EXTENDED JOSEPH POLYNOMIALS, QUANTIZED CONFORMAL BLOCKS, AND A q-SELBERG TYPE INTEGRAL

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To the memory of Yu. Stroganov

Abstract. We consider the tensor product $V = (\mathbb{C}^N)^{\otimes n}$ of the vector representation of $\mathfrak{gl}_N$ and its weight decomposition $V = \oplus_{\lambda=(\lambda_1,\dots,\lambda_N)} V[\lambda]$. For $\lambda = (\lambda_1 \geqslant \dots \geqslant \lambda_N)$, the trivial bundle $V[\lambda]\times\mathbb{C}^n \to \mathbb{C}^n$ has a subbundle of q-conformal blocks at level ℓ, where $\ell = \lambda_1 - \lambda_N$ if $\lambda_1 - \lambda_N > 0$ and $\ell = 1$ if $\lambda_1 - \lambda_N = 0$. We construct a polynomial section $I_\lambda(z_1,\dots,z_n,h)$ of the subbundle. The section is the main object of the paper. We identify the section with the generating function $J_\lambda(z_1,\dots,z_n,h)$ of the extended Joseph polynomials of orbital varieties, defined in [DFZJ05, KZJ09].

For $\ell = 1$, we show that the subbundle of q-conformal blocks has rank 1 and $I_\lambda(z_1,\dots,z_n,h)$ is flat with respect to the quantum Knizhnik–Zamolodchikov discrete connection.

For $N = 2$ and $\ell = 1$, we represent our polynomial as a multidimensional q-hypergeometric integral and obtain a q-Selberg type identity, which says that the integral is an explicit polynomial.

1. Introduction

The bundle of conformal blocks was introduced in conformal field theory. The bundle has a projectively flat Knizhnik–Zamolodchikov (KZ) connection which is a flat connection for conformal blocks on the sphere, see, for example, [KZ84, KL93]. The equations for flat sections of the bundle of conformal blocks on the sphere are called the KZ differential equations. In [SV91, FSV94a, FSV94b] solutions of the KZ differential equations were constructed as multidimensional hypergeometric integrals.

The qKZ difference equations were introduced in [FR92]. The $\mathfrak{gl}_N$ q-conformal blocks were defined in [MV98, MV99], cf [EF99]. The bundle of q-conformal blocks has a discrete flat connection defined by qKZ operators, see [MV98, MV99]. In [TV97] solutions of the $\mathfrak{gl}_2$ qKZ equations were constructed as multidimensional q-hypergeometric integrals.

We consider the tensor product $V = (\mathbb{C}^N)^{\otimes n}$ of the vector representation of $\mathfrak{gl}_N$ and its weight decomposition $V = \oplus_{\lambda=(\lambda_1,\dots,\lambda_N)} V[\lambda]$. For $\lambda = (\lambda_1 \geqslant \dots \geqslant \lambda_N)$, the trivial bundle $V[\lambda]\times\mathbb{C}^n \to \mathbb{C}^n$ has a subbundle of q-conformal blocks at level ℓ, where $\ell = \lambda_1 - \lambda_N$ if $\lambda_1 - \lambda_N > 0$ and $\ell = 1$ if $\lambda_1 - \lambda_N = 0$. We construct a polynomial section $I_\lambda(z_1,\dots,z_n,h)$ of the subbundle. The section is the main object of the paper. We identify the section with

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the generating function $J_\lambda(z_1, \dots, z_n, h)$ of the extended Joseph polynomials of the orbital varieties, defined in [DFZJ05, KZJ09].

For $\ell = 1$, we show that the subbundle of q-conformal blocks has rank 1 and $I_\lambda(z_1, \dots, z_n, h)$ is flat with respect to the qKZ discrete connection.

For $N = 2$ and $\ell = 1$, we represent our polynomial as a multidimensional q-hypergeometric integral and obtain a q-Selberg type identity, which says that the integral is an explicit polynomial. The simplest of these identities is

$$\int_{-i\infty}^{i\infty} \Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\, ds \;=\; 2\pi i\, \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\,, \tag{1.1}$$

which is a formula for the Barnes integral in [WW27]. The integral representation for I_λ gives an integral representation for the extended Joseph polynomials if $N = 2$ and $\ell = 1$.

For $N = 2$ we also give a presentation for I_λ as a multiple residue of a suitable rational function.

The fact that the generating function $J_\lambda(z_1, \dots, z_n, h)$ with $\lambda_1 - \lambda_N \leqslant 1$ satisfies the qKZ equations at level 1 was conjectured in [DFZJ05] and proved in [KNST09] by a different method, by relating the generating function with non-symmetric Jack polynomials.

The results of this paper may be considered as a "quantization" of the results of [Var10, RV11, RSV10], where the bundle of (non-quantum) conformal blocks at level 1 in $(\mathbb{C}^N)^{\otimes n}$ was considered. The bundle is of rank 1 and has a flat connection defined by the KZ differential operators. A rational flat section of the bundle was constructed. The section was interpreted as a generating function of the Euler classes of the fixed points of the GL_n-action on a suitable space of partial flags in $\mathbb{C}^n$. A Selberg type identity was obtained that equates the rational section and a multidimensional hypergeometric integral.

In Section 2 we introduce q-conformal blocks and qKZ equations. In Section 3 we define our main object – the polynomial $I_\lambda(z_1, \dots, z_n, h)$. In Section 4 we identify the polynomial with the generating function of the extended Joseph polynomials of orbital varieties. In Section 5 we prove all properties of the generating function. In Section 6 we prove a q-Selberg type identity. In Section 7 we give an alternative integral formula for $I_\lambda(z_1, \dots, z_n, h)$, if $N = 2$.

2. Quantum conformal blocks and qKZ equations

2.1. **Operators on representation-valued functions.** Let $N \geqslant 2$ be a positive integer. Let $e_{i,j}$, for $i, j = 1, \dots, N$, be the standard generators of the complex Lie algebra $\mathfrak{gl}_N$ satisfying the relations $[e_{i,j}, e_{s,k}] = \delta_{j,s} e_{i,k} - \delta_{i,k} e_{s,j}$. Consider the standard vector representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$, and its n-th tensor product $V = (\mathbb{C}^N)^{\otimes n}$. The space V splits into the direct sum of weight subspaces

$$V = \bigoplus_{\lambda = (\lambda_1, \dots, \lambda_N)} V[\lambda],$$

where $\sum_{i=1}^N \lambda_i = n$, and $V[\lambda] = \{ v \in V \mid e_{i,i} v = \lambda_i v \}$.

In this paper we will be concerned with V-valued functions of $z_1, \dots, z_n$ also depending on a complex parameter h. Now we recall some operators acting on the space of such functions.

- Elements of $\mathfrak{gl}_N$ act naturally on any factor of the tensor product. When $x \in \mathfrak{gl}_N$ acts in the i-th factor, we denote its action by $x^{(i)}$.

- Following [MV98] define an operator

$$e(z)=\sum_{j=1}^{n}\Big(z_j-he^{(j)}_{N,N}+h\sum_{s=j+1}^{n}\big(e^{(s)}_{1,1}-e^{(s)}_{N,N}\big)\Big)\,e^{(j)}_{1,N}+h\sum_{j=2}^{N-1}\sum_{1\leqslant r<s\leqslant N}e^{(r)}_{j,N}\,e^{(s)}_{1,j}.$$

- Let $P^{(i,j)}$ be the permutation of the i-th and j-th factors of $\big(\mathbb{C}^N\big)^{\otimes n}$.

- *The deformed S_n-action on V-valued functions of $z_1,\dots,z_n$.* The i-th elementary transposition $s_i\in S_n$ acts by the formula

$$\begin{aligned}s_i : I(z_1,\dots,z_n)\ &\mapsto \\ &\frac{(z_i-z_{i+1})\,P^{(i,i+1)}+h}{z_i-z_{i+1}}\,I(\dots,z_{i+1},z_i,\dots)-I(\dots,z_i,z_{i+1},\dots)\frac{h}{z_i-z_{i+1}}\end{aligned}\tag{2.1}$$

  This defines an action of S_n. Observe that, despite the presence of denominators, polynomials are mapped to polynomials by elements of S_n. In the whole paper an S_n-action will always mean this deformed action unless otherwise stated.

- Let u be a new variable. We define the following R-matrix operator

$$R^{(i,j)}(u)=\frac{u-hP^{(i,j)}}{u+h}.$$

  Observe that $R^{(i,j)}(u)R^{(i,j)}(-u)=1$ and

$$R^{(i,j)}(u-v)\,R^{(i,k)}(u)\,R^{(j,k)}(v)\ =\ R^{(j,k)}(v)\,R^{(i,k)}(u)\,R^{(i,j)}(u-v)\,.$$

2.2. **Yangian $Y(\mathfrak{gl}_N)$.** The Yangian $Y(\mathfrak{gl}_N)$ is a unital associative algebra with generators $T^{\{s\}}_{i,j}$, $i,j=1,\dots,N$, $s\in\mathbb{N}$. Organize them into generating series

$$T_{i,j}(u)\ =\ \delta_{i,j}+\sum_{s=1}^{\infty}T^{\{s\}}_{i,j}\,u^{-s},\qquad i,j=1,\dots,N\,.$$

The defining relation in $Y(\mathfrak{gl}_N)$ have the form

$$(u-v)\,\big[T_{i,j}(u),T_{k,l}(v)\big]\ =\ T_{k,j}(v)\,T_{i,l}(u)-T_{k,j}(u)\,T_{i,l}(v)\,,\tag{2.2}$$

for all $i,j,k,l=1,\dots,N$.

The Yangian $Y(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a subalgebra. The embedding is given by $e_{i,j}\mapsto T^{\{1\}}_{j,i}$ for any $i,j=1,\dots,N$.

Let

$$T(u)\ =\ \sum_{i,j=1}^{N}E_{i,j}\otimes T_{i,j}(u)$$

where $E_{i,j}$ is the image of $e_{i,j}\in\mathfrak{gl}_N$ in $\mathrm{End}(\mathbb{C}^N)$. Relations (2.2) can be written as the equality of series with coefficients in $\mathrm{End}(\mathbb{C}^N\otimes\mathbb{C}^N)\otimes Y(\mathfrak{gl}_N)$:

$$(u-v+P)\,T^{(1)}(u)T^{(2)}(v)\ =\ T^{(2)}(v)\,T^{(1)}(u)\,(u-v+P)\,,$$

where P is the permutation of the $\mathbb{C}^N$ factors, $T^{(1)}(u)\ =\ \sum_{i,j=1}^{N}E_{i,j}\otimes 1\otimes T_{i,j}(u)$ and $T^{(2)}(u)=1\otimes T(u)\,.$

More information on the Yangian $Y(\mathfrak{gl}_N)$ can be found in [Mol07]. Notice that the series $T_{i,j}(u)$ here corresponds to the series $T_{j,i}(u)$ in [Mol07].

The assignment

$$T(u) \mapsto R^{(0,1)}(z_1 - hu) \dots R^{(0,n)}(z_n - hu) \prod_{i=1}^{n} \frac{z_i - hu + h}{z_i - hu}$$

defines an action of the Yangian $Y(\mathfrak{gl}_N)$ on V-valued functions of $z_1, \dots, z_n$. We identify here the space $(\mathbb{C}^N)^{\otimes(n+1)}$ with $\mathbb{C}^N \otimes V$ and count the tensor factors by $0, 1, \dots, n$.

The action of $e(z)$ on V-valued functions coincide with that of $h\big(T^{\{2\}}_{N,1} - T^{\{1\}}_{N,N} T^{\{1\}}_{N,1}\big)$.

Lemma 2.1. *The Yangian action commutes with the S_n action* (2.1).

Proof. The commutativity with the first term in (2.1) follows from the Yang-Baxter equation for $R(u)$, the last formula in Section 2.1. The commutativity with the second term in (2.1) is the commutativity with multiplication by functions of $z_1, \dots z_n$. □

2.3. **Singular vectors, q-conformal blocks, and qKZ equations.** Let λ be a partition, i.e. assume that $\lambda_1 \geqslant \dots \geqslant \lambda_N$. Define $d(\lambda) = \lambda_1 - \lambda_N$.

A vector $v \in V[\lambda]$ is a *singular vector*, if $\sum_{a=1}^N e^{(a)}_{i,j} v = 0$ for all $i < j$.

Let $\ell \geqslant d(\lambda)$ be a positive integer, and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Following [MV98] we call $v \in V[\lambda]$ a *level ℓ q-conformal block*, if it is a singular vector and

$$e(z)^{\ell - d(\lambda)+1} v = 0.$$

Note that if $v \in V[\lambda]$ is a level ℓ q-conformal block, then v is a level ℓ' q-conformal block for any $\ell' > \ell$.

For $i = 1, \dots, n$, define the qKZ operators at level 1 by the formula

$$\begin{aligned} K_i(z_1, \dots, z_n) = R^{(i,i-1)}(z_i - z_{i-1} - (N+1)h) \cdots R^{(i,1)}(z_i - z_1 - (N+1)h) \times \\ \times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}). \end{aligned}$$

The qKZ difference equations at level 1 for a $V[\lambda]$-valued function I is the system of equations

$$I(z_1, \dots, z_i - (N+1)h, \dots, z_n) = K_i(z_1, \dots, z_n) I(z_1, \dots, z_i, \dots, z_n), \qquad i = 1, \dots, n. \tag{2.3}$$

Lemma 2.2. *Let $d(\lambda) \leqslant 1$. For generic $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the space of q-conformal blocks at level 1 is at most one-dimensional.*

Proof. The space of conformal blocks for $h = 0$ is defined as

$$CB_\lambda(z) = \Big\{ v \in V[\lambda] \text{ is a singular vector and } \Big(\sum_{j=1}^{N} z_a e^{(j)}_{1,N} \Big)^{\ell - d(\lambda)+1} v = 0 \Big\}.$$

For generic $z \in \mathbb{C}^n$ the dimension of $CB_\lambda(z)$ is calculated by the Verlinde formula. For $\ell = 1$ and $d(\lambda) \leqslant 1$ the Verlinde formula gives 1, so in this case the space of (non-quantum) conformal blocks for generic z is one-dimensional. The space of q-conformal blocks specializes to $CB_\lambda(z)$ at $h = 0$. At this specialization the dimension may only increase, hence the dimension of q-conformal blocks is at most 1. □

Below we will show that for generic z the dimension is equal to 1.

3. The minimal degree skew-symmetric polynomial I_λ

Recall that $\lambda \in \mathbb{N}^N$ is a partition of n. Define $k(\lambda) = \sum_{i=1}^N \lambda_i(\lambda_i - 1)/2$.

Let $v_1, \dots, v_N$ be the standard basis in $\mathbb{C}^N$, $e_{i,j} v_k = \delta_{j,k} v_i$. For a multi-index $L = (l_1, \dots, l_n)$ define $v_L = v_{l_1} \otimes \dots \otimes v_{l_n}$. A $V[\lambda]$-valued function I can be expressed as

$$I = \sum_L f_L(z_1, \dots, z_n, h)\, v_L$$

for multi-indices $L = (l_1, \dots, l_n)$ with $|\{j : l_j = i\}| = \lambda_i$. Denote the multi-index

$$L_0 = (\underbrace{1, \dots, 1}_{\lambda_1}, \underbrace{2, \dots, 2}_{\lambda_2}, \dots, \underbrace{N, \dots, N}_{\lambda_N}). \tag{3.1}$$

In what follows we will be concerned with the degree of polynomials and rational functions in z_i, h. Our convention is that $\deg z_i = \deg h = 1$. With this convention, the deformed S_n-action of Section 2.1 is homogeneous. Hence, if I is a skew-symmetric $V[\lambda]$-valued polynomial, then its homogeneous parts are also such. Now we study what the homogeneous degrees of skew-symmetric polynomials can be.

Define another S_n-action on functions of $z_1, \dots, z_n$, where the i-th elementary transposition $s_i \in S_n$ is acting by the formula

$$s_i : f \;\mapsto\; \hat{s}_i f\,, \tag{3.2}$$

$$\hat{s}_i f(z_1, \dots, z_n) \;=\; \frac{z_i - z_{i+1} + h}{z_i - z_{i+1}}\, f(\dots, z_{i+1}, z_i, \dots) - \frac{h}{z_i - z_{i+1}}\, f(\dots, z_i, z_{i+1}, \dots)\,.$$

For a permutation $\sigma \in S_n$ and a multi-index $L = (l_1, \dots, l_n)$ set $\sigma(L) = (l_{\sigma^{-1}(1)}, \dots, l_{\sigma^{-1}(n)})$. The following lemma is obvious.

Lemma 3.1. *A $V[\lambda]$-valued function I is skew-symmetric with respect to action* (2.1) *if and only if* $f_{s_i(L)} = -\,\hat{s}_i f_L$ *for every multi-index L and every* $i = 1, \dots, n-1$.

Lemma 3.2. *If a polynomial $f(z_1, \dots, z_n)$ is skew-symmetric with respect to the S_n-action* (3.2), *then it is divisible by*

$$\prod_{1 \leqslant i < j \leqslant n} (z_i - z_j + h).$$

Proof. Skew-symmetry with respect to $\hat{s}_i$ implies

$$(z_i - z_{i+1} + h)\, f(\dots, z_{i+1}, z_i, \dots) \;=\; (z_{i+1} - z_i + h)\, f(\dots, z_i, z_{i+1}, \dots)\,. \tag{3.3}$$

Therefore $z_i - z_{i+1} + h$ divides f. This further implies that $z_{i-1} - z_{i+1} + h$ divides $f(\dots, z_i, z_{i-1}, \dots)$, which using (3.3) again yields that $z_{i-1} - z_{i+1} + h$ divides f. Iterating this idea we obtain the statement of the lemma. □

Lemma 3.3.

(i) *If $I \neq 0$ is a $V[\lambda]$-valued skew-symmetric polynomial, then its degree is at least $k(\lambda)$.*
(ii) *A $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$ is unique up to multiplication by a number.*
(iii) *There exists a nonzero $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$.*

Proof. (i) By Lemma 3.1, a $V[\lambda]$-valued skew-symmetric polynomial I is uniquely determined by the coefficient f_{L_0}, and $\deg I = \deg f_{L_0}$.

Denote by $S_{\lambda_1} \times \ldots \times S_{\lambda_N} \subset S_n$ the isotropy subgroup of L_0. By Lemma 3.1, for $s_i \in S_{\lambda_1} \times \ldots \times S_{\lambda_N}$ we have $\hat{s}_i f_{L_0} = -f_{L_0}$. Using Lemma 3.2 for each S_{λ_i} we obtain that f_{L_0} is divisible by

$$D_0 = \prod_{1 \leqslant a<b \leqslant \lambda_1} (z_a - z_b + h) \prod_{\lambda_1 < a<b \leqslant \lambda_1+\lambda_2} (z_a - z_b + h) \cdots \prod_{n-\lambda_N < a<b \leqslant n} (z_a - z_b + h) \tag{3.4}$$

and has degree at least $k(\lambda)$.

(ii) If f_{L_0} has degree $k(\lambda)$, then it is proportional to D_0.

(iii) Define

$$I_\lambda = \sum_{\sigma \in S_n / S_{\lambda_1} \times \ldots \times S_{\lambda_N}} \operatorname{sgn}(\sigma)\, \hat{\sigma}(D_0)\, v_{\sigma(L_0)}\,, \tag{3.5}$$

where $\operatorname{sgn}(\sigma)$ is the sign of the shortest permutation in the coset, and $\hat{\sigma}$ denotes action (3.2). Then I_λ is a nonzero $V[\lambda]$-valued skew-symmetric polynomial of homogeneous degree $k(\lambda)$. □

The $V[\lambda]$-valued polynomial I_λ, defined by (3.5), is the main object of this paper. Now we reformulate its definition.

Definition 3.4. *Let $\lambda \in \mathbb{N}^N$ be a partition of n. Let I_λ be the $V[\lambda]$-valued skew-symmetric polynomial of degree $k(\lambda)$ normalized in such a way that the coefficient of v_{L_0} is D_0, see* (3.1), (3.4).

Example. We have

$$\begin{aligned}
I_{(1,1)} &= v_{12} - v_{21}\,,\\
I_{(2,1)} &= (z_1 - z_2 + h) v_{112} + (z_3 - z_1 - 2h) v_{121} + (z_2 - z_3 + h) v_{211}\,,\\
I_{(2,2)} &= (z_1 - z_2 + h)(z_3 - z_4 + h)(v_{1122} + v_{2211}) + (z_1 - z_4 + 2h)(z_2 - z_3 + h)(v_{1221} + v_{2112})\\
&\quad + \big(-(z_1 - z_2 + h)(z_3 - z_4 + h) - (z_1 - z_4 + 2h)(z_2 - z_3 + h)\big)(v_{1212} + v_{2121}).
\end{aligned}$$

Note that the last coefficient function in the third formula does not factor.

Remark. In the quasiclassical limit $h = 0$, the vector I_λ is the minimal degree skew-symmetric polynomial under the S_n-action

$$s_i^{h=0} : I \mapsto P^{(i,i+1)} I(\ldots, z_{i+1}, z_i, \ldots).$$

Its explicit form is

$$I_\lambda^{h=0} = \sum_L \Big(\operatorname{sgn}(L) \prod_{a<b,\, l_a = l_b} (z_a - z_b) \Big) v_L\,, \tag{3.6}$$

with an appropriately defined $\operatorname{sgn}(L)$. Rescaled by the the discriminant $\prod_{a<b}(z_a - z_b)$, this function was studied in [RV11], see also [RSV10]. It is shown there that this function satisfies (non-quantum) conformal block properties and (non-quantum) KZ differential equations.

4. Geometric description of I_λ

In this section we provide the connection of the I_λ to geometry which was advertised in the introduction.

4.1. **Orbital varieties and Joseph representation.**

4.1.1. *Orbital varieties.* Consider some conjugacy class of nilpotent elements inside $\mathfrak{g} = \mathfrak{gl}_n$. Such a conjugacy class is characterized by the unordered set of sizes of the Jordan blocks, which form a *partition* $\lambda' = (\lambda'_1 \geqslant \lambda'_2 \geqslant \cdots \geqslant \lambda'_K)$ of n. It is more convenient to use instead of λ' its *conjugate partition* $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N)$. If we depict partitions as Young diagrams, then the diagram of λ is the transpose of that of λ': the lengths of its *columns* are the sizes of Jordan blocks. For example, for one block of size 3 and one block of size 1, we use $\lambda = (2, 1, 1)$, that is $\yng(2,1,1)$.

Let $\bar{D}_\lambda \subset \mathfrak{g}$ be the closure of the conjugacy class D_λ associated to the partition λ. $\bar{D}_\lambda$ is known to be an irreducible algebraic variety, but if we denote by $\mathfrak{n}$ the space of strict upper triangular matrices, then the intersection $\mathcal{O}_\lambda := \bar{D}_\lambda \cap \mathfrak{n}$ is in general reducible: its geometric components (i.e., reduced irreducible components) are called *orbital varieties.*

Given an element of $x \in \mathcal{O}_\lambda$, note that x leaves stable the natural flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ associated to the standard basis. So the restriction of x to $\mathbb{C}^i$, $i = 0, \ldots, n$, is a nilpotent element to which can be attached a partition of i as described above, say $\varphi_i(x)$. Note that generically, $\varphi_n(x) = \lambda$. The following results were found:

Theorem (Spaltenstein [Spa82])**.** *Let λ be a partition of n, and $x \in \mathcal{O}_\lambda$.*

- *The sequence $\varphi_i(x)$ forms an increasing chain of Young diagrams, so there is a map φ from $\mathcal{O}_\lambda$ to the set of* standard Young tableaux *with n boxes (i.e., fillings of Young diagrams with numbers $\{1, \ldots, n\}$ which are increasing along rows and columns) such that the subdiagram of $\varphi(x)$ made of the boxes labelled from 1 to i is $\varphi_i(x)$.*
- *The irreducible components $\mathcal{O}_{\lambda;\alpha}$ of $\mathcal{O}_\lambda$ are the closures of $\varphi^{-1}(\alpha)$, where α runs over* $\mathrm{SYT}(\lambda)$*, the set of standard Young tableaux of shape λ.*
- *The $\mathcal{O}_{\lambda;\alpha}$ all have the same dimension which is one half of that of D_λ.*

The dimension of D_λ is easily calculated by computing the stabilizer of any element of the orbit and its dimension $\sum_{i,j} \min(\lambda'_i, \lambda'_j) = \sum_i \lambda_i^2$ (cf [Hum95, p. 11]), so that we find:

$$\dim \mathcal{O}_{\lambda;\alpha} = \frac{n(n-1)}{2} - \frac{1}{2} \sum_{i=1}^{n} \lambda_i(\lambda_i - 1)\,. \tag{4.1}$$

When there is no risk of confusion, we shall drop the index λ: $\mathcal{O}_{\lambda,\alpha} = \mathcal{O}_\alpha$.

4.1.2. *The case $\lambda_1 - \lambda_N \leqslant 1$.* Define the *dominance order* on partitions by $\lambda \prec \mu$ iff $\sum_{i\leqslant k} \lambda_i \leqslant \sum_{i\leqslant k} \mu_i$ for all k. Then one has [Hum95, p. 139] $D_\mu \subset \bar{D}_\lambda$ iff $\lambda \prec \mu$. The next proposition gives a more explicit description of $\bar{D}_\lambda$ and $\mathcal{O}_\lambda$ in a special case which is important for our purposes:

Proposition 4.1. *Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_N)$ be a partition such that $\lambda_1 - \lambda_N \leqslant 1$. Then*

$$\bar{D}_\lambda = \{x \in \mathfrak{g} : x^N = 0\},$$
$$\mathcal{O}_\lambda = \{x \in \mathfrak{n} : x^N = 0\}.$$

Proof. According to the discussion above, the first equality amounts to saying that among all the partitions μ of n with at most N parts, λ is the *smallest* for the dominance order. By direct computation, if $n = Nq + r$, $\lambda = (\underbrace{q+1,\dots,q+1}_{r},\underbrace{q,\dots,q}_{N-r})$; and assuming a $\mu = (\mu_1,\dots,\mu_N)$, $\sum_i \mu_i = n$, breaks one of the inequalities $\sum_{i\leqslant k}\lambda_i \leqslant \sum_{i\leqslant k}\mu_i$ leads to a contradiction with μ being decreasing.

The second equality follows immediately from the first. ◻

Example. If $\lambda = (1,1)$ there is only one tableau $\begin{smallmatrix}1\\2\end{smallmatrix}$, and $\mathcal{O}_{\begin{smallmatrix}1\\2\end{smallmatrix}} = \mathfrak{n} = \left\{\begin{pmatrix}\cdot & \star\\ \cdot & \cdot\end{pmatrix}\right\}$, where $\star$ denotes a free entry and $\cdot$ a zero in the lower triangle.

Next,

$$\mathcal{O}_{(2,1)} = \left\{\begin{pmatrix}\cdot & x_{12} & x_{13}\\ \cdot & \cdot & x_{23}\\ \cdot & \cdot & \cdot\end{pmatrix} : x_{12}x_{23} = 0\right\} = \mathcal{O}_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} \cup \mathcal{O}_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}$$

$$\mathcal{O}_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \left\{\begin{pmatrix}\cdot & 0 & \star\\ \cdot & \cdot & \star\\ \cdot & \cdot & \cdot\end{pmatrix}\right\} \qquad \mathcal{O}_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \left\{\begin{pmatrix}\cdot & \star & \star\\ \cdot & \cdot & 0\\ \cdot & \cdot & \cdot\end{pmatrix}\right\}.$$

Similarly, one computes

$$\mathcal{O}_{(2,2)} = \{x \;\; 4\times 4 : x^2 = 0\} = \mathcal{O}_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} \cup \mathcal{O}_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}}$$

$$\mathcal{O}_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} = \left\{\begin{pmatrix}\cdot & 0 & \star & \star\\ \cdot & \cdot & \star & \star\\ \cdot & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}\right\} \qquad \mathcal{O}_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} = \left\{\begin{pmatrix}\cdot & x_{12} & x_{13} & \star\\ \cdot & \cdot & 0 & x_{24}\\ \cdot & \cdot & \cdot & x_{34}\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix} : x_{12}x_{24} + x_{13}x_{34} = 0\right\}.$$

4.1.3. *Hotta's construction of the Joseph representation.* In the rest of Section 4 we fix a partition λ. Define W_λ to be the finite-dimensional space of maps from $\mathrm{SYT}(\lambda)$ to $\mathbb{C}$. Its dimension is that of the irreducible representation of S_n associated to λ. Orbital varieties provide us with a natural action of S_n on W_λ, which we describe now following [Hot84].

Given $i = 1,\dots,n-1$, define $\mathfrak{n}_i$ to be the subspace of $x \in \mathfrak{n}$ whose entry $x_{i,i+1}$ vanishes; and P_i to be the parabolic subgroup of GL_n made of invertible matrices x which are upper triangular *except* possibly at $x_{i+1,i}$. Note that the map $f : P_i \times \mathfrak{g} \to \mathfrak{g}$, $f(p,x) = pxp^{-1}$ sends $P_i \times \mathfrak{n}_i$ to $\mathfrak{n}_i$. We shall now describe the action of the elementary transposition $(i,i+1) \in S_n$ by giving its matrix elements.

Given a $\alpha \in \mathrm{SYT}(\lambda)$, two situations can occur:

(1) Either $\mathcal{O}_\alpha \subset \mathfrak{n}_i$, in which case set $m_{i;\alpha,\beta} = -\delta_{\alpha,\beta}$ for all β.
(2) Or $\mathcal{O}_\alpha \not\subset \mathfrak{n}_i$, in which case consider the scheme-theoretic intersection (i.e., with multiplicities) $\mathcal{O}_\alpha \cap \mathfrak{n}_i$, and then its image by f, i.e., $f(P_i \times (\mathcal{O}_\alpha \cap \mathfrak{n}_i))$ (again keeping

track of the degree of the map on each irreducible component of $\mathcal{O}_\alpha \cap \mathfrak{n}_i$); Clearly $f(P_i \times (\mathcal{O}_\alpha \cap \mathfrak{n}_i)) \subset \mathcal{O}_\lambda \cap \mathfrak{n}_i$, so its top-dimensional components are again orbital varieties (necessarily distinct from α). Then set

$$m_{i;\alpha,\beta} = \begin{cases} 1 & \beta = \alpha \\ \text{multiplicity of } \mathcal{O}_\beta \text{ in } f(P_i \times (\mathcal{O}_\alpha \cap \mathfrak{n}_i)) & \beta \neq \alpha \end{cases}$$

Finally, if $(e_\alpha)_{\alpha \in \mathrm{SYT}(\lambda)}$ is the standard basis of W_λ: $e_\beta(\alpha) = \delta_{\alpha,\beta}$, then define

$$\rho^{(i,i+1)} e_\beta = - \sum_{\alpha \in \mathrm{SYT}(\lambda)} m_{i;\alpha,\beta} e_\alpha \,. \tag{4.2}$$

Theorem 4.2 (Hotta)**.** *The $(m_{i;\alpha,\beta})_{\alpha,\beta \in \mathrm{SYT}(\lambda)}$, $i = 1, \ldots, n-1$, satisfy the symmetric group relations; and equipped with the action $\rho^{(i,i+1)}$ above, W_λ is the standard S_n-module associated to the partition λ.*

Example. For the three cases $(1,1)$, $(2,1)$, $(2,2)$, we find:

$$\lambda = (1,1): \qquad \rho^{(1,2)} = \begin{array}{c|c} & \begin{smallmatrix}1\\2\end{smallmatrix} \\ \hline \begin{smallmatrix}1\\2\end{smallmatrix} & \begin{pmatrix} -1 \end{pmatrix} \end{array},$$

$$\lambda = (2,1): \qquad \rho^{(1,2)} = \begin{array}{c|cc} & \begin{smallmatrix}1&2\\3\end{smallmatrix} & \begin{smallmatrix}1&3\\2\end{smallmatrix} \\ \hline \begin{smallmatrix}1&2\\3\end{smallmatrix} & 1 & 0 \\ \begin{smallmatrix}1&3\\2\end{smallmatrix} & -1 & -1 \end{array} \qquad \rho^{(2,3)} = \begin{array}{c|cc} & \begin{smallmatrix}1&2\\3\end{smallmatrix} & \begin{smallmatrix}1&3\\2\end{smallmatrix} \\ \hline \begin{smallmatrix}1&2\\3\end{smallmatrix} & -1 & -1 \\ \begin{smallmatrix}1&3\\2\end{smallmatrix} & 0 & 1 \end{array},$$

$$\lambda = (2,2): \qquad \rho^{(1,2)} = \rho^{(3,4)} = \begin{array}{c|cc} & \begin{smallmatrix}1&2\\3&4\end{smallmatrix} & \begin{smallmatrix}1&3\\2&4\end{smallmatrix} \\ \hline \begin{smallmatrix}1&2\\3&4\end{smallmatrix} & 1 & 0 \\ \begin{smallmatrix}1&3\\2&4\end{smallmatrix} & -1 & -1 \end{array} \qquad \rho^{(2,3)} = \begin{array}{c|cc} & \begin{smallmatrix}1&2\\3\end{smallmatrix} & \begin{smallmatrix}1&3\\2\end{smallmatrix} \\ \hline \begin{smallmatrix}1&2\\3&4\end{smallmatrix} & -1 & -1 \\ \begin{smallmatrix}1&3\\2&4\end{smallmatrix} & 0 & 1 \end{array}.$$

4.2. **Extended Joseph polynomials.** Now consider the (complex) torus $T = (\mathbb{C}^\times)^{n+1}$ acting on $\mathfrak{n}$ as follows: the first n variables correspond to conjugation by diagonal matrices, whereas the last variable corresponds to scaling. Explicitly, if $x \in \mathfrak{n}$ has entries x_{ij} and $t = (t_1, \ldots, t_n, q) \in T$, then $(t \cdot x)_{ij} = q\, t_i t_j^{-1} x_{ij}$.

Observe that $\mathcal{O}_\lambda$, and therefore its irreducible components $\mathcal{O}_{\lambda,\alpha}$, are invariant by T-action. Thus, they have natural Poincaré-dual classes in equivariant cohomology. It is convenient to describe them in the language of multidegrees (see [MS05]).

4.2.1. *Multidegrees.* Given a torus T acting linearly on a complex vector space W, we assign to a closed T-invariant sub-scheme $X \subseteq W$ its *multidegree* $\mathrm{mdeg}_W X \in \mathrm{Sym}(T^*)$ (here T^* is viewed as a lattice inside the dual of the Lie algebra of T), which can be computed inductively using the following properties (as in [Jos97]):

(1) If $X = W = \{0\}$, then $\mathrm{mdeg}_W X = 1$.
(2) If the scheme X has top-dimensional components X_i, where $m_i > 0$ denotes the multiplicity of X_i in X, then $\mathrm{mdeg}_W X = \sum_i m_i \ \mathrm{mdeg}_W X_i$.
(3) Assume X is a variety, and H is a T-invariant hyperplane in W.
 (a) If $X \not\subset H$, then $\mathrm{mdeg}_W X = \mathrm{mdeg}_H(X \cap H)$.

(b) If $X \subset H$, then $\operatorname{mdeg}_W X = [W/H]_T \operatorname{mdeg}_H X$, where $[\cdot]_T \in T^\star$ denotes the weight of the T-action.

One can readily see from these properties that $\operatorname{mdeg}_W X$ is homogeneous of degree $\operatorname{codim}_W X$, and is a positive sum of products of the weights of T on W.

In our case, $\operatorname{Sym}(T^\star) \cong \mathbb{Z}[z_1, \dots, z_n, h]$, and the weights on $\mathbb{C}[x_{ij}]_{1 \leqslant i<j \leqslant n}$ of the T-action are defined by

$$[x_{ij}]_T = h + z_i - z_j$$

The multidegree of $\mathcal{O}_\alpha$ with respect to this T-action, $J_\alpha := \operatorname{mdeg}_{\mathfrak{n}} \mathcal{O}_\alpha$, is called the *extended Joseph polynomial* of $\mathcal{O}_\alpha$. J_α is by definition a homogeneous polynomial in $\mathbb{Z}[z_1, \dots, z_n, h]$, of degree the codimension of $\mathcal{O}_\alpha$, which is nothing but $k(\lambda) = \frac{1}{2}\sum_i \lambda_i(\lambda_i - 1)$ defined in Section 3, according to Eq. (4.1). The reason for the name, first given in [DFZJ05], is that if we remove the scaling equivariance, i.e., set the variable $h = 0$, these polynomials reduce to the ones Joseph introduced in [Jos84].

Example. All the examples of orbital varieties given above are complete intersections (the number of equations is equal to the codimension); their multidegree is therefore the product of weights of the equations:

$$\begin{aligned}
J_{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}} &= 1\,,\\
J_{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} &= h + z_1 - z_2\,,\\
J_{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} &= h + z_2 - z_3\,,\\
J_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}} &= (h + z_1 - z_2)(h + z_3 - z_4)\,,\\
J_{\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}} &= (h + z_2 - z_3)(2h + z_1 - z_4)\,.
\end{aligned}$$

4.2.2. *Divided Differences.* The geometric construction given in Section 4.1.3 has a direct counterpart for multidegrees. Here our reference is [KZJ09, Sect. 5.1.1].

Define the *divided difference operator* $\partial_i = \frac{1}{z_i - z_{i+1}}(\tau_i - 1)$ where τ_i is permutation of variables z_i and z_{i+1}. Note that both ∂_i and τ_i are operators leaving $\mathbb{Z}[z_1, \dots, z_n]$ stable.

Let B be the group of invertible upper triangular matrices of size n. We use the following special case of [Jos84] (see also [KZJ09, Lemma 8]):

Lemma 4.3. *Let $X \subset \mathfrak{n}$ be a B-invariant variety such that $f(P_i \times X) \subset \mathfrak{n}$. Let k be the degree of the map $f|_X : (P_i \times X)/B \to \mathfrak{n}$, or zero if the generic fiber is infinite (i.e., X is P_i-invariant). Then*

$$-\frac{1}{h + z_{i+1} - z_i}\partial_i((h + z_{i+1} - z_i) \operatorname{mdeg}_{\mathfrak{n}} X) = k \operatorname{mdeg}_{\mathfrak{n}} f(P_i \times X)\,.$$

The proof is a standard equivariant localization argument which we shall not repeat here.

We now discuss separately the two cases of the construction of Section 4.1.3. Given a $\alpha \in \operatorname{SYT}(\lambda)$,

(1) If $\mathcal{O}_\alpha \subset \mathfrak{n}_i$, then $f(P_i \times \mathcal{O}_\alpha) \subset \mathfrak{n}_i \cap \mathcal{O}_\lambda$, is irreducible and contains $\mathcal{O}_\alpha$; therefore it is equal (set-theoretically) to $\mathcal{O}_\alpha$, i.e., $\mathcal{O}_\alpha$ is P_i-invariant. Lemma 4.3 implies

$$\partial_i((h + z_{i+1} - z_i)J_\alpha) = 0\,. \tag{4.3}$$

(2) If $\mathcal{O}_\alpha \not\subset \mathfrak{n}_i$, we have $\mathrm{mdeg}_{\mathfrak{n}}(\mathcal{O}_\alpha \cap \mathfrak{n}_i) = (h+z_i-z_{i+1})J_\alpha$ by property (3b) of multidegrees, and then by applying Lemma 4.3 to each irreducible component of $\mathcal{O}_\alpha \cap \mathfrak{n}_i$ we find:

$$-(h + z_i - z_{i+1})\partial_i J_\alpha = \sum_{\beta\neq\alpha} m_{i;\alpha,\beta} J_\beta\,. \tag{4.4}$$

Adding the diagonal term to the sum, Eq. (4.4) can be rewritten under the equivalent form

$$\hat{s}_i J_\alpha = \sum_{\beta} m_{i;\alpha,\beta} J_\beta \tag{4.5}$$

where we used the S_n-action $\hat{s}_i = \tau_i - h\partial_i$ of Eq. (3.2). Note now that Eq. (4.3) is a special case of Eq. (4.5) where $m_{i;\alpha,\beta} = -\delta_{\alpha,\beta}$. So Eq. (4.5) is valid in all cases. At $h = 0$, which is the case that Joseph considered in [Jos84], $\hat{s}_i$ reduces to the action of S_n on $\mathbb{C}[z_1, \dots, z_n]$ by permutation of variables.

4.3. **Identification with I_λ.** There is a natural object in $W_\lambda \otimes \mathbb{C}[z_1, \dots, z_n, h]$, namely the map $J_\lambda : \alpha \in \mathrm{SYT}(\lambda) \mapsto J_\alpha$. Combining Eqs. (4.2) and (4.5), we find:

$$\rho^{(i,i+1)} J_\lambda = -\hat{s}_i J_\lambda\,. \tag{4.6}$$

According to Theorem 4.2, W_λ carries the structure of S_n-module which is the same, by Schur–Weyl duality, as that of the space of singular vectors in $V[\lambda]$ (where S_n acts by permutation $P^{(i,i+1)}$ of tensors). Tensoring with $\mathbb{C}[z_1, \dots, z_n, h]$ (on which we do not make S_n act), we obtain an S_n-intertwiner $\phi : W_\lambda \otimes \mathbb{C}[z_1, \dots, z_n, h] \to V[\lambda]_{sing} \otimes \mathbb{C}[z_1, \dots, z_n, h]$. The equation above becomes

$$P^{(i,i+1)}\phi(J_\lambda) = -\hat{s}_i \phi(J_\lambda)$$

which means $\phi(J_\lambda)$ satisfies the hypothesis of Lemma 3.1. Note that ϕ is only defined up to a non-zero multiplicative constant.

We now want to identify $\phi(J_\lambda)$ with I_λ by using Lemma 3.3. By definition the entries of $\phi(J_\lambda)$ are linear combinations of those of J_λ and therefore are homogeneous polynomials of degree $k(\lambda)$ in the variables $z_1, \dots, z_n, h$. We have just derived the skew-symmetry of $\phi(J_\lambda)$ from Lemma 3.1. Therefore, we have proved:

Theorem 4.4. *Let λ be a partition of n. Then there exists a unique intertwiner ϕ such that*

$$\phi(J_\lambda) = I_\lambda\,.$$

In particular, all properties that we shall prove for I_λ are true for J_λ as well.

Example. By comparing the formulae for I_λ and J_λ, we find

$\lambda = (1,1)$: $\quad \phi(e_{\begin{smallmatrix}1\\2\end{smallmatrix}}) = v_{12} - v_{21}\,.$

$\lambda = (2,1)$: $\quad \phi(e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}}) = v_{112} - v_{121}\,,\qquad \phi(e_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}) = v_{211} - v_{121}\,.$

$\lambda = (2,2)$: $\quad \phi(e_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}) = v_{1122} + v_{2211} - v_{1212} - v_{2121}\,,\qquad \phi(e_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}}) = v_{1221} + v_{2112} - v_{1212} - v_{2121}\,.$

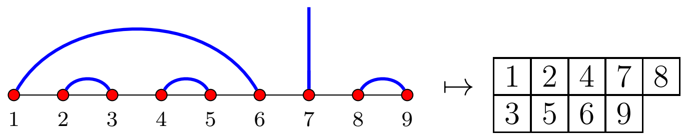


FIGURE 1. From link patterns to standard Young tableaux.

We next investigate in more detail two special cases for which everything can be worked out explicitly; the reader is invited to check all the results on our running examples, which belong to both.

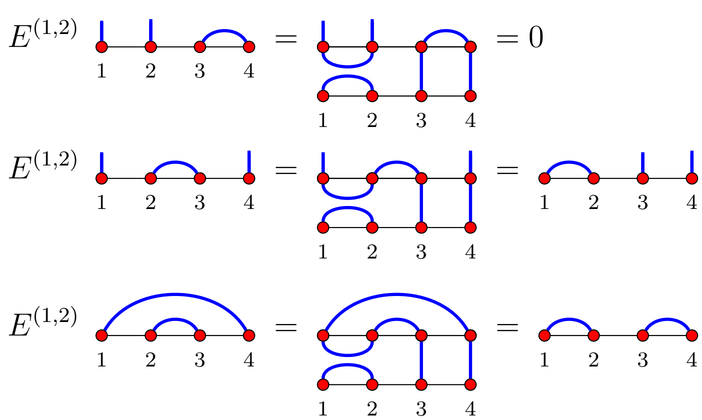
4.4. **Case of two rows.** In this section, we assume that the partition λ has only two rows: $\lambda = (n-p, p)$. This case was investigated in detail in the paper [KZJ09], so that we shall omit proofs of the results that were already contained in it.

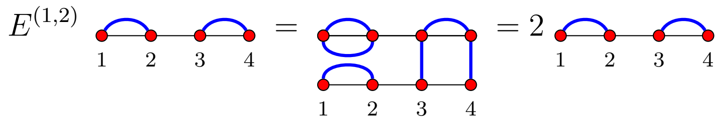
4.4.1. *Link patterns.* Call *link pattern* (or non-crossing matching) an unordered collection of disjoint pairs of $\{1, \ldots, n\}$ such that if $\{1, \ldots, n\}$ are represented as ordered vertices on a line, then the two elements of each pair can simultaneously be connected in the upper half plane in a non-crossing fashion (we sometimes call these connecting lines *arches*) and unpaired elements can be connected to upwards infinity (i.e., they must be outside all arches); cf Fig. 1 left.

There is a simple bijection from link patterns with p arches to standard Young tableaux of shape $\lambda = (n-p, p)$, obtained by recording in the first row the locations of openings of arches and of empty spots, and on the second row the locations of closings of arches. see Fig. 1.

Given $\alpha \in \mathrm{SYT}(\lambda)$, we can therefore consider its associated link pattern, and in particular, we shall use the following notation: if i and j are paired in the link pattern, write $\alpha(i) = j$, $\alpha(j) = i$; if i is unpaired, write $\alpha(i) = \varnothing$.

There is an action of the *Temperley–Lieb algebra* on the space of linear combinations of link patterns, which we identify with W_λ by identifying e_α with the corresponding link pattern. It is defined graphically by the action of the generators $E^{(i,i+1)}$, $i = 1, \ldots, n-1$, of the Temperley–Lieb algebra which corresponds to reconnecting vertices i and $i+1$, e.g.,





with the additional rules that reconnecting two unpaired points produces zero and that closed loops must be erased at a cost of multiplication by 2.

The Temperley–Lieb algebra (at loop weight 2) is a quotient of the symmetric group algebra, and in fact it is shown in [KZJ09, Sect. 5.1.2] that if one sets $\rho^{(i,i+1)} = 1 - E^{(i,i+1)}$, then the action defined above coincides with the Joseph representation defined in Section 4.1.3.

4.4.2. *Description of orbital varieties.* When λ has only two rows, $\mathcal{O}_\lambda$ is a "spherical variety", i.e., the group of invertible upper triangular matrices B acts on it by conjugation with a *finite* number of orbits. This gives us a first description of its irreducible components as B-orbit closures. Define $\alpha_<$ to be the upper triangular matrix with values in $\{0,1\}$ such that $(\alpha_<)_{ij} = 1$ iff $j = \alpha(i) > i$. Then it is easy to show that $\mathcal{O}_\alpha = \overline{B \cdot \alpha_<}$, where $\cdot$ denotes conjugation action.

An alternative description is in terms of equations:

Proposition 4.5. $\mathcal{O}_\alpha$ *is defined by the following equations:*

(1) $x^2 = 0$.
(2) *The rank of any lower-left submatrix of x is lower or equal to the rank of the same submatrix of $\alpha_<$.*

This statement can be extracted with some effort from [Mel06] or can be deduced directly from the results of [KZJ09].

4.4.3. *Exchange relation.* We rewrite explicitly the dichotomy of the Hotta construction (Sections 4.1.3 and 4.2.2) since it will be needed in the next section.

Consider $\alpha \in \mathrm{SYT}(\lambda)$ and its associated link pattern. The two cases are:

(1) Either $\alpha(i) \neq i+1$ (there is no arch connecting i and $i+1$ in the link pattern), which means $(\alpha_<)_{i,i+1} = 0$ and according to Proposition 4.5 (2), among the equations of $\mathcal{O}_\alpha$ there is $x_{i,i+1} = 0$, i.e., $\mathcal{O}_\alpha \subset \mathfrak{n}_i$, and Eq. (4.3) holds, or equivalently, J_α is $h + z_i - z_{i+1}$ times a symmetric polynomial in z_i, z_{i+1}.
(2) Or $\alpha(i) = i+1$ (there is an arch connecting i and $i+1$ in the link pattern), in which case $(\alpha_<)_{i,i+1} = 1$, which implies $\mathcal{O}_\alpha \not\subset \mathfrak{n}_i$. Then we can rewrite Eq. (4.4) as:
$$-(h + z_i - z_{i+1})\partial_i J_\alpha = \sum_{\beta : E^{(i,i+1)}\beta = \alpha} J_\beta$$
where, to keep notations simple, we have identified standard Young tableaux and link patterns when we write "$E^{(i,i+1)}\beta = \alpha$".

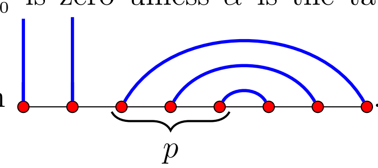
4.4.4. *Change of basis.* Finally, we investigate the intertwiner ϕ. In fact, in the case of two-row diagrams, there is a well-known explicit formula for ϕ, which was rediscovered many times and dates back (at least in the special case $p = n/2$) to [RTW32] (see [SZJ10] for more references and background). Roughly speaking, a pairing between i and j corresponds to a $\mathfrak{sl}(2)$ singlet $v_1^{(i)} \otimes v_2^{(j)} - v_2^{(i)} \otimes v_1^{(j)}$, whereas an unpaired i is a $v_1^{(i)}$ (where superscripts are as usual locations in the tensor product $(\mathbb{C}^2)^{\otimes n}$). With the present sign conventions, the exact statement is:

Proposition 4.6. *The intertwiner ϕ is given by*

$$\phi(e_\alpha) = \sum_{\substack{L=(l_1,\ldots,l_n)\in\{1,2\}^n \\ l_i\neq l_j \text{ if } j=\alpha(i) \\ l_i=1 \text{ if } \alpha(i)=\varnothing}} (-1)^{\lfloor\frac{n-p}{2}\rfloor+\#\{i \text{ even: } l_i=1\}} v_L \,.$$

Proof. Checking that the ϕ thus defined intertwines the actions of the symmetric group is a routine exercise. Because of the conditions on the conditions on the multi-index L, there are k 2's and $n-k$ 1's, so $\phi(e_\alpha)\in V_\lambda$. So the only issue is normalization of ϕ, which is fixed by Theorem 4.4. Consider the multi-index $L_0 = (\underbrace{1,\ldots,1}_{n-p},\underbrace{2,\ldots,2}_{p})$. We have

$$I_{L_0} = D_0 = \prod_{1\leqslant i<j\leqslant n-p} (h+z_i-z_j) \prod_{n-p+1\leqslant i<j\leqslant n} (h+z_i-z_j)\,.$$

On the other hand, by inspection the entry $\phi(e_\alpha)_{L_0}$ is zero unless α is the tableau $\alpha_0 =$ $\begin{array}{|c|c|c|c|c|}\hline 1 & & \cdots & & n-p \\ \hline {n-p \atop +1} & \cdots & n \\ \cline{1-3}\end{array}$, corresponding to the link pattern . J_{α_0} is the multidegree of the orbital variety indexed by α_0, which according to Proposition 4.5 is a linear subspace of the form

$$\mathcal{O}_{\alpha_0} = \left\{ \begin{pmatrix} \cdot & 0 & \cdots & 0 & \overbrace{\star & \cdots & \cdots & \star}^{p} \\ \cdot & \cdot & \ddots & \vdots & \vdots & & & \vdots \\ \cdot & \cdot & \cdot & 0 & \vdots & & & \vdots \\ \cdot & \cdot & \cdot & \cdot & \star & \cdots & \cdots & \star \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}\star\\ \vdots\\ 0\\ \star\end{array}}\right\} n-p \\ \vphantom{\begin{array}{c}0\\ \vdots\\ 0\\ \cdot\end{array}} \end{array} \right\}$$

so that $J_{\alpha_0} = I_{L_0}$. We conclude that the normalization of ϕ is fixed by $\phi(e_{\alpha_0})_{L_0} = 1$. This fits with the formula of the proposition. □

4.4.5. *Cyclicity.* Consider the special case $n = 2p$, i.e., the Young diagram is rectangular, and the corresponding link patterns have no unpaired vertices. One can then define a rotation of link patterns in the natural way, i.e., move the vertices cyclically $1\to 2\to\cdots\to n\to 1$ keeping the pairings intact. Via the one-to-one correspondence from link pattens to $\mathrm{SYT}(\lambda)$, this defines a bijection ρ from $\mathrm{SYT}(\lambda)$ to $\mathrm{SYT}(\lambda)$. One observes empirically the following relation:

$$J_{\rho\alpha}(z_1,\ldots,z_n) = (-1)^{p-1} J_\alpha(z_2,\ldots,z_n,z_1-3h) \tag{4.7}$$

It is well-known that if Eq. (4.5) is satisfied, then Eq. 4.7 is equivalent to the qKZ equation (2.3) (in which $N = 2$). Indeed we shall prove in Section 5 that the case $\lambda = (p, p)$ is among the cases where I_λ and therefore J_λ satisfy the qKZ equation.

The case $\lambda = (p, p)$ will be considered again in Section 6.

4.5. **Case of two columns.** Let $\lambda = (\underbrace{2, \dots, 2}_{p}\underbrace{1, \dots, 1}_{N-p})$, $n = N + p$. In this section we show that all orbital varieties $\mathcal{O}_\alpha$ for such λ are complete intersections and we describe explicitly their equations as well as the extended Joseph polynomials J_α.

Note that the codimension of $\mathcal{O}_\lambda$ is simply $k(\lambda) = p$. Furthermore, λ satisfies the hypothesis of Proposition 4.1, so that

$$\mathcal{O}_\lambda = \{x \;\; n \times n : x^N = 0\} \qquad n = N + p, \quad p \leqslant N\,.$$

There is a general duality of orbital varieties which corresponds to conjugation of partitions and standard Young tableaux (related to the duality of [Spa82, Chapter 3]). It means that the cases of two rows and two columns are dual to each other.

4.5.1. *The dual symmetric group action.* As mentioned above, there is a bijection $'$ from $\mathrm{SYT}(\lambda)$ to $\mathrm{SYT}(\lambda')$ which is just conjugation (reflection through the diagonal) of Young tableaux. Using this bijection we shall define a new "dual" action on W_λ starting from that on $W_{\lambda'}$ as defined in Section 4.4.1. Recall that the action is defined by the $m_{i;\alpha,\beta}$, cf Eq. (4.2).

Given $\alpha \in \mathrm{SYT}(\lambda)$, define the sign of α to be

$$\varepsilon_\alpha = (-1)^{\#\{i<j\,:\;\; j \text{ strictly south-west of } i \text{ in } \alpha\}}\,. \tag{4.8}$$

Now given $\alpha, \beta \in \mathrm{SYT}(\lambda)$ and their conjugate α', β', define

$$m_{i;\alpha,\beta} = -\varepsilon_\alpha \varepsilon_\beta m_{i;\beta',\alpha'}\,. \tag{4.9}$$

Due to the easy lemma the $m_{i;\alpha,\beta} \neq 0$, $\alpha \neq \beta$, implies $\varepsilon_\alpha \neq \varepsilon_\beta$, we can write equivalently

$$m_{i;\alpha,\beta} = (-1)^{\delta_{\alpha,\beta}} m_{i;\beta',\alpha'}$$

i.e., negate the diagonal entries and transpose.

4.5.2. *Defining equations of the orbital varieties.* There is again a bijection between $\mathrm{SYT}(\lambda)$ and the set of link patterns in size n with p arches, obtained by composing the bijection of Section 4.4.1 with conjugation of Young tableaux. So we shall use the same notations $\alpha(i) = j$, $\alpha(i) = \varnothing$, for paired i, j and unpaired i, respectively.

For $\alpha \in \mathrm{SYT}(\lambda)$ and $1 \leqslant i < j \leqslant 2n$, denote

$$p_\alpha(i, j) := j - i + 1 - \#\{k : \;\; i \leqslant k < \alpha(k) \leqslant j\}\,.$$

We have $p_\alpha(i, j) \geqslant \frac{j-i+1}{2}$, with equality if and only if all elements of $[i, j]$ are paired between themselves; in particular,

$$p_\alpha(i, j) = \frac{j - i + 1}{2} \qquad i < j = \alpha(i)\,.$$

An important property is that if $i \leqslant i' \leqslant j' \leqslant j$, then $p_\alpha(i', j') \leqslant p_\alpha(i, j)$ (enlarging an interval by one can either leave p_α unchanged if a new pairing has been absorbed in the interval, or increase p_α by 1 otherwise). This implies that $p_\alpha(i, j) \leqslant p_\alpha(1, n) = n - p = N$ for all $i < j$.

Define

$$\hat{\mathcal{O}}_\alpha := \left\{ x \in \mathfrak{n} : \left(x^{p_\alpha(i,j)}\right)_{i,j} = 0,\ i < j = \alpha(i) \right\} . \tag{4.10}$$

We want to show that $\hat{\mathcal{O}}_\alpha = \mathcal{O}_\alpha$. First we prove the following lemma:

Lemma 4.7. *If $x \in \hat{\mathcal{O}}_\alpha$ then $(x^a)_{i,j} = 0$ for all $i < j$ and $a \geqslant p_\alpha(i,j)$.*

In fact, these are all the $(x^a)_{i,j}$ in the ideal of equations of $\hat{\mathcal{O}}_\alpha$.

Proof. By induction on $j - i$.

If $j = i+1$ then either $a > 1$ in which case $(x^a)_{i,i+1} = 0$ because x is strict upper triangular; or $a = 1 = p_\alpha(i, i+1)$ which implies $i + 1 = \alpha(i)$, in which case $x_{i,i+1} = 0$ is part of the defining equations of $\hat{\mathcal{O}}_\alpha$.

Next, assume $j > i + 1$. We are going to divide into cases depending on the position of $\alpha(i)$ (and similarly for $\alpha(j)$). If $\alpha(i) = j$ and $a = p_\alpha(i,j)$, once again $(x^a)_{i,j} = 0$ is part of the defining equations of $\hat{\mathcal{O}}_\alpha$. If $\alpha(i) \notin [i,j]$ or $\alpha(i) = j$ and $a > p_\alpha(i,j)$, consider

$$(x^a)_{i,j} = \sum_{i<k<j} x_{i,k} \left(x^{a-1}\right)_{k,j} .$$

We claim that every term in the sum is zero. Indeed if $\alpha(i) \notin [i,j]$, $p_\alpha(i+1,j) = p_\alpha(i,j) - 1$ so that $a - 1 \geqslant p_\alpha(i,j) - 1 = p_\alpha(i+1,j) \geqslant p_\alpha(k,j)$ and we apply the induction to $(x^{a-1})_{k,j}$. Similarly if $\alpha(i) = j$ and $a > p_\alpha(i,j)$, $a - 1 \geqslant p_\alpha(i,j) = p_\alpha(i+1,j) \geqslant p_\alpha(k,j)$.

So we can assume in what follows that $\alpha(i) \in]i,j[$. The exact same reasoning applied to j allows to conclude that $\alpha(j) \in]i,j[$, so that $i < \alpha(i) < \alpha(j) < j$.

We now come to the crucial remark that

$$\begin{aligned} p_\alpha(i,j) &= j - i + 1 - \#\{\text{pairings of } \alpha \text{ inside } [i,j]\} \\ &= j - \alpha(j) + 1 - \#\{\text{pairings of } \alpha \text{ inside } [\alpha(j),j]\} \\ &\quad + \alpha(j) - \alpha(i) - 1 - \#\{\text{pairings of } \alpha \text{ inside } [\alpha(i)+1, \alpha(j)-1]\} \\ &\quad + \alpha(i) - i + 1 - \#\{\text{pairings of } \alpha \text{ inside } [i, \alpha(i)]\} \\ &= p_\alpha(i,\alpha(i)) + p_\alpha(\alpha(i)+1, \alpha(j)-1) + p_\alpha(\alpha(j), j) \end{aligned}$$

where in the last line, if $\alpha(i)+1 = \alpha(j)$ then conventionally $p_\alpha(\alpha(i)+1, \alpha(j)-1) = 0$. Indeed the configuration does not allow for mixed pairings between the three intervals $[i,\alpha(i)]$, $[\alpha(i)+1, \alpha(j)-1]$, $[\alpha(j), j]$. So we can write

$$(x^a)_{i,j} = \sum_{i<k<\ell<j} \left(x^{p_\alpha(i,\alpha(i))}\right)_{i,k} \left(x^{a-p_\alpha(i,\alpha(i))-p_\alpha(\alpha(j),j)}\right)_{k,\ell} \left(x^{p_\alpha(\alpha(j),j)}\right)_{\ell,j} .$$

The first factor is zero if $k \leqslant \alpha(i)$ by the induction hypothesis, noting that $p_\alpha(i,k) \leqslant p_\alpha(i,\alpha(i))$ and similarly the third factor is zero if $\ell \geqslant \alpha(j)$. If $\alpha(i) + 1 = \alpha(j)$ the proof is finished; otherwise note that $p_\alpha(k,\ell) \leqslant \alpha_\alpha(\alpha(i)+1, \alpha(j)-1) \leqslant a - p_\alpha(i,\alpha(i)) - p_\alpha(\alpha(j), j)$ and the second factor vanishes for the same reason. $\square$

Taking $a = N \geqslant p_\alpha(i,j)$ in Lemma 4.7, we find that $\hat{\mathcal{O}}_\alpha \subset \mathcal{O}$ (set-theoretically).

Now observe that $\hat{\mathcal{O}}_\alpha$ is defined by p equations, but $\mathcal{O}$ is equidimensional of codimension p, so that $\hat{\mathcal{O}}_\alpha$ is (a complete intersection) of pure codimension p, and so is a union of irreducible components of $\mathcal{O}$.

One could with more effort conclude geometrically that $\mathcal{O}_\alpha = \hat{\mathcal{O}}_\alpha$, but instead we shall use multidegrees and the uniqueness property of Lemma 3.3. Define $\hat{J}_\alpha := \operatorname{mdeg}_{\mathfrak{n}} \hat{\mathcal{O}}_\alpha$. Since the $\hat{\mathcal{O}}_\alpha$ are complete intersections, one can calculate directly

$$\hat{J}_\alpha = \prod_{i<j=\alpha(i)} \left(\frac{j-i+1}{2} h + z_i - z_j \right). \tag{4.11}$$

Note that an identical formula (really, at $h=0$, but this can be absorbed in the shift of the z's) appears in [KL00] in an indirectly related context.

We can then check

Lemma 4.8. $\hat{J}_\lambda = \sum_\alpha \hat{J}_\alpha e_\alpha$ *satisfies Eq.* (4.6).

Proof. Recall that Eq. (4.6) amounts to saying that the entries $\hat{J}_\alpha$ of $\hat{J}_\lambda$ must satisfy Eq. (4.5). Taking into account that we use the *dual* action defined above, we find the same dichotomy as in Section 4.4.3, but inverted:

(1) If $\alpha(i) = i+1$, then according to Section 4.4.3 case (2), $m_{i;\alpha',\alpha'} = +1$, so $m_{i;\alpha,\alpha} = -1$, i.e. we are in case (1) of Section 4.1.3; so Eq. (4.5) can be rewritten as Eq. (4.3), which is trivially satisfied by $\hat{J}_\alpha = C(h + z_i - z_{i+1})$ where C does not depend on z_i, z_{i+1}.

(2) If $\alpha(i) \neq i+1$, then according to Section 4.4.3 case (2), $m_{i;\alpha',\alpha'} = -1$, so $m_{i;\alpha,\alpha} = +1$, i.e. we are in case (2) of Section 4.1.3; so Eq. (4.5) can be rewritten as Eq. (4.4), or more explicitly,

$$-(h + z_i - z_{i+1})\partial_i \hat{J}_\alpha = \begin{cases} 0 & \alpha(i) = \alpha(i+1) = \varnothing \\ \hat{J}_{E^{(i,i+1)}\alpha} & \text{otherwise} \end{cases}$$

where again we have identified standard Young tableaux and link patterns when we write "$E^{(i,i+1)}\alpha$".

This equation can also be checked directly case by case:

- If $\alpha(i) = \alpha(i+1) = \varnothing$, $\hat{J}_\alpha$ does not depend on z_i, z_{i+1}, so $\partial_i \hat{J}_\alpha = 0$.
- If $\alpha(i) = \varnothing$, $\alpha(i+1) = j > i+1$, $\hat{J}_\alpha = C(\frac{j-i}{2}h + z_{i+1} - z_j)$, so $-(h+z_i-z_{i+1})\partial_i \hat{J}_\alpha = C(h + z_{i+1} - z_i)$ which is indeed $\hat{J}_{E^{(i,i+1)}\alpha}$ since $E^{(i,i+1)}\alpha$ differs from α only in pairing $i, i+1$ and having j unpaired.
- The case $\alpha(i) = j < i$, $\alpha(i+1) = \varnothing$ can be treated similarly.
- If $\alpha(i) = j < i$, $\alpha(i+1) = k > i$, $\hat{J}_\alpha = C(\frac{i-j+1}{2}h + z_j - z_i)(\frac{k-i}{2}h + z_{i+1} - z_k)$, so $-(h + z_i - z_{i+1})\partial_i \hat{J}_\alpha = C(h + z_i - z_{i+1})(\frac{k-j+1}{2}h + z_j - z_k)$ which again coincides with $\hat{J}_{E^{(i,i+1)}\alpha}$, since $E^{(i,i+1)}\alpha$ pairs $i, i+1$ and j, k.
- The other two cases $i < i+1 < \alpha(i+1) < \alpha(i)$ and $\alpha(i+1) < \alpha(i) < i < i+1$ can be treated similarly.

□

Applying the intertwiner ϕ and then Theorem 4.4 and Lemma 3.3, we conclude that the $\hat{J}_\alpha$ coincide up to normalization with the multidegrees J_α of the orbital varieties: $\hat{J}_\alpha = cJ_\alpha$ for some $c \neq 0$.

Now according to the above, $\hat{\mathcal{O}}_\alpha$ is a union of a certain certain subset of $\mathcal{O}_\beta$, so we can write at the level of multidegrees $\hat{J}_\alpha = \mathrm{mdeg}_{\mathfrak{n}} \hat{\mathcal{O}}_\alpha = \sum_\beta k_{\alpha,\beta} J_\beta$ where $k_{\alpha,\beta}$ is the multiplicity of $\mathcal{O}_\beta$ in $\hat{\mathcal{O}}_\alpha$ (or zero if $\mathcal{O}_\beta \not\subset \hat{\mathcal{O}}_\alpha$). In order to conclude, we only need to note that according to Eq. (4.5) and Theorem 4.2, the J_β, $\beta \in \mathrm{SYT}(\lambda)$, generate a subspace of $\mathbb{C}[z_1, \dots, z_n, h]$ which is an irreducible representation of the symmetric group under the action $\hat{s}_i$ (associated to the conjugate partition λ', with our sign convention), and so in particular are linearly independent. So $cJ_\alpha = \sum_\beta k_{\alpha,\beta} J_\beta$ implies $k_{\alpha,\beta} = c\delta_{\alpha,\beta}$ and $\hat{\mathcal{O}}_\alpha = \mathcal{O}_\alpha$ (set-theoretically). In other words, we have proved:

Theorem 4.9. *$\mathcal{O}_\alpha$ is defined by the equations*

$$\left(x^{\frac{j-i+1}{2}}\right)_{i,j} = 0, \qquad i < j = \alpha(i)\,.$$

In fact, since all coefficients of $\hat{J}_\alpha = k_{\alpha,\alpha} J_\alpha$ at $h = 0$ are ± 1 and $J_\alpha \in \mathbb{Z}[z_1, \dots, z_n, h]$, we have $k_{\alpha,\alpha} = c = 1$, i.e., $\hat{J}_\lambda = J_\lambda$ (and $\hat{\mathcal{O}}_\alpha$ being a complete intersection, the equations above define $\mathcal{O}_\alpha$ as a reduced scheme). In particular,

$$J_\alpha = \prod_{i<j=\alpha(i)} \left(\frac{j-i+1}{2} h + z_i - z_j\right). \tag{4.12}$$

4.5.3. *Change of basis.* We can use again duality (conjugation of partition and Young tableaux) to find the intertwiner:

Lemma 4.10. *The intertwiner is:*

$$\phi(e_\alpha) = (-1)^{p(p-1)/2}\, \varepsilon_\alpha \sum_{\substack{L=(l_1,\dots,l_n) \\ \text{permutation of } (1,\dots,p,1,\dots,n-p)}} J_{\alpha'}\big|_{z_1^{l_1-1}\cdots z_n^{l_n-1}}\, v_L$$

where $|_{z_1^{l_1-1}\cdots z_n^{l_n-1}}$ denotes the given coefficient of a polynomial.

Proof. We need to check that this ϕ intertwines the symmetric group action. Note that $\deg J_{\alpha'} = p(p-1)/2 + (n-p)(n-p-1)/2$ so $z_1^{l_1} \cdots z_n^{l_n}$ exhausts the degree and we might as well set $h = 0$ in $J_{\alpha'}$.

$$\begin{aligned}
P^{(i,i+1)}\phi(e_\beta) &= (-1)^{p(p-1)/2}\varepsilon_\beta \sum_L J_{\beta'}\big|_{z_1^{l_1-1}\cdots z_n^{l_n-1}}\, v_{l_1,\dots,l_{i+1},l_i,\dots,l_n} \\
&= (-1)^{p(p-1)/2}\varepsilon_\beta \sum_L (\tau_i J_{\beta'})\big|_{z_1^{l_1-1}\cdots z_n^{l_n-1}}\, v_L \\
&= (-1)^{p(p-1)/2}\varepsilon_\beta \sum_L \sum_{\alpha'} m_{i;\beta',\alpha'} J_{\alpha'}\big|_{z_1^{l_1-1}\cdots z_n^{l_n-1}}\, v_L \qquad \text{by Eq. (4.5) at } h=0 \\
&= \sum_{\alpha'} \varepsilon_\beta\varepsilon_\alpha m_{i;\beta',\alpha'}\phi(e_\alpha) \\
&= -\sum_\alpha m_{i;\alpha,\beta}\phi(e_\alpha) \qquad \text{by Eq. (4.9)} \\
&= \phi(\rho^{(i,i+1)}e_\beta) \qquad \text{by Eq. (4.2)}
\end{aligned}$$

Next we check the normalization, which is fixed by Theorem 4.4. Consider the same tableau α_0 that was used in the proof of Proposition 4.6, i.e., $\alpha_0 = \begin{array}{|c|c|c|c|c|}\hline 1 & & \cdots & & n-p \\ \hline \begin{smallmatrix} n-p \\ +1 \end{smallmatrix} & \cdots & n \\ \cline{1-3}\end{array}$, which is a tableau of λ'. Then we have as before $J_{\alpha_0} = \prod_{1\leqslant i<j\leqslant n-p}(h + z_i - z_j)\prod_{n-p+1\leqslant i<j\leqslant n}(h + z_i - z_j)$, so that $\phi(e_{\alpha'_0}) = (-1)^{(n-p)(n-p-1)/2}A_{n-p}\otimes A_n$ where A_k is the antisymmetrizer $\sum_{\sigma\in S_k}(-1)^\sigma v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(k)}$.

Now consider the multi-index $L = (n-p,\dots,1,1,\dots,p)$. According to Eq. (3.6), $I_L = (z_n - z_{n-2p+1})(z_{n-1} - z_{n-2p+2})\cdots(z_{n-p+1} - z_{n-p})$. But this is also $J_{\alpha'_0}|_{h=0}$ according to Eq. (4.12). Recalling that Theorem 4.2 implies that the $J_\alpha|_{h=0}$ are linearly independent, we conclude that the coefficient of v_L in $\phi(e_{\alpha'_0})$ must be 1, which is consistent with the formula above. □

4.5.4. *Cyclicity.* In order to simplify the discussion, we assume now that $n = 2p = 2N$, i.e., the Young diagram is rectangular, and use again the rotation ρ of Section 4.4.5 (in principle the content of the present section is valid for $p < n/2$ but more work would be needed to define ρ).

Then it is obvious from the explicit form (4.12) that the J_α satisfy the extra "cyclicity" relation

$$J_{\rho\alpha}(z_1,\dots,z_n) = -J_\alpha(z_2,\dots,z_n,z_1-(N+1)h)$$

Indeed if n is paired to say i, there is a factor $\frac{n-i+1}{2}h + z_n - z_i$, but once rotated, the pairing $1, i+1$ produces a factor $\frac{i+1}{2}h + z_{i+1} - z_1$ which corresponds to the substitution $z_i \to z_{i+1}$, $z_n \to z_1 - \frac{n+2}{2}h$ and a change of sign.

This together with Eq. (4.5) implies the qKZ equation.

We shall see in next section the condition on λ for I_λ (and therefore J_λ) to satisfy qKZ, a condition which is satisfied if λ has two columns.

5. I_λ is a q-conformal block, I_λ satisfies the qKZ equations if $d(\lambda) \leqslant 1$.

Theorem 5.1. *Let $\lambda \in \mathbb{N}^N$ be a partition of n. If $d(\lambda) > 0$ then I_λ is a q-conformal block at level $d(\lambda)$. If $d(\lambda) = 0$ then I_λ is a q-conformal block at level 1.*

Proof. First we prove that I_λ is a singular vector. Observe that the $e_{i,j}$-image of a $V[\lambda]$-valued function is a $V[\mu]$-valued function with $\mu_k = \lambda_k$ except $\mu_i = \lambda_i + 1$ and $\mu_j = \lambda_j - 1$. The action of $e_{i,j}$ and the deformed action of S_n commute, hence $e_{i,j} I_\lambda$ is skew-symmetric. If $i < j$, then the degree $k(\lambda)$ of $e_{i,j} I_\lambda$ is strictly less than $k(\mu) = k(\lambda) + \lambda_i - \lambda_j + 1$. Hence $e_{i,j} I_\lambda$ must be 0 by Lemma 3.3.

The operation $e(z)$ also commutes with the deformed action of S_n, hence we can argue for the q-conformal block property similarly.

First let $d(\lambda) > 0$. The function $e(z)I_\lambda$ is a skew-symmetric $V[\mu]$-valued function of degree $k(\lambda) + 1$, where $\mu_k = \lambda_k$ except $\mu_1 = \lambda_1 + 1$, $\mu_N = \lambda_N - 1$. Calculation shows that $k(\mu) = k(\lambda) + d(\lambda) + 1$ which is strictly greater than the degree $k(\lambda) + 1$. Hence $e(z)I_\lambda = 0$ by Lemma 3.3.

Now let $d(\lambda) = 0$. The function $e(z)^2 I_\lambda$ is a skew-symmetric $V[\mu]$-valued function of degree $k(\lambda) + 2$, where $\mu_k = \lambda_k$ except $\mu_1 = \lambda_1 + 2$, $\mu_N = \lambda_N - 2$. Calculation shows that $k(\mu) = k(\lambda) + 4$ which is strictly greater than the degree $k(\lambda) + 2$. Hence $e(z)^2 I_\lambda = 0$ by Lemma 3.3. □

Corollary 5.2. *Let $d(\lambda) \leqslant 1$. For generic $z \in \mathbb{C}^n$ the space of q-conformal blocks at level 1 is one-dimensional.*

Proof. We recalled in Lemma 2.2 that for generic z the dimension of q-conformal blocks is at most 1. We proved in Theorem 5.1 that I_λ is a (generically nonzero) q-conformal block. Hence, for generic z this space is one-dimensional. □

Consider $z \in \mathbb{C}^n$ for which Corollary 5.2 holds and for which the qKZ operators have no singularities, e.g. $z_1 - z_2 + h \neq 0$ etc. Over the configuration space of these z's one may consider the bundle of singular vectors. The q-conformal blocks form a rank 1 subbundle. It is proved in [MV98] that the subbundle of q-conformal blocks is preserved by the qKZ connection. In our language this means the following theorem.

Theorem 5.3. [MV98, Theorem 2] *If, for the z's defined above, the space of q-conformal blocks at level 1 is spanned by a $V[\lambda]$-valued function I, then a scalar function multiple of I satisfies the qKZ difference equations* (2.3).

Theorem 5.4. *Let $d(\lambda) \leqslant 1$. Then I_λ satisfies the qKZ equations* (2.3).

The rest of this section is the proof of this theorem.

Proof. The function I_λ is a q-conformal block at level 1. By Theorem 5.3, there is a scalar function $f(z_1, \dots, z_n)$ such that $f I_\lambda$ satisfies the qKZ equations. We obtain

$$f(\dots, z_i - (N+1)h, \dots)\, I_\lambda(\dots, z_i - (N+1)h, \dots) \;=\; K_i f I_\lambda, \qquad i = 1, \dots, n.$$

After rearrangement we have

$$\frac{f(\dots, z_i - (N+1)h, \dots)}{f} I_\lambda(\dots, z_i - (N+1)h, \dots) = K_i I_\lambda, \qquad i = 1, \dots, n. \tag{5.1}$$

Since $I_\lambda(\dots, z_i - (N+1)h, \dots)$ and $K_i I_\lambda$ are rational functions of $z_1, \dots, z_n$, the ratios

$$g_i = \frac{f(\dots, z_i - (N+1)h, \dots)}{f}, \qquad i = 1, \dots, n,$$

are rational functions of $z_1, \dots, z_n$ of degree 0.

Our first goal is to show that g_1 is a constant function equal to 1. We start with two lemmas.

Lemma 5.5. *Let I be a $V[\lambda]$-valued skew-symmetric function (for example $I = I_\lambda$). Then*

$$R^{(i,i+1)}(z_i - z_{i+1})\, I \;=\; -\,P^{(i,i+1)}\, I(z_i \leftrightarrow z_{i+1})\,.$$

Proof. Skew symmetry with respect to the transposition s_i implies the formula by direct calculation. □

Lemma 5.6. *$K_1 I_\lambda$ is a polynomial.*

Proof. By Lemma 5.5, $K_1 I_\lambda \;=\; (-1)^{n-1}\, P^{(1,n)} \cdots P^{(1,2)}\, I_\lambda(z_2, \dots, z_n, z_1)\,.$ □

We claim that the components of I_λ do not have a common polynomial factor of degree $\geqslant 1$. Indeed, if a polynomial, necessarily homogeneous, divides all components of I_λ, then in the quasiclassical limit $h = 0$ a polynomial would divide all components of $I_\lambda^{h=0}$. One sees from the explicit form of $I_\lambda^{h=0}$ in the Remark in Section 3 that this is not the case.

This claim together with Lemma 5.6 implies that the denominator of g_1 is a constant function, and since g_1 is a rational function of degree 0, g_1 must be a constant function. The $h = 0$ limit of equation (5.1) then implies that $g_1 = 1$. In other words, we proved that I_λ satisfies the first qKZ equation.

Our next goal is to show that the first qKZ equation implies the others for skew-symmetric functions.

For brevity we will write p for $(N+1)h$. Assume that the i-th qKZ equation

$$I(z_i \to z_i - p) = R^{(i,i-1)}(z_i - z_{i-1} - p) \cdots R^{(i,1)}(z_i - z_1 - p) R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}) I$$

holds. At the right end of the formula we can use Lemma 5.5 to obtain

$$\begin{aligned} I(z_i \to z_i - p) = {} & R^{(i,i-1)}(z_i - z_{i-1} - p) \cdots R^{(i,1)}(z_i - z_1 - p) \times \\ & \times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+2)}(z_i - z_{i+2})(-P^{(i,i+1)} I(z_i \leftrightarrow z_{i+1})). \end{aligned}$$

Applying $-P^{(i,i+1)}$ to this equation, together with the iterated application of

$$P^{(i,i+1)} R^{(i,m)}(u) = R^{(i+1,m)}(u) P^{(i,i+1)}$$

we get

$$\begin{aligned} -P^{(i,i+1)} I(z_i \to z_i - p) = {} & R^{(i+1,i-1)}(z_i - z_{i-1} - p) \cdots R^{(i+1,1)}(z_i - z_1 - p) \times \\ & \times R^{(i+1,n)}(z_i - z_n) \cdots R^{(i+1,i+2)}(z_i - z_{i+2}) I(z_i \leftrightarrow z_{i+1}). \end{aligned}$$

To this equation we substitute $z_i \leftrightarrow z_{i+1}$ and obtain

$$\begin{aligned} -P^{(i,i+1)}\, I(z_i \to z_{i+1} - p, z_{i+1} \to z_i) \;=\; {} & R^{(i+1,i-1)}(z_{i+1} - z_{i-1} - p) \cdots R^{(i+1,1)}(z_{i+1} - z_1 - p) \\ & \times\; R^{(i+1,n)}(z_{i+1} - z_n) \cdots R^{(i+1,i+2)}(z_{i+1} - z_{i+2})\, I\;. \end{aligned}$$

We can use Lemma 5.5 to write the left hand side in the form of

$$R^{(i,i+1)}(z_i - z_{i+1} + p) I(z_{i+1} \to z_{i+1} - p).$$

Then applying the $R^{(i,i+1)}(z_{i+1} - z_i - p)$ operator to both sides results in the $i+1$-st qKZ equation. This finishes the proof of Theorem 5.4. □

6. A q-Selberg type integral

According to the general principle in [MV00], if a KZ-type equation has a one-dimensional space of solutions, then the hypergeometric or q-hypergeometric integrals representing the solutions can be calculated explicitly, see demonstrations of that principle in [FSV03, RSV10, TV97, TV03, Var10, War09, War10]. In this section we give another example of this type.

In the rest of the paper we fix $N = 2$ and consider the bundle of the $\mathfrak{gl}_2$ q-conformal blocks at level 1 over $(\mathbb{C}^2)^{\otimes n}$. That bundle is of rank 1. The qKZ operators define a discrete flat connection on that bundle. In Section 3 we constructed a vector-valued polynomial on $\mathbb{C}^n$ that generates the space of flat sections of the discrete connection. Moreover, in Section 4 we identified that polynomial with the generating function of extended Joseph polynomials of orbital varieties associated with nilpotent $n \times n$-matrices. On the other hand in [TV97, MV98, MV99] flat sections of the same connection were constructed as multidimensional q-hypergeometric integrals. In this section we identify the q-hypergeometric flat sections constructed in [TV97, MV98, MV99] with the polynomial section constructed in Section 3 and obtain a q-Selberg type identity that a multidimensional q-hypergeometric integral equals a polynomial.

6.1. **Quantized conformal blocks at level 1 and qKZ equations.** First we recall some earlier definitions specialized for $\mathfrak{gl}_2$, and with the substitution $h = 1$. We consider the vector representation $\mathbb{C}^2$ of $\mathfrak{gl}_2$ with the standard basis v_1, v_2 and denote $V = (\mathbb{C}^2)^{\otimes n}$. The space V has a basis of vectors $v_{i_1} \otimes \cdots \otimes v_{i_n}$, where $i_j \in \{1, 2\}$. Every such a sequence $(i_1, \ldots, i_n)$ defines a decomposition $L = (L_1, L_2)$ of $\{1, \ldots, n\}$ into disjoint subsets, $L_j = \{l \mid i_l = j\}$. The basis vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$ is denoted by v_L. We have $V = \bigoplus_{\lambda=(\lambda_1,\lambda_2)} V[\lambda]$, where $\lambda_1 + \lambda_2 = n$, and $V[\lambda] = \{v \in V \mid e_{i,i}v = \lambda_i v, i = 1, 2\}$. Denote by $\mathcal{L}_\lambda$ the set of all indices L with $|L_j| = \lambda_j,\ j = 1, 2$. The vectors $\{v_L \mid L \in \mathcal{L}_\lambda\}$ form a basis of $V[\lambda]$.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define an operator $e(z) : V \to V$, by the formula

$$e(z) = \sum_{j=1}^{n} (z_j - e^{(j)}_{2,2} + \sum_{s=j+1}^{n} (e_{1,1} - e_{2,2})^{(s)}) e^{(j)}_{1,2}.$$

For $\lambda = (\lambda_1, \lambda_2)$ with $1 \geqslant \lambda_1 - \lambda_2 \geqslant 0$, we define the space of quantized conformal blocks at level 1 as

$$CB_\lambda(z) = \{ v \in V[\lambda] \mid e_{1,2}v = 0,\ e(z)^{2+\lambda_2-\lambda_1} v = 0 \}.$$

Note that for any n, the space V has a unique subspace $V[\lambda]$ with $1 \geqslant \lambda_1 - \lambda_2 \geqslant 0$. If $n = 2\ell$, then $\lambda = (\ell, \ell)$ and if $n = 2\ell+1$, then $\lambda = (\ell+1, \ell)$. That λ will be called the *middle weight*. The middle weight λ is determined by n and we will denote the subspace $CB_\lambda(z)$ just by $CB(z)$.

Corollary 5.2 claims that for generic $z \in \mathbb{C}^n$, $\dim CB(z) = 1$.

For $i = 1, \ldots, n$, the qKZ operators at level 1 on V are

$$\begin{aligned} K_i(z_1, \ldots, z_n) = R^{(i,i-1)} (z_i - z_{i-1} - 3) \cdots R^{(i,1)} (z_i - z_1 - 3) \times \\ \times R^{(i,n)}(z_i - z_n) \cdots R^{(i,i+1)}(z_i - z_{i+1}). \end{aligned}$$

The qKZ operators define on V a discrete flat connection,

$$K_j(z_1, \ldots, z_i - 3, \ldots, z_n) K_i(z_1, \ldots, z_n) = K_i(z_1, \ldots, z_j - 3, \ldots, z_n) K_j(z_1, \ldots, z_n)$$

for all i, j, see [FR92]. A V-valued function $I(z)$ is a *flat section* if it satisfies the qKZ equations,

$$I(z_1, \dots, z_i - 3, \dots, z_n) = K_i(z_1, \dots, z_n) I(z_1, \dots, z_i, \dots, z_n), \qquad i = 1, \dots, n. \tag{6.1}$$

The subbundle of conformal blocks at level 1 is invariant with respect to the qKZ connection,

$$K_i(z_1, \dots, z_n) : CB(z_1, \dots, z_n) \to CB(z_1, \dots, z_i - 3, \dots, z_n)$$

for all i, see [MV98, MV99].

Recall the polynomial I_λ from Section 3, with the substitution $h = 1$. In notation we will not indicate the $h = 1$ substitution. Hence in the rest of the paper we have e.g.

$$I_{(2,1)} = (z_1 - z_2 + 1)v_{112} + (z_3 - z_1 - 2)v_{121} + (z_2 - z_3 + 1)v_{211}.$$

Results of the first part of the paper, in our present conventions, are as follows.

- I_λ is skew symmetric with respect to the S_n-action (2.1) with $h = 1$.
- I_λ has degree $k(\lambda) = \frac{1}{2}\lambda_1(\lambda_1 - 1) + \frac{1}{2}\lambda_2(\lambda_2 - 1)$ (the minimal degree skew symmetric polynomial).
- Given λ, let $L = (L_1, L_2)$ be the partition of $\{1, \dots, n\}$ with $L_1 = \{1, \dots, \lambda_1\}$. Then I_λ is normalized in such a way that its L-th coordinate is

  $$\prod_{1 \leqslant a < b \leqslant \lambda_1} (z_a - z_b + 1) \prod_{\lambda_1 < a < b \leqslant n} (z_a - z_b + 1).$$

  That polynomial I_λ will be called *minimal*.
- If $\lambda = (\lambda_1, \lambda_2)$ is the middle weight, i.e. $1 \geqslant \lambda_1 - \lambda_2 \geqslant 0$, then $I_\lambda \in CB(z)$ and I_λ satisfies the qKZ equations (6.1).

6.2. **An integral representation for quantized conformal blocks at level 1.** In this section λ is the middle weight for n, $\lambda = (\ell, \ell)$ if $n = 2\ell$ and $\lambda = (\ell + 1, \ell)$ if $n = 2\ell + 1$.

Define the *master function*

$$\Phi(t_1, \dots, t_\ell, z_1, \dots, z_n) = \prod_{n \geqslant j > i \geqslant 1} \frac{\Gamma((z_j - z_i + 1)/3)}{\Gamma((z_j - z_i - 1)/3)} \prod_{\ell \geqslant j > i \geqslant 1} \frac{\Gamma((t_j - t_i + 1)/3)}{\Gamma((t_j - t_i - 1)/3)} \times$$
$$\times \prod_{i=1}^{n} \prod_{j=1}^{\ell} \frac{\Gamma((z_i - t_j - 1)/3)}{\Gamma((z_i - t_j)/3)}.$$

For $L = (L_1, L_2) \in \mathcal{L}_\lambda$ with $L_2 = \{i_1 < \dots < i_\ell\}$, define the function $w_L(t_1, \dots, t_\ell, z_1, \dots, z_n)$ by the formula

$$w_L = \sum_{\sigma \in S_\ell} \prod_{j=1}^{\ell} \frac{1}{t_{\sigma_j} - z_{i_j}} \prod_{m=1}^{i_j - 1} \frac{t_{\sigma_j} - z_m + 1}{t_{\sigma_j} - z_m} \prod_{1 \leqslant i < j \leqslant \ell,\ \sigma_i > \sigma_j} \frac{t_{\sigma_i} - t_{\sigma_j} + 1}{t_{\sigma_i} - t_{\sigma_j} - 1}$$

Define the $V[\lambda]$-valued *weight function* by the formula

$$w(t_1, \dots, t_\ell, z_1, \dots, z_n) = \sum_{L \in \mathcal{L}_\lambda} w_L(t_1, \dots, t_\ell, z_1, \dots, z_n) v_L.$$

Define the *trigonometric weight function* $W(t_1,\dots,t_\ell,z_1,\dots,z_n)$ by the formula

$$W=\pi^\ell\prod_{j=1}^{\ell}\frac{\sin(\pi(z_{2j}-z_{2j-1}+1)/3)}{\sin(\pi(t_j-z_{2j-1})/3)\sin(\pi(t_j-z_{2j})/3)}\prod_{m=1}^{2j-2}\frac{\sin(\pi(t_j-z_m+1)/3)}{\sin(\pi(t_j-z_m)/3)}.$$

Using the formula $\Gamma(1-x)\,\Gamma(x)=\frac{\pi}{\sin(\pi x)}$, we can write

$$\begin{aligned}\Phi W=\prod_{n\geqslant j>i\geqslant 1}\frac{\Gamma((z_j-z_i+1)/3)}{\Gamma((z_j-z_i-1)/3)}\prod_{\ell\geqslant j>i\geqslant 1}\frac{\Gamma((t_j-t_i+1)/3)}{\Gamma((t_j-t_i-1)/3)}\times\\ \times\prod_{j=1}^{\ell}\prod_{m=0}^{1}\Gamma((z_{2j-m}-t_j-1)/3)\Gamma(1-(z_{2j-m}-t_j)/3)\times\\ \times\prod_{j=1}^{\ell}\prod_{i=1}^{2j-2}\frac{\Gamma(1-(z_i-t_j)/3)}{\Gamma(1-(z_i-t_j-1)/3)}\prod_{i=2j+1}^{n}\frac{\Gamma((z_i-t_j-1)/3)}{\Gamma((z_i-t_j)/3)}\times\\ \times\pi^{-\ell}\prod_{j=1}^{\ell}\sin(\pi(z_{2j}-z_{2j-1}+1)/3).\end{aligned}$$

The function ΦWw is a meromorphic function of $t_1,\dots,t_\ell$ with first order poles at the hyperplanes

$$\begin{aligned}t_i-t_j&=1+3s, && i<j,\quad s=1,2,...\\ t_j-z_m&=-1+3s, && m\leqslant 2j,\quad s=0,1,...\\ t_j-z_m&=-3s, && m\geqslant 2j-1,\quad s=0,1,...\end{aligned}$$

Define an oriented unbounded one-chain $\mathcal{C}_n\subset\mathbb{C}$. It consists of the vertical line $-\frac12+\sqrt{-1}\mathbb{R}$, oriented from $-\frac12-\sqrt{-1}\infty$ to $-\frac12+\sqrt{-1}\infty$, and $2n$ circles $C_1,\dots,C_{2n}$ of radius $\frac14$. The circle C_j for $1\leqslant j\leqslant n$ is centered at $j\sqrt{-1}$ and oriented counterclockwise, while the circle C_j for $n<j\leqslant 2n$ is centered at $-1+j\sqrt{-1}$ and oriented clockwise. Define the *integration cycle*

$$\mathcal{C}_n^\ell=\{(t_1,\dots,t_\ell)\in\mathbb{C}^\ell\mid t_i\in\mathcal{C}_n\}.$$

For $(z_1,\dots,z_n)\in\mathbb{C}^n$ such that $|z_j-j\sqrt{-1}|<\frac14$, define a $V[\lambda]$-valued *q-hypergeometric integral* by the formula

$$\Psi_\lambda(z_1,\dots,z_n)=\int_{\mathcal{C}_n^\ell}\Phi(t,z)w(t,z)W(t,z)\,dt_1\dots dt_\ell.$$

Theorem 6.1 ([TV97])**.** *The function $\Psi_\lambda(z_1,\dots,z_n)$ is well-defined and extends to a meromorphic function on $\mathbb{C}^n$. Moreover, the function $\Psi_\lambda(z_1,\dots,z_n)$ is a solution of the qKZ equations* (6.1).

Theorem 6.2 ([MV98])**.** *For generic $z\in\mathbb{C}^n$, we have $\Psi_\lambda(z_1,\dots,z_n)\in CB(z)$.*

Theorem 6.3. *Let $n=2\ell$, $\lambda=(\ell,\ell)$ or $n=2\ell+1$, $\lambda=(\ell+1,\ell)$. Let $I_\lambda(z_1,\dots,z_n)$ be the minimal skew-symmetric $V[\lambda]$-valued polynomial as above. Then the q-hypergeometric* *integral $\Psi_\lambda(z_1,\dots,z_n)$ equals the polynomial $c_n I_\lambda(z_1,\dots,z_n)$, where*

$$c_2 = 2\pi\sqrt{-1}\frac{\Gamma(2/3)\Gamma(-1/3)}{\Gamma(1/3)}, \qquad c_{2\ell} = 3^{-\ell(\ell-1)}c_2^\ell, \qquad c_{2\ell+1} = (-1)^\ell 3^{-\ell^2}c_2^\ell. \tag{6.2}$$

By Section 4 the polynomial I_λ is the generating function of the extended Joseph polynomials of the orbital varieties associated with nilpotent $n\times n$-matrices. Hence, Theorem 6.3 gives an integral representation for those extended Joseph polynomials.

Proof. For $n=2$ the middle weight is $(1,1)$ and we have $I_{(1,1)} = v_{12} - v_{21}$, $\Psi_{(1,1)} = c_2 I_{(1,1)}$, where

$$\begin{aligned} c_2 &= \pi^{-1}\sin(\pi(z_{2j}-z_{2j-1}+1)/3)\frac{\Gamma((z_2-z_1+1)/3)}{\Gamma((z_2-z_1-1)/3)}\times \\ &\qquad\times\int_{\mathcal{C}_2}\prod_{m=0}^{1}\Gamma((z_{2j-m}-t_j-1)/3)\Gamma(1-(z_{2j-m}-t_j)/3)\frac{dt}{t-z_1} \\ &= 2\pi\sqrt{-1}\frac{\Gamma(2/3)\Gamma(-1/3)}{\Gamma(1/3)}, \end{aligned}$$

see (1.1).

For arbitrary n we have $\Psi_\lambda(z_1,\dots,z_n) = c_n(z_1,\dots,z_n)I_\lambda(z_1,\dots,z_n)$, where $c_n(z_1,\dots,z_n)$ is a scalar function 3-periodic with respect to every variable. Indeed, both $\Psi_\lambda(z_1,\dots,z_n)$ and $I_\lambda(z_1,\dots,z_n)$ are quantized conformal blocks at level 1 and both satisfy the qKZ equations. To check that c_n is given by (6.2) we consider the asymptotic zone:

(i) $|z_{2i}-z_{2i-1}|\leqslant 1$, $|\mathrm{Im}(z_{2i})|\leqslant 1$ for $i=1,\dots,\ell$,

(ii) $\mathrm{Re}(z_{2i+2}-z_{2i})\gg 1$ for $i=1,\dots,\ell-1$ and $\mathrm{Re}(z_n - z_{n-2})\gg 1$ if n is odd,

use the Stirling formula for the Gamma functions,

$$\frac{\Gamma((x+\alpha)/p)}{\Gamma((x+\beta)/p)} = (x/p)^{\alpha-\beta}(1+o(1)), \qquad |\arg(x/p)|<\pi,$$

and similarly to the proof of Theorem 6.7 in [TV97] observe that

$$c_{2\ell} = 3^{-\ell(\ell-1)}c_2^\ell(1+o(1)), \qquad c_{2\ell+1} = (-1)^\ell 3^{-\ell^2}c_2^\ell(1+o(1)).$$

This proves the theorem. □

7. An alternative integral for $N=2$

Finally, we formulate an integral representation in the two-row case $\lambda=(n-p,p)$, distinct from that of the previous paragraph, and which generalizes that of [RSZJ07] (which was the case n odd, $n=2p+1$).

Expanding $I_\lambda = \sum_L I_L v_L$, the multi-indices that contribute to the sum have $n-p$ 1s and p 2s. Let us parameterize them as follows: denote $L(a)$ the multi-index whose 2s are located at indices $a=(a_1<\dots<a_p)$.

Theorem 7.1.

$$I_{L(a)} = (-1)^{p(n-p+1)}h^p\prod_{1\leqslant i<j\leqslant n}(h+z_i-z_j)\oint\prod_{k=1}^{p}\frac{dw_k}{2\pi\sqrt{-1}}\frac{\prod_{1\leqslant k<\ell\leqslant p}(w_\ell-w_k)(h+w_k-w_\ell)}{\prod_{k=1}^p\left(\prod_{i=1}^{a_k}(w_k-z_i)\prod_{i=a_k}^{n}(h+w_k-z_i)\right)}$$

The integration cycle is the product of p identical 1-dimensional cycles. The 1-dimensional cycle is any contour that surrounds once counterclockwise each of the $z_1, \dots, z_n$ but none of the $z_1 - h, \dots, z_n - h$.

Note that the integrals have no pole at infinity (the integrand behaves as $w_k^{2(p-1)-(n+1)}$ as $w_k \to \infty$) so we may as well consider that the contour surrounds clockwise the $z_1 - h, \dots, z_n - h$ but none of the $z_1, \dots, z_n$.

Proof. We are going to apply Lemma 3.3. Denote by $\hat{I}_{L(a)}$ the r.h.s. of the formula above.

First one needs to check that $\hat{I}_{L(a)}$ is a polynomial in $z_1, \dots, z_n, h$. This is a routine calculation based on the application of the residue formula for the w_k integrals and the check that would-be poles in the variables z_i have vanishing residue (see a similar calculation in [FZJ08]); since the formula is homogeneous in $z_1, \dots, z_n, h$ this leaves only a power of h in the denominator which is cancelled by the factor h^p. The degree of $\hat{I}_{L(a)}$ is then (as a homogeneous polynomial in $z_1, \dots, z_n, h$) $p + n(n-1)/2 + p + 2p(p-1)/2 - p(n+1) = p(p-1)/2 + (n-p)(n-p-1)/2 = k(\lambda)$.

Next, we check that $\hat{I}_\lambda = \sum_L \hat{I}_L v_L$ is skew-symmetric by use of Lemma 3.1. Fixing $i = 1, \dots, n-1$, there are four possibilities:

- If $L_i = L_{i+1} = 1$, the integrand (including the prefactor in front of the integral) is $h + z_i - z_{i+1}$ times a symmetric function of z_i, z_{i+1}. This implies that $\hat{s}_i I_L = -I_L$.
- If $L_i = L_{i+1} = 2$, say $a_k = i$, $a_{k+1} = i+1$, then the integrand minus itself with $z_i \leftrightarrow z_{i+1}$ is skew-symmetric in w_k, w_{k+1} and therefore its integral is zero. This implies again that $\hat{s}_i I_L = -I_L$.
- If $L_i = 2$, $L_{i+1} = 1$, say $a_k = i$, $a_{k+1} > i+1$, then the integrand is $\frac{h+z_i-z_{i+1}}{w_k - z_i}$ times a symmetric function of z_i, z_{i+1}. Applying $\hat{s}_i$ results in $-\frac{(h+w_k-z_i)(h+z_i-z_{i+1})}{(w_k-z_i)(w_k-z_{i+1})}$ times the same function, which is nothing but minus the integrand with $a_k \to i+1$, which is precisely $L \to s_i(L)$. That is, $\hat{s}_i I_L = -I_{s_i(L)}$.
- The case $L_i = 1$, $L_{i+1} = 2$ is treated similarly.

Therefore all the hypotheses of Lemma 3.3 are satisfied, and $\hat{I}_\lambda$ is proportional to I_λ. In order to fix the normalization, we consider the case $a = (n-p+1, \dots, n)$, i.e., $L(a) = L_0$. Then the integrals can be computed one by one as follows. The integral over w_p has only one pole outside the contour, at $z_n - h$. Next, the integral over w_{p-1} has two poles, at $z_n - h$ and $z_{n-1} - h$, but the first one is cancelled by the factor $w_p - w_{p-1}$ in the numerator (since we have taken the residue at $w_p = z_n - h$). So there is only one contribution, the residue at $w_{p-1} = z_{n-1} - h$; and so on. In the end, evaluating the residues at $w_k = z_{k+n-p} - h$ results in: $\hat{I}_{L_0} = \prod_{1 \leqslant i < j \leqslant n-p} (h + z_i - z_j) \prod_{n-p+1 \leqslant i < j \leqslant n} (h + z_i - z_j)$, which coincides with I_{L_0}, so that $\hat{I}_\lambda = I_\lambda$. □

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