On Modeling the Performance and Reliability of Multi-Mode Computer Systems*

V. G. Kulkarni†, V. F. Nicola‡ and K. S. Trivedi‡

Abstract

We present an effective technique for the combined performance and reliability analysis of multi-mode computer systems. A reward rate (or a performance level) is associated with each mode of operation. The switching between different modes is characterized by a continuous time Markov chain. Different types of service-interruption interactions (as a result of mode switching) are considered. We consider the execution time of a given job on such a system and derive the distribution of its completion time. A useful dual relationship, between the completion time of a given job and the accumulated reward up to a given time, is noted. We demonstrate the use of our technique by means of a simple example.

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† Curriculum in Operations Research and Systems Analysis, University of North Carolina, Chapel Hill, NC 27514.

‡ Department of Computer Science, Duke University, Durham, NC 27706.

1. *Introduction*

We consider a model for the combined evaluation of performance and reliability of a multi-mode computer system. Performance (e.g., throughput, response time, instruction execution rate) changes from mode to mode and a mode change occurs in response to an event such as a failure or a repair. The stochastic process representing the modes (structure-states) and mode changes can be thought of as a reward process by associating a reward (performance index) with each mode [4,10]. We can then study the the distribution of the accumulated reward until time t by time domain methods[10] or by transform techniques[4,14].

The authors who have taken such a system-oriented view do not consider the effect of a fault occurring during the execution of a program. A task(job or program)-oriented view of such a sytsem recognizes the fact that it is possible for a system failure to occur before the completion of a task [7] and that even if the task is completed, its completion time is likely to be different from its execution time in a given mode [3,5,12]. The job in service is interrupted with each mode change and the type of service -interruption interaction depends upon the mode just entered. For example, the occurrence of a fault during the execution of a job preempts the job and a later system recovery may allow the job to resume from the point of interruption (the preemptive-resume (*prs*) discipline) or the job may have to be repeated from the beginning. In the latter case, the repeated job may have the identical work requirement as the original preempted job (the preemptive-repeat-identical (*pri*) discipline) or a different work requirement sampled from the same distribution (the preemptive-repeat-different (*prd*) discipline).

The purpose of this paper is to develop a model that unifies and extends the efforts of these two groups of researchers. In particular, we show that if all interruptions are of the preemptive-resume type then the completion time of a given task and the accumulated reward until a given time are dual measures, so that the distribution of one of them allows us to compute the distribution of the other. In fact, our model is even more general - in that both acyclic (closed or non-repairable) and cyclic (open or repairable) systems are modeled.

Our model provides an exact analysis of the completion time distribution of a program (job) executing in a multi-mode system. It is also possible to incorporate the effect of queueing in our model. If the time spent in each structure-state is large compared with the interarrival and processing times of jobs, then we can use steady- state performance measures as reward rates for each structure-state. Such approximate decomposition methods have been considered by several authors [4,7,10,15]. If the assumption of a wide separation between the structure-state holding times and job processing times does not hold, then a more complex analysis is required [1,5,12].

We develop the basic model in the next section. In sections 3, 4 and 5 , we consider the individual cases where all structure-states are of the same type, that is, preemptive-resume, preemptive-repeat-identical, or preemptive-repeat-different, respectively.

2. *The Basic Model*

Consider a single server (e.g., a computer) serving a single job (e.g., a program). The job is characterized by its work requirement, B. For example, the work requirement of a computer program can be measured in terms of the number of instructions to be executed. We assume that B is a random variable with cumulative distribution function $G(x) = P(B \leq x)$ and LST $G^{\sim}(s)=E(e^{-sB})$. To avoid trivialities we assume $G(0+) = 0$.

The rate at which the server performs work is assumed to change with time according to the following model: At any time the server is in one (and only one) of the $n+1$ states (modes) numbered $0,1,2,...,n$. In state i the server performs work at rate $r_i \geq 0$, $1 \leq i \leq n$, work units per unit time (e.g., the instruction execution rate). The state

0 is an absorbing "failure" state, i.e., once the server is in state 0, it stays there forever and the work rate in this state is zero ($r_0=0$). We allow absorbing non-failure states among the states $1,\ldots,n$ with reward rates greater than zero so that if the server enters such a state, the job will eventually complete. Let $Z(t)$ be the state of the server at time t. $\{Z(t),t \geq 0\}$ is called the ***structure-state process***. We shall assume that the structure-state process is a stochastic process with piecewise constant paths with finite number of jumps in finite intervals of time. Furthermore, the structure-state process is assumed to be independent of the work requirement B of the job.

The states $i = 1,2,\ldots,n$ are classified as (i) ***prs***: preemptive-resume, (ii) ***pri***: preemptive-repeat-identical or (iii) ***prd***: preemptive-repeat-different.

The following quantities have been analyzed before in the literature for some special $\{Z(t),t \geq 0\}$ processes:

I. ***The job completion time (T(x)):*** defined to be the total time the server takes to complete a job that requires x units of work. T denotes the unconditional completion time of a job that requires a random amount of work, say B units. Gaver[5] studied the distribution of the r.v. T for a system subject to one type of failure and repair, in which the operating state is Markovian and the failure state is semi-Markovian. Nicola[12] extended Gaver's model to allow for mixed types of failures and repairs. Castillo and Siewiorek [3] condsidered a system with two types of failures in which the preemptive-repeat type failure could occur during the repair-time of the preemptive-resume type failure.

II. ***The probability of dynamic failure (η):*** defined to be the probability that the system fails before the job is completed, i.e. the server enters state 0 before completing B units of work[7].

III. ***The cumulative reward upto time t ($Y(t)$):*** defined to be the total amount of work done by the system up to time t. Y is the total accumulated work during the system's lifetime; it is the limit of $Y(t)$ as $t \to \infty$. The r.v. $Y(t)$ was first studied by Puri [14] for Markovian $Z(t)$ processes. Meyer[10] and Donatiello and Iyer[4] studied the distribution of $Y(t)$ for an acyclic Markovian $Z(t)$ process. Beaudry [2] studied the r.v. Y for a Markovian $Z(t)$ process, while Osaki and Nishio [13] studied the r.v. Y for a semi-Markovian $Z(t)$ process.

To present a unifying view of the quantities defined above, we introduce the cumulative measure, $W(t)$, defined as follows: Suppose that at time $t = 0$ the server starts processing a job with infinite work requirement. $W(t)$ is the amount of useful work completed by the server until time t (thus, excluding the work done prior to the last visit to a *pri* or a *prd* state). The following properties of the cumulative measure, $W(t)$, are immediately obvious:

(i) $W(0) = 0$,

(ii) $Z(t) = i \Rightarrow dW(t)/dt = r_i$,

(iii) If there is a transition in the structure-state process at time t and $Z(t+) = i$, then $W(t+) = 0$ if i is a *pri* or a *prd* state and $W(t+) = W(t-)$ if i is a *prs* state.

Typical sample paths of the structure-state process and the cumulative measure, $W(t)$, are shown in figure 1, for the following case: Set of states = $\{0,1,2,3\}$, state 1 is *prs* with $r_1 = 1$, states 2 and 3 are *pri* or *prd* with $r_2 = 2$ and $r_3 = 0$, state 0 is the absorbing failure state.

The following theorem shows how the quantities T, η, $Y(t)$ and Y can be related to each other via the cumulative measure, $W(t)$.

Theorem 1.

(i) $T = \min\{t \geq 0 : W(t) = B\}$.

(ii) The dynamic failure probability, $\eta = P(T = \infty)$.

(iii) If all states are *prs*, then

$$P(Y(t) \le x) = 1 - P(T(x) < t)$$

and

$$P(Y \le x) = 1 - P(T(x) < \infty).$$

Proof: (i) Let T be the job completion time. It is clear that

$$\{T > t\} \Leftrightarrow \{W(u) < B, \text{ for } all \ 0 \le u \le t\},$$

since $W(u)$ represents the useful work done upto time u. As $W(t)$ has piecewise continuous paths with only downward jumps, T is given by (i).

(ii) It is clear that

$$\{\text{ Dynamic Failure }\} \Leftrightarrow \{\text{ system fails before job completion }\}$$
$$\Leftrightarrow \{W(t) < B \text{ for } all \ \ t \ge 0\} \Leftrightarrow \{T = \infty\}.$$

Hence $\eta = P(T = \infty)$.

(iii) Let $T(x) = \min\{t \ge 0 : W(t) = x\}$. If all states are *prs*, then

$$\{Y(t) > x\} \Leftrightarrow \{W(t) > x\} \Leftrightarrow \{T(x) < t\}.$$

Hence

$$P(Y(t) > x) = P(T(x) < t). \qquad Q.E.D.$$

It is apparent from the above theorem that

$$T = \min\{t \ge 0 : W(t) = B\} \tag{2.1}$$

is the unifying random variable. This paper is devoted to the study of this random variable. Define the following distribution functions:

$$F_i(t,x) = P(T \le t \mid B = x, Z(0) = i), \ \ x \ge 0, \ \ 1 \le i \le n,$$
$$F(t,x) = P(T \le t \mid B = x), \ \ x \ge 0,$$
$$F_i(t) = P(T \le t \mid Z(0) = i), \ \ 1 \le i \le n,$$

$$F(t) = P(T \le t)$$

and the corresponding LSTs (Laplace Stieltjes Transforms),

$$F_i^{\sim}(s,x) = E(e^{-sT} \mid B = x, Z(0) = i), \quad x \ge 0,\ 1 \le i \le n, \tag{2.2}$$

$$F^{\sim}(s,x) = E(e^{-sT} \mid B = x), \quad x \ge 0, \tag{2.3}$$

$$F_i^{\sim}(s) = E(e^{-sT} \mid Z(0) = i), \quad 1 \le i \le n, \tag{2.4}$$

$$F^{\sim}(s) = E(e^{-sT}). \tag{2.5}$$

From the independence of $\{Z(t), t \ge 0\}$ and B it follows that

$$F^{\sim}(s,x) = \sum_{i=1}^{n} F_i^{\sim}(s,x) P(Z(0) = i), \quad x \ge 0, \tag{2.6}$$

$$F_i^{\sim}(s) = \int_0^{\infty} F_i^{\sim}(s,x) dG(x), \quad 1 \le i \le n, \tag{2.7}$$

$$F^{\sim}(s) = \sum_{i=1}^{n} F_i^{\sim}(s) P(Z(0) = i). \tag{2.8}$$

From equations (2.6) - (2.8) it is clear that the conditional LSTs $F_i^{\sim}(s,x)$ are of central importance to the analysis of T. In order to obtain explicit formulae for $F_i^{\sim}(s,x)$ it is necessary to make some further assumptions about the structure-state process. In the remaining paper we make the assumption that $\{Z(t), t \ge 0\}$ is a time homogeneous continuous time Markov chain (CTMC). The results derived here can be extended in a straight forward manner to the case when the structure-state process is assumed to be semi- Markov. Let q_{ij}, $1 \le i \ne j \le n$, be infinitesimal transition rate from state i to j and q_{i0} be the absorbing failure rate from state i. Let $Q = [q_{ij}]$, $1 \le i, j \le n$, be the n by n generator matrix where $q_i = \sum_{\substack{j=0 \\ j \ne i}}^{n} q_{ij} = -q_{ii}$. Note that row sums of Q are ≤ 0. We mention one property of the CTMC for future reference. Define

$$H = \min\{t \ge 0 : Z(t) \ne Z(0)\} \tag{2.9}$$

as the holding (or sojourn) time in the initial state. Then we have

($^{\sim}$) denotes LST, i.e., the Laplace transform of a probability density function.

$$P(H \le x, Z(H+) = j \mid Z(0) = i) = \frac{q_{ij}}{q_i}(1-e^{-q_i x}), \quad (i \ne j). \tag{2.10}$$

In the next section we treat the case where all states $i = 1,2,..n$ are preemptive-resume (*prs*) and in sections 4 and 5 we consider the case where all states are preemptive-repeat (*pri* and *prd*, respectively). The mixed cases where some states are *prs* and some are *pri* or *prd* have been studied in [8].

3. *The Preemptive-resume Case*

In this section we assume that the states $1,2,..,n$ are all preemptive-resume states. Note that state 0 does not have to be classified since it is a failure state. Theorem 2 below gives a method of computing the conditional *LST*s defined by equation (2.2). First, some notation:

$$F_i^{\sim *}(s,u) = \int_0^\infty e^{-ux} F_i^{\sim}(s,x)dx \ , \quad 1 \le i \le n, \tag{3.1}$$

$$\underline{F}^{\sim *}(s,u) = [F_1^{\sim *}(s,u), F_2^{\sim *}(s,u), \dots, F_n^{\sim *}(s,u)]^T, \tag{3.2}$$

$$R = diag[r_1, r_2, \dots, r_n], \tag{3.3}$$

$$\underline{r} = [r_1, r_2, \dots, r_n]^T, \tag{3.4}$$

where the superscript T denotes transpose.

Theorem 2. $F_i^{\sim *}(s,u)$, for $1 \le i \le n$, is given by

$$F_i^{\sim *}(s,u) = \frac{r_i}{s + r_i u + q_i} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{s + r_i u + q_i} F_j^{\sim *}(s,u), \; 1 \le i \le n. \tag{3.5}$$

Proof: Conditioning on the sojourn time H in the initial state we get

$$E(e^{-sT} \mid H=h, B=x, Z(0)=i) = \begin{cases} e^{-sx/r_i}, & \text{if } h \ge x/r_i \\ e^{-sh} \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{q_i} F_j^{\sim}(s, x - r_i h), & \text{if } h < x/r_i \end{cases}$$

(*) denotes the Laplace transform of a function

Unconditioning yields

$$F_i^{\sim}(s,x) = \int_0^{\infty} E(e^{-sT} \mid H = h, B = x, Z(0) = i) q_i e^{-q_i h}\, dh$$

$$= e^{-(s+q_i)x/r_i} + \sum_{\substack{j=1 \\ j \neq i}}^{n} q_{ij} \int_0^{x/r_i} e^{-(s+q_i)h} F_j^{\sim}(s, x - r_i h) dh$$

Multiplying both sides by e^{-ux} and integrating we get equation (3.5). *Q.E.D.*

Equation (3.5) can be put in a matrix form as follows:

$$[sI + uR - Q]\, \underline{F}^{\sim *}(s,u) = \underline{r},$$

where I is the identity matrix. As it is well known that $[sI + uR - Q]$ is invertible, we get

$$\underline{F}^{\sim *}(s,u) = [sI + uR - Q]^{-1}\, \underline{r} \; . \tag{3.5a}$$

A direct inversion with respect to s yields

$$d_t\, \underline{F}^{*}(t,u) = e^{(Q-uR)t}\, \underline{r} \; .$$

After integration and some manipulations, we get

$$\underline{F}^{*}(t,u) = \frac{1}{u}[I - e^{(Q-uR)t}]\, \underline{e} \; . \tag{3.6}$$

We now describe how we can use the above theorem to compute $F_i^{\sim}(s,x)$. Using Cramer's rule we can write

$$F_i^{\sim *}(s,u) = A_i(s,u) / C(s,u)$$

where $C(s,u) = det[sI + uR - Q]$ and $A_i(s,u)$ are appropriate n by n subdeterminants of the augmented matrix $[sI + uR - Q; \underline{r}]$. It is obvious that both $A_i(s,u)$ and $C(s,u)$ are polynomials in s and u. Hence one can use partial fractions to invert $F_i^{\sim *}(s,u)$ with respect to u. Let $d = |\{i : r_i > 0\}|$, i.e. d is the number of states in which work rate is positive. Then $C(s,u)$ is a d-degree polynomial in u for a fixed value of s. Let $-u_1(s), \ldots, -u_d(s)$ be the roots of $C(s,u)$. In the special case when these roots are distinct, we can write

$$F_i^{\sim *}(s,u) = \sum_{j=1}^{d} \frac{A_{ij}(s)}{u+u_j(s)}, \qquad 1 \le i \le n \ . \tag{3.7}$$

where

$$A_{ij}(s) = \lim_{u \to -u_j(s)} \frac{A_i(s,u)}{C(s,u)}(u + u_j(s)), \qquad 1 \le j \le d. \tag{3.8}$$

Inverting with respect to u, we get

$$F_i^{\sim}(s,x) = \sum_{j=1}^{d} A_{ij}(s)\, e^{-u_j(s)x}, \qquad 1 \le i \le n. \tag{3.9}$$

Hence from equation (2.4)

$$F_i^{\sim}(s) = \sum_{j=1}^{d} A_{ij}(s)\, G^{\sim}(u_j(s)), \qquad 1 \le i \le n, \tag{3.10}$$

(recall that $G^{\sim}(s) = \int_0^{\infty} e^{-sx} dG(x)$), and

$$F^{\sim}(s) = \sum_{j=1}^{d} [\sum_{i=1}^{n} \pi_i A_{ij}(s)]\, G^{\sim}(u_j(s)). \tag{3.11}$$

where $\pi_i = P(Z(0) = i)$, $1 \le i \le n$.

It is interesting to note that the *LST* of T for a given s is simply a linear combination of the *LST* of B evaluated at $u_1(s), \ldots, u_d(s)$.

Now, assuming that state 0 is reachable from every other state, the probability of dynamic failure can be computed easily from Theorem 1 as

$$\eta = P(T = \infty) = 1 - \lim_{s \to 0} F^{\sim}(s). \tag{3.12}$$

The following corollary indicates how the *LST* of the cumulative reward $Y(t)$, for a given t, can be obtained from the $F_i^{\sim *}(s,u)$ functions.

Corollary 1. For a given $t \ge 0$, let $Y(t)$ be the cumulative reward upto time t. Let

$$Y_i(x,t) = P(Y(t) \le x \mid Z(0) = i),$$

$$Y_i^{\sim}(u,t) = E(e^{-uY(t)} \mid Z(0) = i)$$

and

$$Y_i^{\sim *}(u,s) = \int_0^{\infty} e^{-st}\, Y_i^{\sim}(u,t)\, dt \quad .$$

Then

$$Y_i^{\sim *}(u,s) = \frac{1}{s}(1 - uF_i^{\sim *}(s,u)) \ , \quad 1 \le i \le n. \tag{3.13}$$

Proof: Part (iii) of Theorem 1 implies that

$$P(Y(t) < x \mid Z(0) = i) = P(T(x) > t \mid Z(0) = i).$$

Now,

$$Y_i^{\sim *}(u,s) = \int_0^{\infty} e^{-st} E(e^{-uY(t)} \mid Z(0)=i)dt$$

$$= \int_{t=0}^{\infty} e^{-st} \int_{x=0}^{\infty} e^{-ux} d_x P(Y(t) \le x \mid Z(0)=i)dt$$

$$= \int_{x=0}^{\infty} e^{-ux} \int_{t=0}^{\infty} e^{-st} d_x P(Y(t) \le x \mid Z(0) = i)dt$$

$$= \int_{x=0}^{\infty} e^{-ux} d_x[\int_{t=0}^{\infty} e^{-st}[1-P(T(x) \le t \mid Z(0) = i)]dt]$$

$$= -\int_{x=0}^{\infty} e^{-ux} d_x\, F_i^{\sim}(s,x)/s = [1 - u\, F_i^{\sim *}(s,u)]/s. \qquad Q.E.D.$$

Using equation (3.5a), we can write in a matrix form

$$\underline{Y}^{\sim *}(u,s) = [sI + uR - Q]^{-1}\, \underline{e} \tag{3.13a}$$

with

$$\underline{Y}^{\sim *}(u,s) = [Y_1^{\sim *}(u,s), Y_2^{\sim *}(u,s), \ldots, Y_n^{\sim *}(u,s)]^T.$$

A direct inversion with respect to s yields

$$\underline{Y}^{\sim}(u,t) = e^{(Q-uR)t}\, \underline{e} \ . \tag{3.14}$$

We end this section with a simple example.

Example 3.1. The switching server

Consider a system that operates in two modes each with a different work rate, say r_1 and r_2 for modes "1" and "2", respectively. The system switches between the two modes according to a Poisson process at different rates, say λ and μ from modes "1" and "2", respectively. A total system failure may occur at any mode of operation at different rates, say λ_0 and μ_0 for modes "1" and "2", respectively. The CTMC representing the switching server is shown in figure 2. In the case where a total system failure may not occur, i.e. $\lambda_0 = \mu_0 = 0$ and if $r_2 = 0$ then the switching server model reduces to the completion time model of job execution in a system subject to breakdowns and repairs considered by Gaver [5].

In this example we consider the case in which both states 1 and 2 are of the preemptive-resume type. We note that if we set $\mu = 0$ in this example we obtain the reward model of a two processor system considered by Meyer [10]. In our example the Q matrix is

$$Q = \begin{bmatrix} -\lambda' & \lambda \\ \mu & -\mu' \end{bmatrix}$$

where $\lambda' = \lambda + \lambda_0$ and $\mu' = \mu + \mu_0$. Then from Equation (3.5a)

$$\begin{bmatrix} F_1^{\sim *}(s,u) \\ F_2^{\sim *}(s,u) \end{bmatrix} = \begin{bmatrix} s + r_1 u + \lambda' & -\lambda \\ -\mu & s + r_2 u + \mu' \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Solving for $F_1^{\sim *}(s,u)$ and $F_2^{\sim *}(s,u)$ we get

$$F_1^{\sim *}(s,u) = \frac{r_1 r_2 u + r_1(s+\mu') + r_2\lambda}{(s+\lambda' + r_1 u)(s+\mu'+r_2 u) - \lambda\mu}$$

$$F_2^{\sim *}(s,u) = \frac{r_1 r_2 u + r_2(s+\lambda') + r_1\mu}{(s+\lambda'+r_1 u)(s+\mu'+r_2 u) - \lambda\mu}.$$

Hence, using eq. (3.9) we get

$$F_1^{\sim}(s,x) = A_{11}(s)\exp(-u_1(s)x) + A_{12}(s)\exp(-u_2(s)x),$$

$$F_2^{\sim}(s,x) = A_{21}(s)exp(-u_1(s)x) + A_{22}(s)exp(-u_2(s)x)$$

where

$$u_1(s) = [r_1(s+\mu')+r_2(s+\lambda')+\sqrt{(r_1(s+\mu')-r_2(s+\lambda'))^2+4\lambda\mu r_1 r_2}]/(2r_1r_2)$$

$$u_2(s) = [r_1(s+\mu')+r_2(s+\lambda')-\sqrt{(r_1(s+\mu')-r_2(s+\lambda'))^2+4\lambda\mu r_1 r_2}]/(2r_1r_2)$$

$$A_{11}(s) = (r_1(s+\mu') + r_2\lambda - r_1r_2u_1(s))/[(u_2(s) - u_1(s))r_1r_2]$$

$$A_{12}(s) = (r_1(s+\mu') + r_2\lambda - r_1r_2u_2(s))/[(u_1(s) - u_2(s))r_1r_2]$$

$$A_{21}(s) = (r_2(s+\lambda') + r_1\mu - r_1r_2u_1(s))/[(u_2(s) - u_1(s))r_1r_2]$$

$$A_{22}(s) = (r_2(s+\lambda') + r_1\mu - r_1r_2u_2(s))/[(u_1(s) - u_2(s))r_1r_2]$$

Then

$$F^{\sim}(s) = [\pi_1A_{11}(s)+\pi_2A_{21}(s)]G^{\sim}(u_1(s))+[\pi_1A_{12}(s)+\pi_2A_{22}(s)]G^{\sim}(u_2(s)) ..$$

And η, the probability of dynamic failure, is given by $1 - F^{\sim}(0)$.

From corollary 1, we have

$$Y_1^{\sim *}(u,s) = \frac{1}{s}[1 - u\ F_1^{\sim *}(s,u)]$$

$$= \frac{(s+\lambda')(s+\mu') + r_2u(s+\lambda_0) - \lambda\mu}{s[(s+\lambda'+r_1u)(s+\mu'+r_2u)-\lambda\mu]}$$

$$= \frac{B_{10}(u)}{s} + \frac{B_{11}(u)}{s+s_1(u)} + \frac{B_{12}(u)}{s+s_2(u)}$$

where

$$s_1(u) = \frac{1}{2}[(\lambda' + r_1u + \mu' + r_2u) + \sqrt{(\lambda'+r_1u-\mu'-r_2u)^2 + 4\lambda\mu}]$$

$$s_2(u) = \frac{1}{2}[(\lambda' + r_1u + \mu' + r_2u) - \sqrt{(\lambda'+r_1u-\mu'-r_2u)^2 + 4\lambda\mu}]$$

$$B_{10}(u) = \frac{\lambda'\mu' + r_2u\lambda_0 - \lambda\mu}{s_1(u)s_2(u)}$$

$$B_{11}(u) = \frac{(\lambda'-s_1(u))(\mu'-s_1(u)) + r_2u(\lambda_0-s_1(u)) - \lambda\mu}{s_1(u)[s_1(u)-s_2(u)]}$$

$$B_{12}(u) = \frac{(\lambda'-s_2(u))(\mu'-s_2(u)) + r_2u(\lambda_0-s_2(u)) - \lambda\mu}{s_2(u)[s_2(u)-s_1(u)]}$$

Inverting with respect to s, yields

$$Y_1^{\sim}(u,t) = B_{10}(u) + B_{11}(u)e^{-s_1(u)t} + B_{12}(u)e^{-s_2(u)t}.$$

In a similar manner we can compute $Y_2^{\sim}(u,t)$. We note that the above *LST*s can be inverted in this case to obtain the distribution function of $Y(t)$ as an infinite sum of Bessel functions owing to the occurrences of radicals in the expresions of $s_1(u)$ and $s_2(u)$. However, in the case that μ=0 (as considered by Meyer), the radicals disappear and the inversion is relatively easy (as derived in [4] for arbitrary number of processors).

4. *The Preemptive-repeat-identical Case.*

In this section we assume that all states are preemptive repeat-identical. The main result is given in the following:

Theorem 3. The conditional *LST*s $F_i^{\sim}(s,x)$, $1 \le i \le n$ as defined in equation (2.2) satisfy the following simultaneous equations:

$$F_i^{\sim}(s,x) = e^{-(s+q_i)x/r_i} + \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{(s+q_i)}(1-e^{-(s+q_i)x/r_i})F_j^{\sim}(s,x), \; 1 \le i \le n. \qquad (4.1)$$

Proof: Conditioning on the holding time H in the initial state we have

$$E(e^{-sT} \mid H=h, B=x, Z(0)=i) = \begin{cases} e^{-sx/r_i} & ,\text{if } h \ge x/r_i \\ e^{-sh} \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{q_{ij}}{q_i} F_j^{\sim}(s,x), & \text{if } h < x/r_i \end{cases}$$

Unconditioning yields equation (4.1). *Q.E.D.*

Solving equations (4.1) we get $F_i^{\sim}(s,x)$, for $1 \le i \le n$. Then equations (2.7) and (2.8) can be used to compute $F^{\sim}(s)$. Finally, $\eta = 1 - F^{\sim}(0)$.

Example 4.1. Consider the switching server of example 3.1, except now we assume that states 1 and 2 are preemptive-repeat-identical.

Equations (4.1) become:

$$(s+\lambda')F_1^{\sim}(s,x) = (s+\lambda')e^{-(s+\lambda')x/r_1} + \lambda(1-e^{-(s+\lambda')x/r_1})F_2^{\sim}(s,x)$$

$$(s+\mu')F_2^{\sim}(s,x) = (s+\mu')e^{-(s+\mu')x/r_2} + \mu(1-e^{-(s+\mu')x/r_2})F_1^{\sim}(s,x).$$

Solving the above equations we get

$$F_1^{\sim}(s,x) = (s+\mu')[s+\lambda')a + \lambda(1-a)b]/\Delta$$

$$F_2^{\sim}(s,x) = (s+\lambda')[s+\mu')b + \mu(1-b)a]/\Delta$$

where $a = exp(-(s+\lambda')x/r_1)$, $b = exp(-(s+\mu')x/r_2)$ and $\Delta = (s+\lambda')(s+\mu') - \lambda\mu(1-a)(1-b)$. $F_i^{\sim}(s)$, for i = 1,2 and $F^{\sim}(s)$ can be obtained from equations (2.7) and (2.8).

5. *The Preemptive-repeat-different Case*

Here we consider the case, where all structure-states of the process are preemptive-repeat-different (*prd*).

The following theorem holds

Theorem 4. The *LST*s $F_i^{\sim}(s)$, for $1 \le i \le n$, as defined in equation (2.4) satisfy the following simultaneous equations

$$F_i^{\sim}(s) = G^{\sim}((s+q_i)/r_i) + \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{q_{ij}}{(s+q_i)}[1-G^{\sim}((s+q_i)/r_i)]F_j^{\sim}(s), \quad 1 \le i \le n. \tag{5.1}$$

Note that when $r_i \to 0$, $G^{\sim}((s+q_i)/r_i) \to 0$, since G (0+) = 0 and hence $\lim_{s\to\infty} G^{\sim}(s) \to 0$.

Proof: Conditioning on the work requirement B of the job to be executed and on the holding time H in the initial state we get

$$E(e^{-sT} \mid B=x, H=h, Z(0)=i) = \begin{cases} e^{-sx/r_i}, & \text{if } h \ge x/r_i \\ e^{-sh} \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{q_{ij}}{q_i} F_j^{\sim}(s), & \text{if } h < x/r_i \end{cases}$$

Note that if a structure state transition occurs before the job is completed then a different job with independent and identical distribution is restarted.

Now, unconditioning on B (the job's work requirement) yields

$$E(e^{-sT} \mid H=h, Z(0)=i) = \int_{x=0}^{r_i h} e^{-sx/r_i}\, dG(x) + \int_{x=r_i h}^{\infty} e^{-sh} \sum_{\substack{j=1 \\ j\ne i}}^{n} \frac{q_{ij}}{q_i} F_j^{\sim}(s)\, dG(x)$$

Unconditioning on H (the holding time in the initial state), yields equation (5.1). *Q.E.D.*

Solving equations (5.1) we get $F_i^{\sim}(s)$, for $1 \le i \le n$. Equation (2.8) can be used to get $F^{\sim}(s)$. The dynamic failure probability (η) follows immediately

$$\eta = P(T=\infty) = 1 - F^{\sim}(0).$$

Note that the preemptive-repeat-different case with a constant (or deterministic) work requirement of a job ($B=x$) corresponds to the preemptive-repeat-identical case.

Example 5.1. Again we consider the switching server of example 3.1 with the states 1 and 2 being preemptive- repeat-different. From equations (5.1) we have

$$F_1^{\sim}(s) = G^{\sim}((s+\lambda')/r_1) + \frac{\lambda}{(s+\lambda')}[1 - G^{\sim}((s+\lambda')/r_1)]F_2^{\sim}(s)$$

$$F_2^{\sim}(s) = G^{\sim}((s+\mu')/r_2) + \frac{\mu}{(s+\mu')}[1 - G^{\sim}((s+\mu')//r_2)]F_1^{\sim}(s)$$

It follows that

$$F_1^{\sim}(s) = \frac{G^{\sim}((s+\lambda')/r_1) + (\frac{\lambda}{s+\lambda'})(1 - G^{\sim}((s+\lambda')/r_1))G^{\sim}((s+\mu')/r_2)}{[1 - (\frac{\lambda}{s+\lambda'})(\frac{\mu}{s+\mu'})(1 - G^{\sim}((s+\lambda')/r_1))(1 - G^{\sim}((s+\mu')/r_2))]}$$

$$F_2^\sim(s) = \frac{G^\sim((s+\mu')/r_2)+(\frac{\mu}{s+\lambda'})(1-G^\sim((s+\mu')/r_2))G^\sim((s+\lambda')/r_1)}{[1-(\frac{\lambda}{s+\lambda'})(\frac{\mu}{s+\mu'})(1-G^\sim((s+\lambda')/r_1))(1-G^\sim((s+\mu')/r_2))]}$$

$F^\sim(s)$ can be obtained from equation (2.8).

6. *Conclusions and Extensions*

We have developed a unified model for the combined evaluation of performance and reliability of multi-mode computer systems. This allows us to compute both system-oriented measures (such as the accumulated reward) and task-oriented measures (such as the completion time and the dynamic failure probability) from a single model. We model preemptive-resume and preemptive-repeat interactions between task execution and mode change (failure/repair) events. It is clearly of interest to allow mixed preemptive-resume and preemptive-repeat interactions in the same model. This and other extensions have been studied and reported recently [8]. The techniques developed in this paper can be exetended to the case where the structure-state process is a semi-Markov process.

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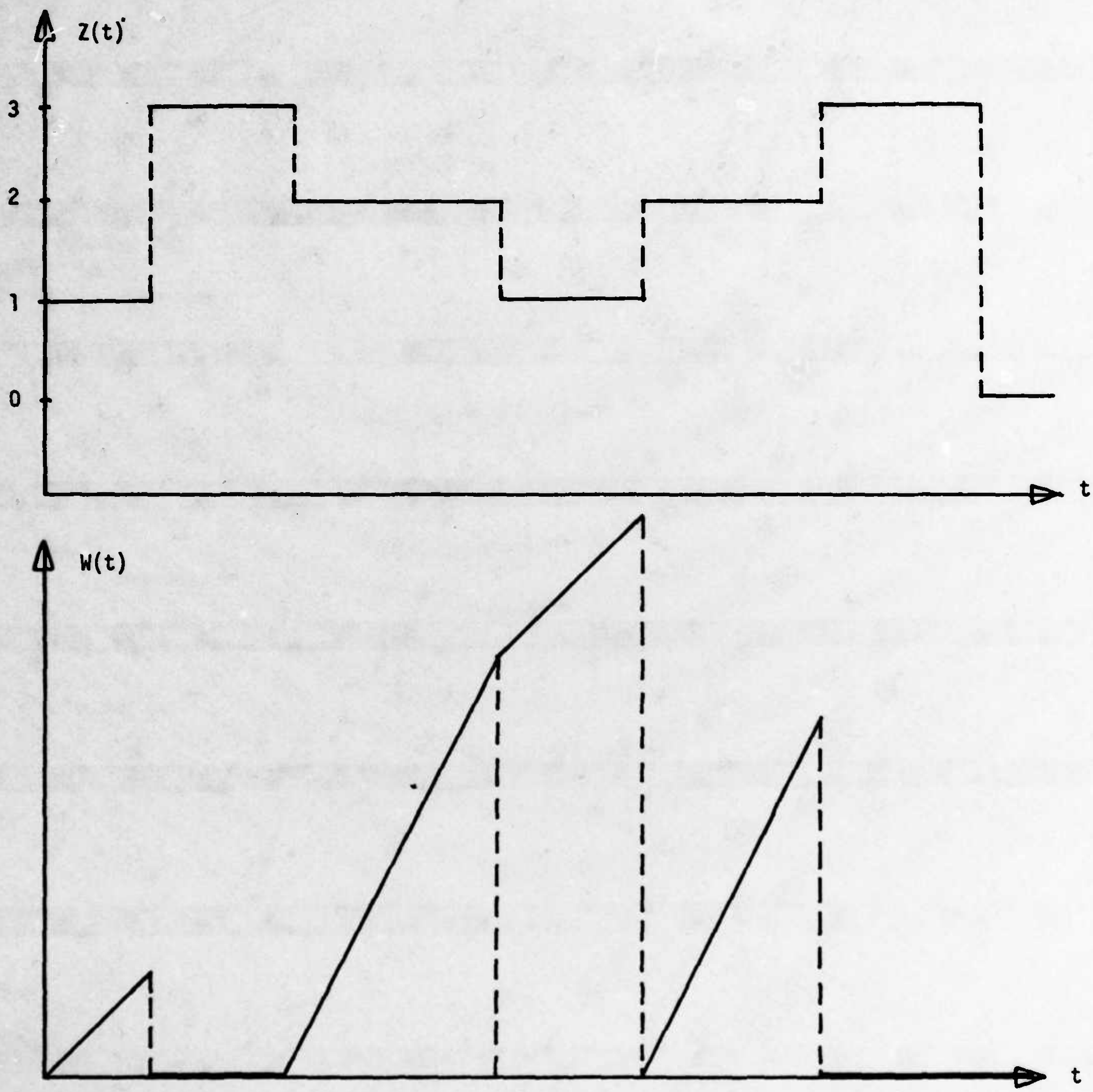

Typical Sample Paths of Z(t) and W(t)

Figure 1

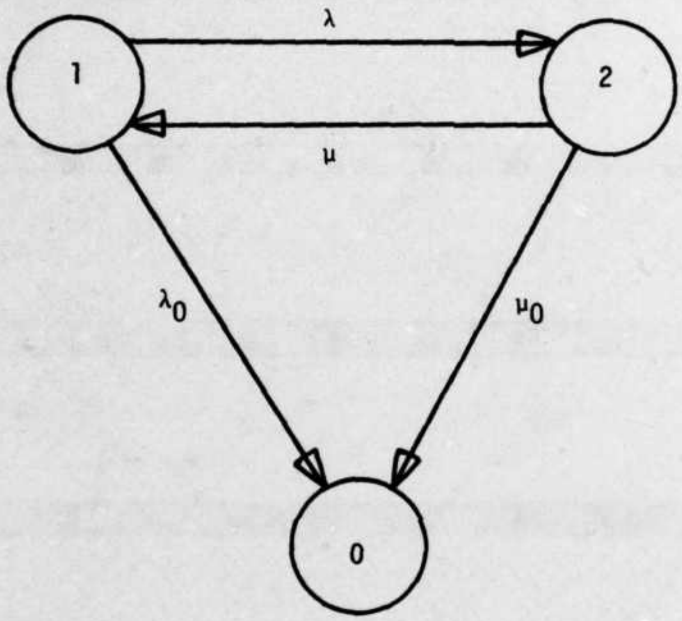

Figure 2

The Switching Server