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Difference Macdonald-Mehta conjecture

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Introduction

In the paper we formulate and check a difference counterpart of the Macdonald-Mehta conjecture and its generalization for the Macdonald polynomials. Namely, we determine the Fourier transforms of the polynomials multiplied by the Gaussian, which is closely connected with the new difference Harish-Chandra theory.

Mehta suggested a formula for the integral of the $\prod_{1\le i<j\le n}(x_i - x_j)^{2k}$ with respect to the Gaussian measure. Macdonald extended it from A_{n-1} to other root systems and verified his conjecture for classical ones by means of the Selberg integrals [M1]. It was established by Opdam in [O1] in full generality using the shift operators.

The integral is an important normalization constant for a k-deformation of the Hankel transform introduced by Dunkl [D]. The generalized Bessel functions [O3] multiplied by the Gaussian are eigenfunctions of this transform. The eigenvalues are given in terms of this constant. See [D,J] for the detail. The Hankel transform is a rational degeneration of the Fourier transform in the Harish-Chandra theory of spherical functions when the symmetric space G/K is replaced by its tangent space $T_e(G/K)$ with the adjoint action of G (see [H]).

The harmonic analysis for G/K is much more complicated than in the rational case. The reproducing kernel of the Fourier transform is not symmetric, the Gaussian is not Fourier-invariant, and so on. The zonal spherical functions for dominant weights, are (trigonometric) polynomials and have no counterparts in the rational theory. They play a great role in mathematics and physics. For $k = 1$ they are the characters of finite dimensional representations of G. Unfortunately the Fourier transforms of spherical polynomials are no good, but for the characters.

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It is the same for the k-deformations, called the Jack - Heckman-Opdam polynomials. They have remarkable combinatorial properties (Macdonald, Stanley, Hanlon and others) and many applications. A variant of the Mehta-Macdonald conjecture for these polynomials is the celebrated Macdonald constant term conjecture [M1] in the differential setup. It was proved by Opdam for all root systems [O1].

A difference generalization of the Harish-Chandra theory was started in [C3,C4]. It fuses together the Fourier transform and the difference spherical functions. From this viewpoint, it is similar to the rational case. The Macdonald q, t-polynomials [M2,M3,M4,C4] form an important class of difference spherical functions, corresponding to the smallest spherical irreducible representation of the double affine Hecke algebra. All spherical representations were classified in [C1]. They are expected to have promising applications in harmonic analysis and combinatorics.

The Macdonald constant term conjecture, as well as the norm, evaluation, and duality conjectures, were justified in the q, t-case in [C2,C3,C4]. In this paper we complete the theory calculating the Fourier transforms of the Macdonald polynomials multiplied by the Gaussian. The Gaussian and its transform are proportional. The coefficent resembles that from the constant term conjecture and the proof is based on the same shift operators. However the Gaussian measure is very much different from the canonical one in the theory of the Macdonald polynomials. It is a new powerful tool.

Since we consider polynomials only, we prefer to stick to the pairing based on the constant term instead of that in terms of Jackson integrals as in [C1]. The general case will be considered in the next paper. Actually different concepts of integration do not influence the main formulas up to minor renormalizations. The shift operators always work well. It holds even when q is a root of unity [C2,C3] or k is special negative (see [C1], [DS]). In these cases the Jackson integrals become finite sums, but it does not change the formulas too much (till they make sense).

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1. Main results

Let $R = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form (z, z') on $\mathbf{R}^n \ni z, z'$, W the Weyl group generated by the the reflections s_α. We assume that $(\alpha, \alpha) = 2$ for long α. Let us fix the set R_+ of positive roots ($R_- = -R_+$), the corresponding simple roots $\alpha_1, ..., \alpha_n$, and their dual counterparts $a_1, ..., a_n, a_i = \alpha_i^\vee$, where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. The dual fundamental weights $b_1, ..., b_n$ are determined from the relations $(b_i, \alpha_j) = \delta_i^j$ for the Kronecker delta. We will also introduce the dual root system $R^\vee = \{\alpha^\vee, \alpha \in R\}, R^\vee_+$, and the lattices

$$A = \oplus_{i=1}^n \mathbf{Z} a_i \subset B = \oplus_{i=1}^n \mathbf{Z} b_i,$$

$A_\pm, B_\pm$ for $\mathbf{Z}_\pm = \{m \in \mathbf{Z}, \pm m \geq 0\}$ instead of $\mathbf{Z}$. In the standard notations, $A = Q^\vee,\ B = P^\vee$ (see [B]).

Later on,

$$\begin{aligned}
\nu_\alpha &= (\alpha,\alpha),\ \nu_i = \nu_{\alpha_i},\ \nu_R = \{\nu_\alpha, \alpha \in R\},\\
\rho_\nu &= (1/2) \sum_{\nu_\alpha = \nu} \alpha = (\nu/2) \sum_{\nu_i = \nu} b_i, \quad \text{for} \quad \alpha \in R_+,\\
r_\nu &= \rho_\nu^\vee = (2/\nu)\rho_\nu = \sum_{\nu_i = \nu} b_i, \quad 2/\nu = 1, 2, 3.
\end{aligned} \tag{1.1}$$

We will mainly use r_ν and $r = \sum_\nu r_\nu$ in the paper. The theory depends on the parameters $q, t_\nu, \nu \in \nu_R$. It is convenient to set

$$q_\nu = q^{2/\nu},\ t_\nu = q_\nu^{k_\nu},\ q_\alpha = q_\nu, t_\alpha = t_\nu \ \text{ for } \ \nu = \nu_\alpha \ \text{ and } \ r_k = \sum_\nu k_\nu r_\nu.$$

Let us put formally

$$x_i = q^{x_i},\ x_b = q^b = \prod_{i=1}^n x_i^{l_i} \ \text{ for } \ b = \sum_{i=1}^n l_i b_i, \tag{1.2}$$

and introduce the algebra $\mathbf{C}(q,t)[x]$ of polynomials in terms of $x_i^{\pm 1}$ with the coefficients belonging to the field $\mathbf{C}(q,t)$ of rational functions.

The coefficient of $x^0 = 1$ (*the constant term*) will be denoted by $\langle\ \rangle$. The following product (the Macdonald truncated θ-function) is a Laurent series in x with coefficients in the algebra $\mathbf{C}[t][[q]]$ of formal meromorphic series in q over polynomials in t:

$$\Delta = \prod_{a \in R_+^\vee} \prod_{i=0}^\infty \frac{(1 - x_a q_\alpha^i)(1 - x_a^{-1} q_\alpha^i)}{(1 - x_a t_\alpha q_a^i)(1 - x_a^{-1} t_\alpha q_\alpha^i)}. \tag{1.3}$$

Here $(1-(\cdot))^{-1}$ are replaced by $1 + (\cdot) + (\cdot)^2 + \ldots$. We note that $\Delta \in \mathbf{C}(q,t)[x]$ if $t_\nu = q_\nu^{k_\nu}$ for $k_\nu \in \mathbf{Z}_+$.

By the *Gaussians* $\gamma^{\pm 1}$ we mean

$$\gamma = \sum_{b \in B} q^{-(b,b)/2} x_b,\ \gamma^{-1} = \sum_{b \in B} q^{(b,b)/2} x_b. \tag{1.4}$$

The multiplication by γ^{-1} preserves the space of Laurent series with coefficients from $\mathbf{C}[t][[q]]$.

Macdonald-Mehta Theorem 1.1.

$$\langle \gamma^{-1}\Delta\rangle \ = \ \prod_{j=0}^{\infty}\Big(\frac{1-q_\alpha^{(r_k,\alpha)+j}}{1-t_\alpha q_\alpha^{(r_k,\alpha)+j}}\Big). \tag{1.5}$$

The *monomial symmetric functions* $m_b \ = \ \sum_{c\in W(b)} x_c$ for $b\in B_-$ form a basis of the space $\mathbf{C}[x]^W$ of all W-invariant polynomials. We introduce the *Macdonald polynomials* $p_b(x),\ b\in B_-$, by means of the conditions

$$\begin{aligned} &p_b - m_b \ \in \ \oplus_c \mathbf{C}(q,t)m_c,\ \langle p_b m_c\Delta\rangle = 0 \ \text{ for } \ c>b,\\ &\text{where } c\in B_-,\ c>b \text{ means that } c-b\in A_+, c\neq b. \end{aligned} \tag{1.6}$$

They can be determined by the Gram - Schmidt process (see [M2,M3]) and form a basis in $\mathbf{C}(q,t)[x]^W$. As it was established by Macdonald, they are pairwise orthogonal for arbitrary $b\in B_-$ for the pairing $\langle f(x)g(x^{-1})\Delta\rangle$.

Fourier-Plancherel Theorem 1.2. *Given $b,c\in B_-$ and the corresponding Macdonald polynomials p_b, p_c,*

$$\begin{aligned} &\langle p_b p_c \gamma^{-1}\Delta\rangle = q^{(b,b)/2+(c,c)/2-(b+c,r_k)} p_c(q^{b-r_k}) p_b(q_\alpha^{r_k})\langle\gamma^{-1}\Delta\rangle \\ &= q^{(b,b)/2+(c,c)/2-(c,r_k)} p_c(q^{b-r_k}) \prod_{\alpha\in R_+}\ \prod_{j=-(\alpha,b)}^{\infty} \Big(\frac{1-q^{(r_k,\alpha)+j}}{1-t_\alpha q_\alpha^{(r_k,\alpha)+j}}\Big), \end{aligned} \tag{1.7}$$

where $x_c(q^b) \stackrel{def}{=} q^{(b,c)}$.

Rewriting the first formula we used the Macdonald evaluation conjecture proved in [C3]:

$$p_b(q^{r_k}) \ = \ q^{(r_k,b)} \prod_{\alpha\in R_+}\ \prod_{j=1}^{-(\alpha,b)} \Big(\frac{1-t_\alpha q_\alpha^{(r_k,\alpha)+j-1}}{1-q_\alpha^{(r_k,\alpha)+j-1}}\Big). \tag{1.8}$$

More general formulas for the non-symmetric Macdonald polynomials will be established below. There is a straightforward passage to non-reduced root systems. One may also take c from the lattice P, substituting $q_\alpha \to q$, $(r_k,\alpha)\to(\rho_k,\alpha^\vee)$, $(b,r_k)\to(b,\rho_k)$ in (1.3), (1.5), and (1.7).

A rational-differential counterpart of (1.7) was verified by Dunkl and de Jeu (see [D], Theorem 3.2). This theorem is the cornerstone of the theory of the Fourier transform (cf. [J], Lemma 4.11). The latter is the map $f(x)\to\hat f(\lambda)=\int p_\lambda(x)f(x)\Delta$, where p_λ is a W-invariant eigenfunction of the generalized Macdonald operators for proper λ and the integration. So (1.7) determines the Fourier transform and its inverse in the space of W-symmetric trigonometric polynomials multiplied by the Gaussian and in various completions.

Both statements seem new. In the case of rank 1, (1.5) resembles the so-called quintuple product identity and the formulas from [AW] (the BC_1 case). So it is likely to be related to known one-dimensional identities.

2. Affine Weyl groups

The vectors $\tilde{\alpha} = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the *affine root system* $R^a \supset R$ ($z \in \mathbf{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{def}{=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta \in R$. The corresponding set R^a_+ of positive roots coincides with $R_+ \cup \{[\alpha, k],\ \alpha \in R,\ k > 0\}$.

We denote the Dynkin diagram and its affine completion with $\{\alpha_j, 0 \leq j \leq n\}$ as the vertices by Γ and Γ^a. Let $m_{ij} = 2, 3, 4, 6$ if α_i and α_j are joined by 0,1,2,3 laces respectively. The set of the indices of the images of α_0 by all the automorphisms of Γ^a will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = r \in O, r \neq 0$. The elements b_r for $r \in O^*$ are the so-called minuscule weights ($(b_r, \alpha) \leq 1$ for $\alpha \in R_+$).

Given $\tilde{\alpha} = [\alpha, k] \in R^a,\ b \in B$, let

$$s_{\tilde{\alpha}}(\tilde{z}) \ = \ \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) \ = \ [z, \zeta - (z, b)] \tag{2.1}$$

for $\tilde{z} = [z, \zeta] \in \mathbf{R}^{n+1}$.

The *affine Weyl group* W^a is generated by all $s_{\tilde{\alpha}}$ (we write $W^a =< s_{\tilde{\alpha}}, \tilde{\alpha} \in R^a_+ >$). One can take the simple reflections $s_j = s_{\alpha_j}, 0 \leq j \leq n$, as its generators and introduce the corresponding notion of the length. This group is the semi-direct product $W \ltimes A'$ of its subgroups $W =< s_\alpha, \alpha \in R_+ >$ and $A' = \{a', a \in A\}$, where

$$a' = \ s_\alpha s_{[\alpha,1]} = \ s_{[-\alpha,1]} s_\alpha \ \text{ for } \ a = \alpha^\vee,\ \alpha \in R. \tag{2.2}$$

The *extended Weyl group* W^b generated by W and B' (instead of A') is isomorphic to $W \ltimes B'$:

$$(wb')([z, \zeta]) \ = \ [w(z), \zeta - (z, b)] \ \text{ for } \ w \in W, b \in B. \tag{2.3}$$

Later on b and b' will not be distinguished.

Given $b \in B$, the decomposition $b = \pi_b \omega_b, \omega_b \in W$ can be uniquely determined from the condition: $\omega_b(b) = b_- \in B_-$ where the length $l(\omega_b)$ of ω in terms of $\{s_1, \ldots, s_n\}$ is the smallest possible. For instance, let $\pi_r = \pi_{b_r}, r \in O$. They leave Γ^a invariant and form a group denoted by Π, which is isomorphic to B/A by the natural projection $\{b_r \to \pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta)$ distinguish the indices $r \in O^*$. Moreover (see e.g. [C2]):

$$W^b = \Pi \ltimes W^a, \quad \text{where } \ \pi_r s_i \pi_r^{-1} = s_j \ \text{ if } \ \pi_r(\alpha_i) = \alpha_j,\ 0 \leq j \leq n. \tag{2.4}$$

We extend the length to W^b. Given $r \in O^*$, $\tilde{w} \in W^a$, and a reduced decomposition $\tilde{w} = s_{j_l} ... s_{j_2} s_{j_1}$ with respect to $\{s_j, 0 \le j \le n\}$, we call $l = l(\hat{w})$ the *length* of $\hat{w} = \pi_r \tilde{w} \in W^b$. Similarly, $l_\nu(\hat{w})$ is the number of s_j with $\nu_j = \nu$.

Let us introduce a partial ordering on B. Here and further b_- is the unique elements from B_- which belong to the orbit $W(b)$. Namely, $b_- = \omega_b(b)$. So the equality $c_- = b_-$ means that b, c belong to the same orbit. Set

$$b \le c, c \ge b \ \text{ for } \ b, c \in B \quad \text{if} \quad c - b \in A_+, \tag{2.5}$$

$$b \preceq c, c \succeq b \quad \text{if} \quad b_- < c_- \ \text{ or } \ b_- = c_- \text{ and } b \le c. \tag{2.6}$$

We use $<, >, \prec, \succ$ respectively if $b \ne c$.

3. Difference operators

We put $m = 2$ for D_{2k} and C_{2k+1}, $m = 1$ for C_{2k}, B_k, otherwise $m = |\Pi|$. We use the parameters $q, t_\nu = q_\nu^{k_\nu}, q_\nu = q^{2/\nu} (\nu \in \nu_R)$ and the variables $x_1, \ldots, x_n$ are from Section 1,

$$\begin{aligned} & t_{[\alpha,k]} = t_\alpha = t_{\nu_\alpha},\ t_j = t_{\alpha_j}, \ \text{ where } \ [\alpha,k] \in R^a,\ 0 \le j \le n, \\ & x_{\tilde{b}} \ = \ \prod_{i=1}^n x_i^{l_i} q^k \ \text{ if } \ \tilde{b} = [b,k], \\ & \text{for } b = \sum_{i=1}^n l_i b_i \in B,\ k \in \frac{1}{m}\mathbf{Z}. \end{aligned} \tag{3.1}$$

Later on $\mathbf{C}_q$ is the field of rational functions in $q^{1/m}$, $\mathbf{C}_q[x] = \mathbf{C}_q[x_b]$ means the algebra of polynomials in terms of $x_i^{\pm 1}$ with the coefficients depending on $q^{1/m}$ rationally. We use $\mathbf{C}_{q,t}$ if the coefficients of functions depend rationally on $\{t_\nu^{1/2}\}$ too.

The elements $\hat{w} \in W^b$ act in $\mathbf{C}_q[x]$ by the formulas:

$$\hat{w}(x_{\tilde{b}}) \ = \ x_{\hat{w}(\tilde{b})}. \tag{3.2}$$

In particular:

$$\pi_r(x_b) \ = \ x_{\omega_r^{-1}(b)} q^{(b_{r^*}, b)} \ \text{ for } \ \alpha_{r^*} \ = \ \pi_r^{-1}(\alpha_0),\ r \in O^*. \tag{3.3}$$

The *Demazure-Lusztig operators* (see [KL, KK, C2])

$$T_j \ = \ t_j^{1/2} s_j \ + \ (t_j^{1/2} - t_j^{-1/2})(x_{a_j} - 1)^{-1}(s_j - 1),\ 0 \le j \le n. \tag{3.4}$$

preserve $\mathbf{C}_{q,t}[x]$. We note that only T_0 involves q:

$$\begin{aligned} T_0 \ &= t_0^{1/2} s_0 \ + \ (t_0^{1/2} - t_0^{-1/2})(qX_\theta^{-1} - 1)^{-1}(s_0 - 1), \\ &\text{where } \ s_0(X_i) \ = \ X_i X_\theta^{-(b_i,\theta)} q^{(b_i,\theta)}. \end{aligned} \tag{3.5}$$

Given $\tilde{w} \in W^a, r \in O,$ the product

$$T_{\pi_r \tilde{w}} \stackrel{def}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \quad \text{where} \quad \tilde{w} = \prod_{k=1}^{l} s_{i_k}, l = l(\tilde{w}), \tag{3.6}$$

does not depend on the choice of the reduced decomposition of $\tilde{w}$ (because $\{T\}$ satisfy the same 'braid' relations as $\{s\}$ do). Moreover,

$$T_{\hat{v}} T_{\hat{w}} = T_{\hat{v}\hat{w}} \quad \text{whenever } l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}) \quad \text{for} \quad \hat{v}, \hat{w} \in W^b. \tag{3.7}$$

In particular, we arrive at the pairwise commutative operators

$$Y_b = \prod_{i=1}^{n} Y_i^{k_i} \quad \text{if} \quad b = \sum_{i=1}^{n} k_i b_i \in B, \quad \text{where} \quad Y_i \stackrel{def}{=} T_{b_i}, \tag{3.8}$$

satisfying the relations

$$\begin{aligned} T_i^{-1} Y_b T_i^{-1} &= Y_b Y_{a_i}^{-1} \quad \text{if} \quad (b, \alpha_i) = 1, \\ T_i Y_b &= Y_b T_i \quad \text{if} \quad (b, \alpha_i) = 0, \ 1 \le i \le n. \end{aligned} \tag{3.9}$$

4. Macdonald polynomials

Recall that $\langle f \rangle$ is the constant term of $f \in \mathbf{C}_{q,t}[x]$. We will switch from Δ to

$$\mu = \mu^{(k)} = \prod_{a \in R_+^\vee} \prod_{i=0}^{\infty} \frac{(1 - x_a q_\alpha^i)(1 - x_a^{-1} q_\alpha^{i+1})}{(1 - x_a t_\alpha q_a^i)(1 - x_a^{-1} t_\alpha q_\alpha^{i+1})}. \tag{4.1}$$

It is considered as a Laurent series with the coefficients in $\mathbf{C}[t][[q]]$. Actually we need a bigger space. Namely, the space of Laurent series $f = \sum_{b \in B} c_b x_b$ with the coefficients $c_b \in q^{-m_b} \mathbf{C}[t][[q]]$ such that $0 \le m_b \le C_f |(b, r)|$ for a constant $C_f > 0$ depending on f. This space is invariant with respect to the action of the affine Weyl group. We may also multiply its elements by the Laurent polynomials from $\mathbf{C}(q,t)[x]$ and by γ^{-1}.

Let $\mu_1 \stackrel{def}{=} \mu / \langle \mu \rangle$, where the formula for the constant term of μ is as follows (see [C2]):

$$\langle \mu \rangle = \prod_{\alpha \in R_+} \prod_{i=1}^{\infty} \frac{(1 - q_\alpha^{(r_k, \alpha) + i})^2}{(1 - t_\alpha q_\alpha^{(r_k, \alpha)+i})(1 - t_\alpha^{-1} q_\alpha^{(r_k,\alpha)+i})}. \tag{4.2}$$

It is a Laurent series with coefficients in $\mathbf{C}(q,t)$, and $\mu_1^* = \mu_1$ with respect to the involution

$$x_b^* = x_{-b}, \ t^* = t^{-1}, \ q^* = q^{-1}.$$

Setting

$$\langle f, g \rangle_1 = \langle \mu_1 f \ g^* \rangle_1 = \langle g, f \rangle_1^* \quad \text{for} \quad f, g \in \mathbf{C}(q,t)[x], \tag{4.3}$$

we introduce the *non-symmetric Macdonald polynomials* $e_b(x) = e_b^{(k)},\ b \in B$, by means of the conditions

$$e_b - x_b \ \in\ \oplus_{c \succ b} \mathbf{C} x_c,\ \langle e_b, x_c\rangle_1 = 0 \ \text{ for }\ B_- \ni c \succ b. \tag{4.4}$$

They are well-defined because the pairing is non-degenerate and form a basis in $\mathbf{C}(q,t)[x]$.

This definition is due to Macdonald (for $t_\nu = q_\nu^k,\ k \in \mathbf{Z}_+$), who extended Opdam's non-symmetric polynomials introduced in the degenerate (differential) case in [O2]. The general case was considered in [C4]. Another approach is based on the Y-operators (see [M4],[C4]):

Proposition 4.1. *The polynomials $\{e_b, b \in B\}$ are eigenvectors of the operators $\{L_f \stackrel{def}{=} f(Y_1, \cdots, Y_n), f \in \mathbf{C}[x]\}$:*

$$L_f(e_b) \ =\ f(q^{-b_\#}) e_b, \quad \textit{where}\ \ b_\# \stackrel{def}{=} b - \omega_b^{-1}(r_k), \tag{4.5}$$

$$x_a(q^{b_\#}) \ =\ q^{(a,b)} \prod_\nu t_\nu^{-(\omega_b^{-1}(\rho_\nu),a)}, \omega_b \textit{ is from Sec. 1.} \tag{4.6}$$

Similarly, the symmetric Macdonald polynomials $p_b = p_b^{(k)}$ from (4.4) are eigenfunctions of the W-invariant operators $L_f = f(Y_1, \cdots, Y_n)$ for $f \in \mathbf{C}_{q,t}[x]^W$:

$$L_f(p_b) \ =\ f(q^{-b+r_k}) p_b,\ b \in B_-. \tag{4.7}$$

They are connected with $\{e\}$ as follows (see [M4,C4] and [O2] in the differential case):

$$\begin{aligned} p_b \ &=\ \mathcal{P}_b^t e_b, \quad b = b_- \in B_-, \\ \mathcal{P}_b^t &\stackrel{def}{=} \sum_{c \in W(b)} \prod_\nu t_\nu^{l_\nu(w_c)/2} \hat{T}_{w_c}, \end{aligned} \tag{4.8}$$

where $w_c \stackrel{def}{=} \omega_c^{-1} w_0$ for the longest $w_0 \in W$.

Following [C2-C3], let us fix a subset $v \in \nu_R$ and introduce the *shift operator* by the formula:

$$\mathcal{G}_v \ =\ \mathcal{G}_v^{(k)} \ =\ (\mathcal{X}_v)^{-1} \mathcal{Y}_v, \tag{4.9}$$

$$\mathcal{X}_v = \prod_{\nu_\alpha \in v} ((t_\alpha x_a)^{1/2} - (t_\alpha x_a)^{-1/2}),\ \mathcal{Y}_v = \prod_{\nu_\alpha \in v} (t_\alpha Y_a^{-1})^{1/2} - (t_\alpha Y_a^{-1})^{-1/2}).$$

Here $a = \alpha^\vee, \alpha \in R_+$, $\mathcal{X}_v = \mathcal{X}_v^{(k)}$ and $\mathcal{Y}_v = \mathcal{Y}_v^{(k)}$ belong to $\mathbf{C}_t[x]$ and $\mathbf{C}_t[Y]$ respectively.

PROPOSITION 4.2. *The operators $\mathcal{G}_v$ are W-invariant and preserve the space $\mathbf{C}_{q,t}[x]^W$. If $t_\nu = 1$ for $\nu \notin v$ then*

$$\begin{aligned} &\mathcal{G}_v^{(k)}(p_b^{(k)}) = g_v^{(k)}(b) p_{b+r_v}^{k+v} \quad \text{for} \\ &g_v^{(k)}(b) \;=\; \prod_{\alpha\in R_+, \nu_\alpha \in v} (q^{(r_k - b,\alpha)/2} - t_\alpha q^{(b-r_k,\alpha)/2}), \end{aligned} \tag{4.10}$$

where $r_v = \sum_{\nu\in v} r_\nu,\ k+v = \{k_\nu + 1, k_{\nu'}\}$ *for* $\nu \in v \not\ni \nu'$, $p_c = 0$ *for* $c \notin B_-$.

5. Fourier transforms

Proofs of the following theorems are based on the analysis of the automorphisms of the double affine Hecke algebras. The technique is similar to that from [C1-C4] and will be exposed in more detail in the next paper.

We will mainly use the *renormalized Macdonald polynomials* :

$$\epsilon_b = e_b/e_b(-r_k) = q^{-(r_k,b_-)} \prod_{[\alpha,j]\in\Lambda_b} \Big(\frac{1 - q_\alpha^{(r_k,\alpha)+j}}{1 - t_\alpha q_\alpha^{(r_k,\alpha)+j}} \Big)\, e_b, \tag{5.1}$$

$$\begin{aligned} &\Lambda_b = \{[\alpha, j],\ 0 < j < -(\alpha, b_-) \ \text{ if } \ (\alpha, b) > 0, \\ &0 < j \le -(\alpha, b_-) \ \text{ if } \ (\alpha, b) < 0\}, \quad b_- = \omega_b(b). \end{aligned} \tag{5.2}$$

Here we applied the Main Theorem from [C4]. This normalization is very convenient in the difference harmonic analysis. For instance, the duality relations are especially simple: $\epsilon_b(q^{c_\#}) = \epsilon_c(q^{b_\#})$.

THEOREM 5.1. *Given $b, c \in B$ and the corresponding renormalized polynomials ϵ_b, ϵ_c,*

$$\langle \epsilon_b \epsilon_c \gamma^{-1} \mu\rangle \;=\; q^{(b_\#,b_\#)/2+(c_\#,c_\#)/2-(r_k,r_k)} \epsilon_c(q^{b_\#}) \langle \gamma^{-1}\mu\rangle, \tag{5.3}$$

$$\langle \epsilon_b \epsilon_c^* \gamma^{-1} \mu\rangle \;=\; q^{(b_\#,b_\#)/2+(c_\#,c_\#)/2-(r_k,r_k)} \epsilon_c^*(q^{b_\#}) \langle \gamma^{-1}\mu\rangle, \tag{5.4}$$

$$\langle \epsilon_b \epsilon_c^* \gamma \mu_1\rangle \;=\; q^{-(b_\#,b_\#)/2-(c_\#,c_\#)/2+(r_k,r_k)} \epsilon_c(q^{b_\#}) \langle \gamma\mu_1\rangle. \tag{5.5}$$

In the last formula, we expand μ_1 in terms of powers of q^{-1} Note that $(\langle\gamma\mu_1\rangle)^* = \langle\gamma^{-1}\mu_1\rangle$. So (5.5) results from (5.4). The products $\mu\gamma^{-1}$ generalizes the (radial) Gaussian measure in the theory of Lie groups and symmetric spaces. Actually (5.5) is a formula for the *Fourier transform* of $\epsilon_c\gamma^{-1}$ (see [C3]).

The following theorem is equivalent to Theorem 5.1 although it does not involves c and any scalar products. We will use the conjugation 'ι' : $q^\iota = q^{-1},\ t^\iota = t^{-1},\ x_b^\iota = x_b$.

THEOREM 5.2. *Given $b \in B$,*

$$\epsilon_b(Y_1^{-1},\cdots,Y_n^{-1})(\gamma) \;=\; q^{(b_\#,b_\#)/2-(r_k,r_k)/2}\epsilon_b\gamma, \tag{5.6}$$

$$\epsilon_b^\iota(Y_1,\cdots,Y_n)(\gamma) \;=\; q^{(b_\#,b_\#)/2-(r_k,r_k)/2}\epsilon_b^*\gamma, \tag{5.7}$$

$$\epsilon_b^\iota(Y_1,\cdots,Y_n)(\gamma^{-1}) \;=\; q^{-(b_\#,b_\#)/2+(r_k,r_k)/2}\epsilon_b\gamma^{-1}. \tag{5.8}$$

The first formula holds for any normalizations of e_b. Moreover, since the coefficient of proportionality $q^{(b_\#,b_\#)/2-(r_k,r_k)/2}$ is the same for all b from the same W-orbit, it can be applied to linear combinations of $e_c, c \in W(b)$. For instance, will can use it for the symmetric Macdonald polynomials or for the t-antisymmetric ones (the next section).

Actually we do not need an exact presentation of $\gamma^{\pm 1}$ as a Laurent series in this theorem. One may take $\gamma^{\pm 1} \;=\; q^{\pm\Sigma_{i=1}^n b_i\alpha_i/2}$ or any W-invariant solution of the following system of difference equations:

$$\begin{aligned} b_j(\gamma) \;&=\; q^{(1/2)\Sigma_{i=1}^n(b_i-(b_j,b_i))(\alpha_i-\delta_i^j)} \;=\\ \gamma q^{-b_j+(b_j,b_j)/2} \;&=\; q^{(b_j,b_j)/2}x_j^{-1}\gamma,\\ b_j(\gamma^{-1}) \;&=\; q^{-(b_j,b_j)/2}x_j\gamma^{-1} \quad \text{for} \;\; 1\le j\le n. \end{aligned} \tag{5.9}$$

The formula (5.8) for the symmetric polynomials was verified in [C3] via the roots of unity. See (4.19) and the end of the proof of Corollary 5.4 (use that $p_b^\iota = p_b$ for the symmetric polynomials). The same method works well in the non-symmetric case and for other identities. A more natural proof will appear in the next paper. To make the formulas complete we need to know the coefficent of proportionality:

THEOREM 5.3.

$$\langle\gamma^{-1}\mu\rangle \;=\; \prod_{j=1}^{\infty}\Big(\frac{1-q_\alpha^{(r_k,\alpha)+j}}{1-t_\alpha q_\alpha^{(r_k,\alpha)+j}}\Big). \tag{5.10}$$

6. Mehta integral

Following [C2,C3] we will use the shift operator to verify Theorem 5.3 in a way similar to that in the differential case [O1].

First we take $k_\nu \in \mathbf{Z}_+$ and replace μ, which is a trigonometric polynomial in this case, by a proportional one:

$$\widetilde{\mu} = \widetilde{\mu}^{(k)} \stackrel{def}{=} \prod_{\alpha\in R_+}((q_\alpha^{k_\alpha-1}x_a)^{1/2} - (q_\alpha^{k_\alpha-1}x_a)^{-1/2})$$
$$\cdots((x_a)^{1/2}-(x_a)^{-1/2})\cdots((q_\alpha^{-k_\alpha}x_a)^{1/2}-(q_\alpha^{-k_\alpha}x_a)^{-1/2}).$$

Then $\widetilde{\mu}^* = \widetilde{\mu}$ and

$$\langle p_b \widetilde{\mu} \gamma^{-1} \rangle \;=\; (\langle p_b^* \,\widetilde{\mu} \gamma \rangle)^*.$$

Let

$$\widetilde{\mu} = \widetilde{\mu}^{(k)},\ \widetilde{\mu}' = \widetilde{\mu}^{(k+v)},\ b' = b - r_v,$$
$$p = p_b^{(k)},\ p' = p_{b-r_v}^{(k+v)} = (g^k(b))^{-1} \mathcal{G}_v^{(k)}(p).$$

The notations are from Proposition 4.2.

KEY LEMMA 6.1.

$$\langle (p')^* \,\widetilde{\mu}' \gamma \rangle \;=\; q^{(r_k,r_k)/2-(b-r_k,b-r_k)/2}(d_{k+v}/d_k) \tag{6.1}$$
$$\times \prod_{\alpha\in R_+,\nu_\alpha\in v} \Big(q_\alpha^{(b-r_k,\alpha)/2+k_\alpha} - q_\alpha^{(r_k-b,\alpha)/2} \Big) \langle (p)^* \,\widetilde{\mu} \gamma \rangle,$$

$$d_k = |W(r_k)|^{-1} \prod_{\alpha\in R_+,k_\alpha\neq 0} \Big(\frac{q_\alpha^{((r_k,\alpha)+k_\alpha)/2} - q_\alpha^{-((r_k,\alpha)+k_\alpha)/2}}{q_\alpha^{(r_k,\alpha)/2} - q_\alpha^{-(r_k,\alpha)/2}} \Big).$$

Proof. One has:

$$\begin{aligned} \langle (p')^* \,\widetilde{\mu}' \gamma \rangle &= (d_{k+v}/d_k) \langle (\mathcal{X}_v^{(k)})^2 (p')^* \,\widetilde{\mu} \gamma \rangle \\ &= (d_{k+v}/d_k) \langle (-1)^{\kappa_v} (\mathcal{X}_v^{(k)} \gamma)(\mathcal{X}_v^{(k)} p')^* \widetilde{\mu} \rangle, \end{aligned} \tag{6.2}$$

where κ_v is the number of roots $\alpha > 0$ with $\nu_\alpha \in v$. See Lemma 4.4 and formula (5.7) from [C2]. We will treat the factors $\mathcal{X}_v^{(k)}$ in two different ways.

First,

$$\mathcal{X}_v^{(k)} \gamma = q^{(r_k,r_k)/2-(b-r_k,b-r_k)/2} \mathcal{Y}_v^{(k)}(\gamma) \tag{6.3}$$

thanks to formula (5.6) applied to $\mathcal{X}_v^{(k)}$, that is a linear combination of $e_c^{(k)}$ for $c \in W(r_v)$. Indeed, $\mathcal{Y}_v^{(k)}(m_c)$ is proportional to $\mathcal{X}_v^{(k)}$ (or zero) for all $B_- \ni c \succeq -r_v$. Applying any Y_a to e_b we get linear combinations of the e-polynomials from the same orbit.

Second,

$$\mathcal{X}_v^{(k)} p' = (g^k(b))^{-1} \mathcal{Y}_v^{(k)}(p).$$

Substituting, let us combine two $\mathcal{Y}$ together using the unitarity of $\{Y\}$ with respect to the pairing $\langle f g^* \,\widetilde{\mu} \rangle$:

$$\langle (p')^* \,\widetilde{\mu}' \gamma \rangle = (d_{k+v}/d_k) \langle \gamma \; (\{\mathcal{Y}_v^{(k)}\}^2 (p))^* \,\widetilde{\mu} \rangle. \tag{6.4}$$

Because the constant term is W-invariant, we may multiply $\mathcal{Y}^2$ on the left by (const)$\mathcal{P}_{r_v}^{(k)}$ for the t-symmetrization from (4.8) with the idempotent normalization. As to $\mathcal{Y}^2$, it can be replaced by $(-1)^{\kappa_v} \mathcal{Y} \bar{\mathcal{Y}}$ for

$$\bar{\mathcal{Y}}_v^{(k)} = \prod_{\alpha\in R_+,\nu_\alpha\in v} ((t_\alpha Y_a)^{1/2} - (t_\alpha Y_a)^{-1/2}),\ a = \alpha^\vee$$

(see [C2], formula (5.10)). Now we use formula (4.7). Collecting all the factors together we get the required. □

The remaining part of the calculation is based on the following chain of the shift operators, that will be applied to $p^{(0)}_{-r_k} = m_{-r_k}$ one after another:

$$\mathcal{G}^{(k-e)}_{\nu_R}\mathcal{G}^{(k-2e)}_{\nu_R}\cdots\mathcal{G}^{(k-se)}_{\nu_R}\mathcal{G}^{(k-se-v)}_{\nu_1}\cdots\mathcal{G}^{(0)}_{\nu_1}, \tag{6.5}$$

where $k_{\nu_1} = s+d,\ k_{\nu_2} = s,\ v = \{v_\nu\},\ v_{\nu_1} = 1,\ v_{\nu_2} = 0$, the set $\{1,1\}$ is denoted by e. We assume that $d \ge 0$ choosing the components properly. So $k - se = dv$.

The product

$$q^{(r_{k-e},r_{k-e})/2-(r_k,r_k)/2}(d_k/d_{k-e})\cdots q^{-(r_v,r_v)/2}(d_v/d_0)$$

is equal to $q^{-(r_k,r_k)/2}d_k$. Taking into consideration that $p_{-r_k}(1) = |W(r_k)|$, we come to the final formula for $\langle\widetilde{\mu}^{(k)}\gamma\rangle$. Conjugating,

$$\langle\gamma^{-1}\widetilde{\mu}^{(k)}\rangle = q^{(r_k,r_k)/2}\prod_{\alpha\in R_+}\prod_{1\le j\le k_\alpha}\Big(q_\alpha^{(r_k,\alpha)/2+1/2} - q_\alpha^{-(r_k,\alpha)/2+1/2-j}\Big). \tag{6.6}$$

Multiplying by $\mu/\widetilde{\mu}$, we verify (5.10) for $k_\alpha \in \mathbf{Z}_+$.

If k are arbitrary we still can use the key lemma. It gives that the ratio $\phi(t)$ of the right and left hand sides of (5.10) is a periodic function of t_ν with respect to the shift $t_\nu \to t_\nu q_\nu$. We may replace the constant term by a contour integral in x for $|t_\alpha q_\alpha| < |x_a| < |t_\alpha^{-1}|,\ \ a = \alpha^\vee, \alpha > 0$, provided that $|t_\nu|, |q_\nu| < 1$. If t has only one component, then $\phi(t)$ is analytic and q-periodic for such q, t and has to be 1. Otherwise, $t = (t_{\nu_1}, t_{\nu_2})$ and we conclude that $\phi(t)$ depends only on $\tau = t_{\nu_1}^{\nu_1/2}t_{\nu_2}^{-\nu_1/2}$. Picking $t_{\nu_2} = 1$ and applying the shift operator for $v = \nu_1$, we conclude that $\phi(\tau) = \phi(q\tau)$ and $\phi = 1$ (cf. [O1]). The calculation is completed.

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References

[AW] R. Askey, and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Memoirs AMS 319 (1985).

[B] N. Bourbaki, Groupes et algèbres de Lie, Ch. **4–6**, Hermann, Paris (1969).

[C1] I. Cherednik, Intertwining operators of double affine Hecke algebras, Selecta Math. (1997).

[C2] ———, Double affine Hecke algebras and Macdonald's conjectures, Annals of Mathematics 141 (1995), 191-216.

[C3] ———, Macdonald's evaluation conjectures and difference Fourier transform, Inventiones Math. 122 (1995),119–145.

[C4] ———, Nonsymmetric Macdonald polynomials , IMRN 10 (1995), 483–515.

[DS] J.F. van Diejen, and J.V. Stockman, Multivariable q-Racah polynomials, Preprint CRM-2369 (1996).

[D] C.F. Dunkl, Hankel transforms associated to finite reflection groups, Contemp. Math. 138 (1992), 123–138.

[J] M.F.E. de Jeu, The Dunkl transform , Invent. Math. 113 (1993), 147–162.

[H] S. Helgason, Groups and geometric analysis, Academic Press, New York (1984).

[KL] D. Kazhdan, and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent.Math. 87(1987), 153–215.

[KK] B. Kostant, and S. Kumar, T-Equivariant K-theory of generalized flag varieties, J. Diff. Geometry 32(1990), 549–603.

[M1] I.G. Macdonald, Some conjectures for root systems, SIAM J.Math. Anal. 13:6 (1982), 988–1007.

[M2] ———, A new class of symmetric functions , Publ.I.R.M.A., Strasbourg, Actes 20-e Seminaire Lotharingen, (1988), 131–171 .

[M3] ———, Orthogonal polynomials associated with root systems,Preprint(1988).

[M4] ———, Affine Hecke algebras and orthogonal polynomials, Séminaire Bourbaki 47:797 (1995), 01–18.

[O1] E.M. Opdam, Some applications of hypergeometric shift operators, Invent.Math. 98 (1989), 1–18.

[O2] ———, Harmonic analysis for certain representations of graded Hecke algebras, Preprint Math. Inst. Univ. Leiden W93-18 (1993).

[O3] ———, Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group, Composito Mathematica 85 (1993), 333–373.