpha} \in \Sigma^a$, we have $w(\tilde{G}_V^{\tilde{\alpha}}) = \tilde{G}_V^{w(\tilde{\alpha})}$.

Let $\{\tilde{G}_w^V \in F_V \mid w \in \mathbb{W}^b\}$ be the 1-cocycle associated with the affine R-matrix G_V^a . Consider the system

$$\prod_{k=1}^n z_k^{-(\omega_i^\vee)^{(k)}} \Phi(z_1, \dots, z_n, \lambda + \kappa\omega_i^\vee) \prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} = \tilde{G}_{t_{\omega_i^\vee}}^V(z_1, \dots, z_n, \lambda) \Phi(z_1, \dots, z_n, \lambda),$$

$i = 1, \dots, r$, of equations (15) associated with the affine R-matrix G_V^a . By Theorem 21 this system is compatible.

Example. For $\mathfrak{g} = sl_N$, this system of equations for an element $\Phi \in F_V$ has the form

$$\prod_{k=1}^n z_k^{-(\omega_i^\vee)^{(k)}} \Phi(z_1, \dots, z_n, \lambda + \kappa\omega_i^\vee) \prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} = \mathbb{B}_{w_{[i]}, V}(\lambda) \Phi(z_1, \dots, z_n, \lambda),$$

$i = 1, \dots, N - 1$, cf. (10).

Introduce *the dynamical difference equations* on a V -valued function $u(z_1, \dots, z_n, \lambda)$ as (16)

$$\prod_{k=1}^n z_k^{-(\omega_i^\vee)^{(k)}} u(z_1, \dots, z_n, \lambda + \kappa\omega_i^\vee) = \tilde{G}_{t_{\omega_i^\vee}}^V(z_1, \dots, z_n, \lambda) u(z_1, \dots, z_n, \lambda),$$

$i = 1, \dots, r$. Notice that the operators $\tilde{G}_{t_{\omega_i^\vee}}^V$ preserve the weight decomposition of V . Notice also that the operators $\tilde{G}_{t_{\omega_i^\vee}}^V$ are well defined on the tensor product of any highest weight \mathfrak{g} -modules according to formula (14).

An easy corollary of the compatibility of system (15) is

Lemma 23. *The dynamical difference equations (16) form a compatible system of equations for a V -valued function $u(z_1, \dots, z_n, \lambda)$.*

In particular, for $\mathfrak{g} = sl_N$, the Lemma says that the system (10) is compatible.

Theorem 24. *Assume that the Lie algebra \mathfrak{g} has a minuscule dual fundamental weight, i.e. \mathfrak{g} is not of type E_8, F_4, G_2 . Then the dynamical equations (16) together with the KZ equations (1) form a compatible system of equations.*

The Theorem is proved in Section 4.4.

We conjecture that the statement of the Theorem holds for any simple Lie algebra.

Let \mathfrak{g} be a simple Lie algebra for which the KZ and dynamical equations are compatible. Let $V = V_1 \otimes \dots \otimes V_n$ be a tensor product of finite dimensional \mathfrak{g} -modules. Fix a basis v_1, \dots, v_d in a weight subspace $V[\nu]$. Suppose that $u_i(z_1, \dots, z_n, \lambda) = \sum_{j=1}^d u_{i,j} v_j$,

$i = 1, \dots, d$, is a set of $V[\nu]$ -valued solutions of the combined system of KZ equations (1) and dynamical equations (16).

Corollary 25.

$$\det(u_{i,j})_{1 \leq i,j \leq d} = C_{V[\nu]}(\lambda) D_{V[\nu]}(z_1, \dots, z_n, \lambda)$$

where $C_{V[\nu]}(\lambda)$ is a function of λ (depending also on V_1, \dots, V_n and ν) such that

$$C_{V[\nu]}(\lambda + \kappa\omega) = C_{V[\nu]}(\lambda)$$

for all $\omega \in P^\vee$ and $D_{V[\nu]}(z_1, \dots, z_n, \lambda)$ is defined in (13).

The Corollary follows from formula (14).

4.4. Proof of Theorem 24. Introduce an action of \mathbb{W}^b on the KZ operators $\nabla_j(\lambda, \kappa)$, $j = 1, \dots, n$. Namely, for any $w \in \mathbb{W}$, set

$${}^w \nabla_j(\lambda, \kappa) = w(\nabla_j(w^{-1}(\lambda), \kappa)) = \kappa z_j \frac{\partial}{\partial z_j} - \sum_{l, l \neq j} w(r(z_j/z_l))^{(j,l)} - \lambda^{(j)}$$

and for any $\omega \in P_t^\vee$ set

$$\begin{aligned} {}^{t\omega} \nabla_j(\lambda, \kappa) &= \prod_{k=1}^n z_k^{\omega^{(k)}} \nabla_j(\lambda - \kappa\omega, \kappa) \prod_{k=1}^n z_k^{-\omega^{(k)}} = \\ &\kappa z_j \frac{\partial}{\partial z_j} - \prod_{k=1}^n z_k^{\omega_i^{(k)}} \left(\sum_{l, l \neq j} r(z_j/z_l)^{(j,l)} \right) \prod_{k=1}^n z_k^{-\omega_i^{(k)}} - \lambda^{(j)}. \end{aligned}$$

The compatibility conditions of the dynamical and KZ equations take the form

$$\tilde{G}_{t_{\omega_i}^\vee}^V(z_1, \dots, z_n, \lambda) \nabla_j(\lambda, \kappa) = {}^{t-\omega_i^\vee} \nabla_j(\lambda, \kappa) \tilde{G}_{t_{\omega_i}^\vee}^V(z_1, \dots, z_n, \lambda)$$

for $i = 1, \dots, r$, $j = 1, \dots, n$.

The compatibility conditions follow from a more general statement.

Theorem 26. Assume that the Lie algebra \mathfrak{g} has a minuscule dual fundamental weight, i.e. \mathfrak{g} is not of type E_8, F_4, G_2 . Then for any $j = 1, \dots, n$ and any $w \in \mathbb{W}^b$ we have

$$\tilde{G}_w^V(z_1, \dots, z_n, \lambda) \nabla_j(\lambda, \kappa) = {}^{w^{-1}} \nabla_j(\lambda, \kappa) \tilde{G}_w^V(z_1, \dots, z_n, \lambda).$$

We conjecture that the statement of the Theorem holds for any simple Lie algebra.

The Theorem follows from the next four Lemmas.

Lemma 27. Let $j = 1, \dots, n$. Assume that

$$\tilde{G}_{s_l}^V \nabla_j(\lambda, \kappa) = {}^{s_l} \nabla_j(\lambda, \kappa) \tilde{G}_{s_l}^V, \quad \pi_i \nabla_j(\lambda, \kappa) = \nabla_j(\lambda, \kappa)$$

for $l = 0, \dots, r$ and $i \in O^*$. Then

$$\tilde{G}_w^V(z_1, \dots, z_n, \lambda) \nabla_j(\lambda, \kappa) = {}^{w^{-1}} \nabla_j(\lambda, \kappa) \tilde{G}_w^V(z_1, \dots, z_n, \lambda)$$

for all $w \in \mathbb{W}^b$.

Proof. If $w = \pi_i s_{m_l} \dots s_{m_1}$ is a reduced presentation, then

$\tilde{G}_w^V = s_{m_1} \dots s_{m_{l-1}} (\tilde{G}_{s_{m_l}}^V) \dots s_{m_1} (\tilde{G}_{s_{m_2}}^V) \tilde{G}_{s_{m_1}}^V$ and

$$\begin{aligned} \tilde{G}_w^V \nabla_j(\lambda, \kappa) &= s_{m_1} \dots s_{m_{l-1}} (\tilde{G}_{s_{m_l}}^V) \dots s_{m_1} (\tilde{G}_{s_{m_2}}^V) \tilde{G}_{s_{m_1}}^V \nabla_j(\lambda, \kappa) = \\ &= s_{m_1} \dots s_{m_{l-1}} (\tilde{G}_{s_{m_l}}^V) \dots s_{m_1} (\tilde{G}_{s_{m_2}}^V) s_{m_1} \nabla_j(\lambda, \kappa) \tilde{G}_{s_{m_1}}^V = \\ &= s_{m_1} \dots s_{m_{l-1}} (\tilde{G}_{s_{m_l}}^V) \dots s_{m_1} s_{m_2} \nabla_j(\lambda, \kappa) s_{m_1} (\tilde{G}_{s_{m_2}}^V) \tilde{G}_{s_{m_1}}^V = \\ &= s_{m_1} s_{m_2} \dots s_{m_l} \nabla_j(\lambda, \kappa) s_{m_1} \dots s_{m_{l-1}} (\tilde{G}_{s_{m_l}}^V) \dots s_{m_1} (\tilde{G}_{s_{m_2}}^V) \tilde{G}_{s_{m_1}}^V = \\ &= w^{-1} \nabla_j(\lambda, \kappa) \tilde{G}_w^V. \end{aligned}$$

□

Lemma 28. *Let $j = 1, \dots, n$ and $w \in \mathbb{W}$. Then*

$$\tilde{G}_w^V \nabla_j(\lambda, \kappa) = w^{-1} \nabla_j(\lambda, \kappa) \tilde{G}_w^V.$$

Proof. For $w \in \mathbb{W}$ we have $\tilde{G}_w^V(z_1, \dots, z_n \lambda) = \mathbb{B}_{w,V}(\lambda)$, and Lemma 28 is equivalent to Lemma 18. □

Lemma 29. *Let $j = 1, \dots, n$ and $i \in O^*$. Then*

$$\pi_i \nabla_j(\lambda, \kappa) = \nabla_j(\lambda, \kappa).$$

Proof. We have $\pi_i = t_{\omega_i^\vee} w_{[i]}^{-1}$. Hence

$$\begin{aligned} \pi_i \nabla_j(\lambda, \kappa) &= t_{\omega_i^\vee} (w_{[i]}^{-1} \nabla_j(\lambda, \kappa)) = t_{\omega_i^\vee} \left(\kappa z_j \frac{\partial}{\partial z_j} - \sum_{l, l \neq j} w_{[i]}^{-1} (r(z_j/z_l))^{(j,l)} - \lambda^{(j)} \right) = \\ &= \kappa z_j \frac{\partial}{\partial z_j} - \prod_{k=1}^n z_k^{(\omega_i^\vee)^{(k)}} \left(\sum_{l, l \neq j} w_{[i]}^{-1} (r(z_j/z_l))^{(j,l)} \right) \prod_{k=1}^n z_k^{-(\omega_i^\vee)^{(k)}} - \lambda^{(j)} = \nabla_j(\lambda, \kappa). \end{aligned}$$

The last equality follows from Lemma 3. □

Lemma 30. *Let $j = 1, \dots, n$. Assume that the Lie algebra \mathfrak{g} has a minuscule dual fundamental weight. Then*

$$\tilde{G}_{s_0}^V \nabla_j(\lambda, \kappa) = s_0 \nabla_j(\lambda, \kappa) \tilde{G}_{s_0}^V.$$

Proof. Let ω_i^\vee be a minuscule dual fundamental weight. We have $s_0 = \pi_i^{-1} s_i \pi_i$ and $\tilde{G}_{s_0}^V = \pi_i^{-1} (\tilde{G}_{s_i}^V)$ according to the 1-cocycle property. Now

$$\begin{aligned} s_0 \nabla_j(\lambda, \kappa) \tilde{G}_{s_0}^V &= \pi_i^{-1} s_i \pi_i \nabla_j(\lambda, \kappa) \pi_i^{-1} (\tilde{G}_{s_i}^V) = \pi_i^{-1} (s_i (\pi_i \nabla_j(\lambda, \kappa))) \tilde{G}_{s_i}^V = \\ &= \pi_i^{-1} (s_i (\nabla_j(\lambda, \kappa))) \tilde{G}_{s_i}^V = \pi_i^{-1} (\tilde{G}_{s_i}^V \nabla_j(\lambda, \kappa)) = \pi_i^{-1} (\tilde{G}_{s_i}^V) \pi_i^{-1} (\nabla_j(\lambda, \kappa)) = \tilde{G}_{s_0}^V \nabla_j(\lambda, \kappa). \end{aligned}$$

□

Theorems 24 and 26 are proved.

REFERENCES

- [ABRR] D.Arnaudon, E.Buffenoir, E.Ragoucy, and Ph.Roche, *Universal Solutions of quantum dynamical Yang-Baxter equations*, q-alg/9712037.
- [BGG] I.N.Bernshtein, I.M.Gelfand, S.I.Gelfand, *Structure of Representations Generated by Vectors of Highest Weight*, *Funct. Anal. Appl.* **5** (1971), 1–8.
- [Ch1] I.Cherednik, *Quantum Knizhnik-Zamolodchikov Equations and Affine Root Systems*, *Commun. Math. Phys.* **150** (1992), 109–136.
- [Ch2] I.Cherednik, *Difference Elliptic Operators and Root Systems*, *Int. Math. Res. Notices* (1995), no. 1, 44–59.
- [Ch3] I.Cherednik, *Integral Solutions of Trigonometric Knizhnik-Zamolodchikov Equations and Kac-Moody Algebras*, *Publ. RIMS* **27** (1991), 727–744.
- [EFK] P.Etingof, I.Frenkel, A.Kirillov, *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, AMS, 1998.
- [ES] P.Etingof, O.Schiffmann, *Lectures on the Dynamical Young-Baxter Equations*, math.QA/9908064.
- [EV1] P.Etingof, A.Varchenko, *Exchange Dynamical Quantum Groups*, *Commun. Math. Phys.* **205** (1999), 19–52.
- [EV2] P.Etingof, A.Varchenko, *Traces of Intertwining Operators for Quantum Groups and Difference Equations, I*, math.QA/9907181.
- [FMTV] G. Felder, Y. Markov, V. Tarasov, A.Varchenko, *Differential Equations Compatible with KZ Equations*, math.QA/0001184.
- [GR] G.Gaspar, M.Rahman, *Basic hypergeometric Series*, Cambridge University Press, 1990.
- [SV] V.Schechtman, A.Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, *Invent. Math.* **106** (1991), 139–194.
- [V] A.Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, *Adv. Ser. Math. Phys.* **21**, World Scientific, 1995.
- [Zh1] D.P.Zhelobenko, *Extremal cocycles on Weyl groups*, *Funct. Anal. Appl.* **21** (1987), 183–192.
- [Zh2] D.P.Zhelobenko, *Extremal projectors and generalized Mickelsson algebras on reductive Lie algebras*, *Math. USSR, Izv.* **33** (1989), 85–100.