FROBENIUS SPLITTING OF COTANGENT BUNDLES OF FLAG VARIETIES

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To the memory of A. Ramanathan

Let G be a semisimple, simply connected algebraic group over an algebraically closed field of characteristic $p > 0$. Let U be the unipotent radical of a Borel subgroup $B \subset G$ and $\mathfrak{u}$ the Lie algebra of U. Springer [16] has shown for good primes, that there is a B-equivariant isomorphism $U \to \mathfrak{u}$, where B acts through conjugation on U and through the adjoint action on $\mathfrak{u}$ (for $G = \mathrm{SL}_n$ one has the well known equivariant isomorphism $A \mapsto A - I$ between unipotent and nilpotent upper triangular matrices). Let p be a good prime for G. Then there is an isomorphism of homogeneous bundles $X = G \times^B U \to G \times^B \mathfrak{u}$, where the latter can be identified with the cotangent bundle $T^*(G/B)$ of G/B.

Motivated in part by [12] we establish a link between the G-invariant form χ on the Steinberg module $\mathrm{St} = \mathrm{H}^0(G/B, (p-1)\rho)$ (cf. §1.8) and Frobenius splittings [15] of the cotangent bundle $T^*(G/B)$: The representation $\mathrm{H}^0(G/B, 2(p-1)\rho)$ is a quotient of the space of functions $\mathrm{H}^0(X, \mathcal{O}_X)$ on X (here $\mathrm{H}^0(G/B, M)$ denotes the G-module induced from the B-module M and ρ half the sum of the roots R^+ opposite to the roots of B) (cf. Corollary 1). There is a natural map

$$\varphi' : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(X, \mathcal{O}_X)$$

such that the multiplication $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$ factors through the projection $\mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(G/B, 2(p-1)\rho)$. In the notation of Corollary 1, $\varphi' = H^0(\varphi)$. Surprisingly the simple situation of [12] generalizes in that $\varphi'(v)$ is a Frobenius splitting of X if and only if $\chi(v) \neq 0$ (if and only if $\mu(v)$ is a Frobenius splitting of G/B) (cf. Theorem 1). In particular, the cotangent bundle $T^*(G/B)$ is Frobenius split (cf. Corollary 2).

Frobenius splitting of the cotangent bundle in this setup has a number of interesting consequences. By filtering the differential forms via a morphism to a suitable partial flag variety and using diagonality of Hodge cohomology and Koszul resolutions, we obtain the vanishing theorem (cf. Theorem 2)

$$\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) = 0, i > 0$$

where λ is any dominant weight and $S\mathfrak{u}^*$ denotes the symmetric algebra of $\mathfrak{u}^*$. This was proved in [1] for large dominant weights and for all dominant weights for groups

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of classical type and G_2 (and large primes). The simple key lemma in the very simple proof of the Borel-Bott-Weil theorem [6] implies that the above vanishing theorem can be extended to weights $\mathcal{C} = \{\lambda \,|\, \langle \lambda, \alpha^\vee \rangle \geq -1, \forall \alpha \in R^+\}$. This vanishing theorem was proved in characteristic zero by Broer [3] using complete reducibility and the Borel-Bott-Weil theorem. As in characteristic zero ([3], Theorem 4.4) it follows that the subregular nilpotent variety is normal, Gorenstein and has rational singularities (cf. Theorem 6).

In the parabolic case we prove the above vanishing theorem for P-regular dominant weights (after proving that the cotangent bundle of partial flag varieties G/P is also Frobenius split) (cf. Corollary 3 and Theorem 5).

By using the Koszul resolution, the vanishing theorem also gives the Dolbeault vanishing:

$$\mathrm{H}^i(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)) = 0$$

for $i > j$ and $\lambda \in \mathcal{C}$ (cf. Theorem 3).

Another consequence is the conjectured isomorphism in ([9], II.12.15), [1] between the group cohomology $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, \mu))^{[-1]}$ of the first Frobenius kernel of G and the space of sections of a homogeneous line bundle on $T^*(G/B)$ (cf. Theorem 8 for a precise statement). Furthermore, by using the B-module structure of $\mathrm{St} \otimes \mathrm{St}$, it follows easily that $T^*(G/B)$ carries a canonical Frobenius splitting [13][10]. This implies that

$$\mathrm{H}^0(G/B, S\mathfrak{u}^* \otimes \lambda)$$

has a good filtration [10] for any weight λ (cf. Theorem 7). One obtains, in particular, that the cohomology of induced representations $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, \mu))^{[-1]}$ has a good filtration [1](for μ dominant and p bigger than the Coxeter number of G).

All of our proofs (and results) work for all groups in a uniform manner.

Our canonical splitting relates to the splitting of Mehta and van der Kallen in the GL_n-case [14] by taking a certain homogeneous component. For now we have ignored the more combinatorial aspects of the methods in this paper, like analyzing compatible Frobenius splitting.

1. Notation and preliminaries

The following notation is used throughout the paper. Fix an algebraically closed field k of characteristic $p > 0$. All schemes and morphisms will be over k.

1.1. **Group data.** Let G be a connected, simply connected, semisimple algebraic group, B a Borel subgroup of G, $T \subset B$ a maximal torus and U the unipotent radical of B. The Lie algebras of G, B and U are denoted $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{u}$ respectively. In the following B will act on U by conjugation and on $\mathfrak{u}$ by the adjoint action. Let B^+ be the opposite Borel subgroup with unipotent radical U^+, $R = R(T, G)$ the root system of G with respect to T, $R^- = R(T, U)$ (the negative roots), $R^+ = R(T, U^+) = \{\alpha_1, \ldots, \alpha_N\}$ (the positive roots), $S \subset R^+$ the simple roots and h the Coxeter number of G. For a parabolic subgroup $P \supset B$ we let U_P denote the unipotent radical of P, U_P^+ the opposite unipotent radical of P, $\mathfrak{u}_P$ the Lie algebra of U_P, $\mathfrak{p}$ the Lie algebra of P and $R_P \supset T$ the Levi factor of P. By $\langle \cdot, \cdot \rangle$ we denote the natural pairing $X(T) \times Y(T) \to \mathbb{Z}$ given by $\langle \lambda, \mu \rangle = \lambda(\mu(1))$, where $X(T)$ is the group of characters (also identified with the weight lattice) and $Y(T)$ the group of one parameter subgroups of T (also identified with the coroot lattice). A simple root $\alpha \in R^+$ defines the (simple) reflection $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, where $\lambda \in X(T)$ and $\alpha^\vee \in Y(T)$ is the coroot associated with α. For a subset $I \subset S$ we let $P = P_I$ denote the associated parabolic subgroup. Recall that the group of characters $X(P)$ of P can be identified with $\{\lambda \in X(T) | \langle \lambda, \alpha^\vee \rangle = 0, \text{for all } \alpha \in I\}$. In particular, $X(B) = X(T)$. A weight $\lambda \in X(B)$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. A dominant weight $\lambda \in X(P)$ is called P*-regular* if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \notin I$, where $P = P_I$ is a parabolic subgroup. A B-regular dominant weight is called *regular*. The Weyl group W of G is generated by the simple reflections. The "dot" action of W on $X(T)$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\langle \rho, \alpha^\vee \rangle = 1$ for every simple root $\alpha \in S$. On the weight lattice $X(T)$ the integral cone $\mathbb{Z}_+ R^+ \subseteq X(T)$ defines the partial order: $\lambda \geq \mu$ iff $\lambda - \mu \in \mathbb{Z}_+ R^+$.

Recall that the prime p is defined to be a *good prime* for G if p is coprime to all the coefficients of the highest root of G written in terms of the simple roots. For simple G, p is a good prime if $p \geq 2$ for type A; $p \geq 3$ for the types B, C and D; $p \geq 5$ for the types F_4, E_6, E_7 and G_2; $p \geq 7$ for the type E_8.

1.2. **Homogeneous bundles.** A P-scheme X gives rise to an associated locally trivial fibration $G \times^P X$ over G/P ([9], I.5.14, II.4.1). If M is a finite dimensional P-representation, we let $\mathcal{L}(M)$ denote the sheaf of sections of the vector bundle $G \times^P M$ on G/P.

1.3. **The relative Frobenius morphism.** The *absolute Frobenius morphism* on a scheme is the identity on point spaces and raising to the p-th power locally on functions. The absolute Frobenius morphism is not a morphism of k-schemes. Let $\pi : X \to \operatorname{Spec}(k)$ be a scheme. Let X' be the scheme obtained from X by base change with the absolute Frobenius morphism on $\operatorname{Spec}(k)$, i.e., the underlying topological space of X' is that of X with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying k-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of X', *the relative Frobenius morphism* $F : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of k-schemes.

1.4. **Frobenius splitting.** Following Mehta and Ramanathan [15] a variety X is called *Frobenius split* if the homomorphism $\mathcal{O}_{X'} \to F_* \mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. A homomorphism $\sigma : F_* \mathcal{O}_X \to \mathcal{O}_{X'}$ is a splitting of $\mathcal{O}_{X'} \to F_* \mathcal{O}_X$ if and only if $\sigma(1) = 1$. By abuse of terminology we will call an $\mathcal{O}_{X'}$-module homomorphism $\sigma : F_* \mathcal{O}_X \to \mathcal{O}_{X'}$ *a Frobenius splitting* if $\sigma(1) \in k \setminus \{0\}$ (so that σ is a splitting up to a constant).

A splitting $\sigma : F_* \mathcal{O}_X \to \mathcal{O}_{X'}$ is said to *split the subvariety* $Y \subseteq X$ *compatibly* if $\sigma(F_* \mathcal{I}_Y) \subseteq \mathcal{I}_{Y'}$, where $\mathcal{I}_Y$ denotes the ideal sheaf of Y.

If X is a smooth variety with canonical line bundle ω_X, the Cartier operator gives an isomorphism ([15], Proposition 5)

$$\mathcal{H}om_{\mathcal{O}_{X'}}(F_* \mathcal{O}_X, \mathcal{O}_{X'}) \cong F_*(\omega_X^{1-p}).$$

In this way global sections of ω_X^{1-p} correspond to homomorphisms $F_* \mathcal{O}_X \to \mathcal{O}_{X'}$. A section of ω_X^{1-p} which corresponds to a Frobenius splitting in this way, is called *a splitting section.* The above isomorphism can be described quite explicitly in local coordinates ([15], Proposition 5)

Proposition 1. *Let P be a closed point of a smooth variety Y over k of dimension n. Choose a system $x_1, \dots, x_n$ of regular parameters in the (regular) local ring $\mathcal{O}_{Y,P}$. Then the isomorphism*

$$F_*(\omega_Y^{1-p}) \to \mathcal{H}om_{\mathcal{O}_{Y'}}(F_*\mathcal{O}_Y, \mathcal{O}_{Y'})$$

is locally described as

$$x^\alpha/(dx)^{p-1} : x^\beta \mapsto x^{((\alpha+\beta+\underline{1})/p)-\underline{1}},$$

for any $\alpha = (\alpha_1, \dots, \alpha_n), \beta \in \mathbb{Z}_+^n$. Here we use the multinomial notation x^α for the element $x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathcal{O}_{Y,P}$, and $\underline{m} = (m, \dots, m) \in \mathbb{Z}_+^n$ for an integer m. If $\gamma = (\gamma_1, \dots, \gamma_n)$ with at least one γ_i nonintegral, we interpret x^γ as zero. Furthermore dx denotes the element $dx_1 \wedge \dots \wedge dx_n$, and $x^\alpha/(dx)^{p-1}$ denotes the local section of ω_Y^{1-p} with value x^α on $(dx)^{p-1}$.

We also have the following well known [15]

Lemma 1. *Let U be an open dense subset of a smooth variety X. If a section $s \in \mathrm{H}^0(X, \omega_X^{1-p})$ restricts to a splitting section $s|_U \in \mathrm{H}^0(U, \omega_U^{1-p})$, then s is a splitting section.*

Lemma 2. *Let X be a Frobenius split variety and $\mathcal{L}$ a line bundle on X. Then there is for each $i \geq 0$ an injection*

$$\mathrm{H}^i(X, \mathcal{L}) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^p)$$

of abelian groups.

1.5. **Volume forms.** Let X be a smooth variety with trivial canonical bundle ω_X. A *volume form* is a nowhere vanishing section θ_X of ω_X (necessarily unique up to scalar multiples if $\mathrm{H}^0(X, \mathcal{O}_X)^* = k$). *A function f on X is said to Frobenius split X* (with respect to θ_X) if $f\,\theta_X^{1-p}$ is a splitting section of ω_X^{1-p}.

Proposition 2. *Let $X = \operatorname{Spec} k[x_1, \dots, x_n]$ be affine n-space. A volume form on X is given by $\theta_X = dx_1 \wedge \dots \wedge dx_n$ and a function $f \in k[X]$ Frobenius splits X if and only if the coefficient of $x^{\underline{p-1}}$ in f is nonzero and the coefficients of the terms $x^{\underline{p-1}+p\alpha}$ are zero for $\alpha \in \mathbb{Z}_{\geq 0}^n \setminus \{0\}$ (in the multinomial notation of Proposition 1).*

Proof: An element $\sigma \in \mathrm{Hom}_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'})$ is a Frobenius splitting if and only if $\sigma(1)$ is a nonzero constant. The proposition now follows from Proposition 1. □

1.6. **Filtration of differentials.** Let $f : X \to Y$ be a smooth morphism between smooth varieties X and Y. Let $\Omega_{X/k}$ (resp. $\Omega_{X/Y}$) be the sheaf of differentials of X (resp. the sheaf of relative differentials of X over Y). Then we have the following

Lemma 3. *There is a short exact sequence*

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0,$$

giving a natural filtration of the sheaf of m-forms $\Omega_{X/k}^m$ for $m \geq 1$ with associated graded object

$$\mathrm{Gr}\, \Omega_{X/k}^m = \bigoplus_{i=0}^m f^*\Omega_{Y/k}^i \otimes \Omega_{X/Y}^{m-i}.$$

1.7. **The induction functor.** Let P be any parabolic subgroup. For a P-module M we let $\mathrm{H}^0(G/P, M)$ denote the induced G-module. Recall that (in algebraic terms) $\mathrm{H}^0(G/P, M) = (k[G] \otimes M)^P$, where P acts on $k[G]$ by right multiplication (it is a G-module with G acting trivially on M and by left multiplication on $k[G]$). This translates into the more familiar

$$\mathrm{H}^0(G/P, M) = \{f : G \to M | f(g\,p) = p^{-1}.f(g)\, \forall g \in G, p \in P\}.$$

In this formulation $\mathrm{H}^0(G/P, M)$ is simply the global sections of the homogeneous vector bundle $\mathcal{L}(M)$ on G/P. The sheaf cohomology $\mathrm{H}^i(G/P, \mathcal{L}(M))$ will also be denoted $\mathrm{H}^i(G/P, M)$ for $i \geq 0$. For $P = B$, the functor $\mathrm{H}^0(G/B, -)$ is also denoted $\mathrm{H}^0(-)$. If M is a G-module, then $i : M \to \mathrm{H}^0(G/P, M)$ given by $i(m)(g) = g^{-1}.m$ is an isomorphism of G-modules.

1.8. **The Steinberg module.** The *Steinberg module* is by definition the induced module $\mathrm{St} = H^0(G/B, (p-1)\rho)$. It is irreducible and selfdual. Fix an isomorphism $\mathrm{St} \to \mathrm{St}^*$ and denote the image of $v \in \mathrm{St}$ in St^* by v^*. This defines a G-invariant form given by $\chi(v \otimes w) = \langle v, w\rangle = v^*(w)$. Let v^+ and v^- denote highest and lowest weight vectors of St.

Let G act on itself by conjugation. Then the map $\mathrm{St} \otimes \mathrm{St} \to k[G]$ given by $(v \otimes w)(g) = \langle v, g\,w\rangle$ is a G-homomorphism. We get, in particular, by restriction a B-homomorphism

$$\varphi : \mathrm{St} \otimes \mathrm{St} \to k[U].$$

The global functions on $G \times^B U$ can be identified with $\mathrm{H}^0(G/B, k[U])$. In this setting we have $\mathrm{H}^0(\varphi)(v \otimes w)(g, u) = \langle v, gug^{-1}w\rangle$ using the identification i from §1.7.

1.9. **The Frobenius kernel.** The relative Frobenius morphism $U \to U'$ is a homomorphism of group schemes. The kernel U_1 is called the (first) *Frobenius kernel* and is a normal (one point) subgroup scheme of U ([9], I.9). If we fix a T-equivariant isomorphism (such that x_i has weight α_i)

$$k[U] \to k[x_1, \ldots, x_N],$$

then $k[U_1] \cong k[x_1, \ldots, x_N]/(x_1^p, \ldots, x_N^p)$. Let γ denote the B-equivariant restriction homomorphism $k[U] \to k[U_1]$. Notice that $k[U_1]$ is a finite dimensional B-representation with all weights $\leq 2(p-1)\rho$. The T-equivariant projection on the highest weight space spanned by the vector $\bar{x}_1^{p-1} \ldots \bar{x}_N^{p-1}$ is in fact a B-homomorphism $\psi : k[U_1] \to 2(p-1)\rho$, where the bar denotes the corresponding element in $k[U_1]$.

2. Frobenius splitting of $G \times^B U$

We begin with the following elementary lemma.

Lemma 4. *For any parabolic subgroup P, the canonical line bundle of the varieties $G \times^P U_P$ and $G \times^P \mathfrak{u}_P$ is G-equivariantly trivial.*

Proof: We give the proof in the case $G \times^P U_P$. The argument for $G \times^P \mathfrak{u}_P$ is similar (in fact this is, for good primes, isomorphic to the cotangent bundle of G/P). Let $n = \dim U_P$. The restriction of the locally free sheaf of relative differentials $\Omega = \Omega_{(G \times^P U_P)/(G/P)}$ on $G \times^P U_P$ to $U_P = P \times^P U_P$ is the sheaf of differentials of U_P, and hence $\Omega^n|_{U_P} = \omega_{U_P}$. Let θ_{U_P} be a volume form on U_P. Since $k[U_P]$ has no nonconstant units, the canonical action of P on θ_{U_P} gives rise to a character β of P, which can be determined by considering the action of P on $\omega_{U_P}|_e$, as the identity

$e \in U_P$ is fixed under P. The cotangent space at e is canonically isomorphic to $\mathfrak{M}_e/\mathfrak{M}_e^2$, where $\mathfrak{M}_e$ denotes the maximal ideal of functions in $k[U_P]$ vanishing at e. Hence $\beta = \sum_{\alpha \in R(T, U_P^+)} \alpha$. Since Ω^n is a G-sheaf, it is the pull back of the line bundle induced by β on G/P. As the canonical line bundle of G/P is induced by $-\beta$, the result follows from Lemma 3. □

Fix T-eigenfunctions $y_1, \dots, y_N$ of weights $-\alpha_1, \dots, -\alpha_N$ respectively, such that $k[U^+] \cong k[y_1, \dots, y_N]$. By Lemma 4, $X = G \times^B U$ carries a volume form θ_X restricting to $dy_1 \wedge \cdots \wedge dy_N \wedge dx_1 \wedge \cdots \wedge dx_N$ on the open subset $U^+ \times U \hookrightarrow G \times^B U$. The following lemma is instrumental in proving Frobenius splitting of $G \times^B U$.

Lemma 5. *The map* $\psi \circ \gamma \circ \varphi : \mathrm{St} \otimes \mathrm{St} \to 2(p-1)\rho$ *is non-zero.*

Proof: It suffices to prove that the monomial $x_1^{p-1} \dots x_N^{p-1}$ occurs with non-zero coefficient in $f \in k[U]$, where $f(x) = \langle v^+, x\, v^+ \rangle$. The functions $x \mapsto \langle v^+, x\, v^- \rangle$ and $x \mapsto \langle v^-, x\, v^- \rangle$ from G to k are highest and lowest weight vectors in $\mathrm{St} = \mathrm{H}^0(G/B, (p-1)\rho)$ respectively. By Theorem 2.3 in [12] the function σ

$$x \mapsto \langle v^+, x\, v^- \rangle \langle v^-, x\, v^- \rangle \in \mathrm{H}^0(G/B, 2(p-1)\rho)$$

is a splitting section of G/B. The restriction of σ to $U^+ \subset G/B$ is given by $x \mapsto \langle v^-, x\, v^- \rangle$. Since f corresponds to this function (which Frobenius splits U^+) under conjugation with w_0 (the longest element in W), the coefficient of $x_1^{p-1} \dots x_N^{p-1}$ in f must be nonzero by Proposition 2. □

If M is a G-module and N a B-module, then by Frobenius reciprocity, restriction followed by evaluation at $e \in G$ is an isomorphism ([9], Proposition I.3.4)

$$\mathrm{Hom}_G(M, \mathrm{H}^0(G/B, N)) \to \mathrm{Hom}_B(M, N).$$

Let $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$ denote the multiplication map.

Corollary 1. *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^0(G/B, k[U]) & \xrightarrow[\mathrm{H}^0(\gamma)]{} & \mathrm{H}^0(G/B, k[U_1]) \\ {\scriptstyle \mathrm{H}^0(\varphi)} \uparrow & & \downarrow {\scriptstyle \mathrm{H}^0(\psi)} \\ \mathrm{St} \otimes \mathrm{St} & \xrightarrow{\mu} & \mathrm{H}^0(G/B, 2(p-1)\rho) \end{array}$$

of G-equivariant homomorphisms.

Proof: By applying the induction functor we get a homomorphism

$$\mathrm{H}^0(\psi) \circ \mathrm{H}^0(\gamma) \circ \mathrm{H}^0(\varphi) : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho),$$

which is non-zero by Lemma 5 (and Frobenius reciprocity). By Frobenius reciprocity μ is (up to a constant) the unique G-homomorphism $\mu : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/B, 2(p-1)\rho)$. Adjusting constants this gives that the diagram is commutative. □

Theorem 1. *Let* $v = \sum_i v_i \otimes w_i$ *be an element of* $\mathrm{St} \otimes \mathrm{St}$. *The function* $f_v = \mathrm{H}^0(\varphi)(v)$ *Frobenius splits* $G \times^B U$ *if and only if* $\mu(v)$ *is a splitting section of* $\omega_{G/B}^{1-p}$.

In particular, the function $f_v : G \times^B U \to k$ *given by*

$$f_v(g, u) = \sum_i \langle v_i, gug^{-1} w_i \rangle$$

for $g \in G$, $u \in U$, *Frobenius splits* $G \times^B U$ *if and only if* $\chi(v)$ *is nonzero.*

Proof: Suppose that $\mu(v)$ is a splitting section of $\omega_{G/B}^{1-p}$. Let $f = \mathrm{H}^0(\varphi)(v)$. We prove that f Frobenius splits $X = G \times^B U$ with respect to the volume form θ_X. Restrict $f\,\theta_X^{1-p}$ to the open subset $U^+ \times U \hookrightarrow G \times^B U$. This leads to a form $f'(dy_1 \wedge \cdots \wedge dy_N \wedge dx_1 \wedge \cdots \wedge dx_N)^{1-p}$ on $U^+ \times U$. By Proposition 2 and Lemma 1, we are done if we prove that the monomial $y^{\underline{p-1}} x^{\underline{p-1}}$ occurs with nonzero coefficient in f' and the monomials $y^{\underline{p-1}+p\alpha} x^{\underline{p-1}+p\beta}$ occur with zero coefficient where $\alpha, \beta \in \mathbb{Z}_{\geq 0}^N$ not simultaneously zero (in the multinomial notation of Proposition 1). We have the following commutative diagram

$$\begin{array}{ccccc}
k[U^+] \otimes k[U] & \xrightarrow{1\otimes\gamma} & k[U^+] \otimes k[U_1] & \xrightarrow{1\otimes\psi} & k[U^+] \otimes 2(p-1)\rho \\
\uparrow & & \uparrow & & \uparrow \\
(k[G] \otimes k[U])^B & \xrightarrow{\mathrm{H}^0(\gamma)} & (k[G] \otimes k[U_1])^B & \xrightarrow{\mathrm{H}^0(\psi)} & (k[G] \otimes 2(p-1)\rho)^B
\end{array}$$

with natural T-equivariant maps. A monomial $y^{\underline{p-1}+p\alpha} x^{\underline{p-1}+p\beta}$ occuring in f' must have $\beta = 0$, as it is the restriction of an element in the image of $(k[G] \otimes \mathrm{St} \otimes \mathrm{St})^B \to (k[G] \otimes k[U])^B$ and since any weight in $\mathrm{St} \otimes \mathrm{St}$ is $\leq 2(p-1)\rho$. Furthermore, by Corollary 1, $(\mathrm{H}^0(\psi) \circ \mathrm{H}^0(\gamma))(f)$ restricted to U^+ is a Frobenius splitting. Chasing through the above diagram this means (using $\beta = 0$) that $\alpha = 0$ and the monomial $y^{\underline{p-1}} x^{\underline{p-1}}$ occurs with nonzero coefficient in f', so that f Frobenius splits $G \times^B U$. On the other hand if $\mathrm{H}^0(\varphi)(v)$ is a Frobenius splitting it is easy to read off the diagram that $\mu(v)$ is a splitting section. The last part of the theorem follows from Theorem 2.3 in [12]. □

Recall that the cotangent bundle $T^*(G/P)$ of G/P is the G-fibration associated to the P-module $(\mathfrak{g}/\mathfrak{p})^*$ under the adjoint action. It is well known that there is an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{u}_P$ of P-modules in good characteristics ([16], Lemma 4.4). Hence in this case $T^*(G/P) \cong G \times^P \mathfrak{u}_P$. We have the following crucial result due to Springer ([16], Proposition 3.5)

Proposition 3. *Let* $\operatorname{char} k$ *be a good prime for* G. *Then there exists a* B*-equivariant isomorphism* $\zeta : U \to \mathfrak{u}$. *Moreover for any parabolic subgroup* P, ζ *restricts to give a* P*-equivariant isomorphism* $\zeta_P : U_P \to \mathfrak{u}_P$.

Corollary 2. *Let* $\operatorname{char} k$ *be a good prime for* G. *Then the cotangent bundle* $T^*(G/B)$ *of* G/B *is Frobenius split.*

Proof: By Proposition 3 we get a G-isomorphism $G \times^B U \to G \times^B \mathfrak{u}$, where the latter can be identified with the cotangent bundle of G/B. The result now follows from Theorem 1. □

Remark 1. For $v \in \mathrm{St} \otimes \mathrm{St}$ define $\tilde{f}_v : G \times^B B \to k$ as in Theorem 1 (where B acts on itself by cojugation). Then $\tilde{f}_v$ Frobenius splits $G \times^B B$ if and only if $\chi(v) \neq 0$. Also the function $g \mapsto \langle v^-, gv^+\rangle\langle v^-, g^{-1}v^+\rangle$ splits G. Furthermore, if char. k is a good prime for G, any such v gives rise to a Frobenius splitting of $G \times^B \mathfrak{b}$, which descends via the map $(g, X) \mapsto Ad(g)X$ to the Lie algebra $\mathfrak{g}$. Since we have no nontrivial applications of these results we do not give any proofs.

3. Vanishing

Let

$$\mathcal{C} = \{\mu \in X(T) | \langle \mu, \alpha^\vee \rangle \geq -1, \forall \alpha \in R^+\}.$$

It is easy to see ([4], Proposition 2) that $\mathcal{C}$ is the set of weights λ such that if μ is a dominant weight with $\lambda \leq \mu \leq \lambda^+$, then $\mu = \lambda^+$ (here λ^+ denotes the dominant weight in the W-orbit of λ). The set $\mathcal{C}$ is precisely the weights of line bundles on G/B in characteristic zero, which have vanishing higher cohomology when pulled back to the cotangent bundle ([3], Theorem 2.4). In this section we prove the analogous vanishing theorem in good prime characteristics.

Andersen and Jantzen ([1], Theorem 3.6) proved the following vanishing theorem under the assumption that $p > h$ and either $\lambda = 0$ or λ strongly dominant (i.e. $\langle \lambda, \alpha^\vee \rangle \geq h-1$ for all $\alpha \in S$). For $p \geq h-1$ and all components of G classical or G_2 they proved the vanishing theorem for λ dominant ([1], Proposition 5.4). Actually the condition $\lambda+\rho$ dominant in ([1], Proposition 5.4) is not sufficient for vanishing as noticed by Graham and Broer - this is also revealed using Lemma 6 in §3.2 coupled with Bott's theorem. Let $\pi : T^*(G/B) \to G/B$ denote the projection.

Theorem 2. *Let* $\operatorname{char} k$ *be a good prime for* G *and suppose that* $\lambda \in \mathcal{C}$. *Then*

$$\mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) = 0$$

when $i > 0$.

Remark 2. By the semicontinuity theorem our result implies the same vanishing theorem over fields of characteristic zero.

3.1. **The Koszul resolution.** Let

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence of vector spaces. For any $n > 0$ one obtains a functorial exact sequence (called the *Koszul resolution*, ([9], II.12.12))

$$\cdots \to S^{n-i}V \otimes \wedge^i V' \to \cdots \to S^{n-1}V \otimes V' \to S^n V \to S^n V'' \to 0.$$

3.2. **A simple lemma.** Let P_α be the minimal parabolic subgroup corresponding to a simple root α. If $\lambda \in X(T)$ is a weight with $\langle \lambda, \alpha^\vee \rangle = -1$ and V a P_α-module, then

$$\mathrm{H}^i(G/B, V \otimes \lambda) = 0$$

for $i \geq 0$. This result is the simple key lemma in Demazure's very simple proof of the Borel-Bott-Weil theorem [6]. It has the following consequence (a similar approach has been used by Broer in [5]).

Lemma 6. *Suppose that* $\lambda \in \mathcal{C}$ *and* $\langle \lambda, \alpha^\vee \rangle = -1$ *for a simple root* α. *Then* $s_\alpha(\lambda) \in \mathcal{C}$ *and*

$$\mathrm{H}^i(G/B, S^n\mathfrak{u}^* \otimes \lambda) \cong \mathrm{H}^i(G/B, S^{n-1}\mathfrak{u}^* \otimes s_\alpha(\lambda))$$

for $i \geq 0$ *and* $n > 0$.

Proof: As s_α permutes $R^+ \setminus \{\alpha\}$ and maps α to $-\alpha$, we get that $s_\alpha(\lambda) \in \mathcal{C}$. The isomorphism follows by applying §3.1 to the short exact sequence of B-modules

$$0 \to \alpha \to \mathfrak{u}^* \to \mathfrak{u}^*_{P_\alpha} \to 0,$$

and then tensoring with λ. □

3.3. **Large dominant weights.** This section contains a proof of a lemma enabling us to turn Frobenius splitting into vanishing for weights, which are not necessarily regular. The key lies in filtering differentials using the fibration $G/B \to G/P$ for a suitable parabolic subgroup $P \supset B$.

Lemma 7. *Let λ be a dominant weight. Then*

$$\mathrm{H}^i(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(m\lambda)) = 0$$

for $i > j$ and all m sufficiently big.

Proof: If $\lambda = 0$, we are done by the fact that $H^i(G/B, \Omega^j_{G/B}) = 0$ for $i \neq j$ ([9], II.6.18). This is usually referred to as the diagonality of Hodge cohomology. If $\lambda \neq 0$, there exists a (unique) parabolic subgroup $P \neq G$, such that λ is a (P-regular) character of P and the induced line bundle $\mathcal{L}(\lambda)$ is ample on G/P. Let f denote the smooth (P/B)-fibration $G/B \to G/P$. Using Lemma 3, we see that it is enough to prove that the cohomology groups

$$\mathrm{H}^i(G/B, f^*\Omega^r_{G/P} \otimes \Omega^{j-r}_{(G/B)/(G/P)} \otimes \mathcal{L}(m\lambda))$$

vanish for all sufficiently big m, where $0 \leq r \leq j$. The E_2-terms in the Leray spectral sequence for f are (using the projection formula)

$$\begin{aligned} E_2^{pq} &= \mathrm{H}^p(G/P, \mathcal{L}(m\lambda) \otimes \Omega^r_{G/P} \otimes R^q f_* \Omega^{j-r}_{(G/B)/(G/P)}) \\ &= \mathrm{H}^p(G/P, \mathcal{L}(m\lambda) \otimes \Omega^r_{G/P} \otimes \mathcal{L}(H^q(P/B, \Omega^{j-r}_{P/B}))). \end{aligned}$$

For all m sufficiently big we get $E_2^{pq} = 0$ for $p > 0$ by Serre vanishing. Diagonality of Hodge cohomology for P/B gives that $E_2^{pq} = 0$ unless $q = j - r$. In particular, for m sufficiently big, combining the two, we get $E_2^{pq} = 0$ unless $p = 0$ and $q = j - r$. Now the result follows by the Leray spectral sequence, since $i > j$ by assumption. $\square$

3.4. **Proof of Theorem 2.** The first isomorphism follows since $\pi : T^*(G/B) \to G/B$ is an affine morphism and $\pi_* \mathcal{O}_{T^*(G/B)} = \mathcal{L}(S\mathfrak{u}^*)$. To prove the vanishing part we may assume that λ is dominant, because of the following argument: Assume by induction on n that $\mathrm{H}^i(G/B, S^j\mathfrak{u}^* \otimes \lambda) = 0$ for $j < n$, $i > 0$ and $\lambda \in \mathcal{C}$. We wish to prove the same result for $j = n$. Take a non dominant weight $\lambda \in \mathcal{C}$. Then there is a simple root α such that $\langle \lambda, \alpha^\vee \rangle = -1$. By Lemma 6, $s_\alpha(\lambda) \in \mathcal{C}$ and

$$\mathrm{H}^i(G/B, S^n\mathfrak{u}^* \otimes \lambda) = \mathrm{H}^i(G/B, S^{n-1}\mathfrak{u}^* \otimes s_\alpha(\lambda)),$$

where the latter group vanishes by induction.

So assume that λ is dominant. Since $(\mathfrak{b}/\mathfrak{u})^*$ is a trivial B-module, it follows from §3.1 (applied to the sequence $0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0$, and breaking the resulting Koszul resolution up into short exact sequences) that the vanishing of $\mathrm{H}^i(G/B, S\mathfrak{b}^* \otimes \lambda)$ implies the vanishing of $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda)$ for $i > 0$. Again using §3.1 for the short exact sequence $0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0$ we get for $n \geq 1$ an exact sequence

$$\cdots \to \wedge^1(\mathfrak{g}/\mathfrak{b})^* \otimes S^{n-1}\mathfrak{g}^* \otimes \lambda \to S^n\mathfrak{g}^* \otimes \lambda \to S^n\mathfrak{b}^* \otimes \lambda \to 0$$

after tensoring with λ. By breaking this up into short exact sequences, we see that the vanishing $\mathrm{H}^i(G/B, S\mathfrak{b}^* \otimes \lambda) = 0$ for any fixed $i > 0$ follows from the vanishing

$$\mathrm{H}^{i+j}(G/B, \wedge^j(\mathfrak{g}/\mathfrak{b})^* \otimes \lambda) = 0$$

for all $j \geq 0$. The B-representation $\wedge^j(\mathfrak{g}/\mathfrak{b})^*$ induces the bundle of j-forms $\Omega^j_{G/B}$ on G/B. By Lemma 7, we get for all large enough r that $H^{i+j}(G/B, \wedge^j(\mathfrak{g}/\mathfrak{b})^* \otimes (p^r\lambda)) = 0$ for $j \geq 0$ and hence $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes (p^r\lambda)) = 0$ for $i > 0$. But by Corollary 2 and Lemma 2, we have an injection of abelian groups

$$\mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(\lambda)) \hookrightarrow \mathrm{H}^i(T^*(G/B), \pi^*\mathcal{L}(p^r\lambda))$$

which translates into an injection $\mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes \lambda) \hookrightarrow \mathrm{H}^i(G/B, S\mathfrak{u}^* \otimes (p^r\lambda))$ for any $r > 0$ (this is where the assumption that p is good for G is used). This proves the theorem.

3.5. **Dolbeault vanishing.** Theorem 2 is in fact equivalent to the following (Dolbeault) vanishing (see [4] for results in characteristic zero and the parabolic case).

Theorem 3. *Let* $\operatorname{char} k$ *be a good prime for* G *and* $\lambda \in \mathcal{C}$*. Then*

$$\mathrm{H}^i(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)) = 0$$

for $i > j$*.*

Proof: Theorem 2 implies that $\mathrm{H}^i(G/B, S^n\mathfrak{b}^* \otimes \lambda) = 0$ for $i > 0$, using induction on n in the Koszul resolution (tensored with λ) coming from the short exact sequence $0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0$. This vanishing now fits in a similar induction on n in the Koszul resolution (tensored with λ) coming from the short exact sequence $0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0$. This gives the desired vanishing. □

4. The parabolic case

In this section we prove that the cotangent bundle of G/P, where P is a parabolic subgroup, is Frobenius split when $\operatorname{char} k$ is a good prime for G.

4.1. **Frobenius splitting of** $G \times^P U_P$**.** Let $P = P_I \supset B$ be the parabolic subgroup given by a subset $I \subset S$. Let R_I denote the root system generated by I. The space of functions $k[(U_P)_1]$ on the Frobenius kernel of U_P is a finite dimensional P-representation with all weights $\leq (p-1)\delta_P$, where $\delta_P := \sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in X(P)$. Observe that $-\delta_P$ is the weight inducing the canonical line bundle of G/P. The canonical line bundle of $G \times^P U_P$ is trivial by Lemma 4. The space of global functions $k[G \times^P U_P]$ can be identified with $\mathrm{H}^0(G/P, k[U_P])$. As in the case of a Borel subgroup $B \subset P$, we have a natural P-equivariant map $\varphi_P : \mathrm{St} \otimes \mathrm{St} \to k[U_P]$. The natural map

$$\mathrm{H}^0(G/P, k[U_P]) \to \mathrm{H}^0(G/P, k[(U_P)_1]) \to \mathrm{H}^0(G/P, (p-1)\delta_P),$$

(where the last map is induced by the T-equivariant projection $k[(U_P)_1] \to (p-1)\delta_P$, which is in fact a P-module map) composed with $\mathrm{H}^0(G/P, \varphi_P)$ gives the G-eqivariant map $\mu_P : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(G/P, (p-1)\delta_P)$.

Theorem 4. *Let* $v = \sum_i v_i \otimes w_i$ *be an element of* $\mathrm{St} \otimes \mathrm{St}$*. The function* $f = \mathrm{H}^0(G/P, \varphi_P)(v)$ *Frobenius splits* $G \times^P U_P$ *if and only if* $\mu_P(v)$ *is a splitting section of* $\omega^{1-p}_{G/P}$*.*

The function $f = f^P_v : G \times^P U_P \to k$ *given by*

$$f^P_v(g, u) = \sum_i \langle v_i, gug^{-1}w_i \rangle$$

for $g \in G$*,* $u \in U_P$*, Frobenius splits* $G \times^P U_P$ *if and only if* $\chi(v)$ *is nonzero.*

Proof: It follows by analogous weight considerations for the restriction of f to $U_P^+ \times U_P$ as in the B-case, that $f = \mathrm{H}^0(G/P, \varphi_P)(v)$ Frobenius splits $G \times^P U_P$ if and only if $\mu_P(v)$ is a splitting section of G/P (since any $\alpha \in R^+ \setminus R_I^+$ contains a simple root outside I with nonzero coefficient when written as a sum of simple roots and $(p-1)\delta_P + \sum_{\beta_i \in R^+ \setminus R_I^+} n_i \beta_i$ can not be a weight of $\mathrm{St} \otimes \mathrm{St}$ for $n_i \geq 0$ unless each n_i is 0).

In order to prove the last part of the theorem, we need to exhibit an element $w \in \mathrm{St} \otimes \mathrm{St}$ such that $\mathrm{H}^0(\varphi_P)(w)$ Frobenius splits $G \times^P U_P$ (because this implies that $\mu_P(w)$ is a Frobenius splitting of G/P, so that μ_P followed by the G-equivariant "evaluation" map [12] $\mathrm{H}^0(G/P, (p-1)\delta_P) \to k$ is a non-zero G-homommorphism $\mathrm{St} \otimes \mathrm{St} \to k$ and hence equals χ up to a non-zero scalar multiple).

As proved in Theorem 1, the function $f(g, u) = \langle v^-, gug^{-1}v^+ \rangle$, Frobenius splits $G \times^B U$. The restriction of this function to $U^+ \times U$ therefore Frobenius splits $U^+ \times U$. Observe that this restriction is given by

$$f(g, u) = \langle v^-, guv^+ \rangle,\ g \in U^+, u \in U.$$

Let w_0' be the longest element of the Weyl group of R_P and let $v_0^+ = w_0' v^+, v_0^- = w_0' v^-$. Index the set of positive roots $\{\alpha_1, \dots, \alpha_N\}$ in such a manner that the first $n := |R_I^+|$ roots are the positive roots of R_P. Let $\mathbf{y}_i : k \to U$ (resp. $\mathbf{x}_i : k \to U^+$) be the root homomorphism corresponding to the root $-\alpha_i$ (resp. α_i).

Write $u = \mathbf{y}_N(t_N) \dots \mathbf{y}_1(t_1)$ and $g = \mathbf{x}_1(s_1) \dots \mathbf{x}_N(s_N)$. Then

$$uv^+ = \mathbf{y}_N(t_N) \dots \mathbf{y}_{n+1}(t_{n+1})\Big(\sum_{l \neq \underline{\text{p-1}}} c_l t_n^{l_n} \dots t_1^{l_1} v_l + c t_n^{p-1} \dots t_1^{p-1} v_0^+ \Big),$$

for some $c, c_l \in k$, where v_l are weight vectors in St and $l = (l_1, \dots, l_n)$. As f Frobenius splits $U^+ \times U$, we see that the coefficient of $t_{n+1}^{p-1} \dots t_N^{p-1} s_N^{p-1} \dots s_{n+1}^{p-1}$ in

$$\langle v_0^-, \mathbf{x}_{n+1}(s_{n+1}) \dots \mathbf{x}_N(s_N) \mathbf{y}_N(t_N) \dots \mathbf{y}_{n+1}(t_{n+1}) v_0^+ \rangle$$

is nonzero. By weight considerations, it therefore easily follows that the function

$$f' : U_P^+ \times U_P \to k, \quad f'(g, u) = \langle v_0^-, guv_0^+ \rangle$$

Frobenius splits $U_P^+ \times U_P$. But f' extends to the function (again denoted by) $f' : G \times^P U_P \to k$ given by $(g, u) \mapsto \langle v_0^-, gug^{-1}v_0^+ \rangle$. (To see this, it suffices to observe that U_P^+ fixes v_0^+.) Hence f' Frobenius splits $G \times^P U_P$. □

Corollary 3. *Let* $\operatorname{char} k$ *be a good prime for* G*. Then the cotangent bundle* $T^*(G/P)$ *of* G/P *is Frobenius split.*

Proof: This follows from Theorem 4 and Proposition 3. □

Theorem 5. *Assume that* $\operatorname{char} k$ *is a good prime for* G*. Let* $\lambda \in X(P)$ *be a* P*-regular weight. Then*

$$\mathrm{H}^i(T^*(G/P), \pi^* \mathcal{L}(\lambda)) = \mathrm{H}^i(G/P, S\mathfrak{u}_P^* \otimes \lambda) = 0$$

for $i > 0$*, where* $\pi = \pi_P : T^*(G/P) \to G/P$ *is the projection.*

Proof: The proof follows §3.4. One applies the Koszul resolution for the short exact sequence of P-modules $0 \to (\mathfrak{g}/\mathfrak{u}_P)^* \to \mathfrak{g}^* \to \mathfrak{u}_P^* \to 0$. We get for $n \geq 1$ an exact sequence

$$\cdots \to S^{n-1}\mathfrak{g}^* \otimes \wedge^1(\mathfrak{g}/\mathfrak{u}_P)^* \otimes \lambda \to S^n \mathfrak{g}^* \otimes \lambda \to S^n \mathfrak{u}_P^* \otimes \lambda \to 0$$

after tensoring with λ. Again the vanishing $\mathrm{H}^i(G/P, S\mathfrak{u}_P^* \otimes \lambda) = 0$ for any fixed $i > 0$ follows from the vanishing

$$\mathrm{H}^{i+j}(G/P, \wedge^j(\mathfrak{g}/\mathfrak{u}_P)^* \otimes \lambda) = 0$$

for all $j \geq 0$. Since λ induces an ample line bundle on G/P this vanishing follows when λ is replaced by $n\lambda$ for all sufficiently large n. In particular, we get the vanishing of $\mathrm{H}^i(T^*(G/P), \pi^*\mathcal{L}(p^r\lambda)) = \mathrm{H}^i(G/P, S\mathfrak{u}_P^* \otimes p^r\lambda)$ for any $i > 0$ and all sufficiently large r. Now the result follows from Corollary 3 and Lemma 2. □

5. The subregular nilpotent variety

Throughout this section we assume that G is simple (and simply connected) and that $\operatorname{char} k$ is good for G.

Let $\mathcal{U}$ be the unipotent variety in G, i.e., the closed subvariety of G consisting of all the unipotent elements. Then the map

$$\varphi : G \times^B U \to \mathcal{U}$$

mapping (g, u) to gug^{-1} is a resolution of singularities ([8], Theorem 6.3) for all prime characteristics. If $P = P_\alpha$ is the minimal parabolic subgroup associated with a short simple root α, then φ restricted to $G \times^B U_P$ factors through

$$\varphi_\alpha : G \times^P U_P \to \mathcal{U}.$$

Lemma 8. *The map*

$$\varphi_\alpha : G \times^P U_P \to S$$

is birational onto its image S, which consists of the closed subvariety of irregular elements (called the subregular unipotent variety).

Proof: It follows by an argument of Tits that φ_α has connected fibres (see [3], Proposition 4.2), so we need to show that φ_α is separable (since $\dim G \times^P U_P = \dim S$). By Richardson's theorem ([17], I 5.1-5.6) the orbit maps for the conjugation action of G on itself are separable for very good primes. This implies the separability of φ_α for good primes, when G is not of type A. In type A the separability of φ_α follows from the GL_n-case, where the orbit maps for the conjugation action are separable for all primes. □

By ([2], Corollary 9.3.4) there is a (Springer) G-isomorphism between the unipotent variety $\mathcal{U}$ and the nilpotent cone $\mathcal{N}$, i.e., the closed subvariety of $\mathfrak{g}$ consisting of all the $ad-$nilpotent elements. In particular, we get that $\mathcal{N}$ is normal by the normality of $\mathcal{U}$ ([8], Theorem 4.24(iii)). As in the unipotent case, the Springer resolution

$$\tilde{\varphi} : G \times^B \mathfrak{u} \to \mathcal{N}, (g, X) \mapsto \operatorname{Ad} g(X),$$

is a resolution of singularities, which gives a resolution (Lemma 8)

$$\tilde{\varphi_\alpha} : G \times^P \mathfrak{u}_P \to \mathcal{S}$$

of singularities of the subregular nilpotent variety $\mathcal{S}$, where $\mathfrak{u}_P$ is the nilpotent radical of the Lie algebra of P. Moreover, all the morphisms $\varphi, \tilde{\varphi}, \varphi_\alpha, \tilde{\varphi_\alpha}$ are projective morphisms. Let $\pi : T^*(G/B) \to G/B$ denote the projection.

Theorem 6. *The subregular nilpotent variety $\mathcal{S}$ is a normal Gorenstein variety with rational singularities.*

Proof: The characteristic zero proof ([3], Theorem 4.4) carries over: The closed subvariety $G \times^B \mathfrak{u}_P$ of the cotangent bundle $G \times^B \mathfrak{u}$ is the zero scheme of a section of the pull back $\pi^*\mathcal{L}(-\alpha)$. So we get an exact sequence

$$0 \to \pi^*\mathcal{L}(\alpha) \to \mathcal{O}_{G\times^B\mathfrak{u}} \to \mathcal{O}_{G\times^B\mathfrak{u}_P} \to 0.$$

By Theorem 2 (since $\alpha \in \mathcal{C}$) and the normality of $\mathcal{N}$, we get a short exact sequence

$$0 \to \mathrm{H}^0(T^*(G/B), \pi^*\mathcal{L}(\alpha)) \to k[\mathcal{N}] \to k[G \times^P \mathfrak{u}_P] \to 0.$$

Let $\tilde{\mathcal{S}}$ denote the normalization of $\mathcal{S}$. The surjection $k[\mathcal{N}] \to k[G \times^P \mathfrak{u}_P]$ factors through the injection $k[S] \to k[\tilde{\mathcal{S}}]$ (followed by the map $k[\tilde{\mathcal{S}}] \to k[G \times^P \mathfrak{u}_P]$ induced by the normalization) via the restriction map $k[\mathcal{N}] \to k[\mathcal{S}]$. This proves that $k[\mathcal{S}] = k[\tilde{\mathcal{S}}]$ so that $\mathcal{S}$ is normal. By Theorem 2 the higher cohomologies of $\mathcal{O}_{G\times^B\mathfrak{u}}$ and $\pi^*\mathcal{L}(\alpha)$ vanish. It follows that $\mathrm{H}^i(G \times^B \mathfrak{u}_P, \mathcal{O}_{G\times^B\mathfrak{u}_P}) = \mathrm{H}^i(G \times^P \mathfrak{u}_P, \mathcal{O}_{G\times^P\mathfrak{u}_P}) = 0$ for $i > 0$, giving that $\mathcal{S}$ has rational singularities (since $\tilde{\varphi}_\alpha$ is birational by Lemma 8). As the canonical line bundle of $G \times^P \mathfrak{u}_P$ is trivial, $\mathcal{S}$ is Gorenstein ([11] p. 49–50). $\square$

6. Good filtrations

Let X be a smooth B-variety. A splitting section (or Frobenius splitting) $\sigma \in \mathrm{H}^0(X, \omega_X^{1-p})$ is called *canonical* [13], ([10], Definition 4.3.5) if σ is T-invariant and for all $\alpha \in S$ and $t \in k$

$$x_\alpha(t).\sigma = \sum_{i=0}^{p-1} t^i \sigma_{i,\alpha}$$

for suitable $\sigma_{i,\alpha} \in \mathrm{H}^0(X, \omega_X^{1-p})$ (of weight $i\,\alpha$), where $x_\alpha : k \to B$ is the root homomorphism corresponding to the root $-\alpha$.

Recall that a filtration $0 = V_0 \subset V_1 \subset \dots$ of a G-module V is called a *good filtration* if V is the union of the G-submodules $V_0, V_1, \dots$ and $V_i/V_{i-1} \cong \mathrm{H}^0(G/B, \lambda_i)$ for λ_i dominant. We have the following weaker version of a result due to Mathieu ([10], Lemma 4.4.2) sufficient for our purposes.

Lemma 9. *Let X be a smooth B-variety and $\mathcal{L}$ a G-equivariant line bundle on $G \times^B X$. Assume that $G \times^B X$ admits a canonical splitting, then the G-module $\mathrm{H}^0(G \times^B X, \mathcal{L})$ has a good filtration.*

For good primes there is a G-equivariant map

$$\varphi' : \mathrm{St} \otimes \mathrm{St} \to \mathrm{H}^0(T^*(G/B), \mathcal{O}_{T^*(G/B)})$$

such that $\varphi'(a \otimes b)$ is a splitting section if $\chi(a \otimes b) \neq 0$ (where $\varphi' := H^0(\varphi)$, cf. §2). Consider the splitting section of the cotangent bundle $T^*(G/B)$ given by $\varphi'(v^+ \otimes v^-)$. It is easy to see that $\varphi'(v^+ \otimes v^-)$ is a canonical Frobenius splitting of $T^*(G/B) = G \times^B \mathfrak{u}$, since the definition can be checked for $v^+ \otimes v^- \in \mathrm{St} \otimes \mathrm{St}$.

Theorem 7. *Suppose that* $\mathrm{char}\, k$ *is a good prime for G. Let $\lambda \in X(T)$ be a weight (not necessarily dominant). Then*

$$\mathrm{H}^0(G/B, S^n\mathfrak{u}^* \otimes \lambda)$$

has a good filtration for $n \geq 0$.

Proof: By the above $T^*(G/B) = G \times^B \mathfrak{u}$ admits a canonical Frobenius splitting. Hence

$$\mathrm{H}^0(T^*(G/B), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^0(G/B, S\mathfrak{u}^* \otimes \lambda)$$

has a good filtration by Lemma 9, where $\pi : T^*(G/B) \to G/B$ denotes the projection. □

Remark 3. Using Theorem 4 it follows in the same way that $T^*(G/P) = G \times^P \mathfrak{u}_P$ admits a canonical Frobenius splitting for any parabolic subgroup $P \supset B$. Mathieu has informed us that $H^0(X, \mathcal{L})$ has a good filtration if X is a smooth G-variety with a canonical Frobenius splitting and $\mathcal{L}$ a G-equivariant line bundle on X. In our case one can prove directly that $G \times^B (G \times^P \mathfrak{u}_P) \cong G/B \times (G \times^P \mathfrak{u}_P)$ admits a canonical Frobenius splitting, so that Lemma 9 implies that $\mathrm{H}^0(G/P, S\mathfrak{u}_P^* \otimes \lambda)$ has a good filtration for (arbitrary) weights $\lambda \in X(P)$.

Theorem 8. *Suppose that $p > h$ and let λ be a dominant weight. Then we have an isomorphism for any $w \in W$ such that $w \cdot 0 + p\lambda$ is dominant*

$$\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, w{\cdot}0{+}p\,\lambda))^{[-1]} \cong \begin{cases} \mathrm{H}^0(G/B, S^{(i-\ell(w))/2}\mathfrak{u}^* \otimes \lambda) & \text{if } i \equiv \ell(w) \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

where $()^{[-1]}$ denotes Frobenius (un)twist of a representation.

In particular, the cohomology of induced representations $\mathrm{H}^i(G_1, \mathrm{H}^0(G/B, w \cdot 0 + p\,\lambda))^{[-1]}$ admits a good filtration.

Proof: The key ingredient in the proof (in [1], §3.3) of the isomorphism is the vanishing Theorem 2, which makes the spectral sequence ([1], 3.3(2)) degenerate. The good filtrations follow from Theorem 7. □

Remark 4. Andersen and Jantzen proved the above theorem for groups not having any components of types E and F([1], §5). For arbitrary G they proved the above theorem under the assumption that λ is strongly dominant ([1], Corollary 3.7(b)).

Remark 5. It follows from the linkage principle that the only dominant μ with

$$\mathrm{H}^\bullet(G_1, \mathrm{H}^0(G/B, \mu)) \neq 0$$

are of the form $w \cdot 0 + p\,\lambda$ for some λ dominant and $w \in W$.

7. Homogeneous Frobenius splittings

We assume that char. k is a good prime for G. The space of functions $(k[G] \otimes k[\mathfrak{u}_P])^P = (k[G] \otimes S\mathfrak{u}_P^*)^P$ on the cotangent bundle $T^*(G/P)$ has a natural grading. Let $\pi_d : (k[G] \otimes S\mathfrak{u}_P^*)^P \to (k[G] \otimes S^d\mathfrak{u}_P^*)^P$ be the projection on the d-th homogeneous factor. Let N_P denote the dimension of G/P. Then a function f Frobenius splits $T^*(G/P)$ implies that $\pi_{N_P(p-1)}(f)$ Frobenius splits $T^*(G/P)$. A homogeneous splitting function (of degree $N_P(p-1)$) descends to give a Frobenius splitting of the projectivization $\mathbb{P}(T^*(G/P))$ (lines in $T^*(G/P)$) of the cotangent bundle. These splittings are in some sense better behaved than the splittings coming directly from $\mathrm{St} \otimes \mathrm{St}$ via Corollaries 2 and 3.

7.1. **The A_n-case.** In type A_n ($G = \mathrm{SL}_{n+1}(k)$) we have the B-equivariant isomorphism $\sigma : A \mapsto I + A$ between the upper triangular nilpotent matrices $\mathfrak{u}$ and the upper triangular unipotent matrices U. In this way we see that the element $v^+ \otimes v^-$ in $\mathrm{St} \otimes \mathrm{St}$ maps to the (splitting) function f

$$(g, A) \mapsto \langle v^+, g(A + I)g^{-1}v^- \rangle$$

on the cotangent bundle $T^*(G/B) = G \times^B \mathfrak{u}$ via $\mathrm{H}^0(\varphi)$ and σ. The function $g \mapsto \langle v^+, gv^- \rangle$ is a highest weight vector in St and equals the $(p-1)$-st power of the highest weight function $f_\rho : g \mapsto \langle w^+, gw^- \rangle$, where w^+ and w^- are highest and lowest weight vectors in $\mathrm{H}^0(G/B, \rho)$. The function f_ρ is a product of certain highest weight functions $f_{\omega_1}, \ldots, f_{\omega_n}$, with weight of $f_{\omega_i} = \omega_i$, where ω_i denotes the i-th fundamental dominant weight. Let $A = (a_{ij})_{1 \leq ij \leq n+1}$ be a matrix in G, then it is well known that

$$f_{\omega_s}(A) = \det((a_{ij})_{1 \leq i,j \leq s})$$

for $1 \leq s \leq n$. In this way the (magical) splitting function of Mehta and van der Kallen [14] on $T^*(G/B)$ is exactly $\pi_{N(p-1)}(f)$, where $N = n(n+1)/2$. One interesting aspect of the Mehta - van der Kallen splitting is that it compatibly splits all $G \times^B \mathfrak{u}_P$, for any parabolic subgroup $P \supseteq B$. Finding a suitable splitting in this context for the other groups would be very interesting.

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