

THE NIL HECKE RING AND SINGULARITY OF SCHUBERT VARIETIES

SHRAWAN KUMAR

Introduction

Let G be a semi-simple simply-connected complex algebraic group and $T \subset B$ a maximal torus and a Borel subgroup respectively. Let $\mathfrak{h} = \text{Lie } T$ be the Cartan subalgebra of the Lie algebra $\text{Lie } G$, and $W := N(T)/T$ the Weyl group associated to the pair (G, T) , where $N(T)$ is the normalizer of T in G . We can view any element $w = \bar{w} \bmod T \in W$ as the element (denoted by the corresponding German character) \mathfrak{w} of G/B , defined as $\mathfrak{w} = \bar{w}B$. For any $w \in W$, there is associated the Schubert variety $X_w := \overline{BwB}/B \subset G/B$ and the T -fixed points of X_w (under the canonical left action) are precisely $I_w := \{\mathfrak{v} : v \in W \text{ and } v \leq w\}$.

We (together with B. Kostant) have defined a certain ring $Q_W(T)$ (which is the smash product of the group algebra $\mathbb{Z}[W]$ with the W -field $Q(T)$ of rational functions on the torus T) and certain elements $y_w \in Q_W(T)$ (for any $w \in W$). Expressing the elements y_w in the $\{\delta_v\}_{v \in W}$ basis:

$$y_w = \sum b_{w^{-1}, v^{-1}} \delta_v,$$

we get the matrix $B = (b_{w^{-1}, v^{-1}})_{w, v \in W}$ with entries in $Q(T)$ (cf. Definition 2.1(d)). Analogously, we defined the nil Hecke ring Q_W (which is the smash product of the group algebra $\mathbb{Z}[W]$ with the W -field $Q(\mathfrak{h})$ of rational functions on the Cartan subalgebra \mathfrak{h}) and certain elements $x_w \in Q_W$. Writing

$$x_w = \sum c_{w^{-1}, v^{-1}} \delta_v,$$

we get another matrix $C = (c_{w^{-1}, v^{-1}})_{w, v \in W}$ with entries in $Q(\mathfrak{h})$ (cf. Definition 3.1(b)).

We prove that the formal T -character of the ring of functions on the scheme theoretic tangent cone $T_{\mathfrak{v}}(X_w)$ (for any $\mathfrak{v} \in I_w$) is nothing but $*b_{w^{-1}, v^{-1}}$ (cf. Theorem 2.2), where $*$ is the involution of $Q(T)$ given by $e^\lambda \mapsto e^{-\lambda}$. This sharpens a result due to Rossmann [R]. In fact this work of Rossmann, and our own work with B. Kostant on the equivariant K-theory of flag varieties, motivated our current work. The proof of Theorem (2.2) requires the Demazure character formula, and occupies §2 of this paper. We use this theorem to prove that $b_{w^{-1}, v^{-1}} \neq 0$ if and only if $v \leq w$, and in this case it has a pole of order exactly equal to $\ell(w)$. Similarly $c_{w^{-1}, v^{-1}} \neq 0$ if and only if $v \leq w$ (cf. Corollaries 3.2).

We study the graded algebra structure on the space of functions $\text{Gr}(\mathcal{O}_{\mathfrak{v}, X_w})$ on the scheme theoretic tangent cone $T_{\mathfrak{v}}(X_w)$ in §4. Our principal result in this direction is Theorem (4.4), which roughly asserts that the graded algebra $\text{Gr}(\mathcal{O}_{\mathfrak{v}, X_w})$

arises from the natural filtration of the Demazure module $v^{-1}V_w(\lambda)$ induced from the standard filtration of the universal enveloping algebra $U(\mathfrak{u}^-)$, where \mathfrak{u}^- is the nil-radical of the opposite Borel subalgebra and $V_w(\lambda)$ is defined in §1. We use this theorem to derive a result due to Carrell-Peterson asserting that for simply-laced G , a point $\mathfrak{v} \in X_w$ is rationally smooth if and only if the reduced tangent cone $T_\theta^{\text{red}}(X_w)$ is an affine space for all $v \leq \theta \leq w$ (cf. Corollary 4.11).

The principal result of our paper is a necessary and sufficient condition for a point $\mathfrak{v} \in X_w$ to be smooth, in terms of the matrix entry $c_{w^{-1},v^{-1}}$ (cf. Theorem 5.5 (b)). This result asserts that for any $v \leq w \in W$, the point $\mathfrak{v} \in X_w$ is smooth \Leftrightarrow

$$c_{w^{-1},v^{-1}} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1},v^{-1})} \beta^{-1},$$

where $S(w^{-1},v^{-1}) := \{\alpha \in \Delta_+ : v^{-1}r_\alpha \leq w^{-1}\}$.

There is a very similar criterion for a point $\mathfrak{v} \in X_w$ to be rationally smooth (cf. Theorem 5.5(a)). This criterion of rational smoothness can be easily deduced by combining some results of Dyer and Carrell-Peterson, but we give a different geometric proof as that proof is used crucially to prove our criterion of smoothness mentioned above (i.e. Theorem 5.5(b)).

It should be mentioned that the elements $c_{w^{-1},v^{-1}}$ (as well as $b_{w^{-1},v^{-1}}$) are defined combinatorially and admit closed expressions (cf. Lemma 3.4).

The nil Hecke ring approach to singularity, developed in this paper, is applied to some specific examples discussed in §§6 and 7. In §6, we determine the precise singular locus of any Schubert variety in any rank-2 group (cf. Proposition 6.1). I believe this result should be well known, but I did not find it explicitly written down in the literature. In §7, we use our Theorem (5.5) to study the smoothness (and rational smoothness) of codimension one Schubert varieties X_i in any G/B . Proposition (7.4) (resp. Corollary 7.6) gives a criterion for a point $\mathfrak{v} \in X_i$ to be smooth (resp. rationally smooth). This criterion is applied to give a complete list of codimension one smooth (as well as rationally smooth) Schubert varieties in any G/B (cf. Proposition 7.8).

Finally in §8, we extend our main result giving the criterion of smoothness to arbitrary (not even symmetrizable) Kac-Moody groups (cf. Theorem 8.9). We also extend our result determining the formal character of the ring of functions on the scheme theoretic tangent cone at any $\mathfrak{v} \in X_w$ to arbitrary Kac-Moody groups (cf. Theorem 8.6). The proofs in the Kac-Moody case are similar to the finite case, and hence we have been brief and outlined only the necessary changes.

There are other criteria for smoothness due to Lakshmibai-Seshadri (for classical groups) [LS] [L], Ryan (for $SL(n)$) [Ry], ...; and for rational smoothness due to Kazhdan-Lusztig [KL], Carrell-Peterson [C], Jantzen [J], ... ; and by works of Deodhar and Peterson rational smoothness implies smoothness for simply-laced groups. It may be mentioned that our criterion for smoothness (as in Theorem 5.5(b)) is applicable to all G uniformly, in contrast to the above mentioned criteria for smoothness. We refer the reader to two survey articles, one by Carrell [C2], and the other by Deodhar [D2].

The main results of this paper were announced in [Ku2].

Acknowledgements. I am grateful to W. Rossmann, D. Peterson, and J. B. Carrell for explaining to me their (then partly unpublished) works. I also thank J.

Wahl and M. Schlessinger for some helpful conversations. This work was partially supported by the NSF grant No. DMS-9203660.

1. Notation

For a complex vector space V (possibly infinite dimensional) , V^* denotes its full vector space dual. For a finite set S , $\#S$ denotes its cardinality.

Let G be a semi-simple simply-connected complex algebraic group, and let B be a fixed Borel subgroup and $T \subset B$ a maximal torus. Let B^- be the (opposite) Borel subgroup such that $B^- \cap B = T$. We denote by U (resp. U^-) the unipotent radical of B (resp. B^-). Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{b}^-, \mathfrak{u}, \mathfrak{u}^-, \mathfrak{h}$ be the Lie algebras of the groups G, B, B^-, U, U^-, T respectively. Let $\Delta \subset \mathfrak{h}^*$ (resp. Δ_+) denote the set of roots for the pair (G, T) (resp. (B, T)). Let $\{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots in Δ_+ and let $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ be the corresponding (simple) coroots (where $n = \text{rank } G$).

Let $W := N(T)/T$ be the Weyl group (where $N(T)$ is the normalizer of T in G) of G . Then W is a Coxeter group, generated by the simple reflections $\{r_1, \dots, r_n\}$ (where r_i is the reflection corresponding to the simple root α_i). In particular, we can talk of the length $\ell(w)$ of any element $w \in W$. We denote the identity element of W by e .

Let $\mathfrak{h}_{\mathbb{Z}}^* := \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \text{ for all } i\}$ be the set of integral weights and $D := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) \geq 0, \text{ for all } i\}$ (resp. $D^\circ := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) > 0, \text{ for all } i\}$) the set of dominant (resp. dominant regular) integral weights. For any $\lambda \in D$ and $w \in W$, we denote by $V(\lambda)$ the irreducible representation of G with highest weight λ , and $V_w(\lambda)$ is the smallest B -submodule of $V(\lambda)$ containing the extremal weight vector $e_{w\lambda}$ (of weight $w\lambda$). Let $R(T) := \mathbb{Z}[X(T)]$ be the group algebra of the character group $X(T)$ of the torus T . Then $\{e^\lambda\}_{\lambda \in \mathfrak{h}_{\mathbb{Z}}^*}$ are precisely the elements of $X(T)$. Let $Q(T)$ be the quotient field of $R(T)$. Clearly W acts on $Q(T)$ and moreover $Q(T)$ admits an involution $*$ (i.e. a field automorphism of order 2) taking $e^\lambda \mapsto e^{-\lambda}$.

For any $w \in W$, the Schubert variety X_w is by definition the closure $\overline{BwB/B}$ of BwB/B in G/B under the Zariski topology (where the notation BwB/B means $B\bar{w}B/B$ for any representative \bar{w} of w in $N(T)$). Then X_w is an irreducible (projective) subvariety of G/B of dimension $\ell(w)$. We can view any element $w = \bar{w} \text{ mod } T \in W$ as the element (denoted by the corresponding German character) \mathfrak{w} of G/B , defined as $\mathfrak{w} = \bar{w}B$. By the Bruhat decomposition, any \mathfrak{v} such that $v \leq w$ belongs to X_w , where \leq is the Bruhat (or Chevalley) partial order in W . The Schubert variety X_w is clearly B -stable (in particular T -stable), under the left multiplication of B on G/B . The T -fixed points of X_w are precisely $I_w := \{\mathfrak{v} : v \in W \text{ and } v \leq w\}$. For any variety X over \mathbb{C} , we denote by $\mathbb{C}[X]$ the ring of global regular functions $X \rightarrow \mathbb{C}$. For any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, let \mathbb{C}_λ be the 1-dimensional representation of B given by the character e^λ and let $\mathfrak{L}(\lambda)$ be the line bundle on G/B associated to the principal B -bundle $G \rightarrow G/B$ via the representation $\mathbb{C}_{-\lambda}$ of B .

2. Character of the ring of functions on the tangent cone of X_w

We follow the notation as in §1.

(2.1) **Definitions.** (a) For any local ring R with maximal ideal \mathfrak{m} , define the

graded R/\mathfrak{m} -algebra:

$$\text{gr } R := \sum_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}.$$

Let X be a scheme of finite type over an algebraically closed field and let x be a closed point of X . Then the *tangent cone* $T_x(X)$ of X at x is, by definition (cf. [M, Chapter 3, §3]), $\text{Spec}(\text{gr } \mathcal{O}_x)$, where $\mathcal{O}_x = \mathcal{O}_{x,X}$ is the local ring at $x \in X$.

(b) Let $\widetilde{R}(T)$ be the set of all the formal sums $\sum_{e^\lambda \in X(T)} n_\lambda e^\lambda$, with arbitrary $n_\lambda \in \mathbb{Z}$

(we allow infinitely many of the n'_λ 's to be non-zero). Even though $\widetilde{R}(T)$ is not a ring, it has a canonical $R(T)$ -module structure (got by the multiplication). We define the $Q(T)$ -module $\widetilde{Q}(T)$ as $Q(T) \otimes_{R(T)} \widetilde{R}(T)$. Since $Q(T)$ is a flat $R(T)$ -module, $Q(T)$ canonically embeds in $\widetilde{Q}(T)$.

(c) A T -module M is said to be a *weight module* if $M = \bigoplus_{e^\lambda \in X(T)} M_\lambda$, where $M_\lambda := \{m \in M : tm = e^\lambda(t)m\}$ is the λ -th weight space. A weight module M is said to be an *admissible T -module* if $\dim M_\lambda < \infty$, for all $e^\lambda \in X(T)$.

For any admissible T -module M , one can define its *formal character* $\text{ch } M := \sum_{e^\lambda \in X(T)} (\dim M_\lambda) e^\lambda$ as an element of $\widetilde{R}(T)$.

(d) *The ring $Q(T)_W$* ([KK2, Section 2]): Let $Q(T)_W$ be the smash product of the W -field $Q(T)$ with the group algebra $\mathbb{Z}[W]$, i.e., $Q(T)_W$ is a free right $Q(T)$ -module with basis $\{\delta_w\}_{w \in W}$ and the multiplication is given by:

$$(1) \quad (\delta_{w_1} q_1) \cdot (\delta_{w_2} q_2) = \delta_{w_1 w_2} (w_2^{-1} q_1) q_2, \text{ for } q_1, q_2 \in Q(T) \text{ and } w_1, w_2 \in W.$$

For any simple reflection $r_i, 1 \leq i \leq n$, define the element $y_{r_i} \in Q(T)_W$ by:

$$(2) \quad y_{r_i} = (\delta_e + \delta_{r_i}) \frac{1}{(1 - e^{-\alpha_i})}.$$

Now, for any $w \in W$, define $y_w \in Q(T)_W$ by

$$(3) \quad y_w = y_{r_{i_1}} \cdots y_{r_{i_p}},$$

where $w = r_{i_1} \cdots r_{i_p}$ is a reduced decomposition. By [KK2, Proposition 2.4], y_w is well defined. Write

$$(4) \quad y_w = \sum_v b_{w^{-1}, v^{-1}} \delta_v,$$

for some (unique) $b_{w^{-1}, v^{-1}} \in Q(T)$. It can be easily seen that $b_{w^{-1}, v^{-1}} = 0$ unless $v \leq w$ (cf. [KK2, Proposition 2.6]).

The ring $Q(T)_W$ has a canonical representation in $Q(T)$ defined by

$$(5) \quad (\delta_w q_1) \cdot q_2 = w(q_1 q_2).$$

It is easy to see that for any $r_i, y_{r_i} \cdot R(T) \subset R(T)$, in particular, $y_w \cdot R(T) \subset R(T)$ for any $w \in W$.

Since $\mathfrak{v} \in X_w$ is fixed under the action of T (cf. §1), the local ring $\mathcal{O}_{\mathfrak{v}, X_w}$ at $\mathfrak{v} \in X_w$ is canonically a T -module.

(2.2) Theorem. *Take any $v \leq w \in W$. Then $gr \mathcal{O}_{\mathfrak{v}, X_w}$ is an admissible T -module and moreover*

$$ch (gr \mathcal{O}_{\mathfrak{v}, X_w}) = *b_{w^{-1}, v^{-1}},$$

as elements of $\widetilde{Q}(T)$, where ch (which is an element of $\widetilde{R}(T)$) is to be thought of as the element $1 \otimes ch$ of $\widetilde{Q}(T) := Q(T) \otimes_{R(T)} \widetilde{R}(T)$.

In particular, $ch(gr \mathcal{O}_{\mathfrak{v}, X_w}) \in Q(T)$.

Before we come the proof of Theorem (2.2), we need the following preparation.

We recall the following very simple lemma without proof.

(2.3) Lemma. *Let Y be an irreducible projective variety with an ample line bundle \mathfrak{L} on Y , together with a non-zero $\sigma \in H^0(Y, \mathfrak{L})$. Define the variety $Y^\circ := Y \setminus Z(\sigma)$, where $Z(\sigma)$ is the zero-set of σ . Then Y° is affine and moreover for any $f \in \mathbb{C}[Y^\circ]$, there exists a $n > 0$ (depending upon f) such that the section $f \cdot \sigma^n$ (of $H^0(Y^\circ, \mathfrak{L}^{\otimes n})$) extends as an element of $H^0(Y, \mathfrak{L}^{\otimes n})$.*

(2.4) Lemma. *Given any $f \in \mathbb{C}[U^-]$, there exists a large enough $\lambda \in D$ (i.e. $\lambda(\alpha_i^\vee) \gg 0$, for all the simple coroots α_i^\vee) and $\theta \in V(\lambda)^*$ such that*

$$f(g) = \langle \theta, ge_\lambda \rangle, \text{ for } g \in U^-,$$

where e_λ is a non-zero highest weight vector of $V(\lambda)$.

Moreover, for any $v \leq w \in W$, f vanishes on $(v^{-1}\overline{BwB}) \cap U^- \Leftrightarrow \theta \in (V(\lambda)/v^{-1}V_w(\lambda))^*$.

Proof. The first part is due to Andersen and also Cline-Parshall-Scott [CPS, §5]. However, for completeness, we give a proof.

By the Borel-Weil theorem (for any $\lambda \in D$), $\chi : V(\lambda)^* \xrightarrow{\sim} H^0(G/B, \mathfrak{L}(\lambda))$, where for any $\phi \in V(\lambda)^*$, $\chi(\phi)$ is given by the section $\chi(\phi)(gB) = (g, g^{-1}\phi|_{\mathbb{C}e_\lambda}) \bmod B$. (Observe that $\mathbb{C}e_\lambda \subset V(\lambda)$ is a one-dimensional representation of B corresponding to the character e^λ and hence $(\mathbb{C}e_\lambda)^*$ corresponds to the character $e^{-\lambda}$.) Let $\phi_\lambda \in V(\lambda)^*$ be the element defined by $\phi_\lambda(e_\lambda) = 1$ and $\phi_\lambda(v) = 0$, for any weight vector $v \in V(\lambda)$ of weight $\mu \neq \lambda$. Consider $U^- \approx U^- \cdot \mathfrak{e} \subset G/B$ as an open subset and take any (ample) line bundle $\mathfrak{L}(\lambda_o)$ on G/B for $\lambda_o \in D^\circ$. Taking the section $\sigma = \chi(\phi_{\lambda_o})$ of $\mathfrak{L}(\lambda_o)$ and applying Lemma (2.3), we get the first part of the lemma for $\lambda = n\lambda_o$ (for some $n > 0$). (Observe that $Z(\sigma) = G/B \setminus U^- \cdot \mathfrak{e}$, since λ_o is regular.)

Let $\tilde{f} : G \rightarrow \mathbb{C}$ be the extension of f given by $\tilde{f}(g) = \langle \theta, ge_\lambda \rangle$. Then, since $v^{-1}\overline{BwB}$ is an irreducible subvariety of G , and by Bruhat decomposition $v^{-1}\overline{BwB} \cap U^- \cdot B$ is non-empty open subset of $v^{-1}\overline{BwB}$,

$$\begin{aligned} f \text{ vanishes on } v^{-1}\overline{BwB} \cap U^- &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}\overline{BwB} \cap (U^- \cdot B) \\ &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}\overline{BwB} \\ &\Leftrightarrow \tilde{f} \text{ vanishes on } v^{-1}BwB \\ &\Leftrightarrow \langle \theta, v^{-1}Bwe_\lambda \rangle = 0 \\ &\Leftrightarrow \langle \theta, v^{-1}V_w(\lambda) \rangle = 0. \end{aligned}$$

For any $\lambda \in D$, define the map

$$\varphi_\lambda : V(\lambda)^* \otimes \mathbb{C}_\lambda \rightarrow \mathbb{C}[U^-]$$

by $\varphi_\lambda(\theta \otimes e_\lambda)(g) = \langle \theta, ge_\lambda \rangle$, for $\theta \in V(\lambda)^*$, $g \in U^-$ and $e_\lambda \in \mathbb{C}_\lambda$; where $\mathbb{C}_\lambda \subset V(\lambda)$ is identified as the highest weight space.

(2.5) Lemma. φ_λ is T -equivariant with respect to the adjoint action of T on U^- , and is an injective map.

Proof. For any $t \in T$,

$$\begin{aligned} \varphi_\lambda(t\theta \otimes te_\lambda)(g) &= \langle t\theta, gte_\lambda \rangle \\ &= \langle \theta, t^{-1}gte_\lambda \rangle \\ &= (t \cdot \varphi_\lambda(\theta \otimes e_\lambda))(g). \end{aligned}$$

This proves the T -equivariance of φ_λ .

To prove the injectivity of φ_λ , take $\theta \otimes e_\lambda \in \ker \varphi_\lambda$, i.e., $\langle \theta, ge_\lambda \rangle = 0$, for all $g \in U^-$. Hence $\langle \theta, gbe_\lambda \rangle = 0$ for all $g \in U^-$ and $b \in B$. In particular, by the density of U^-B in G and the irreducibility of $V(\lambda)$, we get $\langle \theta, V(\lambda) \rangle = 0$, i.e., $\theta = 0$, proving the injectivity of φ_λ . \square

§**(2.6)** For any $\gamma \in D$, let us choose a highest weight vector $e_\gamma \in V(\gamma)$, and define (for any $\lambda, \mu \in D$)

$$V(\lambda + \mu) \xrightarrow{i_{\lambda, \mu}} V(\lambda) \otimes V(\mu) \xrightarrow{\text{Id} \otimes \pi_\mu} V(\lambda) \otimes \mathbb{C}_\mu,$$

where $i_{\lambda, \mu}$ is the unique G -module map taking $e_{\lambda+\mu} \mapsto e_\lambda \otimes e_\mu$ and $\pi_\mu : V(\mu) \rightarrow \mathbb{C}_\mu$ is the T -equivariant projection onto the highest weight space $\mathbb{C}_\mu = \mathbb{C}e_\mu \subset V(\mu)$. We denote the composite map $(\text{Id} \otimes \pi_\mu) \circ i_{\lambda, \mu} : V(\lambda + \mu) \rightarrow V(\lambda) \otimes \mathbb{C}_\mu$ by $\hat{\delta}_{\lambda, \mu}$. Dualizing the above, we get the map

$$\bar{\delta}_{\lambda, \mu} : V(\lambda)^* \otimes \mathbb{C}_{-\mu} \rightarrow V(\lambda + \mu)^* ,$$

and hence the map

$$\delta_{\lambda, \mu} = \bar{\delta}_{\lambda, \mu} \otimes \text{Id} : V(\lambda)^* \otimes \mathbb{C}_\lambda \approx V(\lambda)^* \otimes \mathbb{C}_{-\mu} \otimes \mathbb{C}_{\lambda+\mu} \rightarrow V(\lambda + \mu)^* \otimes \mathbb{C}_{\lambda+\mu}.$$

It is easy to see that $\delta_{\lambda, \mu}$ is injective. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} V(\lambda)^* \otimes \mathbb{C}_\lambda & \xrightarrow{\delta_{\lambda, \mu}} & V(\lambda + \mu)^* \otimes \mathbb{C}_{\lambda+\mu} \\ \varphi_\lambda \searrow & & \swarrow \varphi_{\lambda+\mu} \\ & \mathbb{C}[U^-] & \end{array}$$

By virtue of Lemma (2.4), for any $\lambda \in D$ and $v \leq w \in W$, we get the injective map

$$\varphi_\lambda(v, w) : (v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda \hookrightarrow \mathbb{C}[(v^{-1}\overline{BwB}) \cap U^-],$$

by restricting the map φ_λ

(2.7) Lemma. For $\lambda, \mu \in D$ and $v \leq w \in W$, $\hat{\delta}_{\lambda, \mu}(v^{-1}V_w(\lambda + \mu)) = v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu$. In particular, there exists a unique map $\delta_{\lambda, \mu}(v, w)$ making the following diagram commutative:

$$\begin{array}{ccc} V(\lambda)^* \otimes \mathbb{C}_\lambda & \longrightarrow & (v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda \\ \downarrow \delta_{\lambda, \mu} & & \downarrow \delta_{\lambda, \mu}(v, w) \\ V(\lambda + \mu)^* \otimes \mathbb{C}_{\lambda + \mu} & \longrightarrow & (v^{-1}V_w(\lambda + \mu))^* \otimes \mathbb{C}_{\lambda + \mu} \end{array}$$

where the horizontal maps are the canonical restriction maps. Moreover, $\delta_{\lambda, \mu}(v, w)$ is injective.

Proof. For $b \in B$,

$$(1) \quad \hat{\delta}_{\lambda, \mu}(\bar{v}^{-1}b\bar{w}e_{\lambda + \mu}) = \bar{v}^{-1}b\bar{w}e_\lambda \otimes [\bar{v}^{-1}b\bar{w}e_\mu]_\mu,$$

where $[x]_\mu$ denotes the component of $x \in V(\mu)$ in the μ^{th} weight space, and \bar{v} is a representative of v in $N(T)$. Define the closed subvariety $Y \subset B$ by $Y = \{b \in B : [\bar{v}^{-1}b\bar{w}e_\mu]_\mu = 0\}$. Then $Y \neq B$, for otherwise $e_\mu \notin v^{-1}V_w(\mu)$, which is a contradiction (since $v \leq w$ by assumption). Hence for $b \in B \setminus Y$, $\hat{\delta}_{\lambda, \mu}(\bar{v}^{-1}b\bar{w}e_{\lambda + \mu}) = \bar{v}^{-1}b\bar{w}e_\lambda \otimes e_\mu$, up to a non-zero scalar. But since $\hat{\delta}_{\lambda, \mu}(v^{-1}V_w(\lambda + \mu))$ is a (closed) linear subspace and $B \setminus Y$ is dense in B ,

$$v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu \subset \hat{\delta}_{\lambda, \mu}(v^{-1}V_w(\lambda + \mu)).$$

The inverse inclusion is clear from (1). This proves the first part of the lemma. The ‘in particular’ statement follows immediately from dualizing the map

$$\hat{\delta}_{\lambda, \mu}|_{v^{-1}V_w(\lambda + \mu)} : v^{-1}V_w(\lambda + \mu) \longrightarrow v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu.$$

The injectivity of $\delta_{\lambda, \mu}(v, w)$ follows from the surjectivity of $\hat{\delta}_{\lambda, \mu}|_{v^{-1}V_w(\lambda + \mu)}$. \square

By virtue of the above lemma, we get the following commutative diagram:

$$\begin{array}{ccc} (v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda & \xrightarrow{\delta_{\lambda, \mu}(v, w)} & (v^{-1}V_w(\lambda + \mu))^* \otimes \mathbb{C}_{\lambda + \mu} \\ \varphi_\lambda(v, w) \searrow & & \swarrow \varphi_{\lambda + \mu}(v, w) \\ & \mathbb{C}[v^{-1}\overline{BwB} \cap U^-]. & \end{array}$$

(2.8) Definition. Define a partial order \prec in D as follows:

$$\lambda \prec \mu \Leftrightarrow \mu - \lambda \in D.$$

Taking the limit of the maps $\varphi_\lambda(v, w)$, we get the T -equivariant map

$$\varphi(v, w) : \lim_{\lambda \in D} ((v^{-1}V_w(\lambda))^* \otimes \mathbb{C}_\lambda) \longrightarrow \mathbb{C}[v^{-1}\overline{BwB} \cap U^-].$$

(2.9) Proposition. *The above map $\varphi(v, w)$ is an isomorphism, for all $v \leq w \in W$.*

Proof. Injectivity of the map $\varphi(v, w)$ is clear from the injectivity of the maps $\varphi_\lambda(v, w)$ and $\delta_{\lambda, \mu}(v, w)$. Surjectivity of $\varphi(v, w)$ follows from Lemma (2.4). \square

(2.10) Definition. For any directed set Λ and any sequence $\theta : \Lambda \rightarrow \widetilde{R(T)}$, given as $\theta(\alpha) = \sum_{e^\lambda \in X(T)} n_\lambda(\alpha) e^\lambda$ with $n_\lambda(\alpha) \in \mathbb{Z}$, we say that $\lim_{\alpha \in \Lambda} \theta(\alpha) = \sum_{\alpha \in \Lambda} n_\lambda e^\lambda$, if for any $e^\lambda \in X(T)$, there exists $\alpha_\lambda \in \Lambda$ such that $n_\lambda(\alpha) = n_\lambda$ for all $\alpha \geq \alpha_\lambda$. Of course $\lim_{\alpha \in \Lambda} \theta(\alpha)$ may not exist in general.

Observe that if $\lim_{\alpha \in \Lambda} \theta(\alpha)$ exists, then so is $\lim_{\alpha \in \Lambda} (p\theta(\alpha))$, for any fixed $p \in R(T)$. Moreover

$$(1) \quad \lim_{\alpha \in \Lambda} (p\theta(\alpha)) = p \lim_{\alpha \in \Lambda} \theta(\alpha) .$$

(2.11) Corollary. $\text{ch}_T \mathbb{C}[v^{-1} \overline{BwB} \cap U^-] = \lim_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} * (y_w \cdot e^\lambda)))$.

Proof. By the previous proposition and the Demazure character formula (cf. [A], [Jo2], [Ra2, Remarks 4.4], [Se], [Ku, Theorem 3.4], [Ma]),

$$\begin{aligned} \text{ch}_T \mathbb{C}[v^{-1} \overline{BwB} \cap U^-] &= \lim_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} \text{ch}_T(V_w(\lambda)^*))) \\ &= \lim_{\lambda \in D} (\delta_{v^{-1}} \cdot (e^{v\lambda} * (y_w \cdot e^\lambda))). \end{aligned}$$

Observe that the existence of the above limit is guaranteed by Proposition (2.9) and the fact that $\mathbb{C}[v^{-1} \overline{BwB} \cap U^-]$ is an admissible T -module (being quotient of $\mathbb{C}[U^-]$). \square

Finally we come to the proof of Theorem (2.2).

§**(2.12) Proof of Theorem (2.2).**

Write (cf. (4) of §2.1)

$$y_w = \sum_{u \leq w} b_{w^{-1}, u^{-1}} \delta_u .$$

Then

$$\begin{aligned} (1) \quad e^{v\lambda} * (y_w \cdot e^\lambda) &= \sum_u (*b_{w^{-1}, u^{-1}}) e^{v\lambda - u\lambda} \\ &= *b_{w^{-1}, v^{-1}} + \sum_{\substack{u \neq v \\ u \leq w}} (*b_{w^{-1}, u^{-1}}) e^{v\lambda - u\lambda} . \end{aligned}$$

For any (regular) weight $\lambda_o \in D^o$, $v\lambda_o - u\lambda_o \neq 0$ for $u \neq v$. From the definition of $b_{w^{-1}, u^{-1}}$, it is easy to see that there exist positive roots $\{\beta_1, \dots, \beta_\ell\}$ depending on w (possibly with repetitions) such that

$$(2) \quad B_{w^{-1}} b_{w^{-1}, u^{-1}} \subset B(T) \quad \text{for all } u \leq w$$

where $P := \prod_{k=1}^{\ell} (1 - e^{-\beta_k})$.

Fix $\lambda_o \in D^o$. Then the subset $\{n\lambda_o\}_{n \geq 1} \subset D$ being cofinal in D under \prec ,

$$(3) \quad \lim_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^\lambda)) = \lim_{n \rightarrow \infty} (e^{nv\lambda_o} * (y_w \cdot e^{n\lambda_o})) .$$

Then by (1) of Definition (2.10) and (1), (2), (3) as above, we get

$$\begin{aligned} P \lim_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^\lambda)) &= \lim_{n \rightarrow \infty} (P(e^{nv\lambda_o} * (y_w \cdot e^{n\lambda_o}))) \\ &= \lim_{n \rightarrow \infty} (P * b_{w^{-1}, v^{-1}} + \sum_{\substack{u \neq v \\ u \leq w}} (P * b_{w^{-1}, u^{-1}}) e^{n(v\lambda_o - u\lambda_o)}) \\ &= P * b_{w^{-1}, v^{-1}} + \sum_{\substack{u \neq v \\ u \leq w}} (P * b_{w^{-1}, u^{-1}}) \lim_{n \rightarrow \infty} (e^{n(v\lambda_o - u\lambda_o)}) \\ &= P * b_{w^{-1}, v^{-1}} . \end{aligned}$$

So, we get (in the $Q(T)$ -module $\widetilde{Q}(T)$)

$$(4) \quad 1 \otimes \lim_{\lambda \in D} (e^{v\lambda} * (y_w \cdot e^\lambda)) = *b_{w^{-1}, v^{-1}} .$$

So, by Corollary (2.11) and Identity (4), we get

$$\text{ch}_T \mathbb{C}[v^{-1} \overline{BwB} \cap U^-] = \delta_{v^{-1}} \cdot (*b_{w^{-1}, v^{-1}}) .$$

But the variety $v^{-1} \overline{BwB} \cap U^-$ provides an affine neighborhood of the point $\mathfrak{e} \in v^{-1} X_w$. In particular,

$$\text{gr } \mathcal{O}_{\mathfrak{e}, v^{-1} X_w} \cong \text{gr } \mathbb{C}[v^{-1} \overline{BwB} \cap U^-] .$$

The theorem now follows from the complete reducibility of the T -module $\mathbb{C}[v^{-1} \overline{BwB} \cap U^-]$, by translating the variety $v^{-1} X_w$ under \bar{v} . \square

(2.13) Remarks. (1) This theorem was obtained by the author in 1987 and privately circulated in the preprint “A connection of equivariant K -theory with the singularity of Schubert varieties”.

(2) A different proof of the Theorem was subsequently given by Bressler [Br]. Even though I have not seen, M. Brion mentioned to me that he also obtained a proof of this theorem (unpublished).

3. Some consequences of Theorem (2.2)

After the following definitions, we give some of the corollaries of Theorem (2.2).

(3.1) Definitions. (a) For any $\ell \in \mathbb{Z}^+ := \{0, 1, 2, \dots\}$ and any $a = \sum n_\lambda e^\lambda \in R(T)$, denote by $(a)_\ell = \sum n_\lambda \frac{\lambda^\ell}{\ell!} \in S^\ell(\mathfrak{h}^*)$, where $S^\ell(\mathfrak{h}^*)$ is the space of homogeneous polynomials of degree ℓ on \mathfrak{h} . Further, denote by $[a]_\ell = (a)_\ell$, where ℓ is the

smallest element of \mathbb{Z}^+ such that $(a)_{\ell_0} \neq 0$. (If a itself is 0, we define $[a] = 0$.) Now for $q = \frac{a}{b} \in Q(T)$, where $a, b \in R(T)$, we define $[q] = \frac{[a]}{[b]} \in Q(\mathfrak{h})$ (the quotient field of the symmetric algebra $S(\mathfrak{h}^*)$). Clearly $[q]$ is well defined.

When $q \neq 0$ and $\deg [a] \leq \deg [b]$, we say that q has a pole (at the identity e) of order $= \deg [b] - \deg [a]$. It is easy to see that $b_{w^{-1}, v^{-1}}$ (cf. (4) of §2.1), when non-zero, has a pole of order $\leq \ell(w)$.

(b) *The nil Hecke ring Q_W* ([KK1, §4]): Let Q_W be the smash product of the W -field $Q(\mathfrak{h})$ with the group algebra $\mathbb{Z}[W]$, with the product given by the same formula (1) in §2.1. For any simple reflection r_i , $1 \leq i \leq n$, define $x_{r_i} \in Q_W$ by $x_{r_i} = -(\delta_{r_i} + \delta_e) \frac{1}{\alpha_i}$. Now, for any $w \in W$, define $x_w = x_{r_{i_1}} \dots x_{r_{i_p}}$, where $w = r_{i_1} \dots r_{i_p}$ is a reduced decomposition. The element x_w is well defined by [KK1, Proposition 4.2]. Write, as in [KK1, Proposition 4.3],

$$x_w = \sum_v c_{w^{-1}, v^{-1}} \delta_v, \text{ for some (unique) } c_{w^{-1}, v^{-1}} \in Q(\mathfrak{h}).$$

(3.2) Corollaries (of Theorem 2.2). *For any $v, w \in W$:*

(a) $b_{w^{-1}, v^{-1}} \neq 0$ if and only if $v \leq w$; and in this case it has a pole of order exactly equal to $\ell(w)$. Further,

$$(1) \quad \left(\prod_{\beta \in \Delta_+} (1 - e^\beta) \right) b_{w^{-1}, v^{-1}} \in R(T).$$

(b) $[*b_{w^{-1}, v^{-1}}] = c_{w^{-1}, v^{-1}}$; and hence for any $v \leq w$,

$$[ch(gr \mathcal{O}_{v, X_w})] = c_{w^{-1}, v^{-1}},$$

as elements of $Q(\mathfrak{h})$.

In particular, $c_{w, v} \neq 0$ if and only if $v \leq w$.

Further

$$(2) \quad \left(\prod_{\beta \in \Delta_+} \beta \right) c_{w, v} \in S(\mathfrak{h}^*).$$

Proof. As observed in §2.1(d), $b_{w, v} = 0$ unless $v \leq w$. So let us assume that $v \leq w$. Set $\mathcal{A}^v = vU^{-\mathfrak{c}} \subset G/B$. Since $\mathcal{A}^v \cap X_w$ is a closed subvariety of the affine space \mathcal{A}^v , $\text{Tor}_{\mathbb{C}[\mathcal{A}^v]}^p(\mathbb{C}[\mathcal{A}^v \cap X_w], \mathbb{C})$ is a finite dimensional vector space over \mathbb{C} for any p and moreover (\mathcal{A}^v being smooth) is 0 for large enough p . Set

$$F = \sum_p (-1)^p \text{ch}(\text{Tor}_p^{\mathbb{C}[\mathcal{A}^v]}(\mathbb{C}[\mathcal{A}^v \cap X_w], \mathbb{C})) \in R(T).$$

Then from the Koszul complex we get,

$$(3) \quad \left(\prod (1 - e^{v\beta}) \right) \text{ch} \mathbb{C}[\mathcal{A}^v \cap X_w] = F, \text{ as elements of } R(T).$$

It can be easily seen that the coefficient of e^0 in the left side of the above identity is non-zero, in particular $F \neq 0$. From (3) we obtain

$$(4) \quad \begin{aligned} 1 \otimes \text{ch } \mathbb{C}[\mathcal{A}^v \cap X_w] &= F \cdot \prod_{\beta \in \Delta_+} (1 - e^{v\beta})^{-1} \\ &= F \prod_{\gamma \in \Delta_+ \cap v\Delta_-} (-e^\gamma) \prod_{\beta \in \Delta_+} (1 - e^\beta)^{-1}, \text{ as elements of } \widetilde{Q}(T). \end{aligned}$$

From (4) it is clear that $1 \otimes \text{ch } \mathbb{C}[\mathcal{A}^v \cap X_w] \neq 0$ as an element of $\widetilde{Q}(T)$. Moreover, since $\mathcal{A}^v \cap X_w$ is an affine neighborhood of \mathfrak{v} in X_w , we get

$$(5) \quad \text{ch } \mathbb{C}[\mathcal{A}^v \cap X_w] = \text{ch}(\text{gr } \mathcal{O}_{\mathfrak{v}, X_w}).$$

But then by (5) and Theorem (2.2), we get that $b_{w^{-1}, v^{-1}} \neq 0$. The assertion that $b_{w^{-1}, v^{-1}}$ has a pole of order exactly equal to $\ell(w)$ (whenever $v \leq w$) follows from a lemma of Joseph [Jo, §2.3]. This proves the first part of Corollary (a). Assertion (1) of part (a) follows immediately from (4) (and Theorem 2.2).

To prove part (b), in view of Theorem (2.2), we only need to show that

$$(6) \quad [*b_{w^{-1}, v^{-1}}] = c_{w^{-1}, v^{-1}} :$$

By induction (on $\ell(w)$) we assume the validity of (6) for any w with $\ell(w) \leq k$ and any $v \in W$ and take $w' = wr_i$ of length $k + 1$, where r_i is a simple reflection such that $\ell(w') > \ell(w)$. (The case $w = e$ is obviously true.)

By Definition 2.1 (d),

$$\begin{aligned} y_{w'} &:= y_w y_{r_i} \\ &= \left(\sum_v b_{w^{-1}, v^{-1}} \delta_v \right) (\delta_e + \delta_{r_i}) \left(\frac{1}{1 - e^{-\alpha_i}} \right) \\ &= \sum_v \frac{b_{w^{-1}, v^{-1}} + b_{w^{-1}, r_i v^{-1}}}{1 - e^{-v\alpha_i}} \delta_v. \end{aligned}$$

This gives for any $v \in W$,

$$(7) \quad b_{r_i w^{-1}, v^{-1}} = \frac{b_{w^{-1}, v^{-1}} + b_{w^{-1}, r_i v^{-1}}}{1 - e^{-v\alpha_i}}.$$

Exactly the same way, using the definitions from §3.1(b), we obtain:

$$(8) \quad c_{r_i w^{-1}, v^{-1}} = \frac{c_{w^{-1}, v^{-1}} + c_{w^{-1}, r_i v^{-1}}}{-v\alpha_i}.$$

By (7) and part (a) of the corollary we get:

$$[*b_{r_i w^{-1}, v^{-1}}] = \frac{[*b_{w^{-1}, v^{-1}}] + [*b_{w^{-1}, r_i v^{-1}}]}{-v\alpha_i}.$$

Hence by the induction hypothesis (using (8)), (6) follows for $w' = wr_i$. This completes the proof of Corollaries (3.2). \square

(3.3) Remarks. (1) The (b)-part of the above corollary is due to Rossmann [R, §3.2]. In fact, this motivated our theorem (2.2).

(2) The assertions (1) and (2) as above can be derived purely algebraically (cf. [KK2, Corollary 4.18 and Remark 4.17(b)]).

The following lemma gives an expression for $b_{w,v}$ (and $c_{w,v}$) and can be easily proved by using the definitions

(3.4) Lemma. *Fix any $v \leq w \in W$, and take a reduced decomposition $w = r_{i_1} \dots r_{i_p}$. Then*

$$b_{w^{-1}, v^{-1}} = \sum ((1 - e^{-r_{i_1}^{\epsilon_1} \alpha_{i_1}})(1 - e^{-r_{i_1}^{\epsilon_1} r_{i_2}^{\epsilon_2} \alpha_{i_2}}) \dots (1 - e^{-r_{i_1}^{\epsilon_1} \dots r_{i_p}^{\epsilon_p} \alpha_{i_p}}))^{-1}.$$

Similarly

$$c_{w^{-1}, v^{-1}} = (-1)^p \sum ((r_{i_1}^{\epsilon_1} \alpha_{i_1})(r_{i_1}^{\epsilon_1} r_{i_2}^{\epsilon_2} \alpha_{i_2}) \dots (r_{i_1}^{\epsilon_1} \dots r_{i_p}^{\epsilon_p} \alpha_{i_p}))^{-1},$$

where both the sums run over all those $(\epsilon_1, \dots, \epsilon_p) \in \{0, 1\}^p$ satisfying $r_{i_1}^{\epsilon_1} \dots r_{i_p}^{\epsilon_p} = v$. (The notation r_i^0 means the identity element.)

**4. Ring of functions on the tangent cone –
the graded algebra structure**

§(4.1) For any $\lambda \in D$, the (finite dimensional) G -module $V(\lambda)$ admits a filtration $\{\mathcal{F}_p(\lambda)\}_{p \geq 0}$ as follows:

Let $\{U_p(\mathfrak{u}^-)\}_{p \geq 0}$ be the standard filtration of the universal enveloping algebra $U(\mathfrak{u}^-)$, where we recall that $U_p(\mathfrak{u}^-)$ is the span of the monomials $X_1 \dots X_m$ for $X_i \in \mathfrak{u}^-$ and $m \leq p$. Now set

$$\mathcal{F}_p(\lambda) = U_p(\mathfrak{u}^-) \cdot e_\lambda,$$

where e_λ is any non-zero highest weight vector in $V(\lambda)$.

Fix $\lambda \in D$, $v \leq w \in W$, $\theta \in V(\lambda)^*$, and a highest weight vector $e_\lambda \in V(\lambda)$. Recall the definition of the function φ_λ from §2.4. we abbreviate $\varphi_\lambda(\theta \otimes e_\lambda)$ by φ^θ . Thus $\varphi^\theta : U^- \rightarrow \mathbb{C}$ is the function

$$\varphi^\theta(g) = \langle \theta, ge_\lambda \rangle, \quad \text{for } g \in U^-.$$

By Lemma (2.4), φ^θ vanishes on $v^{-1}\overline{BwB} \cap U^- \Leftrightarrow \langle \theta, v^{-1}V_w(\lambda) \rangle = 0$. Identify U^- with the affine space \mathfrak{u}^- under the exponential map. This gives rise to a gradation on $\mathbb{C}[U^-]$. Now let φ_d^θ be the d^{th} graded component of φ^θ (for any $d \geq 0$), i.e.,

$$(*) \quad \varphi_d^\theta(X) = \frac{1}{d!} \langle \theta, X^d \cdot e_\lambda \rangle, \quad \text{for } X \in \mathfrak{u}^-.$$

The following lemma follows immediately from (*), if we use the fact that for any vector space V , its p^{th} symmetric power $S^p(V)$ is spanned by $\{v^p\}_{v \in V}$.

(4.2) Lemma. *Fix $p \geq 1$. Then for any θ as above (i.e. $\langle \theta, v^{-1}V_w(\lambda) \rangle = 0$), $\varphi_d^\theta \equiv 0$ for all $0 \leq d < p$ if and only if $\langle \theta, v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda) \rangle = 0$. \square*

For any $p \geq 1$ and any subvariety $0 \in Y \subset \mathbb{A}^n$, let $\mathcal{I}_p(Y)$ denote the set of degree p^{th} components of all those functions f in the ideal $\mathcal{I}(Y)$ of $Y \subset \mathbb{A}^n$, such that the d^{th} homogeneous component f_d of f is 0 for all $d < p$.

As an immediate consequence of the above lemma, we get the following

(4.3) Corollary. *For any $p \geq 0$ the map $\theta \otimes e_\lambda \mapsto (\varphi_\lambda(\theta \otimes e_\lambda))_p$ induces a T -equivariant injective map*

$$\left(\frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda \xrightarrow{f_p(v,w;\lambda)} \mathcal{I}_p(v^{-1}\overline{BwB} \cap U^-),$$

where $\mathbb{C}_\lambda \subset V(\lambda)$ is the highest weight subspace, and $\mathcal{F}_{-1}(\lambda)$ is defined to be 0. \square

It is easy to see that under the map $\hat{\delta}_{\lambda,\mu} : V(\lambda + \mu) \rightarrow V(\lambda) \otimes \mathbb{C}_\mu$ (of §2.6), the image $\hat{\delta}_{\lambda,\mu}(\mathcal{F}_p(\lambda + \mu)) \subset \mathcal{F}_p(\lambda) \otimes \mathbb{C}_\mu$. Moreover, by Lemma (2.7),

$$\hat{\delta}_{\lambda,\mu}(v^{-1}V_w(\lambda + \mu)) \subset v^{-1}V_w(\lambda) \otimes \mathbb{C}_\mu.$$

In particular, $\hat{\delta}_{\lambda,\mu}$ gives rise to a T -module map $\delta_{\lambda,\mu}(v, w; p)$ making the following diagram commutative (for any $\lambda, \mu \in D$, $v \leq w \in W$ and $p \geq 0$):

$$\begin{array}{ccc} \left(\frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda & \xrightarrow{\delta_{\lambda,\mu}(v,w;p)} & \left(\frac{v^{-1}V_w(\lambda + \mu) + \mathcal{F}_p(\lambda + \mu)}{v^{-1}V_w(\lambda + \mu) + \mathcal{F}_{p-1}(\lambda + \mu)} \right)^* \otimes \mathbb{C}_{\lambda+\mu} \\ f_p(v, w; \lambda) \searrow & & \swarrow f_p(v, w; \lambda + \mu) \end{array}$$

By the injectivity of the map $f_p(v, w; \lambda)$, we see that $\delta_{\lambda, \mu}(v, w; p)$ is injective. Thus

$$\left\{ \left(\frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda \right\}_{\lambda \in D}$$

forms a directed system of T -modules and there is an induced T -module map

$$f_p(v, w) : \lim_{\lambda \in D} \left(\left[\frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \right]^* \otimes \mathbb{C}_\lambda \right) \rightarrow \mathcal{I}_p(v^{-1}\overline{BwB} \cap U^-).$$

(4.4) Theorem. *The above map $f_p(v, w)$ is a T -equivariant isomorphism for all $p \geq 0$ and all $v \leq w \in W$. In particular, there is a T -equivariant isomorphism*

$$(1) \quad \text{gr}_p(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w}) \approx \lim_{\lambda \in D} \left(\left[\frac{v^{-1}V_w(\lambda) \cap \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) \cap \mathcal{F}_{p-1}(\lambda)} \right]^* \otimes \mathbb{C}_\lambda \right),$$

where $\text{gr}_p(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})$ is the p^{th} graded component of $\text{gr}(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})$.

Proof. Since $f_p(v, w; \lambda)$ is injective for all $\lambda \in D$, $f_p(v, w)$ is clearly injective. The surjectivity of $f_p(v, w)$ follows from Lemma (2.4) and Lemma (4.2).

We now come to the proof of (1): Observe first that by [Ha, Lecture 20],

$$(2) \quad \text{gr}_p(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w}) \approx S^p(\mathfrak{u}^-) / \mathcal{I}_p(v^{-1}\overline{BwB} \cap U^-),$$

where S^p is the p^{th} symmetric power.

Now for any (fixed) p , if we take λ to be sufficiently large, then the map

$$U_p(\mathfrak{u}^-) \otimes \mathbb{C}_\lambda \rightarrow \mathcal{F}_p(\lambda) \text{ given by } X \otimes e_\lambda \mapsto Xe_\lambda,$$

for $e_\lambda \in \mathbb{C}_\lambda \subset V(\lambda)$, is a T -module isomorphism. In particular, by the Poincare-Birkhoff-Witt theorem,

$$(3) \quad \mathcal{F}_p(\lambda) / \mathcal{F}_{p-1}(\lambda) \approx S^p(\mathfrak{u}^-) \otimes \mathbb{C}_\lambda \quad (\text{for large enough } \lambda).$$

Consider the exact sequence

$$0 \rightarrow \frac{v^{-1}V_w(\lambda) \cap \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) \cap \mathcal{F}_{p-1}(\lambda)} \rightarrow \frac{\mathcal{F}_p(\lambda)}{\mathcal{F}_{p-1}(\lambda)} \rightarrow \frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \rightarrow 0.$$

Dualizing this sequence and using (3) we get (for large enough λ)

$$0 \leftarrow \left(\frac{v^{-1}V_w(\lambda) \cap \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) \cap \mathcal{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda \leftarrow S^p((\mathfrak{u}^-)^*) \leftarrow \left(\frac{v^{-1}V_w(\lambda) + \mathcal{F}_p(\lambda)}{v^{-1}V_w(\lambda) + \mathcal{F}_{p-1}(\lambda)} \right)^* \otimes \mathbb{C}_\lambda \leftarrow 0.$$

Now the isomorphism (1) is established from (2) and the isomorphism $f_p(v, w)$. \square

§(4.5) For any variety X and a closed point $x \in X$, let $Z_x(X)$ denote the Zariski tangent space of X at x . For any subvariety $Y \subset G/B$ containing the base point \mathfrak{e} , we get the induced inclusion $Z_\mathfrak{e}(Y) \hookrightarrow Z_\mathfrak{e}(G/B)$. But $Z_\mathfrak{e}(G/B)$ can be canonically identified with \mathfrak{u}^- (since U^- is an open neighborhood around \mathfrak{e} in G/B), in particular, $Z_\mathfrak{e}(Y)$ can be canonically viewed as a subspace of \mathfrak{u}^- . For any $\alpha \in \Delta$, let $r_\alpha \in W$ be the reflection defined by $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$.

The following result is due to Polo [P, Theorem 3.2]. It may be recalled that a different description of the Zariski tangent space in the case of classical groups was given by Lakshmibai-Seshadri (cf. [LS] [L]). Observe that by virtue of the automorphism of G/B , given by $gB \mapsto \bar{v}gB$ (for $g \in G$), $Z_\mathfrak{e}(v^{-1}X_w)$ is isomorphic with $Z_{\mathfrak{v}}(X_w)$.

The first part of the following result follows immediately from Theorem (4.4) and the second part follows from the fact that $X_w \subset G/B$ is defined by linear equations

(4.6) Corollary. *For any $v \leq w$,*

$$(1) \quad Z_{\mathfrak{e}}(v^{-1}X_w) = \{X \in \mathfrak{u}^- : Xe_\lambda \in v^{-1}V_w(\lambda), \text{ for all } \lambda \in D\}.$$

In fact,

$$(2) \quad Z_{\mathfrak{e}}(v^{-1}X_w) = \{X \in \mathfrak{u}^- : Xe_{\lambda_0} \in v^{-1}V_w(\lambda_0)\}, \text{ for any one regular } \lambda_0 \in D,$$

where e_λ is a non-zero highest weight vector of $V(\lambda)$.

Proof. The identity (1) follows from Theorem (4.4) immediately, since $Z_{\mathfrak{e}}(v^{-1}X_w) = \text{gr}_1(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})^*$. However, we give the following direct proof:

Fix $\lambda \in D$ and take $\theta \in (V(\lambda)/v^{-1}V_w(\lambda))^*$, and consider the corresponding function $\varphi^\theta : U^- \rightarrow \mathbb{C}$ defined by

$$\varphi^\theta(\exp X) = \langle \theta, \exp Xe_\lambda \rangle = \langle \theta, Xe_\lambda \rangle + \text{order two and higher terms.}$$

(Observe that $\langle \theta, e_\lambda \rangle = 0$ by assumption.) So the linear part $L(\varphi^\theta) \in (\mathfrak{u}^-)^*$ (under the identification $\exp : \mathfrak{u}^- \rightarrow U^-$) of φ^θ is given by

$$(3) \quad L(\varphi^\theta)X = \langle \theta, Xe_\lambda \rangle, \quad \text{for } X \in \mathfrak{u}^-.$$

Let $\mathcal{I}(v^{-1}\overline{BwB} \cap U^-)$ denote the ideal of the closed subvariety $v^{-1}\overline{BwB} \cap U^-$ of U^- . Then, by the definition of the Zariski tangent space,

$$\begin{aligned} Z_{\mathfrak{e}}(v^{-1}X_w) &= \{X \in \mathfrak{u}^- : L(f)X = 0, \text{ for all } f \in \mathcal{I}(v^{-1}\overline{BwB} \cap U^-)\}, \\ &= \{X \in \mathfrak{u}^- : Xe_\lambda \in v^{-1}V_w(\lambda), \text{ for all } \lambda \in D\}, \text{ by (3) and Lemma (2.4).} \end{aligned}$$

This proves (1).

We now prove (2): The tensor product of sections gives rise to an algebra structure on the space $R := \bigoplus_{m \geq 0} H^0(G/B, \mathfrak{L}(m\lambda_o))$. Let K_m be the kernel of the restriction map $H^0(G/B, \mathfrak{L}(m\lambda_o)) \rightarrow H^0(X_w, \mathfrak{L}(m\lambda_o)|_{X_w})$. Then by a result of Ramanathan [Ra2, Theorem 3.11], the kernel $K := \sum_{m \geq 0} K_m$ of the surjective map $\bigoplus_{m \geq 0} H^0(G/B, \mathfrak{L}(m\lambda_o)) \rightarrow \bigoplus_{m \geq 0} H^0(X_w, \mathfrak{L}(m\lambda_o)|_{X_w})$ is generated as an ideal in the ring R by K_1 (i.e. X_w is linearly defined in G/B with respect to $\mathfrak{L}(\lambda_o)$). This, in particular, implies (by translating via \bar{v}^{-1} and using Lemma 2.3) that the ideal $\mathcal{I}(v^{-1}\overline{BwB} \cap U^-)$ is generated by the functions $\{\varphi^\theta\}$ where θ ranges over $(V(\lambda_o)/v^{-1}V_w(\lambda_o))^*$. Now by an argument identical to the proof of (1), we get (2). \square

(4.7) Lemma. *Let \mathfrak{g} be simply-laced. Assume that there exist integers $p, p_1, \dots, p_k \geq 1$ and roots $\beta, \beta_1, \dots, \beta_k \in \Delta_+$ such that*

$$(1) \quad p\beta = \sum_{j=1}^k p_j \beta_j$$

and

$$(2) \quad \sum_{j=1}^k p_j \leq p$$

Then $\beta_j = \beta$, for all $1 \leq j \leq k$.

Proof. We can assume without loss of generality that no $\beta_j = \beta$. Now by (1) we get

$$(3) \quad p\langle \beta, \beta^\vee \rangle = \sum_{j=1}^k p_j \langle \beta_j, \beta^\vee \rangle.$$

But \mathfrak{g} being simply-laced, $\langle \beta_j, \beta^\vee \rangle \leq 1$ (since $\beta_j \neq \beta$), and hence by (2) and (3) we get

$$2p \leq \sum_{j=1}^k p_j \leq p.$$

This contradiction proves the lemma. \square

(4.8) Definition. For any $v \leq w \in W$, define $S(w, v) = \{\alpha \in \Delta_+ : vr_\alpha \leq w\}$. Then as is easy to see $\#S(w, v) = \#S(w^{-1}, v^{-1})$.

(4.9) Proposition. *Let \mathfrak{g} be simply-laced. Fix $v \leq w \in W$. Then for any $\alpha \in \Delta_+$ such that $\alpha \notin S(w, v)$ but $E_{-\alpha} \in Z_{\mathfrak{e}}(v^{-1}X_w)$, there exists a non-zero element $\theta_\alpha \in \text{gr}_1(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})$ of weight α satisfying $\theta_\alpha^{\langle \rho, \alpha^\vee \rangle} = 0$ as an element of $\text{gr}_{\langle \rho, \alpha^\vee \rangle}(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})$, where $E_{-\alpha}$ is a non-zero root vector of \mathfrak{g} corresponding to the negative root $-\alpha$, and ρ is the half sum of positive roots. In particular, the tangent cone $T_{\mathfrak{e}}(v^{-1}X_w)$ is non-reduced in this case.*

Proof. By Lemma (4.7), the weight space of $U_p(\mathfrak{u}^-)$ corresponding to the weight $-p\alpha$ (for any $p \geq 1$) is one dimensional, and is spanned by $E_{-\alpha}^p$. Since $\text{gr}_1(\mathcal{O}_{y, Y})$ is canonically isomorphic with the dual space $Z_{y, Y}^*$ (for any variety Y and $y \in Y$), and $E_{-\alpha} \in Z_{\mathfrak{e}, v^{-1}X_w}$, there exist a non-zero element $\theta_\alpha \in \text{gr}_1(\mathcal{O}_{\mathfrak{e}, v^{-1}X_w})$ of weight α . Under the embedding $Z_{\mathfrak{e}, v^{-1}X_w} \hookrightarrow \mathfrak{u}^-$ (cf. §4.5), we can identify the element θ_α with the element of $(\mathfrak{u}^-)^*$ defined by $\theta_\alpha(E_{-\beta}) = \delta_{\alpha, \beta}$, for all $\beta \in \Delta_+$.

By virtue of Theorem (4.4), to prove that $\theta_\alpha^p = 0$ (where $p := \langle \rho, \alpha^\vee \rangle$), it suffices to show that (for all large enough $\lambda \in D$)

$$\theta_\alpha^p|_{(v^{-1}V_w(\lambda) \cap \mathcal{F}_p(\lambda))} \equiv 0 :$$

Since θ_α^p is of weight $p\alpha$ and the weight space of $U_p(\mathfrak{u}^-)$ corresponding to the weight $-p\alpha$ is spanned by $E_{-\alpha}^p$, it suffices to show that $E_{-\alpha}^p e_\lambda \notin v^{-1}V_w(\lambda)$ (for all large enough $\lambda \in D$):

For otherwise, assume that $E_{-\alpha}^p e_{\lambda_0} \in v^{-1}V_w(\lambda_0)$ (for some $\rho \preceq \lambda_0$). Then by Lemma (2.7), $E_{-\alpha}^p e_\rho \in v^{-1}V_w(\rho)$. But since $\bar{r}_\alpha e_\rho = E_{-\alpha}^p e_\rho$ (up to a non-zero scalar multiple), $\bar{r}_\alpha e_\rho \in v^{-1}V_w(\rho)$ and hence by [BGG, Theorem 2.9] $vr_\alpha \leq w$, which contradicts the assumption and proves the proposition. \square

(4.10) Remark. The ‘in particular’ statement of the above Proposition can also be deduced from [C, Theorem G(2)].

For a closed point x of a scheme X , recall the definition of the tangent cone $T_x(X)$ as $\text{Spec}(\text{gr } \mathcal{O}_x)$ from §2.1. Define the *reduced tangent cone* $T_x^{\text{red}}(X)$ as $\text{Spec}(\text{gr}^{\text{red}} \mathcal{O}_x)$, where $\text{gr}^{\text{red}} \mathcal{O}_x = (\text{gr } \mathcal{O}_x)/N$ and N is the ideal consisting of all the nilpotent elements in $\text{gr } \mathcal{O}_x$.

The following result is due to Carrell-Peterson [C, Theorem EG], proved by different methods

(4.11) Corollary. *Let \mathfrak{g} be an arbitrary semisimple Lie algebra and fix $v \leq w \in W$. Assume that $T_{\mathfrak{e}}^{\text{red}}(\theta^{-1}X_w)$ is an affine space for all $v \leq \theta \leq w$. Then the point $\mathfrak{v} \in X_w$ is rationally smooth.*

Conversely, in the case when \mathfrak{g} is simply-laced, if the point $\mathfrak{v} \in X_w$ is rationally smooth, then $T_{\mathfrak{e}}^{\text{red}}(\theta^{-1}X_w)$ is an affine space for all $v \leq \theta \leq w$.

Proof. As follows from [C, Theorem F] (cf. also [P, Proposition 4.2]), for any $\alpha \in S(w, \theta)$, $E_{-\alpha} \in Z_{\mathfrak{e}}(\theta^{-1}X_w) \cong \text{gr}_1(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})^*$. Choose a non-zero element θ_{α} of weight α in $\text{gr}_1(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})$. Then $\theta_{\alpha}^p \neq 0$ in $\text{gr}_p(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})$ (for any $p \geq 1$): To prove this, it suffices to show that $E_{-\alpha}^p e_{\lambda} \in \theta^{-1}V_w(\lambda)$, for any $\lambda \in D$ such that $p \leq \langle \lambda, \alpha^{\vee} \rangle$ (cf. proof of Proposition 4.9):

By the $sl(2)$ -theory, $E_{-\alpha}^{\langle \lambda, \alpha^{\vee} \rangle} e_{\lambda} = \bar{r}_{\alpha} e_{\lambda}$ (up to non-zero scalar multiples). If $\theta_{\alpha} \in \Delta_{-}$, clearly $E_{-\alpha}^p e_{\lambda} \in \theta^{-1}V_w(\lambda)$. So assume that $\theta_{\alpha} \in \Delta_{+}$. Then (upto non-zero scalar multiples)

$$E_{-\alpha}^p e_{\lambda} = E_{\alpha}^{\langle \lambda, \alpha^{\vee} \rangle - p} E_{-\alpha}^{\langle \lambda, \alpha^{\vee} \rangle} e_{\lambda} = E_{\alpha}^{\langle \lambda, \alpha^{\vee} \rangle - p} \bar{r}_{\alpha} e_{\lambda} \in \theta^{-1}V_w(\lambda),$$

thereby proving the claim.

We come to the proof of the first part of the Corollary. Since the dimension of the tangent cone is the same as the local dimension of the variety at that point (cf. [Ha, Lecture 20]), and (by assumption) $T_{\mathfrak{e}}^{\text{red}}(\theta^{-1}X_w)$ is an affine space, $\dim T_{\mathfrak{e}}^{\text{red}}(\theta^{-1}X_w) = \ell(w) \geq \#S(w, \theta)$. But, by Deodhar's conjecture (see Theorem 5.1), $\ell(w) \leq \#S(w, \theta)$. Hence $\ell(w) = \#S(w, \theta)$, for all $v \leq \theta \leq w$. So the first part of the Corollary follows from [C, Theorem E]. (Observe that for any $\theta \in W$, $\#\{\alpha \in \Delta_{+} : r_{\alpha}\theta < \theta\} = \ell(\theta)$.)

In the simply-laced case, by Proposition (4.9) and the above argument,

$$(1) \quad \dim(\text{gr}_1^{\text{red}}(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})) = \#S(w, \theta) = \ell(w),$$

since $\mathfrak{v} \in X_w$ is assumed to be rationally smooth. But since gr^{red} is generated (as an algebra) by gr_1^{red} , we get a surjective map $\gamma : S(\text{gr}_1^{\text{red}}(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})) \rightarrow \text{gr}^{\text{red}}(\mathcal{O}_{\mathfrak{e}, \theta^{-1}X_w})$ (where S is the symmetric algebra). But since $T_{\mathfrak{e}}^{\text{red}}(\theta^{-1}X_w)$ is of dim $\ell(w)$, surjectivity of γ and (1) force γ to be an isomorphism. This proves the corollary. \square

(4.12) Remark. The converse statement of the above corollary is not true in general for non simply-laced \mathfrak{g} . Take, e.g., \mathfrak{g} to be of type C_2 or G_2 and $w = r_1 r_2 r_1$, $v = e$. Since \mathfrak{g} is of rank 2, (as is well known; and can also be proved by using Lemma 6.2 and Theorem 5.5 (a)) $\mathfrak{e} \in X_w$ is rationally smooth. But it can be easily seen that $T_{\mathfrak{e}}^{\text{red}}(X_w)$ is not an affine space.

5. Smoothness criterion of Schubert varieties

For any $v \leq w \in W$, recall the definition of $S(w, v)$ from Definition (4.8). We recall the following very interesting conjecture of Deodhar [D], which was proved by Carrell-Peterson [C], Dyer [Dy], and Polo [P].

(5.1) Theorem. *For any $v \leq w \in W$, $\#S(w, v) \geq \ell(w)$.*

Even though the following proposition follows immediately by combining our Corollary 3.2(b) with [Dy, Proposition §3], we give a different (geometric) proof (as that proof is crucially used in the proof of Theorem 5.5(b)).

(5.2) Proposition. *Let $v \leq w \in W$. Then*

$$\#S(w^{-1}, v^{-1}) = \ell(w) \Leftrightarrow [\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{v}, X_w})] = d(-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1},$$

for some $d \in \mathbb{C}$.

Proof. By Corollary 3.2 (b), $[\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{v}, X_w})] = c_{w^{-1}, v^{-1}} \neq 0$ and, moreover, it can be easily seen from the definition of $c_{w^{-1}, v^{-1}}$ that $\deg c_{w^{-1}, v^{-1}} = -\ell(w)$, where $\deg \frac{P}{Q} := \deg P - \deg Q$ for non-zero $P, Q \in S(\mathfrak{h}^*)$. Hence the implication ‘ \Leftarrow ’ of the above proposition follows.

Now we come to the implication ‘ \Rightarrow ’:

Let $\exp : \mathfrak{u}^- \xrightarrow{\sim} U^-$ be the exponential map, where \mathfrak{u}^- is the Lie algebra of U^- . (Observe that U^- being a unipotent group, \exp is an algebraic morphism.) Let $Y := \exp^{-1}(U^- \mathfrak{e} \cap v^{-1} X_w)$ be the closed irreducible subvariety of \mathfrak{u}^- , where we identify U^- with $U^- \mathfrak{e}$. Fix non-zero root vectors $E_{-\beta}$ (corresponding to the negative root $-\beta$) for $\beta \in \Delta_+$. For any $\alpha \in \Delta_+$, let $f_\alpha : \mathfrak{u}^- \rightarrow \mathbb{C}$ be the linear map defined by $\sum_{\beta \in \Delta_+} t_\beta E_{-\beta} \mapsto t_\alpha$, and let f_α^Y be the restriction of f_α to Y . Define the subvariety (with the reduced structure)

$$Z_S = \{x \in Y : f_\alpha^Y(x) = 0, \text{ for all } \alpha \in S := S(w, v)\}.$$

Clearly $0 \in Z_S$. We claim that any irreducible component Z_S^o of Z_S through 0 is 0-dimensional:

The varieties $Z_S^o \subset Z_S$ are clearly T -stable under the adjoint action of the maximal torus T on \mathfrak{u}^- . Further, Z_S^o does not contain any 1-dimensional T -stable closed irreducible subvariety R : It is easy to see that any 1-dimensional T -stable closed irreducible subvariety of \mathfrak{u}^- is of the form $\mathbb{C}E_{-\beta} \subset \mathfrak{u}^-$, for some $\beta \in \Delta_+$. In particular, $R = \mathbb{C}E_{-\beta_0}$ (for some $\beta_0 \in \Delta_+$). This gives that $\exp(\mathbb{C}E_{-v\beta_0})\mathfrak{v} \subset X_w$. Now if $-v\beta_0 \in \Delta_+$, then by [BGG, Corollary 2.3] $vr_{\beta_0} < v \leq w$, so $\beta_0 \in S$. If $v\beta_0 \in \Delta_+$, then clearly $\exp(\mathbb{C}E_{v\beta_0})\exp(\mathbb{C}E_{-v\beta_0})\mathfrak{v} \subset X_w$. In particular, for the subgroup $\mathcal{S}_{vr_{\beta_0}v^{-1}} \subset G$ generated by $\exp(\mathbb{C}E_{-v\beta_0})$ and $\exp(\mathbb{C}E_{v\beta_0})$, $\mathcal{S}_{vr_{\beta_0}v^{-1}}\mathfrak{v} \subset X_w$. Again this gives, by [C, Theorem F(2)], that $\beta_0 \in S$. So, in either case, $R = \mathbb{C}E_{-\beta_0}$, for some $\beta_0 \in S$. But, by the definition of Z_S , such a R is not contained in Z_S . This contradiction establishes the claim that Z_S^o does not contain any 1-dimensional T -stable closed irreducible subvariety.

Embed $i : \mathfrak{u}^- \hookrightarrow G/B$ via the map $X \mapsto (\exp X)\mathfrak{e}$. The map i is clearly T -equivariant open immersion. Take the Zariski closure $\overline{Z_S^o}$ of $i(Z_S^o)$ in G/B . Now applying [C, Lemma of §2] to the T -stable projective variety $\overline{Z_S^o} \subset G/B$, we get that $\dim Z_S^o = 0$ (since Z_S^o does not contain any 1-dimensional T -stable closed irreducible subvarieties). Since any irreducible component of Z_S is T -stable (and closed) in \mathfrak{u}^- and any closed T -stable subset of \mathfrak{u}^- contains 0, we get that any irreducible component of Z_S passes through 0. In particular, $Z_S = \{0\}$.

Since the variety X_w is Cohen-Macaulay (cf. [Ra], [Ku, Theorem 2.23], [Ma]), the variety Y is Cohen-Macaulay. Assume now that $\#S(w^{-1}, v^{-1}) = \#S(w, v) = \ell(w) = \dim Y$, and enumerate the elements of $S(w, v)$ as $\{\gamma_1, \dots, \gamma_\ell\}$, where $\ell = \ell(w)$. By [F, Lemma(a), §2.4] (since $\dim Z_S = 0$), the elements $\{f_{\gamma_j}^Y\}_{1 \leq j \leq \ell}$ considered as elements of the local ring $\mathcal{O}_{0, Y}$ form a regular sequence in $\mathcal{O}_{0, Y}$. Let I be the ideal generated by $\{f_{\gamma_j}^Y\}_{1 \leq j \leq \ell}$ inside the local ring $\mathcal{O}_{0, Y}$. Then there exists

an integer $d > 0$ such that $\mathfrak{m}^d \subset I \subset \mathfrak{m}$, where $\mathfrak{m} \subset \mathcal{O}_{0,Y}$ is the maximal ideal (since $Z_S = \{0\}$). Moreover, by [F, Lemma (b), §2.4], the canonical ring homomorphism

$$(1) \quad \frac{\mathcal{O}_{0,Y}}{I}[X_1, X_2, \dots, X_\ell] \xrightarrow{\sim} \sum_{m \geq 0} I^m / I^{m+1},$$

which takes X_j to the image of $f_{\gamma_j}^Y$ in I/I^2 , is an isomorphism. In particular,

$$(2) \quad \begin{aligned} \text{ch}(\text{gr}(\mathcal{O}_{\epsilon, v^{-1}X_w})) &= \text{ch}(\text{gr}(\mathcal{O}_{0,Y})) = \text{ch}(\mathbb{C}[Y]) \quad (Y \text{ being affine}) \\ &= \text{ch}\left(\mathcal{O}_{0,Y}/I\right) \prod_{j=1}^{\ell} (1 - e^{\gamma_j})^{-1}, \quad \text{by (1)}. \end{aligned}$$

But since $\mathcal{O}_{0,Y}/I$ corresponds to the 0-dimensional variety, it is finite dimensional vector space over \mathbb{C} and hence

$$(3) \quad [\text{ch}(\mathcal{O}_{0,Y}/I)] = \dim(\mathcal{O}_{0,Y}/I).$$

By (2) and (3) we get

$$(4) \quad [\text{ch}(\text{gr}(\mathcal{O}_{\epsilon, v^{-1}X_w}))] = (-1)^\ell d \prod_{j=1}^{\ell} \gamma_j^{-1},$$

where $d := \dim(\mathcal{O}_{0,Y}/I)$. Thus

$$(5) \quad \begin{aligned} [\text{ch}(\text{gr}(\mathcal{O}_{\mathbf{v}, X_w}))] &= (-1)^\ell d \prod_{j=1}^{\ell} (v\gamma_j)^{-1} \\ &= (-1)^\ell d (-1)^{\#\{\gamma_j: v\gamma_j \in \Delta_-\}} \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1} \\ &= (-1)^{\ell(w) - \ell(v)} d \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1}. \end{aligned}$$

This proves the proposition. \square

(5.3) Remark. When the equivalent condition as in the above Proposition (5.2) is satisfied, d in fact is an integer > 0 (as is clear from the above proof).

We recall the definition of a rationally smooth point in a variety Y (cf.[KL, Appendix]).

(5.4) Definition. A variety Y of $\dim d$ is said to be *rationally smooth* if for all $y \in Y$, the singular cohomology $H^i(Y, Y \setminus y, \mathbb{Q}) = 0$ if $i \neq 2d$ and $H^{2d}(Y, Y \setminus y, \mathbb{Q})$ is one-dimensional. A point $y_0 \in Y$ is said to be *rationally smooth* if there exists an open (in the Zariski topology) rationally smooth neighborhood of $y_0 \in Y$.

A smooth point $y_0 \in Y$ is clearly rationally smooth.

The (b) part of the following theorem is the main result of this paper.

(5.5) Theorem. Fix $v \leq w \in W$.

(a) The point $\mathfrak{v} \in X_w$ is rationally smooth \Leftrightarrow
For all $v \leq \theta \leq w$, we have

$$(1) \quad c_{w^{-1}, \theta^{-1}} = d_\theta (-1)^{\ell(w) - \ell(\theta)} \prod_{\beta \in S(w^{-1}, \theta^{-1})} \beta^{-1},$$

for some constants $d_\theta \in \mathbb{C}$.

(b) The point $\mathfrak{v} \in X_w$ is smooth \Leftrightarrow

$$(2) \quad c_{w^{-1}, v^{-1}} = (-1)^{\ell(w) - \ell(v)} \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1}.$$

Proof. (a) By [C, Theorem E], $\mathfrak{v} \in X_w$ is rationally smooth if and only if for all $v \leq \theta \leq w$, $\#S(w^{-1}, \theta^{-1}) = \ell(w)$. By Proposition (5.2), this is equivalent to the requirement that for all $v \leq \theta \leq w$,

$$\text{ch}(\text{gr}(\mathcal{O}_{\theta, X_w})) = d_\theta (-1)^{\ell(w) - \ell(\theta)} \prod_{\beta \in S(w^{-1}, \theta^{-1})} \beta^{-1},$$

for some $d_\theta \in \mathbb{C}$. Now the (a)-part follows from Corollary 3.2(b).

(b) The point $\mathfrak{v} \in X_w$ is smooth if and only if the graded algebra $\text{gr}(\mathcal{O}_{\mathfrak{v}, X_w})$ is isomorphic with the symmetric algebra $S[\text{gr}_1(\mathcal{O}_{\mathfrak{v}, X_w})]$. We first prove the ‘ \Rightarrow ’ implication: So assume that $\mathfrak{v} \in X_w$ is smooth. Then

$$\text{ch}(\text{gr}(\mathcal{O}_{\mathfrak{v}, X_w})) = \text{ch}(S[\text{gr}_1(\mathcal{O}_{\mathfrak{v}, X_w})]) = \prod_{\gamma \in S} (1 - e^\gamma)^{-1},$$

if $\text{ch}(\text{gr}_1(\mathcal{O}_{\mathfrak{v}, X_w})) = \sum_{\gamma \in S} e^\gamma$. It is easy to see that $S \subset v\Delta_+$ and moreover all the weight spaces of $\text{gr}_1(\mathcal{O}_{\mathfrak{v}, X_w})$ are one-dimensional. In particular,

$$(3) \quad c_{w^{-1}, v^{-1}} = [\text{ch}(\text{gr}(\mathcal{O}_{\mathfrak{v}, X_w}))] = \prod_{\gamma \in S \subset v\Delta_+} (-\gamma)^{-1}.$$

But since $\mathfrak{v} \in X_w$ is smooth, in particular, it is rationally smooth. So by the (a)-part of the theorem,

$$(4) \quad c_{w^{-1}, v^{-1}} = (-1)^{\ell(w) - \ell(v)} d_v \prod_{\beta \in S(w^{-1}, v^{-1})} \beta^{-1},$$

for some positive integer d_v (see Remark 5.6(2)).

Equating (3) and (4), we get

$$(5) \quad d_v \prod_{\gamma \in S} \gamma = \pm \prod_{\beta \in S(w^{-1}, v^{-1})} \beta.$$

Let $Q \subset \mathfrak{h}^*$ be the root lattice and let $Q_p := \mathbb{F}_p \otimes_{\mathbb{Z}} Q$ be the reduction mod p (for any prime p) of Q , where \mathbb{F}_p is the prime field of order p . Reducing the equation

(5) mod p (for any prime divisor p of d_v) and observing that no root mod p is 0 in \mathcal{O}_p , we get that $d_v = 1$. This proves the implication ‘ \Rightarrow ’ of the (b)-part.

Conversely, assume that $c_{w-1,v-1} = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w-1,v-1)} \beta^{-1}$. By Corollary 3.2 (b), this gives

$$(6) \quad [\text{ch}(\text{gr}(\mathcal{O}_{\mathfrak{v},X_w}))] = (-1)^{\ell(w)-\ell(v)} \prod_{\beta \in S(w-1,v-1)} \beta^{-1}.$$

By (5) of the proof of Proposition (5.2), we get that

$$(7) \quad [\text{ch}(\text{gr}(\mathcal{O}_{\mathfrak{v},X_w}))] = (-1)^{\ell(w)-\ell(v)} d \prod_{\beta \in S(w-1,v-1)} \beta^{-1},$$

where $d = \dim(\mathcal{O}_{0,Y}/I)$ (the notation is as in the proof of Proposition 5.2).

By comparing (6) and (7), we get that $d = 1$, i.e., I is the maximal ideal of $\mathcal{O}_{0,Y}$. In particular, by (1) of the proof of Proposition (5.2), $\text{gr}(\mathcal{O}_{0,Y})$ is graded isomorphic with the polynomial ring $\mathbb{C}[X_1, \dots, X_\ell]$. So we get that the point $0 \in Y$ is smooth, and hence the point $\mathfrak{v} \in X_w$ is smooth. This proves the theorem completely. \square

(5.6) Remarks. (1) The (a) part of the above theorem can also be proved immediately by combining a result of Dyer [Dy, Proposition §3] with a result of Carrell-Peterson [C, Theorem E], i.e., we can avoid the use of Corollary 3.2(b). But our proof has the advantage that a similar argument (as seen above) gives our criterion for smoothness as in the (b)-part of the above theorem.

(2) In the case (a) as above (i.e. if $\mathfrak{v} \in X_w$ is rationally smooth), the constants d_θ are in fact positive integers for any $v \leq \theta \leq w$ (cf. Remark 5.3).

(3) There are some examples of $\mathfrak{v} \in X_w$ (where X_w is even a codimension one Schubert variety in G/B) such that $c_{w-1,v-1}$ satisfies condition (1) of the above theorem, but \mathfrak{v} is not a rationally smooth point of X_w (cf. Remark 7.9(a)). In particular, to check the rational smoothness of a point $\mathfrak{v} \in X_w$, it is not sufficient (in general) to check the validity of condition (1) only for $\theta = v$.

(4) It is a result of V. V. Deodhar [D] that any rationally smooth Schubert variety is in fact smooth for $G = \text{SL}(n)$. This result has recently been extended for any simply-laced G by D. Peterson. As is well known, this result is false in general for non simply-laced G .

6. Singular locus of Schubert varieties in rank-2 groups

As an immediate corollary of Theorem (5.5), we obtain the following result determining the singular locus of all the Schubert varieties in the case of any rank two group. I believe it should be well known, but I did not find it explicitly written down in the literature. We follow the indexing convention as in Bourbaki [B].

(6.1) Proposition. *The following is a complete description of the singular locus of the Schubert varieties in the case of rank two groups:*

Case I. G of type A_2 : *In this case all the six Schubert varieties are smooth.*

Case II. G of type C_2 : *There are, in all, eight Schubert varieties. Out of these only $X_{r_1 r_2 r_1}$ is singular and it has singular locus $= X_{r_1}$.*

Case III. G of type G_2 : *There are, in all, twelve Schubert varieties. Following is the complete list of singular ones and their singular loci :*

Singular locus

- (1) $X_{r_1 r_2 r_1} - X_{r_1}$
- (2) $X_{r_1 r_2 r_1 r_2} - X_{r_1 r_2}$
- (3) $X_{r_2 r_1 r_2 r_1} - X_{r_2 r_1}$
- (4) $X_{r_1 r_2 r_1 r_2 r_1} - X_{r_1 r_2 r_1}$
- (5) $X_{r_2 r_1 r_2 r_1 r_2} - X_{r_2}$

Proof. As is well known, for any rank-2 group G , any $\mathfrak{v} \in X_w$ is rationally smooth. (This can also be obtained from Theorem 5.5(a) and the following Lemma 6.2.) In particular, $c_{w^{-1}, v^{-1}}$ satisfies identity (1) of Theorem (5.5). Now the proposition follows immediately by combining Theorem (5.5)(b) and the following lemma. \square

The following lemma can be easily proved by a straightforward calculation using the definition of the elements x_{r_i} in the nil Hecke ring Q_W (cf. Definition 3.1(b)).

(6.2) Lemma. *For any group G and any simple reflections $r_1, r_2 \in W$, we have the following (as elements of Q_W):*

$$(a) \quad x_{r_1} x_{r_2} = \frac{1}{\alpha_1} \left(\frac{1}{\alpha_2} \delta_e - \frac{1}{\alpha_2} \delta_{r_2} - \frac{1}{r_1 \alpha_2} \delta_{r_1} + \frac{1}{r_1 \alpha_2} \delta_{r_1 r_2} \right)$$

$$(b) \quad x_{r_1} x_{r_2} x_{r_1} = \frac{1}{\alpha_1} \left(\frac{\alpha_2(\alpha_1^\vee)}{\alpha_2(r_1 \alpha_2)} (\delta_e - \delta_{r_1}) + \frac{1}{\alpha_2(r_2 \alpha_1)} (\delta_{r_2} - \delta_{r_2 r_1}) - \frac{1}{(r_1 \alpha_2)(r_1 r_2 \alpha_1)} (\delta_{r_1 r_2} - \delta_{r_1 r_2 r_1}) \right)$$

$$(c) \quad x_{r_1} x_{r_2} x_{r_1} x_{r_2} = \frac{1}{\alpha_1} \left(\frac{(m-1)}{\alpha_2(r_1 \alpha_2)(r_2 \alpha_1)} (\delta_e - \delta_{r_2}) - \frac{(m-1)}{\alpha_2(r_1 \alpha_2)(r_1 r_2 \alpha_1)} (\delta_{r_1} - \delta_{r_1 r_2}) + \frac{1}{\alpha_2(r_2 \alpha_1)(r_2 r_1 \alpha_2)} (\delta_{r_2 r_1} - \delta_{r_2 r_1 r_2}) - \frac{1}{(r_1 \alpha_2)(r_1 r_2 \alpha_1)(r_1 r_2 r_1 \alpha_2)} (\delta_{r_1 r_2 r_1} - \delta_{r_1 r_2 r_1 r_2}) \right)$$

$$(d) \quad x_{r_1} x_{r_2} x_{r_1} x_{r_2} x_{r_1} = \frac{1}{\alpha_1} \left(\frac{(m-1)(2-m)}{\alpha_2(r_1 \alpha_2)(r_2 \alpha_1)(r_1 r_2 \alpha_1)} (\delta_e - \delta_{r_1}) + \frac{(2-m)\alpha_2(\alpha_1^\vee)}{\alpha_2(r_2 \alpha_1)(r_1 \alpha_2)(r_2 r_1 \alpha_2)} (\delta_{r_2} - \delta_{r_2 r_1}) + \frac{(m-2)\alpha_2(\alpha_1^\vee)}{\alpha_2(r_1 \alpha_2)(r_1 r_2 \alpha_1)(r_1 r_2 r_1 \alpha_2)} (\delta_{r_1 r_2} - \delta_{r_1 r_2 r_1}) + \frac{1}{\alpha_2(r_2 \alpha_1)(r_2 r_1 \alpha_2)(r_2 r_1 r_2 \alpha_1)} (\delta_{r_2 r_1 r_2} - \delta_{r_2 r_1 r_2 r_1}) - \frac{1}{(r_1 \alpha_2)(r_1 r_2 \alpha_1)(r_1 r_2 r_1 \alpha_2)(r_1 r_2 r_1 r_2 \alpha_1)} (\delta_{r_1 r_2 r_1 r_2} - \delta_{r_1 r_2 r_1 r_2 r_1}) \right),$$

where $m := \alpha_1(\alpha_2^\vee)\alpha_2(\alpha_1^\vee)$.

7. Singularity of codimension one Schubert varieties in G/B

Let w_o be the longest element of the Weyl group W (of G). As is well known, the codimension one Schubert varieties in G/B are precisely of the form Y

where $w = w_o r_i$ for a simple reflection r_i . In particular, the number of such Schubert varieties in G/B is equal to $n := \text{rank } G$. We denote the Schubert variety $X_{w_o r_i}$ ($1 \leq i \leq n$) by X_i . Let $\chi_i \in \mathfrak{h}_{\mathbb{Z}}^*$ be the i^{th} ($1 \leq i \leq n$) fundamental weight, defined by $\chi_i(\alpha_j^\vee) = \delta_{i,j}$.

(7.1) Proposition. *Fix any $1 \leq i \leq n$. Then for any $v \in W$ such that $v \leq w_o r_i$,*

$$(1) \quad c_{r_i w_o, v^{-1}} = [\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{v}, X_i})] = (-1)^{|\Delta_+| - \ell(v)} \frac{1}{\prod_{\beta \in \Delta_+} \beta} (w_o \chi_i - v \chi_i),$$

where $[]$ is as in § 3.1(a).

Proof. Consider the i^{th} fundamental representation $V(\chi_i)$ (with highest weight χ_i) and define the function

$$\varphi = \varphi_{i,v} : \mathfrak{u}^- \rightarrow \mathbb{C} \text{ by } \varphi(X) = \langle \exp X \cdot e_{\chi_i}, \bar{v}^{-1} e_{w_o \chi_i}^* \rangle, \text{ for } X \in \mathfrak{u}^-;$$

where \bar{v} is a representative of v in $N(T)$, e_{χ_i} (resp. $e_{w_o \chi_i}$) is a non-zero vector in $V(\chi_i)$ of weight χ_i (resp. $w_o \chi_i$) and $e_{w_o \chi_i}^* \in V(\chi_i)^*$ is defined by $e_{w_o \chi_i}^*(e_{w_o \chi_i}) = 1$ and $e_{w_o \chi_i}^*(v_\mu)$, for any weight vector $v_\mu \in V(\chi_i)$ of weight $\mu \neq w_o \chi_i$. Let Y be the closed subvariety of the affine space \mathfrak{u}^- defined as $Y = \exp^{-1}(U^{-\mathfrak{e}} \cap v^{-1} X_i)$ (cf. proof of Proposition 5.2). It is easy to see that $Y \subset \mathfrak{u}^-$ is defined set-theoretically by the vanishing of the function $\varphi : \mathfrak{u}^- \rightarrow \mathbb{C}$ (use Lemma 7.2). Moreover φ is obtained by restricting the section $\chi(\bar{v}^{-1} e_{w_o \chi_i}^*) \in H^0(G/B, \mathfrak{L}(\chi_i))$ to $U^{-\mathfrak{e}}$ (and using the identification $\exp: \mathfrak{u}^- \rightarrow U^{-\mathfrak{e}} \subset G/B$), where χ is the Borel-Weil homomorphism (cf. Proof of Lemma 2.4). But the line bundle $\mathfrak{L}(\chi_i)$ on G/B corresponds to the irreducible divisor $X_i \subset G/B$ with multiplicity 1 (use, e.g., the Chern class calculation for the line bundle $\mathfrak{L}(\chi_i)$). This, in particular, implies that the ideal I of the irreducible hypersurface $Y \subset \mathfrak{u}^-$ (with the reduced structure) is generated by the function φ (cf. also [C2, Proposition 4.6]). This gives that (as graded T -algebras),

$$(2) \quad \text{gr}(\mathcal{O}_{\mathfrak{e}, v^{-1} X_i}) \approx S(\mathfrak{u}^{-*}) / \langle [\varphi] \rangle,$$

where (as earlier) $S(\mathfrak{u}^{-*})$ is the symmetric algebra of \mathfrak{u}^{-*} and $\langle [\varphi] \rangle$ denotes the (homogeneous) ideal generated by the least degree non-zero homogeneous component $[\varphi]$ of φ . From the definition of φ , it is easy to see that $[\varphi]$ is a weight vector for the adjoint action of T on \mathfrak{u}^- with weight $\chi_i - v^{-1} w_o \chi_i$. So by (2),

$$\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{e}, v^{-1} X_i}) = (1 - e^{\chi_i - v^{-1} w_o \chi_i}) \prod_{\beta \in \Delta_+} (1 - e^\beta)^{-1},$$

and hence

$$(3) \quad [\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{e}, v^{-1} X_i})] = (-1)^{\#\Delta_+} \frac{(v^{-1} w_o \chi_i - \chi_i)}{\prod \beta}.$$

(Observe that by Lemma (7.2), $v^{-1}w_0\chi_i - \chi_i \neq 0$, since by assumption $v \leq w_0r_i$.) By applying v to (3) we get

$$[\text{ch}(\text{gr } \mathcal{O}_{\mathfrak{v}, X_i})] = (-1)^{\#\Delta_+ - \ell(v)} \frac{(w_0\chi_i - v\chi_i)}{\prod_{\beta \in \Delta_+} \beta}.$$

This proves the second equality of (1). First equality of (1) of course follows from Corollary 3.2(b). \square

(7.2) Lemma. *For any simple reflection r_i and any $v \in W$, $v \leq w_0r_i$ if and only if $\chi_i \neq v^{-1}w_0\chi_i$.*

Proof. Let $Z \subset G$ be the zero set of the function $\hat{\varphi} : G \rightarrow \mathbb{C}$ given by $\hat{\varphi}(g) = \langle ge_{\chi_i}, e_{w_0\chi_i}^* \rangle$ (where e_{χ_i} and $e_{w_0\chi_i}^*$ are as in the proof of Proposition 7.1). Then clearly Z is B -stable under the left as well as right multiplication. In particular, $Z/B = \cup X_j$, where j runs over some subset $S \subset \{1, \dots, n\}$. Clearly $i \in S$, whereas for $j \neq i$, $j \notin S$, and hence $Z/B = X_i$. Hence $v \leq w_0r_i \Leftrightarrow v \in X_i = Z \Leftrightarrow v\chi_i \neq w_0\chi_i$. \square

(7.3) Lemma. *Assume that $v \leq w_0r_i$. Then $\chi_i - v^{-1}w_0\chi_i$ is multiple of a root β if and only if $\pm v\beta \notin S(r_iw_0, v^{-1})$. In particular, $\chi_i - v^{-1}w_0\chi_i$ is multiple of a root if and only if $\#S(r_iw_0, v^{-1}) = N - 1$, where $N := \#\Delta_+$.*

Proof. If $\pm v\beta \notin S(r_iw_0, v^{-1})$, then by the above Lemma (7.2), $r_\beta v^{-1}w_0\chi_i = \chi_i$. In particular, $\chi_i - v^{-1}w_0\chi_i$ is a multiple of β .

Conversely, assume that

$$(1) \quad \chi_i - v^{-1}w_0\chi_i = n\beta,$$

for some number n and $\beta \in \Delta$. By Lemma (7.2), $n \neq 0$. To prove that $\pm v\beta \notin S(r_iw_0, v^{-1})$, it suffices to show (again by Lemma 7.2) that $r_\beta v^{-1}w_0\chi_i = \chi_i$: By (1),

$$(2) \quad \langle \chi_i - v^{-1}w_0\chi_i, \beta^\vee \rangle = 2n, \quad \text{and}$$

$$(3) \quad \langle \chi_i + v^{-1}w_0\chi_i, \beta^\vee \rangle = \frac{2}{n\langle \beta, \beta \rangle} \langle \chi_i + v^{-1}w_0\chi_i, \chi_i - v^{-1}w_0\chi_i \rangle = 0.$$

Combining (2) and (3) we get $\langle -v^{-1}w_0\chi_i, \beta^\vee \rangle = n$; and hence $r_\beta v^{-1}w_0\chi_i := v^{-1}w_0\chi_i - \langle v^{-1}w_0\chi_i, \beta^\vee \rangle \beta = v^{-1}w_0\chi_i + n\beta = \chi_i$ (by (1)).

The ‘in particular’ statement of the lemma follows from Deodhar’s conjecture (cf. Theorem 5.1). \square

By virtue of Proposition (7.1), Lemma (7.3), and Theorem 5.5(b), we get the following characterization of the smooth points in the Schubert varieties X_i .

(7.4) Proposition. *Let X_i ($1 \leq i \leq n$) be a codimension one Schubert variety. Then, for any $v \leq w_0r_i \in W$, the following are equivalent:*

- (a₁) $\mathfrak{v} \in X_i$ is smooth.
- (a₂) $c_{r_iw_0, v^{-1}} = (-1)^{N-1-\ell(v)} \frac{1}{\beta_1 \cdots \beta_{N-1}}$, for some positive roots $\{\beta_1, \dots, \beta_{N-1}\}$ (where $N = \dim G/B$).
- (a₃) $v\chi_i - v^{-1}w_0\chi_i$ is a root.

In particular, X_i is smooth if and only if $\chi_i - w_0\chi_i$ is a root.

(7.5) Remark. If $\mathfrak{v} \in X_i$ is smooth, then the set $\{\beta_1, \dots, \beta_{N-1}\}$, as in (a₂) above, coincides with the set $S(r_i w_0, v^{-1})$ (by Theorem 5.5 (b)).

Proof (of Proposition 7.4). As follows from Theorem 5.5(b), (a₁) \Rightarrow (a₂). The implication (a₂) \Rightarrow (a₃) follows from Proposition (7.1). So we come to the proof of (a₃) \Rightarrow (a₁):

By Theorem 5.5(b), we need to show that

$$(1) \quad c_{r_i w_0, v^{-1}} = (-1)^{N-1-\ell(v)} \prod_{\beta \in S(r_i w_0, v^{-1})} \beta^{-1}.$$

By (a₃), $\gamma := v\chi_i - w_0\chi_i$ is a root (and in fact is positive since $v \leq w_0$). In particular, by Proposition (7.1),

$$(2) \quad c_{r_i w_0, v^{-1}} = (-1)^{N-1-\ell(v)} \frac{\gamma}{\prod_{\beta \in \Delta_+} \beta}.$$

But by Lemma (7.3), $S(r_i w_0, v^{-1}) = \Delta_+ \setminus \{\gamma\}$, and hence (1) follows from (2). This proves the implication (a₃) \Rightarrow (a₁).

The ‘in particular’ statement of the proposition follows from the equivalence of (a₁) and (a₃) since X_i is smooth if and only if $\mathfrak{e} \in X_i$ is smooth. \square

By the same proof as above for the implication (a₃) \Rightarrow (a₁) (alternatively, by using Lemma (7.3) with [C, Theorem E]) we obtain the following:

(7.6) Corollary. *With the notation as in Proposition (7.4), $\mathfrak{v} \in X_i$ is rationally smooth if and only if for all $v \leq \theta \leq w_0 r_i$, $\chi_i - \theta^{-1} w_0 \chi_i$ is multiple of a root β_θ (depending upon θ).* \square

We follow the indexing convention of simple roots as in [B, Planche I-IX]. The following lemma follows easily from the explicit knowledge of roots, coroots, fundamental weights etc. as given in loc. cit.

(7.7) Lemma. *Let G be a simple algebraic group. Then for any fundamental weight χ_i ($1 \leq i \leq n$),*

(a) $\chi_i - w_0\chi_i$ is a (positive) root precisely in the following cases (A_n etc. denotes the type of G):

$$\begin{aligned} (a_1) \quad & A_n \quad (n \geq 1) \quad ; \quad i = 1, n \\ (a_2) \quad & C_n \quad (n \geq 2) \quad ; \quad i = 1. \end{aligned}$$

(b) $\chi_i - w_0\chi_i$ is multiple of a root but not a root itself, precisely in the following cases:

$$\begin{aligned} (b_1) \quad & B_n \quad (n \geq 3) \quad ; \quad i = 1, 2 \\ (b_2) \quad & C_n \quad (n \geq 2) \quad ; \quad i = 2 \\ (b_3) \quad & D_n \quad (n \geq 4) \quad ; \quad i = 2 \\ (b_4) \quad & E_6 \quad ; \quad i = 2 \\ (b_5) \quad & E_7 \quad ; \quad i = 1 \\ (b_6) \quad & E_8 \quad ; \quad i = 8 \\ (b_7) \quad & F_4 \quad ; \quad i = 1, 4 \\ (b_8) \quad & G_2 \quad ; \quad i = 1, 2. \end{aligned}$$

As a consequence of the above lemma, we get the following complete list of codimension-1 Schubert varieties which are smooth or rationally smooth.

We assume that G is a simple group in the following proposition

(7.8) Proposition. (c) *The following is a complete list of codimension one Schubert varieties X_i which are smooth:*

$$\begin{aligned} (c_1) \quad A_n \quad (n \geq 1) & : i = 1, n \\ (c_2) \quad C_n \quad (n \geq 2) & : i = 1. \end{aligned}$$

(d) *The following is a complete list of codimension one Schubert varieties X_i which are rationally smooth but not smooth:*

$$\begin{aligned} (d_1) \quad C_2 : i = 2 \\ (d_2) \quad G_2 : i = 1, 2 \\ (d_3) \quad B_n \quad (n \geq 3) : i = 1. \end{aligned}$$

Proof. The (c)-part follows immediately by combining Proposition (7.4) with Lemma (7.7).

To prove the (d)-part, in view of Corollary (7.6) and Lemma (7.7), it suffices to show that in all the cases covered by (b) of Lemma (7.7) but not in the list (d) above, there exists a $\theta \in W$ such that $\chi_i - \theta^{-1}w_0\chi_i$ is not a multiple of any root (by Lemma 7.2, such a θ will automatically satisfy $\theta \leq w_0r_i$), whereas in the cases covered by (d), $\chi_i - \theta^{-1}w_0\chi_i$ is indeed multiple of a root for any $\theta \in W$:

We freely use the notation without explanation from [B; Planche I-IX]. In the cases $(B_{n \geq 3}; i = 2)$, $(C_{n \geq 3}; i = 2)$ and $(D_{n \geq 4}; i = 2)$ take any $\theta \in W$ satisfying $\theta(\epsilon_1) = \epsilon_1$, $\theta(\epsilon_3) = \epsilon_2$. Then $\chi_i - \theta^{-1}w_0\chi_i$ is not a multiple of any root.

In the cases $(E_6; i = 2)$, $(E_7; i = 1)$, and $(E_8; i = 8)$, χ_i is the highest root α_0 . In these cases take any $\theta \in W$ satisfying $\theta(\alpha_2) = \alpha_0$ (observe that $-w_0\alpha_0 = \alpha_0$ and the W -orbit $W \cdot \alpha_0$ consists of all the roots), then $\chi_i - \theta^{-1}w_0\chi_i$ is not a multiple of any root.

In the case $(F_4; i = 1)$, (resp. $F_4; i = 4$), χ_1 (resp. χ_4) is the highest (resp. a short) root, in particular, $W \cdot \chi_1$ consists of all the long (resp. short) roots. Take any $\theta \in W$ satisfying $\theta(\epsilon_2 + \epsilon_3) = \chi_1$ (resp. $\theta(\frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4}{2}) = \chi_4$), then $\chi_i - \theta^{-1}w_0\chi_i$ is not a multiple of any root.

For $(C_2; i = 2)$ and $(G_2; i = 1, 2)$, it is easy to see that $\chi_i - \theta^{-1}w_0\chi_i$ is multiple of a root, for all $\theta \in W$.

So finally we come to $(B_{n \geq 3}; i = 1)$: In this case, $-w_0 = \text{Id.}$, $\chi_1 = \epsilon_1$ (a short root), and hence $W \cdot \chi_1 = \{\pm\epsilon_i\}_{1 \leq i \leq n}$. In particular, $\chi_1 - \theta^{-1}w_0\chi_1$ is multiple of a root for all $\theta \in W$. This finishes the proof of the (d)-part of the proposition. \square

(7.9) Remarks. (a) In all the cases covered by Lemma 7.7(b) but not contained in Proposition 7.8(d), identity (1) of Theorem 5.5(a) is satisfied for $w = w_0r_i$ and $\theta = e$ but is violated for some $e \leq \theta \leq w$ (use Proposition 7.1, Lemma 7.3 and Theorem 5.5(a)).

(b) I am informed that the (c) part of the above proposition, as well as the equivalence of (a_1) and (a_3) in Proposition (7.4) for $v = e$ was contained in an earlier longer version of [C] (cf. [C2, §4]). Of course (d_1) , (d_2) are very well known, and example (d_3) was known to be rationally smooth due to Boe [Bo].

8. Extension of results to the Kac-Moody case

(8.1) Notation. We will follow the notation (often without explaining) from [Ku; §1]: In particular, throughout this section $G = G(A)$ denotes the complex Kac-Moody group associated to an arbitrary $n \times n$ generalized Cartan matrix A (we

do not put symmetrizability restriction on A), with the standard Borel subgroup B , and the standard maximal torus $T \subset B$. There is a Weyl group $W \approx N(T)/T$ associated to the pair (G, T) (where $N(T)$ is the normalizer of T in G). The Weyl group W is a Coxeter group with the simple reflections $\{r_i\}_{1 \leq i \leq n}$ as Coxeter generators (r_i is nothing but the reflection through the simple root α_i). Hence, for any $w \in W$, we can talk of its length $\ell(w)$ and also have Bruhat partial ordering \leq in W .

The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ admits the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+ \subset \mathfrak{h}^*} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [h, X] = \alpha(h)X, \text{ for all } h \in \mathfrak{h}\}$ is the α -th root space, $\mathfrak{h} := \text{Lie } T$ is the standard Cartan subalgebra of \mathfrak{g} , and $\Delta_+ := \{\alpha \neq 0 \in \sum_{i=1}^n \mathbb{Z}_+ \alpha_i : \mathfrak{g}_\alpha \neq 0\}$ is the set of positive roots. We denote $\Delta_- = -\Delta_+$ and $\Delta := \Delta_+ \cup \Delta_-$. The Weyl group W preserves Δ . The set of real roots $\Delta^{\text{re}} \subset \Delta$ is defined to be $W \cdot \{\alpha_1, \dots, \alpha_n\}$ and the set of imaginary roots $\Delta^{\text{im}} := \Delta \setminus \Delta^{\text{re}}$. We set $\Delta_+^{\text{re}} = \Delta_+ \cap \Delta^{\text{re}}$ (resp. $\Delta_-^{\text{re}} = \Delta_- \cap \Delta^{\text{re}}$); Δ_+^{im} and Δ_-^{im} have similar meanings. We denote by $\tilde{\Delta}_+$ (resp. $\tilde{\Delta}_-$) the indexed set of positive (resp. negative) roots such that each root occurs exactly as many times as the dimension of the corresponding root space. Recall that the real root spaces are of dimension one.

The group G (in particular the torus T) acts on G/B by the left multiplication. For any $w \in W$, the Schubert variety X_w is by definition the closure of $B\bar{w}B/B$ in G/B , where \bar{w} is a preimage of w in $N(T)$ and G/B is endowed with the Zariski topology as in [S]. Of course, X_w is T -stable. As is well known (by the Bruhat decomposition), $X_w = \bigcup_{v \leq w} B\bar{v}B/B$. In particular, for any $v \leq w$, $\mathfrak{v} := \bar{v}B \in X_w$ and it is a T -fixed point. We will always endow X_w with the stable variety structure as given in [Ku; §1]. With this structure X_w is an irreducible projective variety of $\dim \ell(w)$.

For any real root β , there exists a unique additive one-parameter subgroup U_β and a homomorphism $u_\beta : \mathbb{C} \rightarrow G$ satisfying $u_\beta(\mathbb{C}) = U_\beta$ and such that

$$tu_\beta(z)t^{-1} = u_\beta(e^\beta(t)z),$$

for any $z \in \mathbb{C}$, and $t \in T$. Furthermore, for any $w \in W$, $\bar{w}U_\beta\bar{w}^{-1} = U_{w\beta}$.

Now let U^- be the subgroup of G generated by the one-parameter groups $\{U_\beta\}_{\beta \in \Delta_-^{\text{re}}}$. Then the map $U^- \rightarrow G/B$, taking $g \mapsto g\mathfrak{e}$ is injective and moreover $U^- \mathfrak{e} \subset G/B$ is an open subset.

For any $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, recall the definition of the line bundle $\mathfrak{L}(\lambda) := G \times_{\mathbb{C}_{-\lambda}} \rightarrow G/B$ from [Ku; §2.2]. For dominant $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, let $V^{\max}(\lambda)$ be the maximal integrable highest weight G -module with highest weight λ (cf. [Ku, §1.5], where it is denoted by $L^{\max}(\lambda)$). Define

$$H^0(G/B, \mathfrak{L}(\lambda)) = \text{Inv limit}_{w \in W} H^0(X_w, \mathfrak{L}(\lambda)|_{X_w}).$$

The highest weight space $\mathbb{C}_\lambda := V^{\max}(\lambda)_{(\lambda)}$ of $V^{\max}(\lambda)$ is one dimensional. Define the map

$$\chi = \chi_\lambda : V^{\max}(\lambda)^* \longrightarrow H^0(G/B, \mathfrak{L}(\lambda))$$

by $\chi(f)(gB) = (g, (g^{-1}f)|_{\mathbb{C}_\lambda}) \bmod B$, for $f \in V^{\max}(\lambda)^*$, and $g \in G$.

The following result is due to Kumar [Ku, Theorem 2.16] (and also Mathieu [M-1])

(8.2) Theorem. *The map χ_λ as above is an isomorphism. Moreover, for any $v \leq w \in W$, it induces an isomorphism*

$$\chi_\lambda(v, w) : (v^{-1}V_w^{\max}(\lambda))^* \xrightarrow{\sim} H^0(v^{-1}X_w, \mathfrak{L}(\lambda)|_{v^{-1}X_w}),$$

making the following diagram commutative:

$$\begin{array}{ccc} V^{\max}(\lambda)^* & \xrightarrow{\chi_\lambda} & H^0(G/B, \mathfrak{L}(\lambda)) \\ \downarrow & & \downarrow \\ (v^{-1}V_w^{\max}(\lambda))^* & \xrightarrow{\chi_\lambda(v, w)} & H^0(v^{-1}X_w, \mathfrak{L}(\lambda)|_{v^{-1}X_w}) \end{array}$$

where $V_w^{\max}(\lambda) \subset V^{\max}(\lambda)$ is the B -submodule generated by the extremal weight space $V^{\max}(\lambda)_{(w\lambda)}$ of weight $w\lambda$, and the vertical maps are the canonical restriction maps.

For any non-zero $e_\lambda \in \mathbb{C}_\lambda$, define $e_\lambda^* \in V^{\max}(\lambda)^*$ as $e_\lambda^*(e_\lambda) = 1$ and $e_\lambda^*(y) = 0$, for any weight-vector y of weight $\mu \neq \lambda$. Now define the section $s_{e_\lambda} \in H^0(G/B, \mathfrak{L}(\lambda))$ by $s_{e_\lambda} = \chi_\lambda(e_\lambda^*)$.

The following lemma follows immediately from the Birkhoff decomposition [KP, §3].

(8.3) Lemma. *The zero set of s_{e_λ} , $Z(s_{e_\lambda}) = G/B \setminus (U^- \mathfrak{e})$, if $\lambda \in D^\circ$, where (as in §1) D° is the set of dominant regular weights. \square*

The line bundle $\mathfrak{L}(\lambda)|_{v^{-1}X_w}$ on the projective variety $v^{-1}X_w$ is ample for any $v \leq w \in W$ and $\lambda \in D^\circ$. In particular, by Lemmas (2.3) and (8.3), $U^- \mathfrak{e} \cap v^{-1}X_w$ is an affine open subset of $v^{-1}X_w$.

Define the T -equivariant map (cf. §2.6)

$$\varphi_\lambda(v, w) : (v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda \rightarrow \mathbb{C}[U^- \mathfrak{e} \cap v^{-1}X_w]$$

$$(\varphi_\lambda(v, w)(f \otimes e_\lambda))(x) s_{e_\lambda}(x) = (\chi_\lambda(v, w)f)(x),$$

for $f \in (v^{-1}V_w^{\max}(\lambda))^*$, $e_\lambda \neq 0 \in \mathbb{C}_\lambda$ and $x \in U^- \mathfrak{e} \cap v^{-1}X_w$. (We set $\varphi_\lambda(v, w)(f \otimes 0) = 0$.) By Lemma (8.3), the map $\varphi_\lambda(v, w)$ is well defined, and is injective by Theorem (8.2). Moreover, as in §2.7, for any $\lambda \in D^\circ$ and $\mu \in D$, the following diagram is commutative:

$$\begin{array}{ccc} (v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda & \xrightarrow{\delta_{\lambda, \mu}(v, w)} & (v^{-1}V_w^{\max}(\lambda + \mu))^* \otimes \mathbb{C}_{\lambda + \mu} \\ \varphi_\lambda(v, w) \searrow & & \swarrow \varphi_{\lambda + \mu}(v, w) \\ & \mathbb{C}[U^- \mathfrak{e} \cap v^{-1}X_w], & \end{array}$$

where the map $\delta_{\lambda, \mu}(v, w)$ is defined as in Lemma (2.7). Taking the limit of the maps $\varphi_\lambda(v, w)$, we get the T -equivariant map

$$\varphi(v, w) : \lim_{\lambda \in D^\circ} \rightarrow ((v^{-1}V_w^{\max}(\lambda))^* \otimes \mathbb{C}_\lambda) \rightarrow \mathbb{C}[U^- \mathfrak{e} \cap v^{-1}X_w].$$

The following proposition follows easily from Lemma (2.2) and Theorem (8.2)

(8.4) Proposition. *The above map $\varphi(v, w)$ is an isomorphism for any $v \leq w \in W$. \square*

Define the Lie subalgebra $\mathfrak{u}^- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ of \mathfrak{g} and (for any $m > 0$) the ideal \mathfrak{u}_m^- of \mathfrak{u}^- by

$$\mathfrak{u}_m^- = \bigoplus_{\substack{\alpha \in \Delta_- \\ |\alpha| \geq m}} \mathfrak{g}_\alpha,$$

where for a root

$$\alpha = \sum m_i \alpha_i, \quad |\alpha| := \left| \sum m_i \right|.$$

The quotient algebra $F_m(\mathfrak{u}^-) := \mathfrak{u}^- / \mathfrak{u}_m^-$ is a finite dimensional nilpotent algebra. Let $F_m(U^-)$ be the associated unipotent complex algebraic group. Corresponding to the Lie algebra homomorphism $\mathfrak{u}^- \rightarrow F_m(\mathfrak{u}^-)$, there is associated a group homomorphism $\theta_m : U^- \rightarrow F_m(U^-)$. We state the following simple lemma without proof.

(8.5) Lemma. *Fix $v \leq w \in W$. Then there exists a positive number $m_0(v, w)$ such that*

$$\theta_m(v, w) : U^- \mathfrak{e} \cap v^{-1} X_w \rightarrow F_m(U^-)$$

(got by restricting the map θ_m) is a closed immersion for all $m \geq m_0(v, w)$.

By an argument identical to the proof of Theorem (2.2) (as given in §2.12), and Corollaries (3.2) (using Proposition 8.4, Lemma 8.5, and [Ku, Theorem 3.4]) we get the following analog of Theorem (2.2) and Corollaries (3.2) for an arbitrary Kac-Moody group G .

(8.6) Theorem. *Let G be an arbitrary Kac-Moody group.*

(a) *For any $v \leq w \in W$, $gr \mathcal{O}_{\mathfrak{v}, X_w}$ is an admissible T -module and moreover*

$$ch (gr \mathcal{O}_{\mathfrak{v}, X_w}) = *b_{w^{-1}, v^{-1}},$$

as elements of $\widetilde{Q}(T)$.

(b) *For any $v \leq w \in W$, $b_{w^{-1}, v^{-1}} \neq 0$ if and only if $v \leq w$, and in this case it has a pole of order exactly equal to $\ell(w)$. Further, there exist $\beta_1, \dots, \beta_N \in \check{\Delta}^+$ (for some $N > 0$) such that*

$$\left(\prod_{j=1}^N (1 - e^{\beta_j}) \right) b_{w^{-1}, v^{-1}} \in R(T).$$

(c) *$[*b_{w^{-1}, v^{-1}}] = c_{w^{-1}, v^{-1}}$; and hence for any $v \leq w$, $[ch (gr \mathcal{O}_{\mathfrak{v}, X_w})] = c_{w^{-1}, v^{-1}}$, as elements of $Q(\mathfrak{h})$.*

In particular, $c_{w, v} \neq 0$ if and only if $v \leq w$. \square

We extend Proposition (5.2) to the Kac-Moody case.

(8.7) Proposition. *Let G be an arbitrary Kac-Moody group and let $v \leq w \in W$. Then*

$$\#S_{w^{-1}, v^{-1}} = \ell(w) \Leftrightarrow [ch (gr \mathcal{O}_{\mathfrak{v}, X_w})] = d(-1)^{\ell(w) - \ell(v)} \prod \beta^{-1},$$

for some $d \in \mathbb{C}$; where $S(w^{-1}, v^{-1}) = \{\alpha \in \Delta_+^{\text{re}} : v^{-1}r_\alpha \leq w^{-1}\}$.

Proof. The proof is very similar to the proof of Proposition (5.2). But we need to make the following modifications:

Define $Y = U^{-\mathfrak{e}} \cap v^{-1}X_w$. Fix any regular $\lambda \in D^o$ and a highest weight vector $e_\lambda \in V^{\max}(\lambda)$ and consider the element $e_\lambda^* \in V^{\max}(\lambda)^*$ as in §8.2. For any root $\alpha \in \Delta_+^{\text{re}}$, choose a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$ and define the map $\theta_\alpha : U^- \rightarrow \mathbb{C}$ by $\theta_\alpha(g) = e_\lambda^*(X_\alpha g e_\lambda)$, for $g \in U^-$. We claim that $\theta_\alpha(g) \neq 0$, for any $g \neq e \in U_{-\alpha}$:

Write $g = \exp(zX_{-\alpha})$, for some $z \neq 0 \in \mathbb{C}$; where $X_{-\alpha}$ is the root vector corresponding to the (real) root $-\alpha$ such that $[X_\alpha, X_{-\alpha}] = \alpha^\vee$ (cf. [K, exercise 5.1]). Then

$$\begin{aligned} \theta_\alpha(g) &= e_\lambda^*(X_\alpha \exp(zX_{-\alpha})e_\lambda) \\ &= e_\lambda^*(zX_\alpha X_{-\alpha}e_\lambda) \\ &= e_\lambda^*(z[X_\alpha, X_{-\alpha}]e_\lambda) \\ &= z\langle \lambda, \alpha^\vee \rangle \\ &\neq 0, \quad \text{since } \lambda \text{ is regular.} \end{aligned}$$

Identifying $U^- \simeq U^{-\mathfrak{e}}$, we can (and do) consider θ_α as a function on Y . Now define

$$Z_S = \{x \in Y : \theta_\alpha(x) = 0, \text{ for all } \alpha \in S := S(w, v)\}.$$

Rest of the argument to prove the proposition is similar to the proof of Proposition (5.2) provided we replace \mathfrak{u}^- by U^- and use the following simple

(8.8) Lemma. *For any $v \leq w \in W$, one dimensional T -orbits in $U^{-\mathfrak{e}} \cap v^{-1}X_w$ are precisely of the form $(U_{-\beta} \setminus e)\mathfrak{e}$, where β ranges over (positive real) roots $\in S(w, v)$.*

Proof. By the Bruhat decomposition

$$X_w = \bigcup_{\theta \leq w} U\theta\mathfrak{e} = \bigcup_{\theta \leq w} \theta(\theta^{-1}U\theta \cap U^-)\mathfrak{e},$$

one-dimensional T -orbits contained in $v^{-1}X_w$ are precisely of the form $I_{\theta, \beta} := v^{-1}\theta(U_{-\beta} \setminus e)\mathfrak{e}$, where $\theta \leq w$ and $\beta \in \Delta_+ \cap \theta^{-1}\Delta_-$. (We are using the fact that any root in $\Delta_+ \cap \theta^{-1}\Delta_-$ is a real root and moreover for any real root β , $d\beta$ is not a root for any $d > 1$.) If $v = \theta$, clearly $I_{\theta, \beta} \subset U^{-\mathfrak{e}}$, and moreover $\beta \in \Delta_+ \cap v^{-1}\Delta_- \Leftrightarrow \beta \in \Delta_+^{\text{re}}$ and $vr_\beta < v$ (by [BG, Corollary 2.3]). So assume that $v \neq \theta$. By Bruhat decomposition for $SL(2)$, we get

$$(U_{-\beta}\mathfrak{e}) \cup \{r_\beta\mathfrak{e}\} = \overline{Br_\beta B/B} \subset G/B,$$

where the closure is taken with respect to the (inductive limit) Zariski topology on G/B . In particular,

$$\overline{I_{\theta, \beta}} \setminus I_{\theta, \beta} = \{\bar{v}^{-1}\bar{\theta}\mathfrak{e}, \bar{v}^{-1}\bar{\theta}r_\beta\mathfrak{e}\},$$

where \bar{v} is a preimage of v in $N(T)$. By Lemma (8.5), it is easy to see that any closed T -stable subset of $U^{-\mathfrak{e}}$ (under the induced subspace topology on $U^{-\mathfrak{e}} \subset G/B$) contains \mathfrak{e} . Hence (if $v \neq \theta$)

i.e. $v = \theta r_\beta$. Again by the Bruhat decomposition for $SL(2)$, it is easy to see that in this case (i.e. $\theta r_\beta = v$) $v^{-1}\theta(U_{-\beta} \setminus e)\mathfrak{e} = (U_{-\beta} \setminus e)\mathfrak{e}$.

But by [BGG, Corollary 2.3],

$$\{\beta \in \Delta_+^{\text{re}} : \beta \in \Delta_+ \cap (r_\beta v^{-1} \Delta_-) \text{ and } vr_\beta \leq w\} = \{\beta \in S(w, v) : v < vr_\beta\}.$$

This proves the lemma. \square

Now by an argument identical to the proof of Theorem (5.5), we obtain the following.

(8.9) Theorem. *Theorem (5.5) is true for an arbitrary Kac-Moody group.* \square

(8.10) Remarks. Even though we have taken the base field to be the field \mathbb{C} of complex numbers, all the results of the paper carry over (with the same proofs) to an arbitrary algebraically closed field of char. 0.

Also, by a result of Polo [P, §4.1], the dimension of the Zariski tangent space $Z_{\mathfrak{v}}(X_w)$ is independent of the char. of the field. In particular, a point $\mathfrak{v} \in X_w$ is smooth in char. 0 if and only if it is smooth in any char. p . So our smoothness criterion (as in Theorem 5.5(b)) works in arbitrary char. p .

REFERENCES

- [A] Andersen, H. H., *Schubert varieties and Demazure character formula*, Invent. Math. **79** (1985), 611-618.
- [BGG] Bernstein, I. N., Gel'fand, I. M., and Gel'fand, S. I., *Schubert cells and cohomology of the spaces G/P* , Russian Math. Surveys **28** (1973), 1-26.
- [Bo] Boe, B. D., *Kazhdan-Lusztig polynomials for hermitian symmetric spaces*, Trans. A.M.S. **309** (1988), 279-294.
- [B] Bourbaki, N., "Groupes et algèbres de Lie, Chap. IV-VI," Hermann, Paris 1968.
- [C] Carrell, J. B., *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Proc. Symp. Pure Math. **56** (1994) (edited by W. J. Haboush and B. J. Parshall), 53-61.
- [C2] Carrell, J. B., *On the singular locus of a Schubert variety: A survey*, Preprint (1994).
- [CPS] Cline, E., Parshall, B., Scott, L., *Cohomology, hyperalgebras and representations*, J. of Algebra **63** (1980), 98-123.
- [D] Deodhar, V. V., *Local Poincaré duality and non-singularity of Schubert varieties*, Comm. in Algebra **13**, no. 6 (1985), 1379-1388.
- [D2] Deodhar, V. V., *A brief survey of Kazhdan-Lusztig theory and related topics*, Preprint (1993).
- [Dy] Dyer, M. J., *The nil Hecke ring and Deodhar's conjecture on Bruhat intervals*, Invent. Math. **111** (1993), 571-574.
- [F] Fulton, W., "Introduction to Intersection Theory in Algebraic Geometry," CBMS regional conference series in Mathematics no. **54** (1984), Am. Math. Soc..
- [Ha] Harris, J., "Algebraic Geometry," Springer-Verlag, Berlin-Heidelberg-New York (1992).
- [H] Hartshorne, R., "Algebraic Geometry," Springer-Verlag, Berlin-Heidelberg-New York (1977).

- [J] Jantzen, J. C., “*Moduln mit einem höchsten Gewicht*,” LNM vol. **750**, Springer–Verlag, Berlin–Heidelberg–New York (1979).
- [Jo] Joseph, A., *On the variety of a highest weight module*, J. of Algebra **88** (1984), 238–278.
- [Jo2] Joseph, A., *On the Demazure character formula*, Ann. Sci. Ec. Norm. Sup. **18** (1985), 389–419.
- [K] Kac, V. G., “*Infinite dimensional Lie algebras*,” Third edition, Cambridge University Press (1990).
- [KP] Kac, V. G., and Peterson, D. H., *Regular functions on certain infinite dimensional groups*, In: “*Arithmetic and Geometry*,” (ed. by M. Artin and J. Tate), Birkhauser (1983), 141–166.
- [KL] Kazhdan, D., and Lusztig, G., *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
- [KK1] Kostant, B., and Kumar, S., *The nil Hecke ring and cohomology of G/P for a Kac–Moody group G* , Advances in Math. **62** (1986), 187–237.
- [KK2] Kostant, B., and Kumar, S., *T –equivariant K –theory of generalized flag varieties*, J. Diff. Geom. **32** (1990), 549–603.
- [Ku] Kumar, S., *Demazure character formula in arbitrary Kac–Moody setting*, Invent. Math. **89** (1987), 395–423.
- [Ku2] Kumar, S., *The nil Hecke ring and singularity of Schubert varieties*, In: “*Lie Theory and Geometry (in honor of Bertram Kostant)*,” (ed. by J. -L. Brylinski et. al.), Progress in Math vol. 123, Birkhauser (1994), 497–507.
- [L] Lakshmibai, V., *Singular loci of Schubert varieties for classical groups*, Bull. A.M.S. **16** (1987), 83–90.
- [LS] Lakshmibai, V., and Seshadri C. S., *Singular locus of a Schubert variety*, Bull. A.M.S. **11** (1984), 363–366.
- [Ma] Mathieu, O., *Formules de caractères pour les algèbres de Kac–Moody générales*, Astérisque **159–160** (1988), 1–267.
- [M] Mumford, D., “*The Red Book of Varieties and Schemes*,” LNM vol. **1358**, Springer–Verlag (1988).
- [P] Polo, P., *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Preprint (1993).
- [Ra] Ramanathan, A., *Schubert varieties are arithmetically Cohen–Macaulay*, Invent. Math. **80** (1985), 283–294.
- [Ra2] Ramanathan, A., *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Publ Math. IHES no. **65** (1987), 61–90.
- [R] Rossmann, W., *Equivariant multiplicities on complex varieties*, Astérisque no. **173–174** (1989), 313–330.
- [Ry] Ryan, K. M., *On Schubert varieties in the flag manifold of $SL(n, \mathbb{C})$* , Math. Annalen **276** (1987), 205–224.
- [Se] Seshadri, C. S., *Line bundles on Schubert varieties*, In: “*Vector Bundles on Algebraic Varieties*,” Tata Institute of Fundamental Research, Bombay (1984), 499–528.
- [S] Slodowy, P., *On the geometry of Schubert varieties attached to Kac–Moody Lie–algebras*, Canad. Math. Soc. Conf. Proc. on ‘*Algebraic Geometry*’ (Vancouver) vol. **6** (1984), 405–442.

University of North Carolina
Chapel Hill, N.C. 27599-3250
U.S.A.