# Macdonald's Evaluation Conjectures, Difference Fourier Transform, and applications 

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## Introduction

Generalizing the characters of compact simple Lie groups, Ian Macdonald introduced in [M1,M2] and other works remarkable orthogonal symmetric polynomials dependent on the parameters $q, t$. He came up with three main conjectures formulated for arbitrary root systems. A new approach to the Macdonald theory was suggested in [C1] on the basis of (double) affine Hecke algebras and related difference operators. In [C2] the norm conjecture (including the celebrated constant term conjecture [M3]) was proved for all (reduced) root systems. This paper contains the proof of the remaining two (the duality and evaluation conjectures), the recurrence relations, and basic results on Macdonald's polynomials at roots of unity. In the next paper the same questions will be considered for the non-symmetric polynomials.

The evaluation conjecture (now a theorem) is in fact a $q, t$-generalization of the classic Weyl dimension formula. One can expect interesting applications of this theorem since the so-called $q$-dimensions are undoubtedly important. To demonstrate deep relations to the representation theory we prove the Recurrence Theorem connected with decomposing of the tensor products of represenations of compact Lie groups in terms of irreducible ones. The arising $q, t$-multiplicities are in fact the coefficients of our difference operators (one needs the duality to establish this). It is likely that we can incorporate the Kac-Moody case as well. The necessary technique was developed in [C4].

The duality theorem (in its complete form) states that the difference zonal $q, t$-Fourier transform is self-dual (its reproducing kernel is symmetric). In this paper we introduce the transform formally in terms of double affine Hecke algebras. The self-duality is directly related to the interpretation of these algebras via the so-called elliptic braid groups (the Fourier involution turns into the transposition of the periods of an elliptic curve). It is not very surprising since this interpretation is actually the monodromy representation of the double affine (elliptic) Knizhnik-Zamolodchikov equation from [C6].

[^0]The classical zonal (spherical) Fourier transform plays one of the main roles in the harmonic analysis on symmetric spaces $G / K$. It sends the radial parts of $K$-invariant differential operators to the corresponding symmetric polynomials (the Harish-Chandra isomorphism) and is not self-dual. The calculation of its inverse (the Plancherel theorem) is important and involved. This transform is the limiting case of our construction when $q \rightarrow 1$ and $t=q^{k}$ for certain special $k$.

Considering the rational theory (replacing the trigonometric coefficients of the operators by their rational degenerations), Charles Dunkl introduced the generalized Hankel transform which appeared to be self-dual [D,J]. Geometrically, it is the case of the tangent space $T_{e}(G / K)$ with the adjoint action of $K$, taken instead of $G / K$ (see $[\mathrm{H}]$ ). We demonstrate in this paper that one can save this very important property in the main (trigonometric) setup going to the difference counterparts of the zonal Laplace operators. At the moment, it is mostly an algebraic observation. The difference-analytical aspects were not studied systematically (see Section 4 where we discuss Schwartz functions).

As to the differential theory, the evaluation conjecture was proved by Eric Opdam [O1] (see also [O2]) together with other Macdonald- Mehta conjectures excluding the duality conjecture which collapses. He used the HeckmanOpdam operators (including the shift operator - see [O1,He]). We use their generalizations from [C1,C2] defined by means of double affine Hecke algebras. We mention that the proof of the norm conjecture from [C2] was based mainly on the relations and properties of affine Hecke algebras. Only in this paper double Hecke algebras work at their full potential ensuring the duality.

Concerning the open questions in the Macdonald theory, we will say a little bit about $A_{n}$. First of all, the Macdonald polynomials (properly normalized) are $q \leftrightarrow t$ symmetric and satisfy various additional relations. Then there are very interesting positivity conjectures (Macdonald [M1], Garsia, Haiman [GH]). Moreover they can be interpreted as generalized characters (Etingof, Kirillov [EK1]). In the differential setting, these polynomials (Jack polynomials) are also quite remarkable (Stanley, Hanlon). By the way, due to Andrews one can add $n$ new $t$-parameters and still the constant term conjecture will hold (Bressoud and Zeilberger), but at the moment we have no definition of the associated orthogonal polynomials. Hopefully some of these properties can be extended to arbitrary roots.

Recently Alexander Kirillov, Jr. established that in the case of $A_{n}$ the action of $S L_{2}(\mathbf{Z})$ appearing in the theory of quantum $S L_{n+1}$ at roots of unity (see $[\mathrm{MS}]$ ) leads to the projective representations of this groop expressed in terms of special values of the Macdonald polynomials (when $q$ is a root of unity, $k \in \mathbf{Z}_{+}$). We demonstrate that this result can be naturally rediscovered in the frameworks of the present paper and generalized to arbitrary root
systems. When $q=t$ the Macdonald polynomials are the characters (Schur functions for $A_{n}$ ) and we arrive at Theorem 13.8 from [K]. Kirillov's work [Ki] is expected to be directly connected with the equivalence of the quantum groups and Kac-Moody algebras due to Kazhdan - Lusztig [KL2] (in the case of $A_{n}$ ). In our setting it should result from the calculation of the monodromy representation of the double affine Knizhnik-Zamolodchikov equation (for arbitrary root systems).

We note that this paper is a part of a new program in the harmonic analysis of symmetric spaces based on certain remarkable representations of Hecke algebras in terms of Dunkl and Demazure operators instead of Lie groups and Lie algebras. It gave already a $k$-parametric deformation of the classical theory (see [O1,He, C5]) directly connected with the so-called quantum manybody problem (Calogero, Sutherland, Moser, Olshanetsky, Perelomov). Then it was extended (in the algebraic context) to the difference, elliptic, and finally to the difference-elliptic case [C4] presumably corresponding to the quantum Kac-Moody algebras. We say presumably because the harmonic analysis for the latter algebras does not exist.

The duality-evaluation conjecture. Let $R=\{\alpha\} \subset \mathbf{R}^{n}$ be a root system of type $A, B, \ldots, F, G$ with respect to a euclidean form $\left(z, z^{\prime}\right)$ on $\mathbf{R}^{n} \ni z, z^{\prime}, W$ the Weyl group generated by the the reflections $s_{\alpha}$. We assume that $(\alpha, \alpha)=2$ for long $\alpha$. Let us fix the set $R_{+}$of positive roots $\left(R_{-}=-R_{+}\right)$, the corresponding simple roots $\alpha_{1}, \ldots, \alpha_{n}$, and their dual counterparts $a_{1}, \ldots, a_{n}, a_{i}=$ $\alpha_{i}^{\vee}$, where $\alpha^{\vee}=2 \alpha /(\alpha, \alpha)$. The dual fundamental weights $b_{1}, \ldots, b_{n}$ are determined from the relations $\left(b_{i}, \alpha_{j}\right)=\delta_{i}^{j}$ for the Kronecker delta. We will also introduce the dual root system $R^{\vee}=\left\{\alpha^{\vee}, \alpha \in R\right\}, R_{+}^{\vee}$, and the lattices

$$
A=\oplus_{i=1}^{n} \mathbf{Z} a_{i} \subset B=\oplus_{i=1}^{n} \mathbf{Z} b_{i}
$$

$A_{ \pm}, B_{ \pm}$for $\mathbf{Z}_{ \pm}=\{m \in \mathbf{Z}, \pm m \geq 0\}$ instead of $\mathbf{Z}$. (In the standard notations, $A=Q^{\vee}, B=P^{\vee}$ - see [B].) Later on,

$$
\begin{align*}
& \nu_{\alpha}=\nu_{\alpha} \vee=(\alpha, \alpha), \nu_{i}=\nu_{\alpha_{i}}, \nu_{R}=\left\{\nu_{\alpha}, \alpha \in R\right\} \\
& \rho_{\nu}=(1 / 2) \sum_{\nu_{\alpha}=\nu} \alpha=(\nu / 2) \sum_{\nu_{i}=\nu} b_{i}, \text { for } \alpha \in R_{+}  \tag{0.1}\\
& r_{\nu}=\rho_{\nu}^{\vee}=(2 / \nu) \rho_{\nu}=\sum_{\nu_{i}=\nu} b_{i}, \quad 2 / \nu=1,2,3 .
\end{align*}
$$

Let us put formally $x_{i}=\exp \left(b_{i}\right), x_{b}=\exp (b)=\prod_{i=1}^{n} x_{i}^{k_{i}}$ for $b=$ $\sum_{i=1}^{n} k_{i} b_{i}$, and introduce the algebra $\mathbf{C}(q, t)[x]$ of polynomials in terms of $x_{i}^{ \pm 1}$ with the coefficients belonging to the field $\mathbf{C}(q, t)$ of rational functions in terms of indefinite complex parameters $q, t_{\nu}, \nu \in \nu_{R}$ (we will put $t_{\alpha}=t_{\nu_{\alpha}}=t_{\alpha \vee}$ ). The coefficient of $x^{0}=1$ (the constant term) will be denoted by $\rangle$. The following product is a Laurent series in $x$ with the coefficients in the algebra
$\mathbf{C}[t][[q]]$ of formal series in $q$ over polynomials in $t$ :

$$
\begin{equation*}
\mu=\prod_{a \in R_{+}^{\vee}} \prod_{i=0}^{\infty} \frac{\left(1-x_{a} q_{a}^{i}\right)\left(1-x_{a}^{-1} q_{a}^{i+1}\right)}{\left(1-x_{a} t_{a} q_{a}^{i}\right)\left(1-x_{a}^{-1} t_{a} q_{a}^{i+1}\right)}, \tag{0.2}
\end{equation*}
$$

where $q_{a}=q_{\nu}=q^{2 / \nu}$ for $\nu=\nu_{a}$. We note that $\mu \in \mathbf{C}(q, t)[x]$ if $t_{\nu}=q_{\nu}^{k_{\nu}}$ for $k_{\nu} \in \mathbf{Z}_{+}$. The coefficients of $\mu_{1} \stackrel{\text { def }}{=} \mu /\langle\mu\rangle$ are from $\mathbf{C}(q, t)$, where the formula for the constant term of $\mu$ is as follows (see [C2]):

$$
\begin{equation*}
\langle\mu\rangle=\prod_{a \in R_{+}^{\vee}} \prod_{i=1}^{\infty} \frac{\left(1-x_{a}\left(t^{\rho}\right) q_{a}^{i}\right)^{2}}{\left(1-x_{a}\left(t^{\rho}\right) t_{a} q_{a}^{i}\right)\left(1-x_{a}\left(t^{\rho}\right) t_{a}^{-1} q_{a}^{i}\right)} \tag{0.3}
\end{equation*}
$$

Here and further $x_{b}\left(t^{ \pm \rho} q^{c}\right)=q^{(b, c)} \prod_{\nu} t_{\nu}^{ \pm\left(b, \rho_{\nu}\right)}$. We note that $\mu_{1}^{*}=\mu_{1}$ with respect to the involution

$$
x_{b}^{*}=x_{-b}, t^{*}=t^{-1}, q^{*}=q^{-1} .
$$

The monomial symmetric functions $m_{b}=\sum_{c \in W(b)} x_{c}$ for $b \in B_{-}$form a base of the space $\mathbf{C}[x]^{W}$ of all $W$-invariant polynomials. Setting,

$$
\begin{equation*}
\langle f, g\rangle=\left\langle\mu_{1} f g^{*}\right\rangle=\langle g, f\rangle^{*} \text { for } f, g \in \mathbf{C}(q, t)[x]^{W} \tag{0.4}
\end{equation*}
$$

we introduce the Macdonald polynomials $p_{b}(x), b \in B_{-}$, by means of the conditions

$$
\begin{align*}
& p_{b}-m_{b} \in \oplus_{c} \mathbf{C}(q, t) m_{c},\left\langle p_{b}, m_{c}\right\rangle=0 \text {, for } c \succ b  \tag{0.5}\\
& \text { where } c \in B_{-}, c \succ b \text { means that } c-b \in A_{+}, c \neq b .
\end{align*}
$$

They can be determined by the Gram - Schmidt process because the (skew Macdonald) pairing (see [M1,M2,C2]) is non-degenerate and form a basis in $\mathbf{C}(q, t)[x]^{W}$. As it was established by Macdonald they are pairwise orthogonal for arbitrary $b \in B_{-}$. We note that $p_{b}$ are "real" with respect to the formal conjugation sending $q \rightarrow q^{-1}, t \rightarrow t^{-1}$. It makes our definition compatible with Macdonald's original one (his $\mu$ is somewhat different).

Main Theorem. Given $b, c \in B_{-}$and the corresponding Macdonald polynomials $p_{b}, p_{c}$,

$$
\begin{align*}
& p_{b}\left(t^{-\rho} q^{c}\right) p_{c}\left(t^{-\rho}\right)=p_{c}\left(t^{-\rho} q^{b}\right) p_{b}\left(t^{-\rho}\right),  \tag{0.6}\\
& p_{b}\left(t^{-\rho}\right)=p_{b}\left(t^{\rho}\right)=x_{b}\left(t^{\rho}\right) \prod_{a \in R_{+}^{\vee}} \prod_{j=1}^{-\left(a^{\vee}, b\right)}\left(\frac{1-q_{a}^{j-1} t_{a} x_{a}\left(t^{\rho}\right)}{1-q_{a}^{j-1} x_{a}\left(t^{\rho}\right)}\right)= \\
& x_{b}\left(t^{\rho}\right) \prod_{a \in R_{+}^{\vee}, 0 \leq j<\infty}\left(\frac{\left(1-q_{a}^{j} x_{a}\left(t^{\rho} q^{-b}\right)\right)\left(1-t_{a} q_{a}^{j} x_{a}\left(t^{\rho}\right)\right)}{\left(1-t_{a} q_{a}^{j} x_{a}\left(t^{\rho} q^{-b}\right)\right)\left(1-q_{a}^{j} x_{a}\left(t^{\rho}\right)\right)}\right) . \tag{0.7}
\end{align*}
$$

The right hand side of (0.7) is a rational function in terms of $q, t\left(a^{\vee}=\right.$ $2 a /(a, a) \in R)$. We mention that there is a straightforward passage to nonreduced root systems, the non-symmetric Macdonald polynomials, and to $\mu$ introduced for $\alpha \in R_{+}$instead of $a \in R_{+}^{\vee}$ (see [C2]). As to the latter case, it is necessary just to replace the indices $a$ by $\alpha\left(q_{a} \rightarrow q, \rho \rightarrow r\right)$ in all final formulas for $\left\{p_{a}\right\}$.

The second formula was conjectured by Macdonald (see (12.10), [M2]). He also formulated an equivalent version of (0.6) in one of his lectures (1991). Both statements were established for $A_{n}$ by Koornwinder in 1988 (his proof was not published) and by Macdonald (to be published). Recently the paper by Etingof and Kirillov [EK2] appeared were they use their interpretation of the Macdonald polynomials to check the above theorem (and the norm conjecture) in the case of $A_{n}$. As to other root systems, it seems that almost nothing was known (excluding $B C_{1}$ and certain special values of the parameters).

Concerning the notations, we use $t$ for Hecke algebras and $q$ for difference operators. So in this paper (in contrast to [C2]) we switch to the standard meaning of these letters in the papers on $q$-orthogonal polynomials and $q$ functions.

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## 1. Double affine Hecke algebras

The vectors $\tilde{\alpha}=[\alpha, k] \in \mathbf{R}^{n} \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the affine root system $R^{a} \supset R\left(z \in \mathbf{R}^{n}\right.$ are identified with $\left.[z, 0]\right)$. We add $\alpha_{0} \stackrel{\text { def }}{=}[-\theta, 1]$ to the simple roots for the maximal root $\theta \in R$. The corresponding set $R_{+}^{a}$ of positive roots coincides with $R_{+} \cup\{[\alpha, k], \alpha \in R, k>0\}$.

We denote the Dynkin diagram and its affine completion with $\left\{\alpha_{j}, 0 \leq\right.$ $j \leq n\}$ as the vertices by $\Gamma$ and $\Gamma^{a}$. Let $m_{i j}=2,3,4,6$ if $\alpha_{i}$ and $\alpha_{j}$ are joined by $0,1,2,3$ laces respectively. The set of the indices of the images of $\alpha_{0}$ by all the automorphisms of $\Gamma^{a}$ will be denoted by $O\left(O=\{0\}\right.$ for $\left.E_{8}, F_{4}, G_{2}\right)$. Let $O^{*}=r \in O, r \neq 0$. The elements $b_{r}$ for $r \in O^{*}$ are the so-called minuscule weights $\left(\left(b_{r}, \alpha\right) \leq 1\right.$ for $\left.\alpha \in R_{+}\right)$.

Given $\tilde{\alpha}=[\alpha, k] \in R^{a}, b \in B$, let

$$
\begin{equation*}
s_{\tilde{\alpha}}(\tilde{z})=\tilde{z}-\left(z, \alpha^{\vee}\right) \tilde{\alpha}, \quad b^{\prime}(\tilde{z})=[z, \zeta-(z, b)] \tag{1.1}
\end{equation*}
$$

for $\tilde{z}=[z, \zeta] \in \mathbf{R}^{n+1}$.

The affine Weyl group $W^{a}$ is generated by all $s_{\tilde{\alpha}}$ (we write $W^{a}=<s_{\tilde{\alpha}}, \tilde{\alpha} \in$ $\left.R_{+}^{a}>\right)$. One can take the simple reflections $s_{j}=s_{\alpha_{j}}, 0 \leq j \leq n$, as its generators and introduce the corresponding notion of the length. This group is the semi-direct product $W \ltimes A^{\prime}$ of its subgroups $W=<s_{\alpha}, \alpha \in R_{+}>$and $A^{\prime}=\left\{a^{\prime}, a \in A\right\}$, where

$$
\begin{equation*}
a^{\prime}=s_{\alpha} s_{[\alpha, 1]}=s_{[-\alpha, 1]} s_{\alpha} \text { for } a=\alpha^{\vee}, \alpha \in R . \tag{1.2}
\end{equation*}
$$

The extended Weyl group $W^{b}$ generated by $W$ and $B^{\prime}$ (instead of $A^{\prime}$ ) is isomorphic to $W \ltimes B^{\prime}$ :

$$
\begin{equation*}
\left(w b^{\prime}\right)([z, \zeta])=[w(z), \zeta-(z, b)] \text { for } w \in W, b \in B \tag{1.3}
\end{equation*}
$$

Given $b_{+} \in B_{+}$, let

$$
\begin{equation*}
\omega_{b_{+}}=w_{0} w_{0}^{+} \in W, \pi_{b_{+}}=b_{+}^{\prime}\left(\omega_{b_{+}}\right)^{-1} \in W^{b}, \omega_{i}=\omega_{b_{i}}, \pi_{i}=\pi_{b_{i}}, \tag{1.4}
\end{equation*}
$$

where $w_{0}$ (respectively, $w_{0}^{+}$) is the longest element in $W$ (respectively, in $W_{b_{+}}$ generated by $s_{i}$ preserving $b_{+}$) relative to the set of generators $\left\{s_{i}\right\}$ for $i>0$.

We will use here only the elements $\pi_{r}=\pi_{b_{r}}, r \in O$. They leave $\Gamma^{a}$ invariant and form a group denoted by $\Pi$, which is isomorphic to $B / A$ by the natural projection $\left\{b_{r} \rightarrow \pi_{r}\right\}$. As to $\left\{\omega_{r}\right\}$, they preserve the set $\left\{-\theta, \alpha_{i}, i>\right.$ $0\}$. The relations $\pi_{r}\left(\alpha_{0}\right)=\alpha_{r}=\left(\omega_{r}\right)^{-1}(-\theta)$ distinguish the indices $r \in O^{*}$. Moreover (see e.g. [C2]):

$$
\begin{equation*}
W^{b}=\Pi \ltimes W^{a}, \quad \text { where } \pi_{r} s_{i} \pi_{r}^{-1}=s_{j} \text { if } \pi_{r}\left(\alpha_{i}\right)=\alpha_{j}, 0 \leq j \leq n . \tag{1.5}
\end{equation*}
$$

We extend the notion of the length to $W^{b}$. Given $\nu \in \nu_{R}, r \in O^{*}, \tilde{w} \in W^{a}$, and a reduced decomposition $\tilde{w}=s_{j_{l}} \ldots s_{j_{2}} s_{j_{1}}$ with respect to $\left\{s_{j}, 0 \leq j \leq n\right\}$, we call $l=l(\hat{w})$ the length of $\hat{w}=\pi_{r} \tilde{w} \in W^{b}$. Setting

$$
\begin{align*}
\lambda(\hat{w})= & \left\{\tilde{\alpha}^{1}=\alpha_{j_{1}}, \tilde{\alpha}^{2}=s_{j_{1}}\left(\alpha_{j_{2}}\right), \tilde{\alpha}^{3}=s_{j_{1}} s_{j_{2}}\left(\alpha_{j_{3}}\right), \ldots\right. \\
& \left.\ldots, \tilde{\alpha}^{l}=\tilde{w}^{-1} s_{j_{l}}\left(\alpha_{j_{l}}\right)\right\}, \tag{1.6}
\end{align*}
$$

one can represent

$$
\begin{align*}
& l=|\lambda(\hat{w})|=\sum_{\nu} l_{\nu}, \text { for } l_{\nu}=l_{\nu}(\hat{w})=\left|\lambda_{\nu}(\hat{w})\right|,  \tag{1.7}\\
& \lambda_{\nu}(\hat{w})=\left\{\tilde{\alpha}^{m}, \nu\left(\tilde{\alpha}^{m}\right)=\nu\left(\tilde{\alpha}_{j_{m}}\right)=\nu\right\}, 1 \leq m \leq l,
\end{align*}
$$

where || denotes the number of elements, $\nu([\alpha, k]) \stackrel{\text { def }}{=} \nu_{\alpha}$.
For instance,

$$
\begin{align*}
& l_{\nu}\left(b^{\prime}\right)=\sum_{\alpha}|(b, \alpha)|, \alpha \in R_{+}, \nu_{\alpha}=\nu \in \nu_{R},  \tag{1.8}\\
& l_{\nu}\left(b_{+}^{\prime}\right)=2\left(b_{+}, \rho_{\nu}\right) \text { when } b_{+} \in B_{+} .
\end{align*}
$$

Here $\left|\mid=\right.$ absolute value. Later on $b$ and $b^{\prime}$ will not be distinguished.

We put $m=2$ for $D_{2 k}$ and $C_{2 k+1}, m=1$ for $C_{2 k}, B_{k}$, otherwise $m=$ $|\Pi|$. The definition involves the parameters $q,\left\{t_{\nu}, \nu \in \nu_{R}\right\}$ and independent variables $X_{1}, \ldots, X_{n}$. Let us set

$$
\begin{align*}
& t_{\tilde{\alpha}}=t_{\nu(\tilde{\alpha})}, t_{j}=t_{\alpha_{j}}, \quad \text { where } \tilde{\alpha} \in R^{a}, 0 \leq j \leq n \\
& X_{\tilde{b}}=\prod_{i=1}^{n} X_{i}^{k_{i}} q^{k} \text { if } \tilde{b}=[b, k]  \tag{1.9}\\
& \text { for } b=\sum_{i=1}^{n} k_{i} b_{i} \in B, k \in \frac{1}{m} \mathbf{Z}
\end{align*}
$$

Later on $\mathbf{C}_{q}$ is the field of rational functions in $q^{1 / m}, \mathbf{C}_{q}[X]=\mathbf{C}_{q}\left[X_{b}\right]$ means the algebra of polynomials in terms of $X_{i}^{ \pm 1}$ with the coefficients depending on $q^{1 / m}$ rationally. We replace $\mathbf{C}_{q}$ by $\mathbf{C}_{q, t}$ if the functions (coefficients) also depend rationally on $\left\{t_{\nu}^{1 / 2}\right\}$.

Let $([a, k],[b, l])=(a, b)$ for $a, b \in B, a_{0}=\alpha_{0}, \nu_{\alpha^{\vee}}=\nu_{\alpha}$, and $\alpha_{r^{*}} \stackrel{\text { def }}{=}$ $\pi_{r}^{-1}\left(\alpha_{0}\right)$ for $r \in O^{*}$.

Definition 1.1. The double affine Hecke algebra $\mathfrak{H}$ (see [C1,C2]) is generated over the field $\mathbf{C}_{q, t}$ by the elements $\left\{T_{j}, 0 \leq j \leq n\right\}$, pairwise commutative $\left\{X_{b}, b \in B\right\}$ satisfying (1.9), and the group $\Pi$ where the following relations are imposed:
(o) $\left(T_{j}-t_{j}^{1 / 2}\right)\left(T_{j}+t_{j}^{-1 / 2}\right)=0,0 \leq j \leq n ;$
(i) $T_{i} T_{j} T_{i} \ldots=T_{j} T_{i} T_{j} \ldots, m_{i j}$ factors on each side;
(ii) $\pi_{r} T_{i} \pi_{r}^{-1}=T_{j}$ if $\pi_{r}\left(\alpha_{i}\right)=\alpha_{j}$;
(iii) $T_{i} X_{b} T_{i}=X_{b} X_{a_{i}}^{-1} \quad$ if $\left(b, \alpha_{i}\right)=1,1 \leq i \leq n$;
(iv) $T_{0} X_{b} T_{0}=X_{s_{0}(b)}=X_{b} X_{\theta} q^{-1}$ if $(b, \theta)=-1$;
(v) $T_{i} X_{b}=X_{b} T_{i}$ if $\left(b, \alpha_{i}\right)=0$ for $0 \leq i \leq n$;
(vi) $\pi_{r} X_{b} \pi_{r}^{-1}=X_{\pi_{r}(b)}=X_{\omega_{r}^{-1}(b)} q^{\left(b_{r^{*}}, b\right)}, r \in O^{*}$.

Given $\tilde{w} \in W^{a}, r \in O$, the product

$$
\begin{equation*}
T_{\pi_{r} \tilde{w}} \stackrel{\text { def }}{=} \pi_{r} \prod_{k=1}^{l} T_{i_{k}}, \quad \text { where } \quad \tilde{w}=\prod_{k=1}^{l} s_{i_{k}}, l=l(\tilde{w}) \tag{1.10}
\end{equation*}
$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$
\begin{equation*}
T_{\hat{v}} T_{\hat{w}}=T_{\hat{v} \hat{w}} \text { whenever } l(\hat{v} \hat{w})=l(\hat{v})+l(\hat{w}) \text { for } \hat{v}, \hat{w} \in W^{b} \tag{1.11}
\end{equation*}
$$

In particular, we arrive at the pairwise commutative elements

$$
\begin{equation*}
Y_{b}=\prod_{i=1}^{n} Y_{i}^{k_{i}} \text { if } b=\sum_{i=1}^{n} k_{i} b_{i} \in B, \quad \text { where } Y_{i} \stackrel{\text { def }}{=} T_{b_{i}} \tag{1.12}
\end{equation*}
$$

satisfying the relations

$$
\begin{align*}
& T_{i}^{-1} Y_{b} T_{i}^{-1}=Y_{b} Y_{a_{i}}^{-1} \text { if }\left(b, \alpha_{i}\right)=1  \tag{1.13}\\
& T_{i} Y_{b}=Y_{b} T_{i} \text { if }\left(b, \alpha_{i}\right)=0,1 \leq i \leq n
\end{align*}
$$

Let us introduce the following elements from $\mathbf{C}_{t}^{n}$ :

$$
\begin{align*}
& t^{ \pm \rho} \stackrel{\text { def }}{=}\left(l_{t}\left(b_{1}\right)^{ \pm 1}, \ldots, l_{t}\left(b_{n}\right)^{ \pm 1}\right), \quad \text { where } \\
& l_{t}(\hat{w}) \stackrel{\text { def }}{=} \prod_{\nu \in \nu_{R}} t_{\nu}^{l_{\nu}(\hat{w}) / 2}, \hat{w} \in W^{b} \tag{1.14}
\end{align*}
$$

and the corresponding evaluation maps:

$$
\begin{equation*}
X_{i}\left(t^{ \pm \rho}\right)=l_{t}\left(b_{i}\right)^{ \pm 1}=Y_{i}\left(t^{ \pm \rho}\right), \quad 1 \leq i \leq n . \tag{1.15}
\end{equation*}
$$

For instance, $X_{a_{i}}\left(t^{\rho}\right)=l_{t}\left(a_{i}\right)=t_{i}$ (see (1.8)).
Theorem 1.2. i) The elements $H \in \mathfrak{H}$ have the unique decompositions

$$
\begin{equation*}
H=\sum_{w \in W} g_{w} T_{w} f_{w}, g_{w} \in \mathbf{C}_{q, t}[X], \quad f_{w} \in \mathbf{C}_{q, t}[Y] . \tag{1.16}
\end{equation*}
$$

ii) The map

$$
\begin{align*}
& \varphi: X_{i} \rightarrow Y_{i}^{-1}, \quad Y_{i} \rightarrow X_{i}^{-1}, T_{i} \rightarrow T_{i}  \tag{1.17}\\
& t_{\nu} \rightarrow t_{\nu}, \\
& \hline
\end{align*}
$$

can be extended to an anti-involution $(\varphi(A B)=\varphi(B) \varphi(A))$ of $\mathfrak{H}$.
iii) The linear functional on $\mathfrak{H}$

$$
\begin{equation*}
\llbracket \sum_{w \in W} g_{w} T_{w} f_{w} \rrbracket=\sum_{w \in W} g_{w}\left(t^{-\rho}\right) l_{t}(w) f_{w}\left(t^{\rho}\right) \tag{1.18}
\end{equation*}
$$

is invariant with respect to $\varphi$. The bilinear form

$$
\begin{equation*}
\llbracket G, H \rrbracket \stackrel{\text { def }}{=} \llbracket \varphi(G) H \rrbracket, G, H \in \mathfrak{H} \tag{1.19}
\end{equation*}
$$

is symmetric $(\llbracket G, H \rrbracket=\llbracket H, G \rrbracket)$ and non-degenerate.
Proof. The first statement is from Theorem 2.3 [C2]. The map $\varphi$ is the composition of the involution (see [C1])

$$
\begin{align*}
& \varepsilon: X_{i} \rightarrow Y_{i}, \quad Y_{i} \rightarrow X_{i}, \quad T_{i} \rightarrow T_{i}^{-1}, \\
& t_{\nu} \rightarrow t_{\nu}^{-1}, q \rightarrow q^{-1}, 1 \leq i \leq n, \tag{1.20}
\end{align*}
$$

and the main anti-involution from [C2], introduced by the condition $\langle H f, g\rangle=$ $\left\langle f, H^{*} g\right\rangle($ see (0.4)):

$$
\begin{align*}
& X_{i}^{*}=X_{i}^{-1}, Y_{i}^{*}=Y_{i}^{-1}, T_{i}^{*}=T_{i}^{-1} \\
& t_{\nu} \rightarrow t_{\nu}^{-1}, q \rightarrow q^{-1}, 0 \leq i \leq n . \tag{1.21}
\end{align*}
$$

The other claims follow directly from the definition of $\llbracket \rrbracket$.
Let us give the explicit formulas for the action of $\varphi, \varepsilon$ on $T_{0}$ :

$$
\begin{align*}
& \varphi\left(T_{0}\right)=Y_{\theta}^{-1} T_{0} X_{\theta}^{-1}=T_{s_{\theta}}^{-1} X_{\theta}^{-1}  \tag{1.22}\\
& \varepsilon\left(T_{0}\right)=X_{\theta} T_{0}^{-1} Y_{\theta}=X_{\theta} T_{s_{\theta}}
\end{align*}
$$

One can extend 【】 to the localization of $\mathfrak{H}$ with respect to all polynomials in $X$ (or in $Y$ ). The algebra becomes the semi-direct product of $\mathbf{C}\left[W^{b}\right]$ and $\mathbf{C}(X)$ after this (see [C3]). Sometimes it is also convenient to involve proper completions of $\mathbf{C}(X)$.

## 2. Difference operators

Setting (see Introduction)

$$
\begin{equation*}
x_{\tilde{b}}=\prod_{i=1}^{n} x_{i}^{k_{i}} q^{k} \quad \text { if } \quad \tilde{b}=[b, k], b=\sum_{i=1}^{n} k_{i} b_{i} \in B, k \in \frac{1}{m} \mathbf{Z}, \tag{2.1}
\end{equation*}
$$

for independent $x_{1}, \ldots, x_{n}$, we will consider $\{X\}$ as operators acting in $\mathbf{C}_{q}[x]=$ $\mathbf{C}_{q}\left[x_{1}^{ \pm 1}, \ldots, x_{n}^{ \pm 1}\right]:$

$$
\begin{equation*}
X_{\tilde{b}}(p(x))=x_{\tilde{b}} p(x), p(x) \in \mathbf{C}_{q}[x] . \tag{2.2}
\end{equation*}
$$

The elements $\hat{w} \in W^{b}$ act in $\mathbf{C}_{q}[x]$ by the formulas:

$$
\begin{equation*}
\hat{w}\left(x_{\tilde{b}}\right)=x_{\hat{w}(\tilde{b})} . \tag{2.3}
\end{equation*}
$$

In particular:

$$
\begin{equation*}
\pi_{r}\left(x_{b}\right)=x_{\omega_{r}^{-1}(b)} q^{\left(b_{\left.r^{*}, b\right)}\right.} \text { for } \alpha_{r^{*}}=\pi_{r}^{-1}\left(\alpha_{0}\right), r \in O^{*} \tag{2.4}
\end{equation*}
$$

The Demazure-Lusztig operators (see [KL1, KK, C1], and [C2] for more detail )

$$
\begin{equation*}
\hat{T}_{j}=t_{j}^{1 / 2} s_{j}+\left(t_{j}^{1 / 2}-t_{j}^{-1 / 2}\right)\left(X_{a_{j}}-1\right)^{-1}\left(s_{j}-1\right), 0 \leq j \leq n \tag{2.5}
\end{equation*}
$$

act in $\mathbf{C}_{q, t}[x]$ naturally. We note that only $\hat{T}_{0}$ depends on $q$ :

$$
\begin{align*}
& \hat{T}_{0}=t_{0}^{1 / 2} s_{0}+\left(t_{0}^{1 / 2}-t_{0}^{-1 / 2}\right)\left(q X_{\theta}^{-1}-1\right)^{-1}\left(s_{0}-1\right), \\
& \text { where } s_{0}\left(X_{i}\right)=X_{i} X_{\theta}^{-\left(b_{i}, \theta\right)} q^{\left(b_{i}, \theta\right)} \tag{2.6}
\end{align*}
$$

Theorem 2.1. The map $T_{j} \rightarrow \hat{T}_{j}, X_{b} \rightarrow X_{b}$ (see (1.9,2.2)), $\pi_{r} \rightarrow \pi_{r}$ (see (2.4)) induces a $\mathbf{C}_{q, t^{-}}$linear homomorphism from $\mathfrak{H}$ to the algebra of linear endomorphisms of $\mathbf{C}_{q, t}[x]$. This representation is faithful and remains faithful
when $q, t$ take any non-zero values assuming that $q$ is not a root of unity (see [C2]). The image $\hat{H}$ is uniquely determined from the following condition:

$$
\begin{align*}
& \hat{H}(f(x))=g(x) \text { for } H \in \mathfrak{H}, \quad \text { if } H f(X)-g(X) \in \\
& I \stackrel{\text { def }}{=}\left\{\sum_{i=0}^{n} H_{i}\left(T_{i}-t_{i}\right)+\sum_{r \in O^{*}} H_{r}\left(\pi_{r}-1\right), \text { where } H_{i}, H_{r} \in \mathfrak{H}\right\} \tag{2.7}
\end{align*}
$$

Due to Theorem 1.2, an arbitrary $H \in \mathfrak{H}$ can be uniquely represented in the form

$$
\begin{align*}
H & =\sum_{b \in B, w \in W} g_{b, w} Y_{b} T_{w}, g_{b, w} \in \mathbf{C}_{q, t}[X] \\
& =\sum_{b \in B, w \in W} T_{w} X_{b} g_{b, w}^{\prime}, g_{b, w}^{\prime} \in \mathbf{C}_{q, t}[Y] \tag{2.8}
\end{align*}
$$

We set:

$$
\begin{align*}
{[H]_{\dagger} } & =\sum_{b \in B, w \in W} g_{b, w} Y_{b} l_{t}(w), \dagger[H]=\sum_{b \in B, w \in W} l_{t}(w) X_{b} g_{b, w}^{\prime} \\
{[H]_{\ddagger} } & =\sum_{b \in B, w \in W} g_{b, w} \llbracket Y_{b} T_{w} \rrbracket, \ddagger[H]=\sum_{b \in B, w \in W} \llbracket T_{w} X_{b} \rrbracket g_{b, w}^{\prime} \tag{2.9}
\end{align*}
$$

One easily checks that

$$
\begin{align*}
& \llbracket H_{1} H_{2} \rrbracket= \llbracket H_{1}\left[H_{2}\right]_{\dagger} \rrbracket= \\
&\left.\llbracket H_{1}\left[H_{2}\right]_{\ddagger} \rrbracket=\llbracket H_{1}\right] H_{2} \rrbracket=  \tag{2.10}\\
& \ddagger\left[H_{1}\right] H_{2} \rrbracket \text { for } H_{1}, H_{2} \in \mathfrak{H} .
\end{align*}
$$

The image $\hat{H}$ of $H$ can be uniquely represented as follows:

$$
\begin{equation*}
\hat{H}=\sum_{b \in B, w \in W} h_{b, w} b w=\sum_{b \in B, w \in W} w b h_{b, w}^{\prime} \tag{2.11}
\end{equation*}
$$

where $h_{b, w}, h_{b, w}^{\prime}$ belong to the field $\mathbf{C}_{q, t}(X)$ of rational functions in $X_{1}, \ldots, X_{n}$. We extend the above operations to arbitrary operators in the form (2.11):

$$
\begin{equation*}
[\hat{H}]_{\dagger}=\sum h_{b, w} b,{ }_{\dagger}[\hat{H}]=\sum b h_{b, w}^{\prime}, \llbracket \hat{H} \rrbracket=\sum h_{b, w}\left(t^{-\rho}\right) \tag{2.12}
\end{equation*}
$$

These operations commute with the homomorphism $H \rightarrow \hat{H}$.
Let us define the difference Harish-Chandra map (see [C2], Proposition 3.1):

$$
\begin{equation*}
\chi\left(\sum_{w \in W, b \in B} h_{b, w} b w\right)=\sum_{b \in B, w \in B} h_{b, w}(\diamond) y_{b} \in \mathbf{C}_{q, t}[y] \tag{2.13}
\end{equation*}
$$

where $\diamond \stackrel{\text { def }}{=}\left(X_{1}=\ldots=X_{n}=0\right),\left\{y_{b}\right\}$ is one more set of variables introduced for independent $y_{1}, \ldots, y_{n}$ in the same way as $\left\{x_{b}\right\}$ were.

Proposition 2.2. Setting

$$
\begin{equation*}
\mathcal{L}_{f}=f(Y), \hat{\mathcal{L}}_{f}=f(\hat{Y}), L_{f}=L_{f}^{q, t} \stackrel{\text { def }}{=}\left[\left(\hat{\mathcal{L}}_{f}\right)\right]_{\dagger} \tag{2.14}
\end{equation*}
$$

for $f=\sum_{b} g_{b} y_{b} \in \mathbf{C}_{q, t}[y]$, one has:

$$
\begin{equation*}
\chi\left(\hat{\mathcal{L}}_{f}\right)=\chi\left(L_{f}\right)=\llbracket f(Y) \rrbracket=\sum_{b \in B} g_{b} \prod_{\nu} t_{\nu}^{\left(b, \rho_{\nu}\right)} y_{b} . \tag{2.15}
\end{equation*}
$$

The proof of the following theorem repeats the proof of Theorem 4.5,[C2] (where the relations $t_{\nu}=q_{\nu}^{k_{\nu}}$ for $k_{\nu} \in \mathbf{Z}_{+}$were imposed). We note that once (2.16) is known for these special $t$ it holds true for all $q, t$ since all the coefficients of difference operators and polynomials are rational in $q, t$.

Theorem 2.3. The difference operators $\left\{L_{f}, f\left(y_{1}, \ldots, y_{n}\right) \in \mathbf{C}_{q, t}[y]^{W}\right\}$ are pairwise commutative, $W$-invariant (i.e $w L_{f} w^{-1}=L_{f}$ for all $w \in W$ ) and preserve $\mathbf{C}_{q, t}[x]^{W}$. The Macdonald polynomials $p_{b}=p_{b}^{q, t}\left(b \in B_{-}\right)$from (0.5) are their eigenvectors:

$$
\begin{equation*}
L_{f}\left(p_{b}^{q, t}\right)=f\left(t^{\rho} q^{-b}\right) p_{b}^{q, t}, y_{i}\left(t^{\rho} q^{-b}\right) \stackrel{\operatorname{def}}{=} q^{-\left(b_{i}, b\right)} \prod_{\nu} t_{\nu}^{\left(b_{i}, \rho_{\nu}\right)} \tag{2.16}
\end{equation*}
$$

We fix a subset $v \in \nu_{R}$ and introduce the shift operator by the formula

$$
\begin{gather*}
\mathcal{G}_{v}=\left(\mathcal{X}_{v}\right)^{-1} \mathcal{Y}_{v}, G_{v}^{q, t}=\left[\hat{\mathcal{G}}_{v}\right]_{\dagger}=\left(\mathcal{X}_{v}\right)^{-1}\left[\mathcal{Y}_{v}\right]_{\dagger}  \tag{2.17}\\
\left.\mathcal{X}_{v}=\prod_{\nu_{a} \in v}\left(\left(t_{a} X_{a}\right)^{1 / 2}-\left(t_{a} X_{a}\right)^{-1 / 2}\right), \mathcal{Y}_{v}=\prod_{\nu_{a} \in v}\left(t_{a} Y_{a}^{-1}\right)^{1 / 2}-\left(t_{a} Y_{a}^{-1}\right)^{-1 / 2}\right)
\end{gather*}
$$

Here $a=\alpha^{\vee} \in R_{+}^{\vee}, \nu_{a}=\nu_{\alpha}, t_{a}=t_{\alpha}$, the elements $\mathcal{X}_{v}=\mathcal{X}_{v}^{t}, \mathcal{Y}_{v}=\mathcal{Y}_{v}^{t}$ belong to $\mathbf{C}_{t}[X], \mathbf{C}_{t}[Y]$ respectively.

Theorem 2.4. The operators $\hat{\mathcal{G}}_{v}$ and $G_{v}^{q, t}$ are $W$-invariant and preserve $\mathbf{C}_{q, t}[x]^{W}$ (their restrictions to the latter space coincide). Moreover, if $t_{\nu}=1$ when $\nu \notin v$ then

$$
\begin{align*}
& G_{v}^{q, t} L_{f}^{q, t}=L_{f}^{q, t q_{v}} G_{v}^{q, t} \text { for } f \in \mathbf{C}_{q, t}[y]^{W}, \\
& G_{v}^{q, t}\left(p_{b}^{q, t}\right)=g_{v}^{q, t}(b) p_{b+r_{v}}^{q, t q_{v}}, \text { for }  \tag{2.18}\\
& g_{v}^{q, t}(b)=\prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(y_{a}\left(t^{\rho / 2} q^{-b / 2}\right)-t_{a} y_{a}\left(t^{-\rho / 2} q^{+b / 2}\right)\right),
\end{align*}
$$

where $r_{v}=\sum_{\nu \in v} r_{\nu}, t q_{v}=\left\{t_{\nu} q^{2 / \nu}, t_{\nu^{\prime}}\right\}$ for $\nu \in v \not \nexists \nu^{\prime}, p_{c}=0$ for $c \notin B_{-}$.

Proof. When $t_{\nu}=q^{2 k_{\nu} / \nu}$ for $k_{\nu} \in \mathbf{Z}_{+}$these statements are in fact from [C2]. They give (2.18) for all $q, t$. Indeed, it can be rewritten as follows:

$$
\begin{equation*}
\left[\hat{\mathcal{L}}_{f}^{q, t} \mathcal{X}_{v}^{t}\right]_{\dagger}=\mathcal{X}_{v}^{t} L_{f}^{q, t q_{v}}, \tag{2.19}
\end{equation*}
$$

where the coefficients of the difference operators on both sides are from $\mathbf{C}_{q, t}[X]$. Here we used that $[\mathcal{L} \mathcal{M}]_{\dagger}=[\mathcal{L}]_{\dagger}[\mathcal{M}]_{\dagger}$ for arbitrary operators $\mathcal{L}, \mathcal{M}$ in the form (2.11) if the second is $W$-invariant. The remaining formulas can be deduced from [ C 2$]$ in the same way (they mean certain identities in $\mathbf{C}_{q, t}$ which are enough to check for $\left.t_{\nu}=q^{2 k_{\nu} / \nu}\right)$. One can use (2.15) as well.

We will also need Proposition 3.4 from [C2]:
Proposition 2.5. Given $b \in B_{-}$, let $m_{b}=\sum_{w \in W / W_{b}} x_{w(b)}$ for the stabilizer $W_{b}$ of $b$ in $W, L_{b} \stackrel{\text { def }}{=} L_{m_{b}}=\left\{L_{b}\right\}+\sum_{c} f_{b}^{c}(X) c, W(c) \succ b$. Then

$$
\begin{equation*}
\left\{L_{b}\right\}=\sum_{w \in W / W_{b}} \prod_{a \in R_{+}^{\vee}, j} \frac{t_{a}^{1 / 2} X_{w(a)} q_{a}^{j}-t_{a}^{-1 / 2}}{X_{w(a)} q_{a}^{j}-1} w(b),-\left(b, a^{\vee}\right)>j \geq 0 . \tag{2.20}
\end{equation*}
$$

If $r \in O^{*}$ then $L_{-b_{r}}=\left\{L_{-b_{r}}\right\}$.

## 3. Duality and evaluation conjectures

First of all we will use Theorem 1.2 to define the Fourier pairing. In the classical theory the latter is the inner product of a function and the Fourier transform of another function. In this and the next sections we will identify the elements $H \in \mathfrak{H}$ with their images $\hat{H}$. The following pairing on $f, g \in \mathbf{C}_{q, t}[x]$ is symmetric and non-degenerate:

$$
\begin{align*}
& \llbracket f, g \rrbracket=\llbracket f(X), g(X) \rrbracket=\llbracket \varphi(f(X)) g(X) \rrbracket= \\
& \llbracket \bar{f}(Y) g(X) \rrbracket=\left\{\mathcal{L}_{\bar{f}}(g(x))\right\}\left(t^{-\rho}\right),  \tag{3.1}\\
& \bar{x}_{b}=x_{-b}=x_{b}^{-1}, \bar{q}=q, \bar{t}=t,
\end{align*}
$$

where $\mathcal{L}$ is from (2.14), and we used the main defining property (2.7) of the representation from Theorem 2.1. The pairing remains non-degenerate when restricted to $W$-invariant polynomials.

Definition 3.1. The Fourier adjoints $\varphi(\mathcal{L}), \varphi(L)$ of $\mathbf{C}_{q, t}$-linear operators $\mathcal{L}, L$ acting respectively in $\mathbf{C}_{q, t}[x]$ or in $\mathbf{C}_{q, t}[x]^{W}$ are defined from the relations:

$$
\begin{align*}
\llbracket \mathcal{L}(f), g \rrbracket & =\llbracket f, \varphi(\mathcal{L})(g) \rrbracket, f, g \in C_{q, t}[x], \\
\llbracket L(f), g \rrbracket & =\llbracket f, \varphi(L)(g) \rrbracket, f, g \in \mathbf{C}_{q, t}[x]^{W} . \tag{3.2}
\end{align*}
$$

If $\mathcal{L}$ preserves $\mathbf{C}_{q, t}[x]^{W}$ then so does $\varphi(\mathcal{L})$ and $\varphi(L)=[\varphi(\mathcal{L})]_{\dagger}$, where $L=$ $[\mathcal{L}]_{\dagger}$ is the restriction of $\mathcal{L}$ to the invariant polynomials.

This anti-involution ( $\varphi^{2}=\mathrm{id}$ ) extends $\varphi$ from (1.17) by construction. If $f \in C_{q, t}[x]^{W}$, then $\varphi\left(L_{f}\right)=[\bar{f}(X)]_{\dagger}$. We arrive at the following theorem:

Duality Theorem 3.2. Given $b, c \in B_{-}$and the corresponding Macdonald's polynomials $p_{b}, p_{c}$,

$$
\begin{equation*}
p_{b}\left(t^{-\rho} q^{c}\right) p_{c}\left(t^{-\rho}\right)=\llbracket p_{b}, p_{c} \rrbracket=\llbracket p_{c}, p_{b} \rrbracket=p_{c}\left(t^{-\rho} q^{b}\right) p_{b}\left(t^{-\rho}\right) . \tag{3.3}
\end{equation*}
$$

To complete this theme we need to calculate $p_{b}\left(t^{-\rho}\right)$. The main step is the formula for $p^{\prime}\left(\left(t q_{v}\right)^{-\rho}\right)$ in terms of $p\left(t^{-\rho}\right)$, where (see (2.18))

$$
p=p_{b}, p^{\prime}=p_{b+r_{v}}^{t q_{v}}, p^{\prime}=\left(g_{v}^{t}(b)\right)^{-1} G_{v}^{t}(p) .
$$

Here and in similar formulas we show the dependence on $t$ omitting $q$ since the latter will be the same for all polynomials and operators. Let

$$
\overline{\mathcal{Y}}_{v}^{t}=\prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(\left(t_{a} Y_{a}\right)^{1 / 2}-\left(t_{a} Y_{a}\right)^{-1 / 2}\right) .
$$

Key Lemma 3.3.

$$
\begin{align*}
d_{v}^{t} p^{\prime}\left(\left(t q_{v}\right)^{-\rho}\right) & =\prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(t_{a}^{-1} y_{a}\left(t^{-\rho / 2} q^{+b / 2}\right)-y_{a}\left(t^{+\rho / 2} q^{-b / 2}\right)\right) p\left(t^{-\rho}\right), \\
\text { 4) } \quad d_{v}^{t} & =\prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(t_{a}^{-1} y_{a}\left(\left(t q_{v}\right)^{-\rho / 2}\right)-y_{a}\left(\left(t q_{v}\right)^{+\rho / 2}\right)\right) m_{-r_{v}}\left(t^{-\rho}\right) . \tag{3.4}
\end{align*}
$$

Proof. Let us use formula (2.19):

$$
\begin{align*}
& \llbracket\left(\mathcal{Y}_{v}^{t} \mathcal{X}_{v}^{t}\right)\left(\mathcal{X}_{v}^{t}\right)^{-1} \mathcal{L}_{\bar{p}^{\prime}} \mathcal{X}_{v}^{t} \rrbracket=\llbracket\left(\mathcal{Y}_{v}^{t} \mathcal{X}_{v}^{t}\right)\left[\left(\mathcal{X}_{v}^{t}\right)^{-1} \mathcal{L}_{\bar{p}^{\prime}}^{t} \mathcal{X}_{v}^{t}\right]_{\dagger} \rrbracket=  \tag{3.5}\\
& \llbracket\left(\hat{\mathcal{Y}}_{v}^{t} \hat{\mathcal{X}}_{v}^{t}\right) \bar{p}^{\prime}\left(\hat{Y}^{t q_{v}}\right) \rrbracket=\llbracket\left(\mathcal{Y}_{v}^{t} \mathcal{X}_{v}^{t}\right) \rrbracket p^{\prime}\left(\left(t q_{v}\right)^{-\rho}\right) .
\end{align*}
$$

On the other hand, it equals:

$$
\begin{align*}
& \llbracket \mathcal{Y}_{v}^{t} p^{\prime}(Y) \mathcal{X}_{v}^{t} \rrbracket=\llbracket \mathcal{Y}_{v}^{t} p^{\prime}\left(X^{t}\right) \mathcal{X}_{v}^{t} \rrbracket= \\
& \llbracket \mathcal{Y}_{v}^{t}\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket= \pm \llbracket \overline{\mathcal{Y}}_{v}^{t}\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket . \tag{3.6}
\end{align*}
$$

Here we applied the anti-involution $\varphi(\varphi(\mathcal{X})=\mathcal{Y}, \varphi(\mathcal{Y})=\mathcal{X})$, then went from the abstract 【】 to that from (2.12), and used Theorem 2.1. The last transformation requires special comment. We will justify it in a moment.

After this, one can use (2.16):

$$
\begin{align*}
& \left.\llbracket \overline{\mathcal{Y}}_{v}^{t}\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket=\llbracket\left(\overline{\mathcal{Y}}_{v}^{t} \mathcal{Y}_{v}^{t}\right) g_{v}^{t}(b)^{-1} p(x)\right) \rrbracket= \\
& g_{v}^{t}(b)^{-1}(\overline{\mathcal{Y}} \mathcal{Y})\left(t^{\rho} q^{-b}\right) \llbracket p(x) \rrbracket=  \tag{3.7}\\
& \prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(t_{a}^{-1} y_{a}\left(t^{-\rho / 2} q^{+b / 2}\right)-y_{a}\left(t^{+\rho / 2} q^{-b / 2}\right)\right) \llbracket p(x) \rrbracket .
\end{align*}
$$

Finally, $d_{v}^{t} \stackrel{\text { def }}{=} \pm \llbracket \mathcal{Y}_{v}^{t} \mathcal{X}_{v}^{t} \rrbracket$ can be determined from (3.7) and the relation $1=$ $p^{\prime}=g_{v}^{t}(b)^{-1} G_{v}^{t}\left(p_{b}^{t}\right)$ for $b=-r_{v}$, where $p_{-r_{v}}$ coincides with the monomial function $m_{-r_{v}}$ (it follows directly from the definition):

$$
\begin{equation*}
d_{v}^{t}=\prod_{a \in R_{+}^{\vee}, \nu_{a} \in v}\left(t_{a}^{-1} y_{a}\left(\left(t q_{v}\right)^{-\rho / 2}\right)-y_{a}\left(\left(t q_{v}\right)^{+\rho / 2}\right)\right) m_{-r_{v}}\left(t^{-\rho}\right) . \tag{3.8}
\end{equation*}
$$

Let us check that

$$
\llbracket\left(\overline{\mathcal{Y}}_{v}^{t}-l_{\epsilon}\left(w_{0}\right) \mathcal{Y}_{v}^{t}\right)\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket=0 \text { for } \quad \text { any } p^{\prime} \in \mathbf{C}[x],
$$

where $l_{\epsilon}\left(w_{0}\right)=\prod_{\nu} \epsilon_{\nu}^{l_{\nu}\left(w_{0}\right)}$,

$$
\epsilon=\left\{\epsilon_{\nu}=-1 \text { if } \nu \in v \text {, otherwise } \epsilon_{\nu}=1\right\}, \nu \in \nu_{R} \text {. }
$$

Following formula (4.18),[C2] we introduce the $t$-symmetrizers, setting

$$
\begin{align*}
\mathcal{P}_{v}^{t} & =\left(\pi_{v}^{t}\right)^{-1} \sum_{w \in W} \prod_{\nu}\left(\epsilon_{\nu} t_{\nu}^{1 / 2}\right)^{\epsilon_{\nu}\left(l_{\nu}(w)-l_{\nu}\left(w_{0}\right)\right)} T_{w}, \\
\pi_{v}^{t} & =\sum_{w \in W} \prod_{\nu}\left(\epsilon_{\nu} t_{\nu}^{1 / 2}\right)^{\epsilon_{\nu}\left(2 l_{\nu}(w)-l_{\nu}\left(w_{0}\right)\right)} . \tag{3.9}
\end{align*}
$$

It results from Proposition 3.5 and Corollary 4.7 (ibidem) that

$$
\mathcal{P}_{v}^{t}\left(\mathcal{X}_{v}^{t} p^{\prime}\right)=\mathcal{X}_{v}^{t} p^{\prime}, \hat{\mathcal{P}}_{v}^{t} \mathcal{P}_{v}^{t=0}=\hat{\mathcal{P}}_{v}^{t}, \text { if } t_{\nu}=1 \text { for } \nu \notin \nu_{R} .
$$

Hence

$$
\begin{aligned}
& \llbracket\left(\overline{\mathcal{Y}}_{v}^{t}-l_{\epsilon}\left(w_{0}\right) \mathcal{Y}_{v}^{t}\right)\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket=\llbracket\left(\overline{\mathcal{Y}}_{v}^{t}-l_{\epsilon}\left(w_{0}\right) \mathcal{Y}_{v}^{t}\right) \mathcal{P}_{v}^{t}\left(\mathcal{X}_{v}^{t} p^{\prime}(x)\right) \rrbracket= \\
& \llbracket\left(\mathcal{Y}_{v}^{t} \bar{p}^{\prime}(Y)\right) \mathcal{P}_{v}^{t}\left(\overline{\mathcal{X}}_{v}^{t}-l_{\epsilon}\left(w_{0}\right) \mathcal{X}_{v}^{t}\right) \rrbracket=\llbracket\left(\mathcal{Y}_{v}^{t} \bar{p}^{\prime}(Y)\right)\left\{\hat{\mathcal{P}}_{v}^{t} \mathcal{P}_{v}^{t=0}\left(\overline{\mathcal{X}}_{v}^{t}-l_{\epsilon}\left(w_{0}\right) \mathcal{X}_{v}^{t}\right)\right\} \rrbracket .
\end{aligned}
$$

The latter equals zero.
Let us take any set $k=\left\{k_{\nu_{1}} \geq k_{\nu_{2}}\right\} \in \mathbf{Z}_{+}$and put

$$
\begin{equation*}
t(k)=\left\{q^{2 k_{\nu} / \nu}\right\}, k \cdot r=\sum_{\nu} k_{\nu} r_{\nu}, p_{b}^{(k)}=p_{b}^{t(k)} . \tag{3.10}
\end{equation*}
$$

The remaining part of the calculation is based on the following chain of the shift operators that will be applied to $p_{b-k \cdot r}^{(0)}=m_{b-k \cdot r}$ one after another:

$$
\begin{equation*}
G_{\nu_{R}}^{(k-1)} G_{\nu_{R}}^{(k-2)} \cdots G_{\nu_{R}}^{(k-s)} G_{\nu_{1}}^{(k-s-e)} \cdots G_{\nu_{1}}^{(0)}, \tag{3.11}
\end{equation*}
$$

where $k_{\nu_{1}}=s+d, k_{\nu_{2}}=s, e=\left\{e_{\nu}\right\}, e_{\nu_{1}}=1, e_{\nu_{2}}=0, k-s=d e$, the set $\{1,1\}$ is denoted by 1 .

Lemma 3.3 gives that for a certain $D^{(k)}$ (which does not depend on $b$ ):

$$
\begin{align*}
& D^{(k)} p_{b}^{(k)}\left(t(k)^{-\rho}\right)=  \tag{3.12}\\
& m_{b-k \cdot r}(1) \prod_{a \in R_{+}^{\cup}, \nu_{a} \in v(i)}^{0 \leq i<s+d}\left(t(i)_{a} y_{a}\left(t(i)^{\rho / 2} q^{-b(i) / 2}\right)-y_{a}\left(t(i)^{-\rho / 2} q^{b(i) / 2}\right)\right),
\end{align*}
$$

where $t(i)=t(k(i)), b(i)=b-(k-k(i)) \cdot r$,

$$
k(i)=i e, v(i)=\nu_{1} \text { if } i<d, k(i)=i-d+d e, v(i)=\nu_{R} \text { if } i \geq d
$$

As to $D^{(k)}$, it equals the right hand side of (3.12) when $b=0$. We note that $t(i)_{a}=q^{2 j / \nu_{a}}$ for $j=k_{a}+i-s-d, k_{a}=k_{\nu_{a}}$, because $i \geq d$ if $\nu_{a} \neq \nu_{1}$ (and $0 \leq j<k_{a}$ ). The relation $(2 / \nu) \rho_{\nu}=r_{\nu}$ leads to the formulas:

$$
\begin{align*}
& t(i)^{\rho / 2} q^{-b(i) / 2}=q^{(k(i) \cdot r-b+(k-k(i)) \cdot r) / 2}=q^{k \cdot r-b / 2}, \\
& y_{a}\left(t(i)^{\rho / 2} q^{-b(i) / 2}\right)=q^{(k \cdot r-b, a) / 2} \tag{3.13}
\end{align*}
$$

Finally, we arrive at the following theorem:
Evaluation Theorem 3.4.

$$
\begin{gather*}
p_{b}^{(k)}\left(t(k)^{-\rho}\right)=  \tag{3.14}\\
\frac{m_{b-k \cdot r}(1)}{m_{-k \cdot r}(1)} \prod_{\alpha \in R_{+}, 0 \leq j<k_{\alpha}}\left(\frac{q^{\{(k \cdot r-b, \alpha)+j\} / \nu_{\alpha}}-q^{-\{(k \cdot r-b, \alpha)+j\} / \nu_{\alpha}}}{q^{\{(k \cdot r, \alpha)+j\} / \nu_{\alpha}}-q^{-\{(k \cdot r, \alpha)+j\} / \nu_{\alpha}}}\right) .
\end{gather*}
$$

Here $m_{b-k \cdot r}(1) / m_{-k \cdot r}(1)=|W(b-k \cdot r)| /|W(k \cdot r)| \in \mathbf{Z}_{+}$. It equals 1 for all $b \in B_{-}$when $\prod_{\nu} k_{\nu} \neq 0$. Assuming this we have:

$$
\begin{align*}
\quad p_{b}^{t(k)}\left(t(k)^{-\rho}\right) & =  \tag{3.15}\\
q^{(k \cdot r, b)} \prod_{\alpha \in R_{+}, 0 \leq j<\infty} & \left(\frac{\left(1-q^{2\{(k \cdot r-b, \alpha)+j\} / \nu_{\alpha}}\right)\left(1-t_{\alpha} q^{2\{(k \cdot r, \alpha)+j\} / \nu_{\alpha}}\right)}{\left(1-t_{\alpha} q^{2\{(k \cdot r-b, \alpha)+j\} / \nu_{\alpha}}\right)\left(1-q^{\left.2\{(k \cdot r, \alpha)+j\} / \nu_{\alpha}\right)}\right)} .\right.
\end{align*}
$$

The limit of (3.15) as one of the $k_{\nu}$ approaches zero exists and coincides with (3.14). Since both sides of this formula are rational functions in $t(k)$ and $q$ we get (0.7) (cf. Theorem 2.4).

We note that actually this paper does not depend very much on the definition of the Macdonald polynomials from the Introduction. We can eliminate $\mu$ introducing these polynomials as the eigenfunctions of the $L$-operators (formula (2.16)). Therefore it is likely that paper [C4] can be extended to give a "difference-elliptic" Weyl dimension formula.

## 4. Discretization, applications

Continuing the same line let us establish the recurrence relations for the Macdonald polynomials generalizing the three-term relation for the $q$ ultraspherical polynomials (Askey, Ismail) and the Pieri rules. We need to go to the lattice version of the considered functions and operators. The discretization of functions $g(x), x \in \mathbf{C}^{n}$ and ( $q$-)difference operators is defined as
follows:

$$
\begin{align*}
& { }^{\delta} g(b)=g\left(q^{b} t^{-\rho}\right),\left({ }^{\delta} a\left({ }^{\delta} g\right)\right)(b)={ }^{\delta} g(b-a), a, b \in B \\
& \left({ }^{\delta} X_{a}\left({ }^{\delta} g\right)\right)(b)=x_{a}\left(q^{b} t^{-\rho}\right){ }^{\delta} g(b)=q^{(a, b)} \prod_{\nu} t_{\nu}^{-\left(a, \rho_{\nu}\right)}{ }^{\delta} g(b) \tag{4.1}
\end{align*}
$$

It is a homomorphism. The image is functions on $B$ and operators acting on such functions.

Given an arbitrary set of functions $\left\{\phi_{b}(), b \in B\right\}$, we can also apply difference operators to the sufficies:

$$
\begin{equation*}
\delta(g a)\left(\sum_{b \in B} c_{b} \phi_{b}()\right)=\sum_{b \in B} c_{b} g\left(q^{b} t^{-\rho}\right) \phi_{b-a}(), c_{b} \in \mathbf{C} . \tag{4.2}
\end{equation*}
$$

It is an anti-homomorphism, i.e.

$$
{ }_{\delta}(G H)={ }_{\delta} H_{\delta} G \text { for difference operators } G, H
$$

From now on we will mostly use the renormalized Macdonald polynomials $\pi_{b}(x) \stackrel{\text { def }}{=} p_{b}(x) / p_{b}\left(t^{-\rho}\right)$ and their discretizations:

$$
\pi_{b}(c) \stackrel{\text { def }}{=}{ }^{\delta} p_{b}(c) /{ }^{\delta} p_{b}(0)=p_{b}\left(q^{c} t^{-\rho}\right) / p_{b}\left(t^{-\rho}\right)=\pi_{c}(b) \text { for } b, c \in B_{-}
$$

Given a symmetric polynomial $f \in \mathbf{C}[x]^{W}$, we construct the operator $L_{f}$, go to its discretization ${ }^{\delta} L_{f}$, and finally introduce the recurrence operator $\Lambda_{f}={ }_{\delta} L_{f}$ acting on the sufficies $b \in B$ of any $\mathbf{C}$-valued functions $\phi_{b}()$. We write $L_{a}, \Lambda_{a}$ when $f$ is the monomial symmetric function $m_{a}, a \in B_{-}$.

Recurrence Theorem 4.1. For arbitrary $a, b \in B_{-}, f \in \mathbf{C}[x]^{W}$,

$$
\begin{equation*}
\Lambda_{f}\left(\pi_{b}(x)\right)=\bar{f}(x) \pi_{b}(x), \quad \Lambda_{a}\left(\pi_{b}(x)\right)=\bar{m}_{a}(x) \pi_{b}(x) \tag{4.3}
\end{equation*}
$$

where $\bar{f}(x)=f\left(x^{-1}\right)$. The operators $\Lambda($ acting on $b)$ do not produce $\pi_{c}$ for $c \notin B_{-}$.

Proof. We can rewrite (2.16) as follows:

$$
\begin{equation*}
{ }^{\delta} L_{f}\left({ }^{\delta} p_{b}\right)=\bar{f}\left(q^{b} t^{-\rho}\right)^{\delta} p_{b} \tag{4.4}
\end{equation*}
$$

Replacing $p$ by $\pi$ and using the duality we yield:

$$
\begin{align*}
{ }^{\delta} L_{f}\left(\pi_{b}(c)\right) & =\bar{f}\left(q^{b} t^{-\rho}\right)^{\delta} \pi_{b}(c) \\
\Lambda_{f}\left(\pi_{b}(c)\right) & ={ }^{\delta} \bar{f}(c)^{\delta} \pi_{b}(c) \tag{4.5}
\end{align*}
$$

and (4.3) if we can ensure that $\Lambda_{f}$ does not create polynomials $p_{c}$ with the indices apart from $B_{-}$. The latter can be checked due to Lemma 2.4 from [C2]. When $a=-b_{r}, r \in O^{*}$ it results directly from Proposition 2.5. We will give below another proof independent of [C2].

To compare the theorem with the classical results about the products of the characters (tensor products of the irreducible representations) take $t=q$. Then corresponding Macdonald polynomials can be obtained from the monomial functions by means of the shift operator for $t=1$ which is very simple and leads to the Weyl formula (this operator was introduced in [BG]). Hence they are the characters of the irreducible representations. For minuscule $-a$ in the case of $A_{n}$ (all dominant weights are their algebraic combinations) the theorem was obtained by Macdonald, Koornwinder, and then in [EK2]. As to other root systems (except $B C_{1}$ and certain special $t, q$ ), it seems to be new.

The theorem has many applications. For instance we can get another prove of formula (0.7) and moreover calculate the norms of $\pi_{b}$. To demonstrate this let

$$
\begin{align*}
L_{a} & =\sum_{b} b g_{a}^{b}(X), \Lambda_{a}=\sum_{b} \delta\left(g_{a}^{b}\right) \delta b, \quad a \in B_{-}, \\
L_{a} & =\sum_{b} f_{a}^{b}(X) b, \Lambda_{a}=\sum_{b} \delta b \delta\left(f_{a}^{b}\right), w_{0}(a) \succeq b \succeq a . \tag{4.6}
\end{align*}
$$

Then $g_{a}^{b}\left(q^{c} t^{-\rho}\right)=f_{a}^{b}\left(q^{c+b} t^{-\rho}\right)$. The $w_{0}$-invariance of $L$ leads to the following relations:

$$
\begin{equation*}
g_{a}^{w_{0}(b)}\left(t^{\rho}\right)=g_{a}^{b}\left(t^{-\rho}\right), \quad f_{a}^{w_{0}(b)}\left(t^{\rho}\right)=f_{a}^{b}\left(t^{-\rho}\right) . \tag{4.7}
\end{equation*}
$$

We mention that $\left\langle\mu_{1} \bar{m}_{a}\right\rangle=g_{a}^{0}\left(t^{ \pm \rho}\right)=f_{a}^{0}\left(t^{ \pm \rho}\right)=\left\langle\mu_{1} m_{a}\right\rangle$ are the coefficients of the symmetrization of $\mu_{1}$. Their good description is one of the main open problems in the Macdonald theory (we will consider it in the next paper).

Proposition 4.2. Setting $b^{o} \stackrel{\text { def }}{=}-w_{0}(b)$,

$$
\begin{align*}
& \sum_{b} f_{a}^{b}\left(t^{\rho}\right) \pi_{b^{o}}=m_{a^{o}}=\sum_{b} g_{a}^{b}\left(t^{-\rho}\right) \pi_{b^{o}}\left\langle\pi_{b}, \pi_{b}\right\rangle^{-1}  \tag{4.8}\\
& \left\langle\pi_{b}, m_{a}\right\rangle=g_{a}^{b}\left(t^{-\rho}\right)=f_{a}^{b}\left(t^{\rho}\right)\left\langle\pi_{b}, \pi_{b}\right\rangle, f_{b}^{b}\left(t^{\rho}\right)=p_{b}\left(t^{-\rho}\right),  \tag{4.9}\\
& \left\langle\pi_{b}, p_{b}\right\rangle=\left\langle\pi_{b}, m_{b}\right\rangle=p_{b}\left(t^{-\rho}\right)\left\langle\pi_{b}, \pi_{b}\right\rangle=g_{b}^{b}\left(t^{-\rho}\right)= \\
& x_{b}\left(t^{-\rho}\right) \prod_{a \in R_{+}^{\vee}} \prod_{1 \leq j \leq-\left(a^{\vee}, b\right)}\left(\frac{1-q_{a}^{j} t_{a}^{-1} x_{a}\left(t^{\rho}\right)}{1-q_{a}^{j} x_{a}\left(t^{\rho}\right)}\right) . \tag{4.10}
\end{align*}
$$

Proof. One has: $\left\langle m_{a^{o}}, \pi_{b^{\circ}}\right\rangle=\left\langle\pi_{b}, m_{a}\right\rangle=\left\langle\mu_{1} \pi_{b} \bar{m}_{a}\right\rangle=\left\langle\mu_{1} \Lambda_{a} \pi_{b}\right\rangle=g_{a}^{b}\left(t^{-\rho}\right)$. On the other hand, $m_{a^{o}}=\pi_{0} m_{a^{o}}=\sum_{b} f_{a}^{b}\left(t^{\rho}\right) \pi_{b^{o}}$. It gives all the relations for $\{g, f\}$. The functions $g_{b}^{b}, f_{b}^{b}$ were calculated in Proposition 2.5.

We note that (4.10) can be directly deduced from the formulas for the norms of the Macdonald polynomials $\left\{p_{b}\right\}$ from [C2] adapted to the present
paper. Indeed, given $b \in B_{-}$,

$$
\begin{align*}
\left\langle\mu p_{b} \bar{p}_{b}\right\rangle= & \prod_{a \in R_{+}^{\vee}}\left(\left(1-t_{a} x_{a}\left(t^{\rho}\right)\right) /\left(1-x_{a}\left(t^{\rho}\right)\right)\right) \\
\prod_{a \in R_{+}^{\vee}, 0 \leq j<\infty} & \left(\frac{\left(1-q_{a}^{j+1} x_{a}\left(t^{\rho} q^{-b}\right)\right)\left(1-q_{a}^{j} x_{a}\left(t^{\rho} q^{-b}\right)\right)}{\left(1-t_{a} q_{a}^{j} x_{a}\left(t^{\rho} q^{-b}\right)\right)\left(1-t_{a}^{-1} q_{a}^{j+1} x_{a}\left(t^{\rho} q^{-b}\right)\right)}\right) . \tag{4.11}
\end{align*}
$$

The necessary analitical continuation is precisely as in [EK2] (the remark after Theorem 3.9). We must divide them by $\langle\mu\rangle$ to go to the scalar product (0.4):

$$
\begin{align*}
& \left\langle p_{b}, p_{b}\right\rangle=\prod_{a \in R_{+}^{\vee}} \prod_{0 \leq j<-\left(a^{\vee}, b\right)} \\
& \left(\frac{\left(1-q_{a}^{j+1} t_{a}^{-1} x_{a}\left(t^{\rho}\right)\right)\left(1-q_{a}^{j} t_{a} x_{a}\left(t^{\rho}\right)\right)}{\left(1-q_{a}^{j} x_{a}\left(t^{\rho}\right)\right)\left(1-q_{a}^{j+1} x_{a}\left(t^{\rho}\right)\right)}\right) . \tag{4.12}
\end{align*}
$$

Then it is necessary just to divide by

$$
\begin{equation*}
p_{b}\left(t^{-\rho}\right)=x_{b}\left(t^{\rho}\right) \prod_{a \in R_{+}^{\vee}} \prod_{0 \leq j<-\left(a^{\vee}, b\right)}\left(\frac{1-q_{a}^{j} t_{a} x_{a}\left(t^{\rho}\right)}{1-q_{a}^{j} x_{a}\left(t^{\rho}\right)}\right) . \tag{4.13}
\end{equation*}
$$

Schwartz functions. We will use Macdonald's polynomials to construct the eigenfunctions of the Fourier transform which are also pairwise orthogonal with respect to the pairing 【, 】. The Gaussian comes naturally in this stuff because of the following theorem.

Theorem 4.3. i) Adding $q^{1 / 2 m}$, the following maps can be uniquely extended to automorphisms of $\mathfrak{H}$, preserving each of $T_{1}, \ldots, T_{n}, t$ and $q$ :

$$
\begin{align*}
& \tau_{+}: X_{b} \rightarrow X_{b}, \quad Y_{r} \rightarrow X_{r} Y_{r} q^{-\left(b_{r}, b_{r}\right) / 2}, \quad Y_{\theta} \rightarrow X_{0}^{-1} T_{0}^{-2} Y_{\theta}, \\
& \tau_{-}: Y_{b} \rightarrow Y_{b}, \quad X_{r} \rightarrow Y_{r} X_{r} q^{\left(b_{r}, b_{r}\right) / 2}, X_{\theta} \rightarrow T_{0} X_{0} Y_{\theta}^{-1} T_{0}, \\
& \omega: Y_{b} \rightarrow X_{b}^{-1}, \quad X_{r} \rightarrow X_{r}^{-1} Y_{r} X_{r} q^{\left(b_{r}, b_{r}\right)}, \quad X_{\theta} \rightarrow T_{0}^{-1} Y_{\theta}^{-1} T_{0},  \tag{4.14}\\
& \text { where } b \in B, \quad r \in O^{*}, \quad X_{0}=q X_{\theta}^{-1} .
\end{align*}
$$

ii) The above maps give automorphisms of $\mathfrak{H}$ and the (elliptic braid) group $\mathfrak{B}$ generated by the elements $\left\{X_{b}, Y_{b}, T_{i}, \pi_{r}, q^{1 / 2 m}\right\}$ satisfying the relations $(i)$ (vi) from Definition 1.1 and (1.12). Let $\boldsymbol{\mathfrak { A }}_{o}$ be the group of its automorphisms modulo the conjugations by the elements from the center $Z(\mathbf{B})$ of the group $\mathbf{B}$ generated by $\left\{T_{1}, \ldots, T_{n}\right\}$. Considering the images of $\varepsilon\left(\right.$ see (1.20)), $\tau_{ \pm}, \omega$ in $\mathfrak{A}_{o}$ we obtain the homomorphism $G L_{2}(\mathbf{Z}) \rightarrow \mathfrak{A}_{o}$ :

$$
\begin{align*}
& \left(\begin{array}{cc}
0 & -1 \\
-1 & 0
\end{array}\right) \rightarrow \varepsilon,\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \rightarrow \tau_{+}, \\
& \left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \rightarrow \omega,\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right) \rightarrow \tau_{-} \tag{4.15}
\end{align*}
$$

iii) The automorphism $\tau_{-}$leaves the left ideal I from 2.7 invariant and therefore acts in $\mathbf{C}_{q, t}[x]$ identified with $\mathfrak{H} / I$.

Proof. The theorem can be deduced from the topological interpretation of $\mathfrak{B}$ from [C1] as the fundamental group of the elliptic configuration space (a proper version of the product of $n$ copies of an elliptic curve without the complexifications of the root hyperplanes and divided by $W$ ). The standard action of the $G L_{2}(\mathbf{Z})$ on the periods of the elliptic curve results in the formulas for $\varepsilon, \tau_{+}$. The remaining elements can be expressed in terms of these two:

$$
\begin{equation*}
\tau_{-}=\varepsilon \tau_{+} \varepsilon=\varphi \tau_{+} \varphi, \quad \omega=\tau_{+}^{-1} \tau_{-} \tau_{+}^{-1}=\tau_{-} \tau_{+}^{-1} \tau_{-} \tag{4.16}
\end{equation*}
$$

Since $G L_{2}(\mathbf{Z})$ preserves the origin of the elliptic curve the relations from this group are fulfilled up to conjugations by the elements from B. However the latter elements must belong to the center because $\tau_{ \pm}, \omega$ fix the elements $T_{1}, \ldots, T_{n}$. In the case of $A_{n}$, the calculations are in fact due to J. Birman.

One can eliminate the topology and check the statements directly. Because we have already (4.16) it suffices to calculate that $\omega^{4}$ is the conjugation by the element $T_{w_{0}}^{2}$, which follows from (4.14). Topologically, a nontrivial step is to see that $\tau_{+}$(which is evidently a pullback of $\left(\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right)$ ) is really an automorphism of $\mathfrak{B}$ (for $A_{n}$ it is not difficult). Fortunately we can make it algebraic as well.

Setting $x_{b}=q^{z_{b}}, z_{a+b}=z_{a}+z_{b}, z_{i}=z_{b_{i}}, a\left(z_{b}\right)=z_{b}-(a, b), a, b \in \mathbf{R}^{n}$, we introduce the Gaussian function $\gamma=q^{\sum_{i=1}^{n} z_{i} z_{\alpha_{i}} / 2}$, which can be considered as a formal series in $x_{i}-1, \log q$ and satisfies the following (defining) difference relations:

$$
\begin{align*}
& b_{j}(\gamma)=q^{(1 / 2) \sum_{i=1}^{n}\left(z_{i}-\left(b_{j}, b_{i}\right)\right)\left(z_{\alpha_{i}}-q_{i}^{j}\right)}= \\
& \gamma q^{-z_{j}+\left(b_{j}, b_{j}\right) / 2}=x_{j}^{-1} \gamma q^{\left(b_{j}, b_{j}\right) / 2} \text { for } 1 \leq j \leq n \tag{4.17}
\end{align*}
$$

The Gaussian function commutes with $T_{j}$ for $1 \leq j \leq n$ because it is $W$-invariant. When $b_{r}$ are minuscule ( $r \in O^{*}$ ), we use directly formulas (1.12, 2.5) to check that

$$
\gamma(X) Y_{r} \gamma(X)^{-1}=X_{r} q^{-\left(b_{r}, b_{r}\right) / 2} Y_{r}=\tau_{+}\left(Y_{r}\right) .
$$

A straightforward calculation gives that

$$
\begin{align*}
& \gamma(X) T_{0} \gamma(X)^{-1}=\tau_{+}\left(T_{0}\right)=X_{0}^{-1} T_{0}^{-1},  \tag{4.18}\\
& \tau_{-}\left(T_{0}\right)=T_{0}, \omega\left(T_{0}\right)=X_{\theta}^{-1} Y_{\theta}^{-1} T_{0},
\end{align*}
$$

which yields (4.14).
Here the multiplication by $\gamma$ is not well defined (only the conjugation). It preserves the bigger space of meromorphic functions in $z_{b}$ which is an $\mathfrak{H}$ module as well. In $\mathbf{C}_{q, t}[x]$ the picture is opposite. The automorphism $\tau_{-}$is inner since it fixes $\{Y, T\}$.

We note that the automorphisms $\left\{\varepsilon, \tau_{ \pm}, \omega\right\}$ are projectively unitary with respect to the Macdonald pairing. Indeed, they send the unitary operators $\{X, Y, T, q, t\}$ generating $\mathfrak{H}$ to unitary ones.

The next claim is that the Schwartz functions $\left\{\gamma^{-1} p_{b}, b \in B_{-}\right\}$defined for the Macdonald polynomials $\left\{p_{b}\right\}$ are eigenfunctions of the Fourier transform and pairwise orthogonal with respect to the Fourier pairing 【, 】. We will reformulate and prove it algebraically representing Schwartz functions as eigenfunctions of self-adjoint operators with respect to $\varphi, \varepsilon$.

Proposition 4.4. The $W$-invariant operators $\mathcal{L}_{f}^{\gamma} \stackrel{\text { def }}{=} \gamma^{-1} \mathcal{L}_{f} \gamma$ defined for $f \in \mathbf{C}[x]^{W}$ obey the following properties (see Theorem 2.3):

$$
\begin{aligned}
& \mathcal{L}_{f}^{\gamma}\left(\gamma^{-1} p_{b}\right)=f\left(t^{\rho} q^{-b}\right)\left(\gamma^{-1} p_{b}\right) \\
& \varphi\left(\mathcal{L}_{f}^{\gamma}\right)=\mathcal{L}_{f}^{\gamma}, \varepsilon\left(\mathcal{L}_{f}^{\gamma}\right)=\mathcal{L}_{f}^{\gamma} .
\end{aligned}
$$

The corresponding eigenvalues (for all $f$ ) distinguish different $\gamma^{-1} p_{b}$.

There is another way to make the statement about the action of the Fourier transform on the Schwartz functions algebraic. The following formula gives this and moreover the exact eigenvalues (up to proportionality):

$$
\begin{equation*}
L_{p_{b}}\left(\gamma^{-1}\right)=q^{-(b, b) / 2} x_{b}\left(t^{\rho}\right) p_{b} \gamma^{-1}, b \in B_{-} . \tag{4.19}
\end{equation*}
$$

The coefficents of proportionality are calculted by means of Proposition 2.5 (see also the end of the paper).

One can introduce formally the Fourier transform $F(f)$ of a function $f$ by means of the relation (see (0.4) and (3.1)):

$$
\begin{equation*}
\langle g, F(f)\rangle=\llbracket g, f \rrbracket=\mathcal{L}_{\bar{g}}(f)\left(t^{-\rho}\right) \tag{4.20}
\end{equation*}
$$

valid for all polynomials $g$ (or even for more general classes of functions). The associated transformation on the operators is exactly $\varepsilon$ from (1.20). When $t=1$ it is the classical Fourier transform:

$$
F(f)(\lambda)=\operatorname{const} \int_{\mathbf{R}^{n}} q^{\Sigma z_{i} \lambda_{\alpha_{i}}} f^{+}(z) d z_{1} \ldots d z_{n}
$$

where ${ }^{+}$is the formal complex conjugation $q^{+}=q^{-1}, t^{+}=t^{-1}$. It is the standard conjugation if $q, t$ belong to the unit circle. The appearance of $f^{+}$ instead of $f$ makes the transform involutive (and is almost inevitable in the difference theory).

Hopefully $\tau_{ \pm}$become inner and the above claims about Schwartz functions can be made rigorous after the discretization (see the beginning of the section). When $q$ is a root of unity the procedure is completely algebraic.

## 5. Roots of unity

Let us assume that $q$ is a primitive $N$-th root of unity for $N \in \mathbf{N}$ and consider $t$ as an indeterminate parameter. More precisely, we will operate over the field $\mathbf{Q}_{t}^{0} \stackrel{\text { def }}{=} \mathbf{Q}\left(q_{0}, t\right)$ where we fix $q_{0}$ such that $q_{0}^{2 m}=q$ ( $q_{0}$ belongs to a proper extension of $\mathbf{Q}$ ). Actually all formulas will hold even over the localization of $\mathbf{Z}\left[q_{0}, t\right]$ by $t^{r_{1}} q^{s_{1}}\left(1-t^{r_{2}} q^{s_{2}}\right) \neq 0, r_{i}, s_{i} \in \mathbf{Z}$. The pairing

$$
\begin{equation*}
B \times B \ni a \times b \rightarrow q^{(a, b)} \stackrel{\text { def }}{=} q_{0}^{2 m(a, b)} \tag{5.1}
\end{equation*}
$$

acts through $B_{N} \times B_{N}$, where $B_{N} \stackrel{\text { def }}{=} B / K_{N}, K_{N}$ is its radical.
Following the previous section we restrict the functions $\left\{m_{b}\right\}$ and the operators $\left\{L_{b}=L_{m_{b}}\right\}$ to the lattice $B$ using the pairing (5.1). We need to consider the quotient $B_{N}$ only. The $L$-operators are well defined over $\mathbf{Q}_{t}^{0}$ since their denominators are products of the binomials $\left(x_{a} q^{k}-1\right)$ for $a \in$ $R^{\vee}, k \in \mathbf{Z}$. The latter remain non-zero when evaluated at $q^{b} t^{-\rho}$ since $(a, \rho) \neq 0$ $\left(x_{a}\left(t^{\rho}\right)\right.$ always contain $\left.t\right)$. More exact information about the properties of these coefficients can be extracted from Proposition 4.2.

Let $B_{-}(N)$ be a fundamental domain of the group $K_{N}$. It means that the map $B_{-}(N) \rightarrow B_{N}$ is an isomorphism. Further we identify these two sets, putting

$$
\begin{equation*}
B_{-}(N)=\left\{\beta^{1}, \ldots, \beta^{d}\right\}=B_{N}, \text { where } d=\left|B_{N}\right|, \beta^{1}=0, \tag{5.2}
\end{equation*}
$$

and denote the image of $b \in B$ in $B_{N}$ by $b^{\prime}$. We note that ${ }^{\delta} m_{a}(c)=m_{a}\left(t^{-\rho} q^{c}\right)$ for $a \in A$ separate $\left\{\beta^{i}\right\}$. It is true since $w\left(t^{\rho} q^{-b}\right) \neq t^{\rho} q^{-c}$ for any $w \in W$ if $b^{\prime} \neq c^{\prime}$ (we remind that $t$ is generic). We assume that $-w_{0}\left(B_{-}(N)\right)=B_{-}(N)$ for the longest element $w_{0}$.

Let us consider (temporarily) the case when $N$ is coprime with the order $|B / A|=|O|$ taking $q_{0}^{2}=q^{1 / m}$ in the $N$-th roots of unity. Then $K_{N}=N P \cap B$ for the weight lattice $P=\oplus_{i=1}^{n} \mathbf{Z} \omega_{i}$ generated by the $\omega_{i}$ (dual to $a_{i}$ ). We can take the following fundamental domain

$$
\begin{align*}
& B_{-}(N)=\left\{b=-\sum_{i=1}^{n} k_{i} b_{i} \in B_{-}\right\} \text {such that }  \tag{5.3}\\
& 0 \leq k_{i}<N \text { if }\left(2 / \nu_{i}, N\right)=1,0 \leq k_{i}<\nu_{i} N / 2 \text { otherwise. }
\end{align*}
$$

We remind that $2 / \nu_{i}=2 /\left(\alpha_{i}, \alpha_{i}\right)=1,2,3$ (see Introduction).
Let us demonstrate that the Macdonald polynomials $p_{b}$ are well defined for $b \in B_{-}(N)$ (later we will see that they always exist). We introduce them directly from (2.16), using that the $L$-operators preserve any subspaces

$$
U_{b}=\oplus_{c \succ b} \mathbf{Q}_{t}^{0} m_{c}, \quad \text { and } U_{b} \oplus \mathbf{Q}_{t}^{0} m_{b}, b, c \in B_{-} .
$$

Here it is necessary to check that given $B_{-} \ni c \succ b$, there exists at least one $a \in B_{-}$such that $m_{a}\left(t^{\rho} q^{-b}\right) \neq m_{a}\left(t^{\rho} q^{-c}\right)$. Then the eigenvalues of $L$ will separate $p_{b}$ from the elements from $U_{b}$ and we can argue by induction. If $A_{+} \ni c-b \in N P$ then there is $a_{i}(1 \leq i \leq n)$ such that $\left(c-b, a_{i}\right)>0$ (the form $($,$) is positive). Since (A, A) \subset\left(2 / \nu_{i}\right) \mathbf{Z}$,

$$
\left(-b, a_{i}\right) \geq-\left(b, a_{i}\right)+\left(c, a_{i}\right) \geq\left(2 N / \nu_{i}\right) /\left(2 / \nu_{i}, N\right)
$$

which contradicts (5.3).
We note that the norms $\left\langle p_{b}, p_{b}\right\rangle$ are non-zero for all these polynomials which results from (4.12). The same holds for $\left\langle\pi_{b^{\prime}}, \pi_{b^{\prime}}\right\rangle, b^{\prime} \in B_{-}(N)$, due to (0.7).

The discretizations of $\pi_{b^{\prime}}$ are well defined too and $\pi_{b^{\prime}}(c)$ depends only on the image $c^{\prime}$ because $\pi_{b^{\prime}}$ is a linear combinations of $m_{a}, a \in B_{-}$. Finally, the $\left\{\pi_{\beta^{i}}\left(c^{\prime}\right)\right\}$ form a basis in the space $V_{N} \stackrel{\text { def }}{=} \operatorname{Funct}\left(B_{N}, \mathbf{Q}_{t}^{0}\right)$ of all $\mathbf{Q}_{t}^{0}$-valued functions on $B_{N}$. Indeed they are non-zero and the action of the $\left\{{ }^{\delta} L_{a}\right\}$ ensures that they are linearly independent.

The end of the proof of Theorem 4.1. First of all, let us rewrite formally relation (4.3) for $f=m_{a}$ as follows:

$$
\begin{equation*}
\bar{m}_{a}(x) \pi_{b}(x)=\Lambda_{a}^{-}\left(\pi_{b}(x)\right)+\sum_{e \notin B_{-}} M_{a b}^{e} \pi_{e}(x) \tag{5.4}
\end{equation*}
$$

Here $e$ form a finite set $E=E(a, b)\left(E \cap B_{-}=\emptyset\right), M_{a b}^{e}$ are rational functions of $q, t$. The truncation $\Lambda_{a}^{-}$of $\Lambda_{a}$ is uniquelly determined by the condition that it does not contain the shifts moving $b$ to elements apart from $B_{-}$. Assuming that $N$ is sufficiently $\operatorname{big}\left(B_{-}(N)\right.$ must contain $b, b+a$ and $c \in B_{-}$such that $\left.c \succeq b-w_{0}(a)\right)$ the discretization gives the relation (see (4.5)):

$$
\begin{equation*}
\bar{m}_{a}(c) \pi_{b}(c)=\Lambda_{a}^{-}\left(\pi_{b}(c)\right)+\sum_{e \notin B_{-}} M_{a b}^{e} \pi_{e}(c), c \in B_{-}(N) \tag{5.5}
\end{equation*}
$$

for $m_{a}(c)={ }^{\delta} m_{a}(c)$. Here $\pi_{e}=\pi_{e^{\prime}}$ for $e \in E$. This substitution was impossible before the discretization. We remind that the formula with $\pi_{c}(e)$ in place of $\pi_{e}(c)$ is always true. Because $c$ is taken from $B_{-}(N)$ the discretization of $\pi_{c}$ exists. Therefore we can go from $e$ to $e^{\prime}$, and then replace $\pi_{c}\left(e^{\prime}\right)$ by $\pi_{e^{\prime}}(c)$. As to $M_{a b}^{e}$, they are the values of the coefficients of ${ }^{\delta} L_{a}$ and are also well-defined when $q^{N}=1$ (enlarging $N$ we can get rid of singularities in $q$ even if $M$ are arbitrary rational).

On the other hand :

$$
\begin{equation*}
\bar{m}_{a}(x) \pi_{b}(x)=\sum_{e \in B_{-}} K_{a b}^{e} \pi_{e}(x) \tag{5.6}
\end{equation*}
$$

where the coefficients $K_{a b}^{e}$ are rational functions of $q, t,\{e\}$ form a finite set $E_{-}=E_{-}(a, b)$. The discretization gives that

$$
\begin{equation*}
\bar{m}_{a}(c) \pi_{b}(c)=\sum_{e \in B_{-}} K_{a b}^{e} \pi_{e}(c), c \in B_{-}(N) \tag{5.7}
\end{equation*}
$$

We pick $N$ to avoid possible singularities.
Since $N$ is sufficiently big, the eigenvalues of the $L$-operators distingwish all $\pi_{e^{\prime}}(c)$ for $e \in E \cup E_{-}$. It holds only for generic $t$ (say, when $t=1$ it is wrong). Comparing (5.5) and (5.7) we conclude that $M_{a b}^{e}=0$ for all $e \in E$, when $q^{N}=1$. Using again that $N$ is arbitrary (big enough, coprime with $|O|$ ) we get that the actions of $\Lambda_{a}^{-}$and $\Lambda_{a}$ coincide on $\pi_{b}$, i.e. the latter operator does not create the indices not from $B_{-}$.

Let us go back to the general case (we drop the condition $(N,|O|)=1$ ). Once the Recurrence Theorem has been established we can use Proposition 4.2 without any reservation. It readily gives that the Macdonald polynomials $p_{b}$ and ${ }^{\delta} \pi_{b}$ are well defined for arbitrary $b \in B_{-}$because $f_{b}^{b}\left(t^{-\rho}\right) \neq 0$ (Proposition 2.5). Moreover, ${ }^{\delta} \pi_{b}={ }^{\delta} \pi_{c}$ if and only if $b^{\prime}=c^{\prime}$, and the restricted Macdonald polynomials $\pi_{\beta^{i}}\left(c^{\prime}\right), 1 \leq i \leq d$ (see (5.2)) form a basis in $V_{N}=\operatorname{Funct}\left(B_{N}, \mathbf{Q}_{t}^{0}\right)$. Indeed, $\pi_{\beta^{i}}\left(c^{\prime}\right)$ are eigenvectors of the ${ }^{\delta} L$-operators separated by the eigenvalues. They are always non-zero since $\pi_{b}(0)=1$. Hence they are linearly independent over $\mathbf{Q}_{t}^{0}$ and form a basis in $V_{N}$. Every ${ }^{\delta} \pi_{b}$ is an $L$-eigenvector and coincides with one of them (when $\beta^{i}=b^{\prime}$ ). Similarly, ${ }^{\delta} L_{\pi_{b}}={ }^{\delta} L_{\pi_{c}}$ if and only if $b^{\prime}=c^{\prime}$ because the latter condition is necessary and sufficient to ensure the coincidence of the sets of eigenvalues. We will also use the basis of the delta-functions $\delta_{\beta^{i}}\left(\beta^{j}\right) \stackrel{\text { def }}{=} \delta_{i j}$ separated by the action of $\left\{{ }^{\delta} m_{a}\right\}$.

Let $\mathcal{A}$ be the algebra of the elements of $\mathfrak{H}$ commuting with $\left\{T_{1}, \ldots, T_{n}\right\}$. All the (anti-)automorphisms under consideration preserve it.

Proposition 5.1. The discretization map $\mathcal{A} \ni A \rightarrow A_{\dagger}=[\hat{A}]_{\dagger} \rightarrow{ }^{\delta} A_{\dagger}$ supplies $V_{N}$ with the structure of an ${ }^{\delta} \mathcal{A}$-module which is irreducible. The algebra ${ }^{\delta} \mathcal{A}$ is generated by the discretizations of $L_{a}, m_{a}$ for $a \in A$. The Fourier pairing is well defined on $V_{N}$ and induces the anti-involution $\varphi\left(\bar{m}_{a} \rightarrow L_{a} \rightarrow\right.$ $\bar{m}_{a}$ ).

Proof. The radical of the Fourier pairing (3.1) contains the kernel of the discretization $\operatorname{map} \mathbf{Q}_{t}^{0}[x]^{W} \rightarrow V_{N}$. Its restriction to $V_{N}$ is non-degenerate since

$$
\begin{equation*}
\Pi=\left(\pi_{i j}\right), \quad \text { where } \quad \pi_{i j}=\llbracket \pi_{\beta^{i}}, \pi_{\beta^{j}} \rrbracket=\pi_{\beta^{i}}\left(\beta^{j}\right) \tag{5.8}
\end{equation*}
$$

is the matrix connecting the bases $\{\pi\}$ and $\{\delta\}$. The coresponding antiinvolution $\varphi$ transposes $\bar{m}_{a}$ and $L_{a}$. Thus $V_{N}$ is semi-simple.

If $V_{N}$ is reducible then $\pi_{\beta^{1}}=1$ generates a proper $\mathcal{A}$-submodule $\left(\neq V_{N}\right)$. But it takes non-zero values at any points of $B_{N}$. Hence its $\mathcal{A}$-span must contain all $\delta_{\beta^{i}}$. We come to a contradiction.

When $t$ are special. Till the end of the paper $t_{\nu}=q_{\nu}^{k_{\nu}}$ for $k_{\nu} \in \mathbf{Z}_{+}, \nu \in \nu_{R}$. The $L$-operators act in $\mathbf{Q}^{0}[x]$ for $\mathbf{Q}^{0} \stackrel{\text { def }}{=} \mathbf{Q}\left(q_{0}\right)$. Let $J \subset \mathbf{Q}^{0}[x]^{W}$ be the radical of the pairing $\llbracket, \rrbracket$. It is an ideal and an $\mathcal{A}$-submodule. The quotient (a ring and an $\mathcal{A}$-module) $\mathcal{V}=\mathbf{Q}^{0}[x]^{W} / J$ is finite dimensional over $\mathbf{Q}^{0}$. It results from Proposition 5.1 as generic $t$ approaches $q^{k}$.

The set $\tilde{B}$ of maximal ideals of $\mathcal{V}$ can be considered as a subset of any fundamental domain (in $B_{-}$) with respect to the action of the groups $K_{N}$ and $W \ni w: b \rightarrow w\left(b+\sum_{\nu} k_{\nu} \rho_{\nu}\right)-\sum_{\nu} k_{\nu} \rho_{\nu}$. We put

$$
\begin{equation*}
\tilde{B}=\left\{\tilde{\beta}^{1}, \ldots, \tilde{\beta}^{\partial}\right\} \subset B_{-}, \tilde{\beta}^{1}=0 \tag{5.9}
\end{equation*}
$$

assuming that $-w_{0}(\tilde{B})=\tilde{B}$ for the longest element $w_{0}$. We remind that the point 0 corresponds to the evaluation at $t^{-\rho}$.

By the construction, all $\left\{L_{a}\right\}$-eigenvectors in $\mathcal{V}$ have pairwise distinct eigenvalues (the difference of any two of them with the same sets of eigenvalues and coinciding evaluations at $t^{-\rho}$ belongs to $\left.J\right)$. Applying this to $\pi_{0}=1$ generating $\mathcal{V}$ as an $\mathcal{A}$-module we establish the irreducibility of $\mathcal{V}$ (use the pairing $\llbracket, \rrbracket$ and follow Proposition 5.1).

Next, we will introduce the restricted Macdonald pairing (cf. (0.2),(0.4)):

$$
\begin{align*}
\langle f(x), g(x)\rangle^{\prime} & \stackrel{\text { def }}{=} \sum_{c \in B_{N}} \mu^{\prime}(c) f(c) \bar{g}(c) \text { for } f, g \in \mathbf{Q}^{0}[x]^{W} \\
\mu^{\prime} & =\prod_{a \in R_{+}^{\vee}} \prod_{i=0}^{k_{a}-1}\left(1-x_{a} q_{a}^{i}\right)\left(1-x_{a}^{-1} q_{a}^{i}\right) \tag{5.10}
\end{align*}
$$

Here $B_{N}=B / K_{N}, \mu^{\prime}(c)={ }^{\delta} \mu^{\prime}(c)=\mu^{\prime}\left(t^{-\rho} q^{c}\right)$. The usage of symmetric $\mu^{\prime}$ in place of $\mu$ (which does not alter $\langle$,$\rangle up to proportionality for generic t$ ) is somewhat more convenient for roots of unity.

The same verification as in [C2], Proposition 4.2 gives that

$$
\begin{equation*}
\langle A f, g\rangle^{\prime}=\left\langle f, A^{*} g\right\rangle^{\prime}, \quad A \in[\hat{\mathcal{A}}]_{\dagger}, \tag{5.11}
\end{equation*}
$$

for the anti-involution * from (1.21) considered on the operators from $\mathcal{A}$ acting on symmetric polynomials. The restricted norms can be calculated by means of the same shift operators till the latter do not vanish (see [C2], Corrolary 5.3 ), which leads to the formulas (4.11) up to a common coefficient of proportionality.

Let us assume that:

$$
\begin{equation*}
q_{a}^{\left(\rho_{k}, a\right)+i} \neq 1 \text { for all } a \in R_{+}^{\vee}, i=-k_{a}+1, \ldots, k_{a}-1, \tag{5.12}
\end{equation*}
$$

where $\rho_{k}=\sum_{\nu} k_{\nu} \rho_{\nu}$. This restriction makes the construction below nonempty. In the simply laced case $(A, D, E)$, it is equivalent to the condition $N \geq k((\rho, \theta)+1)$.

LEMMA 5．2．The natural map $\mathcal{V} \rightarrow \tilde{V} \stackrel{\text { def }}{=} \operatorname{Funct}\left(\tilde{B}, \mathbf{Q}^{0}\right)$ is an isomor－ phism which supplies $\tilde{V}$ with the structure of a non－zero irreducible $\mathcal{A}$－module． Both pairings 【，】，〈，＞are well defined and non－degenerate on $\tilde{V}$ ．

Proof．The radical $J^{\prime}$ of the pairing $\langle,\rangle^{\prime}$ in $\mathbf{Q}^{0}[x]^{W}$ is an $\mathcal{A}$－submodule．It equals the space of all functions $f(x)$ such that ${ }^{\delta} f(c)=0$ for $c$ from the subset $B^{\prime} \subset \tilde{B}_{-}(N)$ where ${ }^{\delta} \mu^{\prime}$ is non－zero．The sets $\tilde{B}$ and $B^{\prime}$ contain 0 ．Indeed， $\llbracket 1, f \rrbracket=f(0)$ and $\mu^{\prime}\left(t^{-\rho}\right) \neq 0$ because of condition（5．12）．Hence the linear span $J+J^{\prime}$（that is an $\mathcal{A}$－submodule）does not coincide with the entire $\mathbf{Q}^{0}[x]^{W}$ ， and the irreducibility of $\mathcal{V}$ results in $J+J^{\prime}=J$ ．

Introducing now the delta－functions $\tilde{\delta}_{i}=\delta_{\tilde{\beta}^{i}}$ ，we can define the $\pi$－functions $\left\{\tilde{\pi}_{i}\right\}$ from the orthogonality and evaluation conditions

$$
\begin{equation*}
\llbracket \tilde{\pi}_{i}, \tilde{\delta}^{j} \rrbracket=C_{i} \delta_{i j} \quad \text { and } \quad \tilde{\pi}_{i}(0)=1,, 1 \leq i, j \leq \partial \tag{5.13}
\end{equation*}
$$

They are eigenvectors of the $L_{a}$－operators with the eigenvalues $\bar{m}_{a}\left(\tilde{\beta}^{i}\right)$ and linearly generate $\tilde{V}$ ．The sets of eigenvalues are pairwise distinct and $\left\langle\tilde{\pi}_{i}, \tilde{\pi}_{i}\right\rangle^{\prime} \neq$ 0 ．

Actually the $\pi$－functions are the discretizations of certain restricted Mac－ donald＇s polynomials and the above scalar products can be calculated explic－ itly but we will not discuss this here．Informally，$\tilde{B}$ is a collection of all weights（up to $K_{N}$ ）corresponding to the Macdonald polynomials of non－zero $q, t$－dimension．Amyway，the restrictions to $\tilde{B}$ of any well defined polynomial $\pi_{b}$ coincide with one of $\tilde{\pi}_{i}$ ．

We will also use that $\left(a, r_{\nu}\right) \in N \mathbf{Z}$ ，where $r_{\nu}=(2 / \nu) \rho_{\nu} \in B$ ，and impose one more restriction：

$$
\begin{equation*}
q^{(a, a) / 2}=q_{0}^{m(a, a)}=1 \text { for } a \in K_{N}, \nu \in \nu_{R} \tag{5.14}
\end{equation*}
$$

If $q_{0}$ is a primitive root of degree $2 m N$ then $K_{N}=N Q \cap B$ for the root lattice $Q=\oplus_{i=1}^{n} \mathbf{Z} \alpha_{i}$（see（5．1））．This condition obviously holds true for even $N$（all roots systems）．For odd $N$ ，it is necessary to exclude $B_{n}, C_{4 l+2}$ ．In the latter case，$B \subset Q, m=1$ and we can pick $q_{0}$ in the roots of unity of degree $N$ ．

Let us equip the matrix algebra $\operatorname{End}_{\mathbf{Q}^{0}}(\tilde{V})$ with the complex conjugation ${ }^{+}: q \rightarrow q^{-1}$ acting on the entries and the hermitian transposition ${ }^{\dagger}$（the composition of the matrix transposition $t r$ and ${ }^{+}$）．

THEOREM 5．3．Introducing $\tilde{\Pi}=\left(\tilde{\pi}_{i}\left(\tilde{\beta}^{j}\right)\right)($ see $(5.8,5.9))$ for $\tilde{\pi}_{i}=\pi_{\tilde{\beta}^{i}}$ ，let

$$
\begin{equation*}
\mathcal{T}_{+}=\operatorname{Diag}\left(q^{\left(\tilde{\beta}^{i}, \tilde{\beta}^{i}\right) / 2} x_{\tilde{\beta}^{i}}\left(t^{-\rho}\right)\right), \mathcal{T}_{-}=\Pi \mathcal{T}_{+}^{-1} \Pi^{-1}, \Omega=\mathcal{T}_{+}^{-1} \mathcal{T}_{-} \mathcal{T}_{+}^{-1} \tag{5.15}
\end{equation*}
$$

Then the following map gives a non－zero projective representation of the group $S L_{2}(\mathbf{Z})$ ：

$$
\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \rightarrow \mathcal{T}_{+},\left(\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right) \rightarrow \mathcal{T}_{-},\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \rightarrow \Omega
$$

Moreover, $\Pi^{+}=\mathcal{W}_{0} \Pi=\Pi \mathcal{W}_{0}, \mathcal{T}_{+}^{\dagger}=\mathcal{T}_{+}^{-1}, \mathcal{T}_{-}^{\dagger}=\mathcal{T}_{-}^{-1}, \Omega^{\dagger}=\Omega^{-1}$, where $\mathcal{W}_{0}=\left(w_{i j}\right), w_{i j}=\delta_{i j^{o}}, \tilde{\beta}^{j^{o}}=-w_{0}\left(\tilde{\beta}^{j}\right)$ for the longest element $w_{0}$.

Proof. We will start with the action of the ${ }^{+}$on $\tilde{\pi}_{i}(c)$ and the deltafunctions:

$$
\begin{equation*}
\left(\tilde{\beta}^{i}\right)^{+}=\tilde{\beta}^{i^{o}}=-w_{0}\left(\tilde{\beta}^{i}\right), \tilde{\delta}_{i}^{+}=\tilde{\delta}_{i^{o}}, \tilde{\pi}_{i}^{+}=\tilde{\pi}_{i} . \tag{5.16}
\end{equation*}
$$

Given an arbitrary $f \in \tilde{V}, f\left(\tilde{\beta}^{i}\right)^{+}=f^{+}\left(\tilde{\beta}^{i^{o}}\right)$. Together with the relation $\tilde{\Pi}^{\mathrm{tr}}=\tilde{\Pi}$ (the duality) it gives the formulas with ${ }^{\dagger}$.

The multiplication by $\gamma$ acts naturally in $\tilde{V}$. The formula for $\mathcal{T}_{+}$describes it (up to a constant factor) in the basis of delta-functions. Really,

$$
\begin{align*}
& \gamma(c)={ }^{\delta} \gamma(c)=q^{\Sigma_{i=1}^{n} \zeta_{i} \zeta_{\alpha_{i}} / 2} \text { for } \\
& \zeta_{i}=\log _{q}\left(x_{i}\left(q^{c} t^{-\rho}\right)\right)=\left(b_{i}, c\right)+\log _{q}\left(\prod_{\nu} t_{\nu}^{-\left(\rho_{\nu}, b_{i}\right)}\right) \tag{5.17}
\end{align*}
$$

Hence $\gamma(c)=g q^{(c, c) / 2} x_{c}\left(t^{-\rho}\right)$ for $g=q^{\left(\rho_{k}, \rho_{k}\right)}$. Since the matrix $\mathcal{T}_{+}$is important up to proportionality one can drop the constant $g$. We see that changing $c$ by any elements from $K_{N}$ does not influence $\gamma(c)$ because of the condition (5.14), which makes the multiplication by $\gamma$ well defined.

Next, the automorphism $\tau_{-}=\varphi \tau_{+} \varphi$ corresponds to $\tilde{\Pi} \mathcal{T}_{+}^{-1} \tilde{\Pi}^{-1}$, and the matrix $\Omega$ from (5.15) induces $\omega=\tau_{+}^{-1} \varphi \tau_{+} \varphi \tau_{+}^{-1}$ in the same delta-basis. Indeed, $\tau_{-}$is the application of $\gamma(Y)$. It multiplies $\pi_{b}$ by $\gamma\left(t^{\rho} q^{-b}\right)$ whereas $\gamma(X)$ multiplies $\delta_{b}$ by $\gamma\left(t^{-\rho} q^{b}\right.$ ) (so we need to inverse $\mathcal{T}_{+}$). More formally, one can use the equation $\llbracket \mathcal{T}_{+}^{-1} f, g \rrbracket=\llbracket f, \mathcal{T}_{-} g \rrbracket$.

Finally, any relations from $S L_{2}(\mathbf{Z})$ hold for these matrices up to proportionality. It results directly from Theorem 4.3 and the irreducibility of $\tilde{V}$.

Actually we have come even to a stronger statement.
Corrolary 5.4. The above map can be extended to a projective representation of $G L_{2}(\mathbf{Z})$ :

$$
\left(\begin{array}{cc}
0 & -1  \tag{5.18}\\
-1 & 0
\end{array}\right) \rightarrow \mathcal{E} \stackrel{\text { def }}{=} \tilde{\Pi} \mathcal{D}^{-1} \tilde{\Pi}^{+} \sigma \tilde{\Pi}^{-1}=\tilde{\Pi} \mathcal{D}^{-1}
$$

where $\mathcal{D}=\operatorname{Diag}\left(\left\langle\tilde{\pi}_{i}, \tilde{\pi}_{i}\right\rangle^{\prime}\right), \mathcal{E}$ belongs to the matrix algebra End ${\mathbf{Q}^{0}}(\tilde{V})$ extended by the complex conjugation ${ }^{+}$denoted by $\sigma$. Modulo proportionality, $\Omega$ equals $\mathcal{D} \tilde{\Pi}^{-1}$.

Proof. We can introduce the Fourier transform (4.20), giving $\varepsilon$ on the operators:

$$
\begin{equation*}
F\left(\tilde{\pi}_{j}\right)=\sum_{i=1}^{\partial} \tilde{\pi}_{i}\left(\tilde{\beta}^{j}\right)^{+} \tilde{\pi}_{i} /\left\langle\tilde{\pi}_{i}, \tilde{\pi}_{i}\right\rangle^{\prime} \tag{5.19}
\end{equation*}
$$

Then $F$ is written in the basis $\{\tilde{\pi}\}$ as $\mathcal{D}^{-1} \tilde{\Pi}^{+} \sigma$. In the basis $\left\{\tilde{\delta}_{i}\right\}$, it will be exactly $\mathcal{E}$ from (5.18). Here we applied the formula

$$
\left(\tilde{\Pi} \mathcal{D}^{-1} \sigma\right)^{2}=\tilde{\Pi}^{+} \mathcal{D}^{-1} \tilde{\Pi} \mathcal{D}^{-1}=\mathrm{const}
$$

resulting directly from the relation $\varepsilon^{2}=1$ and the irreducibility of $\tilde{V}$. Adding $\mathcal{E}$ to (5.15) we get a projective representation of $G L_{2}(\mathbf{Z})$.

Let us check that $\Omega$ is proportional to $\mathcal{D} \Pi^{-1}$. We use that the Fourier transform is diagonal in the basis $\gamma^{-1} \tilde{\pi}_{i}$ (which follows from Proposition 4.4). It means that $\tau_{+} F \tau_{+}^{-1}$ is diagonal in the basis $\{\tilde{\pi}\}$ and gives that

$$
\begin{align*}
& \left(\tilde{\Pi}^{-1} \mathcal{T}_{+} \tilde{\Pi}\right)\left(\mathcal{D}^{-1} \tilde{\Pi}^{+}\right)\left(\tilde{\Pi}^{-1} \mathcal{T}_{+}^{-1} \tilde{\Pi}\right)^{+}=\mathcal{F} \stackrel{\text { def }}{=} \operatorname{Diag}\left(\phi_{\tilde{\beta}^{i}}\right) \text { for }  \tag{5.20}\\
& F\left(p_{b} \gamma^{-1}\right)=\phi_{b} p_{b} \gamma^{-1}=\phi_{0} L_{p_{b}}\left(\gamma^{-1}\right), \phi_{b} / \phi_{0}=x_{b}\left(t^{\rho}\right) q^{-(b, b) / 2} .
\end{align*}
$$

The first expression is proportional to $\tilde{\Pi}^{-1} \mathcal{T}_{+} \tilde{\Pi} \mathcal{T}_{+} \tilde{\Pi}^{-1} \mathcal{D}$ which is $(\Omega \tilde{\Pi})^{-1} \mathcal{T}_{+}^{-1} \mathcal{D}$. Then we observe that $\mathcal{T}_{+} \mathcal{F}$ is a constant matrix. Here the coefficients $\phi_{b} / \phi_{0}$ can be easily calculated (once we have the proportionality). By the way, taking rather $\operatorname{big} N, k$ we come to the proof of formula (4.19).

The theorem generalizes the construction due to Kirillov [Ki] (in the case of $A_{n}$ ). His approach is different and based on quantum groups at roots of unity. As to arbitrary roots, the simplest case of our theorem when $t=q$ is directly related to Theorem 13.8 from $[\mathrm{K}]$. The weights $\left\{\tilde{\beta}^{j}\right\}$ exactly correspond to the representations of non-zero $q$-dimension. The Macdonald polynomials are the characters and always exist in this case.

We expect that the above considerations can be applied with proper changes to generic $q, t$ in the analitic setting and hope that they could be useful to renew elliptic functions towards the Ramanujan theories.

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