Finiteness of the number of compatibly-split subvarieties

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1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let X be a scheme over k (always assumed to be separated of finite type over k). The following is the main theorem of this note and we give here its complete and self-contained proof.

(1.1) Theorem. *Assume that X is Frobenius split by a splitting $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, where F is the absolute Frobenius morphism (cf. [BK, Section 1.1]). Then, there are only finitely many closed subschemes of X which are compatibly split (under σ).*

2 Proof of Theorem 1.1

We first prove the following proposition which is of independent interest. By a variety we mean a reduced but not necessarily irreducible scheme over k.

(2.1) Proposition. *Let X be a nonsingular irreducible variety which is Frobenius split by $\sigma \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq H^0(X, F_*(\omega_X^{1-p}))$, where ω_X is the dualizing sheaf of X (cf. [BK, Proposition 1.3.7]), and let $Y \subsetneq X$ be a compatibly-split closed subscheme of X. Then,*

$$Y \subset Z(\bar{\sigma}),$$

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where $Z(\bar\sigma)$ denotes the set of zeroes of $\bar\sigma$ and $\bar\sigma$ is the section of $F_*(\omega_X^{1-p})$ obtained from σ via the above identification.

Proof. Since any irreducible component of a compatibly-split closed subscheme is compatibly split (cf. [BK, Proposition 1.2.1]), we can assume that Y is irreducible. Assume, if possible, that $Y \cap (X \backslash Z(\bar\sigma)) \neq \emptyset$. Then, $Y^{\mathrm{reg}} \cap (X\backslash Z(\bar\sigma)) \neq \emptyset$, where Y^{reg} is the nonsingular locus of Y.

Take $y \in Y^{\mathrm{reg}} \cap (X\backslash Z(\bar\sigma))$. Choose a system of local parameters $\{t_1, \ldots, t_m, t_{m+1}, \ldots, t_n\}$ at $y \in X$ such that $\{t_1, \ldots, t_m\}$ is a system of local parameters at $y \in Y$ and $\langle t_{m+1}, \ldots, t_n\rangle$ is the completion of the ideal of Y in X at y. (This is possible since both X and Y are nonsingular at y.) By assumption, $\bar\sigma$ is a unit in the local ring $\mathcal{O}_{X,y}$. Moreover, σ induces a splitting $\hat\sigma$ of the power series ring $k[[t_1, \ldots, t_n]]$ compatibly splitting the ideal $\langle t_{m+1}, \ldots, t_n\rangle$. Now, since $\bar\sigma$ does not vanish at y, $\hat\sigma\big((t_1 \cdots t_n)^{p-1}\big)$ is a unit in the ring $k[[t_1, \ldots, t_n]]$. In particular, $\hat\sigma$ does not keep the ideal $\langle t_{m+1}, \ldots, t_n\rangle$ stable. This is a contradiction to the assumption. Hence, $Y \subset Z(\bar\sigma)$, proving the proposition. □

(**2.2**) ***Proof of Theorem 1.1***. By [BK, Proposition 1.2.1], we can assume without loss of generality that X is irreducible. We prove Theorem 1.1 by induction on the dimension of X . If dim $X = 0$, then the theorem is clear. So assume that dim $X = n$ and the theorem is true for varieties of dimension $< n$. Let $Y \underset{\neq}{\subset} X$ be a compatibly-split irreducible closed subscheme. Then, either $Y \subset X^{\mathrm{sing}}$ (where X^{sing} is the singular locus of X) or $Y \cap X^{\mathrm{reg}} \neq \emptyset$. In the latter case, by Proposition 2.1,

$$Y \cap X^{\mathrm{reg}} \subset Z(\bar{\sigma^o}),$$

where $Z(\bar{\sigma^o})$ denotes the set of zeroes of the splitting $\bar{\sigma^o}$ of the open subset X^{reg} of X viewed as a section of $F_*\Big(\omega_{X^{\mathrm{reg}}}^{1-p}\Big)$. Thus, in this case,

$$Y \subset \overline{Z(\bar{\sigma^o})},$$

$\overline{Z(\bar{\sigma^o})}$ being the closure of $Z(\bar{\sigma^o})$ in X. Hence, in either case,

$$Y \subset \overline{Z(\bar{\sigma^o})} \cup X^{\mathrm{sing}}. \tag{1}$$

Considering the irreducible components, the same inclusion (1) holds for any compatibly-split closed subscheme $Y \subset X$ such that $Y \neq X$.

Let $\{Y_i\}_{i\in I}$ be the collection of all the distinct compatibly-split closed subschemes $Y_i \subsetneq X$ and let $Y := \overline{\bigcup_{i\in I} Y_i}$. Since the ideal sheaf $\mathcal{I}_Y = \bigcap_{i\in I} \mathcal{I}_{Y_i}$ and each $\mathcal{I}_{Y_i}$ is stable under the splitting σ of X, the closed subscheme Y is compatibly split. In particular, by (1), for each $i \in I$,

$$Y_i \subset Y \subset \overline{Z(\bar{\sigma^o})} \cup X^{\text{sing}}.$$

Since $\dim\left(\overline{Z(\bar{\sigma^o})} \cup X^{\text{sing}}\right) < \dim X$; in particular, one has $\dim Y < \dim X$. Thus, by the induction hypothesis (applying the theorem with X replaced by Y), I is a finite set. This completes the proof of the theorem.

□

(2.2) Remark. Karl Schwede has also obtained the above theorem via 'F-purity' in a recent preprint [S]. As pointed out by Schwede, when X is projective, the theorem also follows from [EH, Corollary 3.2] again via 'F-purity'. (See also, [Sh] for another proof via 'tight closure'.)

References

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